Matrix φ^4 Models on the Fuzzy Sphere and their Continuum Limits

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Abstract

We demonstrate that the UV/IR mixing problems found recently for a scalar φ^4 theory on the fuzzy sphere are localized to tadpole diagrams and can be overcome by a suitable modification of the action. This modification is equivalent to normal ordering the φ^4 vertex. In the limit of the commutative sphere, the perturbation theory of this modified action matches that of the commutative theory.

1 Introduction

Fuzzy models have been proposed, [1]-[4], as a potential method of doing "lattice" field theory. The basic idea is to take a classical phase space of finite volume, quantize it and thus obtain a space with a finite number of degrees of freedom. Things are of course a little more complicated but essentially

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this idea works when the phase space is a co-adjoint orbit, the simplest such example being the two sphere S^2, with the resulting quantized space known as the fuzzy sphere [5]. Field theory models on the fuzzy sphere then possess only a finite number of modes. The simplest field theory model of a scalar field with φ^4 interaction was proposed in [1] and has proved to be an excellent testing ground for the idea of using fuzzy spaces for doing 'lattice' studies.

A potential problem with using these fuzzy spaces to get a finite approximation to field theories and thus do 'lattice' physics has emerged due to the phenomenon of UV/IR mixing [6]. The problem was discussed in the Moyal plane in [7] where the fields possess an infinite number of modes, and consequently a regularization procedure is needed. This phenomenon appears to be generic for field theories in non-commutative spaces.

In a recent article Vaidya [8] pointed out that this UV/IR mixing phenomenon is present in the φ^4 model of [1] on the fuzzy sphere. His work was followed by that of Chu, Madore and Steinacker,[9], who calculated explicitly the one-loop contribution to the two point vertex function and found that in the commutative limit, the non-planar diagram retains a residual finite contribution over and above the expected commutative term. The additional term, can be seen as a nonlocal rotationally invariant contribution to the effective action. Though this term is non-singular for the fuzzy sphere they showed that in the planar limit it incorporates the UV/IR mixing singularity of the Moyal plane.

The implications of this result are very serious for the program of using the matrix model approximations to continuum field theories to study the non-perturbative continuum behaviour. One implication is that the scalar action considered by these authors cannot be the correct fuzzy action for the 'lattice' program outlined above.

We therefore return to the problem and study how serious it in fact is and whether it has a natural solution. We find that indeed it has a quite natural solution and that the problem disappears when the interaction term in the matrix action is "normal ordered", i.e. when the appropriate subtractions associated with Wick contractions of the φ^4 term are included in the action. The action (55) with normal ordered vertex is therefore the correct starting point for the fuzzy lattice physics program.

The paper is organized as follows: We begin by briefly reviewing the $*$-product, that realizes matrix multiplication at the level of functions on the fuzzy sphere. We then review the one-loop calculation of the two-point function, repeating the calculation using the $*$-product. Next we discuss the 4-point vertex functions and demonstrate that the problem anomalous contributions in that commutative limit does not arise here. We further show, in fact, that this problem is localized to the case of a propagator returning to the same vertex that it leaves, i.e. that it is associated with tadpole diagrams. We finally propose our solution for the elimination of these unwanted contributions—normal ordering the φ^4 vertex. The article concludes with a discussion and conclusions where we define the matrix model which has as limiting theory the standard continuum model.

2 Star products on the fuzzy sphere

At the level of the fundamental representation the fuzzy sphere [5] can be defined as the orbit the adjoint action of $SU(2)$ on a rank one projection operator, ρ, in a 2-dimensional Hilbert space, [3] [10], with $\rho^2 = \rho$, $\rho^\dagger = \rho$ and $\mathrm{Tr}\rho = 1$. At a point on the sphere parameterized by the unit vector $\mathbf{n}$ in $\mathbf{R}^3$, obtained by rotating the north pole by $g \in SU(2)$, we have

$$\rho(\mathbf{n}) = g\rho_0 g^\dagger \tag{1}$$

Here $\rho_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is the projector at the north pole, $\mathbf{n}_0 = (0,0,1)$. At the level of the $(L+1)$-dimensional representation we construct a rank one projector $\mathcal{P}_L(\mathbf{n})$ as the L-fold symmetric tensor product of $\rho(\mathbf{n})$.

Associated with every $(L+1)\times(L+1)$ matrix is a function on the fuzzy sphere, defined by

$$F_L(\mathbf{n}) = \mathrm{Tr}\left(\mathcal{P}_L(\mathbf{n})\hat{F}\right), \tag{2}$$

and an associative $*$-product between two such functions is given by

$$(F_L * G_L)(\mathbf{n}) = \mathrm{Tr}\left(\mathcal{P}_L(\mathbf{n})\hat{F}\hat{G}\right). \tag{3}$$

In terms of derivatives

$$(F_L * G_L)(\mathbf{n}) = \sum_{k=0}^{L} \frac{2^k (L-k)!}{L!k!}\left(\partial_{A_1}\cdots\partial_{A_k}F_L(\mathbf{n})\right)K^{A_1B_1}\cdots K^{A_kB_k}\left(\partial_{B_1}\cdots\partial_{B_k}G_L(\mathbf{n})\right), \tag{4}$$

with $K^{AB} = \frac{1}{2}(P^{AB} + iJ^{AB})$, where $P^{AB} = \delta^{AB} - n^A n^B$ and $J^{AB} = \epsilon^{AB}{}_C n^C$ so $P^2 = P$ and $J^2 = -P$ (indices $A, B, \ldots$ are raised and lowered with the Euclidean metric , δ_{AB}, in $\mathbf{R}^3$).

The action of the $SU(2)$ generators on a function on the fuzzy sphere can be written as

$$\mathcal{L}_A F_L(\mathbf{n}) = \mathrm{Tr}\left(\mathcal{P}_L(\mathbf{n})[\hat{F}, L_A]\right), \tag{5}$$

where L_A are the generators in the $(L+1)$-dimensional matrix representation and $\mathcal{L}_A = i\epsilon_{AB}{}^C n^B \partial_C$ are the generators in differential form.

The $L+1$ dimensional matrices can be expanded in terms of an orthonormal basis of matrices $\hat{Y}_{l,m}$ normalized so that

$$\frac{4\pi}{(L+1)} \mathrm{Tr}\Big((\hat{Y}_{l',m'})^\dagger \hat{Y}_{l,m}\Big) = \delta_{l'l}\delta_{m'm}. \tag{6}$$

The image of these matrices under the map to functions (2) we denote $Y^L_{l,m}(\mathbf{n}) = \mathrm{Tr}(\mathcal{P}_L(\mathbf{n})\hat{Y}_{l,m})$. These functions are proportional to the usual spherical harmonics $Y_{l,m}(\mathbf{n})$ with

$$Y^L_{l,m}(\mathbf{n}) = T_L^{1/2}(l)\, Y_{l,m}(\mathbf{n}), \quad \text{where} \quad T_L(l) = \frac{L!(L+1)!}{(L-l)!(L+l+1)!} \tag{7}$$

and are polynomials in $\mathbf{n}$ of order l, [2]. For a general function $F_L(\mathbf{n})$ which is the image of the matrix $\hat{F}$ the relation (7) takes the form

$$F_L(\mathbf{n}) = \mathcal{T}_L^{1/2}(\mathcal{L}^2) F(\mathbf{n}), \tag{8}$$

where the rotationally invariant operator $\mathcal{T}_L^{1/2}(\mathcal{L}^2)$ has eigenvalues $T_L^{1/2}(l)$. Eq. (8) defines the relation between functions which we denote with a subscript L and those without this subscript. The operator $\mathcal{T}_L(\mathcal{L}^2)$ was introduced in [11], where an alternative expression for it can be found.

3 Fuzzy action

Using the fact that

$$\int_{S^2} d^2\mathbf{n}\, (F_L * G_L)(\mathbf{n}) = \frac{4\pi}{(L+1)} \mathrm{Tr}(\hat{F}\hat{G}), \tag{9}$$

the integral of the $*$-product in equation (9) can be related to the integral over an ordinary commutative product by decomposing the functions F_L

and G_L into harmonics, (note they only contain $l \leq L$), and rescaling each component by an l-dependent factor $T_L^{1/2}(l)$, then

$$\int d^2\mathbf{n}(F_L * G_L)(\mathbf{n}) = \int d^2\mathbf{n} F(\mathbf{n}) G(\mathbf{n}). \tag{10}$$

where the pairs $F(\mathbf{n})$, $F_L(\mathbf{n})$ and $G(\mathbf{n})$, $G_L(\mathbf{n})$ are related as in (8).

The naïve matrix action for a real scalar φ^4 theory on S_F^2 at level L is

$$S_L[\Phi] = \frac{4\pi}{(L+1)} \mathrm{Tr}\left\{\frac{1}{2}[L_A,\Phi][L_A,\Phi] + \frac{1}{2} r\Phi^2 + \frac{\lambda}{4!}\Phi^4\right\} \tag{11}$$

$$= \int_{S^2} d^3\mathbf{n}\left\{\frac{1}{2}\left(\mathcal{L}_A\varphi_L * \mathcal{L}_A\varphi_L + r\varphi_L * \varphi_L\right) + \frac{\lambda}{4!}(\varphi_L * \varphi_L * \varphi_L * \varphi_L)\right\}, \tag{12}$$

where $\varphi_L(\mathbf{n}) = \mathrm{Tr}\left(\mathcal{P}_L(\mathbf{n})\Phi\right)$ and $\Phi = \Phi^\dagger$, so that φ_L is a real field. The parameters r and λ are real with λ positive.

The coherent state representation of the free-field propagator between two points $\mathbf{n}$ and $\mathbf{n}'$ on the fuzzy sphere is

$$\mathcal{G}_L(\mathbf{n},\mathbf{n}') = \sum_{l=0}^{L}\sum_{m=-l}^{l} \frac{\overline{Y}^L_{l,m}(\mathbf{n}) Y^L_{l,m}(\mathbf{n}')}{l(l+1)+r}. \tag{13}$$

It is symmetric under interchange of its arguments, $\mathcal{G}_L(\mathbf{n},\mathbf{n}') = \mathcal{G}_L(\mathbf{n}',\mathbf{n})$ because $\overline{Y}^L_{l,m} = (-1)^m Y^L_{l,-m}$, and can be expressed in terms of Legendre polynomials:

$$\mathcal{G}_L(\mathbf{n},\mathbf{n}') = \frac{1}{4\pi}\sum_{l=0}^{L} \frac{T_L(l)(2l+1)}{l(l+1)+r} P_l(\mathbf{n}.\mathbf{n}') \tag{14}$$

In the continuum limit ($L \to \infty$) the massless free-field propagator reads (omitting the $l = 0$ term)

$$\mathcal{G}'_L(\mathbf{n},\mathbf{n}') = \frac{1}{4\pi}\sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} P_l(\mathbf{n}.\mathbf{n}') = -\frac{1}{4\pi}\ln\left(\frac{1-\mathbf{n}.\mathbf{n}'}{2}\right). \tag{15}$$

Note that the propagator in (13) is not defined with a $*$-star product, since the two points $\mathbf{n}$ and $\mathbf{n}'$ are independent and in general distinct.

It was pointed out in [8] and [9] that there is a subtlety in the one-loop correction to the two-point function for this theory. For the commutative theory the only infinities arise in the tadpole diagram and are accounted for by a mass renormalization, but the fuzzy model (or fuzzy regularization) introduces new momentum dependent terms into the effective action even at one-loop. These are not just wave-function renormalizations but are new momentum dependent two-point interactions that do not disappear in the continuum $L \to \infty$ limit. These terms were interpreted in [8] as being due to UV/IR mixing. Their presence however means that the fuzzy action (11) does not reduce to the usual φ^4 theory in the continuum limit. It is therefore unsatisfactory as a matrix approximation of the commutative model.

4 Fuzzy 2-point functions at one Loop

The problem arises because in non-commutative φ^4 there are two different diagrams at one loop which contribute to the quadratic term in the effective action, a planar diagram and a non-planar diagram, see Fig 1.

Using cyclic symmetry of the vertex it is easy to see that the 8 ways of connecting the legs in the planar diagram contribute $\frac{\lambda}{3} I_P[\varphi_L]$ to the effective action where

$$I_P[\varphi_L] = \int d^2\mathbf{n} \sum_{l,m} \left\{ \frac{\varphi_L(\mathbf{n}) * \overline{Y}^L_{l,m}(\mathbf{n}) * Y^L_{l,m}(\mathbf{n}) * \varphi_L(\mathbf{n})}{l(l+1)+r} \right\}, \tag{16}$$

and $\sum_{l,m} := \sum_{l=0}^{L} \sum_{m=-l}^{l}$. Similarly the 4 ways of connecting the legs in the non-planar diagram contribute $\frac{\lambda}{6} I_{NP}[\varphi_L]$ where

$$I_{NP}[\varphi_L] = \int d^2\mathbf{n} \sum_{l,m} \left\{ \frac{\varphi_L(\mathbf{n}) * \overline{Y}^L_{l,m}(\mathbf{n}) * \varphi_L(\mathbf{n}) * Y^L_{l,m}(\mathbf{n})}{l(l+1)+r} \right\}, \tag{17}$$

The planar diagram can be evaluated rather straightforwardly by noting that

$$\sum_{m=-l}^{l} \hat{Y}^\dagger_{lm} \hat{Y}_{lm} = \frac{(2l+1)}{4\pi} \mathbf{1}, \tag{18}$$

so that $\sum_m \overline{Y}^L_{l,m}(\mathbf{n}) * Y^L_{l,m}(\mathbf{n}) = \frac{2l+1}{4\pi}$ and the final $*$ can be eliminated by replacing $\varphi_L(\mathbf{n}) = \mathcal{T}_L^{1/2}(\mathcal{L}^2)\varphi(\mathbf{n})$ with $\varphi(\mathbf{n})$. So we have

$$I_P[\varphi_L] = \bigcirc\!\!\!\bullet\,(L,r) \int d^2\mathbf{n}\ \varphi^2(\mathbf{n}), \tag{19}$$

where

$$\bigcirc\!\!\!\!\bullet\,(L,r) = \frac{1}{4\pi}\sum_{l=0}^{L}\frac{2l+1}{l(l+1)+r} \tag{20}$$

is a logarithmically divergent function in the limit of large L.

The difference between the planar and non-planar diagrams is

$$\Delta I_2[\varphi] := I_P[\varphi] - I_{NP}[\varphi] = \int d^2\mathbf{n}\sum_{l,m}\left\{\frac{\varphi_L(\mathbf{n}) * \overline{Y}^L_{l,m}(\mathbf{n}) * [Y^L_{l,m}(\mathbf{n}), \varphi_L(\mathbf{n})]_*}{l(l+1)+r}\right\}, \tag{21}$$

where $[F_L, G_L]_* := F_L * G_L - G_L * F_L$.

Using the trace formula (9) for the integral, and the fact that $(\hat{Y}^L_{l,m})^\dagger = (-1)^m \hat{Y}^L_{l,-m}$, cyclicity of the trace allows the sum and integral to be re-arranged as

$$\Delta I_2[\varphi_L] = \frac{1}{2}\int d^2\mathbf{n}\sum_{l,m}\left\{\frac{[\varphi_L(\mathbf{n}), \overline{Y}^L_{l,m}(\mathbf{n})]_* * [Y^L_{l,m}(\mathbf{n}), \varphi_L(\mathbf{n})]_*}{l(l+1)+r}\right\}. \tag{22}$$

The explicit expression for the $*$-product in (4) shows that the commutators start at order $1/L$ in the large L-limit, so one might expect that ΔI_2 would be of order $1/L^2$ and hence vanish as $L \to \infty$. But the problem is that the derivatives in the $*$-product introduce factors of momenta into the numerator, which are summed to L, rendering the result of order one. Expanding φ_L in spherical harmonics, $\varphi_L(\mathbf{n}) = \sum_{l=0}^{L} a_{l,m} Y^L_{l,m}(\mathbf{n})$, shows that there is a momentum dependent contribution to the quadratic term in the effective action. For example if we restrict ourselves to fields φ_L that are linear functions of $\mathbf{n}$, i.e. fields with angular momentum $l=0$ and $l=1$ we see from (4) that the derivatives involving φ_L terminate at $k=1$, so we can write

$$[\varphi_L, \overline{Y}^L_{l,m}]_* = \frac{2i}{L}(\partial_A\varphi_L)J^{AB}(\partial_B\overline{Y}^L_{l,m}) = -\frac{2}{L}(\partial_A\varphi_L)(\mathcal{L}_A\overline{Y}^L_{l,m}), \tag{23}$$

and we get

$$\Delta I_2[\varphi_L] = \frac{2}{L^2}\int d^2\mathbf{n}\sum_{l,m}\frac{\{(\partial_A\varphi_L)(\mathcal{L}_A\overline{Y}^L_{l,m})\} * \{(\mathcal{L}_B Y^L_{l,m})(\partial_B\varphi_L)\}}{l(l+1)+r}. \tag{24}$$

Since $\partial_A\varphi_L$ are constants, in accordance with equations (10) and (7) the remaining $*$ in this expression can be replaced with an ordinary product provided we replace $Y^L_{l,m}$ with $Y_{l,m}$. Thus

$$\Delta I_2[\varphi_L] = -\frac{2}{L^2}\int d^2\mathbf{n}\sum_{l,m}\frac{(\partial_A\varphi_L)(\mathcal{L}_A\overline{Y}_{l,m})(\mathcal{L}_BY_{l,m})(\partial_B\varphi_L)}{l(l+1)+r}. \tag{25}$$

The problem now reduces to the evaluation of

$$\sum_{m=-l}^{l}\Big(\mathcal{L}_A\overline{Y}_{l,m}(\mathbf{n})\Big)\Big(\mathcal{L}_BY_{l,m}(\mathbf{n})\Big). \tag{26}$$

Its exact form can be determined by noting that

$$\sum_{m=-l}^{l}\overline{Y}_{l,m}(\mathbf{n})Y_{l,m}(\mathbf{n}) = \left(\frac{2l+1}{4\pi}\right), \tag{27}$$

so

$$\mathcal{L}^2\left(\sum_m\overline{Y}_{l,m}Y_{l,m}\right) = 2\left\{l(l+1)\left(\frac{2l+1}{4\pi}\right)+\sum_m\Big(\mathcal{L}_A\overline{Y}_{l,m}\Big)\Big(\mathcal{L}^AY_{l,m}\Big)\right\} = 0. \tag{28}$$

from which we deduce that

$$\sum_{m=-l}^{l}\Big(\mathcal{L}_A\overline{Y}_{l,m}(\mathbf{n})\Big)\Big(\mathcal{L}_BY_{l,m}(\mathbf{n})\Big) = -\frac{1}{2}\frac{(2l+1)}{4\pi}l(l+1)P_{AB}. \tag{29}$$

Finally, noting that $\partial_A\varphi_L$ only involves $l=1$ we can write $\partial_A\varphi_L = \sqrt{\frac{L}{L+2}}\partial_A\varphi$, with φ expanded in terms of the ordinary Y_{lm} so that we obtain

$$\Delta I_2[\varphi_L] = \int d^2\mathbf{n}\mathcal{L}_A\varphi\frac{1}{L(L+2)}\sum_{l=1}^{L}\left(\frac{l(l+1)(2l+1)}{l(l+1)+r}\right)\mathcal{L}_A\varphi, \tag{30}$$

In particular we have

$$\Delta I_2[Y^L_{1,0}] = \frac{2}{L(L+2)}\sum_{l=1}^{L}\left(\frac{l(l+1)(2l+1)}{l(l+1)+r}\right), \tag{31}$$

which is clearly finite as $L\to\infty$. The coefficient $\frac{2}{L(L+2)}\sum_{l=1}^{L}\left(\frac{l(l+1)(2l+1)}{l(l+1)+r}\right)$ was derived previously using other methods in [9].

For higher l the $*$-products in the commutators in (22) involve terms up to $k=l$ which look as though they should be of order $1/L^{2l}$, but again the derivatives in the numerator conspire to give a finite answer. The full expression can be obtained by noting that the linear operator

$$\mathcal{R}^L_{l_1}\Phi = \frac{4\pi}{2l_1+1}\sum_{m_1=-l_1}^{l_1} \hat{Y}^\dagger_{l_1,m_1}\Phi\hat{Y}_{l_1,m_1} \tag{32}$$

is rotationally invariant and has eigenvectors $\hat{Y}_{l_2,m_2}$ and m_2 independent eigenvalues $\lambda^L_{l_1,l_2}$ given by

$$\lambda^L_{l_1,l_2} = \frac{(4\pi)^2}{(2l_1+1)(2l_2+1)(L+1)}\sum_{m_1,m_2} \mathrm{Tr}[\hat{Y}^\dagger_{l_1,m_1}\hat{Y}_{l_2,m_2}\hat{Y}_{l_1,m_1}\hat{Y}^\dagger_{l_2,m_2}] \tag{33}$$

Using expressions from Varshalovich et al [12] (equations (24) on page 46 and equation (12) page 236) we obtain an intermediate expression in terms of Wigner 6j-symbols which can be summed to give

$$\lambda^L_{l_1,l_2} = (L+1)(-1)^{l_1+l_2+L}\left\{\begin{array}{ccc} l_1 & \frac{L}{2} & \frac{L}{2} \\ l_2 & \frac{L}{2} & \frac{L}{2}\end{array}\right\}. \tag{34}$$

Using an explicit expression for the 6j-symbols (equation 6, page 294 of [12]) allows us to express the eigenvalues in terms of the eigenvalues of $\mathcal{T}_k(\mathcal{L}^2)$ so that we find

$$\lambda^L_{l_1,l_2} = (L+1)(-1)^{l_1+l_2}\sum_{k=0}^{L}(-1)^k\begin{pmatrix} L+k+1 \\ k+1\end{pmatrix}\begin{pmatrix} L \\ k\end{pmatrix} T_k(l_1)T_k(l_2). \tag{35}$$

We can therefore express operator $\mathcal{R}_{l_1}$ in the form

$$\mathcal{R}^L_{l_1}(\mathcal{L}^2) = (L+1)(-1)^{l_1}\sum_{k=0}^{L}(-1)^{k+\mathcal{N}}\begin{pmatrix} L+k+1 \\ k+1\end{pmatrix}\begin{pmatrix} L \\ k\end{pmatrix} T_k(l_1)\mathcal{T}_k(\mathcal{L}^2), \tag{36}$$

where $\mathcal{N}$ is defined by $\mathcal{N}Y_{l,m}(\mathbf{n}) = lY_{l,m}(\mathbf{n})$ and $\mathcal{L}^2 = \mathcal{N}(\mathcal{N}+1)$. Finally we find the contribution to the effective action of the difference of planar and non-planar diagrams can be written as

$$\Delta_L[\varphi_L] = \int d^2\mathbf{n}\varphi_L(\mathbf{n})\mathcal{Q}_L(\mathcal{L}^2)\varphi_L(\mathbf{n}), \tag{37}$$

where the operator $\mathcal{Q}_L(\mathcal{L}^2)$ is given by

$$\mathcal{Q}_L(\mathcal{L}^2) = \frac{1}{4\pi}\sum_{l=0}^{L}\frac{2l+1}{l(l+1)+r}\left(\mathcal{R}^L_l(\mathcal{L}^2)-1\right). \tag{38}$$

The eigenvalues of this operator agree with those found in [9]. This operator itself can clearly be expressed as the trace of a function of $\mathcal{L}^2$.

It is argued in [9] that the one-loop contribution to the two-point function for external momentum l is given, to a very good approximation, by

$$\Delta I_2[Y^L_{l,m}] = 2\sum_{k=1}^{l}\frac{1}{k} + o(1/L). \tag{39}$$

We can recover this estimate by noting that Y^L_{lm} are eigenfunctions of our operator $\mathcal{Q}_L$ and using the expansions of [9].

5 Fuzzy 4-point functions at one loop

In this section we argue that the problem described in the last section is specific to the 2-point function and does not affect the 4-point function at one loop. The one-loop four-point function has two vertices and in the non-commutative case these can be either planar or non-planar, giving rise to four distinct contributions to the quartic term in the effective action. The four distinct diagrams are given in figures 2 and 3. Stripping off irrelevant factors these diagrams are:

$$\begin{aligned} I_{P,P}[\varphi] : \;=\; & \int d^2\mathbf{n}' d^2\mathbf{n}'' \sum_{l',m'}\sum_{l'',m''}\left\{\frac{\left(\varphi_L * \overline{Y}^L_{l',m'} * Y^L_{l'',m''} * \varphi_L\right)_{\mathbf{n}'}}{l'(l'+1)+r}\right. \\ & \left. \times\frac{\left(\varphi_L * \overline{Y}^L_{l'',m''} * Y^L_{l',m'} * \varphi_L\right)_{\mathbf{n}''}}{l''(l''+1)+r}\right\} \end{aligned} \tag{40}$$

$$\begin{aligned} I_{P,\overline{P}}[\varphi] : \;=\; & \int d^2\mathbf{n}' d^2\mathbf{n}'' \sum_{l',m'}\sum_{l'',m''}\left\{\frac{\left(\varphi_L * \overline{Y}^L_{l',m'} * Y^L_{l'',m''} * \varphi_L\right)_{\mathbf{n}'}}{l'(l'+1)+r}\right. \\ & \left. \times\frac{\left(\varphi_L * \overline{Y}^L_{l',m'} * Y^L_{l'',m''} * \varphi_L\right)_{\mathbf{n}''}}{l''(l''+1)+r}\right\} \end{aligned} \tag{41}$$

$$
\begin{aligned}
I_{N,P}[\varphi]: \quad &= \int d^2\mathbf{n}' d^2\mathbf{n}'' \sum_{l',m'} \sum_{l'',m''} \left\{ \frac{\left(\varphi_L * \overline{Y}^L_{l',m'} * \varphi_L * Y^L_{l'',m''}\right)_{\mathbf{n}'}}{l'(l'+1)+r} \right. \\
&\qquad \left. \times \frac{\left(\varphi_L * \overline{Y}^L_{l'',m''} * Y^L_{l',m'} * \varphi_L\right)_{\mathbf{n}''}}{l''(l''+1)+r} \right\}
\end{aligned}
\tag{42}
$$

and

$$
\begin{aligned}
I_{N,N}[\varphi]: \quad &= \int d^2\mathbf{n}' d^2\mathbf{n}'' \sum_{l',m'} \sum_{l'',m''} \left\{ \frac{\left(\varphi_L * \overline{Y}^L_{l',m'} * \varphi_L * Y^L_{l'',m''}\right)_{\mathbf{n}'}}{l'(l'+1)+r} \right. \\
&\qquad \left. \times \frac{\left(\varphi_L * \overline{Y}^L_{l'',m''} * \varphi_L * Y^L_{l',m'}\right)_{\mathbf{n}''}}{l''(l''+1)+r} \right\}.
\end{aligned}
\tag{43}
$$

Surprisingly diagrams with the interchange in the second factor of $(l', m') \leftrightarrow (l'', m'')$ (which we indicate with a bar on the diagram label) are generally not equal to their unbarred counterpart, even though this corresponds to a spatial twist of π in third dimension out of the plane of the paper. However only the above four are in fact distinct, the one new contribution being $I_{P,\overline{P}}$, the others are not new but related those given by the identities $I_{N,N} = I_{N,\overline{N}}$ and $I_{N,P} = I_{N,\overline{P}}$.

Our aim is to show that these diagrams have no anomalous contributions i.e. that their limiting form for $L \to \infty$ is that of the commutative theory. We begin by arguing that the large L limit of the planar contribution $I_{P,P}$ is the commutative one. For this we look at the contribution to I_{PP} comming from the process $\hat{Y}_{l_1,m_1}\hat{Y}_{l_2,m_2} \to \hat{Y}_{l_3,m_3}\hat{Y}_{l_4,m_4}$ which after simplifying yields:

$$
\begin{aligned}
\mathbb{P}^{l_1,m_1,l_2,m_2}_{l_3,m_3;l_4,m_4} \quad &= \sum_{l,l',l'',m} \frac{(2l+1)(2l'+1)(2l''+1)(-1)^{l+m}}{(l'(l'+1)+r)(l''(l''+1)+r)} \\
&\qquad \times \begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} l_3 & l_4 & l \\ m_3 & m_4 & -m \end{pmatrix} I^{L,l,l',l''}_{l_1,l_2,l_3,l_4}
\end{aligned}
\tag{44}
$$

where we have defined

$$I^{L,l,l',l''}_{l_1,l_2,l_3,l_4} = \frac{(L+1)^2}{(4\pi)^2}\sqrt{(2l_1+1)(2l_2+1)(2l_3+1)(2l_4+1)} \times \left\{ \begin{array}{ccc} l_1 & l_2 & l \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2} \end{array} \right\} \left\{ \begin{array}{ccc} l_3 & l_4 & l \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2} \end{array} \right\} \left\{ \begin{array}{ccc} l' & l'' & l \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2} \end{array} \right\}^2 . \quad (45)$$

Because of the triangular inequalities

$$|l'-l''| \leq l \leq min(l_1+l_2, l_3+l_4)$$

it is first established that

$$|I_{P,\overline{P}}| < C \sum_{l'=0}^{L} \frac{(2l'+1)^2}{(l'(l'+1)+r)^2} \quad (46)$$

for a positive constant C, depending on the external momenta. This established that the $L \to \infty$ limit of $I_{P,P}$ is finite and we can therefore interchange the process of taking the large L limit and the summations. The limiting form of $I_{P,P}$ is then obtained by noting that for $L \to \infty$ we have

$$\left\{ \begin{array}{ccc} l_1 & l_2 & l \\ \frac{L}{2} & \frac{L}{2} & \frac{L}{2} \end{array} \right\} \sim \frac{(-1)^{l_1+l_2+l}}{\sqrt{L+1}} \left(\begin{array}{ccc} l_1 & l_2 & l \\ 0 & 0 & 0 \end{array} \right) + O(L^{-3/2}) \quad (47)$$

Then substituting this into (45) we obtain the leading asymptotic, which is readily seen to be the commutative expression.

The contribution to the diagram $I_{P,\overline{P}}$ can be obtained from that for $I_{P,P}$ (44) but the corresponding $I^{L,l,l',l''}_{l_1,l_2,l_3,l_4}$ acquires the additional phase factor $(-1)^{l+l'+l''}$.

This procedure can be followed for the remaining diagrams, to establish that they also reduce to the commutative limit. However, this is not very insightful so we shall take a different tact. We shall show that the difference of diagrams vanishes as $L \to \infty$, and so establish that they all have the same limit, which from the above we know is the commutative limit.

Consider, therefore, difference between (40) and (42),

$$\Delta I_4[\varphi] : \;=\; I_{P,P}[\varphi] - I_{N,P}[\varphi] = \int d^2\mathbf{n}' d^2\mathbf{n}'' \sum_{l',m'} \sum_{l'',m''} \left\{ \frac{\Big(\varphi_L * \overline{Y}^L_{l',m'} * [Y^L_{l'',m''}, \varphi_L]_*\Big)_{\mathbf{n}'} \Big(\varphi_L * \overline{Y}^L_{l'',m''} * Y^L_{l',m'} * \varphi_L\Big)_{\mathbf{n}''}}{\Big(l'(l'+1)+r\Big)\Big(l''(l''+1)+r\Big)} \right\}. \tag{48}$$

Again one can re-arrange traces and sums to express this as

$$\Delta I_4[\varphi] : \;=\; \frac{1}{2}\int d^2\mathbf{n}' d^2\mathbf{n}'' \sum_{l',m'} \sum_{l'',m''} \left\{ \frac{\Big([\varphi_L, \overline{Y}^L_{l',m'}]_* * [Y^L_{l'',m''}, \varphi_L]_*\Big)_{\mathbf{n}'} \Big(\varphi_L * \overline{Y}^L_{l'',m''} * Y^L_{l',m'} * \varphi_L\Big)_{\mathbf{n}''}}{\Big(l'(l'+1)+r\Big)\Big(l''(l''+1)+r\Big)} \right\}. \tag{49}$$

Naïvely one might infer from the commutators to that this difference is of order $1/L^2$, but the analysis of the two point function shows that one must be more careful. In fact we shall show that, for the four-point function, this expectation is indeed correct. The crucial difference from the two-point function is that neither of the propagators here closes on itself.

Expanding the commutators to first order, as was done for the 2-point function, gives

$$\begin{aligned}\Delta I_4[\varphi] \;\approx\; & \frac{2}{L^2}\int d^2\mathbf{n}' d^2\mathbf{n}'' \Big\{ \Big(\mathcal{L}_A \varphi_L(\mathbf{n}')\Big) J^{AB}(\mathbf{n}') \Big(\mathcal{L}'_B \mathcal{G}_L(\mathbf{n}',\mathbf{n}'')\Big) \\ & \times \Big(\mathcal{L}_D \varphi_L(\mathbf{n}')\Big) J^{DC}(\mathbf{n}') \Big(\mathcal{L}'_C \mathcal{G}_L(\mathbf{n}',\mathbf{n}'')\Big) \varphi_L^2(\mathbf{n}'') \Big\}, \end{aligned} \tag{50}$$

plus corrections down by further factors of $1/L$.

The simplest way to study the structure of $\Delta I_4[\varphi]$ as $L \to \infty$ is to write the propagators using (14) and observe that

$$\mathcal{L}'_B \mathcal{G}_L(\mathbf{n}'.\mathbf{n}'') = i(\mathbf{n}' \times \mathbf{n}'')_B \sum_{l'=0}^{L} \left(\frac{(2l'+1)P'_{l'}(\mathbf{n}'.\mathbf{n}'')(1+O(1/L^2))}{l'(l'+1)+r} \right), \tag{51}$$

where $P'_{l'}$ is the derivative of the Legendre polynomial. Any singularity in the integrand in (50) is coming from terms with l' and l'' near L and most of the support of the Legendre polynomials for such high orders is concentrated within a region of width $1/L$ at the edges of the interval $-1 \leq \mathbf{n}'.\mathbf{n}'' \leq 1$. This allows us to set $\mathbf{n}' \approx \mathbf{n}''$ in the argument of the φ_L terms in (50). Since $P'_{l'}(1) = \frac{l'(l'+1)}{2}$ we can approximate the sum in (51) by $\frac{1}{2}\sum_{l'=0}^{L}\left(\frac{(2l'+1)l'(l'+1)}{l'(l'+1)+r}\right) \approx L^2/2$.

Finally we split the double integral up into centre-of-mass co-ordinates $\mathbf{n}_1$ and relative co-ordinates $\mathbf{n}_2$. The relative co-ordinates are integrated over a set of measure $1/L^2$ on the fuzzy sphere and in this region $|\mathbf{n}' \times \mathbf{n}''| \approx 1/L$. This implies that (50) is approximately

$$\Delta I_4[\varphi] \approx -\frac{L^2}{2}\int d^2\mathbf{n}_1\Big(\mathcal{L}_A\varphi_L^2(\mathbf{n}_1)\Big)J^{AB}(\mathbf{n}_1)\Big(\mathcal{L}_D\varphi_L^2(\mathbf{n}_1)\Big)J^{DC}(\mathbf{n}_1) \times \int_{1/L^2} d^2\mathbf{n}_2(\mathbf{n}_1\times\mathbf{n}_2)_B(\mathbf{n}_1\times\mathbf{n}_2)_C, \qquad (52)$$

where the integral over $\mathbf{n}_2$ is over a region of area $1/L^2$ in which $|\mathbf{n}_1 \times \mathbf{n}_2| \approx 1/L$. The second integral above is therefore of order $1/L^4$ hence the whole expression is of order $1/L^2$ and so vanishes in the continuum limit, $L \to \infty$. Thus, in the $L \to \infty$ limit, there is no difference between the planar and non-planar contributions to the four-point coupling at one-loop. A similar conclusion holds for the differences (40) and (41), (40) and (43), and between (42) and (43).

We have have only examined the first term in the $*$-star product expansion, but the result can be similarly established for higher derivatives as well, since the crucial observation was that all derivatives of Green functions only involve Green functions with different arguments and only the double integral ever allows these points to co-incide.

It is relatively easy to extend the above analysis to higher vertex functions and higher loop orders. One can, in fact, see that, for this two dimensional theory on the fuzzy sphere, as long as the free propagator entering the diagram does not return to the same vertex from which it departs the diagram will limit to the commutative on and there will be no anomalous contribution. In other words for this theory anomalous contributions are restricted to tadpole diagrams.

6 Continuum φ^4-theory

We have seen that the naïve non-commutative action (9) does not reproduce standard φ^4 theory in the $L \to \infty$ limit, but instead gives a continuum theory with a more complicated momentum dependence in the quadratic term. However the only source of this deviation from the standard theory is the tadpole diagrams and this allows it to be removed in a standard manner, by using a normal ordered vertex. By a normal ordered vertex we mean one that has subtracted all possible self contractions with a suitable propagator. To this end we define a modified bare action in matrix form,

$$\tilde{S}_L[\Phi] = \frac{4\pi}{(L+1)} \mathrm{Tr} \left\{ \frac{1}{2}[L_A, \Phi][L_A, \Phi] + \frac{1}{2} t \Phi^2 + \frac{\lambda}{4!} : \Phi^4 : \right\}, \tag{53}$$

where the normal ordered interaction can be written

$$\mathrm{Tr} : \Phi^4 := \mathrm{Tr} \left\{ \Phi^4 - 12 \sum_{l,m} \left(\frac{\Phi \hat{Y}^\dagger_{lm} \hat{Y}_{lm} \Phi}{l(l+1)+t} \right) + 2 \sum_{l,m} \left(\frac{[\Phi, \hat{Y}_{lm}]^\dagger [\Phi, \hat{Y}_{lm}]}{l(l+1)+t} \right) \right\}. \tag{54}$$

The middle term is the usual tadpole subtraction that renders the two dimensional theory finite and corresponds to normal ordering the commutative vertex. The last term is the additional, manifestly positive, noncommutative subtraction that is necessary to obtain the correct commutative limit. A more transparent form of this new action is given by

$$\tilde{S}_L[\Phi] = \frac{4\pi}{(L+1)} \mathrm{Tr} \left\{ \frac{1}{2} \Phi \left(\hat{\mathcal{L}}^2 - \frac{\lambda}{2} \mathcal{Q}_L(\hat{\mathcal{L}}^2) + t - \frac{\lambda}{2} \bigcirc\!\!\!\bullet\,(L,t) \right) \Phi + \frac{\lambda}{4!} \Phi^4 \right\}, \tag{55}$$

where $\hat{\mathcal{L}}_i \Phi = [L_i, \Phi]$, $\mathcal{Q}_L$ is defined in (38) and $\bigcirc\!\!\!\bullet\,(L,t)$ in (20). Note that the first terms can be interpreted as a momentum dependent wavefunction renormalization since $\mathcal{Q}_L(\hat{\mathcal{L}}^2)$ is a power series in $\hat{\mathcal{L}}^2$ which starts at order $\hat{\mathcal{L}}^2$ and therefore we could have written

$$\hat{\mathcal{L}}^2 - \frac{\lambda}{2} \mathcal{Q}_L(\hat{\mathcal{L}}^2) = \mathcal{L}^2 \mathcal{Z}_L(\hat{\mathcal{L}}^2). \tag{56}$$

This theory, (55), is therefore the correct matrix model that represents a lattice regularization of commutative theory. Note that we have further replaced the parameter r with t, as the two are different. In a purely commutative theory the replacement $r = t - \frac{\lambda}{2}\bigcirc\!\!\bullet\,(L,t)$ establishes the relationship between the two parameters. This can be iterated to generate all tadpole contributions.

The new quadratic terms exactly cancel all the unwanted momentum dependent quadratic terms in the effective action arising from non-planar diagrams in the non-commutative theory and the continuum limit of this theory is the standard φ^4 theory in 2 dimensions.

Curiously, for this new model the direct large L limit of the action does not give the commutative action but rather gives a non-local action. However this non-locality is precisely what is need for the full quantum field theory to reproduce the commutative limit.

For a complex scalar field on the fuzzy sphere there are two potential vertices: $\mathrm{Tr}(\Phi^\dagger\Phi\Phi^\dagger\Phi)$ and $\mathrm{Tr}(\Phi^\dagger\Phi^\dagger\Phi\Phi)$. The first has only a planar tadpole and so will have no anomalous contribution, however the second has a non-planar and therefore anomalous tadpole. Again the normal ordering prescription removes the unwanted contribution. In fact for any two dimensional φ^4 model with global $O(N)$ symmetry normal ordering the vertex will guarantee the commutative limit is recovered.

We have here only established the correct model for the two dimensional case. A case that warrants further attention is that of φ^4 for a four dimensional fuzzy space. The simplest example is to take the case $S_F^2 \times S_F^2$. In this case it is easily seen from the above techniques that the problems are more severe and that there are additional residual non-local differences for the two and four-point functions. This is due to the divergences that appear in the commutative limit. It therefore appears to be difficult to establish precisely what fuzzy model will reproduce the commutative limit. We leave this question open for the moment but hope to return to it in the near future.

As a final comment we observe that the normal ordering prescription proposed here removes the UV/IR mixing problems for the two dimensional φ^4 model on the Moyal plane. This is readily seen since the this latter model can be recovered as a particular scaling limit of the model considered here [9].

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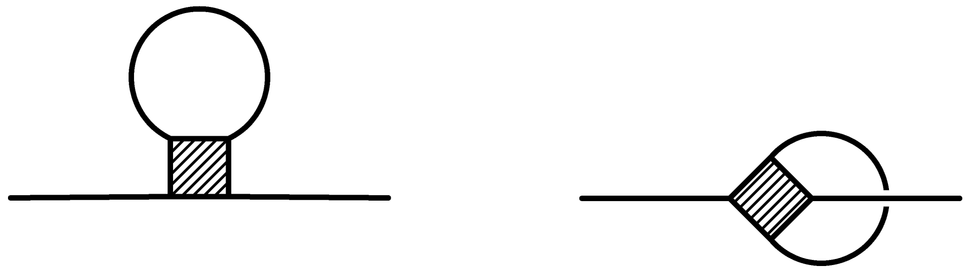

Figure 1: One-loop diagrams contributing to the 2-point function. The planar diagram is on the left and the non-planar on the right.

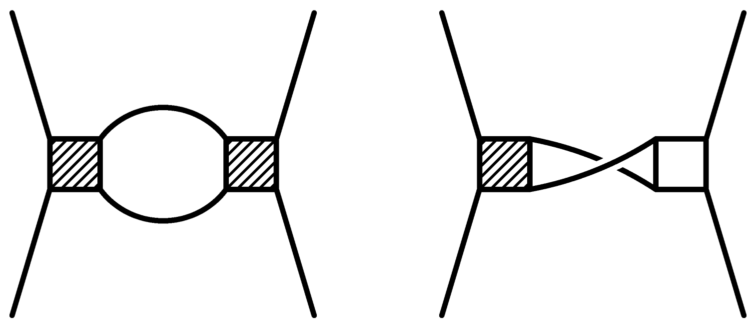

Figure 2: One-loop planar diagrams contributing to the 4-point function. The diagram on the left has two planar vertices, the one on the right has the right vertex rotated by π.

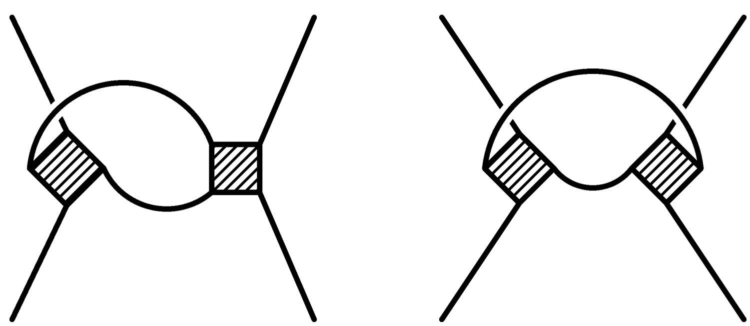

Figure 3: The non planar diagrams: The left one has one planar and one non-planar vertex while the diagram on the right has two non-planar vertices.