On the Kamke-Müller conditions, monotonicity and continuity for bi-modal piecewise-smooth systems

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Abstract

We show that the Kamke-Müller conditions for bimodal piecewise-smooth systems are equivalent to simple conditions on the vector fields defining the system. As a consequence, we show that for a specific class of such systems, monotonicity is equivalent to continuity. Furthermore, we apply our results to derive a stability condition for piecewise positive linear systems.

Keywords: Piecewise-smooth systems; monotone systems; positive systems stability; Kamke-Müller conditions.

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1. Introduction

Systems with non-smooth or piecewise smooth dynamics are of importance in a variety of applications, ranging from the mechanics of dry friction and impacting systems to models of pacemaker cells in the heart [7]. On the other hand, Positive Systems, in which the state variables are constrained to remain nonnegative given nonnegative initial conditions, arise in Ecology, Economics, Biology and Communications. For an overview of the theory of positive linear time-invariant (LTI) systems, see the monograph [6]. Driven by practical considerations, there has recently been considerable activity aimed at extending this theory to more general classes of positive systems; particularly to positive nonlinear systems [5] and switched positive linear systems [2, 3, 4].
Positive LTI systems possess a number of special properties: one such property is that they are naturally order-preserving or monotone [9]. A monotone dynamical system on a normed vector space equipped with a partial order is one for

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which an ordering of the initial states is preserved throughout the evolution of the system (we provide formal definitions of monotone systems in our context in the next section). In a sense, monotone dynamical systems provide a nonlinear generalisation of positive linear systems and they arise as mathematical models in Biology, Ecology and Epidemiology [9, 8]. The monotonicity of a system has strong implications for its long-term dynamics, and monotone systems possess many desirable properties, particularly in regard to convergence to equilibria [9].

In the recent paper [12], conditions for monotonicity for a class of piecewise positive systems that arises in the modelling of gene regulatory networks were presented. The system class considered in [12] was piecewise affine with the state partition defined by affine hyperplanes. We shall derive related conditions here for bi-modal nonlinear systems with partitions defined by nonlinear surfaces.

As discussed in [12], piecewise monotone systems arise in the study of genetic regulatory networks and in traffic control. Results characterising the monotonicity of such systems may provide insight into their behaviour and guide the design of control strategies. Furthermore, monotone systems frequently arise in population and epidemic dynamics [9]. It is reasonable in many situations to assume that the parameters of such systems may vary depending on the state of the system; for instance in the case of epidemic models, this could be due to government intervention and public health policy. Piecewise systems whose local dynamics are monotone provide a potential modelling framework for these scenarios. It should also be noted that related results for piecewise linear Neural Networks have recently appeared in [13].

The structure of the paper is as follows. We formally define the system class under study in Section 2. Essentially, we consider a piecewise smooth nonlinear system with two component systems, each of which is monotone within its own domain of definition. The primary concern in the paper is with the question of when such a system will itself be monotone. Motivated by the work of [12], we introduce piecewise Kamke-Müller (PKM) conditions in Section 3 and provide a simple characterisation of these in Theorem 3.1. We then use this result to highlight a connection between monotonicity and continuity for a class of piecewise smooth systems in Section 4. In Section 5, our results are applied to derive a stability result for bi-modal piecewise linear positive systems. Finally, in Section 6, we present our concluding remarks.

2. Preliminaries

Throughout the paper, $\mathbb{R}$ and $\mathbb{R}^n$ denote the field of real numbers and the vector space of all n-tuples of real numbers, respectively. $\mathbb{R}^{n\times n}$ denotes the space of $n \times n$ matrices with real entries. For $x \in \mathbb{R}^n$ and $i = 1, \dots, n$, x_i denotes the i^{th} coordinate of x. Similarly, for $A \in \mathbb{R}^{n\times n}$, a_{ij} denotes the $(i, j)^{th}$ entry of A. $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n\}$. For vectors $x, y \in \mathbb{R}^n$, we write: $x \geq y$ if $x_i \geq y_i$ for $1 \leq i \leq n$; $x > y$ if $x \geq y$ and $x \neq y$; $x \gg y$ if $x_i > y_i, 1 \leq i \leq n$. A^T denotes the transpose of the matrix A, and the notation A^{-T} denotes the inverse of A^T when A^T is non-singular.

$\{e_1, \ldots, e_n\}$ denotes the standard basis vectors of $\mathbb{R}^n$.
A real $n \times n$ matrix $A = (a_{ij})$ is *Metzler* if and only if its off-diagonal entries a_{ij}, $i \neq j$ are nonnegative.
A matrix A is *Hurwitz* if all of its eigenvalues lie in the open left half plane.
Throughout, $\|.\|$ denotes the infinity norm on $\mathbb{R}^n$, $\|x\| = \max\{|x_i| : 1 \leq i \leq n\}$.
For a subset X of $\mathbb{R}^n$, the notation $\overline{X}$ denotes the topological closure of X while $\text{int}(X)$ denotes its interior.
For a C^1 mapping $\phi : \mathbb{R}^n \to \mathbb{R}$, $\nabla\phi$ denotes the gradient of ϕ.

Monotone systems
Let a C^1 vector field $f : \mathcal{D} \to \mathbb{R}^n$ defined on an open set $\mathcal{D} \subseteq \mathbb{R}^n$ be given and consider the associated dynamic system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x^{(0)} \in \mathcal{D}. \tag{1}$$

As f is C^1, for any $x^{(0)} \in \mathcal{D}$ there exists a unique solution $x(t, x^{(0)})$ of (1) defined on a maximal interval $I(x^{(0)}) := [0, T_{max}(x^{(0)}))$. The system (1) is said to be *monotone* if for any $x^{(0)}, y^{(0)}$ in $\mathcal{D}$ satisfying $x^{(0)} \leq y^{(0)}$, it follows that $x(t, x^{(0)}) \leq x(t, y^{(0)})$ for all $t \in I(x^{(0)}) \cap I(y^{(0)})$.
The vector field f is said to satisfy the Kamke-Müller (KM) conditions if for each $i \in \{1, \ldots n\}$, x, y in $\mathcal{D}$, $x \leq y$, and $x_i = y_i$ implies that $f_i(x) \leq f_i(y)$. The system (1) is monotone if and only if f satisfies the KM conditions on $\mathcal{D}$ [9]. Furthermore, if the domain $\mathcal{D}$ is convex, then the KM conditions are equivalent to the requirement that

$$\frac{\partial f_i}{\partial x_j}(a) \geq 0 \ \text{ for all } i \neq j \quad \text{and all } a \in \mathcal{D}. \tag{2}$$

A vector field satisfying (2) is said to be cooperative on $\mathcal{D}$.

Piecewise-smooth systems
For the remainder of the paper, the following notation is adopted. $\phi : \mathcal{D} \to \mathbb{R}$ is a C^2 function. The sets D_f and D_g are defined by

$$D_f = \{x \in \mathcal{D} : \phi(x) < 0\} \quad D_g = \{x \in \mathcal{D} : \phi(x) > 0\}.$$

$f : U_f \to \mathbb{R}^n$, $g : U_g \to \mathbb{R}^n$ are C^1 vector fields defined on open neighbourhoods $U_f \subseteq \mathcal{D}$, $U_g \subseteq \mathcal{D}$ of $\overline{D_f}$ and $\overline{D_g}$ respectively.

$$S := \{x \in \mathcal{D} : \phi(x) = 0\}$$

denotes the surface separating D_f and D_g.
Our primary concern throughout is with the piecewise-smooth system

$$\dot{x}(t) = \begin{cases} f(x) & \text{if } x \in D_f \\ g(x) & \text{if } x \in \overline{D_g}. \end{cases} \tag{3}$$

Throughout the paper, we make the following technical assumption on the function ϕ defining the surface S.

Assumption A: For any $a \in S$, there exist two distinct indices i, j in $\{1, \dots, n\}$ such that $(\nabla\phi(a))_i \neq 0$, $(\nabla\phi(a))_j \neq 0$.

Remark: Intuitively the above assumption requires that the tangent vector to S at any point is not parallel to an axis. As an illustration, we describe two simple examples of the type of surface that will satisfy Assumption A.

(i) If $\mathcal{D} = \text{int}(\mathbb{R}^3_+)$, then

$$\phi(x_1, x_2, x_3) = x_1 - 2x_2^2 + x_3^{4/3}$$

satisfies Assumption A.

(ii) For $\mathcal{D} = \mathbb{R}^n$, a linear functional $\phi(x) = c^T x$ satisfies Assumption A if and only if the vector c has at least two non-zero components.

The right hand side of (3) may be discontinuous, which leads to subtle difficulties concerning the existence and uniqueness of solutions. In general, it is necessary to consider solutions to (3) in the sense of Filippov [1]. The equation (3) is replaced with a differential inclusion

$$\dot{x}(t) \in F(x(t))$$

where, in our situation the set $F(x)$ is given by

$$F(x) = \begin{cases} \{f(x)\} & \text{if } x \in D_f \\ \{g(x)\} & \text{if } x \in D_g \\ \{\alpha f(x) + (1-\alpha)g(x) : \alpha \in [0,1]\} & \text{if } x \in S. \end{cases}$$

It follows from the remarks after Theorem 4 on page 81 of [1], that for any $x^{(0)}$ in $\mathcal{D}$, there exists a solution $x(t, x^{(0)})$ of (3) in the sense of Filippov satisfying $x(0, x^{(0)}) = x^{(0)}$.
In order for a *unique* solution to exist, further restriction must be placed on the vector fields defining (3). If, for all $a \in S$, either $(\nabla\phi(a))^T f(a) > 0$ or $(\nabla\phi(a))^T g(a) < 0$, then it follows from Theorem 2 on page 110 of [1] that there exists a unique solution $x(t, x^{(0)})$ to (3) satisfying $x(0, x^{(0)}) = x^{(0)}$ for all $x^{(0)} \in \mathcal{D}$.

Monotonicity concepts for piecewise-smooth systems
In the recent paper [12], two types of monotonicity were introduced for a class of piecewise affine systems that arise in the study of genetic regulatory networks. The definitions given in [12] only required that the system preserved order for *almost all* initial conditions. Formally, the system (3) is said to be monotone almost everywhere if there exists a set $N \subset \mathcal{D}$ of measure zero such that for all $x^{(0)}, y^{(0)}$ in $\mathcal{D} \backslash N$ with $x^{(0)} \leq y^{(0)}$, $x(t, x^{(0)}) \leq x(t, y^{(0)})$ for all $t \in I(x^{(0)}) \cap I(y^{(0)})$.
The reason given in [12] for requiring the order preserving property to hold only for almost all initial conditions is that it is possible for (3) to admit non-unique

solutions for some initial conditions. We use slightly different definitions of monotonicity here. First, we define what is to be understood by local monotonicity.

Definition 2.1. *The system (3) is locally monotone if for all* $x^{(0)}, y^{(0)}$ *in* $\mathcal{D}\backslash S$ *with* $x^{(0)} \leq y^{(0)}$*, there exists some* $\delta > 0$ *such that* $x(t, x^{(0)}) \leq x(t, y^{(0)})$ *for all* $t \in [0, \delta]$.

Note that for any $x^{(0)} \in D_f$ ($y^{(0)} \in D_g$), there is some $\delta > 0$ such that the Filippov solution of (3) coincides with the solution of the (C^1) differential equation $\dot{x} = f(x)$, $x(0) = x^{(0)}$ ($\dot{x} = g(x)$, $x(0) = y^{(0)}$). Hence, for initial conditions in $\mathcal{D}\backslash S$, solutions of (3) are locally unique and Definition 2.1 makes sense.
We also consider a stronger form of monotonicity for (3) than that of local monotonicity.

Definition 2.2. *The system (3) is monotone if:*

(i) for all $x^{(0)} \in \mathcal{D}$*, there exists a unique solution* $x(t, x^{(0)})$ *of (3) defined on a maximal interval of existence* $I(x^{(0)})$*;*

(ii) for all $x^{(0)}, y^{(0)}$ *in* $\mathcal{D}$ *with* $x^{(0)} \leq y^{(0)}$*,* $x(t, x^{(0)}) \leq x(t, y^{(0)})$ *for all* $t \in I(x^{(0)}) \cap I(y^{(0)})$*.*

We have incorporated the uniqueness of solutions into the definition of monotonicity. We shall show in Section 4 that for a significant class of systems of the form (3), local monotonicity is equivalent to monotonicity.

3. The Piecewise-Kamke-Müller (PKM) conditions

It is clear that in order for the system (3) to be monotone in either the sense of Definition 2.2 or Definition 2.1, it must satisfy the following:

(i) if x, y are in D_f with $x \leq y$ and $x_i = y_i$ for some i, then $f_i(x) \leq f_i(y)$;

(ii) if x, y are in D_g with $x \leq y$ and $x_i = y_i$ for some i, then $g_i(x) \leq g_i(y)$.

It follows from (i) and (ii) that a necessary condition for (3) to be monotone is that f and g are cooperative on D_f, D_g respectively. For this reason, from now on we assume that f and g are cooperative on U_f, U_g respectively.
Inspired by Hypothesis 1 in [12], we introduce the following Piecewise Kamke-Müller (PKM) conditions for the system (3), where we are assuming that f and g are cooperative on U_f, U_g respectively.

PKM Conditions:

(i) For $x \in D_f$, $y \in D_g$ with $x \leq y$ and $x_i = y_i$ for some i, we have $f_i(x) \leq g_i(y)$;

(ii) for $x \in D_g$, $y \in D_f$ with $x \le y$ and $x_i = y_i$ for some i, we have $g_i(x) \le f_i(y)$.

When (i) and (ii) hold, we say that the system (3) satisfies the PKM conditions. We shall characterise the PKM conditions in terms of the values of the vector fields f, g along the separating surface $S = \{x \in \mathcal{D} : \phi(x) = 0\}$. Later in Section 4, we shall use this characterisation to highlight that for certain classes of surface S, the monotonicity of (3) is equivalent to the continuity of its right hand side.
We associate three subsets $I_0(a)$, $I_+(a)$, $I_-(a)$ of $\{1, \dots, n\}$ with each $a \in S$ as follows.

$$I_+(a) := \{i : (\nabla\phi(a))_j \ge 0, \ \forall j \ne i\};$$

$$I_-(a) := \{i : (\nabla\phi(a))_j \le 0, \ \forall j \ne i\};$$

$$I_0(a) := \{1, \dots n\} \backslash (I_+(a) \cup I_-(a)).$$

We can now state the main result of this section.

Theorem 3.1. *Consider the system (3) and assume that f and g are cooperative on U_f and U_g respectively. Then (3) satisfies the PKM conditions if and only if the following three statements hold for all $a \in S$:*

$$\begin{array}{rcll} f_i(a) & = & g_i(a) & \forall i \in I_0(a); \\ f_i(a) & \le & g_i(a) & \forall i \in I_+(a); \\ f_i(a) & \ge & g_i(a) & \forall i \in I_-(a). \end{array} \tag{4}$$

Proof:
PKM $\Rightarrow$ (4):
Assume that (3) satisfies the PKM conditions. Let $a \in S$ and $i \in I_+(a)$ be given. Then $\frac{\partial \phi}{\partial x_j}(a) \ge 0$ for all $j \ne i$. Furthermore, it follows from Assumption A that there exists some $j \ne i$ for which $\frac{\partial \phi}{\partial x_j}(a) > 0$. This implies that for $s > 0$ sufficiently small, we have

$$\phi(a + se_j) > 0, \ \ \phi(a - se_j) < 0.$$

In particular, this means that $a - se_j \in D_f$, $a + se_j \in D_g$. Clearly

$$a - se_j \le a + se_j$$

and $(a - se_j)_i = (a + se_j)_i$ and hence from the PKM conditions, we must have

$$f_i(a - se_j) \le g_i(a + se_j).$$

As f, g are continuous on neighbourhoods of D_f, D_g respectively, it follows that $f_i(a) \le g_i(a)$.

If $i \in I_-(a)$, it follows similarly that there exists some index $j \neq i$ such that for sufficiently small $s > 0$, $a - se_j \in D_g$, $a + se_j \in D_f$. The same argument as above shows that $g_i(a) \leq f_i(a)$.
Finally, suppose that $i \in I_0(a)$. As i is not in $I_+(a)$, there exists some index $j \neq i$ for which $\frac{\partial \phi}{\partial x_j}(a) < 0$. Similarly, as i is not in $I_-(a)$, it follows that there exists some $k \neq i$ for which $\frac{\partial \phi}{\partial x_k}(a) > 0$. The arguments given above then imply that $f_i(a) \leq g_i(a)$ and $g_i(a) \leq f_i(a)$. It follows that $f_i(a) = g_i(a)$.

(4) ⇒ PKM:
Conversely, assume that (4) holds. Suppose that $x \in D_f$, $y \in D_g$, $x \leq y$ and that $x_i = y_i$ for some $i \in \{1, \ldots, n\}$. Hence, $\phi(x) < 0$, $\phi(y) > 0$ and there exists some $\alpha \in [0, 1]$ such that $\phi(\alpha y + (1 - \alpha)x) = 0$.
Define

$$\alpha_0 := \sup\{\alpha \in [0, 1] : \phi(\alpha y + (1 - \alpha)x) = 0\}$$

and put $a = \alpha_0 y + (1 - \alpha_0)x$. If there are only finitely many α satisfying $\phi(\alpha y + (1 - \alpha)x) = 0$, it follows trivially that $\phi(a) = 0$. If infinitely many such α exist, then it follows from the continuity of ϕ that $\phi(a) = 0$.
From the definition of α_0 and the fact that $\phi(y) > 0$, it follows that for all $\alpha > \alpha_0$,

$$\alpha y + (1 - \alpha)x \in D_g. \tag{5}$$

We claim that $i \in I_0(a) \cup I_+(a)$. Otherwise, if $i \in I_-(a)$, it would follow that for small enough $t > 0$, we have $\phi(a + t(y - x)) < 0$ which contradicts (5). As $i \in I_0(a) \cup I_+(a)$, it follows from (a), (b) that $f_i(a) \leq g_i(a)$. However, f, g are both cooperative on neighbourhoods of D_f, D_g and $x \leq a \leq y$ with $x_i = a_i = y_i$. Hence, it follows that

$$f_i(x) \leq f_i(a) \leq g_i(a) \leq g_i(y). \tag{6}$$

It remains to show that if $x \in D_g$, $y \in D_f$, $x \leq y$ and $x_i = y_i$, then $g_i(x) \leq f_i(y)$. Define α_0 as above and once again write $a = \alpha_0 y + (1 - \alpha_0)x$. Then $\phi(a) = 0$ and $\phi(\alpha y + (1 - \alpha)x) < 0$ for $\alpha_0 < \alpha \leq 1$. By similar reasoning to above, this implies that $i \in I_0(a) \cup I_-(a)$ and hence from (a), (c), it follows that $f_i(a) \geq g_i(a)$. Combining this with the cooperativity of f, g on neighbourhoods of D_f, D_g respectively, we see that

$$g_i(x) \leq g_i(a) \leq f_i(a) \leq f_i(y). \tag{7}$$

This completes the proof.

Remark: It follows immediately from Theorem 3.1 that if f and g agree on the surface S, so that the right hand side of (3) is continuous, then the PKM conditions are satisfied.

4. Monotonicity and continuity

In the previous section, we introduced the PKM conditions for the system (3) (assuming f and g are cooperative on U_f, U_g respectively), and showed that

these are equivalent to the vector fields f, g satisfying (4). In this section, we examine some implications of Theorem 3.1.
We first note that the PKM conditions are necessary for the system (3) to be locally monotone.

Proposition 4.1. *If the system (3) is locally monotone, then it satisfies the PKM conditions.*

Proof: By way of contradiction, suppose that (3) does not satisfy the PKM conditions. We assume that there exists some $x^{(0)} \in D_f$, $y^{(0)} \in D_g$ with $x^{(0)} \le y^{(0)}$, $x_i^{(0)} = y_i^{(0)}$ but $f_i(x^{(0)}) > g_i(y^{(0)})$. (The case where $x^{(0)} \in D_g$, $y^{(0)} \in D_f$ with $x^{(0)} \le y^{(0)}$, $x_i^{(0)} = y_i^{(0)}$ but $g_i(x^{(0)}) > f_i(y^{(0)})$ is identical.) It follows that the solutions $x(t, x^{(0)})$, $x(t, y^{(0)})$ of (3) satisfy

$$\frac{d}{dt} x_i(t, x^{(0)})\Big|_{t=0} > \frac{d}{dt} x_i(t, y^{(0)})\Big|_{t=0}. \tag{8}$$

Furthermore, as D_f, D_g are open, it follows that $x(t, x^{(0)})$, $x(t, y^{(0)})$ are C^1 for sufficiently small $t > 0$. It follows immediately from (8) that $x_i(t, x^{(0)}) > x_i(t, y^{(0)})$ for t in some interval $(0, \delta)$, which contradicts the monotonicity of (3). This completes the proof.

It is a simple consequence of Theorem 3.1 that the PKM conditions are automatically satisfied if $f(a) = g(a)$ for all $a \in S$. In this case, the right hand side of (3) is continuous on $\mathcal{D}$. In fact, it satisfies a local Lipschitz condition so that classical existence and uniqueness results for ordinary differential equations can be applied.

Proposition 4.2. *Consider the system (3). Define $h : \mathcal{D} \to \mathbb{R}^n$ by*

$$h(x) = \begin{cases} f(x) & \text{if } x \in D_f \\ g(x) & \text{if } x \in \overline{D_g}. \end{cases} \tag{9}$$

Suppose that $f(a) = g(a)$ for all $a \in S$. Then for any $x \in \mathcal{D}$, there exists some neighbourhood N of x and a constant $K > 0$ such that for all y, z in N

$$\|h(y) - h(z)\| \le K\|y - z\|.$$

Proof: As f and g are C^1 and hence locally Lipschitz on neighbourhoods of $\overline{D_f}$, $\overline{D_g}$ respectively, the result is immediate if $x \in D_f$ or $x \in D_g$. So assume that $x \in S$. Then as f is locally Lipschitz on a neighbourhood of $\overline{D_f}$, there exists some convex neighbourhood N_f of x and some constant $K_f > 0$ such that for all y, z in N_f,

$$\|f(y) - f(z)\| \le K_f\|y - z\|. \tag{10}$$

Similarly, there exists some convex neighbourhood N_g of x and some constant $K_g > 0$ such that for all y, z in N_g,

$$\|g(y) - g(z)\| \le K_g\|y - z\|. \tag{11}$$

Set $N = N_f \cap N_g$ and $K = \max\{K_f, K_g\}$. Let y, z be any two points in N. Then if y, z are both in $\overline{D_f}$ or both in $\overline{D_g}$, it follows immediately from (10) and (11) and the continuity of h on S that $\|h(y) - h(z)\| \leq K\|y - z\|$. Assume without loss of generality that $y \in D_f$, $z \in D_g$. There exists some $\alpha \in (0, 1)$ such that $a := y + \alpha(z - y) \in S$. Further, as N is convex $a \in N$. Then from (10), (11) and the continuity of h, we know that

$$\begin{aligned} \|h(y) - h(z)\| &= \|f(y) - g(z)\| \qquad (12) \\ &\leq \|f(y) - f(a)\| + \|g(a) - g(z)\| \\ &\leq K\|y - a\| + K\|a - z\| \\ &= K\|y - z\|. \end{aligned}$$

This completes the proof.

The following result now shows that that when $f(a) = g(a)$ for all $a \in S$, the system (3) is monotone. The proof is a straightforward adaptation of the proof of Proposition 3.1.1 in [9], and for this reason we only highlight what is required for the argument given in [9] to work in this context.

Theorem 4.1. *Consider the system (3) where f and g are cooperative on U_f, U_g respectively. Suppose that $f(a) = g(a)$ for all $a \in S$. Then (3) is monotone in the sense of Definition 2.2*

Proof: Note the following points.

(i) From Proposition 4.2, the right hand side h of (3) is locally Lipschitz on $\mathcal{D}$. This guarantees the existence of classical solutions to (3); more specifically, for any $x^{(0)} \in \mathcal{D}$, there exists a unique C^1 solution $x(t, x^{(0)})$ such that $x(0, x^{(0)}) = x^{(0)}$ and $\frac{d}{dt}x(t, x^{(0)}) = h(x(t, x^{(0)}))$ for all t in its maximal interval of existence.

(ii) Theorem 3.1 implies that (3) satisfies the PKM conditions. As h is continuous this means that for all x, y in $\mathcal{D}$ with $x \leq y$, if $x_i = y_i$ for some $i \in \{1, \ldots, n\}$, we have $h_i(x) \leq h_i(y)$.

Points (i) and (ii) ensure that an identical argument to that used to prove Proposition 3.1.1 in [9] yields the present result.

Remarks: It is worth noting that point (i) in the above proof is important in order to apply the argument of [9]. For general systems of the form (3), classical solutions may not exist. For general Filippov solutions to (3), there is no guarantee that the solution satisfies the equation for all t on its interval of existence. Furthermore, in general the solution will not be C^1. The argument used in [9] establishes monotonicity by contradiction. The contradiction arises from comparing left and right derivatives of solutions, using the defining differential equation to evaluate the right derivative. If the solution need not satisfy the equation at all times, and moreover is not necessarily C^1 (so that left and right derivatives need not agree), no direct contradiction will arise in this manner.

Example 4.1. *Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be given by $\phi(x_1, x_2) = x_1 - 2x_2^2$. Let $\mathcal{D} = int(\mathbb{R}^2_+)$ and let f and g be given by*

$$f(x_1,x_2) = \begin{pmatrix} -\frac{1}{1+x_1} + x_2^2 \\ -\frac{1}{1+x_2} + x_1 \end{pmatrix}$$
$$g(x_1,x_2) = \begin{pmatrix} \frac{x_1^2+x_1-2}{2(1+x_1)} \\ -\frac{1}{1+x_2} + x_1 + x_1^2 - 2x_1x_2^2 \end{pmatrix}.$$

Define $D_f := \{x \in \mathcal{D} : \phi(x) < 0\}$, $D_g := \{x \in \mathcal{D} : \phi(x) > 0\}$. Then it is easy to verify that f and g are cooperative on neighbourhoods of D_f, D_g respectively and moreover that $f(a) = g(a)$ for all $a \in \mathcal{D}$ satisfying $\phi(a) = 0$. It follows from Theorem 4.1 that the piecewise smooth system

$$\dot{x} = \begin{cases} f(x) & x \in D_f \\ g(x) & x \in \overline{D_g} \end{cases}$$

is monotone.

The next result shows that for a class of systems of the form (3) monotonicity, local monotonicity and continuity are equivalent. The key property of this class relates to the separating surface S. For example, a system in $\mathbb{R}^4$ whose separating surface is defined by a hyperplane $\{x : c^T x = 0\}$, where c has 2 strictly positive and 2 strictly negative entries, will satisfy the conditions of the theorem.

Theorem 4.2. *Consider the system (3) where f and g are cooperative on U_f, U_g respectively. Assume that $I_0(a) = \{1, \dots, n\}$ for all a in the separating surface $S = \{x \in \mathcal{D} : \phi(x) = 0\}$. Then the following are equivalent:*

(a) (3) is locally monotone;

(b) $f(a) = g(a)$ for all $a \in S$;

(c) (3) is monotone.

Proof: (a) $\Rightarrow$ (b): If (3) is locally monotone, it follows from Proposition 4.1 that it satisfies the PKM conditions. Theorem 3.1 implies that for all $a \in S$, $i \in I_0(a)$ we must have $f_i(a) = g_i(a)$. By assumption $I_0(a) = \{1, \dots, n\}$ for all $a \in S$. Hence $f(a) = g(a)$ for all $a \in S$.
(b) $\Rightarrow$ (c): This is shown in Theorem 4.1
(c) $\Rightarrow$ (a): This is trivial.
Remark:
As noted in an example presented at the end of Section III in [12], for discontinuous vector fields, it is possible for systems such as (3) to admit multiple solutions for initial conditions on the separating surface S. This of course leads

to serious difficulties in the definition of monotonicity we have adopted. For example, consider $\mathcal{D} = \mathbb{R}^2$, $c = (1, -1)^T$, $f(x) = Ax$, $g(x) = Bx$ with

$$A = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} B = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}.$$

Then it is easy to see that there are multiple (Filippov) solutions corresponding to the initial condition $(1, 1)^T$. In the light of the difficulties with discontinuous vector fields, it is worth emphasising that the result of Theorem 4.2 establishes the equivalence of local monotonicity (which only considers initial conditions not on the separating surface) with continuity for separating surfaces satisfying the hypotheses of the theorem.

5. Monotonicity and stability of piecewise linear positive systems

In this section, we present an application of Theorem 4.2 to piecewise linear positive systems. Throughout the section, A, B are Metzler matrices in $\mathbb{R}^{n\times n}$ and c is a vector in $\mathbb{R}^n$ for which there are two distinct indices i, j in $\{1, \dots, n\}$ with $c_i \neq 0$, $c_j \neq 0$. We consider the piecewise linear system

$$\dot{x}(t) = \begin{cases} Ax & \text{if } c^T x < 0 \\ Bx & \text{if } c^T x \geq 0. \end{cases} \tag{13}$$

We alter our notation slightly from that employed in the previous three sections. Specifically, we use D_A, D_B and S to denote the sets

$$D_A := \{x \in \mathbb{R}^n_+ : c^T x < 0\}, \quad D_B := \{x \in \mathbb{R}^n_+ : c^T x > 0\}$$

$$S := \{x \in \mathbb{R}^n_+ : c^T x = 0\}.$$

Note that in order for both D_A and D_B to be non-empty, we require that at least one component of c is negative ($c_i < 0$ for some $i \in \{1, \dots, n\}$) and at least one component of c is positive ($c_j > 0$ for some $j \in \{1, \dots, n\}$). We shall make this assumption from now on.

As A and B are Metzler, the orthant $\mathbb{R}^n_+$ is invariant under (13). For this reason, the definitions of monotonicity and stability for (13) are understood with respect to the state space $\mathbb{R}^n_+$.

For (13) the mapping ϕ is given by $\phi(x) = c^T x$. For $a \in S$, the sets $I_0(a)$, $I_+(a)$, $I_-(a)$ are independent of a in this case and we simplify our notation accordingly.

$$I_+ := \{i \in \{1, \dots, n\} : c_j \geq 0 \ \forall j \neq i\};$$

$$I_- := \{i \in \{1, \dots, n\} : c_j \leq 0 \ \forall j \neq i\};$$

$$I_0 := \{1, \dots, n\} \backslash (I_+ \cup I_-).$$

The following result on monotonicity of (13) follows readily from Theorem 4.2.

Corollary 5.1. *Consider the system (13). If* $B = A + bc^T$ *for some vector* $b \in \mathbb{R}^n$*, then (13) is monotone. Furthermore, if (13) is monotone and* $I_0 = \{1, \ldots, n\}$*, then* $B = A + bc^T$ *for some vector* $b \in \mathbb{R}^n$*.*

Proof: If $B = A + bc^T$, then it follows immediately that $Ax = Bx$ for all x satisfying $c^T x = 0$. It follows from Theorem 4.1 that (13) is monotone.
Conversely, if (13) is monotone, then Theorem 4.2 implies that $Ax = Bx$ for all $x \in \mathbb{R}^n_+$ such that $c^T x = 0$. By linearity, it follows that this is true for all x with $c^T x = 0$. This implies that $B - A$ has rank 1 (or 0 in which case $A = B$ and we are done). As the kernel of $B - A$ coincides with $\{x : c^T x = 0\}$, it follows that $B - A = bc^T$ for some vector b. This concludes the proof.

Remarks: The preceding result shows that a piecewise linear positive system of the form (13) with $I_0 = \{1, \ldots, n\}$ is monotone if and only if the matrices A, B differ by a rank one matrix bc^T. In the remainder of this Section, we assume that $B = A + bc^T$ for some $b \in \mathbb{R}^n$. A simple adaptation of the proof of Proposition 4.2 will show that under this assumption, the right hand side of (13) is in fact globally Lipschitz, ensuring the existence and uniqueness of solutions for all $t \geq 0$.
We next record a simple fact concerning pairs of matrices of this form. This result was given in [14].

Lemma 5.1. *Let* $A \in \mathbb{R}^{n \times n}$*,* $B \in \mathbb{R}^{n \times n}$ *be Hurwitz matrices. Assume that there exist vectors,* $b, c \in \mathbb{R}^n$ *with* $B = A + bc^T$*. Then the matrix product* $A^{-1}B$ *has no negative real eigenvalues.*

Lemma 5.2. *Consider the system (13). Assume that* $B = A + bc^T$ *for some* $b \in \mathbb{R}^n$*. Suppose that* $Av \ll 0$ *for some* $v \in D_A$*. Then the solution* $x(t, v)$ *of (13) with* $x(0, v) = v$ *satisfies* $x(t, v) \leq v$ *for all* $t \geq 0$*. Similarly, if* $Bv \ll 0$ *for some* $v \in D_B$*, the solution* $x(t, v)$ *of (13) with* $x(0, v) = v$ *satisfies* $x(t, v) \leq v$ *for all* $t \geq 0$*.*

Proof: As $v \in D_A$, it is immediate that

$$\left. \frac{d}{dt} x(t, v) \right|_{t=0} \ll 0.$$

This implies that for some $\delta > 0$, $x(t, v) \ll v$ for $t \in (0, \delta]$. Corollary 5.1 implies that (13) is monotone. As $x(\delta, v) \ll v$, it follows that for $s \in (0, \delta]$,

$$x(\delta + s, v) = x(s, x(\delta, v)) \leq x(s, v) \ll v.$$

Hence, $x(t, v) \ll v$ for all $t \in (0, 2\delta]$. Iterating, we see that $x(t, v) \ll v$ for all $t \geq 0$. The proof for $v \in D_B$ with $Bv \ll 0$ is identical.
Corollary 5.2 below provides a sufficient condition for the origin to be a globally exponentially stable equilibrium of (13). We first recall the relevant definition.

Definition 5.1. *The origin is a globally exponentially stable equilibrium of (13) if there exist positive constants $K > 0$, $\alpha > 0$ such that*

$$\|x(t, x^{(0)})\| \leq Ke^{-\alpha t}\|x^{(0)}\|$$

for all $t \geq 0$.

Corollary 5.2. *Consider the system (13) and assume that $B = A + bc^T$ for some $b \in \mathbb{R}^n$. If there exists $v \in D_A$ with $Av \ll 0$ or $v \in D_B$ with $Bv \ll 0$, then the origin is a globally exponentially stable equilibrium of (13).*

Proof: Assume that there is some $v \in D_A$ with $Av \ll 0$ (the case $v \in D_B$ with $Bv \ll 0$ is identical). Then, it follows that there is some $\alpha > 0$ such that $(A+\alpha I)v \ll 0$. Define $\hat{A} = A+\alpha I$, $\hat{B} = B+\alpha I$ and for $x^{(0)} \in \mathbb{R}^n_+$, let $\hat{x}(t, x^{(0)})$ denote the solution of

$$\dot{x}(t) = \begin{cases} \hat{A}x & \text{if } x \in D_A \\ \hat{B}x & \text{if } x \in \overline{D_B} \end{cases}, \tag{14}$$

with $\hat{x}(0, x^{(0)}) = x^{(0)}$. Note that $\hat{A}$, $\hat{B}$ are still Metzler, and moreover $\hat{B} = \hat{A}+bc^T$ so that (14) is monotone. Furthermore, if $x(t, x^{(0)})$ denotes the solution of (13) with $x(0, x^{(0)}) = x^{(0)}$, then

$$\hat{x}(t, x^{(0)}) = e^{\alpha t}x(t, x^{(0)}). \tag{15}$$

Let $x^{(0)} \in \mathbb{R}^n_+$ be arbitrary. As A is Metzler and $v \geq 0$, $Av \ll 0$ implies that $v \gg 0$. Let v_{max}, v_{min} be the maximal and minimal components of v respectively. If we choose $\lambda = \frac{\|x^{(0)}\|}{v_{min}}$, then $x^{(0)} \leq \lambda v$, $\lambda v \in D_A$ and $\hat{A}(\lambda v) \ll 0$ by our choice of α. It follows from Lemma 5.2 applied to (14) that $\hat{x}(t, x^{(0)}) \ll \lambda v$ for $t \geq 0$. This implies that

$$\|\hat{x}(t, x^{(0)})\| \leq \|\lambda v\| = \frac{v_{max}}{v_{min}}\|x^{(0)}\|. \tag{16}$$

Combining (16) with (15), we see that

$$\|x(t, x^{(0)})\| \leq Ke^{-\alpha t}\|x^{(0)}\|$$

with $K = \frac{v_{max}}{v_{min}}$. This completes the proof.

In the proof of Theorem 5.1 below, we will make use of the following two known facts; the former of which concerns the separation of convex cones in $\mathbb{R}^n$.

Proposition 5.1. *[10] Let C_1, C_2 be non-empty convex cones in $\mathbb{R}^n$. Further, assume that $C_1 \cap C_2$ is empty. Then there exists a non-zero vector $v \in \mathbb{R}^n$ such that*

$$\begin{aligned} v^T x &\geq 0 \text{ for all } x \in C_1, \\ v^T x &\leq 0 \text{ for all } x \in C_2. \end{aligned}$$

Lemma 5.3. *[11] Let $A \in \mathbb{R}^{n \times n}$ be Metzler and Hurwitz. Then there exists no non-zero vector $w \in \mathbb{R}^n$ with $w \geq 0$, $Aw \geq 0$.*

Our next result shows that provided both A and $B = A + bc^T$ are Hurwitz, the piecewise linear system (13) has a globally exponentially stable equilibrium at the origin.

Theorem 5.1. *Consider the piecewise linear system (13) and assume that $B = A + bc^T$. If A and B are Hurwitz, then the origin is a globally exponentially stable equilibrium of (13).*

Proof: The result follows from Corollary 5.2 if there exists some $v \in D_A$ with $Av \ll 0$ or some $v \in D_B$ with $Bv \ll 0$. We shall show that at least one of these conditions must be true. By way of contradiction, suppose that:

(i) there is no $v \in D_A$ with $Av \ll 0$;

(ii) there is no $v \in D_B$ with $Bv \ll 0$.

It follows from (i) that there exists no $v \gg 0$ with $Av \ll 0$, $c^T v < 0$. Hence the cone in $\mathbb{R}^{n+1}$ given by

$$C_A := \{ \begin{pmatrix} A \\ c^T \end{pmatrix} v : v \in \mathbb{R}^n,\ v \gg 0\}$$

and the cone

$$P_{n+1} := \{x \in \mathbb{R}^{n+1} : x \ll 0\}$$

are disjoint. It follows from Proposition 5.1 that there exists some non-zero vector $w' \in \mathbb{R}^{n+1}$ with $w'^T = (w_1'^T, t_1)^T$ such that

$$w'^T x \leq 0 \quad \forall x \in P_{n+1}; \tag{17}$$

$$w'^T x \geq 0 \quad \forall x \in C_A. \tag{18}$$

It follows from (17) that $w_1' \geq 0$, $t_1 \geq 0$, while (18) implies that

$$A^T w_1' + t_1 c \geq 0. \tag{19}$$

As A^T is Metzler and Hurwitz, Lemma 5.3 implies that $t_1 > 0$.
Write $p = A^T w_1' + t_1 c$ and define $w_1 = w_1' - A^{-T} p$. Then as A^T is Metzler and Hurwitz, $A^{-T} \leq 0$ and hence $w_1 \geq 0$. It follows from (19) that

$$A^T w_1 + t_1 c = 0. \tag{20}$$

Using similar reasoning, it follows from (ii) that there exists a non-zero $w_2 \geq 0$ in $\mathbb{R}^n$ and a non-zero real number $t_2 > 0$ such that

$$B^T w_2 + t_2 c = 0. \tag{21}$$

As $B^T = A^T + cb^T$ it follows that:

$$A^T x = \lambda c$$

for some $\lambda \in \mathbb{R}$ if and only if

$$B^T x = (\lambda + b^T x)c.$$

This implies that the preimages of $\{\lambda c : \lambda \in \mathbb{R}\}$ under A^T and B^T coincide. As A^T and B^T are Hurwitz, and hence non-singular, by assumption, this observation together with (20), (21) implies that there is some $\kappa > 0$ such that $w_2 = \kappa w_1$.
It now follows from (20), (21) that

$$(A^T + \frac{\kappa t_1}{t_2} B^T) w_1 = 0. \tag{22}$$

As κ, t_1 and t_2 are all positive, (22) implies that $A^{-T}B^T$ has a real negative eigenvalue. As A^T, B^T are Hurwitz and differ by a rank one matrix, Lemma 5.1 shows that this is a contradiction. This contradiction shows that either there exists some $v \in D_A$ with $Av \ll 0$ or there exists some $v \in D_B$ with $Bv \ll 0$. This completes the proof.

Example 5.1. *Let $A \in \mathbb{R}^{4\times 4}$ be given by*

$$A = \begin{pmatrix} -2 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \\ 1 & 0 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{pmatrix}.$$

Further, let $b^T = (1/2,\ 1/2,\ 1/2,\ 1/2)$, $c^T = (-1,\ 1,\ -1,\ 1)$. Define $B = A + bc^T$. It can be verified by direct computation that A and B are both Metzler and Hurwitz. Consider the associated piecewise positive linear system (13). Theorem 5.1 implies that the origin is globally exponentially stable for this system.

6. Conclusions and outlook

The piecewise Kamke-Müller (PKM) conditions described in Section 3 are necessary for bi-modal piecewise smooth systems of the form (3) (where each component system is monotone) to be monotone. We have characterised the PKM conditions in terms of the values of the vector fields of (3) along the separating surface defining the system. This characterisation led to the result, given in Theorem 4.2, that for a significant class of systems of the form (3), monotonicity is equivalent to continuity. Theorem 4.2 has also been applied to positive switched linear systems and a stability result for this class has been derived. There are a number of possible extensions of the work discussed here: these include considering systems with more than 2 component systems; extending the stability analysis of Section 5 to nonlinear systems; clarifying the relationship between monotonicity and the PKM conditions for the discontinuous case.

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