BRANCHING RULES FOR SPECHT MODULES

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Abstract. Let Σ_n be the symmetric group of degree n, and let F be a field of characteristic distinct from 2. Let S_F^λ be the Specht module over $F\Sigma_n$ corresponding to the partition λ of n. We find the indecomposable components of the restricted module $S_F^\lambda \downarrow_{\Sigma_{n-1}}$ and the induced module $S_F^\lambda \uparrow^{\Sigma_{n+1}}$. Namely, if b and B are block idempotents of $F\Sigma_{n-1}$ and $F\Sigma_{n+1}$ respectively, then the modules $S_F^\lambda \downarrow_{\Sigma_{n-1}} b$ and $S_F^\lambda \uparrow^{\Sigma_{n+1}} B$ are 0 or indecomposable. We give examples to show that the assumption $\operatorname{char} F \neq 2$ cannot be dropped.

1. Introduction

Let n be a positive integer and let Σ_n be the symmetric group of degree n. For any field F and any partition λ of n, the Specht module S_F^λ is defined to be the submodule of the permutation module $F_{\Sigma_\lambda} \uparrow^{\Sigma_n}$ spanned by certain elements called polytabloids, where Σ_λ is the Young subgroup associated to λ and F_{Σ_λ} is the principal $F\Sigma_\lambda$-module. (See [1] for definitions.) Specht modules play a central role in the representation theory of the symmetric group, because in characteristic 0 the Specht modules are the simple $F\Sigma_n$-modules, while in characteristic p the heads of the Specht modules S_F^λ such that λ is p-regular are the simple $F\Sigma_n$-modules. When the field F has charactersitic 0, the structure of the restriction of S_F^λ to Σ_{n-1} is given by the Classical Branching Rule: the module $S_F^\lambda \downarrow_{\Sigma_{n-1}}$ is a direct sum $\bigoplus_\mu S_F^\mu$, where μ runs through all partitions of $n-1$ obtained from λ by removing a node from its Young diagram. In 1971, Peel [4] gave the first characteristic p version of the branching rule. He showed that there is a series of submodules such that the successive quotients are the Specht modules S_F^μ, where μ runs through the same set. Nevertheless, the structure of the restriction $S_F^\lambda \downarrow_{\Sigma_{n-1}}$ is not well understood. For example, the problem of finding a composition series is open and very difficult, and the socle is not known. See Kleshchev [2] for an introduction to recent work on $S_F^\lambda \downarrow_{\Sigma_{n-1}}$.

In this paper, we find the indecomposable components of $S_F^\lambda \downarrow_{\Sigma_{n-1}}$, when the characteristic of F is not 2. These are given by Theorem 3.4: if b is a block idempotent of $F\Sigma_{n-1}$, then $S_F^\lambda \downarrow_{\Sigma_{n-1}} b$ is 0 or indecomposable. Thus there is a bijection between the set of indecomposable components of $S_F^\lambda \downarrow_{\Sigma_{n-1}}$ and the set of p-cores that can be obtained from λ by removing first one node and then a sequence of rim p-hooks. We also prove the analogous theorem for the induced module $S_F^\lambda \uparrow^{\Sigma_{n+1}}$. The two proofs are almost identical. We give examples to show that the assumption $\operatorname{char} F \neq 2$ cannot be dropped.

The combinatorial part of the proof is in section 2. Here we find the minimal polynomials for the actions of E_{n-1} on $S_F^\lambda \downarrow_{\Sigma_{n-1}}$ and E_{n+1} on $S_F^\lambda \uparrow^{\Sigma_{n+1}}$, where

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E_k is the sum of all the transpositions in Σ_k. These polynomials have degrees m and $m+1$ respectively, where m is the number of distinct parts of λ. The results of section 2 are valid for all fields, not just those of odd characteristic.

In section 3, we investigate the algebras $\mathcal{E} = \mathrm{End}_{F\Sigma_{n-1}}(S_F^\lambda \downarrow_{\Sigma_{n-1}})$ and $\mathcal{F} = \mathrm{End}_{F\Sigma_{n+1}}(S_F^\lambda \uparrow^{\Sigma_{n+1}})$. Under the assumption that $\mathrm{char}\, F \neq 2$, we use the results from section 2 to show that the natural maps $Z(F\Sigma_{n-1}) \to \mathcal{E}/J(\mathcal{E})$ and $Z(F\Sigma_{n+1}) \to \mathcal{F}/J(\mathcal{F})$ are surjective, where $J(\mathcal{E})$ and $J(\mathcal{F})$ are the Jacobson radicals of $\mathcal{E}$ and $\mathcal{F}$. The main theorem follows easily.

2. The minimal polynomials of the sum of all transpositions acting on the restriction and induction of a Specht module

Throughout this paper n is a fixed positive integer and λ is a fixed partition of n. We orient the Young diagram $[\lambda]$ left to right and top to bottom. This means that the first row is the one at the top and the first column is the one at the left. The (i,j) node is in the ith row and the jth column. We will use $\widehat{n}$ to denote the set $\{1,\ldots,n\}$ and let Σ_n denote the group of permutations of $\widehat{n}$. Permutations and homomorphisms will generally act on the right. The *Murphy element* L_n is the sum of all transpositions in Σ_n that are not in Σ_{n-1} (with $L_1 := 0$). We use E_n to denote the sum of all transpositions in Σ_n. So E_n is the 1-st elementary symmetric function in the Murphy elements.

Let F be any field and let S^λ denote the Specht module, defined over F, corresponding to λ. We use the notation

$$\mathcal{R} \quad \text{for the restriction of } S^\lambda \text{ to } \Sigma_{n-1} \text{ and}$$
$$\mathcal{I} \quad \text{for the induction of } S^\lambda \text{ to } \Sigma_{n+1}.$$

The purpose of this section is to compute the minimal polynomial of E_{n-1} acting on $\mathcal{R}$ and the minimal polynomial of E_{n+1} acting on $\mathcal{I}$.

We consider a λ-*tableau* to be a bijective map $t : [\lambda] \to \widehat{n}$. The value of t at a node (r,c) is denoted by t_{rc}. The group Σ_n acts on the set of all λ-tableaux by functional composition; $(t\pi)_{rc} = t_{rc}\pi$, for each $\pi \in \Sigma_n$.

Suppose that λ has l nonzero parts $[\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l]$. We regard a λ-*tabloid* as an ordered partition $\mathcal{P} = (\mathcal{P}_1,\ldots,\mathcal{P}_l)$ of $\widehat{n}$ such that the cardinality of $\mathcal{P}_u$ is λ_u, for $u = 1,\ldots,l$. Each λ-tableau t determines the λ-tabloid $\{t\}$ whose u-th part is the set of entries in the u-th row of t. If s is a λ-tableau, then $\{t\} = \{s\}$ if and only if $s = t\pi$, for some π in the row stabilizer R_t of t. We denote the column stabilizer of t by C_t. We denote by M^λ the $F\Sigma_n$-module consisting of all formal F-linear combinations of λ-tabloids.

Adapting the notation of James [1], let $(r_1,c_1),\ldots,(r_m,c_m)$ be the removable nodes of $[\lambda]$, ordered so that $r_1 < \ldots < r_m$ and $c_1 > \ldots > c_m$. Set $r_0 = 0 = c_{m+1}$. The addable nodes of $[\lambda]$ are the $(m+1)$ nodes $(r_u+1, c_{u+1}+1)$, for $u = 0,\ldots,m$. We use $\lambda{\downarrow_u}$ to denote the partition of $n-1$ obtained by decrementing the r_u-th part of λ by 1, for $u \in \widehat{m}$. In addition, we use $\lambda{\uparrow^u}$ to denote the partition of $n+1$ obtained by incrementing the (r_u+1)-th part of λ by 1, for $u \in \widehat{m+1}$.

We need special notation for certain subsets of a λ-tableau t. For the rest of the paper, suppose that λ has parts of m different nonzero lengths. For any $u \in \widehat{m}$, let $H_u(t)$ be the set of entries in the union of the top r_u rows of t, and let $V_u(t)$ be the set of entries in the union of columns of t numbered from $c_{u+1}+1$ to c_u (inclusive). Clearly $H_1(t) \subset \ldots \subset H_m(t)$, while $V_m(t),\ldots,V_1(t)$ forms a partition of t. Also

$V_u(t) \subseteq H_v(t)$ if and only if $u \leq v$. As $H_u(t)$ depends only on the rows of t, we may define $H_u(\{t\}) := H(t)$.

By Theorem 9.3 in [1], $\mathcal{R}$ has a Specht series

$$0 \subset \mathcal{R}_1 \subset \mathcal{R}_2 \subset \ldots \subset \mathcal{R}_m = \mathcal{R},$$

with $\mathcal{R}_u/\mathcal{R}_{u-1} \cong S^{\lambda\downarrow_u}$, for $u \in \widehat{m}$. Also, by 17.14 in [1], $\mathcal{I}$ has a Specht series

$$\mathcal{I} = \mathcal{I}_1 \supset \mathcal{I}_2 \supset \ldots \supset \mathcal{I}_{m+1} \supset \mathcal{I}_{m+2} = 0,$$

with $\mathcal{I}_u/\mathcal{I}_{u+1} \cong S^{\lambda\uparrow^u}$, for $u \in \widehat{m+1}$. Each factor $\mathcal{I}/\mathcal{I}_{u+1}$ is isomorphic to a submodule of the permutation module $M^{\lambda\uparrow^u}$.

Lemma 2.1. *Suppose that the $F\Sigma_n$-module M has a Specht series $0 = M_0 \subset M_1 \subset \ldots \subset M_m = M$. Let $z \in Z(F\Sigma_n)$ and let $u \in \widehat{m}$. Then there is a scalar z_u in F such that the map $M_u/M_{u-1} \to M_u/M_{u-1}$ given by multiplication by z is equal to z_u times the identity map.*

Proof. If $\operatorname{char} F = 0$, then M_u/M_{u-1} is an irreducible $F\Sigma_n$-module (a Specht module), and the conclusion is obvious. If $\operatorname{char} F = p$ is positive, then M_u/M_{u-1} is the p-modular reduction of an irreducible module defined over a suitable discrete valuation ring of characteristic 0. The conclusion follows in this case from the characteristic zero case. □

This lemma allows us to give the following upper bound on the degrees of the minimal polynomials of E_{n-1} and E_{n+1}.

Corollary 2.2. *The minimal polynomial of E_{n-1} acting on $\mathcal{R}$ has degree at most m, while the minimal polynomial of E_{n+1} acting on $\mathcal{I}$ has degree at most $m+1$.*

Proof. Let $u \in \widehat{m}$. Lemma 2.1 shows that $\mathcal{R}_u(E_{n-1} - z_u) \subseteq \mathcal{R}_{u-1}$, for some scalar z_u. It follows from a simple inductive argument that $\mathcal{R}\prod_{u=1}^{m}(E_{n-1} - z_u) = 0$. A similar argument deals with the action of E_{n+1} on $\mathcal{I}$. □

It will turn out that the polynomials given in the proof of Corollary 2.2 are minimal. Before we prove this, we will identify the scalars z_u in terms of Young diagrams.

The residue of a node (r, c) is the scalar $(c - r)1_F$. We set $E(\lambda)$ as the sum of the residues of all nodes in $[\lambda]$. So $E(\lambda)$ is the 1-st elementary symmetric function in the residues. An easy calculation shows that $E(\lambda) = \sum_{i=1}^{l} \frac{1}{2}\lambda_i(\lambda_i + 1 - 2i)1_F$. The next lemma is a special case of a more general result proved by G. E. Murphy [3]: 1-st elementary symmetric function can be replaced by any symmetric function in n variables.

Lemma 2.3. *E_n acts as the scalar $E(\lambda)$ on S^λ.*

Proof. Let t be a λ-tableau, let $(r, c) \in [\lambda]$ and let $i = t_{rc}$. Fix $1 \leq c' < c$. Then by a simple Garnir relation (section 7 of [1]), $e_t \sum_j (i, j) = e_t$, where j runs over all entries in the c'-th column of t. Also $e_t(i, j) = -e_t$, for each entry j above i in column c of t. It follows that

$$e_t \sum_j (i, j) = (c - r)e_t,$$

where j runs over those elements of $\widehat{n}$ that lie in t in columns strictly left of i or in the same column as i but strictly above i. If we sum over all $(r, c) \in [\lambda]$, each transposition (i, j) occurs exactly once on the left hand side, while the coefficient of e_t on the right hand side is $E(\lambda)$. $\square$

If t is a λ-tableau, the polytabloid e_t is the following element of M^λ:

$$e_t := \sum_{\pi \in C_t} \operatorname{sgn} \pi \{t\pi\}.$$

It is well known that the polytabloids span the Specht module S^λ. James' description of $\mathcal{R}$, and the Garnir relations, show that e_t lies in $\mathcal{R}_u \backslash \mathcal{R}_{u-1}$ if $n \in V_u(t) \backslash H_{u-1}(t)$ (although we do not use this fact).

We next describe the induced module $\mathcal{I}$. Suppose that $u \in \widehat{m+1}$. Let T be a $\lambda{\uparrow}^u$-tableau, and let t denote the restriction of T to $[\lambda]$. Then the (λ, T)-polytabloid e_T^λ is the following element of $M^{\lambda{\uparrow}^u}$:

$$e_T^\lambda := \sum_{\pi \in C_t} \operatorname{sgn} \pi \{T\pi\}.$$

In Section 17 of [1], James has shown that when $u = m+1$, the corresponding (λ, T)-polytabloids span an $F\Sigma_{n+1}$-submodule of $M^{\lambda{\uparrow}^{m+1}}$, which is isomorphic to the induced module $\mathcal{I}$. We will always work with this copy of $\mathcal{I}$.

When we are showing that the polynomials given in the proof of 2.2 are minimal, it will be convenient to look at the action of the Murphy elements L_n and L_{n+1} rather than E_{n-1} and E_{n+1}. The following lemma provides a link between these actions. If t is a λ-tableau, its *extension to* $[\lambda{\uparrow}^{m+1}]$ is the $\lambda{\uparrow}^{m+1}$-tableau that is obtained from t by appending $n+1$ to the bottom of the first column.

Lemma 2.4. *Let t be a λ-tableau and let T be its extension to $[\lambda{\uparrow}^{m+1}]$. Suppose that $f(x) \in F[x]$. Then*

$$\begin{aligned} e_t\, f(E_{n-1}) &= e_t\, f(E(\lambda) - L_n); \\ e_T^\lambda\, f(E_{n+1}) &= e_T^\lambda\, f(E(\lambda) + L_{n+1}). \end{aligned}$$

Proof. Lemma 2.3 shows that E_n acts as the scalar $E(\lambda)$ on $\mathcal{R}$. The first statement then follows from $E_{n-1} = E_n - L_n$.

Consider the subspace V of $M^{\lambda{\uparrow}^{m+1}}$ spanned by all e_U^λ such that U is a $\lambda{\uparrow}^{m+1}$-tableau with $n+1$ in the unique entry of its last row. The subspace V is a direct summand of the restriction of $\mathcal{I}$ to Σ_n, and as an $F\Sigma_n$-submodule, V is clearly isomorphic to S^λ. Thus e_T^λ lies in a direct summand of the restriction of $\mathcal{I}$ to Σ_n that is isomorphic to S^λ. So Lemma 2.3 shows that $e_T^\lambda E_n = E(\lambda) e_T^\lambda$. The second statement now follows from $E_{n+1} = E_n + L_{n+1}$, and the fact that $E_n L_{n+1} = L_{n+1} E_n$. $\square$

When we are showing that the polynomials given in the proof of 2.2 are minimal, we will want to show that there is a λ-tableau t such that the set of vectors $\{e_t (L_n)^i \mid 0 \le i \le m-1\}$ is linearly independent. This will be accomplished using the following technical lemma concerning the action of L_n on $\mathcal{R}$.

Lemma 2.5. *Let t be a λ-tableau such that $n \in V_m(t) \backslash H_{m-1}(t)$. For each $u \in \widehat{m-1}$, choose $x_u \in V_u(t) \backslash H_{u-1}(t)$. Set $s = t\,(n, x_{m-1}, x_{m-2}, \ldots, x_1)$. Let i be a positive integer with $i \le m-1$. Then the coefficient of $\{s\}$ in the expansion of*

$e_t(L_n)^i$ into tabloids is

$$0, \quad \textit{when } 0 \le i \le m-2;$$
$$1, \quad \textit{when } i = m-1.$$

Proof. Clearly $(L_n)^i = \sum (w_i, n)(w_{i-1}, n) \dots (w_1, n)$, where $(w_1, \dots, w_i)$ ranges over all functions $\widehat{i} \to \widehat{n-1}$. Let $(y_1, \dots, y_i)$ be a function $\widehat{i} \to \widehat{n-1}$, let $\theta = (y_i, n)(y_{i-1}, n) \dots (y_1, n)$, and assume that $\{s\}$ appears with nonzero coefficient in the expansion of $e_t\theta$. We have two goals: (a) to show that $i = m-1$, and (b) to show that when $i = m-1$, the sequence $(y_1, \dots, y_{m-1})$ is equal to the sequence $(x_1, \dots, x_{m-1})$. The second part of the lemma follows easily from this second goal, as we now show. In the sum $\sum e_t(w_i n) \dots (w_1 n)$, $\{s\}$ can appear in only one term, namely $e_t(x_{m-1}, n) \dots (x_1, n)$. Since this term is equal to $e_t(n, x_{m-1}, x_{m-2}, \dots, x_1) = e_s$, $\{s\}$ appears with coefficient 1.

Since $e_t\theta = e_{t\theta}$, there exists π in the column stabilizer of $t\theta$ such that $\{s\} = \{t\,\theta\,\pi\}$. Let $u \in \widehat{m-1}$. Then by construction $x_u \in V_{u+1}(s) \backslash H_u(s)$; since $\{s\} = \{t\theta\pi\}$, it follows that $x_u \notin H_u(t\,\theta\,\pi)$. As π^{-1} is a column permutation of $t\theta$, we have $x_u \in V_{u+1}(t\theta) \cup \dots \cup V_m(t\theta)$. Thus

$$\text{(1)} \qquad \forall u \in \widehat{m-1}, \quad x_u\theta^{-1} \in V_{u+1}(t) \cup \dots \cup V_m(t).$$

In particular, θ does not fix any of the $m-1$ distinct symbols $x_1, \dots, x_{m-1} \in \widehat{n-1}$.

In this paragraph, we will show that θ does not fix n. Assume that θ does fix n. If the symbols in the list $y_1, \dots, y_i$ were distinct, θ would be the cycle $(y_i, y_{i-1}, \dots, y_1, n)$; since θ fixes n, it follows that there is some repetition in the list $y_1, \dots, y_i$. Since $\theta = (y_i, n)(y_{i-1}, n) \dots (y_1, n)$ and θ fixes n, the only symbols potentially moved by θ are on the list $y_1, \dots, y_i$. Since this list contains a repeat, θ moves at most $i-1$ symbols. The previous paragraph shows that θ moves at least $m-1$ symbols. Therefore $m \le i$. But by hypothesis $i \le m-1$. This contradiction shows that θ moves n.

We now know that θ moves all the m symbols in $\{x_1, \dots, x_{m-1}, n\}$. Since $\theta = (y_i, n)(y_{i-1}, n) \dots (y_1, n)$, θ can only move symbols on the list $y_1, y_2, \dots, y_i, n$. By hypothesis, $i \le m-1$. It follows that $i = m-1$, which is part (a) of our goal. It also follows that the sets $\{x_1, \dots, x_{m-1}\}$ and $\{y_1, \dots, y_{m-1}\}$ coincide and that the elements on the list $y_1, y_2, \dots, y_{m-1}$ are distinct. Hence θ is equal to the m-cycle $(y_{m-1}, y_{m-2}, \dots, y_1, n)$. In particular, $y_{m-1}\theta^{-1} = n$. From (1) applied with $u = m-1$, $x_{m-1}\theta^{-1} = n$. (This is because n is the only symbol moved by θ that is in $V_m(t)$.) Hence $y_{m-1} = x_{m-1}$. From this fact and (1) applied with $u = m-2$, it follows that $x_{m-2}\theta^{-1} = x_{m-1}$. Hence $y_{m-2} = x_{m-2}$. Continuing in this way, by reverse induction on u, it follows that for all $u \in \widehat{m-1}$, $y_u = x_u$. This gives goal (b) above, and completes the proof. □

The corresponding result for the action of L_{n+1} on $\mathcal{I}$ is:

Lemma 2.6. *Let t be a λ-tableau and let T be its extension to $[\lambda{\uparrow}^{m+1}]$. For each $u \in \widehat{m}$, choose $x_u \in V_u(t) \backslash H_{u-1}(t)$. Set $S = T\,(n+1, x_m, x_{m-1}, \dots, x_1)$. Let i be a positive integer with $i \le m$. Then the multiplicity of $\{S\}$ in the expansion of $e^{\lambda}_T(L_{n+1})^i$ into tabloids is*

$$0, \quad \textit{when } 0 \le i \le m-1;$$
$$1, \quad \textit{when } i = m.$$

Proof. Clearly we have $(L_{n+1})^i = \sum(w_i, n+1)(w_{i-1}, n+1)\ldots(w_1, n+1)$, where $(w_1, \ldots, w_i)$ ranges over all functions $\widehat{i} \to \widehat{n}$. Let $(y_1, \ldots, y_i)$ be a function $\widehat{i} \to \widehat{n}$, let $\theta = (y_i, n+1)(y_{i-1}, n+1)\ldots(y_1, n+1)$, and assume that $\{S\}$ appears with nonzero multiplicity in the expansion of $e_T^\lambda \theta$ as a linear combination of tabloids. Then there exists π in the column stabilizer of $t\theta$ such that $\{S\} = \{T\,\theta\,\pi\}$.

As π fixes the single entry in the last row of $T\theta$, and x_m occupies this node in S, it follows that $(n+1)\theta = x_m$. Let $u \in \widehat{m-1}$ and let s denote the restriction of S to λ. Then $x_u \in V_{u+1}(s)\backslash H_u(s)$, whence $x_u \notin H_u(t\,\theta\,\pi)$. As π^{-1} is a column permutation of $t\theta$, we have $x_u \in V_{u+1}(t\theta) \cup \ldots \cup V_m(t\theta)$. Thus

$$x_u\theta^{-1} \in V_{u+1}(t) \cup \ldots \cup V_m(t). \tag{2}$$

In particular, θ does not fix x_u.

From its definition, θ moves at most $i+1$ elements of $\widehat{n+1}$. But θ does not fix any of the $m+1$ distinct symbols $n+1, x_m, \ldots, x_1$, and $i \leq m$. So we must have $i = m$. Together with (2), this implies that $x_u\theta^{-1} \in \{x_{u+1}, \ldots, x_m\}$. Reverse induction on u shows that $x_u\theta^{-1} = x_{u+1}$. Thus θ coincides with the $(m+1)$-cycle $(n{+}1, x_m, x_{m-1}, \ldots, x_2, x_1)$. We conclude that $x_u = y_u$, for $u \in \widehat{m}$. This shows that θ occurs with multiplicity 1 in the expansion of $(L_{n+1})^m$ as a linear combination of group elements, whence $\{S\}$ appears with multiplicity 1 in the expansion of $e_T^\lambda(L_{n+1})^m$ as a linear combination of tabloids in $M^{\lambda\uparrow^{m+1}}$. □

We can now prove the main result of this section.

Theorem 2.7. *The minimal polynomial of E_{n-1} acting on $\mathcal{R}$ is*

$$\prod_{u=1}^{m}(x - E(\lambda{\downarrow_u})),$$

while the minimal polynomial of E_{n+1} acting on $\mathcal{I}$ is

$$\prod_{u=1}^{m+1}(x - E(\lambda{\uparrow^u})).$$

Proof. First, we will prove the result on $\mathcal{R}$. Let t be as in Lemma 2.5. Then Lemma 2.5 implies that the set of vectors $\{e_t(L_n)^i \mid 0 \leq i \leq m-1\}$ is linearly independent. It follows from Lemma 2.4 that the set $\{e_t(E_{n-1})^i \mid 0 \leq i \leq m-1\}$ is linearly independent. So the minimal polynomial of E_{n-1} has degree at least m. But Lemma 2.3 and the proof of Corollary 2.2 show that $\mathcal{R}\prod_{u=1}^{m}(E_{n-1} - E(\lambda{\downarrow_u})) = 0$.

The result on $\mathcal{I}$ follows from an identical argument using Lemma 2.6 in place of Lemma 2.5. □

3. The indecomposable components of the restriction and induction of a Specht module

The purpose of this section is to compute the indecomposable components of $\mathcal{R}$ and $\mathcal{I}$, when the characteristic of F is not 2. It is convenient to consider an $F\Sigma_n$-module M that shares the following properties in common with $\mathcal{R}$ and $\mathcal{I}$:

(1) M has a Specht series
$$0 = M_0 \subset M_1 \subset \ldots \subset M_m = M,$$
such that $M_u/M_{u-1} \cong S^{\lambda_u}$, where λ_u is a partition of n, for each $u \in \widehat{m}$.

(2) The labelling partitions satisfy $\lambda_1 \lhd \ldots \lhd \lambda_m$.

(3) There exists $z \in Z(F\Sigma_n)$ such that the minimal polynomial of z acting on M has degree m.

Looking at the proof of Corollary 2.2, we see that z has minimal polynomial $\prod_{u=1}^{m}(x - z_u)$, where z acts as the scalar z_u on the Specht factor M_u/M_{u-1}.

Lemma 3.1. *There exists $\tau \in M$ such that for all $u \in \widehat{m}$, $\tau \prod_{i=u+1}^{m}(z - z_i)$ lies in $M_u \backslash M_{u-1}$.*

Proof. The hypothesis on the degree of the minimal polynomial of z implies that there exists $\tau \in M$ such that τz^{m-1} does not lie in the span of the vectors $\{\tau, \tau z, \ldots, \tau z^{m-2}\}$. Set $\tau_u = \tau \prod_{i=u+1}^{m}(z - z_i)$. Repeated application of Lemma 2.1 shows that $\tau_u \in M_u$. We claim that $\tau_u \notin M_{u-1}$. Suppose otherwise. Then $\tau_u \prod_{i=1}^{u-1}(z - z_u) \subseteq M_{u-1} \prod_{i=1}^{u-1}(z - z_u) = 0$, again using Lemma 2.1. Thus $\tau \prod_{i=1, i \neq u}^{m}(z - z_i) = 0$. This contradicts our choice of τ. □

We now consider the endomorphism ring of a module that has properties (1) and (2) in common with M.

Lemma 3.2. *Suppose that* $\operatorname{char} F \neq 2$. *Let θ be a $F\Sigma_n$-endomorphism of M. Then*

1. *for all $u \in \widehat{m}$, $M_u\theta \subseteq M_u$;*
2. *for all $u \in \widehat{m}$, there is a well-defined Σ_n-endomorphism $\theta_u : M_u/M_{u-1} \to M_u/M_{u-1}$ given by $(v + M_{u-1})\theta_u = v\theta + M_{u-1}$;*
3. *the map $\Phi : \operatorname{End}_{F\Sigma_n}(M) \to \bigoplus_u \operatorname{End}_{F\Sigma_n}(M_u/M_{u-1})$ given by $(\theta)\Phi = (\theta_1, \ldots, \theta_m)$ is an algebra homomorphism;*
4. *the kernel of Φ is the Jacobson radical of* $\operatorname{End}_{F\Sigma_n}(M)$.

Proof. First, we prove (i). By induction, we may assume that $M_{u-1}\theta \subseteq M_{u-1}$. Suppose that $M_u\theta \not\subseteq M_u$. Choose v so that $m \geq v > u$ and v is maximal so that $M_u\theta \not\subseteq M_{v-1}$. Then $M_u\theta \subseteq M_v$, and applying θ to elements of M_u induces a well-defined nonzero Σ_n-homomorphism

$$M_u/M_{u-1} \to M_v/M_{u-1} \twoheadrightarrow M_v/M_{v-1}.$$

But $\lambda_u \lhd \lambda_v$. This, together with the fact that $\operatorname{char} F \neq 2$, contradicts 13.17 of [1], proving (i). Part (ii) follows easily from part (i).

It is immediate from the definition of θ_u that Φ is an algebra homomorphism. As $\operatorname{char} F \neq 2$, the only Σ_n-endomorphisms of M_u/M_{u-1} are scalar multiples of the identity, by 13.17 of [1]. It follows that the codomain of Φ is commutative and semisimple. Any element of the kernel must send M_u to M_{u-1} for all u; therefore the kernel is nilpotent. □

We now compute the indecomposable summands of M.

Proposition 3.3. *Assume that* $\operatorname{char} F \neq 2$. *Let B be a block idempotent of $F\Sigma_n$. Then the $F\Sigma_n$-module MB is 0 or indecomposable.*

Proof. Assume that $MB \neq 0$. Let A be the algebra $\operatorname{End}_{F\Sigma_n}(MB)$. Identify the algebra A in the natural way with a direct summand of the algebra $\operatorname{End}_{F\Sigma_n}(M)$. We will use the notation and results from Lemma 3.2 throughout this proof. Our goal is to show that $A/J(A)$ has dimension 1 over F.

Suppose then that $\theta \in A$. Let w be maximal such that the Specht module M_w/M_{w-1} belongs to B. Our task is to show that if $\theta_w = 0$, then $\theta_u = 0$ for all u such that M_u/M_{u-1} belongs to B. (The proposition follows easily from this. Let ϕ

be in A. Then there is a scalar c such that the map ϕ_w is c times the identity. Let $\theta = \phi - c1_A$. Then $\theta_w = 0$. Since θ_u is also 0 for all u with M_u/M_{u-1} belonging to B, it follows from the last part of Lemma 3.2 that $\theta \in J(A)$. Hence $A/J(A)$ has dimension 1.)

Now assume that $\theta_w = 0$, and let u be an integer such that M_u/M_{u-1} belongs to B. Let $\tau \in M$ be as in Lemma 3.1, set $\tau_u := \tau \prod_{i=u+1}^{m}(z - z_i)$, and set $\tau_w := \tau \prod_{i=w+1}^{m}(z - z_i)$. The lemma states that $\tau_u \in M_u \setminus M_{u-1}$ and $\tau_w \in M_w \setminus M_{w-1}$. Since $u \leq w$, we have

$$\begin{aligned}
\tau_u\theta &= \left(\tau_w \prod_{i=u+1}^{w}(z - z_i)\right)\theta \\
&= \tau_w\theta \prod_{i=u+1}^{w}(z - z_i), \quad \text{as } \theta \text{ is in } \mathrm{End}_{F\Sigma_n}(M), \\
&\in M_{w-1} \prod_{i=u+1}^{w}(z - z_i), \quad \text{as } \theta_w = 0 \text{ implies that } \tau_w\theta \in M_{w-1}, \\
&= \left(M_{w-1} \prod_{i=u+1}^{w-1}(z - z_i)\right)(z - z_w) \\
&\subseteq M_u(z - z_w), \quad \text{using Lemma 2.1 repeatedly.}
\end{aligned}$$

Now M_u/M_{u-1} and M_w/M_{w-1} both belong to B. So $z_u = z_w$, since both scalars are equal to the image of z under the central character of B. Lemma 2.1 and the last inclusion displayed above then show that $\tau_u\theta \in M_{u-1}$. But $\tau_u \notin M_{u-1}$, as proved in Lemma 3.1, and $\mathrm{End}_{F\Sigma_n}(M_u/M_{u-1})$ is one-dimensional, by 13.17 of [1]. We conclude that $\theta_u = 0$, as required. □

We have now done all the work to prove the main result of this paper.

Theorem 3.4. *Assume that* $\mathrm{char}\, F \neq 2$. *Let* b *be a block idempotent of* $F\Sigma_{n-1}$. *Then the* $F\Sigma_{n-1}$*-module* $(S^\lambda{\downarrow_{S_{n-1}}})\, b$ *is* 0 *or indecomposable. Let* B *be a block idempotent of* $F\Sigma_{n+1}$. *Then the* $F\Sigma_{n+1}$*-module* $(S^\lambda{\uparrow^{S_{n+1}}})\, B$ *is* 0 *or indecomposable.*

Proof. We know that $\mathcal{R}$ and $\mathcal{I}$ satisfy properties (1) and (2) of M. That they also satisfy property (3) is a consequence of Theorem 2.7. The result now follows from Proposition 3.3. □

We will finish by giving examples to show that the assumption $\mathrm{char}\, F \neq 2$ cannot be dropped in Theorem 3.4.

Assume that $\mathrm{char}\, F = 2$. Consider the Specht module $S^{(6,1,1,1)}$. The decomposition matrix for Σ_9 given in [1] shows that $S^{(8,1)}$ and $S^{(6,3)}$ are simple and that $S^{(6,1,1,1)}$ has a composition series with factors $S^{(8,1)}$ and $S^{(6,3)}$. By 23.8 in [1], $S^{(6,1,1,1)}$ is self-dual, so there is another composition series in which the factors appear in the other order. It follows that $S^{(6,1,1,1)} \cong S^{(8,1)} \oplus S^{(6,3)}$.

Now consider the restriction of $S^{(6,1,1,1)}$ to Σ_8. All components of the restriction belong to the principal 2-block of Σ_8, which is the block with empty core. Since $S^{(6,1,1,1)}$ is decomposable, so is its restriction to Σ_8.

For the other counterexample, let $M = S^{(6,1,1)}{\uparrow^{\Sigma_9}}$. The module M has a Specht series with factors $S^{(7,1,1)}$, $S^{(6,2,1)}$, and $S^{(6,1,1,1)}$. These factors belong to 2-blocks with cores (1), (1), and $(2,1)$ respectively. It follows that if B is the block idempotent corresponding to 2-core $(2,1)$, then $MB \cong S^{(6,1,1,1)}$; thus MB is decomposable.

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