The Large Deviation Principle for the On/Off Weibull Sojourn Process

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Abstract

This article proves that the on-off renewal process with Weibull sojourn times satisfies the large deviation principle on a non-linear scale. Unusually, its rate function is not convex. Apart from on a compact set, the rate function is infinite, which enables us to construct natural processes that satisfy the LDP with non-trivial rate functions on more than one time scale.

1 Introduction

Let $v(\cdot) : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function that diverges to infinity. A real-valued process $\{Z_t, t \in T\}$ (where T is $\mathbb{N}$ or $\mathbb{R}$) satisfies the *Large Deviation Principle* (LDP) on the scale $v(\cdot)$ with rate function $I : \mathbb{R} \to [0, \infty]$ if I is lower semi-continuous and for all Borel sets $B \subset \mathbb{R}$

$$-\inf_{x \in B^\circ} I(x) \leq \liminf_{t\to\infty} \frac{1}{v(t)} \log \mathbb{P}(Z_t \in B) \leq \limsup_{t\to\infty} \frac{1}{v(t)} \log \mathbb{P}(Z_t \in B) \leq -\inf_{x\in\bar{B}} I(x),$$

where B° denotes the interior of B and $\bar{B}$ denotes the closure of B. A rate function is *good* if its level sets $\{x : I(x) \leq \beta\}$ are compact for all $\beta < \infty$.

The averages $\{n^{-1}S_n = n^{-1}\sum_{i=1}^n Y_i\}$ of many real-valued processes are known to satisfy the LDP, including $\{Y_i\}$ being i.i.d. random variables or satisfying mixing conditions that are broad enough to encompass Doeblin recurrent Markov chains (e.g. Bryc and Dembo [1]).

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If the tail of the distribution of Y_1 decays no slower than an exponential, then the scale for the LDP is the number of summands, n, and, typically, the rate function is convex. If the summands have a semi-exponential (Weibull) tail $\mathbb{P}(Y_1 > y) = \exp(-y^\alpha)$, where $\alpha \in (0,1)$, then they satisfy the LDP on the scale n^α with the concave rate function given in Theorem 1 (e.g. Nagaev [4]) that is finite for all arguments greater than or equal to the mean*. Rate functions that are not convex are interesting as one of the main tools in the theory of large deviations, the duality between the rate function and its Legendre-Fenchel transform, the scaled cumulant generating function, does not hold.

Here we prove the LDP for the on-off renewal process with Weibull sojourn times on the scale t^α. Its rate function is not convex and, moreover, is only finite on a compact set. This provides a new example of a natural process whose properties cannot be deduced from Gartner-Ellis style theorems. Moreover, as its rate function is infinite off a compact set we can readily construct simple processes that have non-trivial rate functions on more than one scale.

2 Main result

Let $\{\xi_i\}$ denote i.i.d. on times and $\{\tau_i\}$ denote i.i.d. off times, where an on time follows an off time which follows an on time. Assume that for $x > 0$, $\mathbb{P}(\xi_1 > x) = \mathbb{P}(\tau_1 > x) = \exp(-x^\alpha)$, where $\alpha \in (0,1)$, and denote $\mu := \mathbb{E}(\xi_1) = \alpha \int_0^\infty x^{\alpha-1}\exp(-x^\alpha)dx = \Gamma(1+\alpha^{-1})$. For each $n \in \mathbb{N}$, $t \in \mathbb{R}_+$ define

$$S_n^\tau := \sum_{i=1}^n \tau_i,\ S_n^\xi := \sum_{i=1}^n \xi_i,\ T_n := S_n^\tau + S_n^\xi,\ N_t := \sup\{n : T_n \le t\}$$

The following theorem is a well known result for the partial sums of semi-exponential distributed random variables (see, for example, Nagaev [4] or Gantert [3]).

Theorem 1 *The process $\left\{S_n^\xi/n\right\}$ satisfies the LDP on the scale n^α with rate function*

$$I(x) = \begin{cases} (x-\mu)^\alpha & \textit{if } x \ge \mu \\ +\infty & \textit{if } x < \mu. \end{cases} \tag{1}$$

Here we are interested in an on/off process whose sojourn times are independent and identically distributed with semi-exponential distribution. Define the on time set $A := \{s : s \in [T_n + \tau_{n+1}, T_{n+1}) \text{ for some } n\}$. The process of interest is the cumulative on time prior to time t:

$$X_t := \int_0^t 1_A(s)\,ds, \qquad 1_A(s) = \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{if } s \notin A. \end{cases}$$

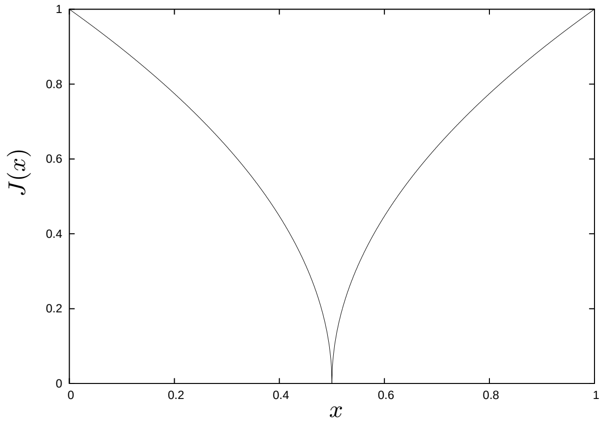

Figure 1: Rate function for $\alpha = 1/2$.

The following theorem is the main result.

Theorem 2 (LDP for Weibull sojourn source) *The process $\{X_t/t\}$ satisfies the LDP in $\mathbb{R}$ on the scale t^α with good rate function*

$$J(x) = \begin{cases} (1-2x)^\alpha & \text{if } x \in [0,1/2] \\ (2x-1)^\alpha & \text{if } x \in [1/2,1] \\ +\infty & \text{if } x \notin [0,1]. \end{cases} \tag{2}$$

The rate function defined in equation (2) is not convex; for example, Figure 1 plots $J(x)$ vs. x for $\alpha = 1/2$. As Gartner-Ellis theorems rely on convexity of the rate-function, Theorem 2 cannot be deduced by that methodology.

PROOF: Theorem 2. Let $B_\epsilon(x)$ denote the open ball of radius ϵ around x. Our approach to proving Theorem 2 is to show that the lower deviation function

$$\lim_{\epsilon \to 0} \liminf_{t \to \infty} \frac{1}{t^\alpha} \log \mathbb{P}\left(\frac{X_t}{t} \in B_\epsilon(x)\right)$$

and the upper deviation function

$$\lim_{\epsilon \to 0} \limsup_{t \to \infty} \frac{1}{t^\alpha} \log \mathbb{P}\left(\frac{X_t}{t} \in B_\epsilon(x)\right)$$

coincide for all x. Once the lower and upper deviation functions are shown to be equal, as X_t/t takes values in the compact set $[0,1]$, the LDP follows from, for example, Theorem 4.1.11

*Fractional Brownian motion is an example of a process that satisfies the LDP on a non-linear scale, but with a rate function that is convex and finite everywhere.

of Dembo and Zeitouni [2]. That the upper and lower deviation functions coincide follows from the following two theorems whose proofs can be found in sections 3 and 4 respectively.

Theorem 3 *For all* $x \in \mathbb{R}$*,*

$$\lim_{\epsilon \to 0} \liminf_{t \to \infty} \frac{1}{t^\alpha} \log \mathbb{P}\left(\frac{X_t}{t} \in B_\epsilon(x)\right) \geq -J(x). \tag{3}$$

Theorem 4 *For all* $x \in \mathbb{R}$*,*

$$\lim_{\epsilon \to 0} \limsup_{t \to \infty} \frac{1}{t^\alpha} \log \mathbb{P}\left(\frac{X_t}{t} \in B_\epsilon(x)\right) \leq -J(x). \tag{4}$$

■

Remark 1, Concave rate functions. Although J is concave (where finite) on either side of its mean, the sum of two independent copies of $\{X_t\}$ satisfies the LDP with a rate function that is not concave on either side of its mean.

Remark 2, Non-trivial large deviations on more than one scale. We say that a rate function I is *non-trivial* if: (i) it is finite at more than a single point; and (ii) it is not zero everywhere where it is finite. A rate function that is not non-trivial is called *trivial*.

As we have constructed a process that satisfies the LDP on a sub-linear scale, t^α, with a rate function that is finite only on a compact interval, we can now construct natural processes that satisfy the LDP with non-trivial rate functions on more than one scale. This leads to the emergence of multiple fundamental time scales for the exponential decay of probability for this process.

We demonstrate this by considering an example constructed by the sum of the Weibull sojourn process with an independent Bernoulli process. First note that from Theorem 2 it is easy to show that $\{X_t/t\}$ also satisfies the LDP on the scale t, but with the trivial good rate function $J_1(x) = 0$ if $x \in [0, 1]$,

$$J_1(x) = \begin{cases} 0 & \text{if } x \in [0,1] \\ +\infty & \text{if } x \notin [0,1]. \end{cases}$$

Next consider a Bernoulli process: let $\{Z_n\}$ be an i.i.d. sequence of random variables taking the values 0 and 1, with $\mathbb{P}(Z_n = 0) = 1 - p$ and $\mathbb{P}(Z_n = 1) = p$, for some $p \in (0, 1)$. With $Y_t = \sum_{i=1}^{[t]} Z_i$, it is well known that $\{Y_t/t\}$ satisfies the LDP on the scale t with the non-trivial good rate function

$$H_1(x) = \begin{cases} x \log(x/p) + (1-x) \log((1-x)/(1-p)) & \text{if } x \in [0,1] \\ +\infty & \text{if } x \notin [0,1]. \end{cases}$$

On the scale t^α, $\{Y_t/t\}$ satisfies the LDP with the trivial good rate function

$$H(x) = \begin{cases} 0 & \text{if } x = p \\ +\infty & \text{if } x \neq p. \end{cases}$$

As the rate functions J and H are both good and addition is continuous, by the contraction principle (e.g. Theorem 4.2.1 of [2]) $\{(X_t + Y_t)/t\}$ satisfies the LDP on the scale t^α with the non-trivial good rate function

$$K(x) = \inf_y \{J(y) + H(x - y)\} = J(x - p).$$

However, J_1 and H_1 are also both good rate functions, so that, by the contraction principle, $\{(X_t + Y_t)/t\}$ also satisfies the LDP on the scale t with the non-trivial good rate function

$$K_1(x) = \inf_y \{J_1(y) + H_1(x - y)\} = \begin{cases} H_1(x) & \text{if } x \in [0, p] \\ 0 & \text{if } x \in [p, 1 + p] \\ H_1(x - 1) & \text{if } x \in [1 + p, 2] \\ +\infty & \text{otherwise.} \end{cases}$$

Figure 2 gives a example of both rate functions with $p = \alpha = 1/2$. Outside $[p, 1 + p]$ large deviations occur on the scale t, but inside $[p, 1 + p]$ they occur non-trivially on the scale t^α. Thus for the sum of the Weibull sojourn process and an independent Bernoulli process, we have the following large deviation approximations for large t: if $x \in [p, 1 + p]$

$$\mathbb{P}(X_t + Y_t \approx xt) \approx \exp(-t^\alpha K(x))$$

and if $x \in [0, p) \cup [1 + p, 2]$

$$\mathbb{P}(X_t + Y_t \approx xt) \approx \exp(-tK_1(x)).$$

That is $\{(X_t + Y_t)/t\}$ satisfies two non-trivial LDPs, with probability decaying on a faster time scale outside $[p, 1 + p]$.

3 Proof of Theorem 3

Recall the statement of Theorem 3: for all $x \in \mathbb{R}$,

$$\lim_{\epsilon \to 0} \liminf_{t \to \infty} \frac{1}{t^\alpha} \log \mathbb{P}\left(\frac{X_t}{t} \in B_\epsilon(x)\right) \geq -J(x). \tag{5}$$

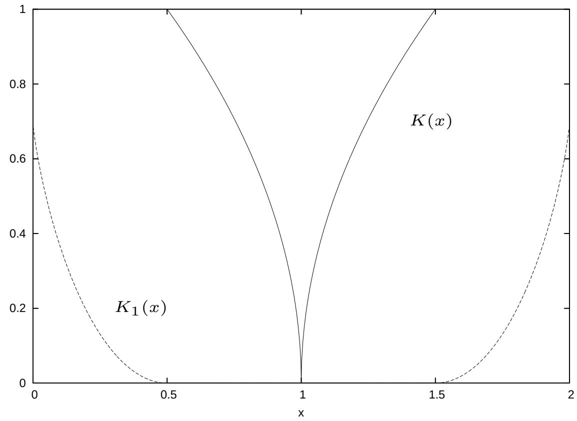

Figure 2: Rate functions for $\{(X_t+Y_t)/t\}$ with $\alpha = p = 1/2$. $K(x)$ is on the scale t^α and is infinite outside [0.5,1.5]. $K_1(x)$ is on the scale t, is infinite outside $[0,2]$, zero in $[0.5,1.5]$ and infinite outside $[0,2]$.

Proof: There are three cases to consider: $x=1/2$, $x\in[0,1/2)$ and $x\in(1/2,1]$. If $x=1/2$ let $\epsilon'=\epsilon$ and if $x\neq 1/2$ let $\epsilon'=\min(\epsilon,|x-1/2|)$. For any n we have:

$$\begin{aligned}\left\{\frac{X_t}{t}\in B_\epsilon(x)\right\} &\supset \left\{\frac{X_t}{t}\in B_{\epsilon'}(x)\right\}\\ &\supset \left\{N_t=n, \frac{X_t}{t}\in B_{\epsilon'}(x)\right\}\\ &\supset \left\{T_n<t, T_n+\tau_{n+1}>t, \frac{X_t}{t}\in B_{\epsilon'}(x)\right\}\\ &\supset \left\{x-\frac{\epsilon'}{2}<\frac{S_n^\xi}{t}<x+\frac{\epsilon'}{2}, 1-x-\epsilon'<\frac{S_n^\tau}{t}\le 1-x-\frac{\epsilon'}{2}, \tau_{n+1}>2\epsilon' t+\mu\right\}.\end{aligned} \tag{6}$$

The final line is an inclusion as members of the set imply that $T_n = S_n^\xi + S_n^\tau < t$, $T_n+\tau_{n+1}>t$ and $X_t/t = S_n^\xi/t \in B_{\epsilon'}(x)$. As the three conditions in (6) correspond to independent events, we have that for any non-decreasing sequence $\{n_t\}$

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t^\alpha}\log P\left(\frac{X_t}{t}\in B_\epsilon(x)\right) &\geq \liminf_{t\to\infty}\frac{1}{t^\alpha}\log P\left(\frac{S_{n_t}^\xi}{t}\in B_{\frac{\epsilon'}{2}}(x)\right)\\ &+\liminf_{t\to\infty}\frac{1}{t^\alpha}\log P\left(\frac{S_{n_t}^\tau}{t}\in B_{\frac{\epsilon'}{4}}\left(1-x-\frac{3\epsilon'}{4}\right)\right)\\ &+\liminf_{t\to\infty}\frac{1}{t^\alpha}\log P(\tau_{n_t}>2\epsilon' t+\mu).\end{aligned} \tag{7}$$

For $x = 1/2$ we choose $n = n_t = \lfloor t/\mu \rfloor$, for $x \in [0, 1/2)$ we choose $n = n_t = \lfloor t(x - \epsilon')/\mu \rfloor$ and for $x \in (1/2, 1]$ we choose $n = n_t = \lfloor t(1 - x - \epsilon')/\mu \rfloor$. As near-identical arguments apply for all three cases, we shall only write out the proof for $x \in (1/2, 1]$. We apply the result of Theorem 1 to lower bound the first term on the right hand side in (7), which gives:

$$\liminf_{t\to\infty} \frac{1}{t^\alpha} \log \mathbb{P}\left(\frac{S^\xi_{n_t}}{t} \in B_{\frac{\epsilon'}{2}}(x)\right) = \frac{(1-x-\epsilon')^\alpha}{\mu^\alpha} \liminf_{n\to\infty} \frac{1}{n^\alpha} \log \mathbb{P}\left(\frac{S^\xi_n}{n} \in B_{\frac{\epsilon'\mu}{2(1-x-\epsilon')}}\left(\frac{\mu x}{1-x-\epsilon'}\right)\right)$$
$$\geq -\left(2x - 1 + \frac{1}{2}\epsilon'\right)^\alpha.$$

For the second term in (6) we again apply Theorem 1:

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{t^\alpha} \log \mathbb{P}\left(\frac{S^\tau_{n_t}}{t} \in B_{\frac{\epsilon'}{4}}\left(1 - x - \frac{3\epsilon'}{4}\right)\right) \\
&= \frac{(1-x-\epsilon')^\alpha}{\mu^\alpha} \liminf_{n\to\infty} \frac{1}{n^\alpha} \log \mathbb{P}\left(\frac{S^\tau_n}{n} \in B_{\frac{\mu\epsilon}{4(1-x-\epsilon')}}\left(\frac{\mu(1-x-3\epsilon/4)}{1-x-\epsilon'}\right)\right) \\
&\geq -\frac{(1-x-\epsilon)^\alpha}{\mu^\alpha} \inf\left\{I(a) : a \in \big(\mu, \mu(1-x-\epsilon'/2)/(1-x-\epsilon')\big)\right\} \\
&= 0,
\end{aligned}$$

as $I(\mu) = 0$. Finally, for the third term, from the Weibull distribution of τ, $\liminf_{t\to\infty} t^{-\alpha} \log \mathbb{P}(\tau > 2\epsilon' t + \mu) = -(2\epsilon')^\alpha$. Hence, from the bound in equation (7) we have

$$\liminf_{t\to\infty} \frac{1}{t^\alpha} \log \mathbb{P}\left(\frac{X_t}{t} \in B_\epsilon(x)\right) \geq -\left(2x - 1 + \frac{1}{2}\epsilon'\right)^\alpha - (2\epsilon')^\alpha.$$

The result follows taking ϵ (and thus ϵ') to zero.

■

4 Proof of Theorem 4

Recall the statement of Theorem 4: For all $x \in \mathbb{R}$,

$$\lim_{\epsilon\to 0} \limsup_{t\to\infty} \frac{1}{t^\alpha} \log \mathbb{P}\left(\frac{X_t}{t} \in B_\epsilon(x)\right) \leq -J(x).$$

Proof: In order to prove this theorem we need the following proposition, which will be deduced from two lemmas that appear later in this section.

Proposition 5 *With J defined in equation (2), both $\{S^\xi_{N_t}/t\}$ and $\{S^\tau_{N_t}/t\}$ satisfy the LDP with good rate function $J(\cdot)$.*

Once Proposition 5 is established, the upper bound on the upper deviation function for $\{X_t/t\}$ can be deduced from the following argument. First note that as X_t is non-decreasing,

$$S^{\xi}_{N_t} \leq X_t \leq t - S^{\tau}_{N_t}. \tag{8}$$

so that we have

$$\mathbb{P}\left(\frac{X_t}{t} \in B_\epsilon(x)\right) \leq \mathbb{P}\left(\frac{X_t}{t} > x - \epsilon\right) \leq \mathbb{P}\left(\frac{S^{\tau}_{N_t}}{t} < 1 - x + \epsilon\right)$$

$$\text{and} \qquad \mathbb{P}\left(\frac{X_t}{t} \in B_\epsilon(x)\right) \leq \mathbb{P}\left(\frac{X_t}{t} < x + \epsilon\right) \leq \mathbb{P}\left(\frac{S^{\xi}_{N_t}}{t} < x + \epsilon\right).$$

Using these inequalities we get that

$$\lim_{\epsilon\to 0}\limsup_{t\to\infty}\frac{1}{t^\alpha}\log\mathbb{P}\left(\frac{X_t}{t} \in B_\epsilon(x)\right) \leq \min\left[\limsup_{t\to\infty}\frac{1}{t^\alpha}\log\mathbb{P}\left(\frac{S^{\xi}_{N_t}}{t} < x + \epsilon\right),\right.$$
$$\left.\limsup_{t\to\infty}\frac{1}{t^\alpha}\log\mathbb{P}\left(\frac{S^{\tau}_{N_t}}{t} < 1 - x + \epsilon\right)\right].$$

Employing the LDP upper bounds for $\{S^{\xi}_{N_t}/t\}$ and $\{S^{\tau}_{N_t}/t\}$ from Proposition 5, we see in the limit $\epsilon \to 0$ that if $x < 1/2$ the first term dominates and we get an upper bound of $-J(x)$. If $x > 1/2$, the second term dominates and we get an upper bound of $-J(1-x) = -J(x)$, which proves the result.

■

All that remains to do is to prove Proposition 5. As $\{S^{\xi}_{N_t}\}$ and $\{S^{\tau}_{N_t}\}$ are equal in distribution, we shall prove the result only for the former. To do this, we employ the same approach as described for Theorem 2. We will show that the lower and upper deviations functions coincide:

$$\lim_{\epsilon\to 0}\liminf_{t\to\infty}\frac{1}{t^\alpha}\log\mathbb{P}\left(\frac{S^{\xi}_{N_t}}{t} \in B_\epsilon(x)\right) = \lim_{\epsilon\to 0}\limsup_{t\to\infty}\frac{1}{t^\alpha}\log\mathbb{P}\left(\frac{S^{\xi}_{N_t}}{t} \in B_\epsilon(x)\right).$$

By replacing X_t with $S^{\xi}_{N_t}$ in the set inclusion (6), it can be seen that the arguments in the proof of Theorem 5 also show that

$$-J(x) \leq \lim_{\epsilon\to 0}\liminf_{t\to\infty}\frac{1}{t^\alpha}\log\mathbb{P}\left(\frac{S^{\xi}_{N_t}}{t} \in B_\epsilon(x)\right)$$

Thus it suffices to prove that

$$\lim_{\epsilon\to 0}\limsup_{t\to\infty}\frac{1}{t^\alpha}\log\mathbb{P}\left(\frac{S^{\xi}_{N_t}}{t} \in B_\epsilon(x)\right) \leq -J(x).$$

Note that when $x = 1/2$, the upper bound is obtained trivially by using 1 in place of the probability. We deduce the upper bound for $x \neq 1/2$ from the following two lemmas and by appealing to the principle of the largest term (e.g. Lemma 1.2.15 of [2]).

Lemma 6 (large n) *For $x \in (0,1)$, define $\bar{x} := \max(x, 1-x)$. If $x \in (0,1)$, for any $0 < \epsilon' < \mu = \mathbb{E}(\tau_1)$ we have*

$$\lim_{\epsilon\to 0}\limsup_{t\to\infty}\frac{1}{t^\alpha}\log\mathbb{P}\left(\frac{S^\xi_{N_t}}{t}\in B_\epsilon(x), \frac{N_t}{t} > \frac{1-\bar{x}}{\mu-\epsilon'}\right) = -\infty.$$

If $x = 0$ or $x = 1$, then for any $0 < \epsilon' < \mu$ we have

$$\lim_{\epsilon\to 0}\limsup_{t\to\infty}\frac{1}{t^\alpha}\log\mathbb{P}\left(\frac{S^\xi_{N_t}}{t}\in B_\epsilon(x), \frac{N_t}{t} > \frac{\epsilon}{\mu-\epsilon'}\right) = -\infty.$$

Proof: There are four cases to consider: $x = 0$, $x = 1$, $x \in (0, 1/2)$ and $x \in (1/2, 1)$. We start with $x = 0$. As S^ξ_n is increasing in n we have that

$$\left\{\frac{S^\xi_{N_t}}{t}\in B_\epsilon(0), \frac{N_t}{t} > \frac{\epsilon}{\mu-\epsilon'}\right\} = \left\{\frac{S^\xi_{N_t}}{t} < \epsilon, \frac{N_t}{t} > \frac{\epsilon}{\mu-\epsilon'}\right\} \subset \left\{S^\xi_{\lceil\frac{t\epsilon}{\mu-\epsilon'}\rceil} < \epsilon t\right\}.$$

Applying the large deviations upper bound from Theorem 1 for this final sequence of sets, with I being defined in equation (1), we get

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t^\alpha}\log P\left(S^\xi_{\lceil\frac{t\epsilon}{\mu-\epsilon'}\rceil} < \epsilon t\right) &= \frac{\epsilon^\alpha}{(\mu-\epsilon')^\alpha}\limsup_{n\to\infty}\frac{1}{n^\alpha}\log P\left(\frac{S^\xi_n}{n} < \mu-\epsilon'\right)\\ &\leq -\frac{\epsilon^\alpha}{(\mu-\epsilon')^\alpha}\inf\left\{I(a) : a < \mu-\epsilon'\right\}\\ &= -\infty\end{aligned}$$

as $I(a) = \infty$ for all $a < \mu$.

If $x \in (0, 1/2)$, then $\bar{x} = 1-x$ and we have apply a similar argument as for the $x = 0$ case, but starting with the following set inclusions

$$\left\{\frac{S^\xi_{N_t}}{t}\in B_\epsilon(x), \frac{N_t}{t} > \frac{x}{\mu-\epsilon'}\right\} \subset \left\{\frac{S^\xi_{N_t}}{t} < x+\epsilon, \frac{N_t}{t} > \frac{x}{\mu-\epsilon'}\right\} \subset \left\{S^\xi_{\lceil\frac{tx}{\mu-\epsilon'}\rceil} < (x+\epsilon)t\right\}.$$

Applying the large deviations upper bound from Theorem 1 for this final sequence of sets, we have

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t^\alpha}\log P\left(S^\xi_{\lceil\frac{tx}{\mu-\epsilon'}\rceil} < (x+\epsilon)t\right) &\leq -\frac{x^\alpha}{(\mu-\epsilon')^\alpha}\inf\left\{I(a) : a < (\mu-\epsilon')\left(1+\frac{\epsilon}{x}\right)\right\}\\ &= -\frac{x^\alpha}{(\mu-\epsilon')^\alpha}I\left((\mu-\epsilon')\left(1+\frac{\epsilon}{x}\right)\right).\end{aligned}$$

As $\epsilon \to 0$, the I argument is strictly less than μ and $I(a) = \infty$ for all $a < \mu$. Thus for all $0 < \epsilon' < \mu$, in the limit as ϵ tends to zero, the right hand side is $-\infty$.

When $x \in (1/2, 1]$ we use the fact that $T_{N_t} = S^{\xi}_{N_t} + S^{\tau}_{N_t} \leq t$ to give us the set inequality

$$\left\{ \frac{S^{\xi}_{N_t}}{t} > x - \epsilon \right\} \subset \left\{ \frac{S^{\tau}_{N_t}}{t} < 1 - x + \epsilon \right\}.$$

If $x \in (1/2, 1)$, then $\bar{x} = x$ and we have the set inclusions

$$\begin{aligned}
\left\{ \frac{S^{\xi}_{N_t}}{t} \in B_\epsilon(x), \frac{N_t}{t} > \frac{1-x}{\mu - \epsilon'} \right\} &\subset \left\{ \frac{S^{\xi}_{N_t}}{t} > x - \epsilon, \frac{N_t}{t} > \frac{1-x}{\mu - \epsilon'} \right\} \\
&\subset \left\{ \frac{S^{\tau}_{N_t}}{t} < (1 - x + \epsilon), \frac{N_t}{t} > \frac{1-x}{\mu - \epsilon'} \right\} \\
&\subset \left\{ S^{\tau}_{\left\lceil \frac{t(1-x)}{\mu - \epsilon'} \right\rceil} < (1 - x + \epsilon)t \right\}.
\end{aligned}$$

Again we apply the LDP upper bound from Theorem 1 for this final sequence of sets and take the limit $\epsilon \to 0$, which gives a rate of $-\infty$. When $x = 1$, we have

$$\begin{aligned}
\left\{ \frac{S^{\xi}_{N_t}}{t} \in B_\epsilon(1), \frac{N_t}{t} > \frac{1-x}{\mu - \epsilon'} \right\} &\subset \left\{ \frac{S^{\xi}_{N_t}}{t} > 1 - \epsilon, \frac{N_t}{t} > \frac{1-x}{\mu - \epsilon'} \right\} \\
&\subset \left\{ \frac{S^{\tau}_{N_t}}{t} < \epsilon, \frac{N_t}{t} > \frac{\epsilon}{\mu - \epsilon'} \right\} \\
&\subset \left\{ S^{\tau}_{\left\lceil \frac{t\epsilon}{\mu - \epsilon'} \right\rceil} < \epsilon t \right\}.
\end{aligned}$$

and the result follows as in the $x = 0$ case.

■

Lemma 7 (small n) *For $x \in (0, 1)$ define $\bar{x} := \max(x, 1 - x)$. For any $x \in (0, 1)$ we have that*

$$\lim_{\epsilon \to 0} \lim_{\epsilon' \to 0} \limsup_{t \to \infty} \frac{1}{t^\alpha} \log \mathbb{P} \left(\frac{S^{\xi}_{N_t}}{t} \in B_\epsilon(x), \frac{N_t}{t} \leq \frac{1 - \bar{x}}{\mu - \epsilon'} \right) \leq -J(x).$$

If $x = 0$ or $x = 1$, we have

$$\lim_{\epsilon \to 0} \lim_{\epsilon' \to 0} \limsup_{t \to \infty} \frac{1}{t^\alpha} \log \mathbb{P} \left(\frac{S^{\xi}_{N_t}}{t} \in B_\epsilon(x), \frac{N_t}{t} \leq \frac{\epsilon}{\mu - \epsilon'} \right) \leq -J(x).$$

Proof: Throughout let $0 < \epsilon' < \mu$. Consider $x = 1$. As S_n^ξ is increasing in n,

$$\left\{\frac{S^\xi_{N_t}}{t} \in B_\epsilon(1), \frac{N_t}{t} \le \frac{\epsilon}{\mu-\epsilon'}\right\} = \left\{\frac{S^\xi_{N_t}}{t} > 1-\epsilon, \frac{N_t}{t} \le \frac{\epsilon}{\mu-\epsilon'}\right\} \subset \left\{S^\xi_{\lceil \frac{\epsilon t}{\mu-\epsilon'}\rceil} > (1-\epsilon)t\right\}.$$

Using the large deviations upper bound from Theorem 1 on this final sequence of sets, we get

$$\begin{aligned}
\limsup_{t\to\infty} \frac{1}{t^\alpha} \log P\left(S^\xi_{\lceil \frac{\epsilon t}{\mu-\epsilon'}\rceil} > (1-\epsilon)t\right) &= \frac{\epsilon^\alpha}{(\mu-\epsilon')^\alpha} \limsup_{n\to\infty} \frac{1}{n^\alpha} \log P\left(\frac{S^\xi_n}{n} > (1-\epsilon)\frac{\mu-\epsilon'}{\epsilon}\right)\\
&\le -\frac{\epsilon^\alpha}{(\mu-\epsilon')^\alpha} I\left(\frac{(1-\epsilon)(\mu-\epsilon')}{\epsilon}\right)\\
&= -\left(1-\epsilon\left(1+\frac{\mu}{\mu-\epsilon'}\right)\right)^\alpha
\end{aligned}$$

and the result follows taking $\epsilon' \to 0$ and then $\epsilon \to 0$.

Next consider $x \in (1/2, 1)$, so that $\bar{x} = x$. With $n_t := \lceil t(1-\bar{x})/(\mu-\epsilon')\rceil$, we have

$$\left\{\frac{S^\xi_{N_t}}{t} \in B_\epsilon(x), \frac{N_t}{t} \le \frac{1-\bar{x}}{\mu-\epsilon'}\right\} \subset \left\{\frac{S^\xi_{N_t}}{t} > x-\epsilon, \frac{N_t}{t} \le \frac{1-\bar{x}}{\mu-\epsilon'}\right\} \subset \left\{\frac{S^\xi_{n_t}}{t} > x-\epsilon\right\}.$$

Now using the LDP upper bound from Theorem 1 we have

$$\begin{aligned}
\limsup_{t\to\infty} \frac{1}{t^\alpha} \log \mathbb{P}\left(\frac{S^\xi_{n_t}}{t} > x-\epsilon\right) &= \frac{(1-x)^\alpha}{(\mu-\epsilon')^\alpha} \limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbb{P}\left(\frac{S^\xi_n}{n} > \frac{(x-\epsilon)(\mu-\epsilon')}{1-x}\right)\\
&\le -\frac{(1-x)^\alpha}{(\mu-\epsilon')^\alpha} \inf\left\{I(a) : a > \frac{(x-\epsilon)(\mu-\epsilon')}{1-x}\right\}\\
&= -\frac{(1-x)^\alpha}{(\mu-\epsilon')^\alpha} I\left(\frac{(x-\epsilon)(\mu-\epsilon')}{1-x}\right)\\
&= -\left(x-\epsilon+(x-1)\frac{\mu}{\mu-\epsilon'}\right)^\alpha.
\end{aligned}$$

Thus the upper bound follows taking the limit $\epsilon' \to 0$ followed by $\epsilon \to 0$.

The results for $x \in [0, 1/2)$ follow analogously using the corresponding constraints on $S^\tau_{N_t}$: for $x = 0$:

$$\left\{\frac{S^\xi_{N_t}}{t} \in B_\epsilon(0), \frac{N_t}{t} \le \frac{\epsilon}{\mu-\epsilon'}\right\} \subset \left\{S^\tau_{\lceil \frac{\epsilon t}{\mu-\epsilon'}\rceil} > (1-\epsilon)t\right\},$$

and for $x \in (0, 1/2)$:

$$\left\{\frac{S^\xi_{N_t}}{t} \in B_\epsilon(0), \frac{N_t}{t} \le \frac{\epsilon}{\mu-\epsilon'}\right\} \subset \left\{\frac{S^\tau_{n_t}}{t} > x-\epsilon\right\}.$$

■

With $x \in (0,1)$, for any $0 < \epsilon' < \mu$, by the principle of the largest term (e.g. Lemma 1.2.15 of [2])

$$\limsup_{t\to\infty} \frac{1}{t^\alpha} \log \mathbb{P}\left(\frac{S^\xi_{N_t}}{t} \in B_\epsilon(x)\right) = \max\left[\limsup_{t\to\infty} \frac{1}{t^\alpha} \log \mathbb{P}\left(\frac{S^\xi_{N_t}}{t} \in B_\epsilon(x), \frac{N_t}{t} > \frac{1-\bar{x}}{\mu-\epsilon'}\right),\right.$$
$$\left.\limsup_{t\to\infty} \frac{1}{t^\alpha} \log \mathbb{P}\left(\frac{S^\xi_{N_t}}{t} \in B_\epsilon(x), \frac{N_t}{t} \le \frac{1-\bar{x}}{\mu-\epsilon'}\right)\right]$$
$$= \limsup_{t\to\infty} \frac{1}{t^\alpha} \log \mathbb{P}\left(\frac{S^\xi_{N_t}}{t} \in B_\epsilon(x), \frac{N_t}{t} \le \frac{1-\bar{x}}{\mu-\epsilon'}\right),$$

where the last line follows as and Lemma 6 proves that the first term in the max is $-\infty$. As this is true for all $0 < \epsilon' < \mu$, the following upper bound follows from Lemma 7 after taking $\epsilon' \to 0$ and then $\epsilon \to 0$:

$$\lim_{\epsilon\to 0} \limsup_{t\to\infty} \frac{1}{t^\alpha} \log \mathbb{P}\left(\frac{S^\xi_{N_t}}{t} \in B_\epsilon(x)\right) \le -J(x)$$

A near identical application of the lemmas suffices for $x = 0$ and $x = 1$. Thus, as the lower and upper deviation functions for $\{S^\xi_{N_t}/t\}$ coincide with $-J(\cdot)$, Proposition 5 is proved.

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