THE CENTRALIZER OF A SUBGROUP IN A GROUP ALGEBRA

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If R is a commutative ring, G is a finite group, and H is a subgroup of G, then the *centralizer algebra* RG^H is the set of all elements of RG that commute with all elements of H. The algebra RG^H is a Hecke algebra in the sense that it is isomorphic to $\mathrm{End}_{RH\times G}(RG) = \mathrm{End}_{RH\times G}({1_{\Delta H}}^{H\times G})$. The authors have been studying the representation theory of these algebras in several recent and not so recent papers [4], [5], [6], [7], [10], [11], mainly in cases where G is p-solvable and H is normal, or when $G = S_n$ and $H = S_m$ for $n-3 \le m \le n$. Part of the original motivation was to see whether there might be a "weight conjecture" for these algebras, one that would simultaneously generalize Alperin's weight conjecture and Brauer's First Main Theorem on Blocks. This idea is explained in more detail in in [4], [5], and [6]. Also, when H is a p-subgroup these algebras play an important role in Green's approach to modular representation theory and in Puig's theory of points. Along the way, several fairly basic and general questions have come up. This paper mainly consists of counterexamples to conjectures that one might be led to make based on the evidence in our earlier papers.

When coefficients belong to an algebraically closed field F of characteristic 0, or of characteristic p where $p \nmid |G|$, the representation theory of a centralizer algebra FG^H is easy to understand.

(i) The algebra FG^H is semisimple.
(ii) If S is a simple FH-module and T is a simple FG-module such that $\mathrm{Hom}_{FH}(S, T\downarrow_H) \neq 0$, then the FG^H-module $\mathrm{Hom}_{FH}(S, T\downarrow_H)$ is simple. (The space of homomorphisms is an FG^H-module via the multiplication $(a\varphi)(v) = a(\varphi(v))$ for all $a \in FG^H$, $\varphi \in \mathrm{Hom}_{FH}(S, T\downarrow_H)$, and $v \in S$.)
(iii) Every simple FG^H-module arises in this way, and appears just once as S and T run through all possibilities.
(iv) The center of FG^H is generated as an F-algebra by the centers of FG and FH.
(v) Every primitive central idempotent of FG^H has the form ef, where e is a primitive central idempotent of FG and f is a primitive central idempotent of FH.

If the characteristic of the field does divide $|G|$, then FG^H is not semisimple, because the one-dimensional space spanned by $\sum_{g\in G} g$ is a nilpotent two-sided ideal. It is natural to ask whether items (ii), (iii), (iv), and (v) are still true in the non-semisimple case. As we will see, none of them is true in general. However, some of the counterexamples were not easy to find. Asking whether they are true or close to true in particular cases has been a useful approach. For example, we show in [4] that (ii), (iii), (iv), and (v) are all true when $G = S_n$ and $H = S_{n-1}$. In the preprint [7], we show that (v) is true when $G = S_n$ and $H = S_{n-2}$ or S_{n-3}.

From now on, F is a field of characteristic p. For any subset A of G, we let $A^+ = \sum_{g \in A} g \in FG$. There is a basis for FG^H consisting of all elements of the form C^+, where C is an orbit for the conjugation action of H on G.

Question 1. *Is FG^H a symmetric algebra?*

This is not true in general. Take $H = G$ so that $FG^H = \mathrm{Z}(FG)$. If G has more p-regular classes than blocks, $\mathrm{Z}(FG)$ is not symmetric. The Reynolds ideal, $\mathrm{Soc}(FG) \cap \mathrm{Z}(FG)$ is spanned by the p-regular section sums (see (39) in [9]). So $\mathrm{Soc}(\mathrm{Z}(FG))$ has dimension greater than or equal to $l(FG)$, the number of simple modules. On the other hand, the dimension of $\mathrm{Hd}(\mathrm{Z}(FG))$ equals the number of p-blocks.

Question 2. *If S is a simple FH-module and T is a simple FG-module such that $\mathrm{Hom}_{FH}(S, T\downarrow_H) \neq 0$, is $\mathrm{Hom}_{FH}(S, T\downarrow_H)$ a simple FG^H-module ?*

This not true in general. To construct a counterexample, we will use the following proposition.

Proposition 1. *Let P be a normal p-subgroup of G such that $C_G(P) \subseteq P$. Let V be the principal FP-module. Let U be a simple FG-module. If $\mathrm{Hom}_{FP}(V, U\downarrow_P)$ is a simple FG^P-module, then U has dimension 1 as a vector space over F.*

Proof. Since V is the unique simple FP-module, it follows from Clifford's Theorem that P acts trivially on U. Pick a non-zero element v of V. It is easily checked that the map $\phi \mapsto \phi(v)$ gives an isomorphism $\mathrm{Hom}_P(V, U) \cong U\downarrow_{FG^P}$ as modules over FG^P. Hence $U\downarrow_{FG^P}$ is simple.

Next, we show that $U\downarrow_{FC_G(P)}$ is simple. Let W be an $FC_G(P)$-submodule of U. We claim that W is also an FG^P-submodule of $U\downarrow_{FG^P}$. To see this, let C be an orbit of the conjugation action of P on G. Since P acts trivially on U, every element of C acts the same way on U. Because $|C|$ is a power of p, it follows that C^+ acts as 0 on U unless $C = \{x\}$ for some $x \in C_G(P)$.

Since $C_G(P) \subseteq P$, and $U\downarrow_{FC_G(P)}$ is simple, it follows that U has dimension 1 as a vector space over F. □

It is now easy to construct a counterexample. Let P be an elementary abelian p-group of order p^2. Let $K = \mathrm{SL}(2,p)$, acting on P by ordinary matrix multiplication. Let G be the semidirect product of K and P. We have $C_G(P) \subseteq P$; it therefore follows from Proposition 1 that for any simple FG-module U with $\dim_F(U) > 1$, $\mathrm{Hom}_{FP}(V, U\downarrow_P)$ is not simple as an FG^P-module. It is easy to find such a module In the book [2], Alperin exhibits all simple FK-modules. They have dimensions $1, 2, \ldots, p$. Any one of these inflated by the projection $G \to K$ is a simple FG-module.

It can, however, easily happen that for a particluar pair G, H, the answer to Question 2 is positive, even when FG^H is not semisimple. For example, Kleshchev's branching rule shows that if $G = S_n$ and $H = S_{n-1}$, then the answer to Question 2 is positive.

Question 3. *Is every simple FG^H-module isomorphic to $\mathrm{Hom}_{FH}(S, T\downarrow_H)$ for some simple FH-module S and simple FG-module T?*

This is not true in general. The following counterexample was communicated to us by Burkhard Külshammer.

If this were true, then it would imply that $l(FG^H)$, the number of simple FG^H-modules is less than or equal to the product $l(G)l(H)$. In order to construct a counterexample to this inequality, let us take H to be a p-subgroup of G, so that $l(H)$ equals one. Then $l(FG^H)$ is at least as big as $l(\mathrm{C}_G(H))$; this follows from the fact that the Brauer homomorphism with respect to H maps FG^H onto $F\mathrm{C}_G(H)$. So a positive answer to Question 3 would imply $l(G) \geq l(\mathrm{C}_G(H))$.

As a counterexample, take G to be the dihedral group of order 80 and H a subgroup of G of order 5, and F of characteristic 5. Then $G/O_5(G)$ is a dihedral group of order 16 and has 7 simple modules in characteristic 5, four of dimension 1 and 3 of dimension 2. On the other hand, $C_G(H)$ is a cyclic group of order 40 and has 8 simple modules in characteristic 5.

As a weak form of Question 3, we ask the following.

Question 4. *Is every simple FG^H-module a composition factor of a module of the form $\mathrm{Hom}_{FH}(S, T\downarrow_H)$, where S is a simple FH-module and T is a simpleFG-module?*

The answer is positive, as we now show. This line of argument was suggested by R. Boltje, B. Külshammer, M.Linckelmann, and L.L.Scott.

First, we point out that the map $\varphi \mapsto \varphi(1)$ gives an isomorphism of FG^H-modules

$$\mathrm{Hom}_{FH}(FH, FG) \cong FG.$$

The positive answer to Question 4 now follows from the next proposition.

Proposition 2. *Let M be an FG-module, and let N be an FH-module. If D is a composition factor of the FG^H-module $\mathrm{Hom}_{FH}(N, M)$, then there are a composition factor S of N and a composition factor T of M such that D is a composition factor of $\mathrm{Hom}_{FH}(S, T)$.*

Proof. Let

$$0 \to M_1 \to M \to M_2 \to 0$$

be a short exact sequence of FG-modules. Left exactness of the functor $\mathrm{Hom}(N, \cdot)$ gives an exact sequence

$$0 \to \mathrm{Hom}_{FH}(N, M_1) \to \mathrm{Hom}_{FH}(N, M) \to \mathrm{Hom}_{FH}(N, M_2),$$

where the last map is not necessarily surjective. It is easily checked that the maps are FG^H-module homomorphisms. Hence each composition factor D of $\mathrm{Hom}_{FH}(N, M)$ is also a composition factor of $\mathrm{Hom}_{FH}(N, T)$ for some composition factor T of M.

Similarly, left exactness of the functor $\mathrm{Hom}(\cdot, T)$ tells us that if

$$0 \to N_1 \to N \to N_2 \to 0$$

is a short exact sequence of FH-modules, then there is an exact sequence

$$0 \to \mathrm{Hom}_{FH}(N_2, T) \to \mathrm{Hom}_{FH}(N, T) \to \mathrm{Hom}_{FH}(N_1, T).$$

The maps are easily checked to be FG^H-homomorphisms.

Thus, if D is a composition factor of $\mathrm{Hom}_{FH}(N, T)$, then there exists a composition factor S of N such that D is a composition factor of $\mathrm{Hom}_{FH}(S, T)$. $\square$

Question 5. *Is the center of FG^H generated as an algebra by $\mathrm{Z}(FG)$ and $\mathrm{Z}(FH)$?*

The answer is negative. We will exhibit a counterexample.

If R is a suitable characteristic 0 local ring with residue field F of characteristic p, we get an algebra epimorphism $RG^H \to FG^H$ that maps $\mathrm{Z}(RG^H)$ into $\mathrm{Z}(FG^H)$. Let $\overline{\mathrm{Z}(RG^H)}$ be the image of this map. Since $\langle \mathrm{Z}(RG), \mathrm{Z}(RH)\rangle \subseteq \mathrm{Z}(RG^H)$, and $\langle \mathrm{Z}(RG), \mathrm{Z}(RH)\rangle$ maps onto $\langle \mathrm{Z}(FG), \mathrm{Z}(FH)\rangle$, we have

$$\langle \mathrm{Z}(FG), \mathrm{Z}(FH)\rangle \subseteq \overline{\mathrm{Z}(RG^H)} \subseteq \mathrm{Z}(FG^H).$$

If it were true that $\langle \mathrm{Z}(FG), \mathrm{Z}(FH)\rangle = \mathrm{Z}(FG^H)$, then it would also be true that $\overline{\mathrm{Z}(RG^H)} = \mathrm{Z}(FG^H)$. Thus in order to produce a counterexample to the conjecture $Z(FG^H) = \langle Z(FG), Z(FH)\rangle$, it is enough to exhibit an element of $Z(FG^H)$ that is not in the image of the map coming from $Z(RG^H)$.

For a counterexample, let $G = S_4$, H the normal Klein-four subgroup containing the products of disjoint 2-cycles, and F of characteristic 2.

The conjugation action of H on G has 12 orbits. The 4 elements of H lie in singleton orbits. The two containing $(1,2,3)$ and $(1,3,2)$ have size 4. In addition, there are 6 orbits of size 2, with representatives $(1,2)$, $(1,3)$, $(1,4)$, $(1,2,3,4)$, $(1,3,4,2)$, $(1,4,2,3)$. Denote the orbit of g by O_g, and the orbit sum in RG^H by ${O_g}^+$, for each representative g.

We claim that ${O_{(1,2,3)}}^+$ is not in $\mathrm{Z}(RG^H)$ but its image is in $Z(FG^H)$.

To see this, note first that ${O_{(1,2,3)}}^+$ commutes with ${O_{(1,2,3)}}^+$ and ${O_{(1,3,2)}}^+$ and all elements of H, as it is a class sum of A_4. Next, ${O_{(1,2,3)}}^+{O_{(1,2)}}^+ = 2({O_{(1,4)}}^+ + {O_{(1,3,4,2)}}^+)$, but ${O_{(1,2)}}^+{O_{(1,2,3)}}^+ = 2({O_{(1,3)}}^+ + {O_{(1,2,3,4)}}^+)$. Conjugating by $(1,2,3)$ and $(1,3,2)$, we get similar equations for the terms ${O_{(1,2,3)}}^+{O_{(1,3)}}^+$, ${O_{(1,3)}}^+{O_{(1,2,3)}}^+$, ${O_{(1,2,3)}}^+{O_{(1,4)}}^+$ and ${O_{(1,4)}}^+{O_{(1,2,3)}}^+$.

Nevertheless, there are also quite a few examples for which the answer to Question 5 is positive. The paper [6] shows that the answer is positive when $G = S_n$ and $H = S_{n-1}$. Computer calculations done by the first author using Magma [3] have shown that the answer is positive when $G = S_n$ and $H = S_m$ for all cases with $m \leq n \leq 8$. Some of these calculations used a fairly recent theorem of Alperin [1]. Alperin shows that $\langle \mathrm{Z}(\mathbb{Z}G), \mathrm{Z}(\mathbb{Z}H)\rangle$ has finite index in $\mathrm{Z}(\mathbb{Z}G^H)$, where as usual $\mathbb{Z}$ denotes the integers.

As a weak form of Question 5, one can ask the following.

Question 6. *Is* $\mathrm{Z}(FG^H) = \langle \mathrm{Z}(FG), \mathrm{Z}(FH)\rangle + \mathrm{J}(\mathrm{Z}(FG^H))$*?*
Equivalently:
Is every block idempotent of FG^H of the form ef, where e is a block idempotent of FG and f is a block idempotent of FH?

The answer is negative in general, although there are many examples in which the answer is positive. The following is a counterexample, originally found by the first author using Magma [3].

The counterexample for Question 5 is not a counterexample for Question 6, as Proposition 3 below shows.

Example. *If $G = S_6$, $H = A_4$, and F is a splitting field of characteristic 5, then $\langle Z(FG), Z(FH)\rangle$ does not contain all primitive idempotents of $Z(FG^H)$. In particular there are block idempotents e for FG and f for FH such that ef is not central primitive in FG^H. The algebra $efFG^H$ is not indecomposable as an F-algebra.*

Proof. We take e to be the principal 5-block idempotent of FG and f to be the principal 5-block idempotent of FH (in fact, we have no choice here). As 5 does not divide $|A_4|$, f has 5-defect zero, hence fFH is semisimple. In fact $f = (1/|H|)H^+$ and $fFH \cong F$, as F-algebras (and as right FH-modules). Note that f is primitive in fFH.

Think of fFG and $fFGe$ as right FG-modules. The first two paragraphs of the proof of Proposition 2.6 of [4] show that there is a natural injective F-algebra map $efFG^H \to \mathrm{End}_{FG}(fFGe)$; the map sends $efx \in efFG^H$ to multiplication on the left by efx. The proof uses the fact that f has p-defect zero and f is primitive in fFH (and not just centrally primitive). In our case, the map happens to be an isomorphism. We can see this by comparing dimensions of domain and range - looking at ordinary character multiplicities we see that both sides are 4-dimensional. Thus $efFG^H \cong \mathrm{End}_{FG}(fFGe)$.

Now we analyse $fFGe$ as a right FG-module. The FG-module fFG is isomorphic to the induced module $(fFH)\uparrow^G \cong (F_H \uparrow^G)$. However consider the chain of groups $H \leq N \leq G$, where $N = S_4$. Then

$$F_H \uparrow^N = F_N \oplus \mathrm{sgn}_N \quad \text{and} \quad F_H \uparrow^G = F_N \uparrow^G \oplus \mathrm{sgn}_N \uparrow^G,$$

where F_N is the trivial module and sgn_N is the sign module.

Consider the principal 5-block eFG of S_6. It has a cyclic defect group, and 5 irreducible characters, labelled by the partitions $[6], [4,2], [3,2,1], [2^2,1^2], [1^6]$ (5-core [1]). There are 4 irreducible eFG-modules, labelled by the 5-regular partitions $[6], [4,2], [3,2,1], [2^2,1^2]$. Denote the corresponding irreducible modules by $D([6])$, $D([4,2])$, etc. With respect to these labellings, the decomposition matrix and the Cartan matrix are

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Thus the Brauer tree is a straight line with 5 nodes and 4 edges. The only bit of information we need here is that $\mathrm{Hom}_{FG}(P([6]), P([2^2,1^2])) = 0$, where $P(\lambda)$ is the projective cover of the simple module $D(\lambda)$. In other words, these PIM's have no composition factor in common. (For all this information, see [8]).)

Back to the displayed equalities. It's easy to see that $F_N \uparrow^G e = P([6])$ and $\mathrm{sgn}_N \uparrow^G e = \mathrm{sgn}_G \otimes F_N \uparrow^G e = P([2^2,1^2])$. Then by the last statement of the last paragraph, $\mathrm{End}_{FG}(efFG) = \mathrm{End}_{FG}(P([6]) \oplus P([2^2,1^2])) = \mathrm{End}_{FG}(P([6])) \oplus \mathrm{End}_{FG}(P([2^2,1^2]))$ decomposes into a direct product of two non-zero F-algebras (each 2-dimensional, each commutative, with a 1-dimensional Jacobson radical). □

Note: From the ordinary character multiplicities we see that $efFG^H$ is commutative (even though FG^H is certainly not commutative). Now with $N = S_4$, it's clear that $Z(FN) \subseteq FG^H$ (for very general reasons). Thus $ef\mathrm{Z}(FN) \subseteq \mathrm{Z}(efFG^H)$. What is happening then is that $ef = aef + bef$ (a nontrivial orthogonal decomposition in the centre of the algebra), where a is the block idempotent of the 5-block of N containing F_N and b is the block idempotent of the 5-block of N containing sgn_N.

Some positive results along these lines of Question 6 are possible. For example, the answer is positive when $G = S_n$ and $H = S_m$ for $m = n-1$, $n-2$, or $n-3$. The following proposition gives another situation in which the answer is positive.

Proposition 3. *Assume that P is a normal p-subgroup of G. Then every central idempotent of FG^P is in $\mathrm{Z}(FG)$.*

Proof. Let e be a primitive central idempotent of FG^P. The Brauer map $\mathrm{Br}_P : FG^P \to FC_G(P)$ is a surjective homomorphism. Its kernel is a nilpotent ideal. (To see this, note that if C is an orbit of the conjugation action of P on $G \backslash C_G(P)$, then C^+ acts as 0 on each simple FG-module, so C^+ is in $\mathrm{J}(FG) \cap FG^P$.) Let $f = \mathrm{Br}_P(e)$. Then $f = e + j$, with $j \in \mathrm{J}(FG^P)$. Pick an n such that $j^{p^n} = 0$. Then $f = f^{p^n} = (e+j)^{p^n} = e^{p^n} + j^{p^n} = e$. Thus $e \in FC_G(P)$. Since e must be central in $FC_G(P)$, it follows that e is a linear combination of elements of $C_G(P)$ of order prime to p.

Assume, for a contradiction, that e is not in the center of FG. Let $g \in G$ such that $g^{-1}eg \neq e$. Then $g^{-1}ege = 0$ since $g^{-1}eg$ is another primitive central idempotent. Let C be the orbit of g under the conjugation action of P. Since P is normal, $C \subseteq gP$. Let $x_1, x_2, \ldots, x_s$ be elements of P such that $C = \{gx_1, gx_2, \ldots, gx_n\}$. Let $a = x_1 + x_2 + \cdots + x_s$. Then $C^+ = ga$. Since e is a central idempotent of FG^P, it follows that $gae = ega$. Since $a \in FP$, $ae = ea$, so $gea = ega$, and hence $ea = g^{-1}ega$. Multiplying from the left by e, we obtain $ea = eea = eg^{-1}ega = 0a = 0$.

However, e is a linear combination of p'-elements of $C_G(P)$, so for each i, ex_i is a linear combination of elements with p-part x_i. Therefore the elements of the set $\{ex_1, \ldots, ex_s\}$ have disjoint support. It follows that the sum $ea = ex_1 + ex_2 + \cdots + ex_s$ cannot be 0. This contradiction completes the proof. □

References

[1] J.L.Alperin, On the center of a Hecke algebra, *J. Algebra* **319** (2008), no. 2, 777–778.

[2] J.L.Alperin, *Local representation theory*, Cambridge Studies in Advanced Mathematics, 11. Cambridge University Press, Cambridge, 1986.

[3] W.Bosma, J.Cannon, C. Playoust, The Magma algebra system. I. The user language, J.Symbolic Comput. **24** (3-4) (1997), 235-265.

[4] H.Ellers, Searching for more general weight conjectures, using the symmetric group as an example, J.Algebra **225** (2000), 602-629.

[5] H. Ellers, The defect groups of a clique, p-solvable groups, and Alperin's conjecture, *J. Reine Angew. Math.* **468** (1995), 1–48.

[6] H. Ellers and J.Murray, Block theory, branching rules, and centralizer algebras, *J. Algebra* **276** (2004), no. 1, 236–258.

[7] H. Ellers and J. Murray, Blocks of ${kS_{n+3}}^{S_n}$ and related algebras, preprint.

[8] G.James and A.Kerber, *The representation theory of the symmetric group,* Encyclopedia of Mathematics and its Applications, 16. Addison-Wesley Publishing Co., Reading, Mass., 1981.

[9] B.Külshammer, Group-theoretical descriptions of ring-theoretical invariants of group algebras, in *Representation theory of finite groups and finite-dimensional algebras* (Bielefeld,1991), 425-442, Progr. Math. 95, Birkhäuser, Basel, 1991.

[10] J. Murray, Generators for the centre of the group algebra of a symmetric group, *J. Algebra* **271** (2004), no. 2, 725–748.

[11] J. Murray, Squares in the centre of the group algebra of a symmetric group, *Bull. London Math. Soc.* **34** (2002), no. 2, 155–164.

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