Quadratic Lyapunov Functions for Systems with State-Dependent Switching*

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Abstract

In this paper, we consider the existence of quadratic Lyapunov functions for certain types of switched linear systems. Given a partition of the state-space, a set of matrices (linear dynamics), and a matrix-valued function $A(x)$ constructed by associating these matrices with regions of the state-space in a manner governed by the partition, we ask whether there exists a positive definite symmetric matrix P such that $A(x)^T P + PA(x)$ is negative definite for all $x(t)$. For planar systems, necessary and sufficient conditions are given. Extensions for higher order systems are also presented.

Keywords: Hybrid Systems; Lyapunov Functions; Quadratic Stability

Mathematical Subject Codes: 15A18; 52A10; 93D05

1 Introduction

Recent years have witnessed great interest in stability problems arising from the study of switched and hybrid systems [1]. In this area, state-dependent switching between linear vector fields represents an important problem that arises frequently in practice, when the rule for switching between the constituent linear systems of a switched system is governed by the (current) state vector of the system, as is the case for piecewise linear systems [2] and for the closely related class of complementarity systems [3]. Loosely speaking, the stability

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problems associated with this type of switching system can be divided into two classes. In the first of these, the state-space is partitioned into a number of regions. This partition determines the mode switches in the system dynamics, and the problem is to analyze the stability of the time-varying system defined in this way. In the second class of problem, one looks for state-dependent rules for switching between a family of unstable systems that will result in stability. Thus, in the former case, a partition of the state-space is specified and the problem is to determine the stability of the piecewise linear system defined by that partition, while in the latter case, the aim is to find stabilizing state-dependent rules for switching between potentially unstable systems. Before proceeding, it is worth pointing out that problems in this latter category have been the subject of some discussion in the hybrid system community. For example, [4, 5] have dealt with this problem with some success. However, aside from some notable numerical approaches based on linear matrix inequalities [6], the former problem has received considerably less attention. Our objective in writing this paper is to begin the task of addressing this problem from a more theoretical perspective. As this is our initial thrust in this direction, our study begins with a somewhat simplified version of the aforementioned general problem. Specifically, we consider planar systems where the state-space partition is constructed using rays passing through the origin and different linear dynamics are active in the regions between these rays. Given this basic set-up, we obtain necessary and sufficient conditions for the existence of quadratic Lyapunov functions and then extend these results to special classes of higher-dimensional systems.

Our paper is structured as follows. We begin by defining the problem of interest and by presenting some background material. Our main results are given in Section 3. After presenting extensions to higher-dimensional systems in Section 4, we give the proofs of our main results in Section 5.

2 Problem Description and Background

The motivation for our work arises from the study of stability of the switched system $\dot{x} = A(x)x$, where $A(x)$ takes discrete values $\{A_1, A_2, \ldots, A_N\}$ depending on the current state $x(t)$. As the vector field defining such a system is discontinuous, solutions in the classical ODE sense are not guaranteed to exist; however, *Filippov* solutions [7], defined via the associated linear differential inclusion, will exist for our system class and it is to this solution concept that our results apply. One method to deduce the stability of such a system is to require the existence of a quadratic Lyapunov function $V(x) = x^T P x$, $P = P^T > 0$, such that $A(x)^T P + P A(x)$ is negative definite for all $x(t) \in \mathbb{R}^{n \times n}$. This latter problem gives rise to the linear algebraic problem considered in this paper.

Notation: Recall that a real matrix A is *Hurwitz* if its eigenvalues have negative real parts. We will denote by $\mathcal{S}_n(\mathbb{R})$ the vector space of $n \times n$ real symmetric matrices, and by $\mathcal{P}_n(\mathbb{R})$ the set of $n \times n$ real symmetric positive definite matrices. For a subset $\Omega \subset \mathbb{R}^n$ we write $\mathrm{Int}(\Omega)$ to denote the (open) interior of Ω. A set $\Omega \subset \mathbb{R}^n$ will be called a *double cone with apex at the origin* if $tx \in \Omega$ for every $x \in \Omega$ and $t \in \mathbb{R}$.

Common quadratic Lyapunov functions: Given $A \in \mathbb{R}^{n\times n}$ and $P \in \mathcal{P}_n(\mathbb{R})$, the function $V(x) = x^T P x$ defines a quadratic Lyapunov function (QLF) for the dynamical system $\dot{x} = Ax$ if $PA + A^T P < 0$. (We will often abuse notation and say that P is a QLF for A, meaning that $V(x)$ is a QLF for $\dot{x} = Ax$.) We define the set of all such QLF matrices as

$$\mathcal{L}(A) := \{P \in \mathcal{P}_n(\mathbb{R}) \,|\, PA + A^T P < 0\}.$$

$\mathcal{L}(A)$ is an open convex pointed cone in the space of $\mathcal{S}_n(\mathbb{R})$ [8]. A matrix $P \in \mathcal{P}_n(\mathbb{R})$ is a common quadratic Lyapunov function (CQLF) for $\{A_1, \ldots, A_N\}$ if P belongs to the intersection of the cones $\{\mathcal{L}(A_1), \ldots, \mathcal{L}(A_N)\}$. A solution to the CQLF existence problem in $\mathbb{R}^2$ is given as follows.

Theorem 1 *[9, 10] Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$ be two Hurwitz matrices. A necessary and sufficient condition for the pair $\{A_1, A_2\}$ to have a CQLF is that the matrices $A_2^{-1}A_1$ and A_2A_1 do not have real negative eigenvalues. An equivalent condition is that all convex combinations of A_1 and A_2, and of A_1 and A_2^{-1}, are Hurwitz.*

The joint quadratic Lyapunov function problem: Given $A \in \mathbb{R}^{n\times n}$ and $\Omega \subset \mathbb{R}^n$, define the QLF set for the pair (A, Ω) as follows:

$$\mathcal{L}(A, \Omega) := \{P \in \mathcal{P}_n(\mathbb{R}) \,|\, x^T(PA + A^T P)x < 0 \;\forall x \in \Omega,\, x \neq 0\}.$$

Note that $\mathcal{L}(A) \subset \mathcal{L}(A, \Omega)$ and hence if A is Hurwitz then $\mathcal{L}(A, \Omega)$ is nonempty. The *joint QLF problem* for a set of matrices A_i and regions Ω_i is to find necessary and sufficient conditions for a nonempty intersection of the sets $\{\mathcal{L}(A_i, \Omega_i)\}$.

Problem statement: Let x_1, x_2 be two vectors in $\mathbb{R}^2$ and define

$$\Omega_1 := \{x = \alpha x_1 + \beta x_2 \,|\, \alpha \text{ and } \beta \in \mathbb{R},\, \alpha\beta \geq 0\}. \tag{1}$$

We will describe this region as the *closed double cone* in $\mathbb{R}^2$ defined by the vectors x_1 and x_2. Assume $A_1, A_2 \in \mathbb{R}^{2\times 2}$ are Hurwitz matrices. We now state the two problems which are solved in this paper.

Problem 1: Let Ω_1 be a closed double cone of the form (1) and let $\Omega_2 = \mathbb{R}^2$. *Solve the joint QLF problem for* (A_1, Ω_1) *and* $(A_2, \mathbb{R}^2)$. Equivalently, find necessary and sufficient conditions for the existence of a $P \in \mathcal{P}_2(\mathbb{R})$ such that the following are simultaneously satisfied:

$$\text{(i) } x^T\left(A_1^T P + PA_1\right)x < 0,\ \forall x \in \Omega_1,\ x \neq 0;$$
$$\text{(ii) } A_2^T P + PA_2 < 0.$$

Problem 2: Let Ω_1 be a closed double cone of the form (1) and let $\Omega_2 = \mathbb{R}^2 \setminus \mathrm{Int}(\Omega_1)$ (note that Ω_2 is also a closed double cone). *Solve the joint QLF problem for* (A_1, Ω_1) *and* (A_2, Ω_2). Equivalently, find necessary and sufficient conditions for the existence of a $P \in \mathcal{P}_2(\mathbb{R})$ such that the following inequalities are simultaneously satisfied:

$$\text{(i) } x^T \left(A_1^T P + P A_1\right) x < 0,\ \forall x \in \Omega_1,\ x \neq 0;$$
$$\text{(ii) } y^T \left(A_2^T P + P A_2\right) y < 0,\ \forall y \in \Omega_2,\ y \neq 0.$$

3 Main Results

Below, we present the solutions to Problems 1 and 2. The proofs will be given in Section 5 along with some other related results. We denote by $A(\Omega)$ the image of the region Ω under the action of the matrix A.

Theorem 2 (Solution to Problem 1) *There exists a joint QLF for the pairs* (A_1, Ω_1) *and* $(A_2, \mathbb{R}^2)$ *if and only if the following conditions are satisfied:*

(a) there is no convex combination of A_1 *and* A_2*, or of* A_1 *and* A_2^{-1}*, which has an eigenvector in* Ω_1 *with non-negative eigenvalue;*

(b) there is no convex combination of A_1^{-1} *and* A_2*, or of* A_1^{-1} *and* A_2^{-1}*, which has an eigenvector in* $A_1(\Omega_1)$ *with non-negative eigenvalue;*

(c) there is no nonzero y *satisfying both equations*

$$\begin{aligned} \left(aA_1 + bA_1^{-1} + cA_2\right) y &= 0, \\ ayy^T + bA_1^{-1} yy^T \left(A_1^{-1}\right)^T &= d_1 x_1 x_1^T + d_2 x_2 x_2^T. \end{aligned}$$

for some non-negative coefficients a, b, c, d_1, d_2.

Remark 1 *Note that Conditions (a) and (b) strongly resemble the singularity conditions for the CQLF problem in Theorem 1.*

Given $x_1, x_2 \in \mathbb{R}^2$, define
$$\mathcal{C}_{12} = \{a x_1 x_1^T + b x_2 x_2^T \mid a, b \geq 0\}.$$

Theorem 3 (Solution to Problem 2) *There exists a joint QLF for the pairs* (A_1, Ω_1) *and* (A_2, Ω_2) *if and only if all of the following conditions are satisfied:*

(a) there is no convex combination of A_1 *and* A_2 *which has an eigenvector in* $\Omega_1 \cap \Omega_2$ *with non-negative eigenvalue;*

(b) there is no convex combination of A_1 *and* A_2^{-1} *which has an eigenvector in* $\Omega_1 \cap A_2(\Omega_2)$ *with non-negative eigenvalue;*

(c) there is no convex combination of A_1^{-1} *and* A_2 *which has an eigenvector in* $A_1(\Omega_1)\cap\Omega_2$ *with non-negative eigenvalue;*

(d) there is no convex combination of A_1^{-1} *and* A_2^{-1} *which has an eigenvector in* $A_1(\Omega_1)\cap A_2(\Omega_2)$ *with non-negative eigenvalue;*

(e) there is no nonzero vector $y\in\Omega_2$ *satisfying both equations*

$$\begin{aligned}(aA_1+bA_2+cA_1^{-1}-dI_n)y &= 0,\\ ayy^T+cA_1^{-1}yy^T(A_1^{-1})^T &\in \mathcal{C}_{12}\end{aligned}$$

for some non-negative coefficients a,b,c,d*;*

(f) there is no nonzero vector $x\in\Omega_1$ *satisfying both equations*

$$\begin{aligned}(aA_1+bA_2+cA_2^{-1}-dI_n)x &= 0,\\ bxx^T+cA_2^{-1}xx^T(A_2^{-1})^T &\in \mathcal{C}_{12}\end{aligned}$$

for some non-negative coefficients a,b,c,d*;*

(g) there is no nonzero vector $z\in A_2(\Omega_2)$ *satisfying both equations*

$$\begin{aligned}(aA_1+bA_1^{-1}+cA_2^{-1}-dI_n)z &= 0,\\ azz^T+bA_1^{-1}zz^T(A_1^{-1})^T &\in \mathcal{C}_{12}\end{aligned}$$

for some non-negative coefficients a,b,c,d*;*

(h) there is no nonzero vector $w\in A_1(\Omega_1)$ *satisfying both equations*

$$\begin{aligned}(aA_1^{-1}+bA_2+cA_2^{-1}-dI_n)w &= 0,\\ bww^T+cA_2^{-1}ww^T(A_2^{-1})^T &\in \mathcal{C}_{12}\end{aligned}$$

for some non-negative coefficients a,b,c,d*;*

(i) denote by S *a* 2×2 *matrix of the form*

$$S=\begin{bmatrix} s_{11} & s_{12}\\ -s_{12} & s_{22}\end{bmatrix},$$

where s_{11},s_{22} *are non-negative. Define* $w_i=A_1x_i,\ z_i=A_2x_i$ *for* $i=1,2$*. Then there are no relations of the forms*

$$\begin{aligned}\begin{bmatrix} ax_i\\ bz_j\end{bmatrix} &= S\begin{bmatrix} x_i\\ x_j\end{bmatrix}\\ \begin{bmatrix} aw_i\\ bz_j\end{bmatrix} &= S\begin{bmatrix} w_i\\ x_j\end{bmatrix}\\ \begin{bmatrix} ax_i\\ bw_j\end{bmatrix} &= S\begin{bmatrix} z_i\\ x_j\end{bmatrix}\\ \begin{bmatrix} ax_i\\ bx_j\end{bmatrix} &= S\begin{bmatrix} w_i\\ z_j\end{bmatrix}\end{aligned}$$

where $a, b \geq 0$.

4 Extensions to the Higher-Dimensional Problem

We now indicate how our results can be extended to some higher-dimensional systems. Of particular interest is the case of nonlinear single-input single-output (SISO) systems; namely, when one has a pair of matrices in the joint QLF problem with rank one difference. While a full treatment of this system class is beyond the scope of the current paper and is given in [11], we provide here a flavour of extensions that are possible. We say that the matrix X is *generated* by Ω if there are vectors $x_1, \ldots, x_k \in \Omega$ such that $X = \sum_{i=1}^{k} x_i x_i^T$.

Theorem 4 *Let $A_1, A_2 \in \mathbb{R}^{3\times 3}$ be Hurwitz matrices with a rank one difference. Suppose that $\Omega_1 \subset \mathbb{R}^3$ is a double cone with apex at the origin and $\Omega_2 = \mathbb{R}^3$. Then there is a joint QLF for (A_1, Ω_1) and (A_2, Ω_2) if and only if there is no linear combination $A_2 A_1 + a^2 A_2^{-1} A_1 + \alpha^2 A_2^2$ with a real negative eigenvalue less than or equal to $-a^2\alpha^2$, whose eigenvector has the form $y = (v^T X v)^{-\frac{1}{2}} X v$ for some matrix X generated by Ω_1, where X satisfies the bound*

$$X \leq (1+\alpha^2) y y^T + k^2 (A_2^2 + a^2)^{-1} \Big(A_2 y y^T A_2^T + a^2 y y^T \Big) \left((A_2^2 + a^2)^T \right)^{-1}. \tag{2}$$

Remark 2 *The conditions of Theorem 4 simplify considerably if $\Omega_1 = \mathcal{C} \cup (-\mathcal{C})$, where $\mathcal{C}$ is a convex cone and where $v^T x$ has the same sign for all $x \in \mathcal{C}$. In this case, Xv is in Ω_1 for every X generated by Ω_1 and thus in the statement of Theorem 4 we may use $X = yy^T$ so that (2) is automatically satisfied.*

The proof of Theorem 4 relies on the following dual formulation.

Lemma 1 *Consider $\Omega_1, \ldots, \Omega_N \subset \mathbb{R}^n$ such that $\Omega_j = \mathbb{R}^n$ for at least one $j = 1, \ldots, N$. Then the collection $\{(A_j, \Omega_j)\}$ has a joint QLF if and only if there do not exist positive semi-definite matrices $X_1, \ldots, X_N$ (not all zero) with X_i generated by Ω_i for $i = 1, \ldots, N$ such that*

$$\sum_{i=1}^{N} A_i X_i + X_i A_i^T = 0. \tag{3}$$

Proof of Lemma 1 ($\Leftarrow$) If (3) holds, then for any positive definite matrix P,

$$0 = \mathrm{Tr} P \Big(\sum_{i=1}^{N} A_i X_i + X_i A_i^T \Big) = \sum_{i=1}^{N} \mathrm{Tr} X_i (P A_i + A_i^T P).$$

Writing $X_i = \sum_j x_{ij} x_{ij}^T$ with $x_{ij} \in \Omega_i$ gives

$$0 = \sum_{i,j} x_{ij}^T (P A_i + A_i^T P) x_{ij}$$

which contradicts the existence of a joint QLF.

($\Rightarrow$) This is well-known for the case $\Omega_j = \mathbb{R}^n$ for all $j = 1, \ldots, N$ [12]. The main idea is to define the dual cones

$$\mathcal{L}^D(A_j, \Omega_j) = \{X \geq 0 \,|\, X \text{ generated by } \Omega_j,\ \mathrm{Tr} X(PA_j + A_j^T P) \leq 0\}$$

and to view $\mathbb{R}^{n\times n}$ as an inner product space by defining $\langle A, B\rangle = \mathrm{Tr}(A^T B)$. Then the existence of a joint QLF for $\{(A_j, \Omega_j)\}$ is equivalent to the existence of a hyperplane in $\mathbb{R}^{n\times n}$ with all cones $\mathcal{L}^D(A_j, \Omega_j)$ on the same side. The condition that $\Omega_j = \mathbb{R}^n$ for at least one j means that the normal vector to this hyperplane is a positive definite matrix. The condition (3) is the obstruction to the existence of such a hyperplane. Hence if (3) is not satisfied then there is a positive definite matrix P such that

$$\mathrm{Tr} P(A_i X_i + X_i A_i^T) < 0 \quad \text{for all } i = 1, \ldots, N$$

for all X_i generated by Ω_i, and this implies the existence of a joint QLF. ■

We now apply Lemma 1 with $N = 2$, $n = 3$, $\Omega_2 = \mathbb{R}^3$ and assume that $A_1 - A_2 = uv^T$ is a rank one matrix. Suppose that there is no joint QLF and consider (3):

$$A_1 X_1 + X_1 A_1^T + A_2 X_2 + X_2 A_2^T = 0.$$

This can be rewritten as

$$u(X_1 v)^T + (X_1 v)u^T + A_2(X_1 + X_2) + (X_1 + X_2)A_2^T = 0.$$

Since A_2 is Hurwitz and $X_1 + X_2 \neq 0$, $X_1 v$ must be nonzero, and so $v^T X_1 v > 0$. Define

$$y = (v^T X_1 v)^{-1/2} X_1 v.$$

It follows that $X_1 v = yy^T v$ and also $X_1 \geq yy^T$. Define $W = X_1 + X_2 - yy^T$, then

$$A_1 yy^T + yy^T A_1^T + A_2 W + W A_2^T = 0 \tag{4}$$

where

$$X_1 \leq yy^T + W. \tag{5}$$

In general, the matrix W in (4) has rank three, hence there is some real α such that $W - \alpha^2 yy^T$ is positive semi-definite with rank two. For this value of α define

$$Z = W - \alpha^2 yy^T.$$

Then (4) can be written where

$$(A_1 + \alpha^2 A_2)yy^T + yy^T(A_1 + \alpha^2 A_2)^T + A_2 Z + Z A_2^T = 0. \tag{6}$$

Since Z is rank two, there are vectors z_1, z_2 such that $Z = z_1 z_1^T + z_2 z_2^T$. Using results of [8], (6) can be solved in the following sense: there are real numbers a, b, c such that

$$\begin{aligned} A_2 z_1 &= a z_2 + b y && (7) \\ A_2 z_2 &= -a z_1 + c y && (8) \\ (A_1 + \alpha^2 A_2) y &= -b z_1 - c z_2. && (9) \end{aligned}$$

The first two equations can be solved to find z_1, z_2 in terms of y:

$$z_1 = (A_2^2 + a^2 I)^{-1}(bA_2 + acI)y, \quad z_2 = (A_2^2 + a^2 I)^{-1}(cA_2 - ab)y. \tag{10}$$

Substituting into (9) leads to

$$(A_1 + \alpha^2 A_2)y = -k^2 (A_2^2 + a^2 I)^{-1} A_2 y, \quad k^2 = b^2 + c^2.$$

This can be recast in the form

$$\left(A_2 A_1 + a^2 A_2^{-1} A_1 + \alpha^2 A_2^2 + (\alpha^2 a^2 + k^2) I\right) y = 0. \tag{11}$$

Thus the non-existence of a joint QLF for this problem is equivalent to the existence of a singular vector $y = X_1 v$ satisfying (11), where X_1 is generated by Ω_1, and where (5) implies that

$$X_1 \leq (1 + \alpha^2) y y^T + z_1 z_1^T + z_2 z_2^T.$$

Using (10) the Lemma follows.

5 Proofs of the Results

In this section, we present proofs of Theorems 2 and 3. Our approach is essentially geometric and we utilize the fact that the sets of Lyapunov functions that we are studying are convex. Central to our approach is the notion of tangential hyperplanes and graphical representations of the sets. We first briefly review these concepts.

Tangent hyperplanes at $\mathcal{L}(A)$: If P_0 is in the boundary of $\mathcal{L}(A)$ then $P_0 A + A^T P_0 \leq 0$ and has a non-empty kernel. Let x_0 be a vector in the kernel of $P_0 A + A^T P_0$, then

$$x_0^T (A^T P_0 + P_0 A) x_0 = 2 x_0^T P_0 A x_0 = 0. \tag{12}$$

The set $H = \{P \in \mathcal{S}_n(\mathbb{R}) \,|\, x_0^T P A x_0 = 0\}$ is a linear subspace in $\mathcal{S}_n(\mathbb{R})$. Since H does not intersect the QLF set $\mathcal{L}(A)$ but does intersect its boundary, it follows that H is tangent to $\mathcal{L}(A)$. Recall that $\mathcal{S}_n(\mathbb{R})$ is isomorphic to $\mathbb{R}^{n(n+1)/2}$, and is equipped with the inner product $\langle A, B\rangle = \mathrm{Tr} A^T B$. Thus the hyperplane H can be described by its normal vector $H^\perp \in \mathcal{S}_n(\mathbb{R})$:

$$H = \{P \in \mathcal{S}_n(\mathbb{R}) \,|\, \langle P, H^\perp \rangle = 0\}.$$

Comparison with (12) shows that $H^{\perp} = Ax_0x_0^T + x_0x_0^TA^T$. Furthermore, the set $\mathcal{L}(A)$, being convex, lies on one side of H, and we note that the normal vector $H^{\perp}$ is directed away from it. For two-dimensional systems, every tangent plane to the set $\mathcal{L}(A)$ has the form described above with a normal vector $Axx^T + xx^TA^T$ for some vector x in the plane. This is because in two dimensions the kernel of the matrix $P_0A + A^TP_0$ can be at most one-dimensional. Thus the situation described above applies and leads to the tangent plane of the stated form.

Separating tangential hyperplanes: If A_1 and A_2 are Hurwitz matrices for which $\mathcal{L}(A_1)$ and $\mathcal{L}(A_2)$ are disjoint then A_1 and A_2 do not have a CQLF. Since $\mathcal{L}(A_1)$ and $\mathcal{L}(A_2)$ are convex sets, there is a separating hyperplane between these sets and this hyperplane may be chosen to be a simultaneous tangent plane for both sets. Supposing that this tangent plane has the form described above then there are vectors x and y such that the normal vector for the plane is $A_1xx^T + xx^TA_1^T$ at $\mathcal{L}(A_1)$ and $A_2yy^T + yy^TA_2^T$ at $\mathcal{L}(A_2)$. Furthermore, since the plane separates the QLF sets, these normals must be oppositely oriented, hence there is a positive constant k such that

$$A_1xx^T + xx^TA_1^T = -k(A_2yy^T + yy^TA_2^T). \tag{13}$$

Equivalently, for all symmetric matrices $Q \in \mathbb{R}^{n\times n}$, we have

$$x^TQA_1x = -ky^TQA_2y.$$

The following result allows us to solve this equation.

Lemma 2 *[13] Let x, y, u, v be four nonzero vectors in $\mathbb{R}^n$ such that for all symmetric matrices $Q \in \mathbb{R}^{n\times n}$, $x^TQy = -ku^TQv$ with $k > 0$. Then, either*

$$x = \alpha u \text{ for some real scalar } \alpha \text{ and } y = -\left(\frac{k}{\alpha}\right)v, \text{ or}$$
$$x = \beta v \text{ for some real scalar } \beta \text{ and } y = -\left(\frac{k}{\beta}\right)u.$$

Lemma 2 implies that some convex combination of A_1 and A_2, or of A_1 and A_2^{-1}, is singular.

Graphical representations of Lyapunov functions in two dimensions: It is possible to represent scaled symmetric matrices $Q \in \mathcal{S}_2(\mathbb{R})$ by points in a plane. Each point (q_{12}, q_{22}) defines a symmetric matrix by

$$Q = \begin{bmatrix} 1 & q_{12} \\ q_{12} & q_{22} \end{bmatrix}. \tag{14}$$

Symmetric matrices whose $(1,1)$ entry is nonzero can be re-scaled to this form and matrices whose $(1,1)$ entry is zero lie in the closure of this set. Figure 1 depicts three points and the parabola $q_{22} = q_{12}^2$. Points on the parabola (eg: Q_3) correspond to positive semi-definite matrices. Points on the positive side of the locus (eg: Q_1) correspond to positive definite matrices and points on the negative side of the locus (eg: Q_2) correspond to indefinite matrices. Under the assumption that A is not a triangular matrix, the projection of a set $\mathcal{L}(A)$ in the (q_{12}, q_{22})-plane corresponds to the interior of an ellipse [9]. Figure 1 also shows a projection of the tangent hyperplane $\{P \,|\, x^T PAx = 0\}$ onto the (q_{12}, q_{22})-plane, where it appears as a tangent line to the ellipse representing $\mathcal{L}(A)$.

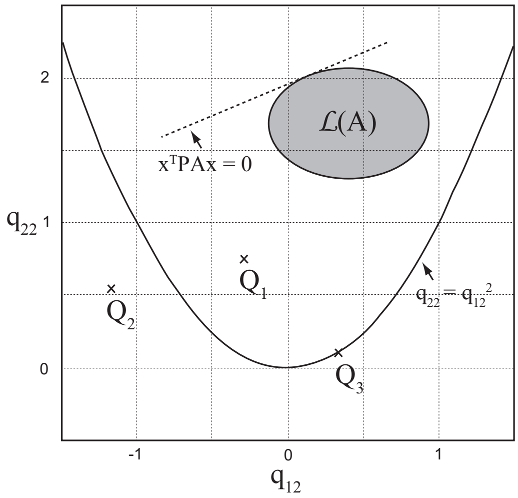

Figure 1: Graphical representation of symmetric matrices.

Tangent hyperplanes at $\mathcal{L}(A, \Omega)$: Now let $\Omega \subset \mathbb{R}^2$ be the closed double cone defined by two vectors x_1, x_2. Then the set $\mathcal{L}(A, \Omega)$ lies between the hyperplanes $H_1 := \{P \,|\, x_1^T PAx_1 = 0\}$ and $H_2 := \{P \,|\, x_2^T PAx_2 = 0\}$, and these hyperplanes are tangent to the set. Examples of possible configurations are shown in Figures 2 and 3. As noted, every tangent line to the ellipse $\mathcal{L}(A)$ has the form $\{P \,|\, x^T PAx = 0\}$ for some $x \in \mathbb{R}^2$. As x varies between x_1 and x_2, the tangent line varies from H_1 to H_2. As a consequence, any line $\{P \,|\, x^T PAx = 0\}$ which is tangent to $\mathcal{L}(A)$ is also tangent to $\mathcal{L}(A, \Omega)$ when $x \in \Omega$. In general, a tangent line may intersect the boundary of a set $\mathcal{L}(A, \Omega)$ in one of three ways, as listed below. In each case we also describe the normal matrix defining the tangent plane:

(i) at a point that lies on the boundary of $\mathcal{L}(A)$. The normal is

$$Axx^T + xx^T A^T \tag{15}$$

for some $x \in \Omega$;

(ii) at the point where the lines defined by H_1 and H_2 intersect. Thus the tangent line is a convex combination of H_1 and H_2 and so its normal is

$$k^2(Ax_1x_1^T + x_1x_1^T A^T) + l^2(Ax_2x_2^T + x_2x_2^T A^T) \tag{16}$$

for some $k, l \in \mathbb{R}$;

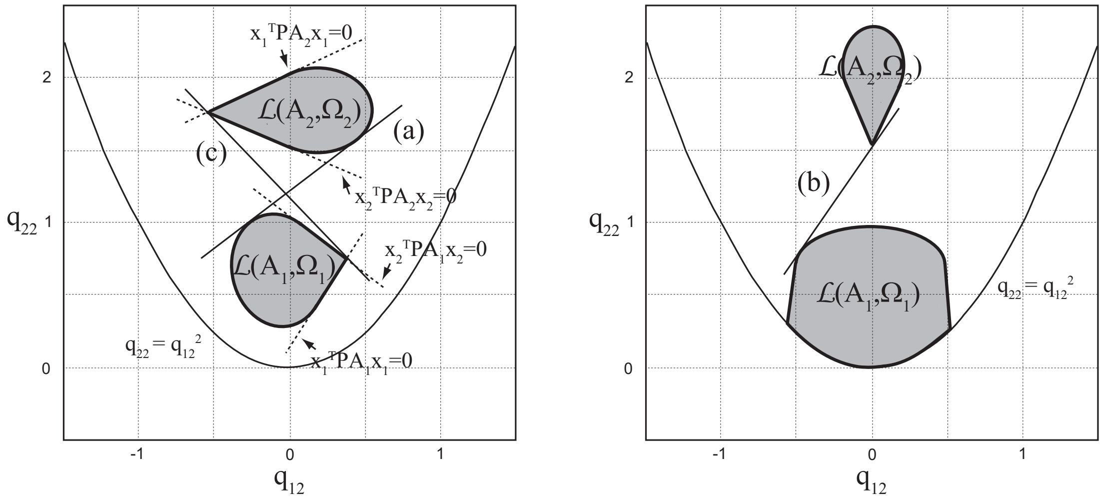

Figure 2: Separating tangential hyperplanes: Types (a), (b) and (c).

(iii) at a point that corresponds to where one of the lines defined by H_1 or H_2 intersects the "parabola" of semi-definite matrices. The tangent line is a convex combination of H_i ($i \in \{1,2\}$) and the tangent to the parabola. The tangent to the parabola is either $x_i^T P x_i = 0$ or $(Ax_i)^T P A x_i = 0$, with corresponding normal vectors $x_i x_i^T$ or $Ax_i x_i^T A^T$ directed towards the positive definite side of the parabola. Hence, the normal is one of the following:

$$\begin{aligned}
&k^2(Ax_1x_1^T + x_1x_1^TA^T) - l^2x_1x_1^T; \\
&k^2(Ax_1x_1^T + x_1x_1^TA^T) - l^2Ax_1x_1^TA^T; \\
&k^2(Ax_2x_2^T + x_2x_2^TA^T) - l^2x_2x_2^T; \\
&k^2(Ax_2x_2^T + x_2x_2^TA^T) - l^2Ax_2x_2^TA^T.
\end{aligned} \tag{17}$$

We now state and prove a preliminary result which contains many of the essential ideas and is of independent interest.

Theorem 5 *Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$ be Hurwitz matrices and Ω_1 be a closed double cone of the form* (1).

(i) Suppose that $\Omega_2 = \mathbb{R}^2$. A necessary condition for the existence of a joint quadratic Lyapunov function $P \in \mathcal{P}_2(\mathbb{R})$ for the pairs (A_1, Ω_1) and (A_2, Ω_2) is that there is no convex combination of A_1 and A_2, and no convex combination of A_1 and A_2^{-1}, which has an eigenvector in Ω_1 with non-negative eigenvalue.

(ii) Suppose that $\Omega_2 = \mathbb{R}^2 \setminus \mathrm{Int}(\Omega_1)$. A necessary condition for the existence of a joint quadratic Lyapunov function $P \in \mathcal{P}_2(\mathbb{R})$ for the pairs (A_1, Ω_1) and (A_2, Ω_2) is that there is no convex combination of A_1 and A_2 which has an eigenvector in $\Omega_1 \cap \Omega_2$ with non-negative eigenvalue, and no convex combination of A_1 and A_2^{-1} which has an eigenvector in $\Omega_1 \cap A_2(\Omega_2)$ with non-negative eigenvalue.

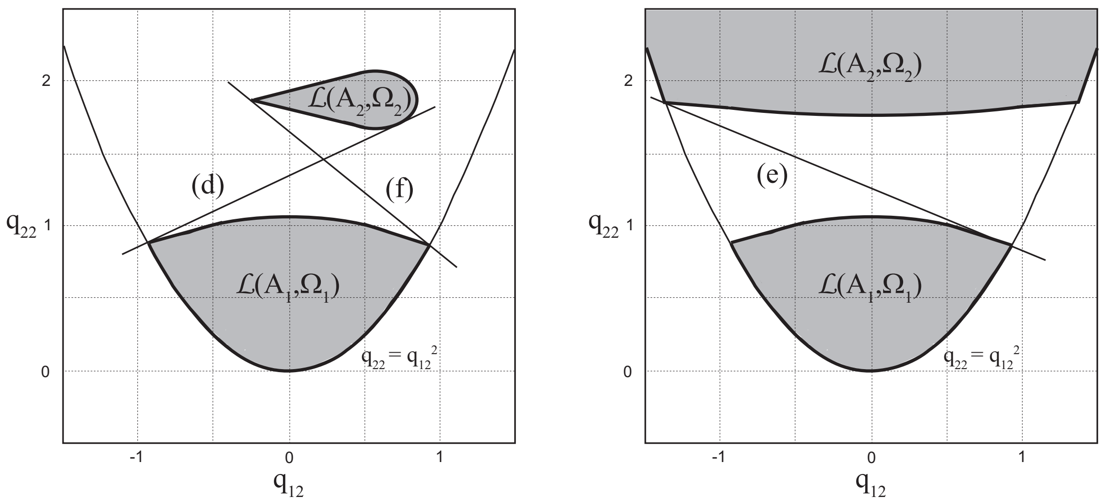

Figure 3: Separating tangential hyperplanes: Types (d), (e) and (f).

Furthermore, these conditions are sufficient in each case to ensure that, for any $x \in \Omega_1$ *and any* $y \in \Omega_2$*, no separating tangential hyperplane of the form described by* (13) *exists between* $\mathcal{L}(A_1, \Omega_1)$ *and* $\mathcal{L}(A_2, \Omega_2)$*.*

Proof of Theorem 5 ($\Rightarrow$) Let $\sigma_\alpha[A_1, A_2] := \alpha A_1 + (1-\alpha)A_2$, where $\alpha \in [0, 1]$. Suppose that P is a joint QLF for the pairs (A_1, Ω_1) and $(A_2, \mathbb{R}^2)$. It follows that P is a QLF for $(\sigma_\alpha[A_1, A_2], \Omega_1)$, for all $\alpha \in [0, 1]$, since:

$$\alpha(A_1^T P + PA_1) + (1-\alpha)(A_2^T P + PA_2) = \sigma_\alpha[A_1, A_2]^T P + P\sigma_\alpha[A_1, A_2]$$

and if $x^T(A_1^T P + PA_1)x < 0$, $\forall x \in \Omega_1$, $x \neq 0$ and $A_2^T P + PA_2 < 0$ then $x^T(\sigma_\alpha[A_1, A_2]^T P + P\sigma_\alpha[A_1, A_2])x < 0$, $\forall x \in \Omega_1$, $x \neq 0$. This immediately implies that there is no convex combination of A_1 and A_2 which has an eigenvector x in Ω_1 corresponding to an eigenvalue $\lambda \geq 0$. To see this, suppose that $\sigma_\alpha[A_1, A_2]x = \lambda x$ for some $\alpha \in [0, 1]$, $x \in \Omega_1$, $\lambda \geq 0$. Then $x^T P\sigma_\alpha[A_1, A_2]x = \lambda x^T Px \geq 0$ which is a contradiction. Necessity in the other instances follows in a similar manner by setting $\hat{y} = A_2 y$, where $y \in \mathbb{R}^2$ is nonzero, for example and noting that the inequalities $y^T(A_2^T P + PA_2)y < 0$ and $\hat{y}^T(A_2^{-T} P + PA_2^{-1})\hat{y} < 0$ are equivalent. Then P is a QLF for $(A_2, \mathbb{R}^2)$ if and only if P is a QLF for $(A_2^{-1}, \mathbb{R}^2)$.

($\Leftarrow$) Suppose that there does not exist a joint QLF for the pairs (A_1, Ω_1) and $(A_2, \mathbb{R}^2)$. Then there exists a separating tangential hyperplane between the sets $\mathcal{L}(A_1, \Omega_1)$ and $\mathcal{L}(A_2, \mathbb{R}^2)$. One of these separating tangential hyperplanes may be of the form described by (13) with x a nonzero vector in Ω_1 and y a nonzero vector in $\mathbb{R}^2$. From Lemma 2, either

$$x = r_1 y \text{ and } A_1 x = -\frac{k}{r_1} A_2 y \tag{18}$$

for some real scalar r_1, or

$$x = r_2 A_2 y \text{ and } A_1 x = -\frac{k}{r_2} y \tag{19}$$

for some real scalar r_2. Suppose that (18) is the solution. Then $\left[A_2^{-1}A_1 + \left(\frac{k}{r_1^2}\right) I\right] x = 0$ which means that $A_2^{-1}A_1$ has eigenvectors in Ω_1 with real negative eigenvalues. Alternatively, $\left[A_1 + \left(\frac{k}{r_1^2}\right) A_2\right] x = 0$ meaning that there exists a convex combination of A_1 and A_2 which has eigenvectors in Ω_1 and is singular. Suppose that (19) is the solution. Then $\left[A_2A_1 + \left(\frac{k}{r_2^2}\right) I\right] x = 0$ which means that A_2A_1 has eigenvectors in Ω_1 with real negative eigenvalues. Alternatively, $\left[A_1 + \left(\frac{k}{r_2^2}\right) A_2^{-1}\right] x = 0$ meaning that there exists a convex combination of A_1 and A_2^{-1} which has eigenvectors in Ω_1 and is singular. Sufficiency for the other case follows in a similar fashion. ■

Remark 3 *The conditions presented in Theorem 5 are necessary but generally not sufficient for determining joint QLF existence.*

Proof of Theorem 2 ($\Leftarrow$) We exploit an idea similar to Theorem 5. Assume that the sets $\mathcal{L}(A_1, \Omega_1)$ and $\mathcal{L}(A_2, \mathbb{R}^2)$ are disjoint. These are open convex sets and we will assume initially that their closures are also disjoint. (At the end of the proof we consider the case where their closures may intersect.) We denote their closures by $\overline{\mathcal{L}(A_1, \Omega_1)}$ and $\overline{\mathcal{L}(A_2, \mathbb{R}^2)}$. Using the representation (14) and the fact that $\overline{\mathcal{L}(A_1, \Omega_1)}$ and $\overline{\mathcal{L}(A_2, \mathbb{R}^2)}$ are disjoint closed convex sets and one of them is bounded, ie: $\overline{\mathcal{L}(A_2, \mathbb{R}^2)}$, it follows that there are infinitely many lines in the plane which separate these sets. Among these separating lines are two extreme cases which are simultaneously tangential to both sets. A line which is simultaneously tangential to the sets $\overline{\mathcal{L}(A_1, \Omega_1)}$ and $\overline{\mathcal{L}(A_2, \mathbb{R}^2)}$ can be described in terms of two normal vectors in the space of symmetric matrices. These normal vectors must be oppositely directed since by assumption there is no joint QLF for the sets. The six possible types of tangent lines to the set $\mathcal{L}(A_1, \Omega_1)$ are described by (15), (16) and (17). Every tangent line to $\mathcal{L}(A_2, \mathbb{R}^2)$ can be described by $A_2yy^T + yy^TA_2^T$ for some $y \in \mathbb{R}^2$. Setting a convex combination of these normal vectors to zero leads to the six possible cases listed below:

(i) $A_2yy^T + yy^TA_2^T + k^2(A_1x_1x_1^T + x_1x_1^TA_1^T) + m^2(A_1x_2x_2^T + x_2x_2^TA_1^T) = 0;$
(ii) $A_2yy^T + yy^TA_2^T + k^2(A_1x_1x_1^T + x_1x_1^TA_1^T) - m^2x_1x_1^T = 0;$
(iii) $A_2yy^T + yy^TA_2^T + k^2(A_1x_1x_1^T + x_1x_1^TA_1^T) - m^2A_1x_1x_1^TA_1^T = 0;$
(iv) $A_2yy^T + yy^TA_2^T + k^2(A_1x_2x_2^T + x_2x_2^TA_1^T) - m^2x_2x_2^T = 0;$
(v) $A_2yy^T + yy^TA_2^T + k^2(A_1x_2x_2^T + x_2x_2^TA_1^T) - m^2A_1x_2x_2^TA_1^T = 0;$
(vi) See (13).

These six equations lead to the singularity conditions of Theorem 2, as follows. Equation (i) can be solved by first writing $X = k^2x_1x_1^T + m^2x_2x_2^T$ so that it becomes

$$A_2yy^T + yy^TA_2^T + A_1X + XA_1^T = 0. \tag{20}$$

If X is semi-definite then this is a special case of Equation (vi), which we discuss shortly. If X is positive definite then there is a unique $\lambda > 0$ and vector w such that

$$X = \lambda yy^T + ww^T. \tag{21}$$

Inserting this into (20) gives

$$(A_2 + \lambda A_1)yy^T + yy^T(A_2 + \lambda A_1)^T + A_1ww^T + ww^TA_1^T = 0.$$

Applying Lemma 2 (and noting that $y \neq w$) yields

$$(A_2 + \lambda A_1 + \alpha A_1^{-1})y = 0$$

for some $\lambda, \alpha > 0$. Together with (21) this leads to Condition (c) of Theorem 2.

Equations (ii) and (iv) are alike and lead to similar conditions. Equation (ii) can be written as

$$A_2yy^T + yy^TA_2^T + \left(\left(k^2A_1 - \frac{m^2}{2}I_n\right)x_1x_1^T + x_1x_1^T\left(k^2A_1 - \frac{m^2}{2}I_n\right)^T\right) = 0$$

where I_n is the $n \times n$ identity matrix. If y is parallel to x_1 this leads to $(k^2A_1 - \frac{m^2}{2}I_n + \alpha A_2)x_1 = 0$ which is a special case of Condition (a). If y and x_1 are not parallel this leads to $(k^2A_1 - \frac{m^2}{2}I_n + \alpha A_2^{-1})x_1 = 0$ which is again a special case of Condition (a). Equation (iv) leads to identical conclusions with x_1 replaced by x_2 and so also leads to Condition (a).

Equations (iii) and (v) are also alike. Equation (iii) can be written as

$$A_2yy^T + yy^TA_2^T + \left(k^2A_1^{-1} - \frac{1}{2}m^2I_n\right)A_1x_1x_1^TA_1^T + A_1x_1x_1^TA_1^T\left(k^2A_1^{-1} - \frac{1}{2}m^2I_n\right)^T = 0.$$

If y and x_1 are parallel this leads to $(k^2A_1^{-1} - \frac{m^2}{2}I_n + \alpha A_2)A_1x_1 = 0$ which is a special case of Condition (b). If y and x_1 are not parallel it leads to $(k^2A_1^{-1} - \frac{m^2}{2}I_n + \alpha A_2^{-1})A_1x_1 = 0$ and this again is a special case of Condition (b). Similar reasoning applies to Equation (v).

Equation (vi) was dealt with in Theorem 5 and leads to Condition (a). This concludes the argument by showing that all possible cases of simultaneous tangent lines are covered by Conditions (a), (b) and (c). Since the existence of these lines is equivalent to the disjointness of the sets $\mathcal{L}(A_1, \Omega_1)$ and $\mathcal{L}(A_2, \mathbb{R}^2)$, this shows that the conditions are sufficient to distinguish the two sets. In the case where $\mathcal{L}(A_1, \Omega_1)$ and $\mathcal{L}(A_2, \mathbb{R}^2)$ are disjoint but with intersecting boundaries again there must be at least one simultaneous tangent hyperplane. So the above reasoning applies again and leads to the same conclusions.

($\Rightarrow$) Necessity follows in a fashion similar to in the proof of Theorem 5. Note that Condition (c) is equivalent to the existence of y, X satisfying (20). ■

Outline of proof of Theorem 3 The strategy is the same as for Theorem 2. That is, we identify lines in the plane which can be simultaneously tangent to both of the sets $\mathcal{L}(A_1, \Omega_1)$ and $\mathcal{L}(A_2, \Omega_2)$. For these lines, we equate the normal vectors to the tangents of both sets, and this leads to the (many) spectral conditions in the Theorem. ■

6 Conclusions and Future Work

In this paper we have presented a framework for solving joint quadratic Lyapunov function problems for continuous time linear state-dependent switching systems. A detailed exposition for planar systems is given, and we indicate how these results can be extended to higher-dimensional systems. These latter conditions will be explored in companion papers.

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