THE ACTION OF THE MURPHY ELEMENT L_n ON THE RESTRICTION OF AN IRREDUCIBLE S_n-MODULE TO S_{n-1}

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1. An S_{n-1}-filtration of irreducible S_n-modules

We study the irreducible representations of the symmetric group Σ_n over a field F of positive characteristic p. For convenience, but no loss of generality, we shall assume that F is algebraically closed. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0)$ be a partition of n. As usual the set of nodes $[\lambda] := \{(i,j) \in \mathbb{Z}^2 \mid 1 \leq i \leq l, 1 \leq j \leq \lambda_i\}$ is called the *Young diagram* of λ. We represent $[\lambda]$ as a set of square boxes in $\mathbb{Z}^2$, by placing a square with opposite corners $(i-1, j-1)$ and (i,j), for each $(i,j) \in [\lambda]$. In accordance with the anglo-american convention, we orient the positive direction of the Y-axis downwards. The *transpose* λ' of λ is the partition of n defined by $\lambda'_i = \#\{j \mid \lambda_j \geq i\}$, for all i. The Young diagram of λ' is obtained from $[\lambda]$ by reflection in the main diagonal.

We use $[1,n]$ to denote the set of integers $\{1,\ldots,n\}$. We fix a λ-tableaux t for the remainder of the paper. So t is a function $[\lambda] \to [1,n]$. We let $(i,j)t$ be the image of $(i,j) \in [\lambda]$ under t. The *transpose of* t is the λ'-tableau t' such that $(i,j)t' = (j,i)t$, for all $(i,j) \in [\lambda']$. The group Σ_n acts on λ-tableau by permuting the contents of the boxes in a tableau. Thus $(i,j)(t\pi) = ((i,j)t)\pi$, for all $(i,j) \in [\lambda]$ and $\pi \in \Sigma_n$. If t is a bijection, we let it denote the node of $[\lambda]$ occupied by i, for

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$i = 1, \ldots, n$. Then $i(t\pi) = (i\pi^{-1})t$. If s is another bijective λ-tableau, we let $\mathrm{sgn}(s/t)$ be the sign of the unique permutation $\pi \in \Sigma_n$ such that $s = t\pi$.

Let R_t be the row stabilizer, and let C_t be the column stabilizer, of t in Σ_n. So $R_t \cong \Sigma_\lambda$ and $C_t \cong \Sigma_{\lambda'}$ are Young subgroups of Σ_n. The relation $t_1 \sim t_2$ if $t_2 = t_1\pi$, for some $\pi \in R_{t_1}$, defines an equivalence relation on λ-tableau. The equivalence class of $\sim$ that contains t is denoted by $\{t\}$ and is called a λ-*tabloid*. The $\mathbb{Z}$-span of the λ-tabloids forms a Σ_n-permutation lattice M^λ. So M^λ is isomorphic to the induced module $\mathbb{Z}_{\Sigma_\lambda}\!\uparrow^{\Sigma_n}$.

The row stabilizer sum of t is $R_t^+ := \sum_{\sigma \in R_t} \sigma$, while the signed column stabilizer sum of t is $C_t^- := \sum_{\sigma \in C_t} (\mathrm{sgn}\ \sigma)\sigma$. Both are elements of $\mathbb{Z}\Sigma_n$. The element $e_t := \{t\}C_t^-$ of M^λ is called a λ-*polytabloid*. The $\mathbb{Z}$-span of all λ-polytabloids forms the *Specht sublattice* S^λ of M^λ. Since $e_t\pi = e_{t\pi}$, for all $\pi \in \Sigma_n$, S^λ is a Σ_n-submodule of M^λ.

The annihilator ideal of a polytabloid e_t in $\mathbb{Z}\Sigma_n$ is generated by the so-called *Garnir elements*. The following result can be used to show that the *standard polytabloids* form a $\mathbb{Z}$-basis for S^λ. Here e_t is standard if $(i,j)t < (k,l)t$, whenever $i \leq k, j \leq l$ and $(i,j) \neq (k,l) \in [\lambda]$.

Lemma 1. *Let X and Y be subsets of the entries in two columns, u and v respectively, of t. Suppose that $|X| + |Y|$ is greater than the length of each of the columns u and v. Then $e_t \sum_\sigma \mathrm{sgn}(\sigma)\sigma = 0$, where σ ranges over the elements of a set of representatives $\Sigma_X \times \Sigma_Y \backslash \Sigma_{X \cup Y}$ for the right cosets of $\Sigma_X \times \Sigma_Y$ in $\Sigma_{X \cup Y}$.*

We say that a relation involving X and Y as above is a *simple Garnir relation* if $|X| = 1$ or $|Y| = 1$. One fact about polytabloids that is obvious from their definition, but not from the Garnir relations, is that

$e_s = e_t$, whenever s is obtained from t by transposing two columns of equal length.

We list the *removable nodes* in $[\lambda]$ as $(r_1, c_1), \dots, (r_m, c_m)$. So for each u we have $c_u = \lambda_{r_u}$ and $r_u = l$ or $\lambda_{r_u} > \lambda_{r_u+1}$. Set

$$\lambda{\downarrow_u} := (\lambda_1 \geq \cdots \geq \lambda_{r_u-1} \geq \lambda_{r_u} - 1 \geq \lambda_{r_u+1} \geq \cdots \geq \lambda_l).$$

So $[\lambda{\downarrow_u}]$ is obtained by removing the node (r_u, c_u) from $[\lambda]$. For convenience we set $r_0 := 0$ and $c_{m+1} := 0$. The *addable nodes* of $[\lambda]$ are then $\{(r_u + 1, c_{u+1} + 1) \mid u = 0, \dots, m\}$. The *residue* of the u-th removable node is the integer $\alpha_u := c_u - r_u$ and the residue of the u-th addable node of λ is $\beta_u := c_{u+1} - r_u$. The *p-residues* are obtained by considering these residues mod p.

As M^λ and S^λ are $\mathbb{Z}$-lattices, we can define $F\Sigma_n$-modules $M_F^\lambda := M^\lambda \otimes_{\mathbb{Z}} F$ and $S_F^\lambda := S^\lambda \otimes_{\mathbb{Z}} F$. There is a Σ_n-invariant symmetric bilinear form defined on M^λ such that $\langle \{t_1\}, \{t_2\} \rangle = \delta_{\{t_1\}\{t_2\}} 1_F$, for all λ-tabloids $\{t_1\}$ and $\{t_2\}$. The restiction of this form to S_F^λ is generally degenerate. Given $X \subseteq S^\lambda$, we use $X^\perp$ to denote the dual space $\{y \in M^\lambda \mid \langle x, y \rangle = 0, \forall x \in X\}$. Set $J^\lambda = (S^\lambda)^\perp \cap S^\lambda$. Then by James [5] the quotient $D_F^\lambda := S_F^\lambda / J_F^\lambda$ is either zero or an irreducible $F\Sigma_n$-module. The former case occurs precisely when λ is p-regular: that is, when no part of λ is repeated p or more times. Moreover $\{D_F^\lambda \mid \lambda \text{ is a } p\text{-regular partition of } n\}$ is a complete set of representatives for the isomorphism classes of irreducible $F\Sigma_n$-modules.

The *sign* Σ_n-module is the 1-dimensional Specht module $S^{[1^n]}$. Let t be the $[1^n]$-tableau such that $(i,1)t = i$, for $i = 1, \dots, n$. The single vector $\Sigma_n^- := e_t$ spans $S^{[1^n]}$ and $\Sigma_n^- \sigma = (\operatorname{sgn} \sigma) \Sigma_n^-$, for all $\sigma \in \Sigma_n$.

Now set $S^{\lambda *} := S^{\lambda'} \otimes S^{[1^n]}$. Then $S^{\lambda *}$ is a Σ_n-module with $\mathbb{Z}$-basis $\{e_t \otimes \Sigma_n^-\}$, where t ranges over the standard λ'-tableau. Here $e_t \otimes \Sigma_n^- \sigma =$

$(\operatorname{sgn}\sigma)e_{t\sigma} \otimes \Sigma_n^-$, for each $\sigma \in \Sigma_n$. Note that each Garnir relation for $\{e_t\}$ gives an identical relation for $\{e_t \otimes \Sigma_n^-\}$. We need the following characterization of J^λ due to G. James:

Lemma 2. *There is an $\mathbb{Z}\Sigma_n$-exact sequence*

$$0 \longrightarrow S^{\lambda\perp} \longrightarrow M^\lambda \xrightarrow{\theta_\lambda} S^{\lambda *} \longrightarrow 0,$$

such that $\{s\}\theta_\lambda := \operatorname{sgn}(s/t)\, e_{s'} \otimes \Sigma_n^-$, for each λ-tabloid $\{s\}$. Moreover, if λ is p-regular, taking images mod p, the above sequence restricts to

$$0 \longrightarrow J_F^\lambda \longrightarrow S_F^\lambda \xrightarrow{\theta_\lambda} D_F^\lambda \longrightarrow 0.$$

Proof. This is the substance of [5, 6.8,8.15]. Note that

$$e_s\theta_\lambda = \operatorname{sgn}(s/t)\, e_{s'} R_{s'}^+ \otimes \Sigma_n^-, \quad \text{for each } \lambda\text{-tableau } s.$$

□

If $\mathrm{X}(\lambda)$ is any region of $[\lambda]$ then $\mathrm{X}\,t$ will denote the image of X under t. So $\mathrm{X}\,t$ is the set of integers that occupy the nodes of $\mathrm{X}(\lambda)$ in t.

We give some names to various regions of $[\lambda]$. The *rim* of $[\lambda]$ is the set of nodes $\operatorname{Rim}(\lambda) := \{(i, \lambda_i) \mid i = 1, \ldots, l\}$. Fix $u \in [1, m]$. Let $\operatorname{Rim}_u(\lambda)$ be the set of nodes in $\operatorname{Rim}(\lambda)$ that belong to column c_u. So $\operatorname{Rim}_u(\lambda) = \{(r_{u-1}+1, c_u), (r_{u-1}+2, c_u), \ldots, (r_u, c_u)\}$. Also $\operatorname{Top}_u(\lambda)$ denotes the set of nodes in the top r_u rows of $[\lambda]$, and $\operatorname{Right}_u(\lambda)$ denotes the set of nodes in the right columns $c_{u+1}, \ldots, c_1$ of $[\lambda]$.

Define

$$M_u^\lambda := \mathbb{Z}\text{-span}\,\{\{t\} \mid n \in \operatorname{Top}_u t\}.$$

Then M_u^λ is a $\mathbb{Z}\Sigma_{n-1}$-submodule of M^λ and

$$M^\lambda = M_m^\lambda \supset M_{m-1}^\lambda \supset \ldots M_1^\lambda \supset M_0^\lambda = 0$$

is a filtration of M^λ, as Σ_{n-1}-module. Moreover $S_u^\lambda := M_u^\lambda \cap S^\lambda$ coincides with $\mathbb{Z}$-span $\{e_t \mid \ t \text{ is a standard } \lambda\text{-tableau and } n \in \mathrm{Right}_u\, t\}$. Thus (as in [5]) $S^\lambda\!\downarrow_{\Sigma_{n-1}}$ has a *Specht filtration*

$$S^\lambda = S_m^\lambda \supset S_{m-1}^\lambda \supset \ldots \supset S_1^\lambda \supset S_0^\lambda = 0,$$

Each S_u^λ is an Σ_{n-1}-submodule of $S^\lambda\!\downarrow_{\Sigma_{n-1}}$ and $S_u^\lambda / S_{u-1}^\lambda \cong S^{\lambda^u}$.

For each u there is a Σ_{n-1}-exact sequence

$$0 \longrightarrow S_{u-1}^\lambda \longrightarrow S_u^\lambda \xrightarrow{\theta_u} S^{\lambda^u} \longrightarrow 0.$$

with θ_u calculated as follows. If $e_t \in S_u^\lambda \backslash S_{u-1}^\lambda$ then $e_t\theta_u = \pm e_s$. Here s is the λ_u-tableau that is obtained from t by transposing, if necessary, n with the entry at the top of its column, interchanging this column with column λ_u, and finally removing n. The sign is $+1$ if n was at the top of its column in t, and -1 otherwise. For example, if $\lambda = [3^2, 1]$ then

$$e_{\begin{smallmatrix} 1 & 7 & 2 \\ 3 & 4 & 5 \\ 6 & & \end{smallmatrix}} \theta_1 = -e_{\begin{smallmatrix} 1 & 2 & 4 \\ 3 & 5 & \\ 6 & & \end{smallmatrix}}.$$

Suppose now that λ is p-regular. Identify D^λ with $S^\lambda\theta_\lambda$. The restricted module $D^\lambda\!\downarrow_{\Sigma_{n-1}}$ has a filtration

$$D^\lambda = D_m^\lambda \supseteq D_{m-1}^\lambda \supseteq \ldots \supseteq D_1^\lambda \supseteq D_0^\lambda = 0,$$

where $D_u^\lambda = S_u^\lambda\theta_\lambda$. Thus $D_u^\lambda / D_{u-1}^\lambda = (S_u^\lambda + J^\lambda)/(S_{u-1}^\lambda + J^\lambda)$. Notice that this coincides with $S_u^\lambda/(S_{u-1}^\lambda + J^\lambda \cap S_u^\lambda)$, which is a quotient of the Specht module S^{λ^u}. While the filtration $\{S_u^\lambda\}$ of S^λ is strictly decreasing, the filtration $\{D_u^\lambda\}$ of D^λ is generally only non increasing.

Set $L_n := (1, n) + (2, n) + \ldots + (n-1, n)$ as the n-th Jucys-Murphy element in $\mathbb{Z}\Sigma_n$. Then L_n commutes with every element of Σ_{n-1}. It follows that L_n acts as Σ_{n-1}-endomorphism on every $F\Sigma_{n-1}$-module, in particular on the Specht module S^λ and on the radical J^λ of the bilinear form on S^λ. If λ is p-regular, then L_n acts on the irreducible S_n-module D^λ. We aim to show that in this case L_n generates the ring $\mathrm{End}_{S_{n-1}}(D^\lambda)$ of all Σ_{n-1}-endomorphisms of D^λ.

2. Symmetric functions

We begin with some results on symmetric functions. These will be required in order to evaluate some inner product expressions obtained in later sections of the paper. The most complicated result is Theorem 5. The proof was emailed to me by Grant Walker. Our original proof was by defining an involution on certain monomials, and cancellation.

Fix positive integers $u > v$. We use the following notation:

$\binom{u}{v}$ is the collection of subsets of $[1, u]$ of size v;

$\left\langle {u \atop v} \right\rangle$ is the collection of multi-subsets of $[1, u]$ of size v.

We regard $\binom{u}{v}$ as the collection of decreasing functions, and $\left\langle {u \atop v} \right\rangle$ as the collection of nonincreasing functions, $[1, v] \to [1, u]$.

Let $X := \{x_1, x_2, \ldots\}$ and $Y := \{y_1, y_2, \ldots\}$ be sets of variables that are finite. Recall the elementary and complete symmetric functions of degree n are

$$\mathrm{E}_n(Y) := \sum_{\mu \in \binom{|Y|}{n}} y_{\mu_1} \cdots y_{\mu_n};$$

$$\mathrm{H}_n(X) := \sum_{\mu \in \left\langle {|X| \atop n} \right\rangle} x_{\mu_1} \cdots x_{\mu_n}.$$

We define the symmetric function HE_n by

$$\mathrm{HE}_n(X; Y) := \sum_{i=0}^{n} (-1)^i \, \mathrm{H}_{n-i}(X) \, \mathrm{E}_i(Y).$$

Then the generating function for HE_n is

$$\sum_{k \geq 0} \mathrm{HE}_k(X; Y) t^k = \prod_{i=1,2,\ldots} (1 - x_i t)^{-1} \prod_{j=1,2,\ldots} (1 - y_j t).$$

Lemma 3. *Let $n, i, j > 0$. Then*

$$\mathrm{HE}_n(X;Y) \;=\; (x_i - y_j)\mathrm{HE}_{n-1}(X; Y\backslash\{y_j\}) + \mathrm{HE}_n(X\backslash\{x_i\}; Y\backslash\{y_j\}).$$

Proof. Considering the generating function of HE_n, this follows from

$$\frac{(1-y_jt)}{(1-x_it)} = \frac{t(a-y_j)}{(1-x_it)} + 1.$$

$\square$

Let u, v be integers. There is a bijection $\left\langle {u \atop v} \right\rangle \leftrightarrow \binom{u+v-1}{v}$ that sends $\mu \in \left\langle {u \atop v} \right\rangle$ to $\tilde{\mu} \in \binom{u+v-1}{v}$, where $\tilde{\mu}_i := \mu_i + i - 1$, for $i = 1, \ldots, v$. This is the basis of the following lemma.

Corollary 4. *Suppose that $|X| < |Y|$. Set $n := |Y| - |X| + 1$. Then*

$$\mathrm{HE}_n(X;Y) \;=\; \sum_{\mu \in \left\langle {|X| \atop n} \right\rangle} \prod_{i=1}^{n} (x_{\mu_i} - y_{\mu_i+i-1}).$$

Proof. For each u, v define $f(u,v) := \sum_{\mu\in\left\langle {u \atop v} \right\rangle} \prod_{i=1}^{v}(x_{\mu_i} - y_{\mu_i+i-1})$. Set $m := |X|$. The $\mu \in \left\langle {m \atop n} \right\rangle$ with $\mu_n = m$ contribute $(x_{|X|} - y_{|Y|}) f(m, n-1)$ to $f(m,n)$. The $\mu \in \left\langle {m \atop n} \right\rangle$ with $\mu_n < m$ contribute $f(m-1, n)$ to $f(m,n)$. Thus

$$f(m,n) = (x_{|X|} - y_{|Y|}) f(m, n-1) + f(m-1, n).$$

The result now follows by induction from the previous lemma. $\square$

Let X, Y be sets of indeterminates, as before, and let $Z = \{z_1, z_2, \ldots\}$ be another set of indeterminates. Suppose that $0 < n \leq m$ and that $\mu \in \left\langle {m \atop n} \right\rangle$. Set $\mu_0 := m$, and for $i \in [1, n]$ define

$$\delta_{\mu,i} := \begin{cases} y_{\mu_i}, & \text{if } \mu_{i-1} < \mu_i; \\ z_i, & \text{if } \mu_{i-1} = \mu_i; \end{cases} \tag{1}$$

So δ_μ keeps track of the places where μ decreases.

Theorem 5. *Suppose that $|X| = |Y| + 1$. Set $n := |Z|$. Then*

$$\mathrm{HE}_n(X; Y \cup Z) \; = \; \sum_{\mu \in \left\langle {|X| \atop n} \right\rangle} \prod_{i=1}^{n} (x_{\mu_i} - \delta_{\mu,i}).$$

Proof. For each u, v define $f(u,v) := \sum_{\mu \in \left\langle {u \atop v} \right\rangle} \prod_{i=1}^{v} (x_{\mu_i} - \delta_{\mu,i})$. Set $m := |X|$. We compare $f(m,n)$ and $f(m-1,n)$. The $\mu \in \left\langle {m \atop n} \right\rangle$ with $\mu_n < m-1$ contribute the same term to $f(m,n)$ and $f(m-1,n)$. The $\mu \in \left\langle {m \atop n} \right\rangle$ with $\mu_n = m-1$ contribute $(x_{m-1} - y_{m-1}) f(m-1, n-1)$ to $f(m,n)$ and $(x_{m-1} - z_n) f(m-1, n-1)$ to $f(m-1,n)$. The $\mu \in \left\langle {m \atop n} \right\rangle$ with $\mu_n = m$ contribute $(x_m - z_n) f(m, n-1)$ to $f(m,n)$ and nothing to $f(m-1,n)$. Thus we have a recursion relation

$$f(m,n) - f(m-1,n) \; = \; (z_n - y_{m-1}) f(m-1, n-1) + (x_m - z_n) f(m, n-1).$$

But $\mathrm{HE}_n(X; Y \cup Z)$ satisfies the same recursion relation and initial conditions, as follows from examination of its generating function and the equality

$$\frac{(1 - y_{m-1}t)(1 - z_n t)}{(1 - x_m t)} - (1 - z_n t) \; = \; t(z_n - y_{m-1}) + \frac{t(x_m - z_n)(1 - y_{m-1}t)}{(1 - x_m t)}.$$

□

Corollary 6. *Suppose that $|X| = |Y| + 1$ and that Z is a subset of X of size n. Then*

$$\mathrm{HE}_n(X \backslash Z; Y) \; = \; \sum_{\mu \in \left\langle {|X| \atop n} \right\rangle} \prod_{i=1}^{n} (x_{\mu_i} - \delta_{\mu,i}).$$

Proof. It follows from Theorem 5 that

$$\mathrm{HE}_n(X; Y \cup Z) \; = \; \sum_{\mu \in \left\langle {|X| \atop n} \right\rangle} \prod_{i=1}^{n} (x_{\mu_i} - \delta_{\mu,i}).$$

The result now follows from Lemma 3. □

3. The action of L_n on a Specht filtration of $S^\lambda\!\downarrow_{S_{n-1}}$

Let u be the largest index such that $n \in \mathrm{Right}_u$. Equivalently n occupies a column of length r_u in t. Set $\alpha_t := c_u - r_u$ as the residue of the u-th removable node of t. Define a transitive relation $\to$ on $[1, n]$, such that $i \to j$ if i occupies a longer column than j in t. G. E. Murphy proved the next result for standard polytabloids in [8, 3.3].

Lemma 7.

$$e_t L_n \; = \; \alpha_t e_t \; + \sum_{n \to w} e_t(n, w).$$

In particular if $m \geq u \geq v \geq 1$ then $S_u^\lambda \prod_{w=v+1}^{u} (L_n - \alpha_w) \subseteq S_v^\lambda$.

Proof. Let u be the largest index such that $n \in \mathrm{Right}_u$ and suppose that n occupies column c in t. Then $c \leq c_u$, and column c has length r_u. Let $1 \leq d \leq c_u$ with $d \neq c$. Then column d of $[\lambda]$ has length $\geq r_u$. By a simple Garnir relation we have $\sum \{e_t(n, w) \mid w$ belongs to column d of $t\} = e_t$. There are $c_u - 1$ such columns. Now $e_t(n, w) = -e_t$, if w belongs to column c in t and $w \neq n$. There are $r_u - 1$ such integers w. Combining these facts we get

$$\sum \{e_t(n, w) \mid n \not\to w \text{ and } n \neq w\} = \alpha_u e_t.$$

The first statement now follows from the fact that $e_t L_n = \sum_{w=1}^{n-1} e_t(n, w)$.

The last statement follows from the first by induction. $\square$

We define a *t-cycle* to be any cyclic permutation in Σ_n of the form $(n, w, x, \ldots, y)$, where

$$n \to w \to x \to \ldots \to y.$$

Clearly if π_1 is a t-cycle and π_2 is a $t\pi_1$-cycle then $\pi_1 \pi_2$ is a t-cycle.

We set $\overleftarrow{t}$ as the λ-tableau got by reversing the rows of t i.e.

$$(i,j)\overleftarrow{t} = (i, \lambda_i - j + 1)t, \quad \text{for } (i,j) \in [\lambda].$$

Let ε_i denote the number of parts of λ that equal i. G. James proved that $S^\lambda/J^\lambda \neq 0$ if λ is *p-regular*, by observing that

$$\langle e_t, e_{\overleftarrow{t}} \rangle = \prod_{i=1}^{n} (\varepsilon_i !)^i. \tag{2}$$

Lemma 8. *Suppose that $n \in \operatorname{Rim} t$, and let π be a t-cycle. Then*

$$\langle e_{t\pi}, e_{\overleftarrow{t}} \rangle = \begin{cases} \operatorname{sgn} \pi \, \langle e_t, e_{\overleftarrow{t}} \rangle, & \text{if } \pi \text{ fixes each integer not in } \operatorname{Rim} t; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Suppose that π fixes each integer not in $\operatorname{Rim} t$. Then π is a permutation of the first column of $\overleftarrow{t}$ and $e_{\overleftarrow{t}} = \operatorname{sgn}(\pi)\, e_{\overleftarrow{t\pi}}$. We conclude from (2) that $\langle e_{t\pi}, e_{\overleftarrow{t}} \rangle = \operatorname{sgn} \pi \, \langle e_t, e_{\overleftarrow{t}} \rangle$.

Conversely, suppose that $w\pi \neq w$, for some integer $w \notin \operatorname{Rim} t$. Let w belong to a column of length r_u in t. Set $\mathrm{X}\, t := \{n\} \cup \mathrm{Right}_u \cap \operatorname{Rim} t$. Now π fixes each integer in $\operatorname{Rim}_u t$ and $\mathrm{Right}_{u-1}\, t \subseteq \mathrm{Right}_u\, t\pi$, as π is a t-cycle. It follows that $\mathrm{X}\, t \subseteq \mathrm{Right}_u\, t\pi$. Let ρ be a column permutation of $t\pi$ and let σ be a column permutation of $\overleftarrow{t}$. The $r_u + 1$ integers in $\mathrm{X}\, t$ are constrained to the first r_u rows of $t\pi\rho$. So two or more of them belong to the same row of $t\pi\rho$. However the first column of $\overleftarrow{t}$ coincides with $\operatorname{Rim} t$, and thus contains $\mathrm{X}\, t$. So the elements of $\mathrm{X}\, t$ belong to different rows of $\{\overleftarrow{t}\,\sigma\}$. This shows that $\{\overleftarrow{t}\,\sigma\} \neq \{t\pi\rho\}$. We conclude that $\langle e_{t\pi}, e_{\overleftarrow{t}} \rangle = 0$. $\square$

We call a t-cycle π *visible* if $\langle e_{t\pi}, e_{\overleftarrow{t}} \rangle \neq 0_F$. Lemma 8 implies that π is visible if and only if π is a permutation of $\operatorname{Rim} t$. Any t-cycle that is not visible is said to be *invisible*.

Lemma 9. *Suppose that π is invisible. Then $e_t\pi L_n$ is a sum of polytabloids $e_t\sigma$, where each σ is invisible.*

Proof. This follows from Lemma 7 and the observation that if ρ is a $t\pi$-cycle then $\pi\rho$ is an invisible t-cycle. $\square$

We now set S_t^λ as the $\mathbb{Z}$-span of all $e_{t\pi}$, where π is a t-cycle. Also set I_t^λ as the sum of the following two subspaces of S_t^λ: the first is the $\mathbb{Z}$-span of all $e_{t\pi}$, where π is an invisible t-cycle; the second is the $\mathbb{Z}$-span of all $(\operatorname{sgn}\pi_1)e_t\pi_1 - (\operatorname{sgn}\pi_2)e_t\pi_2$, where π_1 and π_2 are visible t-cycles such that $n \in \operatorname{Rim}_v t\pi_i$, for $i = 1, 2$, for some v.

Suppose that n belongs to $\operatorname{Rim}_u t$. Set $t_u := t$ and $e_u := e_t$. For $i = 1, \dots, u-1$, define $e_i := -e_t(n, (r_i, c_i)t)$. Clearly

$$e_u, e_{u-1}, \dots, e_1 \quad \text{form a basis for } S_t^\lambda \text{ modulo } I_t^\lambda. \tag{3}$$

Now identify each e_i with its image in $S_t^\lambda / I_t^\lambda$.

Lemma 10. *There is a well-defined action of L_n on S_t^λ/I_t^λ. The matrix of L_n with respect to $e_u, e_{u-1}, e_{u-2}, \dots, e_2, e_1$ is*

$$\begin{bmatrix}
\alpha_u & (\alpha_{u-1}-\beta_{u-1}) & (\alpha_{u-2}-\beta_{u-2}) & \dots & (\alpha_2-\beta_2) & (\alpha_1-\beta_1) \\
0 & \alpha_{u-1} & (\alpha_{u-2}-\beta_{u-2}) & \dots & (\alpha_2-\beta_2) & (\alpha_1-\beta_1) \\
0 & 0 & \alpha_{u-2} & \dots & (\alpha_2-\beta_2) & (\alpha_1-\beta_1) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & \alpha_2 & (\alpha_1-\beta_1) \\
0 & 0 & 0 & \dots & 0 & \alpha_1
\end{bmatrix}$$

Proof. Fix $i = 1, \dots, u$. It follows from Lemma 7 that

$$e_i L_n = \alpha_i e_i + \sum_{j=1}^{i-1} \sum_{w_j} e_i(n, w_j), \qquad \text{modulo } I_t^\lambda.$$

Here w_j ranges over all integers in $\mathrm{Rim}_j\, t$. But Rim_j contains $r_j - r_{j-1} + 1 = \beta_j - \alpha_j$ nodes. The result now follows from the fact that $e_i(n, w_j) = -e_j$ modulo I_t^λ. □

Let v be a positive integer. Specialize sets of variables X and Y such that $x_i := \alpha_i$, for $i \in [1, u]$ and $y_j := \beta_j$, for $j \in [1, u-1]$. Let $Z := \{z_1, \dots, z_v\}$ be a set of variable scalars, and define δ_μ as in (1). Define $F_Z := \prod_{i=1}^{v}(L_n - z_i)$.

Corollary 11. $\langle e_t F_Z, e_{\overleftarrow{t}} \rangle = \sum_{\mu \in \left\langle {u \atop v} \right\rangle} \prod_{i=1}^{v} (x_{\mu_i} - \delta_{\mu,i})$.

Proof. Lemma 10 and an inductive argument show that

$$e_t F_Z = \sum_{j=1}^{u} \sum_{\substack{\mu \in \left\langle {u \atop v} \right\rangle \\ \mu_v = j}} \prod_{i=1}^{v} (x_{\mu_i} - \delta_{\mu,i}) e_i, \quad \mathrm{mod}\ I_t^\lambda.$$

Then Lemma 8 implies that

$$\langle e_t F_Z, e_{\overleftarrow{t}} \rangle = \sum_{\mu \in \left\langle {u \atop v} \right\rangle} \prod_{i=1}^{v} (x_{\mu_i} - \delta_{\mu,i}). \tag{4}$$

□

As an immediate consequence of Theorem 5 and Corollary 11, we get

Corollary 12. $\langle e_t F_Z, e_{\overleftarrow{t}} \rangle = \mathrm{HE}_{|Z|}(X; Y \cup Z)$.

4. A lower bound on the degree of the minimal polynomial of L_n acting on D^λ

It is known that the p-block of Σ_n that contains S^λ is determined by the multiset of p-residues of the nodes in $[\lambda]$. Thus for all u, v, the Specht modules S^{λ^u} and S^{λ^v} belong to the same p-block of Σ_{n-1} if and only if $\alpha_u \equiv \alpha_v$ modulo p. Given $u \in [1, m]$ we use B_u to denote the primitive idempotent in the centre of $F\Sigma_{n-1}$ that acts as the identity on S^{λ^u}. Note that the image of B_u is a primitive idempotent in the centre of $\mathrm{End}_{\Sigma_{n-1}}(S^\lambda)$.

A. Kleshchev defines the *normal* and *good* removable nodes of λ as follows. Fix u. Consider the sequence $S_u := (\alpha_{u-1}, \beta_{u-1}, \alpha_{u-1}, \ldots, \alpha_1, \beta_1)$ that alternates between the residues of the addable and removable nodes of $[\lambda]$ above the u-th removable node. Let $T_u = (t_1, t_2, \ldots)$ be the sign sequence that is obtained from S_u by removing those terms not congruent to α_u modulo p, and replacing each remaining α_i by $+1$ and each remaining β_i by -1. Then the u-th removable node is normal if $\sum_{i=1}^{j} t_i \geq 0$, for each $j \geq 1$. The u-th removable node is good if it is the lowest normal node with p-residue α_u.

Now set ε_u as the number of $i \in [1, u-1]$ such that the i-th removable nodes of $[\lambda]$ is normal and $\alpha_i \equiv \alpha_u$ modulo p.

Lemma 13. *If the u-th removable node of $[\lambda]$ is normal then*

$$D_u^\lambda (L_n - \alpha_u)^{\varepsilon_u} B_u \neq 0.$$

Proof. We claim that $E_u := \prod_{\substack{i=1 \\ \alpha_i \not\equiv \alpha_u}}^{u-1} (L_n - \alpha_i)$ is a unit in $\mathrm{End}_{\Sigma_{n-1}}(S_u^\lambda) B_u$. For, E_u annihilates $S_u^\lambda(1 - B_{\alpha_u})$, as this module has a Specht series with successive quotients S^{λ^i} on which $(L_n - \alpha_i)$ acts as zero. Moreover by [4], the subalgebra $F[L_n]$ of $\mathrm{End}_{\Sigma_{n-1}}(S^\lambda)$ is unital and local. The claim now follows from the fact that E_u is not nilpotent, as it acts as the nonzero scalar $\prod_{\substack{i=1 \\ \alpha_i \not\equiv \alpha_u}}^{u-1} (\alpha_u - \alpha_i)$ on the quotient $S_u^\lambda / S_{u-1}^\lambda$. So it is enough to show that $D_u^\lambda (L_n - \alpha_u)^{\varepsilon_u} E_u \neq 0$.

Assume that n occupies the u-th removable node in t. Let σ be the set of indices $i \in [1, u-1]$ such that either the i-th removable node of $[\lambda]$ is normal or its p-residue is different to α_u. Suppose that σ has cardinality v. So $\sigma \in \binom{u-1}{v}$. Now specialize sets of variables X, Y and Z such that $x_i := \alpha_i$, for $i \in [1, u]$ and $y_j := \beta_j$, for $j \in [1, u-1]$ and $z_k := \alpha_{\sigma_k}$, for $k = [1, v]$. Then $(L_n - \alpha_u)^{\varepsilon_u} E_u$ coincides with Z_F, in the notation of Corollary 12. Now that Corollary implies that $\langle e_t F_Z, e_{\overleftarrow{t}} \rangle = \mathrm{HE}_v(X; Y \cup Z)$. We complete the proof by showing that $\mathrm{HE}_v(X; Y \cup Z) \not\equiv 0$ modulo p.

As the u-th removable node is normal, there is a bijection between the addable nodes with p-residue α_u above the u-th removable node and the non-normal nodes with p-residue α_u above the u-th removable node. Let $i \in [1, u-1]$. If the i-th removable node is non-normal and of p-residue α_u, then $x_i \equiv y_j \equiv \alpha_u$, where the j-th addable node is the corresponding one of p-residue α_u. Otherwise there exists k so that $\sigma_k = i$ and $x_i = z_k$. In this way we get an injective map $f : X \backslash \{x_u\} \to Y \cup Z$ such that $x_i \equiv f(x_i)$, modulo p. Now $Y \cup Z \backslash f(X \backslash \{x_u\})$ consists of the v elements Y_1 of Y whose values do not equal $x_u \equiv \alpha_u$. Applying Lemma 3 repeatedly, and working modulo p, we can remove the v equal pairs in the graph of f from $\mathrm{HE}_v(X; Y \cup Z)$. Thus

$$\mathrm{HE}_v(X; Y \cup Z) \equiv \mathrm{HE}_v(\{x_u\}; Y_1) \equiv \prod_{y_i \in Y_1} (x_u - y_i) \not\equiv 0 \quad \text{modulo } p.$$

$\square$

5. An upper bound on the degree of the minimal polynomial of L_n acting on D^λ

Let $R_{t,1}$ be the group of permutations in R_t that fix each entry in the row of t that contains n and let $R_{t,2}$ be the group of permutations in R_t that fix each entry *not* in this row. So R_t is the internal direct product

$$R_t = R_{t,1} \times R_{t,2}.$$

We write $\to_t$ if we need to point out the dependence of the relation $\to$ (defined in Section 3) on t. Define a subrelation $\Rightarrow$ of $\to$ on $[1, n]$ by $i \Rightarrow j$ if $i \to j$ and if i and j occupy the *same row* of t. The motivation for the following lemma comes from considering the images of polytabloids under the map $\theta_\lambda : S^\lambda_F \to D^\lambda_F$, when λ is p-regular. Compare it with Lemma 7.

Lemma 14. $e_t\, R^+_{t,1}\, L_n \;=\; \left(\alpha_t e_t + \sum_{n \Rightarrow i} e_t(n,i)\right) R^+_{t,1}.$

Proof. As each $\pi \in R_{t,1}$ fixes all entries in the same row as n in t, $\Rightarrow$ has the same meaning for t and $t\pi$. Moreover, if $n \Rightarrow i$ then $\pi(n,i) = (n,i)\pi$. Lemma 7 gives

$$e_t\, R^+_{t,1}\, L_n \;=\; \alpha_t e_t R^+_{t,1} + \sum_{\pi \in R_{t,1}} \sum_{n \to_{t\pi} i} e_t \pi(n,i).$$

Our proof is completed by showing that those polytabloids $e_t\pi(n,i)$ such that n and i belong to different rows of $t\pi$ cancel in pairs. Fix $\pi \in R_{t,1}$ and suppose that $n \to_{t\pi} i$ but n and i belong to different rows of $t\pi$. Let j be the integer in $t\pi$ in the same column as n and the same row as i. Then $j \neq i, n$, and $\pi(i,j) \in R_{t,1}$ and $n \to_{t\pi(i,j)} j$. Since $e_t\pi(i,j)(n,j) = e_t\pi(n,i)(i,j)$, and (i,j) is a column permutation of $t\pi(n,i)$, the sum $e_t\pi(i,j)(n,j) + e_t\pi(n,i)$ is zero. The Lemma follows from this. $\square$

Now set τ_u as the number of $i \in [1, u]$ such that the i-th removable nodes of $[\lambda]$ is normal and $\alpha_i \equiv \alpha_u$ modulo p. So $\tau_u = \varepsilon_u + 1$, if the u-th removable node of $[\lambda]$ is normal.

Lemma 15. $D^\lambda_u (L_n - \alpha_u)^{\tau_u} B_u = 0.$

Proof. Adopt the notation and assumptions of Lemma 13. In particular n occupies the u-th removable node of t. We change the definition of

σ so that $\sigma \in \binom{u}{v+1}$ is the set of indices $i \in [1,u]$ such that either the i-th removable node of $[\lambda]$ is normal or its p-residue is not α_u. Recall that θ_λ is a Σ_n-homomorphism $S^\lambda \to S^{\lambda'} \otimes S^{[1^n]}$ whose image is the irreducible module D^λ. Now specialize the values of Z as $z_k := \alpha_{\sigma_k}$, for $k = [1, v+1]$. Set $F_Z := \prod_{i=1}^{v+1}(L_n - \alpha_{\sigma_i})$. We will show that, in the notation of Corollary 12, $e_t F_Z \theta_\lambda = 0$.

Since L_n is a sum of permutations in Σ_n, it commutes with θ_λ. Moreover, the permutations are odd. So

$$
\begin{aligned}
e_t F_Z \theta_\lambda = e_t \theta_\lambda F_Z &= \left(e_{t'} R_{t'}^+ \otimes \Sigma^-\right) \prod_{i=1}^{v+1}(L_n - \alpha_{\sigma_i}) \\
&= e_{t'} R_{t'}^+ \prod_{i=1}^{v+1}(-L_n - \alpha_{\sigma_i}) \otimes \Sigma^-.
\end{aligned}
$$

So it is enough to show that $e_{t'} R_{t'}^+ \prod_{i=1}^{v+1}(-L_n - \alpha_{\sigma_i}) = 0$.

Now n occupies the c_u-th row and r_u-th column of the transpose t' of t. Considering the dimensions of $[\lambda']$, given $\pi \in R_{t'}$, the symbol n occupies a column of length $r_1, r_2, \ldots, r_{u-1}$ or r_u in $t'\pi$.

Let $i, j \in [1,u]$, with $i < j$, and let $\pi \in R_{t'}$ be such that n occupies a column of length r_j in $t\pi$. Suppose that k is a symbol in the c_u-th row of $t'\pi$ and in a column of length r_i. So $\pi(n,k) \in R_{t'}$. Then n occupies the same row as k in $t\pi(n,k)$, but a column of longer length. So $e_{t'}\pi$ is one of the polytabloids that occurs in the expansion of $e_{t'}\pi(n,k)R_{t',1}^+ L_n$ given by Lemma 14. Moreover, this is the only way that $e_{t'}\pi$ can occur in the expansion of $e_{t'}\eta R_{t',1}^+ L_n$, for any $\eta \in R_{t'}$. Note that there are $(\beta_i - \alpha_i)$ choices for k, as $[\lambda']$ has $\beta_i - \alpha_i$ columns of length r_i.

For $i \in [1,u]$, let f_i denote the sum, in $S^{\lambda'}$, of all polytabloids $e_{t'}\pi$, where n occupies a column of length r_i in $t'\pi$. Note that $\alpha_{t'\pi} = -\alpha_i$, for each such π. Set $t_i := t(n, (r_i, c_i)t)$, and let $\overleftarrow{t_i}$ be the row reversal of t_i.

Lemma 14 and the previous paragraph implies that

$$f_i(-L_n) \;=\; \alpha_i f_i + \sum_{j=i+1}^{u} (\alpha_i - \beta_i) f_j.$$

Now $e_{t'} R^+_{t'} = f_1 + \ldots f_u$ and we have

$$(f_1 + \ldots + f_u)(-L_n) \;=\; \sum_{i=i+1}^{u} \left(\alpha_i + \sum_{j=1}^{i-1} (\alpha_j - \beta_j) \right) f_i.$$

Using Corollary 11 and induction, it can be seen that

$$(f_1 + \ldots + f_u) \prod_{i=1}^{v+1} (-L_n - \alpha_{\sigma_i}) \;=\; \sum_{i=1}^{n} \langle e_{t_i} F_Z, e_{\overleftarrow{t_i}} \rangle f_i.$$

For $i \in [1, u]$, we get from Corollary 12 that

$$\langle e_{t_i} F_Z, e_{\overleftarrow{t_i}} \rangle = \mathrm{HE}_{v+1}(\{\alpha_i, \ldots, \alpha_1\}; \{\beta_{i-1}, \ldots, \beta_1\} \cup Z).$$

Arguing as in the proof of Lemma 3, there exists an injective map $f_i : \{\alpha_i, \ldots, \alpha_1\} \to \{\beta_{i-1}, \ldots, \beta_1\} \cup Z$ such that $f_i(\alpha_j) \equiv \alpha_j$, modulo p, for all j. Thus $\mathrm{HE}_{v+1}(\{\alpha_i, \ldots, \alpha_1\}; \{\beta_{i-1}, \ldots, \beta_1\} \cup Z) \equiv 0$, modulo p. The lemma follows from this. □

Corollary 16. *Suppose that there are no normal removable nodes of p-residue α_u at or above the u-th removable node of $[\lambda]$. Then $D^\lambda_u = 0$.*

Proof. Our hypothesis is that $\tau_u = 0$. Now apply the previous lemma. □

Our main result here is

Theorem 17. *Suppose that the u-th removable node of $[\lambda]$ is good. Then $(x - \alpha_u)^{\tau_u}$ is the minimal polynomial of L_n acting on $D^\lambda B_u$.*

Proof. By definition, the u-th removable node is the lower normal removable node of $[\lambda]$ that has p-residue α_u. So $\tau_u = \varepsilon_u + 1$. Thus

$D_u^\lambda B_u(L_n - \alpha_u)^{\tau_u - 1} \neq 0$, by Lemma 13. But τ_u equals the number of normal removable nodes of $[\lambda]$ that have p-residue α_u. So $D^\lambda B_u(L_n - \alpha_u)^{\tau_u} = 0$, by Lemma 15. □

6. Additional remarks, unfinished section

The next result is well known, being a special case of results of Carter and Payne [2]. Recently several authors [7], [4] have shown that in this situation $\mathrm{Hom}_{\Sigma_n}(S^\alpha, S^\beta)$ is 1-dimensional. Here we prove existence and give a simple algorithm to compute the image of a polytabloid under the One-Box Shift homomorphism.

Theorem 18. *Suppose that α and β are partitions of n such that $[\beta]$ is obtained by removing a node from $[\alpha]$ and adding it back on in a lower row, so that the removed and added positions have the same p-residue. Then $\mathrm{Hom}_{\Sigma_n}(S^\alpha, S^\beta) \neq 0$.*

Proof. Note that α and β have the same p-core. Also, there exists positive integers $u > v$, and a partition λ of $n+1$ whose diagram can be obtained by adding a node to the end of row u of $[\alpha]$ or to the end of row v of β.

We use the notation for the residue of λ and the Specht filtration $\{S_u^\lambda\}$ of $S^\lambda\downarrow_{\Sigma_n}$ as described above, noting that here λ is a partition of $n+1$, and not n as before. For integers $i > j$ define $\theta_{i,j} := \prod_{i=v+1}^{u}(L_{n+1} - \alpha_i)$ Then $\theta_{u,v} = \theta_{u-1,v-1}$, as $\alpha_u = \alpha_v$. So by the previous corollary, we have $S_u^\lambda \theta_{u,v} \subseteq S_v^\lambda$ and also $S_{u-1}^\lambda \theta_{u,v} \subseteq S_{v-1}^\lambda$. Moreover $S_u^\lambda \theta_{u,v} \nsubseteq S_{v-1}^\lambda$ by a result in [3]. Thus $\theta_{u,v}$ induces a non-zero Σ_n-homomorphism, denoted $f_{u,v}$, from $S^\alpha = S_u^\lambda / S_{u-1}^\lambda$ into $S^\beta = S_v^\lambda / S_{v-1}^\lambda$. As $\mathrm{Hom}_{\Sigma_n}(S^\alpha, S^\beta)$ is 1-dimensional, $f_{u,v}$ must be a nonzero multiple of the homomorphism defined in [2].

Recall that there is a short exact sequence of $F\Sigma_n$-modules

$$0 \longrightarrow S^{\lambda}_{u-1} \longrightarrow S^{\lambda}_{u} \xrightarrow{\theta_u} S^{\lambda^u} \longrightarrow 0.$$

Let θ_u^{-1} be a F-retraction of θ_u. Then $f_{u,v} = \theta_u^{-1}\theta_{u,v}\theta_v$, by the previous paragraph. $\square$

Example 19. *For $p = 5$ there is a one-box shift $f : S^{[4,2]} \to S^{[3,2,1]}$. Here $\lambda = [4,2,1]$ has removable residues $\alpha_1 = 3, \alpha_2 = 0$ and $\alpha_3 = 3$. Using the previous theorem, and Lemma 7 we compute*

$$e_{\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6\\\cline{1-2}\end{array}}\theta_3^{-1}(L_9-3)L_9\theta_1 = e_{\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6\\\cline{1-2} 7\\\cline{1-1}\end{array}}(L_9-3)L_9\theta_1$$

$$= (e_{\begin{array}{|c|c|c|c|}\hline 1&7&3&4\\\hline 5&6\\\cline{1-2} 2\\\cline{1-1}\end{array}} + e_{\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&7\\\cline{1-2} 6\\\cline{1-1}\end{array}} + e_{\begin{array}{|c|c|c|c|}\hline 1&2&7&4\\\hline 5&6\\\cline{1-2} 3\\\cline{1-1}\end{array}} + e_{\begin{array}{|c|c|c|c|}\hline 1&2&3&7\\\hline 5&6\\\cline{1-2} 4\\\cline{1-1}\end{array}})L_9\theta_1$$

$$= (e_{\begin{array}{|c|c|c|c|}\hline 1&3&7&4\\\hline 5&6\\\cline{1-2} 2\\\cline{1-1}\end{array}} + e_{\begin{array}{|c|c|c|c|}\hline 1&4&3&7\\\hline 5&6\\\cline{1-2} 2\\\cline{1-1}\end{array}} + e_{\begin{array}{|c|c|c|c|}\hline 1&2&7&4\\\hline 5&3\\\cline{1-2} 6\\\cline{1-1}\end{array}} + e_{\begin{array}{|c|c|c|c|}\hline 1&2&3&7\\\hline 5&4\\\cline{1-2} 6\\\cline{1-1}\end{array}} + 3e_{\begin{array}{|c|c|c|c|}\hline 1&2&7&4\\\hline 5&6\\\cline{1-2} 3\\\cline{1-1}\end{array}} + 3e_{\begin{array}{|c|c|c|c|}\hline 1&2&3&7\\\hline 5&6\\\cline{1-2} 4\\\cline{1-1}\end{array}})\theta_1$$

$$= e_{\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 5&6\\\cline{1-2} 2\\\cline{1-1}\end{array}} + e_{\begin{array}{|c|c|c|}\hline 1&4&3\\\hline 5&6\\\cline{1-2} 2\\\cline{1-1}\end{array}} + e_{\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 5&3\\\cline{1-2} 6\\\cline{1-1}\end{array}} + e_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 5&4\\\cline{1-2} 6\\\cline{1-1}\end{array}} + 3e_{\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 5&6\\\cline{1-2} 3\\\cline{1-1}\end{array}} + 3e_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 5&6\\\cline{1-2} 4\\\cline{1-1}\end{array}}$$

Theorem 20. *Suppose that $1 \leq v < u \leq m$ are such that $a_v = \alpha_u$, but none of $\alpha_v, \alpha_{v+1}, \dots, \alpha_{u-1}$ equals α_u. Then $D_u = D_{u-1}$.*

Proof. The elements $\{e_t \in S^{\lambda}_u \backslash S^{\lambda}_{u-1}\}$ generate S^{λ}_u as S^{λ}_{u-1}-module. Moreover Lemma 7 implies that $e_t \prod_{i=v}^{u}(L_n - \alpha_i) \equiv \prod_{i=v}^{u}(\alpha_u - \alpha_i)e_t$ (mod S^{λ}_{u-1}), for each $e_t \in S^{\lambda}_U \backslash S^{\lambda}_{u-1}$. As $\prod_{i=v}^{u}(\alpha_u - \alpha_i) \neq 0_F$, it is enough to show that $e_t \prod_{i=v}^{u}(L_n - \alpha_i) \in J^{\lambda} = \ker(\theta)$. $\square$

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