Gain-Scheduled & Nonlinear Systems: Dynamic Analysis by Velocity-Based Linearisation Families

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Abstract

A family of velocity-based linearisations is proposed for a nonlinear system. In contrast to the conventional series expansion linearisation, a member of the family of velocity-based linearisations is valid in the vicinity of *any* operating point, not just an equilibrium operating point. The velocity-based linearisations facilitate dynamic analysis far from the equilibrium operating points and enable the transient behaviour of the nonlinear system to be investigated. Using velocity-based linearisations, stability conditions are derived for both smooth and non-smooth nonlinear systems which avoid the restrictions, to trajectories lying within an unnecessarily, perhaps excessively, small neighbourhood about the equilibrium operating points, inherent in existing frozen-input theory. For systems where there is no restriction on the rate of variation, the velocity-based linearisation analysis is *global* in nature. The analysis techniques developed, whilst quite general, are motivated by the gain-scheduling design approach and have the potential for direct application to the analysis of gain-scheduled systems.

1. Introduction

Whilst nonlinear dynamic systems are widespread, the analysis and design of such systems remains relatively difficult. In contrast, techniques for the analysis and design of linear time-invariant systems are rather better developed even though systems with genuinely linear time-invariant dynamics do not, in reality, exist. It is, therefore, attractive to adopt a divide and conquer strategy whereby the analysis/design of a nonlinear system is decomposed into the analysis/design of a family of linear time-invariant systems. This type of strategy forms the basis of one of the most widely, and successfully, applied techniques for the design of nonlinear controllers; namely, gain-scheduling.

Gain-scheduled controllers are linked by the design approach employed, whereby a nonlinear controller is constructed by interpolating, in some manner, between the members of a family of linear time-invariant controllers. In the conventional, and most common, gain-scheduling design approach (see, for example, Astrom & Wittenmark 1989, Hyde & Glover 1993), each linear controller is typically associated with a specific equilibrium operating point of the plant and is designed to ensure that, locally to the equilibrium operating point, the performance requirements are met. (The existence of a family of equilibrium operating points, which spans the envelope of plant operation, is central to most gain-scheduling arrangements and it is not sufficient to restrict consideration to a single, isolated, equilibrium operating point). By employing a first-order linear approximation which, locally to the equilibrium operating point, has similar dynamics to the plant, linear techniques may be applied to this local design task. However, whilst nonlinear controllers designed by this gain-scheduling approach are widely employed, the theoretical tools for the analysis and design of gain-scheduled controllers are rather poorly developed.

In this paper, the analysis of nonlinear dynamic systems in terms of associated velocity-based families of linear systems is investigated. Emphasis is placed on establishing a consistent, unified and conceptually clear framework for the local and non-local dynamic analysis of nonlinear systems. Whilst motivated by the gain-scheduling design methodology, the analysis is quite general. The paper is organised as follows. In section 2, the existing theory regarding the relationships between the dynamics of nonlinear systems and associated linear systems is reviewed. In section 3, velocity-based linearisation families are derived for a broad class of nonlinear systems and the ability of the former to approximate the latter is investigated. In section 4, the relationship between the stability properties of a nonlinear system and those of its associated family of velocity-based linearisations is investigated. The conclusions are summarised in section 5.

2. Extended Review

There exists a wide variety of long-standing theoretical results which, for a broad class of nonlinear systems, relate the dynamic characteristics of a member of the class to those of an associated family of linear systems. However, many of these results have been developed in specific contexts, often independently of one another. Moreover, despite this diversity, there is a notable absence from the literature of a formal survey which considers the relationships between these results. It is appropriate, therefore, to present in this section a somewhat extended review. In sections 2.1 and 2.2, the primary results from, respectively, series expansion linearisation theory and frozen-time theory are reviewed. The theory concerning the analysis of the dynamics relative to a family of equilibrium operating points is reviewed in section 2.3.

Before proceeding, the following two stability definitions are stated.

Definition *Exponential Stability* (see, for example, Khalil 1992 p168)

An unforced dynamic system,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t)$$

where $\mathbf{x} \in \Re^n$, is locally exponentially stable if there exist strictly positive constants γ, a and c such that

$$|\mathbf{x}(t)| \le \gamma e^{-a(t-t_o)} |\mathbf{x}(t_o)| \quad \forall t \ge t_o \ge 0,\ |\mathbf{x}(t_o)| < c$$

where $|\cdot|$ denotes an appropriate norm. The system is globally exponentially stable if this inequality is satisfied with c unbounded.

Definition *Bounded Input-Bounded Output (BIBO) Stability*

A forced dynamic system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r}, t), \qquad \mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{r}, t)$$

where $\mathbf{r} \in \Re^m$, $\mathbf{y} \in \Re^p$, $\mathbf{x} \in \Re^n$, is locally BIBO stable if there exist positive constants γ, c and d, with $\gamma<\infty$, such that, for $\mathbf{r}\in L_p^m$,

$$|\mathbf{y}|_p \le \gamma|\mathbf{r}|_p + \beta(|\mathbf{x}(t_o)|, t) \qquad \forall\ t \ge t_o > 0,\ |\mathbf{x}(t_o)| < c,\ |\mathbf{r}(t)| < d$$

where $p\in[1,\infty]$, $|\cdot|_p$ denotes the p-norm, L_p^m denotes the normed linear space of functions $\mathbf{r}:[0,\infty)\to\Re^m$ with finite p-norm and $\beta(|\mathbf{x}(t_o)|, t)$ is a class *KL* function (β is strictly increasing with respect to $|\mathbf{x}(t_o)|$ for each fixed t and zero when $|\mathbf{x}(t_o)|$ is zero, and β is strictly decreasing with respect to t for each fixed $|\mathbf{x}(t_o)|$ and $\beta\to 0$ as $t\to\infty$). The system is globally BIBO stable if this inequality is satisfied with c and d unbounded. This definition of BIBO stability is closely related to that of input-to-state stability (Sontag 1989) and differs slightly from other definitions of BIBO stability (Desoer & Vidyasagar 1975, Vidyasagar & Vannelli 1982) which require $\mathbf{x}(t_o)$ to be zero.

In the context of the present paper, all references to BIBO stability denote systems where β is of the exponential form $\gamma e^{-a(t-t_o)}|\mathbf{x}(t_o)|$, a>0. It is noted that such systems are exponentially stable when the input, **r**, is zero. Hence, this form of BIBO stability may be interpreted as a direct generalisation of exponential stability.

2.1 Series expansion linearisation theory

Consider the nonlinear system,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r}, t) \tag{1a}$$

$$\mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{r}, t) \tag{1b}$$

where $\mathbf{r} \in \Re^m$, $\mathbf{y} \in \Re^p$, $\mathbf{x} \in \Re^n$, $\mathbf{F}(\cdot,\cdot,\cdot)$ and $\mathbf{G}(\cdot,\cdot,\cdot)$ are differentiable with bounded, Lipschitz continuous derivatives. Let ($\tilde{\mathbf{x}}$(t), $\tilde{\mathbf{r}}$(t), $\tilde{\mathbf{y}}$(t)) denote a specific trajectory of the nonlinear system, (1); that is,

$$\dot{\tilde{\mathbf{x}}} = \mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}}, t), \qquad \tilde{\mathbf{y}} = \mathbf{G}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}}, t) \tag{2}$$

The trajectory, ($\tilde{\mathbf{x}}$(t), $\tilde{\mathbf{r}}$(t), $\tilde{\mathbf{y}}$(t)), could simply be an equilibrium operating point of (1), in which case $\mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}}, t)$ is identically zero and $\tilde{\mathbf{x}}$ is a constant. The nonlinear system, (1),can be reformulated, relative to the trajectory ($\tilde{\mathbf{x}}$(t), $\tilde{\mathbf{r}}$(t), $\tilde{\mathbf{y}}$(t)), as,

$$\delta\dot{\mathbf{x}} = \nabla_x\mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}}, t)\delta\mathbf{x} + \nabla_r\mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}}, t)\delta\mathbf{r} + \varepsilon_\mathbf{F} \tag{3a}$$

$$\delta\mathbf{y} = \nabla_x\mathbf{G}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}}, t)\delta\mathbf{x} + \nabla_r\mathbf{G}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}}, t)\delta\mathbf{r} + \varepsilon_\mathbf{G} \tag{3b}$$

$$\delta\mathbf{r} = \mathbf{r} - \tilde{\mathbf{r}}, \quad \mathbf{y} = \delta\mathbf{y} + \tilde{\mathbf{y}}, \quad \delta\mathbf{x} = \mathbf{x} - \tilde{\mathbf{x}} \tag{3c}$$

where,

$$\varepsilon_F = \mathbf{F}(\mathbf{x}, \mathbf{r}, t) - \mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}}, t) - \nabla_x\mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}}, t)\delta\mathbf{x} - \nabla_r\mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}}, t)\delta\mathbf{r} \qquad (4a)$$

$$\varepsilon_G = \mathbf{G}(\mathbf{x}, \mathbf{r}, t) - \mathbf{G}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}}, t) - \nabla_x\mathbf{G}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}}, t)\delta\mathbf{x} - \nabla_r\mathbf{G}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}}, t)\delta\mathbf{r} \qquad (4b)$$

The dynamics, (3a) and (3b), cannot be considered in isolation but must be considered together with the input, output and state transformations, (3c), in order to maintain the relationship between (3) and (1). However, the transformations, (3c), are fixed when consideration is confined to a specific trajectory, ($\tilde{\mathbf{x}}$(t), $\tilde{\mathbf{r}}$(t), $\tilde{\mathbf{y}}$(t)). Hence, the nonlinear system, (1), is stable provided the linear time-varying dynamics,

$$\delta\dot{\hat{\mathbf{x}}} = \nabla_x\mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}}, t)\delta\hat{\mathbf{x}} + \nabla_r\mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}}, t)\delta\mathbf{r} \qquad (5a)$$

$$\delta\hat{\mathbf{y}} = \nabla_x\mathbf{G}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}}, t)\delta\hat{\mathbf{x}} + \nabla_r\mathbf{G}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}}, t)\delta\mathbf{r} \qquad (5b)$$

are robustly stable with respect to the perturbation terms, ε_F and ε_G. Consequently, analysis of the nonlinear system, (1), may be reformulated as the analysis of the robust stability of the associated linear system, (5), which is simply the first-order Taylor series expansion of the nonlinear system, (1), relative to the trajectory, ($\tilde{\mathbf{x}}$(t), $\tilde{\mathbf{r}}$(t), $\tilde{\mathbf{y}}$(t)).

From Lyapunov theory, when $\delta\mathbf{r}$ is zero the internal nonlinear dynamics, (3a), are locally exponentially stable if and only if the unforced linear time-varying system,

$$\delta\dot{\hat{\mathbf{x}}} = \nabla_x\mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}}, t)\delta\hat{\mathbf{x}} \qquad (6)$$

is stable (see, for example, Khalil 1992 p184). When $\delta\mathbf{r}$ is non-zero, the nonlinear dynamics, (3), are locally BIBO stable provided (6) is stable, $\delta\mathbf{x}$ is initially zero, $\delta\mathbf{r}$ is sufficiently small and the derivatives $\nabla_r\mathbf{F}$, $\nabla_x\mathbf{G}$, $\nabla_r\mathbf{G}$ are uniformly bounded (Vidyasagar & Vannelli 1982, Vidyasagar 1993 section 6). Since this holds for all time, it is straightforward to show that the initial conditions need not be restricted to be zero and the result may be extended to encompass other initial conditions, provided that they are sufficiently close to the origin. Of course, for the special case when the system, (6), is in fact linear time-invariant, simple necessary and sufficient conditions for its stability are well-known (see, for example, Vidyasagar 1993). However, in the time-varying case, the stability analysis of (6) is, in general, not so straightforward.

In addition, the peak absolute difference between the solution, $\delta\hat{\mathbf{x}}$, of the approximate system, (5a), and the solution, $\delta\mathbf{x}$, of the nonlinear system, (3a), is bounded provided the approximate system, (5a), is stable, $\delta\mathbf{r}$ is sufficiently small and the initial conditions, $\delta\mathbf{x}(0)$ and $\delta\hat{\mathbf{x}}(0)$, are zero (Desoer & Wong, 1968, Desoer & Vidyasagar 1975 section 4.9). Once again, it is straightforward to extend this result to encompass non-zero initial conditions which are sufficiently close to the origin. Nonetheless, even with this extension, this result is quite weak and is, essentially, a restatement of local BIBO stability; that is, simply that the solutions of (3a) and (5a) both remain within a bounded region enclosing the origin provided the input and the initial conditions are sufficiently small. If the dynamics, (5a), are a genuine approximation to the nonlinear dynamics, (3a), then it might be expected that, when starting from the same initial conditions, the solutions of (3a) and (5a) remain correlated for some time; that is, the difference grows, in some sense, gradually over time. However, despite the fundamental nature, and considerable importance, of series expansion approximations, it is emphasised that there do not appear to be any published results regarding this anticipated stronger property.

2.2 Frozen-time theory

Whilst series expansion theory enables a relationship between the stability properties of the nonlinear system, (1), and the associated linear time-varying system, (6), to be established, frozen-time theory enables the stability of a broad class of the linear time-varying systems, (6), to be analysed in a relatively straightforward manner.

Consider the unforced linear time-varying system,

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} \qquad (7)$$

where $\mathbf{x} \in \Re^n$. Let the constant matrix, $\mathbf{A}_\tau$, denote the value of $\mathbf{A}$(t) at time, τ. Assume that $\mathbf{A}(\cdot)$ is bounded, differentiable and the eigenvalues of $\mathbf{A}_\tau$ lie in the left-half complex plane and are uniformly bounded away from the imaginary axis for every value of τ, then the linear time-varying system, (7), is globally exponentially stable provided $\sup_{t\geq 0} |\dot{\mathbf{A}}(t)|$ is sufficiently small (Desoer 1969). It should be noted that this result only establishes a sufficient condition for stability. The differentiability condition on $\mathbf{A}(\cdot)$ may be relaxed to a requirement for Lipschitz continuity and the restriction on $\sup_{t\geq 0} |\dot{\mathbf{A}}(t)|$ may be replaced by a restriction on the moving average, $\frac{1}{T}\int_t^{t+T}|\dot{\mathbf{A}}(s)|ds$ (see, for example, Ilchmann *et al.* 1987, Khalil 1992 section 4.5). Furthermore, the requirement for the continuity of $\mathbf{A}(t)$ may be relaxed provided that any discontinuities in $\mathbf{A}(t)$ occur sufficiently infrequently (Zhang 1993, Morse 1995 p97). However, referring back to section 2.1, it should be noted the available results relating the stability of the nonlinear system, (1), to that of the unforced system, (5), require Lipschitz continuity of $\nabla_x\mathbf{F}$.

The analysis is extended by Barman (1973) to unforced smoothly-nonlinear time-varying systems with a *single* equilibrium operating point (see also Desoer & Vidyasagar 1975 section 4.8, Vidyasagar 1993 section 5.8.2). By applying this result to the perturbations

$$\dot{\hat{\mathbf{x}}} = \mathbf{F}(\hat{\mathbf{x}}, \tau) \tag{8}$$

of the family of linear time invariant systems,

$$\dot{\mathbf{x}} = \mathbf{A}_\tau\mathbf{x} \tag{9}$$

where the frozen-time nonlinear systems, (8), are uniformly exponentially stable and $\mathbf{F}(0, \tau)$ is zero (so that the equilibrium point is uniformly the origin), it follows immediately that the time-varying perturbed system,

$$\dot{\hat{\mathbf{x}}} = \mathbf{F}(\hat{\mathbf{x}}, t) \tag{10}$$

is also exponentially stable provided the rate of time variation is sufficiently slow (in an appropriate sense). Consequently, provided the rate of variation is sufficiently slow, the linear time-varying system, (7), inherits the stability robustness of the family of linear time invariant systems, (8), to smooth nonlinear dynamic perturbations of arbitrary finite dimension which preserve the origin as the equilibrium point. In addition, it is shown by Shamma & Athans (1991) that, provided the rate of variation is sufficiently slow, the linear time-varying system, (7), inherits the stability robustness of the linear time-invariant systems, (8), to infinite dimensional linear time-invariant perturbations in the dynamics.

The foregoing results can be applied to the system, (6), which has the same form as (7). Provided $\nabla_r\mathbf{F}$, $\nabla_x\mathbf{G}$ and $\nabla_r\mathbf{G}$ are uniformly bounded, local exponential stability of the unforced system ensures local BIBO stability when the system is forced; that is, BIBO stability for $\delta\mathbf{r}$ and $\delta\mathbf{x}(0)$ sufficiently small (see, for example, Vidyasagar & Vannelli, 1982, Vidyasagar 1993 section 6). Consequently, provided the rate of variation is sufficiently slow, the linear time-varying system, (5), inherits certain stability properties of the members of the family of linear time-invariant systems,

$$\delta\dot{\hat{\mathbf{x}}} = \nabla_x\mathbf{F}(\tilde{\mathbf{x}}_\tau, \tilde{\mathbf{r}}_\tau, \tau)\delta\hat{\mathbf{x}} + \nabla_r\mathbf{F}(\tilde{\mathbf{x}}_\tau, \tilde{\mathbf{r}}_\tau, \tau)\delta\mathbf{r} \tag{11a}$$

$$\delta\hat{\mathbf{y}} = \nabla_x\mathbf{G}(\tilde{\mathbf{x}}_\tau, \tilde{\mathbf{r}}_\tau, \tau)\delta\hat{\mathbf{x}} + \nabla_r\mathbf{G}(\tilde{\mathbf{x}}_\tau, \tilde{\mathbf{r}}_\tau, \tau)\delta\mathbf{r} \tag{11b}$$

where $\tau \geq 0$ is a constant, $\tilde{\mathbf{x}}_\tau = \tilde{\mathbf{x}}(\tau)$, $\tilde{\mathbf{r}}_\tau = \tilde{\mathbf{r}}(\tau)$. The family, (11), consists of the so-called frozen-time linearisations of the linear time-varying dynamics, (5).

2.3 Frozen-input theory

A relationship between certain local stability properties of the nonlinear system, (1), and the stability properties of the associated family of linear time-invariant systems, (11), is established by the results of sections 2.1 and 2.2. However, the results are confined to the dynamic behaviour local to a single trajectory or equilibrium operating point, which is a significant limitation of the series expansion linearisation theory . In particular, within a gain-scheduling context, it is almost always required to consider the behaviour of a system relative to a family of operating points, which spans the envelope of operation, rather than relative to a single operating point. The existing theory regarding the behaviour relative to a family of equilibrium operating points, rather than just the behaviour relative to a single equilibrium operating point, stems primarily from an early lemma by Hoppensteadt (1966) originally derived in the context of singular perturbation theory. The conditions required for this result are satisfied by various combinations of assumptions; the statement of the result presented below is based on that of Khalil (1992, section 5.3).

Consider the smooth nonlinear system,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r}) \tag{12}$$

where $\mathbf{r} \in \Gamma \subset \Re^m$, $\mathbf{x} \in \Re^n$, $\mathbf{F}(\cdot,\cdot)$ is continuously differentiable on $\Gamma\times\Re^n$ and has a family of equilibrium operating points, $(\mathbf{x_o}, \mathbf{r_o})$, for which

$$\mathbf{x_o} = \mathbf{H}(\mathbf{r_o}), \qquad \mathbf{F}(\mathbf{H}(\mathbf{r_o}), \mathbf{r_o}) = 0 \qquad \forall\, \mathbf{r_o} \in \Gamma \tag{13}$$

Assume that there exists an open ball, X, about the origin in $\Re^n$, such that each member of the family of frozen-input nonlinear systems,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r_o}), \qquad \mathbf{r_o} \in \Gamma \tag{14}$$

is uniformly exponentially stable for initial conditions, $\mathbf{x}(0)$, which satisfy $\mathbf{x}(0)-\mathbf{H}(\mathbf{r_o}) \in X$. In addition, assume that $\mathbf{H}(\cdot)$ is differentiable with $\nabla\mathbf{H}$ uniformly bounded on Γ and that $\nabla_x\mathbf{F}(\mathbf{z}+\mathbf{H}(\mathbf{r}),\mathbf{r})$, $\nabla_r\mathbf{F}(\mathbf{z}+\mathbf{H}(\mathbf{r}),\mathbf{r})$ are uniformly bounded for all $\mathbf{r}\in\Gamma$, $\mathbf{z}\in X$. It follows that there exists a neighbourhood, $X_o \subset X$, such that the forced nonlinear system, (12), is locally BIBO stable for initial conditions satisfying $\mathbf{x}(0)-\mathbf{H}(\mathbf{r}(0)) \subset X_o$ and inputs, $\mathbf{r}(t)\in\Gamma$, provided $\mathbf{r}$ is differentiable and $\sup_{t\geq 0} |\dot{\mathbf{r}}|$ is sufficiently small (Khalil 1992, section 5.3). It should be noted that the restrictions on the rate of variation of the input and on the initial condition play two roles: firstly, they restrict the rate at which the frozen-input systems, (14), are traversed and, secondly, they ensure that the state trajectories remain uniformly within the neighbourhood, X.

Clearly, the foregoing stability analysis is not confined to the behaviour in the vicinity of a single trajectory or equilibrium operating point. However, the members of the family of frozen-input systems, (14), are *nonlinear*. From series expansion theory, there exists a neighbourhood, $\hat{X}(\mathbf{r_o})$, such that, for initial conditions satisfying $\mathbf{x}(0)-\mathbf{H}(\mathbf{r_o}) \in \hat{X}(\mathbf{r_o})$, the trajectories of a member of the family of frozen-input nonlinear systems, (14), are locally exponentially stable if and only if the corresponding series expansion linearisation,

$$\delta\dot{\hat{\mathbf{x}}} = \nabla_x\mathbf{F}(\mathbf{H}(\mathbf{r_o}), \mathbf{r_o})\delta\hat{\mathbf{x}} \qquad (15)$$

is stable. The neighbourhoods, $\hat{X}(\mathbf{r_o})$, may be different for each member of the family of frozen-input nonlinear systems. Select a region, X, which is sufficiently small that it is encompassed by every $\hat{X}(\mathbf{r})$; for example, let X be the intersection over all $\mathbf{r_o}\in\Gamma$ of the $\hat{X}(\mathbf{r_o})$. Some additional conditions are required to ensure that X contains an open ball about the origin and, thereby, establish a relationship between the local stability properties of the nonlinear system, (12), and the stability of the family of unforced linear time-invariant systems, (15) (Lawrence & Rugh 1990, Khalil & Kokotovic 1991). The additional conditions are relatively mild; for example, Γ is bounded and $\nabla_x\mathbf{F}(\mathbf{H}(\mathbf{r_o}), \mathbf{r_o})$ is differentiable for all $\mathbf{r_o}\in\Gamma$ (Lawrence & Rugh 1990), or the somewhat weaker requirement that $\nabla_x\mathbf{F}(\mathbf{H}(\mathbf{r_o}), \mathbf{r_o})$ is uniformly Lipschitz (Khalil 1992, section 4.5.1). Having imposed sufficient additional conditions that X contains an open ball, a relationship between certain local stability properties of the nonlinear system, (12), and the stability of the family of unforced linear time-invariant systems, (15), can be established (Lawrence & Rugh 1990, Khalil & Kokotovic 1991), for example, by directly applying the result of Hoppensteadt (1966). (Also, it is immediately evident that, provided the input varies sufficiently slowly and the initial conditions of the state are suitably restricted, the nonlinear system, (12), inherits the stability robustness of the frozen-input systems, (15), to smooth nonlinear dynamic perturbations of arbitrary finite dimension).

The foregoing stability result requires the family of equilibrium operating points to be parameterised by the system input, $\mathbf{r}$, but this is not unduly restrictive in an analysis context (although it is undesirable in the gain-scheduling design context, where it is more natural to parameterise the equilibrium operating points by the scheduling variable). However, the systems are also required to be smooth, thereby excluding discontinuous dynamics. In addition, conditions on the rate of variation of the input and on the initial condition are required in order to restrict the rate at which the frozen-input systems are traversed and also to ensure that the state trajectories remain uniformly within X. The latter constraint is required because the information about the plant dynamics employed in the analysis is derived from the family of series expansion linearisations, (15), relative to the equilibrium operating points. Furthermore, X can be no larger than the smallest of the neighbourhoods, $\hat{X}(\mathbf{r})$, within which the series expansion linearisations, (15), are valid. The analysis is, therefore, *inherently* confined to a small, perhaps excessively small, neighbourhood enclosing the equilibrium operating points and consequently is quite conservative.

Shamma & Athans (1990 section 4) apparently attempt to extend this type of analysis to a class of systems with a particular feedback structure and for which the family of equilibrium operating points is parameterised by the system output, $\mathbf{y}$, rather than the input, $\mathbf{r}$. The output, $\mathbf{y}$, and the system dynamics mutually interact with one another whereas the input, $\mathbf{r}$, is independent of the system dynamics. Consequently, the analysis of an output-scheduled nonlinear system is more difficult than the input-scheduled case. Shamma & Athans (1990) establish stability conditions requiring that $|\dot{\mathbf{y}}|$ is sufficiently small and, in addition, that the magnitude of the input, $\mathbf{r}$, and initial condition of a certain transformed state vector, ξ, are sufficiently small. The latter conditions confine the analysis, in general, to trajectories, $(\xi(t), \mathbf{r}(t))$ which lie within a small region enclosing the origin (see, for example, Shamma & Athans 1990, theorem 4.4). Since $\mathbf{y}$, which parameterises the equilibrium operating point, is a subset of the transformed states, ξ, it follows that $\mathbf{y}(t)$ is also confined to a region enclosing the origin. Hence, the analysis cannot be applied to an extended family of equilibrium operating points.

3. Velocity-based linearisation families

It is evident from the foregoing survey that the existing theory, relating the dynamic properties of a nonlinear system to those of an associated family of linear time-invariant systems, is rather poorly developed. Series expansion linearisation theory is well established but is strictly confined to the dynamic analysis, locally to a single trajectory or equilibrium operating point, of smooth nonlinear systems. Frozen-input techniques cater for the analysis of smooth nonlinear systems relative to a family of equilibrium operating points and relate the stability of a nonlinear system to the stability of a family of frozen-input nonlinear systems. A slow variation requirement is necessary which seems to be inherent to this type of analysis, implicitly restricting the class of allowable inputs and initial conditions; that is, implicitly restricting the trajectories to remain sufficiently close to the equilibrium operating points. In order to relate the stability of the nonlinear system to the properties of a family of *linear* time-invariant systems, a further explicit restriction on the allowable trajectories is necessary to ensure they remain sufficiently close to the equilibrium operating points that series expansion linearisations are valid. This latter restriction is not *a priori* necessary yet may be very strong since the neighbourhoods within which the series expansion linearisations are valid may, in general, be excessively small. The utility of frozen-input theory is, thus, somewhat diminished since it may imply a high degree of unnecessary conservativeness. Series expansion linearisation theory and frozen-input theory consider only the stability properties of the nonlinear system and provide little direct insight into other dynamic properties, such as the transient response. When the scheduling is not continuous, few techniques, other than extensive simulation testing, appear to be available for analysing the dynamic behaviour of the controlled system.

The requirement is to develop techniques for the dynamic analysis of nonlinear and gain-scheduled systems which address the main deficiencies of the existing theory. In particular, the conservativeness of the analysis techniques should be minimal. Although, a slow variation requirement of some sort seems inevitable, it should be as weak as possible; that is, there should be no unnecessary restriction to small neighbourhoods of the equilibrium operating points. In addition, whilst stability is essential, other dynamics properties are also usually important. Hence, the analysis should apply to other aspects of the dynamic behaviour, such as the transient response. Motivated by the requirement to accommodate gain-scheduled systems which switch between local controller designs, the analysis should not be confined to smooth nonlinear systems but should also cater for discontinuous nonlinear systems.

In this section, velocity-based linearisation families are proposed with the aim of developing an appropriate framework for addressing these issues. Nonlinear systems with dynamics,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r}), \qquad \mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{r}) \tag{16}$$

are considered, where $\mathbf{F}(\cdot,\cdot)$ and $\mathbf{G}(\cdot,\cdot)$ are continuous with Lipschitz continuous first derivatives, $\mathbf{r} \in \Re^m$ denotes the input to the system, $\mathbf{y} \in \Re^p$ the output and $\mathbf{x} \in \Re^n$ the states. The set of equilibrium operating points of the nonlinear system, (16), consists of those points, $(\mathbf{x_o}, \mathbf{y_o}, \mathbf{r_o})$, for which

$$\mathbf{F}(\mathbf{x_o}, \mathbf{r_o}) = 0, \qquad \mathbf{y_o} = \mathbf{G}(\mathbf{x_o}, \mathbf{r_o}) \tag{17}$$

Let $\Phi{:}\Re^n\times\Re^m$ denote the space consisting of the union of the states, $\mathbf{x}$, with the inputs, $\mathbf{r}$. Assume $[\nabla_\mathbf{r}\mathbf{F}\ \nabla_\mathbf{x}\mathbf{F}\nabla_\mathbf{r}\mathbf{F}\ \ldots(\nabla_\mathbf{x}\mathbf{F})^{n-1}\nabla_\mathbf{r}\mathbf{F}]$ is rank n $\forall\mathbf{x},\mathbf{r}$. The set of equilibrium operating points of the nonlinear system, (16), forms a locus of points, $(\mathbf{x_o}, \mathbf{r_o})$, in Φ and the response of the system to a general time-varying input, $\mathbf{r}(t)$, is depicted by a trajectory in Φ.

3.1 Approximation by first-order series expansion about an equilibrium operating point

When the dynamic behaviour of a nonlinear system is investigated by analysing an associated family of linear time-invariant systems, the family is most commonly chosen to consist of the first-order series expansions relative to the equilibrium operating points of the nonlinear system. In this section. the extent, to which the solutions of the nonlinear system are approximated by the solutions of the members of this family, is investigated.

3.1.1 Local to a single equilibrium operating point

Consider the situation when the solutions to the nonlinear system, (16), are restricted to the vicinity of a single equilibrium operating point, $(\mathbf{x_o}, \mathbf{r_o})$. Employing a standard series expansion approach relative to $(\mathbf{x_o}, \mathbf{r_o})$, the nonlinear system, (20), can be approximated by the linear dynamics

$$\delta\dot{\hat{\mathbf{x}}} = \nabla_x\mathbf{F}(\mathbf{x_o}, \mathbf{r_o})\,\delta\hat{\mathbf{x}} + \nabla_r\mathbf{F}(\mathbf{x_o}, \mathbf{r_o})\,\delta\mathbf{r} \tag{18a}$$

$$\delta\hat{\mathbf{y}} = \nabla_x\mathbf{G}(\mathbf{x_o}, \mathbf{r_o})\,\delta\hat{\mathbf{x}} + \nabla_r\mathbf{G}(\mathbf{x_o}, \mathbf{r_o})\,\delta\mathbf{r} \tag{18b}$$

together with the algebraic input, output and state transformation

$$\delta\mathbf{r} = \mathbf{r} - \mathbf{r_o}, \quad \hat{\mathbf{y}} = \mathbf{y_o} + \delta\hat{\mathbf{y}}, \quad \hat{\mathbf{x}} = \delta\hat{\mathbf{x}} + \mathbf{x}_o, \; \dot{\hat{\mathbf{x}}} = \delta\dot{\hat{\mathbf{x}}} \tag{18c}$$

provided $\mathbf{x_o}+\delta\mathbf{x} \subseteq N_\mathbf{x}$ $\mathbf{r_o}+\delta\mathbf{r} \subseteq N_\mathbf{r}$, where the neighbourhoods, $N_\mathbf{x}$ and $N_\mathbf{r}$, of, respectively, $\mathbf{x_o}$ and $\mathbf{r_o}$, are sufficiently small. Since the transformations, (18c), are fixed, the system dynamics are, locally, completely described by the linear time-invariant dynamics, (18a,b). As observed in section 2.1, even though the stability of the nonlinear system, (16), can be determined from the linear system, (18) ((16) being locally exponentially stable if and only if (18) is stable), no indication is given of the extent to which the solution, $\hat{\mathbf{x}}(t)$, of (18) approximate the solution, $\mathbf{x}(t)$, of (16).

Expanding, with respect to time, $\mathbf{x}(t)$ relative to an initial time, t_1,

$$\mathbf{x}(t) = \mathbf{x}(t_1) + \dot{\mathbf{x}}(t_1)\delta t + ½\,\ddot{\mathbf{x}}(t_1)\delta t^2 + \varepsilon_x, \; \delta t = t - t_1 \tag{19a}$$

where,

$$\varepsilon_x = \mathbf{x}(t) - \{\mathbf{x}(t_1) + \dot{\mathbf{x}}(t_1)t + ½\,\ddot{\mathbf{x}}(t_1)t^2\} \tag{19b}$$

$$\dot{\mathbf{x}}(t_1) = \mathbf{F}(\mathbf{x}(t_1), \mathbf{r}(t_1)) \tag{19c}$$

$$\ddot{\mathbf{x}}(t_1) = \nabla_x\mathbf{F}(\mathbf{x}(t_1), \mathbf{r}(t_1))\,\dot{\mathbf{x}}(t_1) + \nabla_r\mathbf{F}(\mathbf{x}(t_1), \mathbf{r}(t_1))\,\dot{\mathbf{r}}(t_1) \tag{19d}$$

It should be noted that the approximate series expansion, $\breve{\mathbf{x}}(t)$, obtained when ε_x is neglected in (19a), satisfies,

$$\breve{\mathbf{x}}(t_1) = \mathbf{x}(t_1), \; \dot{\breve{\mathbf{x}}}(t_1) = \dot{\mathbf{x}}(t_1), \; \ddot{\breve{\mathbf{x}}}(t_1) = \ddot{\mathbf{x}}(t_1) \tag{20}$$

and is, therefore, a second-order approximation to (19).

The solution, $\hat{\mathbf{x}}(t)$, to (18) may also be expanded with respect to time as,

$$\hat{\mathbf{x}}(t) = \hat{\mathbf{x}}(t_1) + \dot{\hat{\mathbf{x}}}(t_1)\delta t + ½\,\ddot{\hat{\mathbf{x}}}(t_1)\delta t^2 + \hat{\varepsilon}_x \tag{21a}$$

where,

$$\hat{\varepsilon}_x = \hat{\mathbf{x}}(t) - \{\hat{\mathbf{x}}(t_1) + \dot{\hat{\mathbf{x}}}(t_1)t + ½\,\ddot{\hat{\mathbf{x}}}(t_1)t^2\} \tag{21b}$$

When $\hat{\mathbf{x}}(t_1)$ equals $\mathbf{x}(t_1)$ so that both (16) and (18) have the same initial conditions, it is evident from (18) that,

$$\dot{\hat{\mathbf{x}}}(t_1) = \nabla_x\mathbf{F}(\mathbf{x_o}, \mathbf{r_o})\,(\mathbf{x}(t_1)-\mathbf{x_o}) + \nabla_r\mathbf{F}(\mathbf{x_o}, \mathbf{r_o})\,(\mathbf{r}(t_1)-\mathbf{r_o}) \tag{22a}$$

$$\ddot{\hat{\mathbf{x}}}(t_1) = \nabla_x\mathbf{F}(\mathbf{x_o}, \mathbf{r_o})\,\dot{\hat{\mathbf{x}}}(t_1) + \nabla_r\mathbf{F}(\mathbf{x_o}, \mathbf{r_o})\,\dot{\mathbf{r}}(t_1) \tag{22b}$$

Clearly, the derivatives, $\dot{\hat{\mathbf{x}}}(t_1)$ and $\ddot{\hat{\mathbf{x}}}(t_1)$, are not equal to $\dot{\mathbf{x}}(t_1)$ and $\ddot{\mathbf{x}}(t_1)$. Consequently, the solutions to (16) and (18) are not tangential at time zero. Indeed, when there is no restriction on $\dot{\mathbf{r}}$, the difference between $\ddot{\mathbf{x}}(t_1)$ and $\ddot{\hat{\mathbf{x}}}(t_1)$ may be unbounded. Therefore, $\hat{\mathbf{x}}(t)$ only provides a zeroeth-order approximation to $\mathbf{x}(t)$. The reason is that, in contrast to $\breve{\mathbf{x}}(t)$, the expansion is carried out relative to the equilibrium operating point, $(\mathbf{x_o}, \mathbf{r_o})$, rather than the actual initial condition of the system, $(\mathbf{x}(t_1), \mathbf{r}(t_1))$. Hence, whilst indicating stability, (18) provides, in general, a somewhat poor indication of the time response of (16).

This result is perhaps a little surprising since the series expansion linearisation relative to a single equilibrium operating point has been in widespread use by control engineers for many years and this body of experience indicates that it is of great utility. Of course, the approximation error depends on the strength of the system nonlinearity. When the nonlinearity is weak, the approximation error may be small even for system trajectories which do not remain particularly close to a specific equilibrium operating point. In this situation, the series expansion linearisation relative to an equilibrium operating point may be quite adequate. Indeed, since linear controller designs are often very successful, it follows that many systems are, in the foregoing sense, weakly nonlinear. When the nonlinearity is stronger, the approximation error also remains bounded and relatively small provided the system trajectories are for the most part confined to a sufficiently small neighbourhood about the specific equilibrium operating point, $(\mathbf{x_o}, \mathbf{r_o})$. Hence, despite its relatively poor approximation ability, the first order series expansion, (18), about a single equilibrium operating point is indeed of utility, particularly since it has the virtue of being linear.

3.1.2 Local to a family of equilibrium operating points

In the context of gain-scheduling it is *not* sufficient to consider the stability behaviour in the vicinity of a single equilibrium operating point. Instead, the input and initial condition are assumed to be restricted such that the solutions to (16) trace trajectories in Φ which remain within a neighbourhood about the locus of equilibrium operating points but are not confined to a neighbourhood about a single equilibrium point; that is, the solution, $\mathbf{x}(t)$, to the nonlinear system moves from the vicinity of one equilibrium operating point to the vicinity of another as time evolves.

Since the solution, $\mathbf{x}(t)$, to the nonlinear system does not stay in the vicinity of a single equilibrium operating point, it is necessary to consider a *family* of associated first-order expansions relative to the equilibrium operating points. By combining the solutions to the members of this family in an appropriate manner, an approximation to $\mathbf{x}(t)$ can be obtained that does not involve solving the nonlinear dynamics. Over a time interval, $[t_1,t_2]$, an approximation is obtained by partitioning the interval into a number of short sub-intervals. Over each sub-interval, the approximate solution is just the local solution of the series expansion linearisation corresponding to a nearby equilibrium operating point (with the initial conditions chosen to ensure continuity of the approximate solution). As the durations of the sub-intervals are reduced, a greater number of local solutions are used and it might be expected that, as the sub-intervals become smaller and the number of local solutions employed increases, the resulting piece-wise continuous approximation might converge to the exact solution, $\mathbf{x}(t)$, of the nonlinear system. However, from the analysis of section 3.1.1, the local solutions are only accurate to zeroeth order; that is, the approximation error over each sub-interval is proportional to its duration. Hence, although the approximation error for each local solution decreases as the duration of the sub-intervals is decreased, it is counter-balanced by the corresponding increase in the number of local approximate solutions employed and the overall approximation error need not decrease (see Appendix A). In contrast to the situation considered in section 3.1.1, both the exact solution to the nonlinear system and the approximate solution are not confined to a small neighbourhood about a single equilibrium operating point. Hence, a large difference between them can develop and the solutions to the family of first-order series expansions relative to the equilibrium operating points are, therefore, a poor approximation to the solution to the nonlinear system (16).

Moreover, the input, output and state transformations, (18c), vary with the equilibrium operating point. Hence, when the solution to (16) traces a trajectory which is not confined to a neighbourhood about a *single* equilibrium operating point, the relationship between the non-local dynamics of the nonlinear system, (16), and the local linear dynamics, (18a,b), is, in fact, no longer straightforward. Indeed, the first-order series expansion systems, (18), may be reformulated as,

$$\dot{\mathbf{x}} = -\{\nabla_x\mathbf{F}(\mathbf{x_o}, \mathbf{r_o})\mathbf{x_o} + \nabla_r\mathbf{F}(\mathbf{x_o}, \mathbf{r_o})\mathbf{r_o}\} + \nabla_x\mathbf{F}(\mathbf{x_o}, \mathbf{r_o})\,\mathbf{x} + \nabla_r\mathbf{F}(\mathbf{x_o}, \mathbf{r_o})\,\mathbf{r} \tag{23a}$$

$$\mathbf{y} = -\{\nabla_x\mathbf{G}(\mathbf{x_o}, \mathbf{r_o})\mathbf{x_o} + \nabla_r\mathbf{G}(\mathbf{x_o}, \mathbf{r_o})\mathbf{r_o}\} + \nabla_x\mathbf{G}(\mathbf{x_o}, \mathbf{r_o})\,\mathbf{x} + \nabla_r\mathbf{G}(\mathbf{x_o}, \mathbf{r_o})\,\mathbf{r} \tag{23b}$$

Each member of the family of first-order representations, (23), now has the same input, output and state at every equilibrium operating point and the input, output and state transformations are no longer required. However, they are no longer linear owing to the presence of an inhomogeneous term. The main advantage of the first-order series expansions relative to the equilibrium operating points, namely the linearisation of the nonlinear dynamics, is, therefore, lost.

3.2 First-order series expansion about a general operating point

When the solutions to the nonlinear system are not confined to the vicinity of a single equilibrium operating point (such as in gain-scheduling applications), it is evident that the family of first-order series expansions relative to the equilibrium operating points, (18), or equivalently, (23), offers neither an accurate approximation to the solution of the nonlinear system nor the benefits of linearity. An alternative approach to local representation of the nonlinear system is, therefore, required.

Consider the behaviour of the nonlinear system, (16), when there are no restrictions on the class of allowable inputs and initial conditions. The solutions to (16) may trace trajectories anywhere in Φ and are not confined to the vicinity of either a single equilibrium operating point or the locus of equilibrium operating points. Suppose that the nonlinear system is evolving along a trajectory, $(\mathbf{x}(t), \mathbf{r}(t))$, in Φ and at time, t_1, the trajectory has reached the point, $(\mathbf{x_1}, \mathbf{r_1})$. It is emphasised that the point, $(\mathbf{x_1}, \mathbf{r_1})$, need not be an equilibrium operating point and, indeed, may lie far from the locus of equilibrium operating points. From Taylor series expansion theory, the subsequent behaviour of the nonlinear system, (16), can be approximated, locally to $(\mathbf{x_1}, \mathbf{r_1})$, by the first order representation,

$$\delta\dot{\hat{\mathbf{x}}} = \mathbf{F}(\mathbf{x_1}, \mathbf{r_1}) + \nabla_x\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\,\delta\hat{\mathbf{x}} + \nabla_r\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\,\delta\mathbf{r} \tag{24a}$$

$$\delta\hat{\mathbf{y}} = \nabla_x\mathbf{G}(\mathbf{x_1}, \mathbf{r_1})\,\delta\hat{\mathbf{x}} + \nabla_r\mathbf{G}(\mathbf{x_1}, \mathbf{r_1})\,\delta\mathbf{r} \tag{24b}$$

$$\delta\mathbf{r} = \mathbf{r} - \mathbf{r_1}, \quad \hat{\mathbf{y}} = \mathbf{y_1} + \delta\hat{\mathbf{y}}, \quad \hat{\mathbf{x}} = \delta\hat{\mathbf{x}} + \mathbf{x_1}, \; \dot{\hat{\mathbf{x}}} = \delta\dot{\hat{\mathbf{x}}} \tag{24c}$$

provided $\mathbf{x_1}+\delta\hat{\mathbf{x}} \subseteq N_x$ $\mathbf{r_1}+\delta\mathbf{r}\subseteq N_r$, where the neighbourhoods, N_x and N_r, of, respectively, $\mathbf{x_1}$ and $\mathbf{r_1}$ are sufficiently small.

When (24) and (16) have the same initial conditions, ($\mathbf{x_1}$, $\mathbf{r_1}$), the solution to (24) satisfies,

$$\dot{\hat{\mathbf{x}}}(t_1) = \mathbf{F}(\mathbf{x_1}, \mathbf{r_1}) = \dot{\mathbf{x}}(t_1) \quad (25a)$$

$$\ddot{\hat{\mathbf{x}}}(t_1) = \nabla_x\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\,\dot{\mathbf{x}}(t_1) + \nabla_r\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\,\dot{\mathbf{r}}(t_1) = \ddot{\mathbf{x}}(t_1) \quad (25b)$$

Hence, the solution to (24) is, initially, tangential to the solution of (16) and, indeed, locally to time t_1, provides a first-order approximation to $\dot{\mathbf{x}}(t)$ and a second-order approximation to $\mathbf{x}(t)$. The reason is that, in contrast to the series expansion linearisation, (18), the series expansion, (24), is performed relative to the actual initial operating point, ($\mathbf{x_1}$, $\mathbf{r_1}$), rather than an adjacent equilibrium operating point.

The solution to the first-order series expansion, (24), provides a valid approximation only while the solution, $\mathbf{x}(t)$, to the nonlinear system remains in the vicinity the operating point, ($\mathbf{x_1}$, $\mathbf{r_1}$). However, the solution, $\mathbf{x}(t)$, to the nonlinear system need not stay in the vicinity of a single operating point and so it is necessary to consider a *family* of first-order series expansions relative to *all* operating points. Following a similar approach to that described in section 3.1.2, consider an approximation to $\mathbf{x}(t)$ over a time interval, $[t_1,t_2]$, obtained by partitioning the interval into a number of short sub-intervals. Over each sub-interval, the approximate solution is the solution to the first-order series expansion relative to the operating point reached at the initial time for the sub-interval (with the initial conditions chosen to ensure continuity of the approximate solution). As before, the number of local solutions employed is dependent on the duration of the sub-intervals. However, the local solutions are now accurate to second order; that is, the approximation error is proportional to the duration of the sub-interval cubed. Hence, as the number of sub-intervals increases, the approximation error associated with each rapidly decreases and the overall approximation error reduces. Indeed, the overall approximation error tends to zero as the number of sub-intervals becomes unbounded (see Appendix A). Hence, in contrast to the series expansion relative to an equilibrium operating point, the first-order series expansion, (24), can provide an accurate approximation to the solution of the nonlinear system. Moreover, this approximation property holds throughout $\mathbf{\Phi}$ and is not confined to the vicinity of a single equilibrium operating point or even of the locus of equilibrium operating points. Provided some care is taken, in many circumstances the potential clearly exists with the family of first-order series expansions, (24), (but not with the family of series expansion linearisations, (18)) to infer the transient response of the nonlinear system, (16), from the responses to a few members, sometimes one member, of the family.

Combining (24a,b) with the local input, output and state transformations, (24c), each member of the family of first-order representations, (24), may be reformulated as,

$$\dot{\hat{\mathbf{x}}} = \{\mathbf{F}(\mathbf{x_1}, \mathbf{r_1}) - \nabla_x\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\,\mathbf{x_1} - \nabla_r\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\,\mathbf{r_1}\} + \nabla_x\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\,\hat{\mathbf{x}} + \nabla_r\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\,\mathbf{r} \quad (26a)$$

$$\hat{\mathbf{y}} = \{\mathbf{G}(\mathbf{x_1}, \mathbf{r_1}) - \nabla_x\mathbf{G}(\mathbf{x_1}, \mathbf{r_1})\,\mathbf{x_1} + \nabla_r\mathbf{G}(\mathbf{x_1}, \mathbf{r_1})\mathbf{r_1}\} + \nabla_x\mathbf{G}(\mathbf{x_1}, \mathbf{r_1})\,\hat{\mathbf{x}} + \nabla_r\mathbf{G}(\mathbf{x_1}, \mathbf{r_1})\,\mathbf{r} \quad (26b)$$

where the state, input and output is the same at every point in Φ. It is evident that (26) subsumes the family of systems, (23), and, also, (18). Whilst the first-order series expansions, (26), are a better approximation than (23), their degree of nonlinearity is no greater.

3.3 Velocity-based linearisation

The first limitation of first-order series expansions relative to the equilibrium operating points, specifically the inability to provide an accurate local approximation to the solution of the nonlinear system, (16), is overcome by the first-order series expansions, (26).. However, the second limitation, namely its lack of linearity, is not. This difficulty may be resolved by appropriate transformation of the system. By differentiating, (26) may be reformulated in the equivalent velocity-based form,

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{w}} \quad (27a)$$

$$\dot{\hat{\mathbf{w}}} = \nabla_x\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\,\hat{\mathbf{w}} + \nabla_r\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\,\dot{\mathbf{r}} \quad (27b)$$

$$\dot{\hat{\mathbf{y}}} = \nabla_x\mathbf{G}(\mathbf{x_1}, \mathbf{r_1})\,\hat{\mathbf{w}} + \nabla_r\mathbf{G}(\mathbf{x_1}, \mathbf{r_1})\,\dot{\mathbf{r}} \quad (27c)$$

With appropriate initial conditions, namely,

$$\hat{\mathbf{x}}(t_1) = \mathbf{x}(t_1),\ \hat{\mathbf{w}}(t_1) = \dot{\hat{\mathbf{x}}}(t_1) = \dot{\mathbf{x}}(t_1) = \mathbf{F}(\mathbf{x_1}, \mathbf{r_1}),\ \hat{\mathbf{y}}(t_1) = \mathbf{y}(t_1) = \mathbf{G}(\mathbf{x_1}, \mathbf{r_1}) \quad (27d)$$

the transformed system, (27), is dynamically equivalent to the original system, (26). In contrast to (26), the transformed system, (27), is linear. Associated with every point in $\mathbf{\Phi}$ is a velocity-based linearisation, (27). Hence, a velocity-based linearisation family, with members defined by (27), can be associated with the nonlinear system, (16).

Differentiating (20), an alternative representation of the nonlinear system is

$$\dot{\mathbf{x}} = \mathbf{w} \quad (28a)$$

$$\dot{\mathbf{w}} = \nabla_x\mathbf{F}(\mathbf{x}, \mathbf{r})\mathbf{w} + \nabla_r\mathbf{F}(\mathbf{x}, \mathbf{r})\,\dot{\mathbf{r}} \tag{28b}$$

$$\dot{\mathbf{y}} = \nabla_x\mathbf{G}(\mathbf{x}, \mathbf{r})\mathbf{w} + \nabla_r\mathbf{G}(\mathbf{x}, \mathbf{r})\,\dot{\mathbf{r}} \tag{28c}$$

Dynamically, (28), with appropriate initial conditions corresponding to (27d), and (16) are equivalent. It should be noted that the transformation relating the system, (28), to the system, (16), maps the locus of equilibrium points, $(\mathbf{x_o}, \mathbf{r_o})$ onto the origin, $\mathbf{w} = 0 = \dot{\mathbf{r}}$. The relationship between (28) and the members, (27), of the associated velocity-based linearisation family is direct; indeed, (27) is simply the frozen form of (28) at the operating point, $(\mathbf{x_1}, \mathbf{r_1})$. (When $\mathbf{w} = \mathbf{F}(\mathbf{x}, \mathbf{r})$, $\mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{r})$ is invertible at every operating point, $(\mathbf{x}, \mathbf{r})$, in an appropriate neighbourhood enclosing the locus of equilibrium operating points, so that $\mathbf{x}$ may be expressed as a function of $\mathbf{w}$, $\mathbf{r}$ and $\mathbf{y}$, then the transformation relating (28) to (16) is, in fact, algebraic). Similarly to the discussion in section 3.2, the solutions to the members of the family of velocity-based linearisations, (27), can be pieced together to approximate the solution to the nonlinear system, (28). In this case, the $\hat{\mathbf{x}}(t)$ are still second-order approximations to the $\mathbf{x}(t)$ but the $\hat{\mathbf{w}}(t)$ are first-order approximations to the $\mathbf{w}(t)$. However, with minor amendments to the analysis of Appendix A, it is straightforward to show that the piece-wise approximation converges to the exact solution.

The members of the family of conventional series expansion linearisations, (18), are individually only valid in the vicinity of an equilibrium operating point. In contrast, a member of the family of velocity-based linearisations, (27), is valid in the vicinity of *any* operating point. Moreover, the time-evolution of the solution of the nonlinear system is indicated by the solution to the velocity-based linearisations. Hence, by means of linearisation at any operating point, the family of velocity-based linearisations, in addition to facilitating non-local dynamic analysis far from the equilibrium operating points, enables the transient behaviour to be investigated .

4. Stability analysis of gain-scheduled & nonlinear systems

In the previous section, the ability of the velocity-based linearisations to provide an indication of the transient behaviour of a nonlinear system is investigated. However, the relationship of the stability of the nonlinear system, (16), to the family of velocity-based linearisations, (27), is yet to be established. In particular, it is required to develop stability results which are not unnecessarily conservative; that is, which do not restrict the trajectories to an unnecessarily, perhaps excessively, small neighbourhood about the locus of equilibrium operating points. Since the velocity-based linearisation is valid at any operating point, not just equilibrium operating points, it might be expected to be of assistance in achieving this objective. In this section, the relationship between the stability properties of the nonlinear system and those of its associated family of velocity-based linearisations is investigated.

The velocity states, $\mathbf{w}$, are related to the states, $\mathbf{x}$, by the nonlinear function, $\mathbf{F}:(\mathbf{x}, \mathbf{r}) \mapsto \mathbf{w}$. It is assumed that the inverse mapping from $\mathbf{w}$ to $(\mathbf{x}, \mathbf{r})$ is bounded; that is, $\mathbf{x}$ is bounded when $\mathbf{w}$ and $\mathbf{r}$ are bounded. Provided $|\mathbf{w}(t)|$ is sufficiently small, it follows that the states, $\mathbf{x}(t)$, remain close to the locus of equilibrium operating points and stability of the full nonlinear system, (28), is guaranteed by stability of the internal dynamics, (28b). It is, therefore, sufficient to confine consideration to the behaviour of the dynamics, (27) and (28b). The assumption, that the inverse mapping is bounded, is quite weak, particularly in a gain-scheduling context. For example, when $\nabla_x\mathbf{F}(\mathbf{x}, \mathbf{r})$ exists and is uniformly invertible, then by the mean value theorem (see, for example, Khalil 1992),

$$|\mathbf{F}(\mathbf{x}, \mathbf{r}) - \mathbf{F}(\mathbf{x_1}, \mathbf{r})|_2 = |\nabla_x\mathbf{F}(\mathbf{z}, \mathbf{r})(\mathbf{x}-\mathbf{x_1})|_2 \geq M|\mathbf{x}-\mathbf{x_1}|_2 \tag{29a}$$

where $\mathbf{z}$ lies on the line segment joining $\mathbf{x}$ and $\mathbf{x_1}$ and M, the minimum singular value of $\nabla_x\mathbf{F}(\mathbf{z}, \mathbf{r})$, satisfies

$$M > 0 \tag{29b}$$

Hence,

$$|\mathbf{x}(t) - \mathbf{x_o}(\mathbf{r}(t))|_2 \leq |\mathbf{w}(t)|_2/M \tag{30}$$

where $\mathbf{x_o}(\mathbf{r}(t))$ is the state associated with the equilibrium operating point at which the input equals $\mathbf{r}(t)$.

As noted in section 3.3, the transformation relating the system, (28), to the system, (16), maps the locus of equilibrium points, $(\mathbf{x_o}, \mathbf{r_o})$ onto the origin, $\mathbf{w} = 0 = \dot{\mathbf{r}}$; that is, onto a single equilibrium operating point in the transformed co-ordinates. Moreover, the velocity-based linearisation is simply the frozen form of the nonlinear system. Hence, the analysis framework reduces to a form which is similar to that employed in conventional frozen-time/frozen-input theory (section 2). Consequently, it might be expected that, by employing the velocity transformation, frozen-time/frozen-input theory, albeit appropriately modified, might be extended to resolve the conservativeness of conventional theory. As expected, the Lyapunov-based analysis employed in frozen-time/frozen-input theory can, indeed, be extended in this manner (Appendix B). A slow variation condition is required which has the form of a restriction on the initial velocity conditions, $\mathbf{w}(0)$, and rate of variation, $\dot{\mathbf{r}}(t)$, of the inputs. It is clear from (28) that this condition also implicitly restricts the rate of variation of the state, $\mathbf{x}(t)$, and input, $\mathbf{r}(t)$. Equivalently, the condition ensures that the state trajectories are confined to an appropriate region enclosing the locus of equilibrium operating points. However, in contrast to the results discussed in section 2.3, this restriction is purely a consequence of the slow variation requirement, with no additional requirement to constrain the trajectories to be sufficiently close to the equilibrium operating points that series expansion linearisations relative to them are valid. In this sense, it is as weak as possible. Indeed, when, for example, a single Lyapunov function exists which is common to every member of the velocity-based linearisation family, there is no restriction on the rate of variation of the system. In this case, the analysis is *global* in nature and indicates that the nonlinear system is stable for any input and initial condition.

However, the Lyapunov-based analysis requires that the nonlinear system is smooth and therefore excludes, for example, gain-scheduled systems which switch discontinuously between local controller designs. Moreover, the stability analysis approach is conceptually quite different from the piecewise-approximation approach utilised transient analysis in section 3. To resolve these issues, it is necessary to adopt an alternative stability analysis approach; in particular, an approach which, in philosophy, is similar to the piecewise approximation methods of section 3 is attractive

4.1 Approximation over an interval

In section 3, a piecewise temporal approximation by the family of velocity-based linearisations, (27), of the nonlinear system, (16), is investigated. However, this approach is not appropriate for analysis of stability properties. Instead, a spatial piecewise approximation is required. Consider the nonlinear dynamics, (28b), and assume that the initial conditions are,

$$\mathbf{x}(t_1) = \mathbf{x_1}, \quad \mathbf{r}(t_1) = \mathbf{r_1}, \quad \mathbf{w}(t_1) = \mathbf{w_1} = \mathbf{F}(\mathbf{x_1}, \mathbf{r_1}) \tag{31}$$

The corresponding velocity-based linearisation is (27b) and the initial conditions are,

$$\hat{\mathbf{x}}(t_1) = \mathbf{x_1}, \quad \hat{\mathbf{w}}(t_1) = \mathbf{w_1} \tag{32}$$

Let α_i denote a positive finite constant and $|\bullet|_p$ denote both the p-norm, p=1,2,…∞, and, where appropriate, the induced p-norm. The p subscript is dropped when any p-norm may be employed. In addition, let $\|\bullet\|_T$ denote $\sup_{t\in[t_1,t_1+T)} |\bullet|$.

The solution to the velocity-based linearisation, (27b), may be explicitly written as

$$\hat{\mathbf{w}}(t) = e^{\nabla_x \mathbf{F}(\mathbf{x_1},\mathbf{r_1})(t-t_1)}\hat{\mathbf{w}}(t_1) + \int_{t_1}^{t} e^{\nabla_x \mathbf{F}(\mathbf{x_1},\mathbf{r_1})(t-s)}\nabla_{\mathbf{r}}\mathbf{F}(\mathbf{x_1},\mathbf{r_1})\dot{\mathbf{r}}(s)ds \tag{33}$$

The nonlinear dynamics, (28b), may be reformulated as the perturbed linear dynamics,

$$\dot{\mathbf{w}} = \nabla_{\mathbf{x}}\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\mathbf{w} + \nabla_{\mathbf{r}}\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\,\dot{\mathbf{r}} + \boldsymbol{\varepsilon}_{\mathbf{F}} \tag{34a}$$

where,

$$\boldsymbol{\varepsilon}_{\mathbf{F}} = \{\nabla_{\mathbf{x}}\mathbf{F}(\mathbf{x}, \mathbf{r}) - \nabla_{\mathbf{x}}\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\}\mathbf{w} + \{\nabla_{\mathbf{r}}\mathbf{F}(\mathbf{x}, \mathbf{r}) - \nabla_{\mathbf{r}}\mathbf{F}(\mathbf{x_1}, \mathbf{r_1})\}\,\dot{\mathbf{r}} \tag{34b}$$

It follows that the solution to the nonlinear dynamics may be expressed as,

$$\mathbf{w}(t) = e^{\nabla_x \mathbf{F}(\mathbf{x_1},\mathbf{r_1})(t-t_1)}\mathbf{w}(t_1) + \int_{t_1}^{t} e^{\nabla_x \mathbf{F}(\mathbf{x_1},\mathbf{r_1})(t-s)}\{\nabla_r \mathbf{F}(\mathbf{x_1},\mathbf{r_1})\dot{\mathbf{r}}(s) + \boldsymbol{\varepsilon}_{\mathbf{F}}(s)\}ds \tag{35}$$

However, owing to the perturbation, $\boldsymbol{\varepsilon}_{\mathbf{F}}$, the solution to the nonlinear dynamics might exhibit characteristics which are quite different from the solution, (33); for example, $\boldsymbol{\varepsilon}_{\mathbf{F}}$ might be unbounded, in which case the solution to the nonlinear dynamics is also unbounded and the nonlinear dynamics are unstable. In order to investigate the characteristics of the solution to the nonlinear dynamics, it is necessary to investigate the characteristics of $\boldsymbol{\varepsilon}_{\mathbf{F}}$.

Consider the time interval, $[t_1, t_1+T)$, and assume that for any δ_{α_1} there exists a δ (which can depend on T) such that provided

$$\|\mathbf{w}\|_{T_o} + \|\dot{\mathbf{r}}\|_{T_o} \le 2\delta \tag{36a}$$

then

$$\|\boldsymbol{\varepsilon}_{\mathbf{F}}\|_{T_o} \le \alpha_1^{T_o}\{\|\mathbf{w}\|_{T_o} + \|\dot{\mathbf{r}}\|_{T_o}\} \le \alpha_1\{\|\mathbf{w}\|_{T_o} + \|\dot{\mathbf{r}}\|_{T_o}\} \quad \text{with} \quad \alpha_1 \in [0, \delta_{\alpha_1}) \tag{36b}$$

where $T_o \in [0, T]$. Note, $\mathbf{x}$, and so $\mathbf{w} = \dot{\mathbf{x}}$, is arbitrary here and not necessarily a solution to the nonlinear dynamics. Whilst $\alpha_1^{T_o}$ may depend on T_o, it is emphasised that the δ and α_1 are uniform bounds which may depend on T but are required to be independent of T_o. The condition, (36), is essentially a smoothness requirement on $\nabla_{\mathbf{x}}\mathbf{F}(\mathbf{x}, \mathbf{r})$ and $\nabla_{\mathbf{r}}\mathbf{F}(\mathbf{x}, \mathbf{r})$. In the context of series expansion analysis, this type of requirement seems unavoidable. In fact, the condition, (36), is rather weak; for example, when $\nabla_x\mathbf{F}$ and $\nabla_r\mathbf{F}$ are *locally* Lipschitz continuous,

$$|\nabla_x\mathbf{F}(\mathbf{x},\mathbf{r}) - \nabla_x\mathbf{F}(\hat{\mathbf{x}},\hat{\mathbf{r}})| \le L\{|\mathbf{x}-\hat{\mathbf{x}}| + |\mathbf{r}-\hat{\mathbf{r}}|\} \qquad \forall\ (\mathbf{x},\mathbf{r}),(\hat{\mathbf{x}},\hat{\mathbf{r}}) \in \hat{\Phi} \tag{37a}$$

$$|\nabla_r\mathbf{F}(\mathbf{x},\mathbf{r}) - \nabla_r\mathbf{F}(\hat{\mathbf{x}},\hat{\mathbf{r}})| \le L\{|\mathbf{x}-\hat{\mathbf{x}}| + |\mathbf{r}-\hat{\mathbf{r}}|\} \qquad \forall\ (\mathbf{x},\mathbf{r}),(\hat{\mathbf{x}},\hat{\mathbf{r}}) \in \hat{\Phi} \tag{37b}$$

where,

$$\hat{\Phi} = \{(\mathbf{x},\mathbf{r}): |\mathbf{x}-\mathbf{x}_1| \le 2\delta T,\ |\mathbf{r}-\mathbf{r}_1| \le 2\delta T\} \tag{37a}$$

then it follows from (34b) that,

$$|\varepsilon_F(t)| \le L\{|\mathbf{x}(t)-\mathbf{x}_1| + |\mathbf{r}(t)-\mathbf{r}_1|\}\{|\mathbf{w}(t)| + |\dot{\mathbf{r}}(t)|\} \qquad \forall\ t \in [t_1, t_1+T_o) \tag{38}$$

$$\le L\{\|\mathbf{w}\|_{T_o} T_o + \|\dot{\mathbf{r}}\|_{T_o} T_o\}\{|\mathbf{w}(t)| + |\dot{\mathbf{r}}(t)|\} \qquad \forall\ t \in [t_1, t_1+T_o) \tag{39}$$

$$\le 2T_oL\delta\{|\mathbf{w}(t)| + |\dot{\mathbf{r}}(t)|\} \qquad \forall\ t \in [t_1, t_1+T_o) \tag{40}$$

$$\le 2TL\delta\{|\mathbf{w}(t)| + |\dot{\mathbf{r}}(t)|\} \qquad \forall\ t \in [t_1, t_1+T_o) \tag{41}$$

and so, provided δ is sufficiently small (less than $\delta_{\alpha_1}/2TL$), (36) is satisfied.

In addition, it is assumed that the solutions to the nonlinear dynamics, (28b), are continuous over the time interval, $[t_1, t_1+T)$. Once again, this is a weak smoothness requirement on $\nabla_x\mathbf{F}(\mathbf{x},\mathbf{r})$ and $\nabla_\mathbf{r}\mathbf{F}(\mathbf{x},\mathbf{r})$ and, from standard theory (see, for example, theorem 2.5, Khalil 1992), it is satisfied by, for example, the local Lipschitz condition, (37). Also, assume that $|\nabla_x\mathbf{F}(\mathbf{x}_1, \mathbf{r}_1)|$ and $|\nabla_r\mathbf{F}(\mathbf{x}_1, \mathbf{r}_1)|$ are uniformly bounded and the eigenvalues of $\nabla_x\mathbf{F}(\mathbf{x}_1, \mathbf{r}_1)$ lie in the left-half complex plane and are uniformly bounded away from the imaginary axis; that is,

$$|\nabla_x\mathbf{F}(\mathbf{x}_1,\mathbf{r}_1)| \le \alpha_2, \quad \mathrm{Re}\{\mathrm{eig}[\nabla_x\mathbf{F}(\mathbf{x}_1,\mathbf{r}_1)]\} \le -\alpha_3 < 0, \quad |\nabla_r\mathbf{F}(\mathbf{x}_1,\mathbf{r}_1)| \le \alpha_4,\ \forall\ \mathbf{x}_1, \mathbf{r}_1 \tag{42}$$

Under these conditions, the dynamics of the velocity-based linearisation, (31b), are uniformly stable and (Desoer 1969),

$$|e^{-\nabla_x\mathbf{F}(\mathbf{x}_1,\mathbf{r}_1)(t-t_1)}| \le \alpha_5 e^{-\alpha_6(t-t_1)} \tag{43}$$

where the amplification factor, α_5, is greater than or equal to one and the exponent, α_6, is strictly greater than zero. However, it is noted once again that there is no assumption that $\boldsymbol{\varepsilon}_\mathbf{F}$ is bounded and so, at this point in the analysis, the nonlinear dynamics, (34), may be unstable. (The requirement that the solutions are continuous over any finite interval excludes pathological instabilities, such as finite escape-time behaviour, associated with non-smooth dynamics. However, as noted previously, in a series expansion theory context some degree of restriction to smooth behaviour seems unavoidable).

Finally, it is assumed that δ_{α_1} is chosen such that

$$\delta_{\alpha_1} \le \alpha_6/\alpha_5 \tag{44a}$$

and that the inputs and initial conditions are restricted to the class satisfying

$$\frac{\alpha_5\,|\mathbf{w}(t_1)| + \frac{\alpha_5}{\alpha_6}(\alpha_4+\alpha_1)\|\dot{\mathbf{r}}\|_T}{1 - \frac{\alpha_5}{\alpha_6}\alpha_1} \le \mu < \delta \tag{44b}$$

and

$$\|\dot{\mathbf{r}}\|_T \le \delta, \tag{44c}$$

It is noted that (44) implies that,

$$|\mathbf{w}(t_1)| < \delta, \qquad \|\dot{\mathbf{r}}\|_{T_o} \le \delta \tag{45}$$

An interval, $[t_1, t_1+T_o)$, is selected with $T_o \in (0, T]$, for which $\|\mathbf{w}\|_{T_o}$ is less than δ. The existence of such a time interval is guaranteed by (45) and the continuity of $\mathbf{w}(t)$. On this interval, $\|\mathbf{w}\|_{T_o} + \|\dot{\mathbf{r}}\|_{T_o}$ is less than 2δ and so, substituting (36) and (43) into (35),

$$\|\mathbf{w}\|_{T_o} \le \alpha_5|\mathbf{w}(t_1)| + \alpha_5/\alpha_6\|\boldsymbol{\varepsilon}_\mathbf{F}\|_{T_o} + \alpha_4\alpha_5/\alpha_6\|\dot{\mathbf{r}}\|_{T_o} \le \mu < \delta \tag{46}$$

Since μ is independent of the time interval, T_o, the analysis may be repeated to obtain successively larger intervals, T_i, for which $\|\mathbf{w}\|_{T_i} + \|\dot{\mathbf{r}}\|_{T_i}$ is less than 2δ and $\|\mathbf{w}\|_{T_i}$ is less than or equal to μ. Consequently,

$$\|\mathbf{w}\|_T \le \mu < \delta \tag{47}$$

and

$$\|\boldsymbol{\varepsilon}_\mathbf{F}\|_T < 2\delta\alpha_1 \tag{48}$$

The bound, (47), is not particularly tight. However, it guarantees that $\|\mathbf{w}\|_T + \|\dot{\mathbf{r}}\|_T$ is less than 2δ and so, on substituting (36) and (43) into (35),

$$|\mathbf{w}(t)| \le \alpha_5 e^{-\alpha_6(t-t_1)}\, |\mathbf{w}(t_1)| + \alpha_5(\alpha_4+\alpha_1)\,(1-e^{-\alpha_6(t-t_1)})\,/\alpha_6 \|\dot{\mathbf{r}}\|_T + \int_{t_1}^{t} \alpha_5 e^{-\alpha_6(t-s)} \alpha_1 |\mathbf{w}(s)| ds \qquad \forall t \in [t_1, t_1+T) \tag{49}$$

Applying the Bellman-Gronwall inequality to (49),

$$|\mathbf{w}(t)| \le \alpha_5 e^{-(\alpha_6-\alpha_5\alpha_1)(t-t_1)}\, |\mathbf{w}(t_1)| + \frac{\frac{\alpha_5}{\alpha_6}(\alpha_4+\alpha_1)(1-e^{-\alpha_6(t-t_1)})}{1-\frac{\alpha_5}{\alpha_6}\alpha_1} \|\dot{\mathbf{r}}\|_T \qquad \forall\, t \in [t_1, t_1+T) \tag{50}$$

The inequality, (50), is a much tighter bound than (47) and, in particular, indicates that, under the foregoing assumptions, when the input, **r**, is constant, the solution, **w**, of the nonlinear dynamics, (28b), is contained within an envelope which decays exponentially over the interval, $[t_1, t_1+T)$.

It follows from (27b) and (28b) that,

$$\dot{\hat{\mathbf{w}}} - \dot{\mathbf{w}} = \nabla_x \mathbf{F}(\mathbf{x_1}, \mathbf{r_1})(\hat{\mathbf{w}} - \mathbf{w}) + \varepsilon_{\mathbf{F}} \tag{51a}$$

with the initial conditions,

$$\hat{\mathbf{w}}(t_1) - \mathbf{w}(t_1) = 0, \tag{51b}$$

Hence, the difference between the solution to the velocity-based linearisation, (27b), and the solution to the nonlinear dynamics, (28b), is simply the residual, $\varepsilon_{\mathbf{F}}$, filtered by the linearised dynamics; that is,

$$\hat{\mathbf{w}}(t) - \mathbf{w}(t) = \int_{t_1}^{t} e^{\nabla_x \mathbf{F}(\mathbf{x_1}, \mathbf{r_1})(t-s)} \varepsilon_{\mathbf{F}}(s) ds \tag{52}$$

It is assumed that the inputs and initial conditions belong to the class satisfying (44). Substituting (36) and (43) into (52),

$$\|\hat{\mathbf{w}} - \mathbf{w}\|_{T_o} \le \frac{\alpha_5}{\alpha_6} \alpha_1^{T_o} (\|\mathbf{w}\|_{T_o} + \|\dot{\mathbf{r}}\|_{T_o}), \qquad \forall\, T_o \in [0, T] \tag{53}$$

and,

$$\|\hat{\mathbf{w}} - \mathbf{w}\|_T \le \frac{\alpha_5}{\alpha_6} \alpha_1 (\|\mathbf{w}\|_T + \|\dot{\mathbf{r}}\|_T) \le \Delta \tag{54}$$

where Δ equals $2\frac{\alpha_5}{\alpha_6}\alpha_1\delta$. The upper bound on the peak difference between the solution to the velocity-based linearisation and the solution to the exact nonlinear dynamics is proportional to the peak value of the exact solution but can be made arbitrarily small by suitably restricting the inputs and initial conditions (and thereby δ).

4.2 Stability analysis

It is established by the analysis of section 4.1 that, over a time-interval of length, T, the solution to the nonlinear dynamics is approximated by the solution to the velocity-based linearisation with an accuracy which depends on the initial conditions and the rate of variation of the input; that is, on the rate of change of the input and the state. Consider, firstly, the situation when the input is zero; that is, the unforced case. Provided that the dynamics of the velocity-based linearisation are stable, then over a sufficiently long time interval the solution to the linearisation, relative to a specific operating point, must decrease in magnitude. Hence, owing to the ability of the velocity-based linearisation to approximate the nonlinear system, when the length of the time interval, T, is sufficiently great, the solution to the nonlinear dynamics must also decrease over the interval. In other words, over the time-interval, T, the 'gain' of the nonlinear system is less than unity. Consequently, over a sequence of such time-intervals, the solutions to the nonlinear dynamics must decay towards the origin; that is, the nonlinear system is stable. This argument may be generalised to the forced situation although the solution to the nonlinear dynamics converges to a region enclosing the origin rather than to the origin itself. Similarly to the Lyapunov-based analysis, the foregoing analysis requires a restriction on the inputs and initial conditions (to ensure that the interval, T, is sufficiently long) which is simply a slow variation condition. This restriction is purely a consequence of the slow variation requirement and, in the same sense as in the preceding discussion of the Lyapunov-based analysis, is as weak as possible. Indeed, when, for example, the magnitude of the solution to the linearisation uniformly decreases monotonically with time (in which case α_5 is unity), the length, T, of the time interval may be zero and there is no restriction on the rate of variation of the system; that is, the analysis is global in nature.

The foregoing argument employs a piecewise-approximation to relate the stability properties of the nonlinear system to those of the associated velocity-based linearisation family. The corresponding rigorous derivation is presented in Appendix C for the situation when the nonlinear system is smooth (in the sense that $\mathbf{F}(\cdot,\cdot,\cdot)$ is differentiable with bounded, Lipschitz continuous, derivatives), and in Appendix D, for the situation when the nonlinear system contains discontinuities ($\nabla_x\mathbf{F}$ and $\nabla_r\mathbf{F}$ need only be piece-wise continuous with respect to time along trajectories of the nonlinear system). The latter analysis accommodates switched and other, discontinuous, forms of scheduling as required. The aspect of the analysis of Appendices C and D which primarily differentiates it from previous work is the use of the linearised dynamics along the solution trajectory of the nonlinear system and not the linearised dynamics at the equilibrium operating points.

The analysis may be extended to investigate stability robustness by suitably augmenting the system to include a nonlinear dynamic perturbation. It then follows immediately from the foregoing analysis that, provided the input and initial condition are appropriately restricted, the nonlinear system is robustly stable with respect to finite-dimensional dynamic perturbations for which the members of the family of velocity-based linearisations of the perturbed system are uniformly stable. Moreover, the robustness extends to a broad class of distributed/infinite-dimensional dynamic perturbations (by straightforward application of the results of Appendix E to the analysis of Appendices C and D). Of course, this robustness analysis is confined to dynamic perturbations which are smooth or for which the discontinuities occur sufficiently slowly. It may be extended to more general perturbations by employing the small gain theorem. The analysis in Appendices C and D indicates that the induced norm, or 'gain', of the nonlinear dynamics is less than or equal to $\hat{\gamma}$ (as defined by (D.5b), Appendix D). Consequently, from the small gain theorem, the dynamics are robustly stable with respect to general perturbations with induced norm less that $1/\hat{\gamma}$. Moreover, it is noted that as the restriction on the class of inputs and initial conditions is tightened (δ reduced), $\hat{\gamma}$ tends to $\alpha_4\alpha_5/\alpha_6$ which is simply the uniform bound on the induced norm of the family of velocity-based linearisations; that is, the nonlinear system inherits the robustness of the linear family to general perturbations. (Of course, the small gain theorem only provides sufficient conditions for stability and both the nonlinear system and the members of the linear family may well be robust to a wider class of perturbations).

5. Conclusions

The existing theory, relating the dynamic properties of a nonlinear system to those of an associated family of linear time-invariant systems, is rather poorly developed. Series expansion linearisation theory is well established but is strictly confined to the dynamic analysis, locally to a single trajectory or equilibrium operating point, of smooth nonlinear systems. Frozen-input techniques cater for the stability analysis of smooth nonlinear systems relative to a family of equilibrium operating points but are confined to an unnecessarily, perhaps excessively, small neighbourhood about the equilibrium operating points. Series expansion theory and frozen-input theory consider only the stability properties of the nonlinear system, providing little direct insight into other dynamic properties such as the transient response.

In this paper, a family of velocity-based linearisations for a nonlinear system is proposed. In contrast to the conventional series expansion linearisation, a member of the family of velocity-based linearisations is valid in the vicinity of *any* operating point, not just an equilibrium operating point. Moreover, unlike series expansion linearisations relative to equilibrium operating points, the solutions to the members of the family of velocity-based linearisations can be pieced together to approximate the solution to a nonlinear system. Hence, the velocity-based linearisations, in addition to facilitating dynamic analysis far from the equilibrium operating points, also enable the transient behaviour of the nonlinear system to be investigated.

The family of velocity-based linearisations is utilised to derive a stability condition for smooth nonlinear systems which avoids the restrictions, to trajectories lying within an unnecessary, perhaps excessively, small neighbourhood about the equilibrium operating points, inherent in existing frozen-input theory. A slow variation condition is required; that is, a restriction on the allowable rate of change of the input and initial condition. However, in contrast to previous results, it is emphasised that this restriction is purely a consequence of the slow variation requirement and, in this sense, is a weak as possible. Indeed, for systems where there is no restriction on the rate of variation, the analysis is *global* in nature. The stability analysis is extended to include nonlinear systems with non-smooth dynamics, and the corresponding conditions for stability are derived.

The analysis techniques developed, whilst quite general, are motivated by the gain-scheduling design approach and clearly have the potential for direct application to the analysis gain-scheduled systems.

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References

ASTROM, K.J., WITTENMARK, B., 1989, *Adaptive Control*. (Addison-Wesley).
BARMAN, J.F., 1973, *Well-posedness of Feedback Systems and Singular Perturbations*. (Ph.D. Thesis, University of California, Berkeley).
DESOER, C.A., 1969, Slowly Varying Controller dx/dt=A(t)x. *IEEE Transactions on Automatic Control*, **14**, 780-781.
DESOER, C.A., VIDYASAGAR, M.,1975, *Feedback Systems: Input-Output Properties.* (London : Academic Press).
DESOER, C.A., WONG, K.K., 1968, Small Signal Behaviour of Nonlinear Lumped Networks. *Proceedings of the IEEE*, **56**, 14-23.
HOPPENSTEADT, F.C., 1966, Singular Perturbations on the Infinite Interval. *Transactions of the American Mathematical Society*, **123**, 521-535.
HYDE, R.A., GLOVER, K., 1993, The Application of Scheduled H_∞ Controllers to a VSTOL Aircraft. *IEEE Transactions on Automatic Control*, **38**, 1021-1039.
ILCHMANN, A., OWENS, D.H., PRATZEL-WOLTERS, D., 1987, Sufficient Conditions for Stability of Linear Time-Varying Systems. *Systems & Control Letters*, **9**, 157-163.
KHALIL, H.K., 1992, *Nonlinear Systems*. (Macmillan, New York).
MORSE, A.S., 1995, Control Using Logic-Based Switching. In *Trends in Control* (London : Springer Verlag)
RUGH, W.J., 1990, On a Stability Theorem for Nonlinear Systems with Slowly Varying Inputs. *IEEE Transactions on Automatic Contol,* **35**, 860-864.
SHAMMA, J.S., ATHANS, M., 1990, Analysis of Gain Scheduled Control for Nonlinear Plants. *IEEE Transactions on Automatic Control*, **35**, 898-907.
SHAMMA, J.S., ATHANS, M., 1991, Guaranteed Properties of Gain-Scheduled Control for Linear Parameter-Varying Plants. *Automatica*, **27**, 559-564.
SONTAG, E.D., 1989, Smooth Stabilization Implies Coprime Factorization. *IEEE Transactions on Automatic Control*, **34**, 435-443.
VIDYASAGAR, M., VANNELLI, A., 1982, New Relationships Between Input-Output and Lyapunov Stability. *IEEE Transactions on Automatic Control*, **27**, 481-483.
VIDYASAGAR, M., 1993, *Nonlinear Systems Analysis.* (Prentice-Hall, New Jersey).
ZHANG, J.F., 1993, General Lemmas for Stability Analysis of Linear Continuous-Time Systems with Slowly Time-Varying Parameters. *International Journal of Control*, **58**, 1437-1444.

Appendix A – Piece-wise approximation

Consider the solution, $\mathbf{x}(t)$, to the nonlinear system, (16), over a time interval [0, T], T>0. Divide this interval into n smaller sub-intervals, $[T_{i-1}, T_i]$, i=1,2,..n, T_i=iT/n, and consider the piece-wise approximation, $\hat{\mathbf{x}}(t)$,

$$\delta\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}(\hat{\mathbf{x}}(T_{i-1}), \mathbf{r}(T_{i-1})) + \nabla_x\mathbf{F}(\hat{\mathbf{x}}(T_{i-1}), \mathbf{r}(T_{i-1}))\,\delta\hat{\mathbf{x}}(t) + \nabla_\mathbf{r}\mathbf{F}(\hat{\mathbf{x}}(T_{i-1}), \mathbf{r}(T_{i-1}))\,\delta\mathbf{r}(t) \quad t\in[T_{i-1}, T_i] \quad \text{(A.1a)}$$

$$\delta\mathbf{r}(t) = \mathbf{r}(t) - \mathbf{r}(T_{i-1}), \quad \hat{\mathbf{x}}(t) = \delta\hat{\mathbf{x}}(t) + \hat{\mathbf{x}}(T_{i-1}), \quad \dot{\hat{\mathbf{x}}}(t) = \delta\dot{\hat{\mathbf{x}}}(t) \qquad t\in[T_{i-1}, T_i] \quad \text{(A.1b)}$$

with the initial condition

$$\hat{\mathbf{x}}(0) = \mathbf{x}(0) \qquad \text{(A.1c)}$$

Let $\chi_i(t)$ denote the state solution to the nonlinear dynamics, (16), starting from time, T_{i-1}, and with the initial condition,

$$\chi_i(T_{i-1}) = \hat{\mathbf{x}}(T_{i-1}) \qquad \text{(A.2)}$$

Integrating (A.1a), it follows that

$$\hat{\mathbf{x}}(t) = \chi_i(T_{i-1}) + \dot{\chi}_i(T_{i-1})(t - T_{i-1}) + \frac{1}{2}\ddot{\chi}_i(T_{i-1})(t - T_{i-1})^2 + \varepsilon \qquad t\in[T_{i-1}, T_i] \qquad \text{(A.3a)}$$

where,

$$\varepsilon = \int_{T_{i-1}}^{t}\Big(\nabla_{\mathbf{x}}\mathbf{F}(\hat{\mathbf{x}}(T_{i-1}),\ \mathbf{r}(T_{i-1}))\ \big(\delta\hat{\mathbf{x}}(s) - \delta\dot{\hat{\mathbf{x}}}(T_{i-1})s\big)\ +\ \nabla_{\mathbf{r}}\mathbf{F}(\hat{\mathbf{x}}(T_{i-1}),\ \mathbf{r}(T_{i-1}))\ \big(\delta\mathbf{r}(s) - \delta\dot{\mathbf{r}}(T_{i-1})s\big)\Big)ds \qquad \text{(A.3b)}$$

Assume that $\mathbf{F}(\bullet,\bullet)$ is twice continuously differentiable. From Taylor series expansion theory, there exists a finite constant, k_o, such that

$$|\delta\hat{\mathbf{x}}(s) - \delta\dot{\hat{\mathbf{x}}}(T_{i-1})s| < k_o(s - T_{i-1})^2, \qquad |\delta\mathbf{r}(s) - \delta\dot{\mathbf{r}}(T_{i-1})s| < k_o(s - T_{i-1})^2 \qquad \text{(A.4)}$$

Hence, there exists a finite constant, k_1, such that

$$|\varepsilon(t)| < k_1(t-T_{i-1})^3 \qquad t\in[T_{i-1}, T_i] \qquad \text{(A.5)}$$

From Taylor series expansion theory, the solution obtained by truncating (A.3a) before ε is a second order approximation to $\chi_i(t)$. Hence, it follows from (A..5) that $\hat{\mathbf{x}}(t)$ is also a second-order approximation to $\chi_i(t)$; that is, there exists a finite constant, k_2 such that

$$|\chi_i(t)- \hat{\mathbf{x}}(t)| < k_2(t-T_{i-1})^3 \qquad \text{(A.6)}$$

Since $\mathbf{F}(\bullet,\bullet)$ is continuous and differentiable, the solutions to the nonlinear system depend continuously on the initial conditions and, for some positive finite constant K (see, for example, Khalil 1992 Theorem 2.5),

$$|\chi_i(t)- \mathbf{x}(t)| < |\chi_i(T_{i-1})-\mathbf{x}(T_{i-1})|\, e^{K(t-T_{i-1})} \qquad t\in[T_{i-1}, T_i] \qquad \text{(A.7)}$$

Hence, applying inequalities, (A.6) and (A.7), to the recursive definition, (A.1),

$$|\hat{\mathbf{x}}(T)-\mathbf{x}(T)| < k_2(T/n)^3(1-e^{KT})/(1-e^{KT/n}) \qquad \text{(A.8)}$$

Since

$$\lim_{n\to\infty} k_2(T/n)^3(1-e^{KT})/(1-e^{KT/n}) = 0 \qquad \text{(A.9)}$$

it follows that,

$$\lim_{n\to\infty} |\hat{\mathbf{x}}(T) - \mathbf{x}(T)| = 0 \qquad \text{(A.10)}$$

and so the piece-wise approximation, (A.1), can be made arbitrarily accurate by increasing the number of sub-intervals employed.

In comparison, consider the situation when a zeroeth-order approximation is employed over each sub-interval, $[T_{i-1}, T_i]$ (*e.g.* when the series expansion linearisation about an adjacent equilibrium operating point is employed); that is,

$$|\chi_i(t)- \hat{\mathbf{x}}(t)| < k(t-T_{i-1}) \qquad t\in[T_{i-1}, T_i] \qquad \text{(A.11)}$$

In this case,

$$|\hat{\mathbf{x}}(T)-\mathbf{x}(T)| < k(T/n)(1-e^{KT})/(1-e^{KT/n}) \qquad \text{(A.12)}$$

and since,

$$\lim_{n\to\infty} k(T/n)(1-e^{KT})/(1-e^{KT/n}) = k(e^{KT}-1)/K \neq 0 \qquad \text{(A.13)}$$

it follows that the approximation error need not converge to zero as the number of sub-intervals is increased.

Appendix B - Lyapunov-based analysis

When the solution, $\mathbf{w}$, to the nonlinear system, (28), is continuous, Lyapunov theory may be employed to analyse the stability. Consider the candidate Lyapunov function, V,

$$V = \mathbf{w}^T\mathbf{P}(\mathbf{x},\mathbf{r})\mathbf{w} \qquad \text{(B.1a)}$$

with $\mathbf{P}(\mathbf{x},\mathbf{r})$ positive definite and

$$\mathbf{P}(\mathbf{x}, \mathbf{r})\ \nabla_{\mathbf{x}}\mathbf{F}(\mathbf{x}, \mathbf{r}) + \nabla_{\mathbf{x}}\mathbf{F}^T(\mathbf{x}, \mathbf{r})\ \mathbf{P}(\mathbf{x}, \mathbf{r}) = -\mathbf{I}\ \ \forall\ \mathbf{x},\mathbf{r} \qquad \text{(B.1b)}$$

Provided the (linear time-invariant) members of the velocity-based linearisation family, (27), are stable, the existence of $\mathbf{P}(\mathbf{x}, \mathbf{r})$ satisfying (B.1b) for each $(\mathbf{x}, \mathbf{r})$ is guaranteed (see, for example, theorem 3.6 in Khalil 1992). Let β_i denote a positive finite constant. Assume $\nabla_{\mathbf{x}}\mathbf{F}(\mathbf{x}, \mathbf{r})$ is continuous, uniformly bounded and that the eigenvalues of $\nabla_{\mathbf{x}}\mathbf{F}(\mathbf{x}, \mathbf{r})$ are uniformly bounded away from the imaginary axis; that is,

$$|\nabla_{\mathbf{x}}\mathbf{F}(\mathbf{x}, \mathbf{r})| \le \beta_1, \quad \mathrm{Re}\{\lambda\,[\nabla_{\mathbf{x}}\mathbf{F}(\mathbf{x}, \mathbf{r})]\} \le -\beta_2 < 0 \qquad \forall \mathbf{x}, \mathbf{r} \qquad \text{(B.2)}$$

These conditions ensure that

$$|e^{\nabla_{\mathbf{x}}\mathbf{F}(\mathbf{x},\, \mathbf{r})\, t}| \le \beta_3\, e^{-\beta_4 t} \qquad \forall\ t > 0 \qquad \text{(B.3)}$$

for some finite constants $\beta_3 \ge 1$, $\beta_4 > 0$ which are independent of $\mathbf{x}$, $\mathbf{r}$ (Desoer 1969). It follows that $\mathbf{P}(\mathbf{x},\mathbf{r})$ is uniformly bounded and V is positive definite, decrescent,

$$|\mathbf{P}(\mathbf{x},\mathbf{r})|_2 \le \beta_5 \qquad \forall\ \mathbf{x}, \mathbf{r} \tag{B.4a}$$

$$\beta_6|\mathbf{w}|_2^2 \le V \le \beta_7|\mathbf{w}|_2^2 \tag{B.4b}$$

with $\beta_6 > 0$. When $\mathbf{P}(\mathbf{x}, \mathbf{r})$ is differentiable, the derivative of V with respect to time, along the trajectories of (28), is

$$\dot{V} = -\mathbf{w}^T(\mathbf{I} - \dot{\mathbf{P}})\mathbf{w} + \mathbf{w}^T\mathbf{P}(\mathbf{x}, \mathbf{r})\nabla_\mathbf{r}\mathbf{F}(\mathbf{x}, \mathbf{r})\,\dot{\mathbf{r}} + \nabla_\mathbf{r}\mathbf{F}^T(\mathbf{x}, \mathbf{r})\,\dot{\mathbf{r}}^{\,T}\mathbf{P}(\mathbf{x}, \mathbf{r})\mathbf{w} \tag{B.5}$$

Expand $\dot{\mathbf{P}}$ as

$$\dot{\mathbf{P}} = \sum_i \nabla_{x_i}\mathbf{P}(\mathbf{x},\mathbf{r})w_i + \sum_j \nabla_{r_j}\mathbf{P}(\mathbf{x},\mathbf{r})\dot{r}_j \tag{B.6}$$

and assume that the partial derivatives of **P** with respect to the elements of the state vector, **x,** and input vector, **r**, are uniformly bounded,

$$\sup_i |\nabla_{x_i}\mathbf{P}(\mathbf{x},\mathbf{r})|_2 \le \beta_8, \qquad \sup_j |\nabla_{r_j}\mathbf{P}(\mathbf{x},\mathbf{r})|_2 \le \beta_9 \qquad \forall\ \mathbf{x}, \mathbf{r} \tag{B.7a}$$

and that $\nabla_\mathbf{r}\mathbf{F}(\mathbf{x}, \mathbf{r})$ is uniformly bounded,

$$|\nabla_\mathbf{r}\mathbf{F}(\mathbf{x}, \mathbf{r})|_2 \le \beta_{10} \qquad \forall\ \mathbf{x},\mathbf{r} \tag{B.7b}$$

It follows that,

$$\dot{V} \le \beta_8|\mathbf{w}|_2^3 - (1-\beta_9|\dot{\mathbf{r}}|_2)|\mathbf{w}|_2^2 + 2\beta_5\beta_{10}|\dot{\mathbf{r}}|_2\,|\mathbf{w}|_2 \tag{B.8}$$

which, using (B.4b) and letting W denote $V^{½}$, may be reformulated as

$$\dot{W} \le \beta_8/(2\beta_6^{3/2})\,W^2 - (1-\beta_9|\dot{\mathbf{r}}|_2)/2\beta_7 W + \beta_5\beta_{10}|\dot{\mathbf{r}}|_2/\beta_6^{1/2} \tag{B.9}$$

where it is assumed, for the moment, that $1-\beta_9|\dot{\mathbf{r}}|_2$ is positive. Provided

$$|\dot{\mathbf{r}}|_2 \le \beta_{11} \le (1-4\beta_7\lambda)/\beta_9 \tag{B.10a}$$

$$|\dot{\mathbf{r}}|_2 \le \beta_{11} \le 2\lambda^2\beta_6^{5/2}/(\beta_5\beta_8\beta_{10}) \tag{B.10b}$$

$$W(0) \le 2\lambda\,\beta_6^{3/2}/\beta_8 \tag{B.10c}$$

for some $\lambda \in [0, 1/4\beta_7)$, then $1-\beta_9|\dot{\mathbf{r}}|_2$ is positive and , from (B.9),

$$\dot{W} \le \beta_8/(2\beta_6^{3/2})\,W^2 - 2\lambda W + \beta_5\beta_{10}|\dot{\mathbf{r}}|_2/\beta_6^{1/2} \le -\lambda W + \beta_5\beta_{10}\beta_{11}/\beta_6^{1/2} \tag{B.11}$$

Conditions (B.10a) and (B.10c) may be interpreted as restricting the rate of variation of the system trajectories, ($\mathbf{x}$(t), $\mathbf{r}$(t)), to ensure the existence, within some region enclosing the origin, of stable solutions to the dynamics, (B.8), with the parameter, λ, quantifying the relationship between the size of the region and the rate of variation of the input (qualitatively, as the input varies more rapidly the region becomes smaller, and *vice versa*). Condition (B.10b) is required to ensure that the input does not drive the state trajectories outside this stability region.

It follows from (B.11) that,

$$W \le W(0)e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)}\beta_5\beta_{10}\beta_{11} / \beta_6^{\frac{1}{2}}\,ds \le W(0)e^{-\lambda t} + \frac{\beta_5\beta_{10}\beta_{11}}{\lambda\beta_6^{1/2}} \tag{B.12}$$

and, therefore,

$$|w|_2 \le \gamma e^{-\lambda t}|w(0)|_2 + \delta \qquad \forall\ t > 0 \tag{B.13}$$

with $\gamma = (\beta_7/\beta_6)^{1/2}$, $\delta = \beta_5\beta_{10}\beta_{11}/\lambda\beta_6^{1/2}$. Hence, provided the family of linear systems, (27), satisfies the mild conditions, (B.2) and (B.7), and the system inputs and initial conditions are restricted to the class satisfying condition (B.10), the states of the nonlinear system, (28), are uniformly ultimately bounded. Moreover, the ultimate bound, δ, is proportional to the rate of change, $\dot{\mathbf{r}}$**,** of the input and the system is, therefore, exponentially stable for constant inputs. It is noted that a similar approach to that of Appendix D may be employed to extend this analysis to encompass non-smooth scheduling.

Appendix C Stability of smooth nonlinear systems

Assume that the nonlinear system satisfies the conditions of section 4.1. The analysis of section 4.1 then establishes a bound on the solution to the nonlinear system over a time interval, T; namely,

$$|\mathbf{w}(t)| \le \alpha_5 e^{-\alpha_8(t-t_1)}|\mathbf{w}(t_1)| + \frac{\alpha_5}{\alpha_7}(\alpha_4 + \alpha_1)\,(1 - e^{-\alpha_6(t-t_1)})\,\|\dot{\mathbf{r}}\|_T \qquad \forall\ t \in [t_1, t_1+T) \tag{C.1}$$

where,

$$\alpha_7 = \alpha_6 - \alpha_5\,\alpha_1$$

provided the initial conditions and inputs satisfy (44). Since the solution, $\mathbf{w}$(t), of the nonlinear system is continuous; (C.1) holds for $t \in [t_1, t_1+T]$.

Let

$$T = 2\frac{\ln\alpha_5}{\alpha_6 - \eta_o} \tag{C.2}$$

for some $\eta_o \in (0, \alpha_6)$. Assume that the inputs and initial conditions satisfy,

$$\|\dot{\mathbf{r}}\| \le \delta_r \le \delta, \qquad |\mathbf{w}(0)| \le \delta_w - \gamma\delta_r \tag{C.3a}$$

(where $\|\bullet\|$ denotes $\sup_t |\bullet|$) with

$$\frac{\alpha_5}{\alpha_7}(\alpha_6\delta_w + (\alpha_4+\alpha_1)\delta_r) < \delta, \qquad \gamma = \frac{\alpha_5}{\alpha_7}(\alpha_4+\alpha_1)\frac{(1-e^{-\alpha_6 T/2})}{(1-e^{-\eta T})}, \ \eta = ½(\eta_o - \alpha_1\alpha_5) \tag{C.3b}$$

and δ, δ_r are chosen sufficiently small that,

$$\delta_{\alpha_1} < \eta_o/\alpha_5, \qquad \delta_w - \gamma\delta_r \ge 0 \tag{C.3c}$$

It is noted that, with these choices, for any $T_o \in [T/2, T]$,

$$\eta > 0, \qquad \delta_{\alpha_1} < \alpha_6/\alpha_5, \qquad \delta_w \le \delta, \qquad \alpha_5 e^{-\alpha_7 T_o} \le e^{-\eta T} < 1 \tag{C.4a}$$

$$e^{-\eta k T}|\mathbf{w}(0)| + \gamma\delta_r(1-e^{-\eta T})\sum_{i=0}^{k-1}\left(e^{-\eta T}\right)^i \le \delta_w \qquad \forall\ k > 0 \tag{C.4b}$$

where in (C.4b) the following identity is employed,

$$\gamma\delta_r(1-e^{-\eta T})\sum_{i=0}^{k-1}\left(e^{-\eta T}\right)^i = \gamma\delta_r\left(1-e^{-\eta k T}\right) \qquad \forall\ k > 0 \tag{C.4c}$$

The conditions, (C.3), ensure that the inputs and initial conditions satisfy (44) for $t \in [0,T]$. Hence, for any $T_o \in [T/2, T]$,

$$|\mathbf{w}(T_o)| \le \alpha_5 e^{-\alpha_7 T_o}|\mathbf{w}(0)| + \gamma\|\dot{\mathbf{r}}\|(1-e^{-\eta T}) \le e^{-\eta T}|\mathbf{w}(0)| + \gamma\delta_r\left(1-e^{-\eta T}\right) \le \delta_w \tag{C.5}$$

and $\|\mathbf{w}\|_{T_o} \le \delta$. It follow that the conditions, (44), are also satisfied for $t \in [T_o, 2T_o]$ and

$$|\mathbf{w}(2T_o)| \le \alpha_5 e^{-\alpha_7 T_o}|\mathbf{w}(T_o)| + \gamma\left(1-e^{-\eta T}\right)\|\dot{\mathbf{r}}\| \le e^{-2\eta T}|\mathbf{w}(0)| + \gamma\delta_r\left(1-e^{-\eta T}\right)(1+e^{-\eta T}) \le \delta_w \tag{C.6}$$

Repeating this analysis for further time intervals, it follows that

$$|\mathbf{w}(kT_o)| \le \delta_w \qquad \forall\ k \ge 0 \tag{C.7}$$

and, consequently,

$$|\mathbf{w}(t)| \le \alpha_5 e^{-\alpha_8(t-kT_o)}|\mathbf{w}(kT_o)| + \gamma\left(1-e^{-\alpha_6(t-kT_o)}\right)\|\dot{\mathbf{r}}\| \qquad \forall\ t \in [kT_o, (k+1)T_o],\ k \ge 0 \tag{C.8}$$

Let k denote the largest integer such that t-kT/2 is positive. When t is less than T/2, k is zero and

$$|\mathbf{w}(t)| \le \alpha_5 e^{-\alpha_7 t}|\mathbf{w}(0)| + \gamma\left(1-e^{-\eta t}\right)\|\dot{\mathbf{r}}\| \qquad \forall\ 0 \le t < T/2 \tag{C.9}$$

Otherwise, k is greater than zero. Selecting T_o equal to t/k ensures that T_o lies in the interval [T/2,T] and

$$\begin{aligned} |\mathbf{w}(t)| &\le \alpha_5 e^{-\alpha_7(kT_o-(k-1)T_o)}\left(\alpha_5 e^{-\alpha_7((k-1)T_o-(k-2)T_o)}|\mathbf{w}((k-2)T_o)| + \gamma\|\dot{\mathbf{r}}\|(1-e^{-\eta T})\right) + \gamma\|\dot{\mathbf{r}}\|(1-e^{-\eta T}) \\ &\le \alpha_5{}^k e^{-\alpha_7 t}|\mathbf{w}(0)| + \gamma\|\dot{\mathbf{r}}\|(1-e^{-\eta T})\sum_{i=0}^{k-1}\left(\alpha_5 e^{-\alpha_7 T_o}\right)^i \le e^{-\eta t}|\mathbf{w}(0)| + \gamma\left(1-e^{-\eta t}\right)\|\dot{\mathbf{r}}\| \qquad \forall\ t \ge T/2 \end{aligned} \tag{C.10}$$

Employing (C.9) and (C.10), it follows that,

$$|\mathbf{w}(t)| \le \alpha_5 e^{-\eta t}|\mathbf{w}(0)| + \gamma\,(1-e^{-\eta t})\,\|\dot{\mathbf{r}}\| \qquad \forall t \ge 0 \tag{C.11}$$

and the nonlinear dynamics, (28b), are, under the foregoing conditions, BIBO stable. Moreover, the nonlinear dynamics are exponentially stable for constant inputs, **r**.

Appendix D Stability of non-smooth nonlinear systems

In Appendix C, it is assumed that the nonlinear system is smooth, in the sense that $\mathbf{F}(\cdot,\cdot,\cdot)$ is differentiable with bounded, Lipschitz continuous, derivatives. The stability analysis is extended in this appendix to encompass non-smooth nonlinear systems for which $\nabla_x\mathbf{F}$ and $\nabla_r\mathbf{F}$ need only be piece-wise Lipschitz continuous with respect to time along the trajectories of the nonlinear system. Hence, switched and other, discontinuous, forms of scheduling may be accommodated.

Assume that $\mathbf{w}(t)$ is continuous and $\nabla_x\mathbf{F}(\mathbf{x}(t),\mathbf{r}(t))$ and $\nabla_r\mathbf{F}(\mathbf{x}(t),\mathbf{r}(t))$ are uniformly bounded and piece-wise Lipschitz continuous with respect to *time*, t. Moreover, assume that the eigenvalues of $\nabla_x\mathbf{F}(\mathbf{x}(t),\mathbf{r}(t))$ lie in the left-half complex plane and are uniformly bounded away from the imaginary axis. Let $\{t_k\}$ denote the sequence of times, with $t_{k+1} > t_k$, at which $\nabla_x\mathbf{F}(\mathbf{x}(t),\mathbf{r}(t))$ and $\nabla_r\mathbf{F}(\mathbf{x}(t),\mathbf{r}(t))$ are discontinuous. In addition, assume that, when $\mathbf{w}(t)$ and $\dot{\mathbf{r}}(t)$ are of finite magnitude, the intervals satisfy,

$$t_{k+1} - t_k \ge \alpha_8 > 0 \qquad \forall\ k \tag{D.1}$$

Condition (D.1) prevents infinite discontinuities occurring in finite time and ensures the existence of solutions to (28b) when $\mathbf{w}(t)$ and $\dot{\mathbf{r}}(t)$ are finite. Moreover, it is assumed that the minimum interval, α_8, increases as the magnitudes of $\mathbf{w}(t)$ and $\dot{\mathbf{r}}(t)$ become smaller, with $\alpha_8 \to\infty$ as $\mathbf{w}(t), \dot{\mathbf{r}}(t) \to 0$; that is, α_8 increases as $\mathbf{x}(t)$and $\mathbf{r}(t)$ vary more slowly. In a gain-scheduling context, these conditions are quite mild; for example, when non-smooth scheduling is achieved by switching between linear time-invariant controllers as some scheduling variable crosses different thresholds, the conditions are satisfied when the thresholds are spaced a finite distance apart, the switching incorporates hysteresis to prevent chatter and the scheduling variable is a continuous function of the inputs, $\mathbf{r}$, and/or states, $\mathbf{x}$, of the system.

Under the foregoing assumptions, $\nabla_x\mathbf{F}(\mathbf{x}(t),\mathbf{r}(t))$, $\nabla_r\mathbf{F}(\mathbf{x}(t),\mathbf{r}(t))$ are Lipschitz continuous only over each open time interval, (t_k, t_{k+1}). Nevertheless, since the nonlinear and approximate systems have the same initial conditions at the state of each interval, the residual, $\varepsilon_F(t_k)$, is zero at the start of each interval and the condition, (36), is satisfied. Consequently, by similar analysis to that of Appendix C, the system states are bounded over each interval, $[t_k, t_{k+1}]$, by.

$$|\mathbf{w}(t)| \le \alpha_5 e^{-\eta(t-t_k)} |\mathbf{w}(t_k)| + \gamma(1-e^{-\eta(t-t_k)})\|\dot{\mathbf{r}}\| \qquad \forall\, t \in [t_k, t_{k+1}] \tag{D.2}$$

provided the input and initial condition are restricted to the class satisfying (C.3). For the class of nonlinear systems considered in this section, there exist constants, α_9 and α_{10}, such that

$$\alpha_8 \ge \frac{\ln \alpha_5}{\eta - \hat{\eta}} \tag{D.3}$$

for any $\hat{\eta} \in (0, \eta)$ provided,

$$\|\mathbf{w}\| \le \alpha_9, \ \|\dot{\mathbf{r}}\| \le \alpha_{10} \tag{D.4}$$

Employing a similar approach to that of Appendix C, it is straightforward to show that, provided the inputs and initial conditions jointly satisfy (C.3) and,

$$\delta \le \alpha_9, \qquad \delta_r \le \alpha_{10}, \qquad |\mathbf{w}(0)| \le \delta_w - \hat{\gamma}\,\delta_r, \qquad \delta_w - \hat{\gamma}\delta_r \ge 0 \tag{D.5a}$$

where,

$$\hat{\gamma} = \frac{\gamma(1-e^{-\eta\alpha_8})}{1-e^{-\hat{\eta}T}} \tag{D.5b}$$

then,

$$|\mathbf{w}(t)| \le \alpha_5 e^{-\hat{\eta}t} |\mathbf{w}(0)| + \hat{\gamma}(1-e^{-\hat{\eta}t})\|\dot{\mathbf{r}}\| \qquad \forall t \ge 0 \tag{D.6}$$

and the class of non-smooth nonlinear systems considered is, under the foregoing conditions, BIBO stable. Moreover, the nonlinear dynamics are exponentially stable for constant inputs, $\mathbf{r}$.

Appendix E Infinite dimensional dynamics

Although the analysis of section 4.1 is restricted to systems with finite dimensional dynamics, it may be easily extended to encompass distributed/infinite dimensional dynamics. In this appendix, the notation, in particular the subscripts for the constants, α_i, is selected to indicate the close relationship between the analysis here and that of section 4.1. However, in order to encompass infinite dimensional systems the constants are defined somewhat differently from those in section 4.1.

Suppose that the velocity-based linearisation is linear time-invariant but may be infinite dimensional; that is, the linearised dynamics can be represented as the convolution operator,

$$\hat{\mathbf{w}}(t) = \int_0^t \phi(t-s)\dot{\mathbf{r}}(s)ds \tag{E.1}$$

where $\hat{\mathbf{w}}$ is the output of the linear dynamics (and need no longer be the state), (E.1) is reachable. Consider the time interval, $[t_1, t_1+T)$, and assume that nonlinear dynamics are sufficiently smooth that for any δ_{α_1} there exists a δ such that,

$$\mathbf{w}(t) = \int_0^t \phi(t-s)(\dot{\mathbf{r}}(s) + \varepsilon(s))ds \qquad \forall\, t \in [t_1, t_1+T_o] \tag{E.2a}$$

provided

$$\|\mathbf{w}\|_{T_o} + \|\dot{\mathbf{r}}\|_{T_o} \le 2\delta \tag{E.2b}$$

with the residual satisfying,

$$|\varepsilon_F(t)| = \alpha_4|\varepsilon(t)| \le \alpha_1\{|\mathbf{w}(t)|+|\dot{\mathbf{r}}(t)|\} \qquad \text{with } \alpha_1 \in [0, \delta_{\alpha_1}) \qquad \forall\, t \in [t_1, t_1+T_o] \tag{E.2c}$$

The foregoing requirement is similar to that in section 4.1 and is simply a second-order condition on the approximation residual which ensures that

$$\left\|\varepsilon_{\mathbf{F}}\right\|_{T_o} \le \alpha_1\{ \|\mathbf{w}\|_{T_o} + \|\dot{\mathbf{r}}\|_{T_o} \} \qquad (E.2)$$

Assume that $\phi(\bullet)$ is exponentially bounded,

$$|\phi(\tau)| \le \alpha_4\alpha_5 e^{-\alpha_6\tau} \qquad (E.4)$$

It then follows from (E.2a) that

$$|\mathbf{w}(t)| \le \alpha_5 e^{-\alpha_6(t-t_1)}\,\mathbf{w^1} + \alpha_4\alpha_5(1-e^{-\alpha_6(t-t_1)})/\alpha_6\|\dot{\mathbf{r}}\|_T + \int_{t_1}^{t}\alpha_5 e^{-\alpha_6(t-s)}|\varepsilon_F(s)|ds \quad \forall t\in[t_1,t_1+T_o] \qquad (E.5a)$$

where,

$$\mathbf{w^1} = \int_0^{t_1}\alpha_4 e^{-\alpha_6(t_1-s)}\{|\dot{\mathbf{r}}(s)|+|\varepsilon(s)|\}ds \qquad (E.5b)$$

embodies the dependence of the present solution on the previous behaviour (owing to the 'memory' of the dynamics) and, essentially, provides a bound on the initial conditions of the dynamics.

Hence,

$$\|\mathbf{w}\|_{T_o} \le \alpha_5\mathbf{w^1} + \alpha_5/\alpha_6\left\|\varepsilon_{\mathbf{F}}\right\|_{T_o} + \alpha_4\alpha_5/\alpha_6\|\dot{\mathbf{r}}\|_{T_o} \qquad (E.6)$$

and so, employing a similar approach to that in section 4.1, it may therefore be shown that, provided the inputs and initial conditions are suitably restricted, the conditions, (E.2b), is satisfied over the interval $[t_1, t_1+T)$. Consequently, substituting (E.2c) into (E.5),

$$|\mathbf{w}(t)| \le \alpha_5 e^{-\alpha_6(t-t_1)}\,\mathbf{w^1} + \alpha_5(\alpha_4+\alpha_1)\,(1-e^{-\alpha_6(t-t_1)})/\alpha_6\|\dot{\mathbf{r}}\|_T + \int_{t_1}^{t}\alpha_5 e^{-\alpha_6(t-s)}|\mathbf{w}(s)|ds \quad \forall t\in[t_1,t_1+T) \qquad (E.7)$$

which has the same form as (49). Applying the Bellman-Gronwall inequality to (E.7), an expression similar to (50) is obtained and the solution, $\mathbf{w}$, of the nonlinear dynamics, (E.2a), is bounded and decays exponentially when $\dot{\mathbf{r}}$ is zero; that is, for constant inputs, $\mathbf{r}$.

Furthermore, it follows from (E.1) and (E.2) that,

$$\hat{\mathbf{w}}(t)-\mathbf{w}(t) = \int_{t_1}^{t}\phi(t-s)\varepsilon(s)ds \quad \forall t\in[t_1,t_1+T) \qquad (E.8)$$

Hence, similarly to the analysis of section 4.2 for finite dimensional dynamics, it may be seen that in the infinite dimensional case the error between the solution to the approximate dynamics, (E.1), and the exact dynamics, (E.2), is also simply the residual, ε, filtered by the linearised dynamics.