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Entanglement entropy of integer quantum Hall states

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We compute the entanglement entropy, in the real space, of the ground state of the integer Quantum Hall states for three different domains embedded in the cylinder, the disk and the sphere. We establish the validity of the area law with a vanishing value of the topological entanglement entropy. The entropy per unit length of the perimeter depends on the filling fraction, but it is independent of the geometry.

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In recent years the notion of entanglement has become a new tool for analyzing the quantum states that arise in condensed-matter systems.[1] This notion has brought a quantum information perspective to traditional problems and techniques in the field, such as quantum phase transitions, numerical simulation methods, and renormalization group. A generic measure of entanglement is given by the von Neumann entropy S_A of the reduced density matrix of a part A of the total system. This quantity measures the amount of quantum entanglement of the subsystem A with its environment, usually denoted as B. For finite systems, one has that $S_A = S_B$, so this quantity reflects a property shared by A and B. If the quantum state has a finite correlation length, then heuristic arguments implies that the entropy is proportional to the area of the common boundary between A and B. This statement is known as the area law, and it seems to be a universal property satisfied by the quantum states appearing in Ref. 2 (see Ref. 2 for some rigorous results). There are logarithmic violations of this law in critical one dimensional systems, and some higher dimensional fermionic systems, but the former ones can be understood using conformal field theory.[3]

Particularly interesting are the two dimensional systems with topological order, where the entropy law becomes $S_A = cL - \gamma + \mathcal{O}(1/L)$, where L is the length of the boundary, c is a nonuniversal constant and γ is a quantity called topological entanglement entropy.[4] The excitations of these systems are anyons and it turns out that γ is the logarithm of the total quantum dimension of these anyons. The paradigmatic system with (abelian) anyons is the fractional Quantum Hall (FQH) state with filling fraction $1/m$, for which $\gamma = \frac{1}{2}\log m$. The area, or rather, perimeter law of the FQH states has been the target of several recent studies,[5–7] in order to confirm its validity and to compute the value of γ predicted in Ref. 4 (see Refs. 5 and 8 for the study of the entanglement entropy for particle partitioning). Reference[7] uses Chern-Simons theory, finding the predicted value of γ, however the linear behavior of S_A, is not captured, due to the purely topological nature of this theory. There are numerical studies using the Laughlin wave function[5] and exact diagonalization,[6] for filling fractions $\nu = 1/3, 1/5$ and the $\nu = 5/2$ Pfaffian state. The approaches of Refs. 5 and 6 use the orbital basis for the Landau levels. The close relationship of this basis to the spatial partitioning of the blocks leads to an area law of the form $S_A = c\sqrt{l_A} - \gamma + \mathcal{O}(1/l_A)$, where l_A is the number of Landau orbitals in the block A. The numerical values of γ computed in the spherical geometry[5] and the torus geometry[6] agree, within some precision, with their theoretical values, despite of the fact that the systems analyzed are not very large. We remark that the previous form of the area law in the orbital basis holds only in the case of fractional fillings. For integer fillings the orbital partitioning entropy is actually zero since the ground state is simply a product state in that basis.

In this Brief Report we address the problem of computing the entanglement entropy S_A directly in real space, for the Integer Quantum Hall states with $\nu \geq 1$, in three different domains: strips in the cylinder, annulus in the disk and casquettes in the sphere. The reason for choosing integer filling fractions is that the ground state is given by free fermions, where standard techniques for computing entanglement entropies are available.[9] We find the area law $S_A \approx c_\nu L - \gamma$, with $\gamma = 0$ in agreement with general arguments.[4] The nonuniversal constant c_ν is computed analytically for $\nu = 1$ and numerically up to $\nu = 5$. We also analyze the crossover from thin to large blocks, finding that the onset of the area law occurs when the width of the boundary is larger that a correlation length. The blocks, whose entropy we have computed, are adapted to the standard gauge choices used to analyze the geometries of the cylinder, the disk and the sphere.

Let us consider the Landau model for a particle in a cylinder of size $L_x \times L_y$. The one particle wave function in the lowest Landau level (LLL), in the gauge $\mathbf{A} = B(0,x)$, is (in units of the magnetic length ℓ equal to one)

$$\phi_{k_y}(x,y) = \frac{1}{\pi^{1/4} L_y^{1/2}} e^{ik_y y} e^{-(x-k_y)^2/2}. \tag{1}$$

On the cylinder, the identification of the wave function along the y direction implies

$$k_y = \frac{2\pi n}{L_y}, \qquad -\frac{n_0}{2} + 1 \leq n \leq \frac{n_0}{2}. \tag{2}$$

The number of LLLs, n_0, is obtained imposing that the particle lives in the strip $|x| \leq L_x/2$, which yields $n_0 = \frac{L_x L_y}{2\pi}$. This value also gives the total number of quantum fluxes through the box. The electron operator can be written as

$$\psi(x,y) = \sum_{k_y} \phi_{k_y}(x,y) c_{k_y} + \text{higher LLs}, \tag{3}$$

where c_{k_y} is the fermionic destruction operator of the LLL labeled by k_y. The extra term in Eq. (3) involves the remain-

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ing Landau levels, which are empty for filling fraction $\nu=1$. Later on, we shall take them into account when considering higher filling fractions ν. The ground state for $\nu=1$ is given by

$$|\Phi_0\rangle = \Pi_{k_y} c^{\dagger}_{k_y}|0\rangle, \tag{4}$$

where $|0\rangle$ is the Fock vacuum. The two-point fermion correlator in this state is

$$C_{\mathbf{r},\mathbf{r}'} = \langle\Phi_0|\psi^{\dagger}(x,y)\psi(x',y')|\Phi_0\rangle. \tag{5}$$

Using Eqs. (3) and (4) one finds

$$C_{\mathbf{r},\mathbf{r}'} = \sum_{k_y} \phi^*_{k_y}(x,y)\phi_{k_y}(x',y'), \tag{6}$$

and plugging Eq. (1)

$$C_{\mathbf{r},\mathbf{r}'} = \frac{1}{\pi^{1/2}L_y}\sum_{k_y} e^{ik_y(y'-y)}e^{-1/2((x-k_y)^2+(x'-k_y)^2)}. \tag{7}$$

The sum in Eq. (7) is over the n_0 values of k_y given in Eq. (2). In the limit $L_x, L_y \to \infty$ correlator (6) becomes

$$C_{\mathbf{r},\mathbf{r}'} = \frac{1}{2\pi}e^{-1/4(x-x')^2-1/4(y-y')^2-i/2(x+x')(y-y')}, \tag{8}$$

and it is short range with a correlation length proportional to the magnetic length $\ell=1$.

We want to compute the entanglement entropy, $S_{\mathcal{D}}$, of the state Φ_0, in the strip

$$\mathcal{D}: -\frac{l_x}{2} \le x \le \frac{l_x}{2}, \quad 0 \le y \le L_y. \tag{9}$$

This entropy is given by the formula $S_{\mathcal{D}}=\mathrm{Tr}_{\mathcal{D}_c}|\Phi_0\rangle\langle\Phi_0|$, where $\mathcal{D}_c$ is the complement of $\mathcal{D}$ in the cylinder. The computation of $S_{\mathcal{D}}$ is done in two steps.[9] First one restricts the correlation matrix $C_{\mathbf{r},\mathbf{r}'}$, to the domain $\mathcal{D}$, i.e.,

$$\tilde{C}_{\mathbf{r},\mathbf{r}'} = C_{\mathbf{r},\mathbf{r}'}, \quad \mathbf{r},\mathbf{r}' \in \mathcal{D}. \tag{10}$$

Next, one diagonalizes $\tilde{C}_{\mathbf{r},\mathbf{r}'}$, i.e.,

$$\int_{\mathcal{D}} d^2\mathbf{r}'\tilde{C}_{\mathbf{r},\mathbf{r}'}f_m(\mathbf{r}') = \lambda_m f_m(\mathbf{r}). \tag{11}$$

The entropy $S_{\mathcal{D}}$ is obtained by means of

$$S_{\mathcal{D}} = \sum_m H(\lambda_m), \tag{12}$$

where $H(x)=-x\log x-(1-x)\log(1-x)$. Eigenvalue problem (11) can be rather difficult for a generic domain $\mathcal{D}$; however, for strip (9) this task simplifies. The basic observation is that $\tilde{C}_{\mathbf{r},\mathbf{r}'}$ only depends on the difference $y-y'$, which suggests the ansatz

$$f_m(\mathbf{r}) = e^{-i\mu_m y}g_m(x). \tag{13}$$

Plugging Eq. (13) into Eq. (11), and taking the limit $L_y \to \infty$ one gets

$$e^{-1/2x^2+\mu_m x-\mu^2}A_m = \lambda_m g_m(x), \tag{14}$$

where

$$A_m = \int_{-l_x/2}^{l_x/2}\frac{dx}{\pi^{1/2}}e^{-1/2(x^2+\mu_m x)}g_m(x). \tag{15}$$

For a nonvanishing eigenvalue λ_m, Eq. (14) fixes the function $g_m(x)$, up to an overall factor. Plugging Eq. (14) into Eq. (15), the constant A_m drops, and one gets the eigenvalue

$$\lambda_m = \int_{-l_x/2}^{l_x/2}\frac{dx}{\pi^{1/2}}e^{-(x-\mu_m)^2}. \tag{16}$$

On the other hand, if $\lambda_m=0$, Eq. (14) yields $A_m=0$, which becomes a condition for the function g_m. However, vanishing eigenvalues do not contribute to entropy (12), so the solution of $A_m=0$ is not required. Recalling that function (13) is defined on domain (9), one obtains a quantization condition similar to Eq. (2)

$$\mu_m = \frac{2\pi m}{L_y}, \quad ,-\frac{n_0}{2}+1 \le m \le \frac{n_0}{2}. \tag{17}$$

In fact, the eigenfunctions f_m of $\tilde{C}_{\mathbf{r},\mathbf{r}'}$ coincide with the conjugate of the LLL eigenfunctions $\phi^*_{k_y}$, under the identification $k_y=\mu_m$. Moreover, Eq. (16) can be written as the norm of Eq. (1) over domain (9); i.e.,

$$\lambda_m = \int_{\mathcal{D}} d^2\mathbf{r}|\phi_{\mu_m}(\mathbf{r})|^2, \tag{18}$$

which means that λ_m is the probability of finding the electron in the state $k_y=\mu_m$ in the domain $\mathcal{D}$. Integrating Eq. (16) yields

$$\lambda_m \equiv \lambda(\mu_m,l_x) = \frac{1}{2}\left[\mathrm{Erf}\left(\mu_m+\frac{l_x}{2}\right)-\mathrm{Erf}\left(\mu_m-\frac{l_x}{2}\right)\right], \tag{19}$$

where Erf(x) is the error function. The function $H(\lambda(\mu,l_x))$ is localized in the regions $|\mu|\sim l_x/2$, associated to the boundaries of $\mathcal{D}$, where it can be approximated as

$$\lambda(\mu,l_x) \sim \frac{1}{2}\left[1-\mathrm{Erf}\left(|\mu|-\frac{l_x}{2}\right)\right] = \frac{1}{2}\mathrm{Erfc}\left(|\mu|-\frac{l_x}{2}\right), \tag{20}$$

where Erfc$(x)=1-$Erf(x) is the complementary error function. In the limit $L_y \gg 1$, one can use Eq. (17) to write Eq. (12) as the integral,

$$S_{\mathcal{D}} \equiv S(l_x,L_y) = \frac{L_y}{2\pi}\int_{-\infty}^{\infty} d\mu H(\lambda(\mu,l_x)). \tag{21}$$

Furthermore, if $l_x \gg 1$, the main contribution to Eq. (21) comes from the values of μ around $\pm l_x/2$, where one can use approximation (20). Shifting the integration variable μ, one finally obtains

$$S(l_x,L_y) = 2c_{\mathrm{cylinder}}L_y, \tag{22}$$

where the constant c_{cylinder} is given by

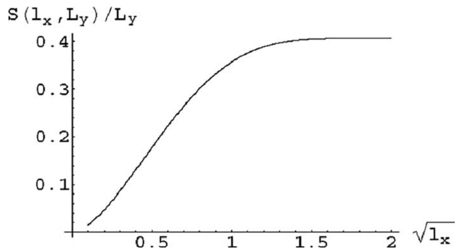

FIG. 1. Plot of $S(l_x,L_y)$ as a function of $\sqrt{l_x}$ for $L_y=20,30,40$.

$$c_{\text{cylinder}} = \int_{-\infty}^{\infty} \frac{d\mu}{2\pi} H\left(\frac{1}{2}\text{Erfc}(\mu)\right) \approx 0.203\,290\,81. \quad (23)$$

Equation (22) is the area law satisfied by $S_{\mathcal{D}}$ in the limit $l_x \gg 1$. The absence of a constant term in Eq. (22) implies the vanishing of the topological entropy γ.[4]

Equations (22) and (23) were derived under the assumption that $l_x \gg 1$, but using Eq. (12) one can estimate the value of l_x above which the area law (22) starts to be valid. In Fig. 1, we plot $S(l_x,L_y)/L_y$ as a function of $\sqrt{l_x}$, for various values of L_y. For $l_x \geq l_{x,c}=2$ the entropy reaches a constant value $2c_{\text{cylinder}}$ given by Eq. (23), which agrees with area law (22). The value of $l_{x,c}$ is easy to understand, it means that the width of $\mathcal{D}$, along the x direction, must be bigger than two magnetic lengths, which guarantees the existence of two boundaries with a width of one correlation length each. Figure 1 also shows that the entropy increases linearly with $\sqrt{l_x}$ for $l_x < l_{x,c}$, and reminds the result of Ref. 6, where the entropy of the $\nu=1/3,1/5$ FQH states was computed for the torus in the orbital basis. These authors found that the entropy of a strip of width ℓ, varies as $\sqrt{\ell}$, where ℓ label the orbital angular momenta. Our computation and that of Ref. 6 differ both in the basis and the filling fraction, so the previous comparison has to be taken with care.

Let us now consider the disk geometry in the Landau gauge where the eigenfunctions of the LLL are given by

$$\phi_m(z) = \frac{z^m}{(2\pi 2^m m!)^{1/2}} e^{-|z|^2/2}, \quad m=0,1,\ldots . \quad (24)$$

The two-point fermion correlator is similar to Eq. (8),

$$C_{z,z'} = \frac{1}{2\pi} e^{-1/4(|z|^2+|z'|^2+2z^* z')}. \quad (25)$$

We want to compute the entanglement entropy in the annulus of radii $r_1 < r_2$,

$$\mathcal{D}: r_1 < |z| < r_2. \quad (26)$$

The procedure follows closely the case of the cylinder. The eigenfunctions $f_m(z)$ of Eq. (11), with nonzero eigenvalues λ_m, are given by $f_m(z)=\phi_m^*(z)(n=0,1,\ldots)$ and the eigenvalues are

$$\lambda_m = \int_{\mathcal{D}} d^2z |\phi_m(z)|^2, \quad n=0,1,\ldots . \quad (27)$$

Plugging Eq. (24) into Eq. (27) and performing the integral over domain (26), one finds

$$\lambda_m(r_1,r_2) = \frac{1}{m!}\left[\Gamma\left(m+1,\frac{r_1^2}{2}\right) - \Gamma\left(m+1,\frac{r_2^2}{2}\right)\right]. \quad (28)$$

The entropy of the annulus is given by

$$S(r_1,r_2) = \sum_{m=0}^{\infty} H(\lambda_m(r_1,r_2)), \quad (29)$$

which, for large values of r_1 and r_2, satisfies the area law

$$S(r_1,r_2) = 2\pi(r_1+r_2)c_{\text{disk}}, \quad (30)$$

with $c_{\text{disk}}=c_{\text{cylinder}}$. Equation (30) can be proved analytically along the same lines as was done before.

Another example which can be solved explicitly is that of an electron moving on a sphere of radius R under the influence of a radial magnetic field created by a monopole at the origin. In the gauge where the vector potential is given by $\mathbf{A}=\hbar Q/eR \cot\phi$, the wave functions of the LLLs are the monopole harmonics,

$$Y_{Q,Q,m} = \left[\frac{2Q+1}{4\pi}\binom{2Q}{Q-m}\right]^{1/2} (-1)^{Q-m} u^{Q+m} v^{Q-m}, \quad (31)$$

where $u=\cos(\theta/2)e^{-i\phi/2}$, $v=\sin(\theta/2)e^{i\phi/2}$, with θ and ϕ the polar and azimuthal angles, and $2Q$ the total quantum flux traversing the sphere. The two-point fermion correlator is given by

$$C_{\mathbf{r},\mathbf{r}'} = \sum_{m=-Q}^{Q} Y^*_{Q,Q,m}(\theta,\phi) Y_{Q,Q,m}(\theta',\phi'). \quad (32)$$

Making the change $n=m+Q$ one can write Eq. (32) as

$$C_{\mathbf{r},\mathbf{r}'} = \frac{2Q+1}{4\pi}\sum_{n=0}^{2Q}\binom{2Q}{n}(\bar{u}u')^n(\bar{v}v')^{2Q-n}. \quad (33)$$

We are interested in computing the entanglement entropy in the spherical segment (i.e., casquette)

$$\mathcal{D}: \theta_a < \theta < \theta_b \quad (34)$$

bounded by the polar angles θ_1 and θ_2. The eigenfunctions of correlator (33) in domain (34), with nonzero eigenvalue, are given by $f_m = Y^*_{Q,Q,m}$, with

$$\lambda_n = \int_{\theta_a}^{\theta_b} d\theta \int_0^{2\pi} d\phi |Y_{Q,Q,n-Q}|^2. \quad (35)$$

Performing the integral one finds,

$$\lambda_n(\theta_a,\theta_b) = (2Q+1)\binom{2Q}{n}\left\{B\left[\cos^2\left(\frac{\pi a}{4Q+2}\right),1+n,1-n+2Q\right] - B\left[\cos^2\left(\frac{\pi b}{4Q+2}\right),1+n,1-n+2Q\right]\right\},$$
$$(n=0,1,\ldots,2Q), \tag{36}$$

where $B(z,n,m)$ is the incomplete beta function and $\theta_{a(b)} = a(b)\pi/(2Q+1)$. The entropy of region (34) is computed from

$$S(\theta_a,\theta_b) = \sum_{n=0}^{2Q} H(\lambda_m(\theta_a,\theta_b)). \tag{37}$$

The perimeter of the casquette of angle θ is given by $P_\theta = 2\pi R \sin\theta$, where the radius is given by $R=\sqrt{Q}$ as follows from computing the number of quantum fluxes. For large values of Q, entropy (37) satisfies the area law

$$S(\theta_a,\theta_b) = (P(\theta_a)+P(\theta_b))c_{\text{sphere}}, \tag{38}$$

with $c_{\text{sphere}}=c_{\text{cylinder}}=c_{\text{disk}}$ so that the three geometries yield the same entropy per unit length of the perimeter.

The previous results can be easily generalized for integer filling fractions $\nu>1$. The correlation matrix $C_{\mathbf{r},\mathbf{r}'}$ is given by

$$C_{\mathbf{r},\mathbf{r}'} = \sum_{n=0,\nu-1}\sum_{m} \phi^*_{n,m}(\mathbf{r})\phi_{n,m}(\mathbf{r}'), \tag{39}$$

where $\phi_{n,m}(\mathbf{r})$ is the wave function of the state m in the n^{th} Landau level. The eigenfunctions of $\tilde{C}_{\mathbf{r},\mathbf{r}'}$ are linear combinations of $\phi^*_{n,m}(\mathbf{r})$ with $n=0,1,\ldots,\nu-1$, and their eigenvalues are those of the $\nu\times\nu$ matrix,

$$\Lambda_m(n,n') = \int_{\mathcal{D}} d^2\mathbf{r}\,\phi^*_{n,m}(\mathbf{r})\phi_{n',m}(\mathbf{r}), \quad n,n'=0,\ldots,\nu-1. \tag{40}$$

The entanglement entropy is computed using Eq. (12), where the summation runs over all the eigenvalues of Λ_m. The area law $S_{\mathcal{D}}\approx c_\nu L-\gamma_\nu$ remains valid, with $\gamma_\nu=0$, and a value of c_ν, which depends on the filling fraction (see Fig. 2).

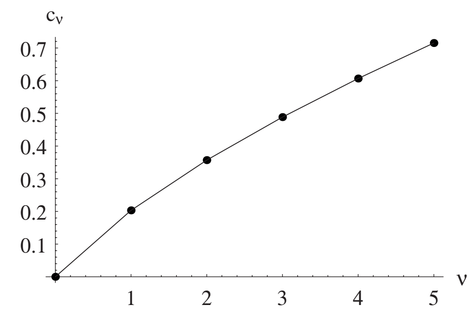

FIG. 2. The points denote the values of the constant c_ν in the area law for integer filling fractions $\nu=1,\ldots,5$. The continuous line is a forth order polynomial fit.

In summary, we have derived in this Brief Report the area law satisfied by the entanglement entropy of the integer Quantum Hall states with filling fraction, ν, for different types of domains in the cylinder, the disk and the sphere. We have computed the nonuniversal constant c_ν of the area law as a function of ν. The topological entanglement entropy vanishes, in agreement with the theoretical results.[4,7] For $\nu=1$, we have found a simple interpretation of the area law. In this case the entanglement entropy is given by the sum over the LLL states, of the Shanon entropies associated to finding an electron or a hole in the domain. The area law arises from the contribution of the LLLs inside a correlation length of the boundary of the domain. Our method allows the computation of the entanglement entropy for more complicated domains. Of special interest are those with curvature singularities wether one may expect deviations from the area law.

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