Using estimated entropy in a queueing system with dynamic routing

K. Duffy(1), E.A. Pechersky(2), Y.M Suhov(2,3) and N.D. Vvedenskaya(2)

November 2005

(1) Hamilton Institute, National University of Ireland, Maynooth, Ireland

(2) Institute for Information Transmission Problems, RAS, Moscow, Russia

(3) Statistical Labarotary, DPMMS, Cambridge University, Cambridge, UK

Abstract

In this article we consider a discrete time two server queueing system with dynamic routing. We prove logarithmic asymptotics for the liklihood that a message from a source that divides its messages between the two servers in a way that minimizes the message's waiting time experiences a large waiting time. We demonstrate the merit of this asymptotic by comparing its predictions with experimental data. We illustrate how estimated entropies of the traffic streams can be used to predict the likelihood of long waiting times and demonstrate the method's accuracy through comparison with simulations.

1 Introduction

Throughout this article we refer to queueing systems in which messages are constrained to arrive at integer times as discrete time systems. The term continous time systems is used for queueing systems that have no such constraint.

Consider a discrete time single server queue with infinite buffer fed by a stationary source of traffic serving at rate s. For each $n \in \mathbf{Z}$, let $a(n)$ denote volume of work required to process message n, which arrives at time n. The amount of time, $w(n)$, that the n^{th} message must wait before service is initiated on it evolves according to Lindley's recursion:

$$w(n+1) = [w(n) + a(n) - s]^+, \tag{1}$$

where $x^+ = \max\{0, x\}$. By a theorem of Loynes [17], if the sequence $\{a(n)\}$ is stationary, then there exists a stationary sequence of random variables that satisfies the recursion defined by (1). Letting W denote an element of the stationary solution, and defining $S(0) = 0$, $S(n) = a(1) + \cdots + a(n)$, W has the same distribution as $\sup_{n\geq 0}(S(n) - sn)$.

It is known (for example, see [14, 13, 6]) that if the process $\{S(n)/n\}$ satisfies the large deviation principle with rate function I (so that, roughly speaking, $\mathbf{P}[S(n) \approx nx] \asymp \exp(-nI(x))$), then the distribution of W has exponential tails:

$$\lim_{w\to\infty} \frac{1}{w} \log \mathbf{P}(W \geq w) = -\inf_{x>0} xI(1/x+s) = -\sup\{\theta \geq 0 : \lambda(\theta) \leq s\theta\} = -\delta, \tag{2}$$

where λ is the scaled Cumulant Generating Function (sCGF) of the input traffic

$$\lambda(\theta) = \lim_{n\to\infty} \frac{1}{n} \log \mathbf{E}[\exp(\theta S(n))], \tag{3}$$

which is the Legendre-Fenchel transform of the rate function I.

One approach to using equation (2) when the statistics of the process $\{a(n)\}$ are unknown was suggested and developed by John T. Lewis and co-workers. They knew that the rate function I plays an analogous to rôle to thermodynamic entropy; a macroscopic function determined from microscopic behavior that succinctly records bulk properties of the system. Taking inspiration from chemical engineers who estimate entropy directly they proposed estimating the Legendre Fenchel transform of the rate function I, the sCGF λ in equation (3), directly. The estimation scheme they settled on is described by Duffield et al. [5]: select a block length B sufficiently large that you believe the blocked sequence $\{Y(n)\}$, where $Y(n) = a((n-1)B+1)+\cdots+a(nB)$, can be treated as i.i.d.; if $\{Y(n)\}$ is i.i.d. then $\lambda(\theta) = B^{-1} \log \mathbf{E}[\exp(\theta Y(1))]$ and thus it is natural to suggest using the maximum likelihood estimator:

$$\lambda_n(\theta) = \frac{1}{B} \log \frac{1}{\lfloor n/B \rfloor} \sum_{i=1}^{\lfloor n/B \rfloor} \exp(\theta Y(i)) \tag{4}$$

to estimate λ in equation (3) when n samples have been observed. They referred to this as estimating a traffic source's entropy; a terminology we adopt.

As an example of this estimator's use, note that δ in equation (2) can be estimated by $\delta_n = \sup\{\theta : \lambda_n(\theta) \leq s\theta\}$. In [5] a central limit theorem is proved for $\{\delta_n\}$ and, for bounded sequences $\{a(n)\}$, a large deviation principle in [8] from which a weak law of large numbers is deduced. For an indication of the success in applying this approach see, for example, Crosby et al. [1] and Lewis et al. [16].

We remark that in general in queueing systems there is a second quantity of interest, one which we will not consider here: the number of customers awaiting service - the queue length. For a general continuous time single server queue, an asymptotic related to (2) holds for the distribution of the number of messages awaiting service after the system has been run for a long time (see [14, 10]).

In this article we consider a discrete time system with two infinite-buffer first-come first-served (FCFS) servers, serving at speed one. There are three arrivals processes, one dedicated to each server and a third that divides each message between the servers in a way which minimizes the time until the message's processing is complete. We assume large deviation assumptions that include processes for which each source's message sizes can be correlated to each other, and correlated in time. We prove a large deviation result for the likelihood that a message from the third, discretionary flow experiences a large waiting time. Clearly allowing a source to dynamically route its messages to the two servers provides it with an advantage. However, we shall see that when long delays occur for a collection of source statistics the routing offers no advantage over a combined FCFS system with no routing. We investigate through simulation the merit of limiting logarithmic asymptotics in the non limiting regime, showing surprisingly good predictions even for correlated sources. We demonstrate the use of estimated entropies in this system and illustrate the value of this methodology's predictions through comparison with simulations.

2 A queueing system with dynamic routing

In the continuous time setting, various problems of long waiting times and large queue lengths in systems with dynamic routing have been investigated by many authors. In particular systems where a subset of the input flows can dynamically choose to join, from a subset of queues, the queue with the least amount of work left to process. The technicalities that are particular to problems of this sort are usually due to interaction between the servers caused by routed flows; in particular in the presence of non-routed flows. This makes these systems difficult to study, even when flows consist of i.i.d. message sizes and inter-arrival times. For examples of work of this sort, see [26, 25, 28, 20, 11] and references therein.

Here we consider a discrete time two FCFS server queueing system with dynamic routing, where messages arrive at integer times. Each server serves at rate 1, has an infinite buffer and a dedicated stream of customers. In addition there is a third set of discretionary customers that divide their messages between the two servers in a way that minimizes each message's processing latency. The flows are assumed to posses joint sample-path large deviation properties, which includes sources whose message size processes are not constructed from i.i.d. random variables and are not independent of each other.

Explicitly, for $i \in \{1,2\}$ let $w_i(n)$ denote the waiting time experienced by a virtual message (a message of length zero) that arrives at time n to queue i and is processed before the other messages that arrive at time n. Let $a_i(n)$ denote the amount of time required to process message n from source $i \in \{0,1,2\}$, where source 0 is discretionary, source 1 is dedicated to server 1 and source 2 dedicated to server 2. The virtual waiting times evolve according to:

$$w_1(n+1) = [w_1(n) + a_1(n) - 1 + \chi(n)a_0(n)]^+, \tag{5}$$

$$w_2(n+1) = [w_2(n) + a_2(n) - 1 + (1-\chi(n))a_0(n)]^+, \tag{6}$$

where for $a_0(n) > 0$

$$\chi(n) = \max\left\{\min\left\{\frac{w_2(n-1) - w_1(n-1) + a_2(n) - a_1(n) + a_0(n)}{2a_0(n)}, 1\right\}, 0\right\},$$

and $\chi(n)$ is arbitrary if $a_0(n) = 0$. The value of $\chi(n)$ determines the division of the message $a_0(n)$ to the two servers, ensuring that the message gets processed as quickly as possible. The possible division of routed messages between the two servers leads to a monotonicity in the waiting times as a function of messages sizes. Our proof relies on this feature, which is not necessarily present if routed messages cannot be divided between servers and join the shortest queue routing is used, but we conjecture the same results holds in that setting.

The object that interests us are logarithmic asymptotics of the form in equation (2), but for the discretionary source of messages. That is, we wish to analyse the likelihood that a virtual discretionary message experiences a long waiting time. In the continuous time case where the three input flows are Poisson, each with a general distribution of service times, a result of this sort has been announced by Pechersky, Suhov and Vvedenskaya [22, 21]. It's predictions are compared with experimental data in [7].

Assuming the process $\{a_0(n), a_1(n), a_2(n)\}$ of message sizes is stationary, the event that interests us is when both servers have a lot of work to process. In this situation a message from the discretionary source must wait a long time before service is initiated on it. In particular set $y_i(0) = 0$ and define for $n \geq 0$

$$y_1(n+1) = \sum_{j=1}^{n} (a_1(j) + \alpha(j) a_0(j) - 1) \text{ and } y_2(n+1) = \sum_{j=1}^{n} (a_2(j) + (1-\alpha(j)) a_0(j) - 1),$$

where for $a_0(n) > 0$

$$\alpha(n) = \max\left\{\min\left\{\frac{y_2(n-1) - y_1(n-1) + a_2(n) - a_1(n) + a_0(n)}{2a_0(n)}, 1\right\}, 0\right\},$$

and $\alpha(n)$ is arbitrary if $a_0(n) = 0$. Defining

$$W_0 = \sup_{n \geq 0} \min\{y_1(n), y_2(n)\}, \tag{7}$$

we are interested in the probability of the event $\{W_0 \geq w\}$ when w is large. We identify conditions under which we prove logarithmic asymptotics for the probability of this event:

$$\lim_{w \to \infty} w^{-1} \log \mathbf{P}(W_0 \geq w) = -\delta_0, \tag{8}$$

where we relate δ_0 to assumed large deviation behavior of the input flows. We use sample-path techniques together with the following monotonicity lemma to deduce the main result.

Lemma 1. *The value of W_0 is monotonic in its inputs. That is, if $a_i'(n) \geq a_i(n)$ for all i and n, then $W_0 \leq W_0'$, where W_0' as in equation (7), but with $a_i'(n)$.*

Proof. Assume that for all $n \leq N$

$$y_1(n) \leq \sum_{j=1}^{n-1} (a_1'(j) + \alpha'(j) a_0'(j) - 1) = y_1'(n)$$

and

$$y_2(n) \leq \sum_{j=1}^{n-1} (a_2'(j) + (1-\alpha'(j)) a_0'(j) - 1) = y_2'(n),$$

where $\alpha'(n)$ is defined analogously to $\alpha(n)$. Monotonicity is clear if $a_0'(N+1) = 0$, so assume $a_0'(N+1) > 0$. Then with $a_i'(N+1) = a_i(N+1) + \epsilon_i$ and $y_i'(N) = y_i(N) + \gamma_i$, where $\epsilon_i, \gamma_i \geq 0$,

$$\begin{aligned}
y_1'(N+1) &= y_1(N) + \gamma_1 + a_1(N) + \epsilon_1 - 1 \\
&\quad + \max\left\{\min\left\{\tfrac{y_2(N-1)+\gamma_2-y_1(N-1)-\gamma_1+a_2(N)+\epsilon_2-a_1(N)-\epsilon_1+a_0(N)+\epsilon_0}{2}, a_0(N)+\epsilon_0\right\}, 0\right\} \\
&\geq y_1(N) + \gamma_1 + a_1(N) + \epsilon_1 - 1 \\
&\quad + \max\left\{\min\left\{\tfrac{y_2(N-1)-y_1(N-1)+a_2(N)-a_1(N)+a_0(N)}{2}, a_0(N), 0\right\}\right\} - \tfrac{\gamma_1+\epsilon_1}{2} \\
&= y_1(N+1) + \tfrac{\gamma_1+\epsilon_1}{2}.
\end{aligned}$$

Similar arguments apply to $y_2(N+1)$. $\square$

3 Large deviation and functional setup

For convenience we first recall the basic facts of the Large Deviation Principle (LDP). More details can be found in one of they standard texts, such as [3]. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability triple and $\mathcal{X}$ be a Hausdorff space with Borel σ-algebra $\mathcal{B}$. Let $\{X_n\}$ be a sequence of random elements taking values in $\mathcal{X}$. We say that $\{X_n, n \in \mathbf{N}\}$ satisfies the *Large Deviation Principle* (LDP) with *rate function* $I : \mathcal{X} \to [0, +\infty]$ if I is lower semi-continuous, and

$$-\inf_{x\in G} I(x) \leq \liminf_{n\to\infty} \frac{1}{n} \log \mathbf{P}(X_n \in G) \text{ and } \limsup_{n\to\infty} \frac{1}{n} \log \mathbf{P}(X_n \in F) \leq -\inf_{x\in F} I(x) \tag{9}$$

for all open G and all closed F. A rate function is *good* if its level sets $\{x : I(x) \leq \alpha\}$ are compact for all $\alpha \geq 0$. The *contraction principle* states that if $\{X_n, n \in \mathbf{N}\}$ satisfies the LDP in $\mathcal{X}$ with good rate function I and $f : \mathcal{X} \to \mathcal{Y}$ is continuous, where $\mathcal{X}$ and $\mathcal{Y}$ are Hausdorff, then $\{f(X_n), n \in \mathbf{N}\}$ satisfies the LDP in $\mathcal{Y}$ with good rate function given by $J(y) = \inf\{I(x) : f(x) = y\}$. A proof can be found in Dembo and Zeitouni [3] Theorem 4.2.1. If $\mathcal{X}$ is a metric space with distance d, two processes $\{X_n\}$ and $\{Y_n\}$ are said to be *exponentially equivalent* if for all $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbf{P}(d(X_n, Y_n) > \epsilon) = -\infty.$$

If two processes are exponentially equivalent they satisfy the same LDP (Theorem 4.2.13 of [3]).

For each source $i \in \{0, 1, 2\}$, we define the *sample-paths* of the message size process $\{a_i(n)\}$ by

$$S_n^{(i)}(t) = \frac{1}{n} \sum_{j=1}^{[nt]} a_i(j),\ t \in [0, \infty),\ n \geq 1, \tag{10}$$

where the empty sum is defined to be zero. For each n, $S_n^{(i)}$ is a CADLAG (right continuous having left hand limits) function. We also define their piecewise linear approximations, which are continuous functions

$$\hat{S}_n^{(i)}(t) = \frac{1}{n} \sum_{j=1}^{[nt]} a_i(j) + \left(t - \frac{[nt]}{n}\right) a_i([nt] + 1),\ t \in [0, \infty),\ n \geq 1. \tag{11}$$

In order to prove the existence of the limit in equation (8) and identify δ_0 we assume the LDP holds for the process defined by the paths defined in equation (10). We shall also assume that the paths $(S_n^{(0)}, S_n^{(1)}, S_n^{(2)})$ and $(\hat{S}_n^{(0)}, \hat{S}_n^{(1)}, \hat{S}_n^{(2)})$ are exponentially equivalent, though we do this for convenience and the result can be pushed through without this assumption.

For $\nu \in \mathbf{R}$, let $\mathcal{X}_\nu$ denote the set of CADLAG functions ϕ on $[0, \infty)$ such that $\lim_{t\to\infty}(1+t)^{-1}\phi(t) = \nu$. Let $\mathcal{A}_\nu \subset \mathcal{X}_\nu$ denote the subset of absolutely continuous functions on $[0, \infty)$ with $\phi(0) = 0$. In particular, the elements of $\mathcal{A}_\nu$ are exactly the integrals of functions that are elements of $\mathcal{L}^1[0, x)$ for all $x > 0$ (for example, see Riesz and Sz.-Nagy [23]). Equip $\mathcal{X}_\nu$ with the topology induced by the norm $||\phi|| = \sup_{t\geq 0} |(1 + t)^{-1}\phi(t)|$. We equip products of $\mathcal{X}_\nu$ spaces with the product topology.

Assumption 1. *The process* $\{S_n^{(0)}, S_n^{(1)}, S_n^{(2)}\}$ *satisfies the large deviation principle in* $\mathcal{X}_{m_0} \times \mathcal{X}_{m_1} \times \mathcal{X}_{m_2}$ *with rate function*

$$I(\phi_0, \phi_1, \phi_2) = \begin{cases} \int_0^\infty I_{\text{local}}(\dot{\phi}_0(s), \dot{\phi}_1(s), \dot{\phi}_2(s))ds; & \text{if } \phi_i \in \mathcal{A}_{m_i}\ \forall i \in \{0, 1, 2\}, \\ +\infty & \text{otherwise.} \end{cases}$$

where $I_{\text{local}} : \mathbf{R} \to [0, \infty]$ *is a convex, lower semi-continuous function with compact level sets satisfying* $I_{\text{local}}(m_0, m_1, m_2) = 0$ *and* $I_{\text{local}}(x_0, x_1, x_2) > 0$ *for all* $(x_0, x_1, x_2) \neq (m_0, m_1, m_2)$*. Moreover, we assume that the sample paths* $(S_n^{(0)}, S_n^{(1)}, S_n^{(2)})$ *and their piecewise linear approximations* $(\hat{S}_n^{(0)}, \hat{S}_n^{(1)}, \hat{S}_n^{(2)})$ *are exponentially equivalent.*

Note that using the contraction principle and Jensen's inequality, a consequence of this assumption is that the process defined by

$$(S_n^{(0)}(1), S_n^{(1)}(1), S_n^{(2)}(1)) = (\hat{S}_n^{(0)}(1), \hat{S}_n^{(1)}(1), \hat{S}_n^{(2)}(1)) = \left(\frac{1}{n} \sum_{i=1}^{n} a_0(i), \frac{1}{n} \sum_{i=1}^{n} a_1(i), \frac{1}{n} \sum_{i=1}^{n} a_2(i) \right)$$

satisfies the LDP in $\mathbf{R}^3$ with rate function I_{local}, hence the subscript "local". As I_{local} is convex, the Legendre Fenchel transform of I_{local} is the scaled cumulant generating function:

$$\lambda_{\text{local}}(\theta_0, \theta_1, \theta_2) = \lim_{n \to \infty} \frac{1}{n} \log \mathbf{E}[\exp(n(\theta_0 S_n^{(0)}(1) + \theta_1 S_n^{(1)}(1) + \theta_2 S_n^{(2)}(1))].$$

We note that it is known that a large class of processes satisfy Assumption 1. Restricting to $[0, 1]$, Dembo and Zajic [2] provide general conditions under which the piecewise linear sample paths satisfy the functional LDP with a good rate function in the space of continuous functions on $[0, 1]$ equipped with the sup norm. Theorem 1 of [13] establishes that if these conditions are met and λ_{local} is differentiable at the origin, then the LDP for the piecewise linear paths is also satisfied in the subset of continuous functions of $\mathcal{X}_{m_i}$. Sufficient conditions for exponential equivalence of the CADLAG and piecewise linear sample-paths can be found [2], which essentially require that the tail of the message size distributions decay faster than exponentially. For a range of other applications of this LDP see, for example, [4, 27, 15, 18, 19, 9].

As well as the large deviations Assumption 1, we make the following stability assumption.

Assumption 2. *We have* $m_1 < 1$*,* $m_2 < 1$ *and* $m_0 + m_1 + m_2 < 2$*.*

Assuming $m_i < 1$ ensures that on the large deviations scale processing dedicated messages alone, each of the two servers is stable. This is not necessary to ensure the routed messages experience a stable system, only $m_0 + m_1 + m_2 < 2$ is. However, if either of the two queues is unstable, that queue can be ignored and the system effectively reduces to a single server queue, making considerations uninteresting.

4 Logarithmic asymptotics

Our main result, which follows from Lemmas 3 and 4 below, is:

Proposition 1. *Under Assumptions 1 and 2*

$$\lim_{w \to \infty} \frac{1}{w} \log \mathbf{P}(W_0 \geq w) = -\delta_0, \tag{12}$$

where

$$\delta_0 = \inf_{\beta \in [0,1]} \inf_{t > 0} \inf_{z \in \mathbf{R}} t I_{\text{local}} \left(\frac{z}{t}, \frac{1 - \beta z}{t} + 1, \frac{1 - (1 - \beta) z}{t} + 1 \right). \tag{13}$$

One method of proof would be to attempt an explicit construction of W_0 as a function of the sample-paths defined in equation (10), then prove this construction is continuous in its inputs and use the contraction principle or one of its extensions. The representations of such a construction, thought clearly algorithmically expressible, do not lend themselves to manipulation.

Instead we identify sets G open in $\mathcal{X}_{m_0} \times \mathcal{X}_{m_1} \times \mathcal{X}_{m_2}$ and F closed in $\mathcal{X}_{m_0} \times \mathcal{X}_{m_1} \times \mathcal{X}_{m_2}$ such that if $(\hat{S}_n^{(0)}, \hat{S}_n^{(1)}, \hat{S}_n^{(2)}) \in G$ then $W_0 \geq n$ and $\{(S_n^{(0)}, S_n^{(1)}, S_n^{(2)}) : W_0 \geq n\} \subset F$. Moreover, applying the LDP bounds (9) to G and F gives an arbitrarily small difference. In Assumption 1 we assume exponential equivalence of the sample paths and their linear approximations, as it is convenient to consider the lower bound with the linear approximation. At the cost of slightly more involved arguments, this assumption can be removed.

We first prove the upper bound, then lower. The set F defined in the next lemma will be used in the proof of the upper bound. We first prove it's closed.

Lemma 2. *Under Assumption 2 the set*

$$F = \{(\phi_0, \phi_1, \phi_2) : \phi_1(t) + \beta\phi_0(t) - t \geq 1, \phi_2(t) + (1-\beta)\phi_0(t) - t \geq 1 \textit{ some } (t, \beta) \in \mathbf{R}_+ \times [0,1]\} \quad (14)$$

is closed.

Proof. Let $(\phi_{0,n}, \phi_{1,n}, \phi_{2,n}) \in F$ for all n and $(\phi_{0,n}, \phi_{1,n}, \phi_{2,n}) \to (\phi_0, \phi_1, \phi_2)$. We wish to prove $(\phi_0, \phi_1, \phi_2) \in F$. We restrict our attention to a compact t range by the following argument. Choosing $0 < \epsilon < 2 - m_0 - m_1 - m_2$, there exists N_ϵ such that for all $t \in [0, \infty)$ and all $n > N_\epsilon$

$$\sum_{i=0}^{2} \phi_{i,n}(t) < \sum_{i=0}^{2} \phi_i(t) + \frac{\epsilon}{2}(1+t). \quad (15)$$

Moreover, there exists $T_\epsilon < \infty$ such that for all $t > T_\epsilon$

$$\sum_{i=0}^{2} \phi_i(t) < \sum_{i=0}^{2} \left(m_i + \frac{\epsilon}{6}\right)(1+t). \quad (16)$$

From equations (15) and (16), we deduce that for all $n > N_\epsilon$ and $t > T_\epsilon$

$$\sum_{i=0}^{2} \phi_{i,n}(t) < \left(\sum_{i=0}^{2} m_i + \epsilon\right)(1+t) < 2t + 2. \quad (17)$$

Note that $\phi_1(t) + \beta\phi_0(t) - t \geq 1$ and $\phi_2(t) + (1-\beta)\phi_0(t) - t \geq 1$ imply $\phi_0(t) + \phi_1(t) + \phi_2(t) - 2t - 2 \geq 0$. Therefore equation (17) implies that for all $n > N_\epsilon$, $t > T_\epsilon$ and all $\beta \in [0,1]$

$$\phi_{1,n}(t) + \beta\phi_{0,n}(t) - t < 1 \qquad \text{and} \qquad \phi_{2,n}(t) + (1-\beta)\phi_{0,n}(t) - t < 1.$$

Thus for all $n > N_\epsilon$ the t that satisfies the condition in the set F of equation (14) for $(\phi_{0,n}, \phi_{1,n}, \phi_{2,n})$ must be in $[0, T_\epsilon]$.

Assume that $(\phi_0, \phi_1, \phi_2) \notin F$ so that there does not exist $t \in [0, T_\epsilon]$ and $\beta \in [0,1]$ such that $\phi_1(t) + \beta\phi_0(t) - t \geq 1$ and $\phi_2(t) + (1-\beta)\phi_0(t) - t \geq 1$. Then there exists $\delta > 0$ such that

$$\sup_{t \in [0, T_\epsilon]} \sup_{\beta \in [0,1]} \min\{\phi_1(t) + \beta\phi_0(t) - t, \phi_2(t) + (1-\beta)\phi_0(t) - t\} < 1 - \delta.$$

On the compact interval $[0, T_\epsilon]$, $(\phi_{0,n}, \phi_{1,n}, \phi_{2,n})$ converges in the sup norm to (ϕ_0, ϕ_1, ϕ_2), so that there exists N_δ so that for all $n > \max(N_\delta, N_\epsilon)$

$$\sup_{t\in[0,T_\epsilon]} \sup_{\beta\in[0,1]} \min\{\phi_{1,n}(t) + \beta\phi_{0,n}(t) - t, \phi_{2,n}(t) + (1-\beta)\phi_{0,n}(t) - t\} < 1,$$

contradicting the assumption that $(\phi_{0,n}, \phi_{1,n}, \phi_{2,n}) \in F$ and thus F is closed. $\square$

Having shown F is closed, we deduce the upper bound.

Lemma 3. *Under Assumptions 1 and 2*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbf{P}[W_0 \geq n] \leq -\delta_0. \tag{18}$$

Proof. Consider a point $\omega \in \Omega$ contained in the event $\{W_0 \geq n\}$. Then for the corresponding sample-paths $(S_n^{(0)}, S_n^{(1)}, S_n^{(2)})$ there exists t and $\beta \in [0,1]$ such that

$$S_n^{(1)}(t) + \beta S_n^{(0)}(t) - t \geq 1 \qquad \text{and} \qquad S_n^{(2)}(t) + (1-\beta)S_n^{(0)}(t) - t \geq 1. \tag{19}$$

However, the existence of t and β such that the inequalities (19) are satisfied do not imply $\{W_0 \geq n\}$ occurred. For example, if

$$S_n^{(0)}(t) = \begin{cases} 0 & \text{if } t \in [0,1) \\ 1.5 & \text{if } t \geq 1 \end{cases} ;\ S_n^{(1)}(t) = \begin{cases} 0 & \text{if } t \in [0,1) \\ 0.5 & \text{if } t \in [1,2) \\ 2.5 & \text{if } t \geq 2 \end{cases} ;\ S_n^{(2)}(t) = \begin{cases} 0 & \text{if } t \in [0,1) \\ 1 & \text{if } t \in [1,2) \\ 2 & \text{if } t \geq 2 \end{cases} ;$$

then $y_1(0) = y_2(0) = 0$, $y_1(n) = y_2(n) = 0.5n$, $y_1(2n) = 1.5n, y_2(2n) = 0.5n$, but with $\beta = 1/3$, $S_n^{(1)}(2) + \beta S_n^{(0)}(2) - 2 = 1 = S_n^{(1)}(2) + (1-\beta)S_n^{(0)}(2) - 2$. Thus

$$\{(S_n^{(0)}, S_n^{(1)}, S_n^{(2)}) : W_0 \geq n\} \subset F,$$

where F is defined in equation (14). Therefore

$$\mathbf{P}(W_0 \geq n) \leq \mathbf{P}((S_n^{(1)}, S_n^{(2)}, S_n^{(0)}) \in F).$$

As Lemma 2 proves that F is closed, the large deviation bounds from equation (9) provides an upper bound on $\limsup n^{-1} \log \mathbf{P}((S_n^{(1)}, S_n^{(2)}, S_n^{(0)}) \in F)$:

$$\begin{aligned}
&\leq -\inf_\beta \inf_{t>0} \{I(\phi_0, \phi_1, \phi_2) : \phi_1(t) + \beta\phi_0(t) - t \geq 1, \phi_2(t) + (1-\beta)\phi_0(t) - t \geq 1\} \\
&= -\inf_\beta \inf_{t>0} \left\{ \int_0^\infty I_{\text{local}}(\dot\phi_0(s), \dot\phi_1(s), \dot\phi_2(s))ds : \phi_1(t) + \beta\phi_0(t) - t \geq 1, \phi_2(t) + (1-\beta)\phi_0(t) - t \geq 1 \right\} \\
&\leq -\inf_\beta \inf_{t>0} \left\{ \int_0^t I_{\text{local}}(\dot\phi_0(s), \dot\phi_1(s), \dot\phi_2(s))ds : \phi_1(t) + \beta\phi_0(t) - t \geq 1, \phi_2(t) + (1-\beta)\phi_0(t) - t \geq 1 \right\} \\
&\leq -\inf_\beta \inf_{t>0} \{tI_{\text{local}}(\phi_0(t)/t, \phi_1(t)/t, \phi_2(t)/t) : \phi_1(t) + \beta\phi_0(t) \geq 1 + t, \phi_2(t) + (1-\beta)\phi_0(t) \geq 1 + t\} \\
&\leq -\inf_{\beta\in[0,1]} \inf_{t>0} \inf_{z\in\mathbf{R}} tI_{\text{local}}\left(\frac{z}{t}, \frac{1-\beta z}{t} + 1, \frac{1-(1-\beta)z}{t} + 1\right) = -\delta_0,
\end{aligned}$$

where we have used Jensen's inequality and the convexity of I_{local} in the transition from the third to second last inequalities. Thus the upper bound is obtained. $\square$

To establish the corresponding lower bound, we identify a set G such that $(S_n^{(0)}, S_n^{(1)}, S_n^{(2)}) \in G$ implies $W_0 \geq n$ and the logarithmic rate of $\mathbf{P}((S_n^{(0)}, S_n^{(1)}, S_n^{(2)}) \in G)$ is arbitrarily close to $-\delta_0$. It is in this lemma that we make use of the monotonicity demonstrated in Lemma 1.

Lemma 4. *Under Assumption 1*

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbf{P}(W_n \geq n) \geq -\delta_0. \tag{20}$$

Proof. Let (t^*, β^*, z^*) be the optimal arguments from equation (13). Define the following absolutely continuous functions which are our candidate paths that give rise to this optimal value

$$\hat{\phi}_0(s) = \begin{cases} (z^* s)/t^* & \text{if } s \leq t^*, \\ (z^*) + (s - t^*)m_0 & \text{if } s \geq t^*, \end{cases} \quad \hat{\phi}_1(s) = \begin{cases} (1 - \beta^* z^*)s/t^* + s & \text{if } s \leq t^*, \\ (1 - \beta^* z^* + t^*) + (s - t^*)m_1 & \text{if } s \geq t^*, \end{cases}$$

and

$$\hat{\phi}_2(s) = \begin{cases} (1 - (1 - \beta^*)z^*)s/t^* + s & \text{if } s \leq t^*, \\ (1 - (1 - \beta^* z^*) + t^*) + (s - t^*)m_2 & \text{if } s \geq t^*. \end{cases}$$

Consider an open ball G of radius $\epsilon > 0$ around the functions $(\hat{\phi}_0(t) + \eta(t), \hat{\phi}_1(t) + \eta(t), \hat{\phi}_2(t) + \eta(t))$, where

$$\eta(s) = \begin{cases} \epsilon(1 + t^*)(t^*)^{-1} s & \text{if } s \leq t^*, \\ \epsilon(1 + t^*) & \text{otherwise.} \end{cases}$$

We wish to show that if $(\hat{S}_n^{(0)}, \hat{S}_n^{(1)}, \hat{S}_n^{(2)}) \in G$ then $W_0 \geq n$. Now for any n let $(\hat{S}_n^{(0)}, \hat{S}_n^{(1)}, \hat{S}_n^{(2)})$ be the smallest triple of functions greater than $(\hat{\phi}_0, \hat{\phi}_1, \hat{\phi}_2)$ that are piecewise linear on intervals of length $1/n$. Effectively, $(\hat{S}_n^{(0)}, \hat{S}_n^{(1)}, \hat{S}_n^{(2)})$ equals $(\hat{\phi}_0, \hat{\phi}_1, \hat{\phi}_2)$, but with the possibility of being larger on the interval $(t^*, [t^* n + 1]/n]$. For n sufficiently large $(\hat{S}_n^{(0)}, \hat{S}_n^{(1)}, \hat{S}_n^{(2)}) \in G$. Using the monotonicity Lemma 1 it suffices to prove that for this path $(\hat{S}_n^{(0)}, \hat{S}_n^{(1)}, \hat{S}_n^{(2)})$ it is the case that $W_0 \geq n$. Note that

$$\hat{\phi}_1(t^*) + \beta^* \hat{\phi}_0(t^*) - t^* = \hat{\phi}_2(t^*) + (1 - \beta^*)\hat{\phi}_0(t^*) - t^* = 1.$$

As $(\hat{S}_n^{(0)}, \hat{S}_n^{(1)}, \hat{S}_n^{(2)}) \geq (\hat{\phi}_0, \hat{\phi}_1, \hat{\phi}_2)$, $\min(y_1([t^* n + 1]), y_2([t^* + 1])) \geq n$ and thus $W_0 \geq n$.

As G is an open ball, the large deviation lower bound (9) implies the following lower bound on $\liminf n^{-1} \log \mathbf{P}((S_n^{(0)}, S_n^{(1)}, S_n^{(2)}) \in G)$:

$$\begin{aligned}
&\geq \; -\inf\{I(\phi_0, \phi_1, \phi_2) : (\phi_0, \phi_1, \phi_2) \in G\} \\
&\geq \; -\int_0^\infty I_{\text{local}} \left(\frac{d}{ds}(\hat{\phi}_0(s) + \eta(s)), \frac{d}{ds}(\hat{\phi}_1(s) + \eta(s)), \frac{d}{ds}(\hat{\phi}_2(s) + \eta(s)) \right) ds \\
&= \; -t^* I_{\text{local}} \left(\frac{z^*}{t^*} + \frac{\epsilon(1 + t^*)}{t^*}, \frac{(1 - \beta^* z^*)}{t^*} + 1 + \frac{\epsilon(1 + t^*)}{t^*}, \frac{(1 - (1 - \beta^*)z^*)}{t^*} + 1 + \frac{\epsilon(1 + t^*)}{t^*} \right).
\end{aligned}$$

On taking limits as $\epsilon \to 0$, this coincides with δ_0 in equation (13). $\square$

5 Comparison with the single server queue

Proposition 1 proves that

$$\lim_{w\to\infty} \frac{1}{w} \log \mathbf{P}(W_0 > w) = -\delta_0,$$

where

$$\delta_0 = \inf_{\beta\in[0,1]} \inf_{t>0} \inf_{z\in\mathbf{R}} tI_{\text{local}}\left(\frac{z}{t}, \frac{1-\beta z}{t}+1, \frac{1-(1-\beta)z}{t}+1\right).$$

The value of δ_0 corresponds to the waiting times at both servers being large simultaneously and should therefore be compared with 2δ from equation (2) for a single server queue fed by all three input flows that serves at rate 2. This can be written as:

$$2\delta = \inf_{t>0} \inf_{y\in\mathbf{R}} \inf_{z\in\mathbf{R}} tI_{\text{local}}\left(\frac{y}{t}, \frac{z}{t}, \frac{2-y-z}{t}+2\right). \tag{21}$$

For comparison, equation (13) can be re-written as

$$\delta_0 = \inf_{t>0} \inf_{y\in\mathbf{R}} \inf_{z\in[1+(1-y)t, 1+t]} tI_{\text{local}}\left(\frac{y}{t}, \frac{z}{t}, \frac{2-y-z}{t}+2\right). \tag{22}$$

With a larger range over which to select z in equation (21), clearly $2\delta \leq \delta_0$. Thus sending all messages to a combined server can only make it more likely that a virtual flow 0 message (which in the routed system would be discretionary) experiences a large waiting time. We have the following convexity property for the function over which the infimums are taken.

Lemma 5. *As I_{local} is convex, the function $J(y,z,t) = tI_{\text{local}}\left(yt^{-1}, zt^{-1}, (2-y-z)t^{-1}+2\right)$ is convex for $y, z \in \mathbf{R}$ and $t > 0$.*

Proof. For $\vec{a} = (a_0, a_1, a_2)$, $\vec{b} = (b_0, b_1, b_2)$ and $\alpha \in [0,1]$, note that

$$\begin{aligned}
&J(\alpha\vec{a} + (1-\alpha)\vec{b}) \\
&\quad = \tfrac{1}{\alpha a_2 + (1-\alpha)b_2} I_{\text{local}}\left(\eta\left(\tfrac{a_0}{a_2}, \tfrac{a_1}{a_2}, \tfrac{2-a_0-a_1}{a_2} - 2\right) + (1-\eta)\left(\tfrac{b_0}{b_2}, \tfrac{b_1}{b_2}, \tfrac{2-b_0-b_1}{b_2} - 2\right)\right) \\
&\quad \leq \alpha I\left(\tfrac{a_0}{a_2}, \tfrac{a_1}{a_0}, \tfrac{2-a_0-a_1}{a_2} - 2\right) + (1-\alpha) I\left(\tfrac{b_0}{b_2}, \tfrac{b_1}{b_2}, \tfrac{2-b_0-b_1}{b_2} - 2\right) \\
&\quad = \alpha J(\vec{a}) + (1-\alpha) J(\vec{b}),
\end{aligned}$$

where we have used the convexity of I_{local} and, as $a_2, b_2 > 0$,

$$\eta = \frac{\alpha a_2}{\alpha a_2 + (1-\alpha)b_2} \in [0,1].$$

□

Thus the infimum in equation (22) is either 2δ or occurs at one the boundary points of the z constraint. These boundary points correspond to β in equation (13) being 0 or 1. This effect can be understood easily. In the system with routing W_0 can have a lighter tail than in the combined system; if one server is prone to being back-logged, discretionary messages join the other server's queue. This happens when the infimum over β in equation (13) occurs at $\beta = 0$ or 1, indicating all discretionary messages are routed to one of the two servers.

As we plan to use estimated sCGFs, we would like a dual form for δ_0 given in equation (13). We present one below to show that such a representation exits, but it is not that practically useful as it cannot be explicitly represented in terms of the sCGF of I_{local}.

Theorem 6. *We have the identity*

$$\delta_0 = \inf_{\beta \in [0,1]} \sup\{\theta \geq 0 : \lambda^{(\beta)}(\theta) \leq 0\}, \tag{23}$$

where $\lambda^{(\beta)}(\cdot)$ is the Legendre-Fenchel transform of the convex function

$$I^{(\beta)}(t) = \inf_{y \geq 0} I_{\text{local}}\left(y, t - \beta y + 1, t - (1-\beta)y + 1\right), \ t > 0. \tag{24}$$

Proof. Consider equation (13) and note it can be written as

$$\delta_0 = \inf_{\beta \in [0,1]} \inf_{t>0} t I^{(\beta)}\left(\frac{1}{t}\right).$$

Observe that for $\theta \geq 0$

$$\begin{aligned} \theta \leq \inf_{t>0} t I^{(\beta)}\left(\frac{1}{t}\right) \quad &\Longleftrightarrow \quad \frac{\theta}{t} \leq I^{(\beta)}\left(\frac{1}{t}\right) \ \forall\, t > 0 \\ &\Longleftrightarrow \quad \sup_{t>0}\left\{\theta t - I^{(\beta)}(t)\right\} \leq 0. \end{aligned}$$

Thus $0 \leq \theta \leq \inf_{t>0} t I^{(\beta)}\left(t^{-1}\right)$ if and only if it's Legendre-Fenchel transform $\lambda^{(\beta)}(\theta) \leq 0$ and therefore

$$\sup\{\theta : \lambda^{(\beta)}(\theta) \leq 0\} = \inf_{t>0} t I^{(\beta)}\left(\frac{1}{t}\right) \text{ and } \delta_0 = \inf_{\beta \in [0,1]} \sup\{\theta : \lambda^{(\beta)}(\theta) \leq 0\}.$$

All that remains to be shown is that $I^{(\beta)}$ is convex. We proceed by proving that the function

$$L_\beta(x, y) = I_{\text{local}}\left(y, x - \beta y + 1, x - (1-\beta)y + 1\right).$$

is convex. Let $\eta \in [0,1]$ and $\vec{a} = (a_1, a_2), \vec{b} = (b_1, b_2)$, then

$$\begin{aligned} L_\beta(\eta\vec{a} + (1-\eta)\vec{b}) \quad &= \quad I_{\text{local}}(\eta a_1 + (1-\eta)b_2, \\ &\qquad \eta(a_1 - \beta a_2 + 1) + (1-\eta)(b_1 - \beta b_2 + 1), \\ &\qquad \eta(a_1 - (1-\beta)a_2 + 1) + (1-\eta)(b_1 - (1-\beta)b_2 + 1)) \\ &\leq \quad \eta L_\beta(\vec{a}) + (1-\eta)L_\beta(\vec{b}). \end{aligned}$$

That $I^{(\beta)}(x) = \inf_y L_\beta(x, y)$ is convex follows from Theorem 5.3 of [24]. ☐

Although the previous representation is not particularly helpful unless we estimate $\lambda^{(\beta)}$ directly, we can limit δ_0 to one of three values that are readily calculated from the entropies of the input flows when they are independent of each other. That is, each individual source may have dependencies in time, but flows are not correlated to each other.

Theorem 7. *Assume the three arrival flows are independent of each other (though possibly dependent in time), so that $I_{\text{local}}(x, y, z) = K_0(x) + K_1(y) + K_2(z)$, where K_i is the local rate function for source i. Let λ_i denote the Legendre-Fenchel transform of K_i. Define*

$$\delta^{(1)} = \sup\{\theta : \inf_{\vartheta}[\lambda_0(\vartheta) + \lambda_1(\vartheta) + \lambda_2(\theta - \vartheta)] - \theta \leq 0\},$$

which is the exponent in the tail probability that the waiting times at both servers are long simultaneously when the first server is fed by sources 0 *and* 1 *and the second by source* 2*, and*

$$\delta^{(0)} = \sup\{\theta : \inf_{\vartheta}[\lambda_0(\vartheta) + \lambda_2(\vartheta) + \lambda_1(\theta - \vartheta)] - \theta \leq 0\},$$

which is the exponent in the tail probability that the waiting times at both servers are long simultaneously when the first server is fed by source 1 *and the second by sources* 0 *and* 2*. When a single server queue, serving at rate 2, is fed with three independent sources equation (2) gives*

$$\delta = \sup\{\theta : \lambda_0(\theta) + \lambda_1(\theta) + \lambda_2(\theta) - 2\theta \leq 0\}.$$

Then δ_0 *is either* $\min(\delta^{(0)}, \delta^{(1)})$ *or* 2δ*, and* $2\delta \leq \delta_0 \leq \min(\delta^{(0)}, \delta^{(1)})$*.*

Proof. Noting that the Legendre Fenchel transform of an inf convolution is the sum of the Legendre Fenchel transforms and vice versa (Theorem 16.4 of Rockafellar [24]), the value of δ is just the standard formula (equation (2)) for the single server queue fed by three independent sources served at rate 2.

The values of $\delta^{(1)}$, $\delta^{(0)}$ correspond to the system where β is 1 and 0. As the logic is similar for both cases, we consider only $\delta^{(1)}$. Let $\square$ denote the infimal convolution operator, so that $f \square g(t) = \inf_y(f(y) + g(t-y))$. Assume that the infimum over β is attained when $\beta = 1$, then equation (13) gives

$$\delta^{(1)} = \inf_{t>0} t \left(K_0 \square K_1(t^{-1}+1) + K_2(t^{-1}+1)\right).$$

As in Theorem 6, this implies

$$\delta^{(1)} = \sup\left\{\theta \geq 0 : \sup_x \left[\theta x - (K_0 \square K_1(x) + K_2(x))\right] - \theta \leq 0.\right\}$$

Note that the interior of this equation is the Legendre-Fenchel transform of a sum of K_2 and the inf-convolution of K_0 and K_1. Thus the form for $\delta^{(1)}$ in the statement of this theorem follows.

The relationships between δ_0, δ, $\delta^{(1)}$ and $\delta^{(2)}$ then follow from the comments after Lemma 5. $\square$

6 Theory compared with simulation

Our interests in this section are twofold:

- To demonstrate the differences between the likelihood of a long waiting time in the single server queueing system fed by all three sources and the queueing system with routing.
- To understand the merit of approximating $\mathbf{P}(W_0 \geq w)$ by $\exp(-w\delta_0)$.

Simulations of both the queueing system with routing (equations (5) and (6)) and the combined single server queueing system (equation (2)), served at rate 2, were constructed. When results regarding both are reported on the same graph, the systems were run with identical input flow traces. As we are interested in the logarithmic asymptotics of equation (12), the data recorded from simulations are "watermark" plots: the logarithm of the empirical frequency with which $W_0 \geq w$ versus w. All simulations in this section were run starting with empty queues for $n = 2,000,000$ time steps.

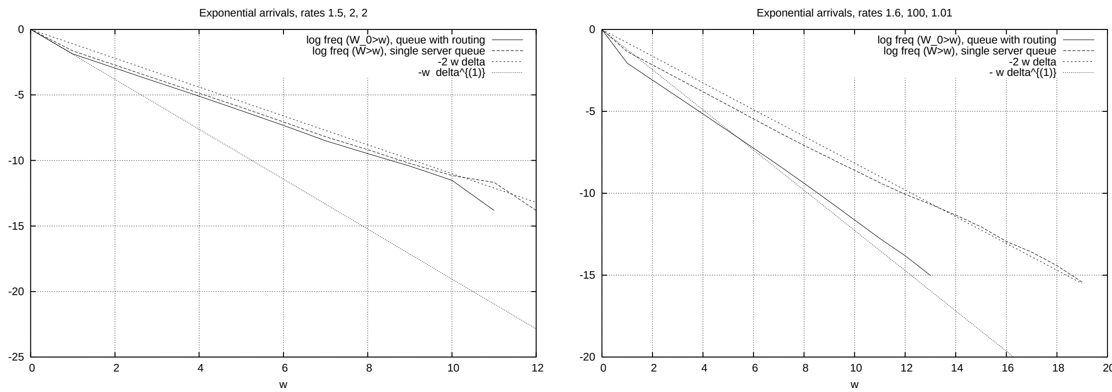

Figure 1: Exponential inputs. Simulated queues, with and without routing, and 2δ and $\delta^{(1)}$ from Theorem 7.

According to the logarithmic asymptotic in equation (12) $\log \mathbf{P}(W_0 \geq w) = -w\delta_0 + o\Big(w\Big)$, as $w \to \infty$. Even having precise knowledge of δ_0, fundamental to determining the relevance of approximating $\mathbf{P}(W_0 \geq w)$ by $\exp(-w\delta_0)$ is understanding the character of the error $o\Big(w\Big)$. In this section first we report on the predictions of Theorem 7 in comparison with results of simulation for i.i.d. exponentially distributed sources and then for Markovian sources.

Assume that each of the three sources offers i.i.d. exponentially distributed message sizes. Source $i \in \{0, 1, 2\}$ messages sizes are distributed with rate ν_i, so that $\mathbf{P}[a_i(n) \geq x] = \exp(-\nu_i x)$ for all n. Then the sCGF for source i is $\lambda_i(\theta) = \log(\nu_i(\nu_i - \theta)^{-1})$ if $\theta < \nu_i$ and $+\infty$ if $\theta \geq \nu_i$. The stability condition is $\nu_1, \nu_2 > 1$, $\nu_0^{-1} + \nu_1^{-1} + \nu_2^{-1} < 2$. Figure 1 shows two representative plots of theory and simulation. In the first plot setting $\nu_0 = 1.5$, $\nu_1 = \nu_2 = 2$ results in the theory predicting $\delta_0 = 2\delta$. That is, the tail of the waiting time distribution experienced by the discretionary source in the system with routing is the same as the tail of the overall waiting time distribution in the single server queue, so that routing offers no advantage in avoiding long waiting times. Although both simulated queues have been fed with the same source traces, because of their different disciplines the waiting times experienced by source 0 messages in each system are not the same. However the tails of their empirical waiting time distribution is seen to have near identical form. Also plotted is $-2w\delta$ and $-w\delta^{(1)}$. Clearly the former makes an accurate prediction, with the later being incorrect. In the second plot of figure 1, where $\nu_1 = 1.6$, $\nu_1 = 100$ and $\nu_2 = 1.01$, the first queue has few dedicated arrivals and the second queue is nearly saturated with its dedicated arrivals. Consequently, during a busy period, in the system with routing, discretionary messages are most likely to be routed to the first server, so that routing offers an advantage. The value 2δ accurately predicts the tail of the single server queue fed by all three sources, whereas $\delta^{(1)}$ predicts the lighter tail seen by discretionary messages in the system with routing.

In the results reported so far each input's message sizes are i.i.d. exponential. Next we report on flows that are independent of each other, but whose messages size processes are not i.i.d.; a setting which is not typically considered in the literature on queues with routing, but is of practical interest as real queueing systems are likely to be fed by sources that have correlations in time. If the message

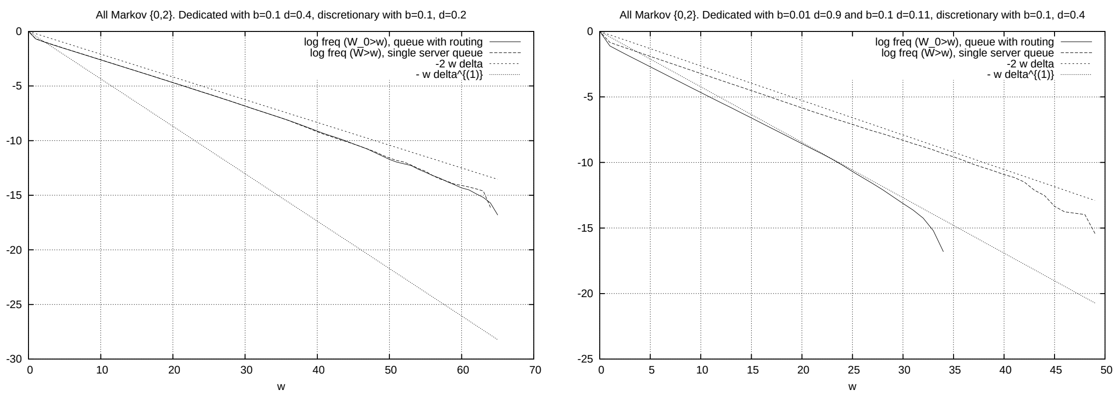

Figure 2: Markovian inputs. Simulated queues, with and without routing, and 2δ and $\delta^{(1)}$ from Theorem 7.

sizes from sources i are 2-state Markovian taking the values $\{0,2\}$ with transition matrix

$$\pi = \begin{pmatrix} 1-b_i & b_i \\ d_i & 1-d_i \end{pmatrix}, \text{ where } b_i, d_i \in (0,1),$$

then its sCGF λ_i can be calculated using techniques described in Section 3.1 of [3]:

$$\lambda_i(\theta) = \log\left(\frac{(1-b_i)+(1-d_i)e^{2\theta}+\sqrt{(1-b_i+(1-d_i)e^{2\theta})^2-4(1-b_i-d_i)e^{2\theta})}}{2}\right).$$

If $b_i + d_i < 1$ the chain is positively correlated. If $b_i + d_i > 1$ the chain is negatively correlated. If $b_i + d_i = 1$, the chain is Bernoulli. In the two graphs of figure 2 all inputs are 2-state Markov. In the first graph, $b_0 = 0.1$ and $d_0 = 0.2$, giving a mean message size for the discretionary flow of $2b_0/(b_0 + d_0) = 2/3$. Dedicated sources have $b_1 = b_2 = 0.1$ and $d_1 = d_2 = 0.4$, giving a mean message size of 0.4. The tails of both the routed waiting time distribution and combined single server queue are similar. In the second graph of figure 2, $b_0 = 0.1$, $d_0 = 0.4$ giving a mean message size of 0.4, but $b_1 = 0.01$, $d_1 = 0.91$ giving a mean message size of approximately 0.022 and $b_2 = 0.1$, $d_2 = 0.11$ giving a mean message size of approximately 0.953. Here the second queue is heavily loaded by its dedicated arrivals and the single server queue has differing asymptotics from the system with routing. The single server queue is predicted to match 2δ whereas the routed system to match $\delta^{(1)}$.

Note that all the experimental results in this paper so far show similar behavior: a non-zero intercept; initial curvature; a straight line; and then noise. The noise is caused due to scarcity of data. If the experiments are run for longer, the place at which the noise occurs moves further to the right. The non-zero intercept and initial curvature are due to the details of the process (for example, the probability that the waiting time is zero moves the intercept). Logarithmic frequencies should, in theory, match the slope of the straight line $-w\delta_0$. Good agreement of slope is seen in figures 1 and 2, demonstrating the accuracy and relevance of the asymptotic from Theorem 7. Indeed, the approximation works well for not only large values of w, but also quite small values. That the difference between prediction and the log-linear approximation is almost constant suggests $\mathbf{P}(W_0 > w) \approx \exp(\mu - w\delta_0)$. That is the error in using the logarithmic asymptotic approximation, $o\big(w\big)$, appears to be constant.

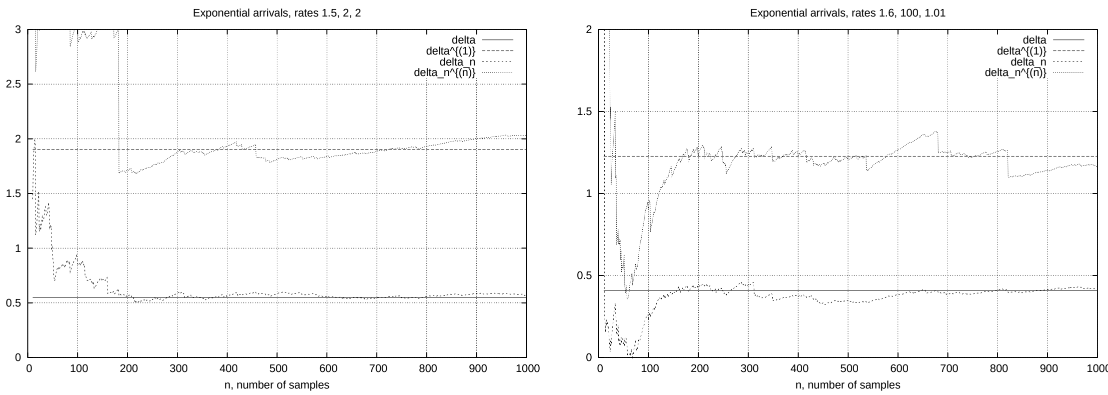

Figure 3: Exponential inputs: estimates vs. real values of δ and $\delta^{(1)}$ from Theorem 7.

7 Estimated entropy, Theorem 7 and simulation

Having demonstrated in the previous section the significance of logarithmic, large deviation style, asymptotics for the two server queueing system with routing, here we illustrate the utility of its use with estimated entropy. Using the estimator described in equation (4) for the entropies of each of three independent sources, we can use the result in Theorem 7 to estimate δ and $\delta^{(1)}$ for each of the four experiments described in Section 6. In these settings, as we have explicit formula for the sCGFs, we determine their real values (numerically), which we compare the estimated values to. We then illustrate the method's accuracy for recorded traffic traces for which no stochastic description of the source can be given, by comparing estimated and simulated behavior.

For each source $i \in \{0,1,2\}$, let B_i be a block length and define the blocked data $Y^{(i)}(n) = a_i((n-1)B_i+1) + \cdots + a_i(nB_i)$. As in equation (4) having seen n samples of data we define the entropy estimator for each of the sources $i \in \{0,1,2\}$ by

$$\lambda_n^{(i)}(\theta) = \frac{1}{B_i} \log \frac{1}{\lfloor n/B_i \rfloor} \sum_{j=1}^{\lfloor n/B_i \rfloor} \exp(\theta Y^{(i)}(j)).$$

We then define the estimators of δ and $\delta^{(1)}$ from Theorem 7 by

$$\delta_n = \sup\{\theta : \lambda_n^{(0)}(\theta) + \lambda_n^{(1)}(\theta) + \lambda_n^{(2)}(\theta) \leq 2\theta\}$$

and

$$\delta_n^{(1)} = \sup\{\theta : \inf_{\vartheta} [\lambda_n^{(0)}(\vartheta) + \lambda_n^{(1)}(\vartheta) + \lambda_n^{(2)}(\theta - \vartheta)] \leq \theta\}.$$

The graphs of figure 3 correspond to the setting with i.i.d. exponential sources in the previous section. As we know the sources are i.i.d, for each $i \in \{0,1,2\}$ the block length B_i is set to 1. As well as the real values of δ and $\delta^{(1)}$ which are determined numerically, δ_n and $\delta_n^{(1)}$ are plotted for n between 1 and 1000. The convergence of the estimates to close to the true values is clear in both plots, even with only a small number of observed message sizes. The δ_n estimates appear to converge faster than

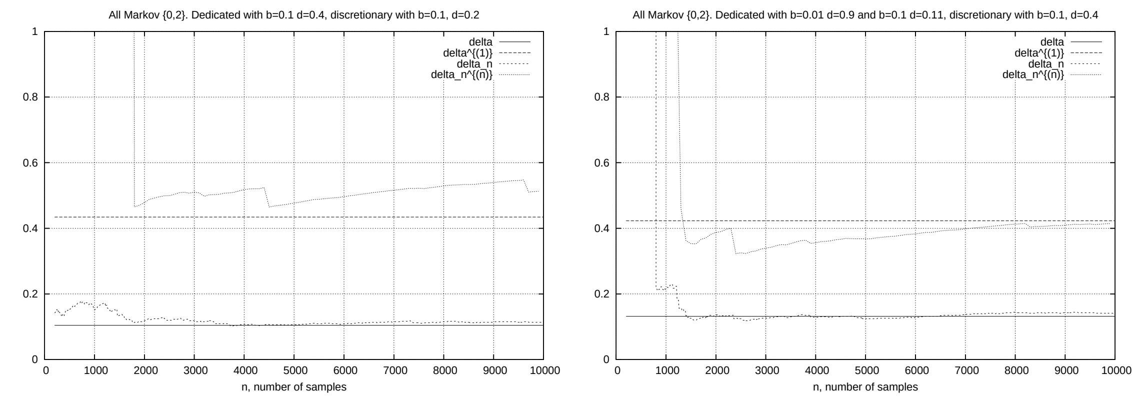

Figure 4: Markov inputs: estimates vs. real values of δ and $\delta^{(1)}$ from Theorem 7.

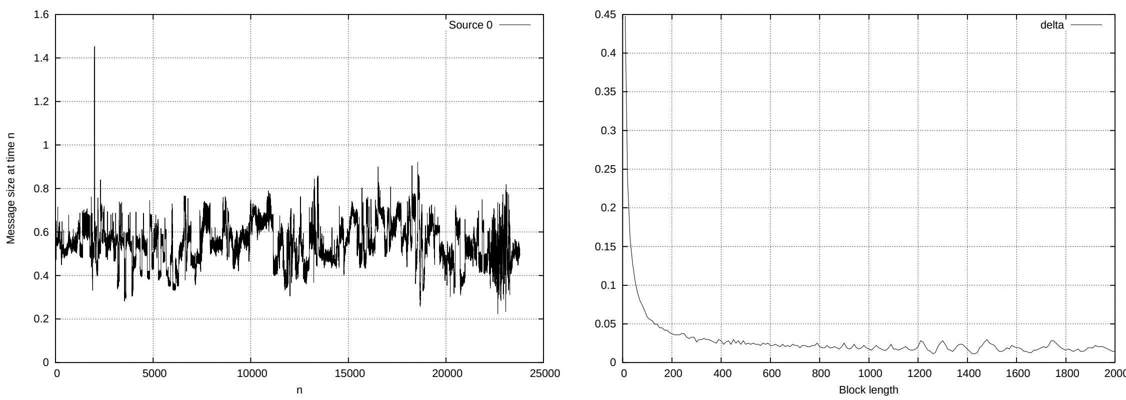

Figure 5: Discretionary source of data and δ vs. block length.

the $\delta_n^{(1)}$, a property we have observed for many other setups not reported on here, suggesting $\delta^{(1)}$ is more sensitive to the accuracy of the estimated entropies.

The graphs of figure 4 correspond to the Markovian system from Section 6. Here the question of choice of block length B is more tricky. It is known [5, 12] that there is a trade off in choosing B. Select it too small, and the estimator is biased, treating the source as independent over time-scales where it is not. Select it too large and the blocks are too homogeneous, requiring significantly more data before estimation becomes accurate. Here we select $B = 20$ for all estimators, as it appears neither too large nor too small for a Markov chain whose correlation at distance n is $(1 - b_i - d_i)^n$. For correlated sources, the quantity of data required to make accurate predictions of entropy is larger than in the independent case, partially because of the blocking of data. This is reflected in the larger n range of the plots. Note that the δ estimates are particularly good, converging quickly. The $\delta^{(1)}$ estimates require more data, but are still convincing.

Finally, as J. T. Lewis felt that applied probability should have more than the potential to be applied,

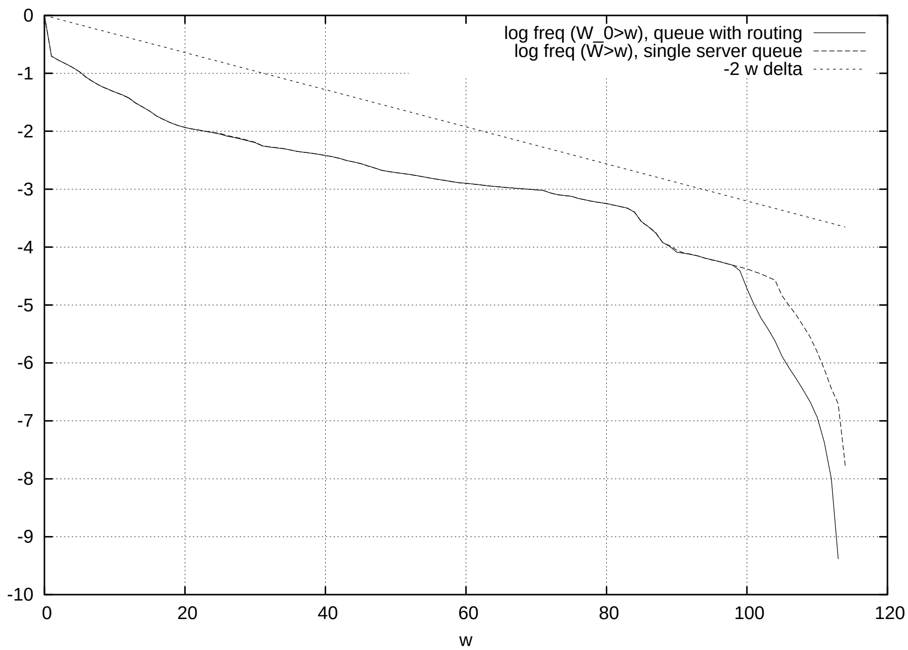

Figure 6: Trace driven simulation and estimate.

it should be applied, here we report on two exploratory experiments. Our stochastic source is the famous Bellcore Starwars trace; a trace of the activity of an MPEG encoded version of the film Starwars - the volume of data required to encode each frame of the film. We first aggregated the frames into messages that contain the volume of data per 10^{th} of a second and then cut the trace into three non-overlapping contiguous pieces. Each of the three distinct pieces is used as the source of traffic for a flow. In the first experiment, we rescaled each piece individually so that each server's service rate, 1, was halfway between the mean and peak value of the trace. This process ensures the correlation structure of the original trace is retained, the queueing system is stable, but waiting times due to queueing are possible.

For example, the first graph in figure 5 shows the activity of our discretionary source versus time. As we have no stochastic description of the source of data, a definitive value of δ_0 is not possible. We can, however, queue the data and estimate δ_0. Here the block length is a real issue, as without further tests we have no feel for the correlation structure in the data. The second graph in figure 5 shows the estimate of δ for a range of block lengths, where the estimates were are based on the entire data set. We selected the value for $B = 1000$ which was then used in the watermark figure 6, where both the system with routing and the single server queue clearly have the same waiting time distribution tail, which is well predicted by the estimates.

In the second experiment, the discretionary flow was rescaled so that its mean is 0.4. The directed flows were rescaled to have means 0.5 and 0.95 respectively. Again the graphs of figure 7 demonstrates the issue of selecting an appropriate block length. For δ we choose a block length of $B = 1000$ for all three λ estimators, but for $\delta^{(1)}$ we choose $B = 200$. In this system, the discretionary messages

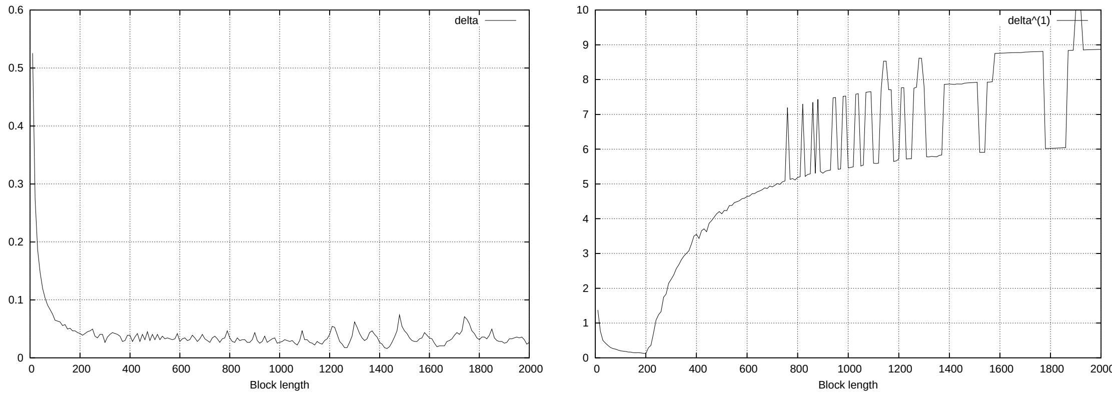

Figure 7: δ and $\delta^{(1)}$ vs. block length.

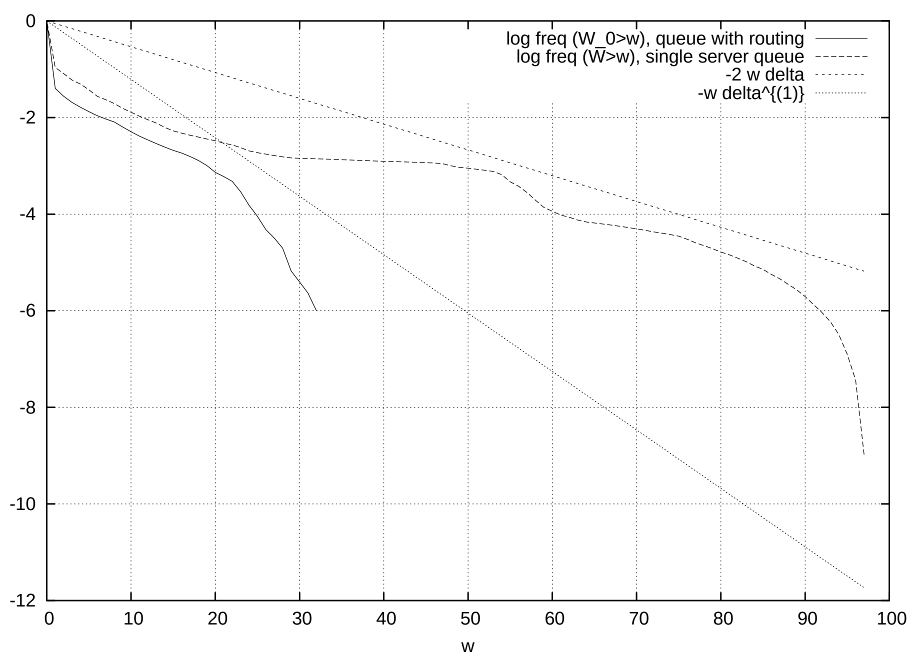

Figure 8: Trace driven simulation and estimates.

in the system with routing experience are less likely to experience a long waiting time than in the single server queue. This can be seen in figure 8 where our estimates of δ and $\delta^{(1)}$ are shown. It is interesting to note that this suggests the selection of B is dependent on the queueing system, not just the source. We cannot propose an explanation for this phenomenon.

Figures 6 and 7 show the accuracy of the estimates and suggests there is value in this approach even for real world data where mathematical assumptions cannot be checked.

Acknowledgment. The work was partly done at the Dublin Institute for Advanced Studies, Ireland. The authors thank Professor T. Dorlas for his hospitality.

K.D is supported by Science Foundation Ireland grant IN3/03/I346. The work of E.A.P. and N.D.V. was partly supported by Russian RFFI grant 02-01-00068.

References

[1] S. Crosby, I. Leslie, B. McGurk, J. T. Lewis, R. Russell, and F. Toomey, *Statistical properties of a near-optimal measurement-based CAC algorithm*, Proceedings of IEEE ATM '97 (Lisbon, Portugal), 1997.

[2] A. Dembo and T. Zajic, *Large deviations: from empirical mean and measure to partial sums*, Stochastic Processes and their Applications **57** (1995), 191–224.

[3] A. Dembo and O. Zeitouni, *Large deviation techniques and applications*, Springer, 1998.

[4] J.-D. Deuschel and D. W. Stroock, *Large deviations*, Pure and Applied Mathematics, vol. 137, Academic Press Inc., Boston, MA, 1989.

[5] N. G. Duffield, J. T. Lewis, N. O'Connell, R. Russell, and F. Toomey, *Entropy of ATM traffic streams: a tool for estimating QoS parameters*, IEEE Journal of Selected Areas in Communications (special issue on Advances in the Fundamentals of Networking) **13** (1995), 981–990.

[6] K. Duffy, J. T. Lewis, and W. G. Sullivan, *Logarithmic asymptotics for the supremum of a stochastic process*, Ann. Appl. Probab. **13** (2003), no. 2, 430–445.

[7] K. Duffy, D. Malone, E. Pechersky, Y. M. Suhov, and N. Vvedenskaya, *Large deviations provide good approximation to queueing system with dynamic routing*, Tech. report, Dublin Insitutute for Advanced Studies, 2004.

[8] K. Duffy and A. P. Metcalfe, *The large deviations of estimating rate functions*, J. Appl. Probab. **42** (2005), no. 1, 267–274.

[9] K. Duffy and M. Rodgers-Lee, *Some useful functions for functional large deviations*, Stoch. Stoch. Rep. **76** (2004), no. 3, 267–279.

[10] K. Duffy and W. G. Sullivan, *Logarithmic asymptotics for unserved messages at a FIFO*, Markov Process. Related Fields **10** (2004), no. 1, 175–189.

[11] R. D. Foley and D. R. McDonald, *Join the shortest queue: stability and exact asymptotics*, Ann. Appl. Probab. **11** (2001), no. 3, 569–607.

[12] A. J. Ganesh, *Bias correction in effective bandwidth estimation*, Performance Evaluation **27** (1996), no. 8, 319–330.

[13] A. J. Ganesh and N. O'Connell, *A large deviation principle with queueing applications*, Stoch. Stoch. Rep. **73** (2002), no. 1-2, 25–35.

[14] P. Glynn and W. Whitt, *Logarithmic asymptotics for steady-state tail probabilities in a single-server queue*, J. Appl. Probab. **31A** (1994), 413–430.

[15] B. M. Hambly, James B. Martin, and Neil O'Connell, *Concentration results for a Brownian directed percolation problem*, Stochastic Process. Appl. **102** (2002), no. 2, 207–220.

[16] J. T. Lewis, R. Russell, F. Toomey, B. McGurk, S. Crosby, and I. Leslie, *Practical connection admission control for ATM networks based on on-line measurements*, Computer Communications **21** (1998), no. 17, 1585–1596.

[17] R. M. Loynes, *The stability of a queue with non-independent inter-arrival and service times*, Proceedings of the Cambridge Philosphical Society **58** (1962), 497–520.

[18] K. Majewski, *Large deviations for multi-dimensional reflected fractional Brownian motion*, Stoch. Stoch. Rep. **75** (2003), no. 4, 233–257.

[19] ———, *Large deviation bounds for single class queueing networks and their calculation*, Queueing Syst. **48** (2004), no. 1-2, 103–134.

[20] D. R. McDonald and S. R. E. Turner, *Resource pooling in distributed queueing networks*, Fields Inst. Communications **28** (2000), 107–131.

[21] E. A. Pechersky, Y. M. Suhov, and N. D. Vvedenskaya, *Large deviations in a two-server system with dynamic routing*, Tech. report, Isaac Newton Institute for Math. Sci., 2003.

[22] ———, *Large deviations in a two-server system with dynamic routing*, 2004 IEEE Internat. Symposium on Inform. Theory (Chicago, USA), 2004.

[23] F. Riesz and B. SZ.-Nagy, *Functional analysis*, Blackie and Son Limited, 1955.

[24] R. T. Rockafellar, *Convex analysis*, Princeton University Press, 1970.

[25] J. S. Sadovski, *The probability of large queue lengths and waiting times in a heterogeneous multiserver queue II: positive recurrence and logarithmic limits*, Adv. Appl. Prob. **27** (1995), 567–583.

[26] J. S. Sadovski and W. Szpankovski, *The probability of large queue lengths and waiting times in a heterogeneous multiserver queue I: tight limits*, Adv. Appl. Prob. **27** (1995), 532–566.

[27] F. Toomey, *Bursty traffic and finite capacity queues*, Annals of Operations Research **79** (1998), 45–62.

[28] S. R. E. Turner, *Large deviations for join the shortest queue*, Fields Inst. Communications **28** (2000), 95–106.