Split Nonthreshold Laplacian Integral Graphs

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Abstract

The aim of this article is to answer a question posed by Merris in *European Journal of Combinatorics*, 24(2003)413 − 430, about the possibility of finding split nonthreshold graphs that are Laplacian integral, i.e., graphs for which the eigenvalues of the corresponding Laplacian matrix are integers. Using Kronecker products, balanced incomplete block designs, and solutions to certain Diophantine equations, we show how to build infinite families of these graphs.

Keywords: Split graph, threshold graph, semiregular graph, Laplacian integral graph, block design.

1 Basic notions

Let $G = (V, E)$ be a *simple graph* such that, for $i, j = 1, 2, \ldots, n$, $v_i \in V$ is a vertex, $\{v_i, v_j\} \in E$ is an edge and its *degree sequence* is $\pi(G) = (d_1, d_2, \cdots, d_n)$, where d_i is the *degree* of v_i. If all the vertices of G have the same degree, the graph is *regular*, while G is *biregular* if its degree sequence is constituted by only two distinct values. A graph G is *bipartite* if its vertices can be partitioned into two sets in such a way that no edge joins two vertices in the same set.

The *Laplacian matrix* of G is $L(G) = D(G)\text{-}A(G)$, where $A(G)$ is the adjacency matrix of G and $D(G)$ is the diagonal matrix of the vertex degrees. The *spectrum of* $L(G)$, $\zeta(G) = (\mu_1, \ldots, \mu_{n-1}, \mu_n)$, is the sequence of eigenvalues of $L(G)$ displayed in non-increasing order: $\mu_1 \geq \ldots \geq \mu_{n-1} \geq \mu_n$. It is well-known that $L(G)$ is a positive semidefinite and singular matrix, so $\mu_n = 0$. When each element of $\zeta(G)$ is an integer, G is called a *Laplacian integral graph*.

A graph G is a *cograph*, also known as a *decomposable graph* if and only if no induced subgraph of G is isomorphic to P_4, [4]. These graphs can be constructed from isolated vertices by a sequence of operations of unions and complements. When a graph is not a cograph, it is called *indecomposable*. In [8], it is proved that any cograph is Laplacian integral. A graph G is a *threshold graph* if and only if it does not have an induced subgraph isomorphic to one of the forbidden graphs P_4, C_4 or $2K_2$. Among the many interesting properties of threshold graphs is the fact that they are uniquely determined by their Laplacian spectra (see Theorem 6.1 of [6]). Because threshold graphs are cographs, their Laplacian eigenvalues are all integers. A *split graph* is one whose vertex set can be partitioned as the disjoint union of an independent set and a clique (either of which may be empty). It is shown in [2] that a graph is split if and only if it does not have an induced subgraph isomorphic to one of the following three forbidden graphs: C_4, C_5, and $2K_2$. It follows from the definition that the complement of a split graph, as well as every induced subgraph of a split graph, is split, [9]. Based on the characterizations above, no nonthreshold split graph is a cograph. In a recent paper, Grone and Merris [7], demonstrate the existence of an infinite number of indecomposable Laplacian integral graphs. This result strengthens the interest of the following open problem posed by Merris in his paper [9]: *There appear, on the other hand, to be many nondecomposable Laplacian integral graphs, and one might think that the natural place to look for them would be among those graphs closest to the thresholds. Yet, preliminary explorations have not yielded a single nonthreshold, Laplacian integral, split graph. Why are they so difficult to find?*

The question above motivates our investigation of graphs with those characteristics. Specifically, in this paper we look for nonthreshold split graphs that are Laplacian integral. We construct infinite families of such graphs, thus partially answering Merris's question.

In the next section, we build biregular split graphs from regular or biregular bipartite graphs. This allows us to obtain a characterization of Laplacian integral biregular split graphs. In Section 3, we use a generalization of the balanced incomplete block design in order to get Theorem 3.1, the main result of this paper. It shows how to obtain an infinite family of biregular split nonthreshold Laplacian integral graphs, for each generalized block design. This section ends with two example of these graphs, one of them with 52 vertices, 714 edges and the maximal clique with the size 28. A note on the complements of biregular split nonthreshold Laplacian integral graphs is presented in the Section 4. In the last section, we show how to obtain biregular split nonthreshold Laplacian integral graphs that are cospectral and nonisomorphic.

2 Characterization of Laplacian integral biregular split graphs

This section is devoted to proving Theorem 2.2, which gives necessary and sufficient conditions for a biregular split graph to be Laplacian integral. First, we review some concepts and introduce a new one, the *splitness* of a split graph, which is useful in the characterization of bipartite split graphs.

DEFINITION 2.1. *Let $H = (V, E)$ be a bipartite graph with a vertex partition $V = V_1 \cup V_2$ such that each vertex in V_1 has degree x and each vertex in V_2 has degree y. If $x \neq y$, we say that H is an (x, y)-semiregular graph, or simply semiregular and, if $x = y$, we say that H is a y-regular bipartite graph, or simply regular bipartite. In both cases, this partition is called a degree partition of H. Although in the second case we can have more than one partition, in the first one, one degree partition is unique.*

It is interesting to note that every semiregular graph is bipartite biregular, but the converse is not necessarily true. For example, P_4 is bipartite biregular, but it is not semiregular.

DEFINITION 2.2. *A graph $G = (V, E)$ is a split graph if there is a partition of $V = V_1 \cup V_2$ such that the induced subgraph $< V_1 >$ is a complete graph and V_2 is an independent set. Such a partition, which may not be unique, is referred to as a split partition set of V.*

EXAMPLE 2.1. *The split graph G showed in Figure 1 has split partitions $V = \{1, 2, 3\} \cup \{4, 5\}$, $V = \{1, 2, 3, 4\} \cup \{5\}$ and $V = \{1, 2, 3, 5\} \cup \{4\}$.*

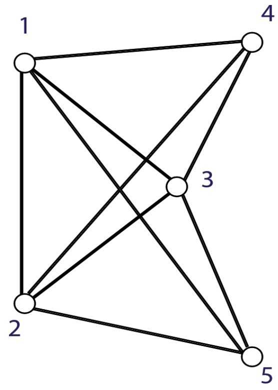

Figure 1: G is a 4,3)-biregular split graph

THEOREM 2.1. *Let G be a (t, y)-biregular connected split graph, where $t > y > 0$. Then $V = V_t \cup V_y$, where $V_t = \{v \in V \mid d(v) = t\}$ and $V_y = \{v \in V \mid d(v) = y\}$, is a split partition set of V.*

Proof: Let G be a (t, y)-biregular connected split graph where $t > y > 0$. Consider a split partition set of $V = U_1 \cup U_2$ such that the induced subgraph $< U_1 >$ is a maximal clique and U_2 is an independent set. Note that, since G is connected, there is a vertex $v \in U_1$ such that $d(v) > |U_1| - 1$. So, $t > |U_1| - 1$. Since $< U_1 >$ is a maximal clique and $t > y$, it follows that $y \leq |U_1| - 1$. Moreover, each vertex in U_2 has degree y, for U_2 is an independent set. So $U_2 \subseteq V_y$. If all vertices in U_1 have degree t then $U_1 = V_t$, $U_2 = V_y$ and the result follows. In the case that not every vertex of U_1 has degree t, then there is some vertex in U_1 with degree y. So, we can take $U = \{u \in U_1 \mid d(u) = y\}$, a non-empty subset of U_1 and also $V_1 = U_1 - U$ and $V_2 = U_2 \cup U$. Since U_1 induces a clique, then $y \geq |U_1| - 1$ and, as $y \leq |U_1| - 1$, we have $y = |U_1| - 1$. So, if $u \in U$ then u is not adjacent to any vertex of U_2. Since all vertices of U_2 have degree y, we have that $y \leq |U_1| - |U|$. So, $|U| = 1$ and V_2 is an independent set. It follows that $V = V_1 \cup V_2$ is a split partition set of V such that $V_1 = V_t$ and $V_2 = V_y$, where $y = |U_1| - 1$. ■

DEFINITION 2.3. *Let G be a (t, y)-biregular connected split graph, where $t > y > 0$. The split partition set $V = V_t \cup V_y$ is called the split degree partition (or sdp) of the graph G.*

EXAMPLE 2.2. *The biregular split graph G in Figure 1 has split degree partition $V = \{1, 2, 3\} \cup \{4, 5\}$.*

Corollary 2.1. *Let G be a (t, y)-biregular connected split graph, where $t > y > 0$. There is a unique bipartite graph H which is either the maximal semiregular spanning subgraph of G, or the maximal regular spanning subgraph of G such that there is a degree partition that is the split degree partition of G.*

Proof: Let G be a (t, y)-biregular connected split graph where $t > y > 0$ and consider its split degree partition $V = V_t \cup V_y$. Let H be the bipartite graph obtained from G taking off all edges between the vertices in V_t. If $t > y + |V_t| - 1$, H is the maximal semiregular spanning subgraph of G. Otherwise, H is the maximal regular bipartite spanning subgraph of G. In both cases, the degree partition of H is the sdp of G. Clearly, H is the unique bipartite spanning subgraph of G satisfying these properties.■

Definition 2.4. *Let G be a (t, y)-biregular connected split graph, where $t > y > 0$. We define the splitness of G to be the bipartite graph H which is either the maximal semiregular spanning subgraph of G, or the maximal regular spanning subgraph of G such that its degree partition is the split degree partition of G.*

Example 2.3. *Figure 2 shows the splitness of the (4,3)-biregular split graph G shown in Figure 1.*

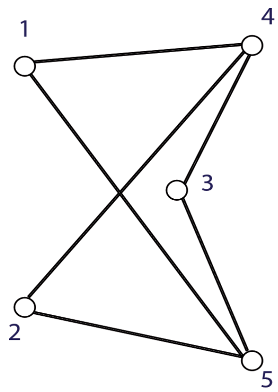

Figure 2: The splitness of G in Figure 1

From Corollary 2.1, we note that each biregular split graph is obtained from a semiregular or a regular bipartite graph H, its splitness, by choosing one of the sets, either V_1 or V_2, and adding all the possible edges between their elements. Conversely, a semiregular bipartite or a regular bipartite graph H can be obtained from a biregular split graph taking off all edges in the clique determined by its sdp.

The following example illustrates the fact that every (x, y)-semiregular graph generates two non-isomorphic biregular split graphs.

EXAMPLE 2.4. *The semiregular graph $K_{2,3}$ generates two biregular split graphs which are shown in Figure 3. Its adjacency matrix can be written as*

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

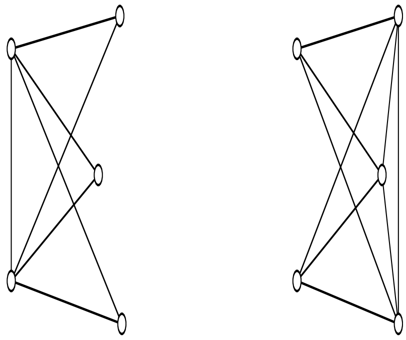

Figure 3: Biregular split graphs generated by $K_{2,3}$

REMARK 2.1. *The adjacency matrix of the splitness H of a split biregular graph $G = (V, E)$, such that its sdp is $V = V_1 \cup V_2$ with $\mid V_1 \mid= p$ and $\mid V_2 \mid= q$, has the form*

$$A(H) = \begin{pmatrix} 0 & \mathbb{X} \\ \mathbb{X}^T & 0 \end{pmatrix},$$

where $\mathbb{X}$ is a $(0, 1)$-submatrix of order $p \times q$. Moreover, there are parameters x, y such that $X satisfies the following equalities : X1_q = x\mathbb{1}_p$; $\mathbb{X}^T \mathbb{1}_p = y\mathbb{1}_q$ and $\mathbb{1}_p^T \mathbb{X} \mathbb{1}_q = px = qy$, where $\mathbb{I}_p$ is the identity matrix of order p and $\mathbb{1}_q$ is the vector of order q with all elements equal to 1.

Henceforth, we denote the $p \times p$ all ones matrix by $\mathbb{J}_p$, and the $p \times q$ all ones matrix by $\mathbb{J}_{p,q}$.

COROLLARY 2.2. *A graph G is split biregular if, and only if, its Laplacian matrix has the form*

$$L(G) = \begin{pmatrix} (p+x)\mathbb{I}_p - \mathbb{J}_p & 0 \\ 0 & y\mathbb{I}_q \end{pmatrix} - A = \begin{pmatrix} (p+x)\mathbb{I}_p - \mathbb{J}_p & -\mathbb{X} \\ -\mathbb{X}^T & y\mathbb{I}_q \end{pmatrix}, \qquad (2.1)$$

where A is the adjacency matrix of the splitness of G, and where the sdp $V = V_1 \cup V_2$ is such that $p =\mid V_1 \mid$ and $q =\mid V_2 \mid$.

Proof: The statement follows directly from Theorem 2.1. ■

Finally, we present the main result of this section, a characterization theorem of biregular split Laplacian integral graphs.

THEOREM 2.2. *Let G be a connected biregular split graph with Laplacian matrix given by (2.1) in Corollary 2.2. If $\mathbb{X} = \mathbb{J}_{p,q}$, then the Laplacian spectrum of G is 0 with multiplicity 1, p with multiplicity $q-1$, and $p+q$ with multiplicity p. If $\mathbb{X} \neq \mathbb{J}_{p,q}$, let r denote the rank of $\mathbb{X}$, and for each nonzero eigenvalue τ of $\mathbb{X}\mathbb{X}^T$, let m_τ be equal to the maximum number of linearly independent τ-eigenvectors of $\mathbb{X}\mathbb{X}^T$ that are orthogonal to $\mathbb{1}_p$. Then the Laplacian spectrum of G is as follows: 0 with multiplicity 1; y with multiplicity $q-r$; $p+x$ with multiplicity $p-r$; $x+y$ with multiplicity 1; and for each nonzero eigenvalue τ of $\mathbb{X}\mathbb{X}^T$, the roots of the equation $\mu^2 - (p+x+y)\mu + (p+x)y - \tau$, each with multiplicity m_τ.*

Proof: If $\mathbb{X} = \mathbb{J}_{p,q}$ then $G = \overline{(\overline{K_p} \cup K_q)}$ and the result follows. Now suppose that $\mathbb{X} \neq \mathbb{J}_{p,q}$. Note that $x+y$ is an eigenvalue of $L(G)$ with

$$\begin{pmatrix} x\mathbb{1}_p \\ -y\mathbb{1}_q \end{pmatrix}$$

as its eigenvector. Further, if $Ker(\mathbb{X}) \neq \{0\}$ and w is a nonzero vector in $Ker(\mathbb{X})$ then

$$u = \begin{pmatrix} 0 \\ w \end{pmatrix}$$

satisfies

$$L(G)u = \begin{pmatrix} -\mathbb{X}w \\ y\mathbb{I}_q w \end{pmatrix} = y \begin{pmatrix} 0 \\ w \end{pmatrix} = yu.$$

In this case, y is an eigenvalue of $L(G)$ with multiplicity equal to $q-r$. Similarly, if z is a non-zero vector in $Ker(\mathbb{X}^T)$ then z is orthogonal to $\mathbb{1}_p$, since G is connected and $\mathbb{X}\mathbb{1}_q = x\mathbb{1}_p$. Then, for

$$v = \begin{pmatrix} z \\ 0 \end{pmatrix},$$

$L(G)v = (p+x)v$. So, $p+x$ is an eigenvalue of $L(G)$ with multiplicity equal to $p-r$.

Finally, we suppose that τ is a non-zero eigenvalue of $\mathbb{X}\mathbb{X}^T$ with a corresponding eigenvector u that is orthogonal to $\mathbb{1}_p$. Then, for $a, b \in \mathbb{R}$,

$$w = \begin{pmatrix} a \\ b \end{pmatrix}$$

is an eigenvector of

$$\begin{pmatrix} p+x & -\tau \\ -1 & y \end{pmatrix}$$

associated with eigenvalue μ if and only if,

$$v = \begin{pmatrix} au \\ b\mathbb{X}^T u \end{pmatrix}$$

is also an eigenvector of $L(G)$ associated with eigenvalue μ. So, for each τ as above, the roots of $\mu^2 - (p+x+y)\mu + (p+x)y - \tau = 0$ are eigenvalues of $L(G)$. Further, the multiplicity of each root μ is equal to m_τ.■

The following is immediate.

COROLLARY 2.3. *Let G be as in Theorem 2.2. Then G is Laplacian integral if and only if, for each non-zero eigenvalue τ of $\mathbb{X}\mathbb{X}^T$ with an associated eigenvector orthogonal to $\mathbb{1}_p$, $(p+x-y)^2 + 4\tau$ is a perfect square.*

3 Construction of Laplacian integral biregular split graphs

The balanced incompleted block design, $BIBD$, is an important combinatorial concept, useful in several areas, especially Applied Statistics, see [1] , [5] and [11]. We begin this section by giving a generalization of this concept that allows us to build an infinite family of Laplacian integral biregular split graphs.

DEFINITION 3.1. *A generalized balanced incomplete block design, $GBIBD$, is a structure consisting of a set $Y = \{y_1, \cdots, y_v\}, v \geq 2$, and b distinct subsets of Y, $\{B_1, \cdots, B_b\}$, called blocks, such that there are parameters $r \geq 1, \lambda \geq 0$, and v with $1 \leq k \leq v-1$ for which the following hold:*

1. *each element of Y belongs to r blocks;*

2. each block contains k elements; and

3. each pair of elements is simultaneously in λ blocks.

The non-negative integers (v, b, r, k, λ) are called the design parameters and they satisfy: $vr = bk$ and $\lambda(v-1) = r(k-1)$. Since these equations hold, the (v, b, r, k, λ)-generalized block design can be denoted more simply as a (v, k, λ)- generalized block design. If $v = b$, we say that a corresponding $GBIBD$ is a symmetric generalized incomplete block design. Observe that our generalization coincides with the usual definition, except that we allow the cases $\lambda = 0$ and $k = v - 1$.

The $v \times b$ incidence matrix M of a $GBIBD$ is given by

$$m_{ij} = \begin{cases} 1, & \text{if } y_i \in B_j; \\ 0, & \text{if } y_i \notin B_j. \end{cases}$$

For this matrix M, the following equations hold:

$$MM^T = (r - \lambda)\mathbb{I}_v + \lambda\mathbb{J}_v, M\mathbb{1}_b = r\mathbb{1}_v$$

and

$$M^T\mathbb{1}_v = k\mathbb{1}_b.$$

We note that each (v, b, r, k, λ)-generalized block design corresponds to another block design called the *complement generalized block design*. In order to obtain this complement, it is enough to take the blocks $\overline{B_i} = Y - B_i, i = 1, \cdots, b$. It is easy to see that if $\overline{M}$ is its incidence matrix then $\overline{M} = \mathbb{J}_{v,b} - M$. When $v \neq 2$, the complement of a (v, b, r, k, λ)-GBBID has parameters $(v, b, b - r, v - k, b - 2r + k)$. If $v = 2$, the GBBID is self-complementary. Since every $BIBD$ is a $GBIBD$, several examples of these structures can be found in [1], [5] and [11]. In Example 3.1, we present an instance of a $GBIBD$ that is not a $BIBD$.

EXAMPLE 3.1. *Let $v = b = 5$, $r = k = 1$ and $\lambda = 0$. We have the following $(5, 1, 0) - GBIBD$ with incidence matrix*

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Its complement is the $(5,4,4)-GBIBD$ *with incidence matrix*

$$\overline{M} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Let M be the incidence matrix of a $(v,k,\lambda)-GBIBD$. For each $\alpha, \beta \in \mathbb{N}$, let $\mathbb{X} = M \otimes \mathbb{J}_{\alpha,\beta}$, where $\otimes$ is the Kronecker product of matrices. Set $p = \alpha v$ and $q = \beta b$. We define the graph $G(M,\alpha,\beta)$ to be the graph on $p+q$ vertices whose Laplacian matrix is given by

$$L(G(M,\alpha,\beta)) = \begin{pmatrix} (p+x)\mathbb{I}_p - \mathbb{J}_p & -\mathbb{X} \\ -\mathbb{X}^T & y\mathbb{I}_q \end{pmatrix}.$$

The next lemma shows that each $(v,k,\lambda)-GBIBD$ yields an infinite family of biregular split nonthreshold graphs.

Lemma 3.1. *Let* M *be the incidence matrix of a* $(v,k,\lambda)-GBIBD$*. For each* $\alpha, \beta \in \mathbb{N}$*, the graph* $G(M,\alpha,\beta)$ *is a biregular split nonthreshold graph.*

Proof: Let $\mathbb{X} = M \otimes \mathbb{J}_{\alpha,\beta}$, and note that we have $\mathbb{X}\mathbb{1}_q = \beta r \mathbb{1}_p$ and $\mathbb{X}^T \mathbb{1}_p = \alpha k \mathbb{1}_q$, where $p = \alpha v$ and $q = \beta b$. Since $G(M,\alpha,\beta)$ has Laplacian matrix

$$\begin{pmatrix} (p+x)\mathbb{I}_p - \mathbb{J}_p & -\mathbb{X} \\ -\mathbb{X}^T & y\mathbb{J}_q \end{pmatrix},$$

it is clear that $G(M,\alpha,\beta)$ is a biregular split graph.

It remains to prove that $G(M,\alpha,\beta)$ is a nonthreshold graph. Since the parameters of a $(v,k,\lambda)-GBIBD$ satisfy $v \geq 2$ and $0 \leq \lambda < r$, for $1 \leq i, i' \leq v$ there are $1 \leq j, j' \leq b$ such that $m_{ij} = m_{i'j'} = 1$ and $m_{ij'} = m_{i'j} = 0$ in the incidence matrix M. Hence, the subgraph induced by the vertices corresponding to indices $\alpha i, \alpha i', \beta j$ and $\beta j'$ is isomorphic to P_4. ■

The next theorem is the main result of this paper. It shows how we can obtain an infinite family of biregular split nonthreshold Laplacian integral graphs, for each generalized block design.

Theorem 3.1. *Let* M *be the incidence matrix of a* $(v,k,\lambda)-GBIBD$*. There are infinitely many Laplacian integral biregular split nonthreshold graphs of the form* $G(M,\alpha,\beta)$*.*

Proof: From Lemma 3.1, for $\alpha, \beta \in \mathbb{N}$, $G(M, \alpha, \beta)$ is a biregular split non-threshold graph. Thus, we need only show how to obtain appropriate parameters α and β in order to construct an infinite family of Laplacian integral graphs. To do this, we set $\mathbb{X} = M \otimes \mathbb{J}_{\alpha,\beta}$, and consider two cases: $\lambda = 0$ and $\lambda > 0$.

1. **Case** $\lambda = 0$:

   For $\lambda = 0$, we have $k = 1 = r$, $v = b$. For each $\alpha, \beta \in \mathbb{N}$ the Laplacian matrix of $G(M, \alpha, \beta)$ can be written as

   $$\begin{pmatrix} (\alpha v + \beta)\mathbb{I}_{\alpha v} - \mathbb{J}_{\beta v} & -\mathbb{I}_v \otimes \mathbb{J}_{\alpha,\beta} \\ -(\mathbb{I}_v \otimes \mathbb{J}_{\alpha,\beta})^T & \alpha \mathbb{I}_{\beta v} \end{pmatrix}.$$

   It is easy to see that $XX^T = \mathbb{I}_v \otimes \mathbb{J}_\alpha$ is a non-negative and reducible matrix with only one non-zero eigenvalue, namely $\tau = \alpha\beta$, with multiplicity equal to v. It follows from Corollary 2.3 that $G(M, \alpha, \beta)$ is Laplacian integral if and only if, $(p+x-y)^2+4xy = (x+(v-1)y)^2+4xy$ is a perfect square, ie, if and only if the Diophantine equation

   $$(\alpha v + \beta - \alpha)^2 + 4\alpha\beta = (\beta + (v-1)\alpha)^2 + 4\alpha\beta - \gamma^2 = 0, \qquad (3.1)$$

   is satisfied for some $\gamma \in \mathbb{Z}$. Note that $(\alpha_0, \beta_0, \gamma_0) = (v+1, v, v^2+v+1)$ is a particular solution of equation (3.1); thus, the split nonthreshold graph $G(M, v+1, v)$ is Laplacian integral.

   According to [3], the general solution of the equation (3.1) on $\mathbb{Q}$ is given by

   $$\alpha = d[-s + (v+1)t][(v+1)s + (v^2+1)t],$$

   and

   $$\beta = d(2v+1)(v+2)[s - \frac{v-1}{2v+1}t][s + \frac{v(v-1)}{v+2}t]$$

   where $s, t \in \mathbb{Z}$ and $d \in \mathbb{Q}$.

   In particular, for each $d \in \mathbb{N}$, positive integer solutions of the equation (3.1) can be obtained as follows:

   (a) for s and t of the same sign, for example, $s > 0$ and $t > 0$, choose $s \in \mathbb{N}$ such that $\frac{v-1}{2v+1}t < s < (v+1)t$;

(b) otherwise, choose $t < 0 < s$ such that $\frac{v(v-1)}{v+2}|t| < s < \frac{v^2+1}{v+1}|t|$.

2. **Case** $\lambda > 0$:

   In this case, $\mathbb{X}\mathbb{X}^T = \beta((r-\lambda)\mathbb{I}_v + \lambda\mathbb{J}_v) \otimes \mathbb{J}_\alpha$ is non-negative and irreducible. This matrix has three distinct eigenvalues, $\tau_1 = \frac{\alpha\beta\ k\lambda(v-1)}{k-1}$, $\tau_2 = \frac{\alpha\beta\lambda(v-k)}{k-1}$ and $\tau_3 = 0$, provided that $\alpha \geq 2$, otherwise it has only two distinct eigenvalues, namely τ_1 and τ_2.

   From Theorem 2.2, $G(M, \alpha, \beta)$ is Laplacian integral if and only if, $(p + x - y)^2 + 4\tau_2$ is a perfect square. This is equivalent to having

   $$\alpha^2(v-k)^2 + 2\frac{\alpha\beta\lambda(v-k)(v+1)}{k-1} + (\frac{\beta\lambda(v-1)}{k-1})^2 - \gamma^2 = 0, \quad (3.2)$$

   for some $\gamma \in \mathbb{Z}$. Note that $(\alpha_0, \beta_0, \gamma_0) = (\lambda(2v-1), 2(k-1)(v-k), \lambda(v-k)(4v-1))$ is a particular solution of equation (3.1) and hence the split nonthreshold graph $G(M, \lambda(2v-1), 2(k-1)(v-k))$ is Laplacian integral.

   According to [3], the general solution of the equation (3.2) on $\mathbb{Q}$ is given by

   $$\alpha = 3d\lambda(2v+1)(v-k)^2[s - \frac{\lambda(v-1)}{3(v-k)(k-1)}t][s + \frac{\lambda(v-1)(2v-1)}{(2v+1)(v-k)(k-1)}t],$$

   and

   $$\beta = -2d(v-k)^3(k-1)[s - \frac{\lambda(3v-1)}{(v-k)(k-1)}t][s + \frac{\lambda v}{(v-k)(k-1)}t]$$

   where $s, t \in \mathbb{Z}$ and $d \in \mathbb{Q}$.

   Thus, for $d \in \mathbb{N}$, positive integer solutions of the equation (3.1) can be obtained as follows:

   (a) for s and t having the same sign, for example, $s > 0$ and $t > 0$, choose $s \in \mathbb{N}$ such that $\frac{\lambda(v-1)}{3(v-k)(k-1)}t < s < \frac{\lambda(3v-1)}{(v-k)(k-1)}t$;

   (b) otherwise, choose $t < 0 < s$ such that $\frac{\lambda(v-1)(2v-1)}{(2v+1)(v-k)(k-1)}|t| < s < \frac{\lambda v}{v-k}|t|$. ■

The following examples present two cases of split nonthreshold Laplacian integral graphs. Figure 4 illustrates the case $\lambda = 0$, while Figure 5 gives an example for which $\lambda > 0$.

EXAMPLE 3.2. ***Case*** $\lambda = 0$***:*** *Let* $v = 2, M = \mathbb{I}_2, \alpha = 2$ *and* $\beta = 3$. *As* $(\beta+(v-1)\alpha)^2+4\alpha\beta = 7^2$, *then* $G(M, 2, 3)$ *is a split nonthreshold Laplacian integral graph. It is shown in Figure 4.*

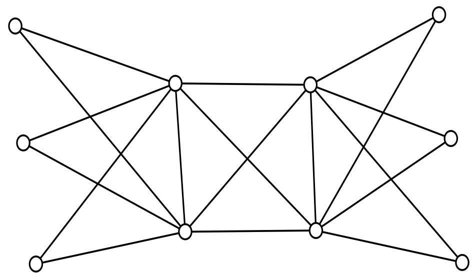

Figure 4: $G(M, 3, 2)$ is a split nonthreshold Laplacian integral graph

EXAMPLE 3.3. ***Case*** $\lambda > 0$: *Consider the* $(4, 2, 1)-GBIBD$ *with incidence matrix*

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

The graph $G(M, 7, 4)$*, displayed in Figure 5, is a split nonthreshold Laplacian integral graph on* 52 *vertices with* 714 *edges and the maximal clique of size* 28.

4 A note on complements

If G is a biregular split Laplacian integral graph, so is its complement, $\overline{G}$. This happens since the spectrum of $L(\overline{G})$ is $\zeta(\overline{G}) = (n-\mu_{n-1}, \ldots, n-\mu_1, 0)$, where $\mu_1 \geq \ldots \geq \mu_{n-1} \geq \mu_n = 0$ are the eigenvalues of $L(G)$.

In the case $\lambda = 0$, it is easy to see that $\overline{G(\mathbb{I}_v, \alpha, \beta)}$ is isomorphic to $G(\overline{M}, \beta, \alpha)$, where $\overline{M} = \mathbb{J}_v - \mathbb{I}_v$ is the incidence matrix of the complement block design.

In the case that we have an incidence matrix M for a GBIBD with $\lambda > 0$, there may not exist another incidence matrix $\tilde{M}$ for a $(v', k', \lambda') - GBIBD$ such that the complement of $G(M, \alpha, \beta)$ is $G(\tilde{M}, \alpha', \beta')$, as we can see in

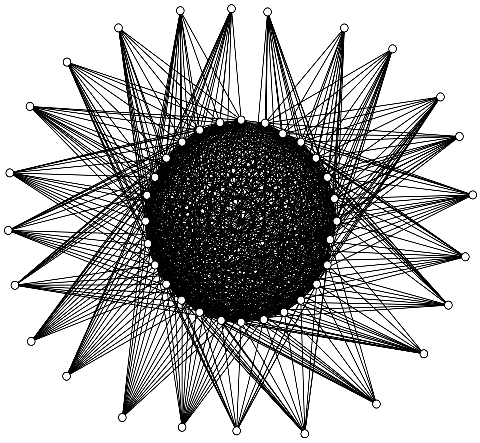

Figure 5: $G(M, 7, 4)$ is a split nonthreshold Laplacian integral graph

Example 3.3. The Laplacian matrix of the complement of the graph in Example 3.3 can be written as

$$L(\overline{G}) = \begin{pmatrix} 38\mathbb{I}_{24} - \mathbb{J}_{24} & -\overline{\mathbb{X}} \\ -\overline{\mathbb{X}}^T & 12\mathbb{J}_{28} \end{pmatrix},$$

where $\overline{\mathbb{X}} = \overline{M} \otimes \mathbb{J}_{4,7}$ and

$$\overline{M} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Since $\overline{M}$ is not the incidence matrix of any GBIBD, we see that $\overline{G}$ can not be written as $G(\tilde{M}, \alpha', \beta')$ for any GBIBD incidence matrix $\tilde{M}$.

5 Biregular split graphs with a common Laplacian spectrum

Suppose that M is the incidence matrix of a (v, k, λ)-GBIBD, where $\lambda > 0$. Select parameters $\alpha, \beta \in \mathbb{N}$, and form $\mathbb{X} = M \otimes J_{\alpha,\beta}$. It is straightforward to determine that the rank of $\mathbb{X}$ coincides with the rank of M; since $rank(M) =$

$rank(M^T) = rank(MM^T)$ and $MM^T = (r-\lambda)\mathbb{I}_v + \lambda\mathbb{J}_v$, we find that the rank of $\mathbb{X}$ is v. Observe that the row sums of $\mathbb{X}$ are equal to $\frac{\beta\lambda(v-1)}{k-1}$, while the column sums of $\mathbb{X}$ are equal to αk. As noted in the proof of Theorem 3.1, $\mathbb{X}\mathbb{X}^T = \beta((r-\lambda)\mathbb{I}_v + \lambda\mathbb{J}_v) \otimes \mathbb{J}_\alpha$, so that $\mathbb{X}\mathbb{X}^T$ has just two nonzero eigenvalues: $\tau_1 = \frac{\alpha\beta\ k\lambda(v-1)}{k-1}$ and $\tau_2 = \frac{\alpha\beta\lambda(v-k)}{k-1}$. Observe that the quantity m_{τ_1} of Theorem 2.2 is 0, since τ_1 is the (simple) Perron value for XX^T, with corresponding eigenvector $\mathbb{1}_{\alpha v}$. Similarly, we have $m_{\tau_2} = v-1$. Appealing to Theorem 2.2, we find that the spectrum (including multiplicities) of the Laplacian matrix of the graph $G(M,\alpha,\beta)$ is completely determined by the parameters v, k, λ, α, and β.

In particular, suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ are two GBIBDs with the same parameters (v,k,λ), with incidence matrices M_1 and M_2, respectively. We can extend the definition of isomorphism of block designs in [11] to GBIBDs as follows: $\mathcal{D}_1$ and $\mathcal{D}_2$ are *isomorphic* if there are permutation matrices R and S, of orders v and b, respectively, such that $M_1 = RM_2S$.

The graphs $G(M_1,\alpha,\beta)$ and $G(M_2,\alpha,\beta)$ will share the same Laplacian spectrum. From Corollary 2.2, it follows that $G(M_1,\alpha,\beta)$ and $G(M_2,\alpha,\beta)$ are isomorphic if and only if there are permutation matrices P and Q of orders αv and βv, respectively, such that $P(M_2 \otimes \mathbb{J}_{\alpha,\beta})Q = M_1 \otimes \mathbb{J}_{\alpha,\beta}$.

We claim that this last condition holds if, and only if, $\mathcal{D}_1$ and $\mathcal{D}_2$ are isomorphic. Certainly if there exist permutation matrices R, S such that $M_1 = RM_2S$, then we can find permutation matrices P, Q so that $P(M_2 \otimes \mathbb{J}_{\alpha,\beta})Q = M_1 \otimes \mathbb{J}_{\alpha,\beta}$.

To see the other direction of the claim, suppose that $P(M_2 \otimes \mathbb{J}_{\alpha,\beta})Q = M_1 \otimes \mathbb{J}_{\alpha,\beta}$ for some permutation matrices P and Q. Observe that both $A_1 \equiv M_1 \otimes \mathbb{J}_{\alpha,\beta}$ and $A_2 \equiv M_2 \otimes \mathbb{J}_{\alpha,\beta}$ can be partitioned as block matrices, with each block equal to either $\mathbb{J}_{\alpha,\beta}$ or the $\alpha \times \beta$ zero matrix. Further, the partitioning of the rows of A_1, A_2 corresponds to the sets of indices $U_i = \{(i-1)\alpha+1, (i-1)\alpha+2, \ldots, i\alpha\}, i = 1, \ldots, v$, while the partitioning of the columns of A_1, A_2 corresponds to the sets of indices $W_j = \{(j-1)\beta + 1, (j-1)\beta+2, \ldots, j\beta\}, j = 1, \ldots, b$. From the fact that both A_1 and A_2 are block matrices, it follows that P can be taken to be a permutation that maps all indices in a single U_i to a common U_{k_i}, for each $i = 1, \ldots, v$, and Q can be take to be a permutation that maps all indices in a single W_j to a common W_{l_j}, for each $j = 1, \ldots, b$. It now follows readily that there are permuations matrices R, S such that $RM_2S = M_1$.

Now, suppose that we have $\lambda > 0$, and that $\mathcal{D}_1$ and $\mathcal{D}_2$ are nonisomorphic. Then for each $\alpha, \beta \in \mathbb{N}$, we find from the considerations above that $G(M_1,\alpha,\beta)$ and $G(M_2,\alpha,\beta)$ are nonisomorphic graphs with the same Laplacian spectrum; moreover, if α, β are selected as in Theorem 3.1, then the nonisomorphic graphs $G(M_1, \alpha, \beta)$ and $G(M_2, \alpha, \beta)$ have the same *integral* Laplacian spectrum.

As an example of this kind of construction, consider the following scenario. Recall that a Hadamard matrix of order $4n + 4$ is a $(1, -1)$ matrix H such that $HH^T = (4n + 4)\mathbb{I}$. By an appropriate sequence of row permutations, column permutations, signing of rows, and signing of columns, any Hadamard matrix H can be put into the following form:

$$\begin{pmatrix} 1 & \mathbb{1}^T \\ \mathbb{1} & \tilde{H} \end{pmatrix}.$$

It now follows readily that the matrix $M = \frac{1}{2}(J + \tilde{H})$ is a $(0, 1)$ matrix of order $4n + 3$ satisfying $MM^T = (n + 1)\mathbb{I} + n\mathbb{J}$. Thus M is the matrix of a $(4n + 3, 2n + 1, n)$ BIBD. Two Hadamard matrices H_1, H_2 are said to be *equivalent* if H_1 can be produced from H_2 by an appropriate sequence of row permutations, column permutations, row signings, and columns signing. It follows then that for nonequivalent Hadamard matrices H_1, H_2 of order $4n+4$, the incidence matrices M_1, M_2 of the corresponding $(4n+3, 2n+1, n)$ BIBD's will have the property that $PM_2Q \neq M_1$ for all $4n + 3n + 3$ permutation matrices P and Q. Consequently, for α, β selected as in Theorem 3.1, $G(M_1, \alpha, \beta)$ and $G(M_2, \alpha, \beta)$ are nonisomorphic split nonthreshold graphs having the same integral Laplacian spectrum.

As a particular instance, it is known (see [10]) that there are exactly five nonequivalent Hadamard matrices of order 16. These give rise to five incidence matrices $M_1, \ldots, M_5$ for $(15, 7, 3)$ BIBD's with the properties that none can be generated from another by row and/or column permutations. For each pair α, β chosen as in Theorem 3.1, this in turn gives rise to five nonisomorphic split nonthreshold graphs, $G(M_1, \alpha, \beta), \ldots, G(M_5, \alpha, \beta)$, all sharing a common integral Laplacian spectrum. Evidently taking different appropriate choices of α, β will generate an infinite family of collections of such graphs.

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