ROUGH CAT(0) SPACES

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Abstract. We investigate various notions of rough CAT(0). These conditions define classes of spaces that strictly include the union of all Gromov hyperbolic length spaces and all CAT(0) spaces.

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1. Introduction

Gromov hyperbolic spaces and CAT(0) spaces have been intensively studied, in particular with regard to their boundary theories, which display many common features as for instance the presence of canonical boundary topologies. It is thus natural to ask whether there is a 'unified theory' including Gromov hyperbolic spaces, CAT(0) spaces, and more, together with as much common boundary theory as possible. In this paper we discuss various possible variants of such a 'unified theory' of so-called rough CAT(0) spaces, also taking into consideration some existing weak notions of nonpositive curvature. We first investigate properties of the interior of such spaces, such as the property of having (roughly) unique geodesics, and then produce non-trivial examples of rough CAT(0) spaces. In a sequel of this paper [6], we investigate the boundary theory within the 'unified theory' of Gromov hyperbolic and CAT(0) spaces introduced here.

Recall that in the context of geodesic metric spaces, δ-hyperbolic spaces, $\delta \geq 0$, are spaces with the property that for every geodesic triangle, each side of the triangle is contained in a δ-neighborhood of the union of the other two sides. On the other hand, CAT(0) spaces are geodesic spaces with metric d having the property that for any two points u and v on a geodesic triangle the comparison points $\bar{u}$ and $\bar{v}$ in some Euclidean comparison triangle satisfy $d(u,v) \leq |\bar{u}-\bar{v}|$. It is thus natural to introduce some amount of 'additive fudge' to this comparison property in order to obtain the notion of a rough CAT(0) space.

We work in length spaces and thus replace geodesic triangles and segments by h-short triangles made of h-short segments, which were introduced by Väisälä [12] in the context of Gromov hyperbolicity: a h-short segment, $h \geq 0$, is a path whose length is larger by at most h than the distance between its endpoints. Comparison triangles can be defined using the distances between vertices, and one could then attempt to define a rough CAT(0) condition by introducing a uniform additive fudge to the CAT(0) condition. There is, however, a problem with choosing a fixed h, since then even the Euclidean plane would not be rough CAT(0): h-short segments are not forced to remain a uniformly bounded distance apart when the distance between their common endpoints increases; see Example 3.3. Thus, h must depend on how far apart are the vertices of a h-short triangle.

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Since the definition can be formulated in this generality, we introduce rough CAT(κ) spaces with $-\infty \le \kappa \le 0$; the case $\kappa > 0$ is trivial and we discard it. We write rCAT(k) as an abbreviation of "rough CAT(k)". Initially we define a notion of rCAT(k) spaces with an explicit upper bound on h which, although useful for many purposes, seems a little contrived. We therefore also define local rCAT(k) spaces, where the positive upper bound on h is an arbitrary function of the vertices of the triangle. This local variant is aesthetically more pleasing, but turns out to be quantitatively equivalent to the original notion of rCAT(k), a fact that will prove to be quite useful in Section 5.

We also define weak and very weak rCAT(k) conditions (and their local variants, each of which turn outs to be equivalent to the corresponding non-local condition). The weak rCAT(k) condition, which is equivalent to the full strength CAT(k) condition at least when $k < 0$, is equivalent to a certain 4-point subembedding condition which makes it clear that it is stable under many limiting processes. The very weak rCAT(0) condition will be seen to be equivalent to the bolicity condition of Kasparov and Skandalis [9], [10] that was introduced in the context of their work on the Baum-Connes and Novikov Conjectures.

Some of the results mentioned above are established in Section 3, and the remaining ones are proven in Section 4 where, motivated by the fact that CAT(0) spaces are uniquely geodesic, we explore a rough unique geodesic property for (local weak) rough CAT(0) spaces (see Theorem 4.2). We also prove in Section 4 that every CAT(0) space is $(2+\sqrt{3})$-rCAT(0).

Knowing that the class of rCAT(0) spaces includes both Gromov hyperbolic length spaces and CAT(0) spaces, it is natural to ask whether there are rCAT(0) spaces that are neither CAT(0) nor Gromov hyperbolic. In Section 5 we give two constructions (products and gluing) for getting new rCAT(0) spaces from old ones, which easily produce such examples.

In Theorem 5.1 we show that the l^2-product of rough CAT(0) spaces is also rough CAT(0). A rough CAT(0) space that is neither CAT(0) nor Gromov hyperbolic is thus obtained by taking the l^2-product of a Gromov hyperbolic space that is not CAT(0) and a CAT(0) space that is not Gromov hyperbolic (e.g. the l^2-product of the unit circle and the Euclidean plane).

Theorem 5.5 shows that gluing rough CAT(0) spaces along bounded isometric subspaces also gives rough CAT(0) spaces, but Example 5.10 shows that this mechanism breaks down as soon as we ask for unbounded gluing sets, even if they are convex. Finally, Proposition 5.11 shows that normed vector spaces do not produce interesting examples, since they must be CAT(0) if they are rough CAT(0).

2. Preliminaries

Let (X,d) be a metric space. We shall not distinguish notationally between paths $\gamma : I \to X$, $I \subset \mathbb{R}$, and their images $\gamma(I)$. Suppose (X,d) is rectifiably connected. We define the *intrinsic metric associated with* d by

$$l(x,y) := \inf\{\mathrm{len}(\gamma) : \gamma \text{ is a path in } X \text{ containing } x,y\}\,.$$

(X,d) is a *length space* if $l = d$. A path γ of length $d(x,y)$ joining $x,y \in X$ is called a *geodesic segment*, and is often denoted $[x,y]$. (X,d) is a *geodesic space* if all pairs of points can be joined by geodesic segments, that is, the above infimum is always attained.

Definition 2.1. A *h-short segment*, $h \geq 0$, in the length space (X, d) is a path $\gamma : [0, L] \to X$, $L \geq 0$, satisfying

$$\operatorname{len}(\gamma) \geq d(\gamma(0), \gamma(1)) \geq \operatorname{len}(\gamma) - h.$$

We denote h-short segments connecting points $x, y \in X$ by $[x, y]_h$. It is convenient to use $[x, y]_h$ also for the image of this path, so instead of writing $z = \gamma(t)$ for some $0 \leq t \leq L$, we often write $z \in [x, y]_h$. Given such a path γ and point $z = \gamma(t)$, we denote by $[x, z]_h$ and $[z, y]_h$ respectively the subpaths $\gamma|_{[0,t]}$ and $\gamma|_{[t,L]}$, respectively; note that both of these are h-short segments. We sometimes write $\gamma[x, z]$ and $\gamma[z, y]$ in place of $[x, z]_h$ and $[z, y]_h$ if we need to specify the short path (or geodesic) of which we are taking a subpath.

The above notation requires further explanation because of its ambiguity: given points x, y in a length space X, there are always many short segments $[x, y]_h$ for each $h > 0$, so the notation $[x, y]_h$ involves a choice. When we use this notation in any part of this paper (by a *part*, we mean a definition or a statement or proof of a result), the choice of such a path does not affect the truth of the underlying statements. However, all subsequent uses of $[x, y]_h$ in the same part of the paper refer to the same choice of short segment, and subsequent uses of $[x, z]_h$ and $[z, y]_h$ for $z \in [x, y]_h$ refer to subpaths of this choice of $[x, y]_h$. Even once we fix $\gamma = [x, y]_h : [0, L] \to X$, the definitions of such subpaths $[x, z]_h$ and $[z, y]_h$ may require a choice of $t \in [0, L]$ for which $z = \gamma(t)$ (since $[x, y]_h$ might not be an arc). The first use of $[x, z]_h$ or $[z, y]_h$ in any part of the paper involves such a choice, and all subsequent uses of either $[x, z]_h$ or $[z, y]_h$ in the same part is consistent with this choice of t.

Note that a 0-short segment is a geodesic segment; in this case, we simply write $[x, y]$ instead of $[x, y]_0$, and we also write (x, y) for the subpath of $[x, y]$ with endpoints removed. Geodesic segments are used in this paper only in the context of the model spaces M^2_κ.

Remark 2.2. The fact that (X, d) is assumed to be a length space ensures that for any $x, y \in X$ and $h > 0$, there exists an h-short segment $[x, y]_h$.

Given a number $\kappa \in \mathbb{R}$, the metric model space M^2_κ is defined as follows. M^2_0 is the Euclidean plane, M^2_κ, $\kappa > 0$, is obtained from the sphere by multiplying the metric with $1/\sqrt{\kappa}$, and M^2_κ, $\kappa < 0$, is obtained from the hyperbolic plane by multiplying the metric with $1/\sqrt{-\kappa}$. For more details we refer for instance to [3, Chapter I.2].

When $\kappa = -\infty$, M^2_κ is the union of the real and imaginary axes of $\mathbb{R}^2$ with the length metric attached. This is a much smaller space than what $M^2_{-\infty}$ would be if it were defined as a cone at infinity of the space M^2_κ for $\kappa \in (-\infty, 0)$. We are, however, only interested in embeddings and subembeddings of three or four points in our model space, and for these our simple definition of $M^2_{-\infty}$ suffices.

Since only the case $\kappa = 0$ will be considered for the bulk of this paper, the distance between $a, b \in M^2_\kappa$ is denoted by $|a - b|$, no matter what value κ has. For $\kappa > 0$, let D_κ denote the diameter of M^2_κ; for $-\infty \leq \kappa \leq 0$, set D_κ to be infinity.

The following result is referred to as Alexandrov's lemma and will be instrumental for the considerations in Section 3.

Lemma 2.3 (Alexandrov's lemma)**.** *Let $\kappa \in \mathbb{R}$ and consider distinct points A, B, B', $C \in M^2_\kappa$; if $\kappa > 0$, we assume that $|B - C| + |C - B'| + |B - A| + |A - B'| < 2D_\kappa$. Suppose that B and B' lie on opposite sides of the line AC. (Note that the triangle inequality and the assumption above imply that $|B - B'| < D_\kappa$.)*

Consider geodesic triangles $T := T(A,B,C)$ and $T' := T(A,B',C)$. Let α, β, γ (resp. α', β', γ') be the vertex angles of T (resp. T') at A,B,C (resp. A,B',C). Suppose that $\gamma+\gamma' \ge \pi$. Then

$$|B-C| + |C-B'| \le |B-A| + |A-B'| \,.$$

Let $\overline{T} \subset M^2_\kappa$ be a geodesic triangle with vertices $\bar{A}$, $\bar{B}$, $\bar{B}'$ such that $|\bar{A}-\bar{B}| = |A-B|$, $|\bar{A}-\bar{B}'| = |A-B'|$, and $|\bar{B}-\bar{B}'| = |B-C| + |C-B'| < D_\kappa$. Let $\bar{C}$ be the point in $[\bar{B},\bar{B}']$ with $|\bar{B}-\bar{C}| = |B-C|$. Let $\bar\alpha, \bar\beta, \bar\beta'$ be the vertex angles of T at vertices $\bar{A}, \bar{B}, \bar{B}'$. Then

$$\bar\alpha \ge \alpha+\alpha', \quad \bar\beta \ge \beta, \quad \bar\beta' \ge \beta', \quad |\bar{A}-\bar{C}| \ge |A-C| \,.$$

Moreover, an equality in any of these implies the equality in the others, and occurs if and only if $\gamma+\gamma'=\pi$.

A *geodesic triangle* $T(x,y,z)$ in a geodesic space X is a collection of three points $x,y,z \in X$ together with a choice of geodesic segments $[x,y]$, $[x,z]$ and $[y,z]$. Given such a geodesic triangle $T(x,y,z)$, a *comparison triangle* is a geodesic triangle in M^2_κ, $T(\bar{x},\bar{y},\bar{z})$, such that corresponding distances coincide: $d(x,y) = |\bar{x}-\bar{y}|$, $d(y,z) = |\bar{y}-\bar{z}|$, $d(z,x) = |\bar{z}-\bar{x}|$. A point $\bar{u} \in [\bar{x},\bar{y}]$ is a *comparison point* for $u \in [x,y]$ if $d(x,u) = |\bar{x}-\bar{u}|$.

For details on the definition and characterizations of CAT(κ) we refer the reader for instance to [3, Chapter II.1]. Let X be geodesic and $\kappa \in \mathbb{R}$. Let $T(x,y,z)$ be a geodesic triangle in X with perimeter less than $2D_\kappa$, and consider a comparison triangle $T(\bar{x},\bar{y},\bar{z})$ for $T(x,y,z)$ in M^2_κ. We say that $T(x,y,z)$ satisfies the *CAT(κ) condition* if for any $u,v \in T(x,y,z)$,

$$d(u,v) \le |u-v|.$$

In the case that $\kappa \le 0$ we call X a *CAT(κ) space* if all geodesic triangles in X satisfy the CAT(κ) condition. For $\kappa > 0$ we say X is a CAT(κ) space if all geodesic triangles of perimeter less than $2D_\kappa$ satisfy the CAT(κ) condition. Equivalently, u can be assumed to be one of the vertices of the triangle $T(x,y,z)$ and v can be assumed to be on the opposite side. Even more, v can be assumed to be a midpoint of the opposing side.

Another way of characterizing geodesic CAT(κ) spaces, $\kappa \in \mathbb{R}$, is by using the so-called 4-point condition. Suppose $x_i \in X$ and $\bar{x}_i \in M^2_\kappa$ for $0 \le i \le 4$, with $x_0 = x_4$ and $\bar{x}_0 = \bar{x}_4$. We say that $(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4)$ is a *subembedding* of (x_1,x_2,x_3,x_4) in M^2_κ if

$$\begin{aligned} &d(x_i,x_{i-1}) = |\bar{x}_i - \bar{x}_{i-1}|\,, \quad 1 \le i \le 4\,, \\ &d(x_1,x_3) \le |\bar{x}_1-\bar{x}_3| \quad \text{and} \quad d(x_2,x_4) \le |\bar{x}_2-\bar{x}_4| \,. \end{aligned}$$

The metric space (X,d) satisfies the 4-*point condition*, if every 4-tuple in X has a subembedding in M^2_κ. When X is geodesic, this turns out to be equivalent to X being CAT(κ).

Also, X is CAT(0) if and only if the *CN inequality* of Bruhat and Tits is satisfied, that is, for all $x,y,z \in X$ and all $m \in X$ with $d(y,m) = d(m,z) = d(y,z)/2$,

$$d(x,y)^2 + d(x,z)^2 \ge 2d(x,m)^2 + \frac{1}{2}d(y,z)^2.$$

We refer the reader to [8], [7], [12], or [3, Part III.H] for the theory of Gromov hyperbolic spaces. A metric space (X,d) is *δ-hyperbolic*, $\delta \ge 0$, if

$$\langle x,z;\,w\rangle \ge \langle x,y;\,w\rangle \wedge \langle y,z;\,w\rangle - \delta\,, \qquad x,y,z,w \in X\,,$$

where $\langle x,z;\,w\rangle$ is the Gromov product defined by

$$2\,\langle x,z;\,w\rangle = d(x,w) + d(y,w) - d(x,y)\,.$$

The following is a version of the well-known *Tripod Lemma*, almost as stated in [12, 2.15], the only minor difference being that it is stated for short arcs rather than short paths.

Lemma 2.4. *Suppose that γ_1 and γ_2 are unit speed h-short paths from o to x_1 and x_2, respectively, in a δ-hyperbolic space. Let $y_1 = \gamma_1(t)$ and $y_2 = \gamma_2(t)$ for some $t \geq 0$, where $d(o, y_1) \leq \langle x_1, x_2; o\rangle$. Then $d(y_1, y_2) \leq 4\delta + 2h$.*

A map $f : (X, d_X) \to (Y, d_Y)$ is an (A, B)-*quasi-isometry* if there are constants $A > 0$, $B \geq 0$ such that

$$\frac{1}{A} d_X(x_1, x_2) - B \leq d_Y(f(x_1), f(x_2)) \leq A\, d_X(x_1, x_2) + B$$

for any $x_1, x_2 \in X$, and such that $\operatorname{dist}(y, f(X)) \leq B$, $y \in Y$. Here, $\operatorname{dist}(x, A) := \inf\{d(x, y) : y \in A\}$ is the distance of a point x from a set A. A B-*rough isometry* is a $(1, B)$-quasi-isometry; B is called the *roughness constant* of f.

We write $A \wedge B$ and $A \vee B$ for the minimum and maximum, respectively, of two numbers A, B.

3. Rough CAT(κ) spaces: basic results

In this section, we define rough CAT(0) spaces and some weaker variants of them, and prove some some basic results involving these conditions.

Definition 3.1. A *h-short triangle* $T := T_h(x_1, x_2, x_3)$ with vertices $x_1, x_2, x_3 \in X$ is a collection of h-short segments $[x_1, x_2]_h$, $[x_2, x_3]_h$ and $[x_3, x_1]_h$. Given such a h-short triangle T, a *comparison triangle* will mean a geodesic triangle $\overline{T} := T(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the comparison space M^2_κ, $-\infty \leq \kappa < \infty$ such that $|\bar{x}_i - \bar{x}_j| = d(x_i, x_j)$, $i, j \in \{1, 2, 3\}$. Furthermore, we say that $\bar{u} \in \overline{T}$ is a *comparison point* for $u \in T$, say $u \in [x_1, x_2]_h$, if

$$|\bar{x} - \bar{u}| \leq \operatorname{len}([x, u]_h) \quad \text{and} \quad |\bar{u} - \bar{y}| \leq \operatorname{len}([u, y]_h)\,.$$

If $\kappa = -\infty$, the comparison triangle is called a *comparison tripod.*

Note that $\bar{u}$ is not uniquely determined by u as in the case of comparison points for triangles in CAT(κ) spaces. Also, it immediately follows from the definition that

$$|\bar{x} - \bar{u}| \geq \operatorname{len}([x, u]_h) - h \quad \text{and} \quad |\bar{u} - \bar{y}| \geq \operatorname{len}([u, y]_h) - h.$$

In order to avoid cluttered notation, we do not specify the comparison space in the notation $T(\cdot, \cdot, \cdot)$; the space will always be clear from the context.

Remark 3.2. Clearly, we can always find comparison triangles in M^2_κ for any h-short triangle $T_h(x, y, z)$ in any length space X, as long as $d(x, y) + d(y, z) + d(z, x) \leq 2D_\kappa$. In fact this amounts to the well-known fact that triangles in M^2_κ can be constructed with arbitrary sidelengths $a \leq b \leq c$, as long as the perimeter $a + b + c$ is at most twice the diameter of M^2_κ and the triangle inequality $c \leq a + b$ holds.

Recall that a CAT(κ) space is a geodesic space in which the distance between any pair of points in a geodesic triangle is at most as large as the distance between comparison points in a comparison triangle in M^2_κ. The natural definition of rough CAT(κ) should therefore involve a similar distance inequality between an arbitrary pair of points in an h-short triangle, and a pair of comparison points in an comparison triangle, for some $h > 0$. Our definition will indeed have this form (and we can work with length spaces rather than geodesic spaces), but for $\kappa = 0$ (the main case that interests us!), the value of h must depend on how far apart are the vertices of the h-short triangle. The following example shows that a fixed $h > 0$ "would not work" when $\kappa = 0$ in the sense that even the Euclidean plane would fail to satisfy such a condition.

Example 3.3. Let $h > 0$ be fixed, and take x, $y = z$ to be the points given in coordinate form as $(-R, 0)$ and $(R, 0)$, respectively, for some $R > 0$. Let $T := T_h(x, y, z)$ be the short triangle consisting of the pair of line segments from x to y and y to z (the latter being degenerate), plus a path from z to x consisting of the two line segments from z to $u := (0, t)$ and u to x, where $t = \sqrt{hR + h^2/4}$; it is clear that T is an h-short triangle. The comparison triangle $\overline{T}$ is the (geodesic) planar triangle with the same vertices. If we take v to be the origin, then t, and so $d(u, v)$, tends to infinity as R tends to infinity, while the distance between any comparison points in $\overline{T}$ remains bounded.

The formula for t in the above example shows that we need to restrict h to be at most some multiple of $1/(d(x, y) \vee d(x, z) \vee d(y, z))$ in order to get $d(u, v) \leq |u - v| + C$ for some fixed C independent of R. On the other hand, easy planar examples with $x = y = z$ show that h should also be bounded. These examples are the motivation behind the choice of h in the following definition. However, any smaller h would give a quantitatively equivalent definition; see Corollary 4.4.

Definition 3.4. Let $-\infty \leq \kappa \leq 0$ and $C > 0$. A length space (X, d) is called *C-rough CAT(κ)*, or simply *C-rCAT(κ)*, if for every $x, y, z \in X$, every h-short triangle $T_h(x, y, z)$ and every comparison triangle $T(\bar{x}, \bar{y}, \bar{z})$ in M_κ^2 associated with $T_h(x, y, z)$, with

$$h = \frac{1}{1 \vee d(x, y) \vee d(x, z) \vee d(y, z)}, \tag{3.5}$$

the *C-rough CAT(κ) condition* is satisfied:

$$d(u, v) \leq |\bar{u} - \bar{v}| + C,$$

whenever u, v lie on different sides of $T_h(x, y, z)$ and $\bar{u}, \bar{v} \in T(\bar{x}, \bar{y}, \bar{z})$ are corresponding comparison points. We say that X is rCAT(κ) if it is C-rCAT(κ) for some $C > 0$. We refer to C as the *roughness constant* of X.

In the above definition, we could allow κ to be positive, as long as we restrict x, y, z so that $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$ (as in the definition of CAT(κ) for $\kappa > 0$). However, it is trivial that every length space is C-rough CAT(κ) for $C > D_\kappa$, so the class of all rough CAT(κ) spaces is of no interest. For this reason, we insist that $-\infty \leq \kappa \leq 0$ from now on.

Let us now mention some variants of rCAT(κ). Example 3.3 told us that h had to be restricted somehow to get a good definition of rCAT(κ), and it also made our choice of h rather natural. However a non-prescriptive definition of h is perhaps aesthetically better than (3.5), leading us to the following definition.

Definition 3.6. Let $-\infty \leq \kappa \leq 0$ and $C > 0$. A length space (X, d) is said to be *locally C-rough CAT(κ)* if the C-rough CAT(κ) condition holds for all h-short triangles $T_h(x, y, z)$ and associated other data, as long as $h \leq H$ for some $H = H(x, y, z) > 0$.

It is well known (and easily shown) that the CAT(κ) condition is equivalent to a weaker version of the same definition where the comparison inequality holds only when one point is a vertex, and one can even restrict the other point to being a midpoint of a side. This leads us to the following definitions.

Definition 3.7. Let $-\infty \le \kappa \le 0$ and $C > 0$. A *weak C-rough CAT(κ) condition* is similar to the C-rough CAT(κ) condition defined in Definition 3.4, except that it is required to hold only when $v = x$ and $u \in [y,z]_h$. A *very weak C-rough CAT(κ) condition* is also similar to the C-rough CAT(κ) condition, except that it is required to hold only when $v = x$ and $u \in [y,z]_h$ is a *h-midpoint* of $[y,z]_h$, that is, if it has the property that the Euclidean midpoint $\bar{u}$ of $[\bar{y}, \bar{z}]$ is a comparison point for u. *Weak* and *very weak C-rCAT(κ) spaces*, and their local variants, are then defined by making the associated changes to the above definitions of (local) C-rCAT(κ) spaces.

By elementary geometry, we see that if x, y, z are points in the Euclidean plane and u lies on the line segment from y to z with $|u - y| = t|z - y|$, then

$$(d(x,u))^2 \le (1-t)(d(x,y))^2 + t(d(x,z))^2 - t(1-t)(d(y,z))^2 \,. \tag{3.8}$$

It follows that the weak C-rCAT(0) condition can be written in the following more explicit form: if $u = \lambda(s)$, where $\lambda : [0,L] \to X$ is a h-short path from y to z parametrized by arclength, h satisfies the usual bound, and we have both $td(y,z) \le s$ and $(1-t)d(y,z) \le L - s$ for some $0 \le t \le 1$, then

$$(d(x,u) - C)^2 \le (1-t)(d(x,y))^2 + t(d(x,z))^2 - t(1-t)(d(y,z))^2 \,. \tag{3.9}$$

The very weak C-rCAT(0) condition can be written in a similar form, but with the restriction $t = 1/2$.

The following result summarizes what we can say about the relationships between all these variants of rCAT(k) spaces.

Theorem 3.10. (a) *For $-\infty \le \kappa < 0$, the classes of rCAT(k), local rCAT(k), weak rCAT(k), and local weak rCAT(k) spaces all coincide with the class of Gromov hyperbolic spaces, and all containment implications hold with quantitative dependence of parameters.*

(b) *The classes of local rCAT(0) spaces and rCAT(0) spaces coincide, again with quantitative control of parameters, and the same is true of local weak rCAT(0) spaces and weak rCAT(0).*

(c) *The class of rCAT(0) spaces is strictly larger than the union of the classes of Gromov hyperbolic and CAT(0) spaces.*

Part (a) of this theorem follows from Theorem 3.19 below, while part (b) follows from Corollary 4.4, and an example to prove part (c) was given in the Introduction (see also Section 5).

There are a few other possible relationships between these variant rCAT(k) spaces whose truth we cannot determine. Specifically we do not know if very weak rCAT(k) spaces are necessarily weak rCAT(k) (either for $k < 0$ or $k = 0$), and we do not know if weak rCAT(0) spaces are necessarily CAT(0). While the class of rCAT(0) spaces is the main focus of our interest in this paper, the (weak) rCAT(κ) characterization of Gromov hyperbolicity in the above theorem may also be of some interest.

We now wish to discuss another connection to existing notions of non-positive curvature. In their work on the Baum-Connes and Novikov Conjectures, Kasparov and Skandalis [9], [10] introduced the class of bolic spaces which, as our class of rCAT(0) spaces, includes both Gromov hyperbolic spaces and CAT(0) spaces. It turns out that in the case of length spaces, bolicity is equivalent to very weak rCAT(0). To see this, we first note that by work of Bucher and Karlsson [5], bolicity is reduced to a condition reminiscent of the CN inequality of Bruhat and Tits ([3, p. 163] and [4]).

Definition 3.11. A metric space X is called δ-*bolic*, for some $\delta > 0$, if there is a map $m : X \times X \to X$ with the property that for all $x, y, z \in X$

$$2d(m(x,y),z) \le \sqrt{2d(x,z)^2 + 2d(y,z)^2 - d(x,y)^2} + 4\delta.$$

Proposition 3.12. *Let X be a length space. If X is very weak C-rCAT(0), $C > 0$, then it is δ-bolic with $\delta = C/2$. If X is δ-bolic, $\delta > 0$, then it is very weak C-rCAT(0) with $C = 4\delta + \sqrt{2}$.*

Proof. Let X be a very weak C-rCAT(0) space. Let $x, y, z \in X$ and let $T_h(x, y, z)$ be some h-short triangle with comparison triangle $T(\bar{x}, \bar{y}, \bar{z})$. Let $m(y, z)$ be some h-midpoint of $[y, z]_h$. This defines a map $m : X \times X \to X$. By definition, the Euclidean midpoint $\bar{m}$ of $[\bar{y}, \bar{z}]$ is a comparison point for $m(y, z)$. Using the comparison triangle property, the Euclidean parallelogram law and the very weak C-rCAT(0) condition, we obtain

$$\begin{aligned} d(x,y)^2 + d(x,z)^2 &= |\bar{x} - \bar{y}|^2 + |\bar{x} - \bar{z}|^2 \\ &= 2\,|\bar{x} - \bar{m}|^2 + \frac{1}{2}|\bar{y} - \bar{z}|^2 \\ &\ge 2(d(x,m) - C)^2 + \frac{1}{2}d(y,z)^2. \end{aligned}$$

Thus X is $C/2$-bolic.

Let now X be a δ-bolic length space with some $\delta > 0$. Let $T_h(x, y, z)$ be some h-short triangle and $T(\bar{x}, \bar{y}, \bar{z})$ a corresponding comparison triangle in the Euclidean plane. Furthermore, let m be some h-midpoint for $[y, z]_h$, that is, m admits the Euclidean midpoint $\bar{m}$ of $[\bar{y}, \bar{z}]$ as a comparison point. By definition we thus obtain that $d(y, m) \le d(y, z)/2+h$ and $d(m, z) \le d(y, z)/2+h$. By applying the bolic inequality for $y, z, m \in X$ and $m(y, z) \in X$, and the fact that $h = 1/(1 \vee d(x, y) \vee d(x, z) \vee d(y, z))$, it follows that

$$\begin{aligned} 2d(m(y,z),m) &\le \sqrt{2d(y,m)^2 + 2d(m,z)^2 - d(y,z)^2} + 4\delta \\ &\le \sqrt{4(d(y,z)/2 + h)^2 - d(y,z)^2} + 4\delta \\ &\le 2\sqrt{2} + 4\delta. \end{aligned}$$

Applying bolicity for $x, y, z \in X$ and $m(y, z) \in X$ now yields

$$\begin{aligned} 2d(x,m) &\le 2d(x,m(y,z)) + 2d(m(y,z),m) \\ &\le \sqrt{2d(x,y)^2 + 2d(x,z)^2 - d(y,z)^2} + 8\delta + 2\sqrt{2}. \end{aligned}$$

By using the comparison triangle property and the Euclidean parallelogram equality we finally deduce

$$\begin{aligned} 2(d(x,m) - 4\delta - \sqrt{2})^2 &\le d(x,y)^2 + d(x,z)^2 - \frac{1}{2}d(y,z)^2 \\ &= |\bar{x} - \bar{y}|^2 + |\bar{x} - \bar{z}|^2 - \frac{1}{2}|\bar{y} - \bar{z}|^2 \\ &= 2\,|\bar{x} - \bar{m}|^2, \end{aligned}$$

which implies the very weak C-rCAT(0) inequality with $C = 4\delta + \sqrt{2}$. □

The CAT(κ) condition (for geodesic spaces X) is normally stated as the $C = h = 0$ variant of our rCAT(κ) definition, but it can also be written as a so-called 4-point condition. We prove an rCAT(κ) analogue of this, but first we need a simple lemma.

Lemma 3.13. *Let x, y be a pair of points in the Euclidean plane $\mathbb{R}^2$, with $l := |x-y| > 0$. Fixing $h > 0$, and writing $L := l + h$, let $\gamma : [0, L] \to \mathbb{R}^2$ be a h-short segment from x to y, parametrized by arclength. Then there exists a map $\lambda : [0, L] \to [x, y]$ such that $\lambda(0) = x$, $\lambda(L) = y$, and*

$$|\lambda(t) - x| \leq |\gamma(t) - x| \,, \qquad 0 \leq t \leq L \,, \tag{3.14}$$

$$|\lambda(t) - y| \leq |\gamma(t) - y| \,, \qquad 0 \leq t \leq L \,, \tag{3.15}$$

$$\delta(t) := \operatorname{dist}(\gamma(t), \lambda(t)) \leq M := \frac{1}{2}\sqrt{2lh + h^2} \,, \qquad 0 \leq t \leq L \,. \tag{3.16}$$

In particular if $h \leq 1/(1 \vee l)$, then $\delta(t) \leq \sqrt{3}/2$ for all $0 \leq t \leq L$.

Proof. The desired result is invariant under isometries of the plane, so we choose the points $x = (x_1, 0)$ and $y = (y_1, 0)$ to be located on the first coordinate axis with $x_1 < y_1$. We also write $\gamma = (\gamma_1, \gamma_2)$ in Euclidean coordinates. Now define $\lambda(t) = (\lambda_1(t), 0)$, where $\lambda_1(t) = (\gamma_1(t) \vee x_1) \wedge y_1$. It is clear that $\lambda(0) = x$, $\lambda(L) = y$, and that λ satisfies (3.14) and (3.15).

It is clear that to maximize $\delta(t) := \operatorname{dist}(\gamma(t), \lambda(t))$, we should pick γ to be the concatenation of two straight line paths, one from x to $\gamma(t)$ of length t and one from $\gamma(t)$ to y of length $L - t$. But then $\gamma(t)$ traces out an ellipse and it is routine to verify that $\delta(t) \leq \delta(L/2) = M$. $\square$

Definition 3.17. Let (X, d) be a metric space, $-\infty \leq \kappa \leq 0$, and $C \geq 0$. Suppose $x_i \in X$ and $\bar{x}_i \in M^2_\kappa$ for $0 \leq i \leq 4$, with $x_0 = x_4$ and $\bar{x}_0 = \bar{x}_4$. We say that $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$ is a *C-rough subembedding* of (x_1, x_2, x_3, x_4) in M^2_κ if

$$\begin{aligned} d(x_i, x_{i-1}) &= |\bar{x}_i - \bar{x}_{i-1}| \,, \quad 1 \leq i \leq 4 \,, \\ d(x_1, x_3) &\leq |\bar{x}_1 - \bar{x}_3| \,, \qquad \text{and} \\ d(x_2, x_4) &\leq |\bar{x}_2 - \bar{x}_4| + C \,. \end{aligned}$$

Definition 3.18. A metric space (X, d) satisfies the *(C, κ)-rough 4-point condition*, where $C \geq 0$ and $-\infty \leq \kappa \leq 0$, if every 4-tuple in X has a C-rough subembedding in M^2_κ. When $\kappa = 0$, we omit it from the notation.

Theorem 3.19. *For a length space (X, d) and $-\infty \leq \kappa \leq 0$, consider the following pair of conditions:*

(a) *X is local weak C-rCAT(κ) for some $C > 0$;*
(b) *X satisfies the (C', κ)-rough 4-point condition for some $C' > 0$.*

Then (a) implies (b) with $C' = 2C$. Moreover if $-\infty \leq \kappa < 0$, then the converse is also true: in fact, both of these conditions are quantitatively equivalent to δ-hyperbolicity and to X being rCAT(κ).

Proof. Suppose X is a local weak C-rCAT(κ) space and let (x_1, x_2, x_3, x_4) be a 4-tuple in X. Let $h := H(x_1, x_3, x_2) \wedge H(x_1, x_3, x_4)$, where H is as in the definition of local (weak) rCAT(κ). Choose h-short triangles $T_2 := T_h(x_1, x_3, x_2)$ and $T_4 := T_h(x_1, x_3, x_4)$, and comparison triangles $\overline{T}_1 := T(\bar{x}_1, \bar{x}_3, \bar{x}_2)$ and $\overline{T}_2 := T(\bar{x}_1, \bar{x}_3, \bar{x}_4)$, such that $\overline{T}_2$ and $\overline{T}_4$ have a common side $[\bar{x}_1, \bar{x}_3]$, and that $\bar{x}_2$ and $\bar{x}_4$ lie on opposite sides of the line through $\bar{x}_1$ and $\bar{x}_3$. Let $\bar{z}$ be the point of intersection of $[\bar{x}_2, \bar{x}_4]$ and the line through $\bar{x}_1$ and $\bar{x}_3$.

Suppose first that $\bar{z} \in [\bar{x}_1, \bar{x}_3]$; this is always the case if $\kappa = -\infty$ but it may fail for finite κ. Picking a h-short segment $[x_1, x_3]_h$, let $z \in [x_1, x_3]_h$ be such that $d(x_1, z) = |\bar{x}_1 - \bar{z}|$;

note that for $i = 2, 4$, the points $\bar z, \bar y_i$ are comparison points in the triangle $\overline{T}_i$ for z, y_i, respectively. By the triangle and weak rough CAT(κ) inequalities,

$$d(x_2, x_4) \le d(x_2, z) + d(z, x_4) \le |\bar x_2 - \bar z| + |\bar z - \bar x_4| + 2C = |\bar x_2 - \bar x_4| + 2C\,.$$

Note that $d(x_1, x_3) = |\bar x_1 - \bar x_3|$. Thus $(\bar x_1, \bar x_2, \bar x_3, \bar x_4)$ is a C'-rough subembedding in M^2_κ of (x_1, x_2, x_3, x_4), with $C' = 2C$.

Alternatively suppose that the segments $[\bar x_1, \bar x_3]$ and $[\bar x_2, \bar x_4]$ do not intersect (and so $\kappa \in \mathbb{R}$). Let Q be the quadrilateral consisting of the union of the four geodesic segments $[\bar x_1, \bar x_2]$, $[\bar x_2, \bar x_3]$, $[\bar x_3, \bar x_4]$, and $[\bar x_4, \bar x_1]$. Then $M^2_\kappa \setminus Q$ has two components: we call the one containing $(\bar x_1, \bar x_3)$ the *inner component*, and we define the *inner and outer angles* at the vertices in Q in the natural way; note that the inner angle at $\bar x_i$, $i = 1, 3$, is the sum of the angles at the same point in the triangles $\overline{T}_2$ and $\overline{T}_4$. If both inner angles were less than π, then it would follow by continuity that there exists a point $\bar u \in [\bar x_1, \bar x_3]$ such that the inner angle at u for the quadrilateral with vertices $\bar x_1, \bar x_2, \bar u, \bar x_4$ is π. But then it follows from Alexandrov's Lemma (Lemma 2.3) that $\bar z = \bar u$, contradicting the fact that $\bar z \notin [\bar x_1, \bar x_3]$.

Thus we may assume without loss of generality that the inner angle at $\bar x_1$ in Q is at least π. It follows that there exists a geodesic triangle in M^2_κ with vertices $\tilde x_2 = \bar x_2$, $\tilde x_3 = \bar x_3$, and $\tilde x_0 \equiv \tilde x_4$, together with a point $\tilde x_1 \in [\tilde x_2, \tilde x_4]$, such that $|\tilde x_i - \tilde x_{i-1}| = d(x_i, x_{i-1})$, $1 \le i \le 4$, and

$$|\tilde x_2 - \tilde x_4| = |\tilde x_2 - \tilde x_1| + |\tilde x_1 - \tilde x_4| = d(x_2, x_1) + d(x_1, x_4) \ge d(x_2, x_4)\,.$$

By Alexandrov's lemma, $|\tilde x_1 - \tilde x_3| \ge |\bar x_1 - \bar x_3| = d(x_1, x_3)$. Thus $(\tilde x_1, \tilde x_2, \tilde x_3, \tilde x_4)$ is a 0-rough subembedding in M^2_κ of (x_1, x_2, x_3, x_4). Putting this case together with the first case, we see that weak C-rough CAT(κ) implies the $(2C, \kappa)$-rough 4-point condition.

Conversely, suppose X satisfies the C'-rough 4-point condition, let $T = T_h(x, y, z)$ be a h-short geodesic triangle, where h satisfies (3.5). We wish to verify the C-rough CAT(κ) condition with $u \in [y, z]_h$ and $v = x$.

Suppose $(\bar y, \bar u, \bar z, \bar x)$ is a C'-rough subembedding for $(x_1, x_2, x_3, x_4) = (y, u, z, x)$. Apply Lemma 3.13, with γ being the piecewise linear path from $\bar y$ to $\gamma(t) := \bar u$ to $\bar z$, to get an associated point $\lambda(t) := \bar u'$ on the line segment $[\bar y, \bar z]$ such that $|\bar u - \bar u'| \le \sqrt{3}/2$. Thus $d(u, x) \le |\bar u' - \bar x| + C_1$, where $C_1 := C' + \sqrt{3}/2$.

The Euclidean triangle $T(\bar x, \bar y, \bar z)$ satisfies $|\bar x - \bar y| = d(x, y)$ and $|\bar x - \bar z| = d(y, z)$, but it is not necessarily a comparison triangle for T because we know only that $|\bar z - \bar y| \ge d(z, y)$. However if we take $T' := T(x', y', z')$ to be a comparison triangle in M^2_0 for T, and define $u' \in [y', z']$ via the equation

$$\frac{|u' - y'|}{|z' - y'|} = \frac{|\bar u' - \bar y|}{|\bar z - \bar y|}\,, \tag{3.20}$$

then it follows from (3.8) that $|\bar u' - \bar x| \le |u' - x'|$. Moreover by combining the subembedding properties, (3.14), (3.15), (3.20), and the fact that $|y' - z'| \le |\bar y - \bar z|$, we see that

$$\begin{aligned} |u' - y'| &\le |\bar u' - \bar y| \le |\bar u - \bar y| = |u - y| \le t\,, \\ |u' - z'| &\le |\bar u' - \bar z| \le |\bar u - \bar z| = |u - z| \le L - t\,, \end{aligned}$$

and so u' is a comparison point for u. Since $d(u, x) \le |\bar u' - \bar x| + C_1$, we have shown the C_1-rCAT(0) condition for this choice of data, comparison triangle, and this particular choice of comparison point u'.

A general comparison point u'' for u on the side $[y',z']$ of T' must satisfy $|u''-u'| \le h \le 1$, and so it follows that the C'-rough 4-point condition implies a weak C-rCAT(0) condition for $C = C' + 1 + \sqrt{3}/2$.

Suppose instead that $-\infty \le \kappa < 0$. The δ-hyperbolicity condition can be written in the form

$$d(x,z) + d(y,w) \le (d(x,y) + d(z,w)) \vee (d(x,w) + d(y,z)) + 2\delta\,,$$

and this condition holds (with $\delta = \log 3/\sqrt{-\kappa}$) for points $x,y,z,w \in M^2_\kappa$; see [7, Theorem 1.5.1]. If instead x,y,z,w lie in a space X that satisfies the (C',κ)-rough 4-point condition, then this condition and the δ-hyperbolicity of M^2_κ immediately imply the $(\delta + C')$-hyperbolicity of X.

Taking $h \le 1$ in Lemma 2.4, it is readily deduced that every δ-hyperbolic space is C-rCAT$(-\infty)$ for $C = 4\delta + 2$, and so a fortiori C-rCAT(κ) for every κ. We note in particular that the (C',κ)-rough 4-point condition implies the C-rCAT(κ) condition for $C = 4C' + 2 + 4\log 3/\sqrt{-\kappa}$. □

It follows rather easily from Theorem 3.19 that CAT(k) spaces are rCAT(κ) with roughness constant $C = C(\kappa)$ when $k < 0$; alternatively, this follows from the well-known *geodesic stability* of Gromov hyperbolic spaces (see for instance [3, Part III.H] or [12]). The fact that every CAT(κ) space is an rCAT(κ) space is also true when $\kappa = 0$: see Corollary 4.6.

Remark 3.21. The CAT(0) analogue of Theorem 3.19 in [3, II.3.9] assumes that X is a complete space with *approximate midpoints*. Such an assumption readily implies that X is a length space, so we use this latter assumption in our theorem and in Corollary 3.23 below, since we do not wish to restrict the theory of rCAT(0) spaces to complete spaces.

It is shown in Bridson and Haefliger [3, II.3.10] that CAT(0) is preserved by various limit operations, including pointed Gromov-Hausdorff limits and ultralimits; in particular both generalized tangent space and asymptotic cones of CAT(0) spaces are CAT(0) spaces (see [3] for the definition of all of these concepts). The trick is to use the 4-point condition and the rather weak limit concept of a 4-point limit. Essentially the same arguments, with the 4-point condition replaced by our rough 4-point condition, give us similar results for rCAT(0) spaces which we now state. We omit the proofs since they are obtained by routine adjustments to the proofs of II.3.9 and II.3.10 in [3]. For completeness, we begin with a definition of 4-point limits.

Definition 3.22. A metric space (X,d) is a 4-point limit of a sequence of metric spaces (X_n,d_n) if for every $x_1,x_2,x_3,x_4 \in X$, and $\varepsilon > 0$, there exist infinitely many integers n and points $x_{i,n} \in X_n$, $1 \le i \le 4$, such that $|d(x_i,x_j) - d_n(x_{i,n},x_{j,n})| < \varepsilon$ for $1 \le i,j \le 4$.

Corollary 3.23. *Suppose the length space (X,d) is a 4-point limit of the weak C_n-rCAT(0) spaces (X_n,d_n). If $C_n \le C$ for all n, then (X,d) is a weak C-rCAT(0) space. If $C_n \to 0$, then (X,d) is a CAT(0) space.*

Corollary 3.24. *Suppose (X,d) is a length space and (X_n,d_n) form a sequence of C-rCAT(0) spaces.*

(a) *If (X,d) is a (pointed or unpointed) Gromov-Hausdorff limit of (X_n,d_n) then (X,d) is a C-rCAT(0) space.*
(b) *If (X,d) is an ultralimit of (X_n,d_n), then (X,d) is a C-rCAT(0) space.*
(c) *The asymptotic cone $Cone_\omega X := \lim_\omega(X,d/n)$ is a CAT(0) space for every non-principal ultrafilter ω.*

In each of the cases above, the existence of an approximate midpoint for arbitrary $x, y \in X$ (meaning a point m such that $d(x,m) \vee d(y,m) \le \varepsilon + d(x,y)/2$ for fixed but arbitrary $\varepsilon > 0$) follows easily from the hypotheses, and so (X,d) is a length space if it is complete.

4. Rough CAT(0) and roughly unique geodesics

In this section, we explore the rough unique geodesic property of a (local weak) rough CAT(0) space. Recall that CAT(0) spaces are uniquely geodesic. The rCAT(0) condition for a h-short triangle $T(x,y,y)$ readily gives the following rough version of this.

Observation 4.1. Let x, y be a pair of points in a C-rCAT(0) space (X,d), and let $h := 1/(1 \vee d(x,y))$. Let $\gamma_i : [0, L_i] \to X$, $i = 1,2$ be a pair of h-short segments from x to y, parametrized by arclength, with $L_1 \le L_2$. Then $d(\gamma_1(t), \gamma_2(t)) \le C$, $0 \le t \le L_1$.

The following theorem improves the above observation.

Theorem 4.2. *Let x, y be a pair of points in a local weak C-rCAT(0) space (X,d), with $L := d(x,y)$. For $i = 1,2$, let $h_i > 0$ and let $\gamma_i : [0, L+h_i] \to X$ be a h_i-short segment from x to y, parametrized by arclength; we assume that $h_1 \le h_2$. Then*

$$d(\gamma_1(t), \gamma_2(t)) \le 2C + h_2 + \frac{\sqrt{2Lh_1 + h_1^2}}{2} + \frac{\sqrt{2Lh_2 + h_2^2}}{2}, \qquad 0 \le t \le L + h_1 ,$$

In particular, if $h_2 \le 1/(1 \vee L)$, then

$$d(\gamma_1(t), \gamma_2(t)) \le 2C + 1 + \sqrt{3}, \qquad 0 \le t \le L + h_1 .$$

Proof. The result follows from the triangle inequality if $t \le h_2$. Let us therefore assume that $h_2 \le L + h_1$ and fix $t \in (0, L+h_1]$. Let $t' := t \wedge L$.

Throughout this proof i can equal either 1 or 2. We write $z_i := \gamma_i(t)$ and choose a path $\gamma_3 : [0, L+h] \to X$ from x to y, parametrized by arclength, for some $0 < h \le H(x,y,z_1) \wedge H(x,y,z_2)$. Let $T^i = T_h(x,y,z_i)$ be a h-short triangle which includes γ_3 as a side, let $\overline{T}^{\,i} = T(\bar{x}, \bar{y}, \bar{z}_i)$ be a corresponding comparison triangles in M_0^2, let $\bar{u}_i$ be a point on $[\bar{x}, \bar{y}]$ that is closest to $\bar{z}_i$, and let $u_i = \gamma_3(t_i)$, where $t_i := |\bar{u}_i - \bar{x}|$. Note that $\bar{u}_i$ is a comparison point for u_i.

By basic geometry, we have $t_i \le |\bar{z}_i - \bar{x}| = d(z_i, x)$ and $L - t_i \le |\bar{z}_i - \bar{y}| = d(z_i, y)$, and so $t_i \in [t - h_i, t]$. Thus $|u_1 - u_2| \le h_2$.

For $i = 1,2$, the concatenation of the two sides of $\overline{T}^{\,i}$ other than $[\bar{x}, \bar{y}]$ forms a h_i-short path, so by local weak C-rCAT(0) and Lemma 3.13,

$$\begin{aligned} d(z_1, z_2) &\le d(z_1, u_1) + d(u_1, u_2) + d(u_2, z_2) \\ &\le |\bar{z}_1 - \bar{u}_1| + h_2 + |\bar{u}_2 - \bar{z}_2| + 2C \\ &\le 2C + h_2 + \frac{\sqrt{2Lh_1 + h_1^2}}{2} + \frac{\sqrt{2Lh_2 + h_2^2}}{2}, \end{aligned}$$

as required. □

Remark 4.3. It is clear from the above proof that the upper bound can be improved if $h_1 \le h$ (where h is as in the proof). In this case, we get

$$d(\gamma_1(t), \gamma_2(t)) \le C + h_2 + \frac{\sqrt{2Lh_2 + h_2^2}}{2}, \qquad 0 \le t \le L + h_1 ,$$

and if $h_2 \le 1/(1 \vee L)$, then

$$d(\gamma_1(t), \gamma_2(t)) \le C + 1 + \frac{\sqrt{3}}{2}, \qquad 0 \le t \le L + h_1 .$$

Using the above theorem and remark, we readily get the part of the following result which says that local rCAT(0) is equivalent to rCAT(0). The weak and very weak variants follow by an examination of the proof of the theorem. The weak rCAT(0) part of this corollary can alternatively be deduced (with the same constant C'') from the proof of Theorem 3.19, since that proof shows quantitatively that local weak rCAT(0) implies a rough 4-point condition, which in turn implies weak rCAT(0).

Corollary 4.4. *A local C-rCAT(0) is C'-rCAT(0), for $C' = 3C + 2 + \sqrt{3}$. A local weak (or local very weak) C-rCAT(0) is weak (or very weak) C''-rCAT(0), for $C'' = 2C + 1 + \sqrt{3}/2$.*

We now state a variant of Theorem 4.2 for CAT(0) spaces; we omit the very similar (but less technical) proof.

Theorem 4.5. *Let x, y be a pair of points in a CAT(0) space (X, d), with $L := d(x, y)$. For $i = 1, 2$, let $h_i \ge 0$ and let $\gamma_i : [0, L + h_i] \to X$ be a h_i-short segment from x to y, parametrized by arclength; we assume that $h_1 \le h_2$. Then*

$$d(\gamma_1(t), \gamma_2(t)) \le h_2 + \frac{\sqrt{2Lh_1 + h_1^2}}{2} + \frac{\sqrt{2Lh_2 + h_2^2}}{2}, \qquad 0 \le t \le L + h_1 ,$$

In particular, if $h_1 = 0$ and $h_2 \le 1/(1 \vee L)$, then

$$d(\gamma_1(t), \gamma_2(t)) \le 1 + \frac{\sqrt{3}}{2}, \qquad 0 \le t \le L .$$

The above theorem has the following easy corollary.

Corollary 4.6. *A CAT(0) space is C-rCAT(0) for $C = 2 + \sqrt{3}$.*

We record here a Rough Convexity lemma for rCAT(0) spaces. This is a rough analogue of [3, Proposition II.2.2], and can be proved in a similar way, so we leave its proof as an exercise.

Lemma 4.7. *Suppose a_1, a_2, b_1, b_2 are points in a C-rCAT(0) space. Let $\gamma_i : [0, 1] \to X$ be constant speed h_i-short paths parametrized by arclength from a_i to b_i, $i = 1, 2$, where $h_i = 1/(1 \vee d(a_i, b_i))$. Then*

$$d(\gamma_1(t), \gamma_2(t)) \le (1 - t)d(a_1, a_2) + td(b_1, b_2) + 2C .$$

If either $a_1 = a_2$ or $b_1 = b_2$, then we can replace $2C$ by C in the above estimate.

Remark 4.8. Note that $\mathbb{R}^2$ with the Euclidean metric is CAT(0), while $\mathbb{Z}^2$ with the ℓ^1-metric is not even very weak rCAT(0). Thus rCAT(0) is not invariant under quasi-isometry. By comparison, we note the well-known facts that Gromov hyperbolicity is invariant under quasi-isometry in the context of geodesic spaces, while the CAT(0) property is only invariant under isometry.

5. Examples

We already know that the class of rCAT(0) spaces include both Gromov hyperbolic (by Theorem 3.19 and the fact that rCAT(κ) implies rCAT(0) for $\kappa < 0$) and CAT(0) spaces (by Corollary 4.6). Here we give two constructions (products and gluing) for getting new rCAT(0) spaces from old ones, making it easy to construct rCAT(0) spaces that are neither CAT(0) nor Gromov hyperbolic.

For metric spaces (X_1, d) and (X_2, d), the *l^2-product* $(Z, |\cdot|)$ is given by $X := X_1 \times X_2$ and

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{(d(x_1, y_1))^2 + (d(x_2, y_2))^2}.$$

It is well known that (X, d) is a metric space. Note that we are using d to indicate three different metrics: in all cases, the reader should infer from the context which one is meant. We also use $\mathrm{len}(\gamma)$ to indicate length of a path γ in any one of these spaces.

The product of CAT(0) spaces is CAT(0). A proof follows immediately from the equivalence of CAT(0) with the CN inequality of Bruhat and Tits. Since it is not clear if rCAT(0) is equivalent to a rough version of the CN inequality (that is, bolicity or very weak rCAT(0), as shown in Proposition 3.12), no such easy proof of the rough analogue of this result is available. Nevertheless it is true according to the following theorem. Quantitative dependence of the roughness constant is most neatly stated in terms of local roughness constants, but note that this gives quantitative dependence of the non-local roughness constant by Corollary 4.4.

Theorem 5.1. *If (X_1, d) and (X_2, d) are rCAT(0) length spaces with the same local roughness constant $C > 0$, then their l^2-product (X, d) is also an rCAT(0) length space with local roughness constant $\sqrt{2}\,C$.*

To prove the above theorem, we first need a lemma.

Lemma 5.2. *Suppose X_1, X_2, and X are as in Theorem 5.1. If $\gamma = (\gamma_1, \gamma_2) : [0, T] \to X_1 \times X_2$ is a path in X, then*

$$\mathrm{len}(\gamma) \ge \sqrt{(\mathrm{len}(\gamma_1))^2 + (\mathrm{len}(\gamma_2))^2}\,, \tag{5.3}$$

with equality if γ_1 and γ_2 are traversed at the same relative rate, i.e. if

$$\mathrm{len}(\gamma_1)\,\mathrm{len}\left(\gamma_2|_{[0,t]}\right) = \mathrm{len}(\gamma_2)\,\mathrm{len}\left(\gamma_1|_{[0,t]}\right), \qquad 0 < t < T\,. \tag{5.4}$$

Proof. The triangle inequality for the Euclidean plane immediately gives the following inequality for non-negative numbers a_i, b_i, $1 \le i \le n$:

$$\sum_{i=1}^{n} \sqrt{a_i^2 + b_i^2} \ge \sqrt{\left(\sum_{i=1}^{n} a_i\right)^2 + \left(\sum_{i=1}^{n} b_i\right)^2}\,.$$

By taking $a_i := d(\gamma_1(t_i), \gamma_1(t_{i-1}))$ and $b_i := d(\gamma_2(t_i), \gamma_2(t_{i-1}))$ in the above inequality, where the numbers $0 = t_0 \le t_1 \le \cdots \le t_n = T$ form a partition of $[0, T]$, we deduce (5.3).

If γ_1 and γ_2 are traversed at the same relative rate, then the vectors (a_i, b_i) defined in the last paragraph are positive scalar multiples of each other, so we get equality in the planar triangle inequality, which upon taking a supremum over all such partitions gives equality in (5.3).

If the paths are not traversed at the same relative rate then we split γ into two subpaths $\gamma^i = (\gamma_1^i, \gamma_2^i)$, $i = 1, 2$, where $\gamma^1 = \gamma|_{[0,T_1]}$, $\gamma^2 = \gamma|_{[T_1,T]}$, and $0 < T_1 < T$ is such that the equation in (5.4) fails for $t = T_1$. Letting $a_i = \operatorname{len}(\gamma_1^i)$ and $b_i = \operatorname{len}(\gamma_2^i)$, it follows that (a_1, b_1) and (a_2, b_2) are not scalar multiples of a single vector, and so

$$\begin{aligned}\operatorname{len}(\gamma) = \sum_{i=1}^{2} \operatorname{len}(\gamma^i) \geq \sum_{i=1}^{2} \sqrt{a_i^2 + b_i^2} &> \sqrt{\left(\sum_{i=1}^{2} a_i\right)^2 + \left(\sum_{i=1}^{2} b_i\right)^2} \\ &= \sqrt{(\operatorname{len}(\gamma_1))^2 + (\operatorname{len}(\gamma_2))^2}\,.\end{aligned}$$

□

We are now ready to prove Theorem 5.1. Note that it follows implicitly from the following proof that the "if" clause for equality in Lemma 5.2 is actually an "if and only if".

Proof of Theorem 5.1. Suppose $a = (a_1, a_2)$, $b = (b_1, b_2)$ are a pair of points in X. Suppose γ_i is a rectifiable path from a_i to b_i of length L_i, $i = 1, 2$. By reparametrization if necessary, we assume that γ_i is of constant speed, and then define $\gamma = (\gamma_1, \gamma_2)$. It follows from Lemma 5.2 that $\operatorname{len}(\gamma) = \sqrt{L_1^2 + L_2^2}$. Since X_i is a length space, we can choose γ_i so that L_i is arbitrarily close to $d(a_i, b_i)$, $i = 1, 2$, and it then follows that $\operatorname{len}(\gamma)$ is arbitrarily close to $d(a, b)$. Thus X is a length space.

Letting $a, b \in X$ be as above, it follows from Lemma 5.2 that if $\gamma = (\gamma_1, \gamma_2)$ is a h-short path from a to b, then γ_i is a h'-short path from a_i to b_i, $i = 1, 2$, where $h' > 0$ depends only on $d(a_1, b_1)$, $d(a_2, b_2)$, and h, with $h' \to 0$ as $h \to 0$ (for fixed a, b).

Suppose now that we are given points $x, y, z \in X$, a h-short triangle $T := T_h(x, y, z)$, and points $u, v \in X$ on different sides of T. By projecting this data onto X_i, $i = 1, 2$, it follows that we get an associated h'-short triangle $T_i := T_h(x_i, y_i, z_i)$, and points $u_i, v_i \in X_i$ on different sides of T_i; here $h' > 0$ depends only on the distances between pairs of vertices of T_i, $i = 1, 2$, and on h, with $h' \to 0$ as $h \to 0$ (for fixed x, y, z). We assume that $h > 0$, and hence h', is sufficiently small to guarantee that the C-rough CAT(0) condition holds for T_1 and T_2.

Suppose now that for each side $\gamma = (\gamma_1, \gamma_2)$ of T, the projected paths γ_1 and γ_2 are traversed at the same relative rate. Let u be on the side $[x, y]_h$ and let v be on the side $[y, z]_h$. The C-rough CAT(0) condition applied to the projected pairs of points gives $d(u_i, v_i) \leq |\bar{u}_i - \bar{v}_i| + C$, where $\bar{u}_i$, $\bar{v}_i$ are comparison points for u_i, v_i on the comparison triangle $\overline{T_i} = T(\bar{x}_i, \bar{y}_i, \bar{z}_i)$, $i = 1, 2$. It follows readily that if we define $\overline{T} = T(\bar{x}, \bar{y}, \bar{z})$, where $\bar{x} = (\bar{x}_1, \bar{x}_2)$, etc., if we define $\bar{u}$, $\bar{v}$ analogously, and if we identify the plane in $\mathbb{R}^4$ containing $\overline{T}$ with M_0^2, then $\overline{T}$ is a comparison triangle for T; $\bar{u}$, $\bar{v}$ are comparison points for u, v; and the triangle inequality implies that $d(u, v) \leq |u - v| + \sqrt{2}\,C$, as required.

In view of the above, the theorem follows readily once we prove the following claim: if we fix a pair of points $x = (x_1, x_2), y = (y_1, y_2) \in X$ with $d(x_i, y_i) > 0$ for $i = 1, 2$, and we pick a h-short path $\gamma = (\gamma_1, \gamma_2) : [0, 1] \to X$ from x to y, then γ_1 and γ_2 are traversed at *almost the same relative rate*. More precisely, if we define $L(t; \gamma_i) := \operatorname{len}(\gamma_i|_{[0,t]}) / \operatorname{len}(\gamma_i)$, then for all numbers $0 \leq t \leq 1$ and for our fixed pair of points x, y, we claim that there exists ε dependent only on h such that $|L(t; \gamma_1) - L(t; \gamma_2)| < \varepsilon$, and such that $\varepsilon \to 0$ as $h \to 0$.

Let $\mathcal{F}$ be the set of all rectifiable paths from x to y, let $D := d(x, y)$, let $D_i := d(x_i, y_i)$, and let $D(t, \gamma_i) = d(\gamma_i(t), x_i)$, for $i = 1, 2$. Since γ is h-short, and so γ_i are h'-short, with

$h' \to 0$ as $h \to 0$, the claim follows if we prove that $(D(t;\gamma_1), D(t;\gamma_2))$ stays uniformly close to the main diagonal of the rectangle $[0,D_1]\times[0,D_2]$.

Given $\gamma = (\gamma_1,\gamma_2) \in \mathcal{F}$, we define a path $\lambda_\gamma : [0,1] \to [0,1]^2$ by the equation $\lambda_\gamma(t) = (D(t;\gamma_1), D(t;\gamma_2))$. Note that λ_γ is a path from $(0,0)$ to (D_1,D_2) and we need to show that this path remains close to the diagonal (with a tolerance tending to 0 as $h \to 0$).

Given $p := (p_1,p_2) \in [0,1]^2$, let $\mathcal{F}_p$ be the set of all paths $\nu \in \mathcal{F}$ such that $\lambda_\nu(t) = p$, for some point $t \in [0,1]$. We denote the associated value of t as $t(p,\nu)$; note that $t(p,\nu)$ may not be unique, but any non-uniqueness corresponds only to a harmless choice of a point in a subinterval of $[0,1]$ on which ν remains stationary so, for the sake of having a fixed definition, we choose $t(p,\nu)$ to be the smallest number with the above defining property. Cutting $\nu \in \mathcal{F}_p$ into two subpaths ν^1,ν^2 at the point $t(p,\nu)$, we see that

$$\mathrm{len}(\nu) = \mathrm{len}(\nu^1) + \mathrm{len}(\nu^2) \ge |(p_1,p_2)| + (D_1-p_1, D_2-p_2)| =: f(p)\,.$$

Note that the function $f : [0,D_1]\times[0,D_2] \to \mathbb{R}$ defined above is continuous and it takes on its minimum value $|(D_1,D_2)| = D$ only when p lies on the main diagonal of its rectangular domain. By compactness it readily follows that the minimum value outside any given neighborhood of the main diagonal is strictly larger than D. The claim follows. $\square$

Since the class of rCAT(0) spaces are preserved by taking l^2-products, it is easy to produce an rCAT(0) space that is neither CAT(0) nor Gromov hyperbolic by taking the l^2-product of a Gromov hyperbolic space that is not CAT(0) and a CAT(0) space that is not Gromov hyperbolic. The simplest such example is the product of the unit circle and the Euclidean plane.

We now consider spaces obtained by gluing a pair of length spaces (X_i,d_i), $i=1,2$, along isometric closed subspaces $S_i \subset X_i$, $i = 1,2$ where $f_i : S \to S_i$ are isometries from some fixed metric space (S,d_S) to $(S_i, d_i|_{S_i})$. This means that we are creating a new space $X = X_1 \sqcup_S X_2$ as the quotient of the disjoint union of X_1 and X_2 under the identification of $f_1(s)$ with $f_2(s)$ for each $s \in S$. The *glued metric* d on X is defined by the equations $d|_{X_i\times X_i} = d_i$, $i=1,2$, and

$$d(x_1,x_2) = \inf_{s\in S}(d_1(x_1,f_1(s)) + d_2(f_2(s),x_2))\,, \qquad x_1 \in X_1,\ x_2 \in X_2\,.$$

Then d is also a length metric [3, I.5.24]. For simplicity of notation, we identify X_1, X_2, and S with the naturally associated subspaces of X, so that $S = X_1 \cap X_2$.

Theorem 5.5. *If $X = X_1 \sqcup_S X_2$ where (S,d_S) is of diameter $D<\infty$ and (X_i,d_i) is a C-rCAT(0) space for $i=1,2$, then X is a C'-rCAT(0) space for some $C' = C'(C,D)$.*

A comparable gluing result for CAT(0) spaces X_1 and X_2 requires that S_i be convex in X_i (meaning that it contains all geodesics in X_i between every pair of points in S_i) and complete for $i=1,2$, but the boundedness of S is dropped. The conclusion is then that X is CAT(0); see [3, II.11.1].

Before proving Theorem 5.5, we need some elementary lemmas concerning planar geometry. The first is a "small perturbation" result.

Lemma 5.6. *Suppose $T(x,y,z)$ and $T(x',y',z')$ are triangles in the Euclidean plane, and that two of $|\,|x-z|-|x'-z'|\,|$, $|\,|y-z|-|y'-z'|\,|$, and $|\,|x-y|-|x'-y'|\,|$ equals zero, with the third being at most $h := 1/(1+(|x-y|\vee|x-z|\vee|y-z|)^2)$. Suppose also that $u\in[x,z]$, $u'\in[x',z']$, $v\in[x,y]$, and $v'\in[x',y']$, with $|x-u| = |x'-u'|$ and $|x-v| = |x'-v'|$. Then $|u'-v| \le |u-v|+2$.*

Proof. We write $a = |x - z|$, $a' = |x' - z'|$, $b = |y - z|$, $b' = |y' - z'|$, $c = |x - y|$, $c' = |x' - y'|$, $d = |z - v|$, $d' = |z' - v'|$, $e = |u - v|$, and $e' = |u' - v'|$, $l = |x - u|$. Since two of the three sidelengths are preserved, we may assume by symmetry between y and z that $c = c'$. Define the numbers $s, t, t' \in [0, 1]$ by $t = l/a$, $t' = l/a'$, and $s = |x - v|/c$. We assume that $a \vee a' \geq 1$, since otherwise the result follows trivially from the triangle inequality. Thus

$$|t - t'| \leq hl/aa' \leq h/(a \vee a') \leq h\,.$$

Using (3.8), we get the following four equations, which we use implicitly in the rest of the proof:

$$\begin{aligned} d^2 &= (1-s)a^2 + sb^2 - s(1-s)c^2\,, \\ (d')^2 &= (1-s)(a')^2 + s(b')^2 - s(1-s)c^2\,, \\ e^2 &= (1-t)(sc)^2 + td^2 - t(1-t)a^2\,, \\ (e')^2 &= (1-t')(sc)^2 + t'(d')^2 - t'(1-t')(a')^2\,. \end{aligned}$$

Note that $t(1-t)a^2 = l(a-l)$ and similarly $t'(1-t')(a')^2 = l(a'-l)$. It is readily verified that $h \leq 1/(1 \vee a' \vee b' \vee c')^2$, and trivially $h \leq 1/(1 \vee a \vee b \vee c)^2$.

If $a = a'$ and $b' \leq b$, then it follows from the above equations that $d' \leq d$ and $e' \leq e$, so we are done. If $a = a'$ and $b < b' \leq b + h$, we see that $(d')^2 = d^2 + 2sbh + sh^2 \leq d^2 + 3$, and hence that $(e') \leq e^2 + 3$. Thus $e' \leq e + \sqrt{3}$ in this case.

Suppose instead that $b = b'$ and $a - h \leq a' \leq a$. Then $(d')^2 \leq d^2$, and so

$$(e')^2 \leq e^2 + (t' - t)d^2 + l(a - a') \leq e^2 + 2\,.$$

In the last inequality, we used the estimate $(t' - t)d^2 \leq 1$, which in turn follows from the earlier estimate $|t' - t| \leq h$ and the fact that $d \leq a \vee b$. We deduce that $e' \leq e + \sqrt{2}$ in this case.

Lastly, suppose that $b = b'$ and $a < a' \leq a+h$. Then $(d')^2 \leq d^2+(1-s)(ah+h^2) \leq d^2+3$, and as in the previous case

$$(e')^2 \leq e^2 + 3t' + (t - t')(sc)^2 \leq e^2 + 4\,.$$

Thus $e' \leq e + 2$ in this case. □

We now state a lemma that we call the *Zipper Lemma* because in the important case $\delta_x = \delta_y > 0$, we get one triangle from another by "zipping up" two sides (shortening them by the same amount).

Lemma 5.7. *Suppose x, y, z, z', u, u' are points in the Euclidean plane and write $\delta_x := |x - z| - |x - z'|$ and $\delta_y := |y - z| - |y - z'|$. Suppose also that $u \in [x, z]$, $u' \in [x, z']$, and $|x - u| = |x - u'|$. Then*

(a) *If $v \in [x, y]$ and $|\delta_x| \leq \delta_y$ then $|u' - v| \leq |u - v|$.*
(b) *If $v \in [y, z]$ and $v' \in [y, z']$ with $|y - v| = |y - v'|$ and $\delta_x = \delta_y \geq 0$, then $|u' - v'| \leq |u - v|$.*

Proof. By the Cosine Rule applied to the triangles $T(x, u', v)$ and $T(x, z', y)$, it is clear that the distance from u' to $v \in [x, y]$ decreases as we move z' directly towards y while keeping x, y and z fixed, since both are associated with the (common) angle at x in both triangles decreasing. Thus it suffices to prove that the angle at x in the triangle

$T(x, z', y)$ is smaller than the angle at x in the triangle $T(x, z, y)$ in the special cases $\delta_y = \delta_x > 0$ and $\delta_y = -\delta_x > 0$.

We first prove (a) for $\delta_x = \delta_y > 0$. Without loss of generality, we assume that x, y are given in Cartesian coordinates by $(c, 0)$ and $(-c, 0)$, respectively. Let

$$2a := \big|\, |z-x| - |z-y| \,\big| = \big|\, |z'-x| - |z'-y| \,\big| \,,$$

so that $a \le c$. The lemma is clear if either $a = c$ or $a = 0$, so we assume that $0 < a < c$ and write $b = \sqrt{c^2 - a^2}$ and $e = c/a$. Thus z and z' both lie on one branch of the hyperbola

$$\frac{w_1^2}{a^2} - \frac{w_2^2}{b^2} = 1 \,,$$

where (w_1, w_2) are the Cartesian coordinates of a point w on this hyperbola.

We assume for now that z, z' lie on the right branch of this hyperbola, i.e. that $|z-x| < |z-y|$. Let $r = |z-x|$ and let θ be the angle at x in the triangle $T(x, y, z)$. Then $z = (z_1, z_2)$ satisfies the equation

$$r^2 = (z_1 - ae)^2 + z_2^2 = (z_1 - ae)^2 + (e^2-1)(z_1^2 - a^2) = (ez_1 - a)^2 \,,$$

and so $r = ez_1 - a$, since we are on the right branch of the hyperbola. Also $z_1 = r\cos(\pi - \theta) + ae = ae - r\cos\theta$, and so $r = e(ae - r\cos\theta) - a$. Rearranging this equation we get

$$r = \frac{a(e^2-1)}{1 + e\cos\theta} \,.$$

It is clear from this equation that the angle θ decreases as r decreases, so we are done.

If instead $|z-x| > |z-y|$, the analysis is similar except that now $r = a - ez_1$, and so we instead get

$$r = \frac{a(e^2-1)}{-1 + e\cos\theta} \,,$$

and again it is clear that θ decreases as r decreases.

We now prove (a) for $\delta_y = -\delta_x > 0$. We could do this in a similar manner to the proof for $\delta_y = \delta_x$ above, but using an ellipse rather than a hyperbola. However we will instead give a slightly shorter calculus proof. Let $a := |x-z|$, $b := |y-z|$, and $c := |x-y|$, and let θ be the angle at x in the triangle $T(x, y, z)$. The desired conclusion is obvious in the degenerate cases $b = a + c$ and $c = a + b$, and the degenerate case $a = b + c$ cannot arise by the triangle inequality since $|x - z'| > a$ and $|y - z'| < b$. We may therefore assume that we are in the non-degenerate case with $\sin\theta > 0$. For the rest of this paragraph prime superscripts indicate derivatives with respect to a parameter t. Specifically, holding c fixed, and considering $a = a(t)$, $b = b(t)$, and $\theta = \theta(t)$ to be functions of t with $a'(t) = 1$ and $b'(t) = -1$, it suffices to show that $\theta'(t) < 0$ for all $0 \le t < \delta_y$. The fact that the triangle is non-degenerate at $t = 0$ implies that it is non-degenerate for all $0 \le t < \delta_y$, and so $\sin\theta(t) > 0$ on $[0, \delta_y)$. Differentiating the equation $b^2 = a^2 + c^2 - 2ac\cos\theta$, we get

$$-b(t) = a(t) - c(t)\cos\theta(t) + a(t)c(t)\sin\theta(t)\theta'(t) \,,$$

and so

$$\theta'(t) = \frac{-a(t) - b(t) + c(t)\cos\theta(t)}{a(t)c(t)\sin\theta(t)} \,.$$

The desired inequality $\theta'(t) < 0$ follows easily for all $0 \le t < \delta_x$.

Finally we prove (b). Let $a := |x-z|$, $b := |y-z|$, and $c := |x-y|$ as before, and also let $p := |z-u|$, $q := |z-v|$, and $e := |u-v|$. Without loss of generality, we assume that

$a, b, p, q > 0$, $p < a$, and $q < b$. Again we use calculus and reserve prime superscripts for t-derivatives below. Holding c fixed and taking $a'(t) = b'(t) = p'(t) = q'(t) = -1$, with $\theta(t)$ being the angle at z for $T(x, y, z)$, it suffices to show that $e'(t) < 0$. Differentiating the Cosine Rule for the triangles $T(x, y, z)$ and $T(u, v, z)$ with respect to t, we get

$$0 = (a(t) + b(t))(\cos\theta(t) - 1) + (a(t)b(t)\sin\theta(t))\theta'(t)\,,$$
$$e(t)e'(t) = (p(t) + q(t))(\cos\theta(t) - 1) + (p(t)q(t)\sin\theta(t))\theta'(t)\,.$$

Combining these equations, we get

$$\frac{e(t)e'(t)}{p(t)q(t)} = \left(\frac{1}{a(t)} + \frac{1}{b(t)} - \frac{1}{p(t)} - \frac{1}{q(t)}\right)(1 - \cos\theta(t))\,,$$

and so it is clear that $e'(t) \le 0$, as required. □

We now state a useful perturbation of the previous lemma. The proof is easy: for (a), first apply Lemma 5.6 to lengthen $|z - x|$ by h, and then apply Lemma 5.7, and for (b), apply Lemma 5.6 twice and then Lemma 5.7.

Lemma 5.8. *Suppose x, y, z, z', u, u' are points in the Euclidean plane and write $\delta_x := |x - z| - |x - z'|$ and $\delta_y := |y - z| - |y - z'|$. Suppose also that $u \in [x, z]$, $u' \in [x, z']$, and $|x - u| = |x - u'|$ and we write and $h := 1/(1 + (|x - y| \vee |x - z| \vee |y - z|)^2)$. Then*

(a) *If $v \in [x, y]$ and $|\delta_x| \le \delta_y + h$ then $|u' - v| \le |u - v| + 2$.*
(b) *If $v \in [y, z]$ and $v' \in [y, z']$ with $|y - v| = |y - v'|$ and $\delta_x + h_1 = \delta_y + h_2 \ge 0$ for some $0 \le h_1, h_2 \le h$, then $|u' - v'| \le |u - v| + 4$.*

Proof of Theorem 5.5. Let d be the glued metric on X. We first claim that any h-short path $\gamma : [0, L] \to X$ for a pair of points $x, y \in X_1$ lies within a distance $D/2 + 2h$ of a h-short path for this pair in X_1.

Suppose without loss of generality that γ is parametrized by arclength and not contained in X_1. The only parts of γ that do not fully lie in X_1 consist of disjoint subpaths γ_i, $i \in I$, where I is a countable index set and the endpoints of every γ_i lie in S.

The distance between these endpoints is the same in either X_1 or X_2, and X_1 is a length space, so we can replace these subpaths by subpaths in X_1 whose combined length is at most the same as the combined length of the γ_i subpaths, as long as least one γ_i is non-geodesic, an assumption that we add for the moment. We therefore get a new h-short path $\gamma' : [0, L'] \to X_1$ from x to y, parametrized by arclength, with $L' \le L$. By the triangle inequality, $d(\gamma(t), \gamma'(t)) \le (D + h)/2 + h$ for all $0 \le t \le L'$. This establishes the claim under the assumption that at least one of the subpaths γ_i is non-geodesic.

The argument when every γ_i is geodesic is similar except that we may not be able to replace them by geodesic subpaths in X_1. As long as $L < d(x, y) + h$, we can choose the replacement subpaths so short as to guarantee that the resulting path $\gamma' : [0, L'] \to X_1$ from x to y is h-short, is parametrized by arclength, and again satisfies $d(\gamma(t), \gamma'(t)) \le (D + h)/2 + h$ for all $0 \le t \le L$.

The only remaining problem is when $L = d(x, y) + h$. It follows that the parts of γ other than the γ_i subpaths cannot all be geodesic, so we can take a non-geodesic subpath of γ that is disjoint from every γ_i and has length at most $h/2$. We replace this non-geodesic subpath by a shorter subpath that remains within X_1. We now have a path of length less than $d(x, y) + h$ and we can proceed as in the last paragraph to construct a h-short path $\gamma' : [0, L'] \to X_1$ from x to y, parametrized by arclength, satisfying

$d(\gamma(t),\gamma'(t)) \le (D+h)/2+h+h/2$ for all $0\le t\le L'$. This finishes the proof of the claim.

In view of the above claim and Corollary 4.4, it suffices to prove the rCAT(0) condition for all h-short triangles with given vertices x,y,z, where $h\le H$ for some $H=H(x,y,z)>0$, and considering only h-short sides within X_i for any pair of vertices that both lie in X_i, $i=1,2$.

Thus it suffices to prove an rCAT(0) condition for a h-short triangle with vertices x,y,z, where $x,y\in X_1$ and $z\in X_2$, and the path in the triangle from x to y is $\gamma_{xy}:[0,L_{xy}]\to X_1$, with similar notation for the other two sides. We assume that $h\le H$, where $H:=1/3(1+(d(x,y)\vee d(x,z)\vee d(y,z))^2)$.

We may further assume that both $\gamma^1_{xz}:=\gamma_{xz}|_{[0,M_{xz}]}$ and $\gamma^1_{yz}:=\gamma_{yz}|_{[0,M_{yz}]}$ lie in X_1, and both $\gamma^2_{xz}:=\gamma_{xz}|_{[M_{xz},L_{xz}]}$ and $\gamma^2_{yz}:=\gamma_{yz}|_{[M_{yz},L_{yz}]}$ lie in X_2, for some choice of numbers M_{xz} and M_{yz}. Of these four subpaths, we call the two with superscript "1" the *initial segments* of the associated side of the triangle, and the other two the *final segments* of the associated side. We write $s_x:=\gamma_{xz}(M_{xz})$ and $s_y:=\gamma_{yz}(M_{yz})$.

Symmetry reduces the task of verifying the rCAT(0) condition for points u,v to the following five cases:

(a) u lies on the initial segment of γ_{xz}, and v lies on γ_{xy}.
(b) u lies on the final segment of γ_{xz}, and v lies on γ_{xy}.
(c) u,v lie on final segments of γ_{xz} and γ_{yz}, respectively.
(d) u,v lie on initial segments of γ_{xz} and γ_{yz}, respectively.
(e) u lies on the final segment of γ_{xz}, and v lies on the initial segment of γ_{yz}.

In Case (a), we first apply the Zipper Lemma Lemma 5.8(a), with all data as in that lemma except for the Lemma's z' and u': we take $z'=s_y$ and the u' is taken to be a point on a h-short path λ from x to s_y whose distance to x is $d(x,u)$, if such a point exists (which we assume for now). Now $d(u,v)\le d(u,u')+d(u',v)$, and by the rCAT(0) condition for the triangle with vertices x,s_x,s_y, we see that $d(u,u')\le D+C$. Combining the rCAT(0) conditions for the triangles with vertices x,y,s_y with the Zipper Lemma and this estimate for $d(u,u')$, we deduce the desired rCAT(0) inequality for the pair u,v in the triangle with vertices x,y,z.

If there is no point u' on λ with $d(u',x)=d(u,x)$, then take $u'=s_y$, and so $d(u',x)<d(u,x)$. As in the last paragraph, we get an rCAT(0) inequality for the pair u'',v, where u'' is a point on γ_{xy} such that $d(u'',x)=d(u',x)$. But since

$$d(u',x)\ge d(s_x,x)-D\ge d(u,x)-D-h\,,$$

and so $d(u',u)\le D+2h$. Since this quantity is bounded, the rCAT(0) condition for u'',v implies an rCAT(0) condition for u,v (with a parameter C that is larger by $2D+4h$).

We next consider Case (b). First construct a "comparison quadrilateral" $\bar Q$ with vertices $\bar x,\bar y,\bar s_y,\bar s_x$ for the quadrilateral Q with vertices x,y,s_y,s_x. Theorem 3.19 ensures that we can do this in a certain sense, but we need less than guaranteed by that: in fact we need only that distances between each of the four pairs of adjacent pairs of adjacent vertices is preserved (such a "comparison quadrilateral" exists for any quadrilateral in any metric space). We form a new metric space space (G,d_G) by gluing a filled Euclidean triangle with sides of length $|\bar s_x-\bar s_y|=d(s_x,s_y)$, $d(y,z)-d(y,s_y)$, and $d(x,z)-d(x,s_x)$, to the Euclidean plane along the line segment from $\bar s_x$ to $\bar s_y$. If we can prove a variant rCAT(0) condition for the triangle with vertices x,y,z, and u,v as in Case (b) where we have all the usual inequalities and equations of Definition 3.1, but with the (geodesic) comparison triangle $\overline{T}$ in G rather than the Euclidean plane, then the usual CAT(0) condition follows by combining this variant rCAT(0) condition with the usual CAT(0) condition for the comparison triangle in G; the fact that G is CAT(0) follows from the CAT(0) gluing theorem referred to after the statement of Theorem 5.5. Since u is on the final segment of γxz and $v \in X_1$, we see that $d(u,v) = d(u,s) + d(s,v)$ for some $s \in S$, and so $d(u,v)$ is within a distance $2D$ of $d(u,s_x) + d(s_x,v)$. Similarly if $\bar{u}, \bar{v}$ are the comparison points for u, v in $\overline{T}$, then $d(\bar{u},\bar{v})$ is within a distance $2D$ of $d(\bar{u},\bar{s}_x) + d(bs_x,\bar{v})$. Since $d(u,s_x)$ and $d(\bar{u},\bar{s}_x)$ differ by at most h, it follows that the desired variant rCAT(0) condition for u, v follows from the usual rCAT(0) condition for the pair of points s_x, v, as proven in Case (a) (once we increase the parameter C by $8D + 2h$).

Case (c) follows easily from the fact that u, v lie on a h-short triangle in X_2 with vertices z, u_0, and v_0, with $d(u_0, v_0) \le D$; we leave the details to the reader.

We next handle Case (d). Suppose first that we can find a point $w \in X_1$ such that $d(w,s_x) \le D + h$, $d(w,s_y) \le D + h$, and

$$d(z,x) - d(w,x) + h_1 = d(z,y) - d(w,y) + h_2 \ge 0\,,$$

for some $0 \le h_1, h_2 \le 3h$. We pick h-short paths γ_{xw} and γ_{yw} from x to w, and from y to w, respectively, and associated points u' on γ_{xw} and v' on γ_{yw} such that $d(u',x) = d(u,x)$ and $d(v',y) = d(v,y)$ (as for u'' in Case (a), we let u' and/or v' equal w if one or other of these last equations cannot be satisfied). Applying Lemma 5.8(b) with $3h$ playing the role of h in the lemma, it is clear that the the distance apart of the comparison points for u, v in the comparison triangle $T_1 = T(\bar{x}, \bar{y}, \bar{z})$ for the triangle with vertices x, y, z is either larger, or smaller by at most 4, than the distance apart of the comparison points for u, v in the comparison triangle $T_2 = T(\bar{x}, \bar{y}, \bar{w})$ for the triangle with vertices x, y, w, assuming that we choose the comparison points so that $d(u,x) = |\bar{u} - \bar{x}|$, and similarly preserve $d(u',x)$, $d(v,y)$, and $d(v',y)$. Putting the rCAT(0) condition for the pair u, u' together with the estimates $d(u,u') \le D + h + C$ and $d(v,v') \le D + h + C$, we get an rCAT(0) condition for u, v, as required.

It remains to find a point w with the desired properties. Let $\lambda : [0,L] \to X_1$ be a h-short path from s_x to s_y, parametrized by arclength. Let $w = \lambda(t)$ for some $t \in [0,L]$. Then $d(w,s_x) \le D + h$ and $d(w,s_y) \le D + h$ whenever $w = \lambda$. Also let $\delta_x := d(x,z) - d(x,w)$ and $\delta_y := d(y,z) - d(y,w)$.

When $t = 0$, the h-shortness of γ_{xz} implies that

$$L_{xz} - M_{xz} + h \le \delta_x + 2h \le L_{xz} - M_{xz} + 2h\,,$$

and the h-shortness of γ_{xz} implies that

$$-D \le -d(s_y,w) \le h + d(z,y) - d(z,s_y) - d(s_y,w) \le \delta_y + h \le L_{xz} - M_{xz} + h\,.$$

In particular, $\delta_x + 2h \ge \delta_y$. Similarly when $t = L$ we get that $\delta_y + 2h \ge \delta_x$. It follows that for some $t \in [0,L]$ we have $\delta_x + h_1 = \delta_y + h_2$, for some non-negative numbers h_1, h_2 with $h_1 + h_2 = 3h$. (In fact we get $\delta_x + h_1 = \delta_y + h_2$ for some non-negative numbers h_1, h_2 satisfying $h_1 + h_2 \le 2h$ and $h_1 h_2 = 0$, but it suits us to increase both numbers so that $h_1 + h_2 = 3h$.)

Note that

$$L \le d(s_x,s_y) + h \le d(s_x,z) + d(z,s_y) + h \le L_{xz} - M_{xz} + L_{yz} - M_{yz} + h\,,$$

and so

$d(w,x)) + d(w,y) \le (M_{xz} + t) + (L - t + M_{yz}) \le L_{xz} + L_{yz} + h \le d(z,x) + d(z,y) + 3h\,.$

It follows that $\delta_x + \delta_y + 3h \ge 0$, and so $\delta_x + h_1 = \delta_y + h_2 \ge 0$, as required.

Case (e) follows from Case (d) in the same way as Case (b) follows from Case (a). □

It is often useful to glue an infinite number of spaces together, sometimes along a single point or set, or sometimes at different places along some base space. The following general gluing theorem says that for either of these types of gluing of C-rCAT(0) spaces along uniformly bounded gluing sets, we get another rCAT(0) space.

Theorem 5.9. *Suppose we have a collection of C-rCAT(0) spaces X_i, $i \in I$, where I is some index set containing* 0 *as an element. We write $I^* = I \setminus \{0\}$. Suppose further that in each X_i, $i \in I^*$, we have a closed subspace S_i that is glued isometrically to a closed subspace T_i of X_0. Suppose further that S_i (and T_i) is of diameter at most $D < \infty$, $i \in I^*$. Then the resulting space X is a C'-rCAT(0) space for some $C' = C'(C, D)$.*

Sketch of proof. Using a similar argument to the proof of the claim at the beginning of the proof of Theorem 5.5, we see that for sufficiently small h, a h-short path between $x \in X_i$ and $j \in X_j$, $i, j \in I$, is within a bounded Hausdorff distance of a h-short path path that only passes through X_i, X_j, and X_0. Thus we may restrict ourselves to examining h-short triangles whose sides are of this type, and an rCAT(0) condition for any pair of points on such a triangle with vertices in X_i, X_j, and X_k follows from at most three appeals to Theorem 5.5 (to glue X_i, X_j, and X_k to X_0). □

As mentioned earlier, if we glue a pair of CAT(0) spaces along a pair of isometric convex subspaces, we get a CAT(0) space. It is tempting therefore to suspect that if we glue a pair of rCAT(0) spaces along a pair of isometric convex subspaces (or even isometric "roughly convex" subspaces, whatever this should mean), we get an rCAT(0) space. However this is false as the following example shows.

Example 5.10. First let $H_1 := \{(x,y) \mid y \ge 1\}$ be given the Riemannian metric $ds^2 = (dx^2 + dy^2)/y^2$, so that H_1 is a closed horodisk in the upper half-plane with the hyperbolic metric attached. At each point $(n,1) \in H_1$, $n \in \mathbb{N}$, we glue the endpoint of a line segment I_n of length $\exp(-|n|)$, and we glue the other endpoint of I_n to the point n on a copy of the real line $\mathbb{R}$ (same copy for each n). We call this copy of $\mathbb{R}$ the *glue line* in view of its use below. Attaching the intrinsic metric to the resulting glued space X, we get a geodesic space which restricts to the hyperbolic metric on H_1 (because the distance from $(n,0)$ to $(n+1,0)$ in H_1 is $\cosh^{-1}(3/2) < 1$). It is a routine matter to prove that X is roughly isometric to H_1. It follows immediately that X is also Gromov hyperbolic, and so rough CAT(0).

We now glue two copies of X along their glue lines by identifying both copies of x for each $x \in \mathbb{R}$. We claim that the resulting glued space Z is not rough CAT(0). To see this note first that within H_1, the distance from $(0,y)$, $y > 1$, to $x_n := (n,1)$ is an even function of $n \in \mathbb{Z}$ and is increasing as a function of $|n|$. Moreover

$$a_n := d((0,y), x_n) - d((0,y), x_0) \to 0 \qquad (y \to \infty)\,,$$

because the metric boundary of H_1 is a horocycle in H. Fixing $n \in \mathbb{N}$ and choosing y so large that $a_n < e^{-n+1} - e^{-n}$, we ensure that one geodesic segment between the two copies of $(y,0)$ in Z must intersect the glue line at a point $(N,0)$ for some $N \in \mathbb{N}$, $N \ge N$. By symmetry, another geodesic segment between the two copies of $(y,0)$ goes via $(-N,0)$. Letting $n\to\infty$, we therefore have a pair of geodesics between the same endpoints such that the distance between their midpoints, $2n$, can be arbitrarily large. Such a configuration is incompatible with the rCAT(0) condition.

Finally, we show that there are no interesting examples among the class of normed real vector spaces. As is well known, such spaces are CAT(0) if and only if they are inner product spaces [3, II.1.14]. It is straightforward to use the dilation structure of such spaces to show that they must be CAT(0) if they are rCAT(0); we give the details for completeness.

Proposition 5.11. *Suppose* $(V,\|\cdot\|)$ *is a normed real vector space with distance* $d(x,y)=\|x-y\|$*. Then* V *is rCAT(0) if and only if it is CAT(0).*

Proof. Suppose (V,d) is C-rCAT(0). Being a normed vector space, V is certainly a geodesic space. We wish to prove the CAT(0) condition for a fixed but arbitrary geodesic triangle T with vertices $x,y,z\in V$. The translation invariance of d allows us to assume without loss of generality that $x=0$. Let $\overline{T}$ be a comparison triangle in M_0^2 with vertex at 0 corresponding to $x=0$, let u,v be points on different sides of T and let $\bar u,\bar v$ be the respective comparison points on $\overline{T}$.

We now exploit the dilation invariance of V. Given a geodesic $\gamma:[0,L]\to V$ from $a\in V$ to $b\in V$, we get a dilated geodesic $R\gamma:[0,L]\to V$ from Ra to Rb for any given $R>0$ by defining $(R\gamma)(t)=R\gamma(t)$. If we dilate our geodesic triangle T in this manner, we get a geodesic triangle which we call RT, and it is clear that the similarly dilated Euclidean triangle $R\overline{T}$ is a comparison triangle for RT, and that $R\bar u,R\bar v$ are respective comparison points for $Ru,Rv\in RT$. Furthermore if $d(u,v)=|\bar u-\bar v|+\varepsilon$ for some $\varepsilon>0$, then $d(Ru,Rv)=|R\bar u-R\bar v|+R\varepsilon$, so by taking $R>C/\varepsilon$ we contradict the rCAT(0) inequality. Thus the rCAT(0) condition can only hold if the CAT(0) condition holds. $\square$

Remark 5.12. It follows from the above theorem that we cannot change the l^2-product in Theorem 5.1 to an l^p-product for any $p\neq 2$, since certainly the l^p-product of two Euclidean lines is rCAT(0) only when $p=2$.

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