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Applications of Linear Co-positive Lyapunov Functions for Switched Linear Positive Systems

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Abstract In this paper we review necessary and sufficient conditions for the existence of a common linear co-positive Lyapunov function for switched linear positive systems. Both the state dependent and arbitrary switching cases are considered and a number of applications are presented.

1 Introduction

Positive systems, that is systems in which each state can only take positive values, play a key role in many and diverse areas such as economics [11, 16], biology [1, 9], communication networks [4, 17], decentralised control [21] or synchronisation / consensus problems [10]. Although these as well as switched systems have been the focus of many recent studies in the control engineering and mathematics literature — to name but a few [2, 3, 12, 20] — there are still many open questions relating to the stability of systems that fall into both categories: switched positive systems.

Proving stability for switched systems involves determining a Lyapunov function that is common to all constituent subsystems, [18]. In that context, work discussed in [14, 15] provides necessary and sufficient conditions for the existence of a particular type of Lyapunov function, namely a linear co-positive Lyapunov function (LCLF). It is the aim of this paper to review these results and provide examples of their use.

Our brief paper is structured as follows. In Section 2 we present a number of examples from various applications to motivate the problem. We then summarise conditions for the existence of a common LCLF for switched systems evolving in the entire positive orthant, as well as when the positive orthant is partitioned into cones. Finally, in Section 4 we apply these results to the examples given at the beginning.

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Notation and mathematical preliminaries

Throughout, $\mathbb{R}$ (resp. $\mathbb{R}_+$) denotes the field of real (resp. positive) numbers, $\mathbb{R}^n$ is the n-dimensional Euclidean space and $\mathbb{R}^{n\times n}$ the space of $n\times n$ matrices with real entries. A closed, pointed convex cone $\mathscr{C}$ is a subset of $\mathbb{R}^n$ if and only if $\alpha\mathbf{x}+\beta\mathbf{y}\in\mathscr{C}$ for any $\mathbf{x},\mathbf{y}\in\mathscr{C}$ and non-negative scalars α,β.

Matrices or vectors are said to be positive (resp. non-negative) if all of their entries are positive (resp. non-negative); this is written as $\mathbf{A}\succ\mathbf{0}$ (resp. $\mathbf{A}\succeq\mathbf{0}$), where $\mathbf{0}$ is the zero-matrix of appropriate dimension. A matrix $\mathbf{A}$ is said to be *Hurwitz* if all its eigenvalues lie in the open left half of the complex plane. A matrix is said to be *Metzler* if all its off-diagonal entries are non-negative.

We use $\Sigma_{\mathbf{A}}$ to denote the linear time-invariant (LTI) system $\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}$. Such a system is called *positive* if, for a positive initial condition, all its states remain in the positive orthant throughout time. A classic result shows that this will be the case if and only if $\mathbf{A}$ is a Metzler matrix, [3]. Similarly, a switched linear positive system is a dynamical system of the form $\dot{\mathbf{x}}=\mathbf{A}_{s(t)}\mathbf{x}$, for $\mathbf{x}(0)=\mathbf{x}_0$ where $s:\mathbb{R}\to\{1,\dots,N\}$ is the so-called *switching signal* and $\{\mathbf{A}_1,\dots,\mathbf{A}_N\}$ are the system matrices of the *constituent systems*, which are Metzler matrices. See [5, 18] for more details on systems of this type. Below we will just write $\dot{\mathbf{x}}=\mathbf{A}(t)\mathbf{x}$ for such a system.

Finally, the function $V(\mathbf{x})=\mathbf{v}^{\mathsf{T}}\mathbf{x}$ is said to be a *linear co-positive Lyapunov function* (LCLF) for the positive LTI system $\Sigma_{\mathbf{A}}$ if and only if $V(\mathbf{x})>0$ and $\dot{V}(\mathbf{x})<0$ for all $\mathbf{x}\succeq\mathbf{0}$ and $\mathbf{x}\neq\mathbf{0}$, or, equivalently, $\mathbf{v}\succ\mathbf{0}$ and $\mathbf{v}^{\mathsf{T}}\mathbf{A}\prec\mathbf{0}$.

2 Motivating examples

To motivate our results we shall first present a few situations to which they can be applied.

1) Classes of switched time-delay systems

Consider the class of n-dimensional linear positive systems with time-delay $\tau\geq 0$, similar to those considered by Haddad *et al.* in [6], but where both the system and the delay matrices may be switching over time:

$$\dot{\mathbf{x}}(t)=\mathbf{A}(t)\mathbf{x}(t)+\mathbf{A}_{\mathrm{d}}(t)\mathbf{x}(t-\tau),\quad \mathbf{x}(\theta)=\phi(\theta),\quad -\tau\leq\theta\leq 0 \tag{1}$$

where we assume that the system matrix $\mathbf{A}(t)\in\left\{\mathbf{A}_1,\dots,\mathbf{A}_N\right\}$ is Metzler, the delay matrix $\mathbf{A}_{\mathrm{d}}(t)\in\left\{\mathbf{A}_{\mathrm{d}1},\dots,\mathbf{A}_{\mathrm{d}M}\right\}$ is non-negative, $\left[\mathbf{A}(t)+\mathbf{A}_{\mathrm{d}}(t)\right]$ is Metzler and Hurwitz for all $t\geq 0$, and where $\phi:[-\tau,0]\to\mathbb{R}^n$ is a continuous, vector valued function specifying the initial condition of the system. How can stability of the system for arbitrary switching and delays be shown?

2) Switched positive systems with multiplicative noise

Consider the class of switched positive systems with feedback quantisation or where the states experience resets. In this type of system, the states on the right hand side are scaled by a (usually time-varying) diagonal matrix:

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{D}(t)\mathbf{x}, \quad \mathbf{A}(t) \in \{\mathbf{A}_1, \dots, \mathbf{A}_N\}$$

where we assume that $\mathbf{A}(t)$ is Metzler and Hurwitz for all t, and the diagonal matrix $\mathbf{D}(t)$ has strictly positive and bounded diagonal entries for all t. Under which conditions would such a system be stable?

3) Robustness of switched positive systems with channel dependent multiplicative noise

An important class of positive systems is the class that arises in certain networked control problems. Here, the system of interest has the form:

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \left[\mathbf{C}^{[1]}(t) + \cdots + \mathbf{C}^{[n]}(t)\right]\mathbf{x}$$

where $\mathbf{A}(t)$ is Metzler and where $\mathbf{C}^{[i]}(t) \succeq \mathbf{0}$ is an $n \times n$ matrix that describes the communication path from the network states to the ith state; namely it is a matrix of unit rank with only one non-zero row. Usually, the network interconnection structure varies with time between N different configurations, so that $\mathbf{A}(t) \in \{\mathbf{A}_1, \dots, \mathbf{A}_N\}$ and $\mathbf{C}^{[i]}(t) \in \{\mathbf{C}_1^{[i]}, \dots, \mathbf{C}_N^{[i]}\}$ for $i = 1, \dots, n$. Again, we assume that $\left[\mathbf{A}(t) + \mathbf{C}^{[1]}(t) + \cdots + \mathbf{C}^{[n]}(t)\right]$ is also Metzler and Hurwitz for all t. What can be said regarding asymptotic stability here?

4) Numerical example

Finally, to provide a more concrete example, assume we are given a switched linear positive system with the following three Metzler and Hurwitz matrices

$$\mathbf{A} = \begin{bmatrix} -16 & 6 & 6 \\ 1 & -18 & 2 \\ 5 & 3 & -20 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -10 & 4 & 0 \\ 8 & -10 & 9 \\ 4 & 3 & -13 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -9 & 2 & 8 \\ 6 & -10 & 4 \\ 8 & 0 & -16 \end{bmatrix} \tag{2}$$

Can we prove that it is stable under arbitrary switching?

3 Common linear co-positive Lyapunov functions

As mentioned in the introduction, work reported in [14] discusses conditions for the existence of a common LCLF for switched linear positive systems comprised of sets of LTI systems, where each of the constituent systems is assumed to be associated with a convex region of the positive orthant of the $\mathbb{R}^n$.

Let us briefly present two results. The first, more general result concerns situations where the state space (the positive orthant) is partitioned into smaller regions, and where only certain subsystems may be active in certain regions (this may be interpreted as state dependent switching). The other result focuses on the special case where each of those regions is the entire positive orthant itself, that is the system can switch to any subsystem in any given point in the state space.

3.1 Switching in partitioned positive orthant

Assume there are N closed pointed convex cones $\mathscr{C}_j$ such that the closed positive orthant can be written as $\mathbb{R}^n_+ = \cup_{j=1}^N \mathscr{C}_j$. Moreover, assume that we are given stable positive LTI systems $\Sigma_{\mathbf{A}_j}$ for $j = 1, \dots, N$ such that the jth system can only be active for states within $\mathscr{C}_j$. The following theorem then gives a necessary and sufficient condition for the existence of a common LCLF in this set-up.

Theorem 1. *Given N Metzler and Hurwitz matrices $\mathbf{A}_1, \dots, \mathbf{A}_N \in \mathbb{R}^{n\times n}$ and N closed, convex pointed cones $\mathscr{C}_1, \dots, \mathscr{C}_N$ such that $\mathbb{R}^n_+ = \cup_{j=1}^N \mathscr{C}_j$, precisely one of the following statements is true:*

1. *There is a vector $\mathbf{v} \in \mathbb{R}^n_+$ such that $\mathbf{v}^\mathsf{T}\mathbf{A}_j\mathbf{x}_j < 0$ for all non-zero $\mathbf{x}_j \in \mathscr{C}_j$ and $j = 1, \dots, N$.*
2. *There are vectors $\mathbf{x}_j \in \mathscr{C}_j$ not all zero such that $\sum_{j=1}^N \mathbf{A}_j\mathbf{x}_j \succeq \mathbf{0}$.*

Proof. $2 \Rightarrow \neg 1$:[1] Assume 2 holds. Then, for any positive vector $\mathbf{v} \succ \mathbf{0}$ we have $\mathbf{v}^\mathsf{T}\mathbf{A}_1\mathbf{x}_1 + \cdots + \mathbf{v}^\mathsf{T}\mathbf{A}_N\mathbf{x}_N \geq 0$ which implies that 1 cannot hold.

$\neg 2 \Rightarrow 1$: Assume 2 does not hold, i. e. there are no vectors $\mathbf{x}_j \in \mathscr{C}_j$ not all zero such that $\sum_{j=1}^N \mathbf{A}_j\mathbf{x}_j \succeq \mathbf{0}$. This means that the following intersection of convex cones is empty:

$$\underbrace{\left\{\textstyle\sum_{j=1}^N \mathbf{A}_j\mathbf{x}_j : \mathbf{x}_j \in \mathscr{C}_j, \text{not all zero}\right\}}_{\mathscr{O}_1} \cap \underbrace{\left\{\mathbf{x} \succeq \mathbf{0}\right\}}_{\mathscr{O}_2} = \emptyset.$$

By scaling appropriately we can see that this is equivalent to:

$$\underbrace{\left\{\textstyle\sum_{j=1}^N \mathbf{A}_j\mathbf{x}_j : \mathbf{x}_j \in \mathscr{C}_j, \sum_{j=1}^N \|\mathbf{x}_j\|_1 = 1\right\}}_{\bar{\mathscr{O}}_1} \cap \underbrace{\left\{\mathbf{x} \succeq \mathbf{0}\right\}}_{\mathscr{O}_2} = \emptyset \tag{3}$$

where $\|\cdot\|_1$ denotes the L_1-norm. Now, $\bar{\mathscr{O}}_1$ and $\mathscr{O}_2$ are disjoint non-empty closed convex sets and additionally $\bar{\mathscr{O}}_1$ is bounded. Thus, we can apply Corollary 4.1.3 from [8] which guarantees the existence of a vector $\mathbf{v} \in \mathbb{R}^n$ such that

$$\max_{\mathbf{y}\in\bar{\mathscr{O}}_1} \mathbf{v}^\mathsf{T}\mathbf{y} < \inf_{\mathbf{y}\in\mathscr{O}_2} \mathbf{v}^\mathsf{T}\mathbf{y} \tag{4}$$

As the zero vector is in $\mathscr{O}_2$, it follows that $\inf_{\mathbf{y}\in\mathscr{O}_2} \mathbf{v}^\mathsf{T}\mathbf{y} \leq 0$. However, as $\mathscr{O}_2$ is the cone $\{\mathbf{x} \succeq \mathbf{0}\}$ it also follows that $\inf_{\mathbf{y}\in\mathscr{O}_2} \mathbf{v}^\mathsf{T}\mathbf{y} \geq 0$. Thus, $\inf_{\mathbf{y}\in\mathscr{O}_2} \mathbf{v}^\mathsf{T}\mathbf{y} = 0$. Hence, $\mathbf{v}^\mathsf{T}\mathbf{y} \geq 0$ for all $\mathbf{y} \succeq \mathbf{0}$ and thus $\mathbf{v} \succeq \mathbf{0}$.

[1] That is, we show that if 2 is true, then 1 cannot hold.

Moreover, from (4), we can conclude that for any $j = 1, \ldots, N$ and any $\mathbf{x}_j \in \mathscr{C}_j$ with $\|\mathbf{x}_j\|_1 = 1$ we have $\mathbf{v}^\mathsf{T}\mathbf{A}_j\mathbf{x}_j < 0$. As $\mathscr{C}_j \cap \{\mathbf{x} \succeq \mathbf{0} : \|\mathbf{x}\|_1 = 1\}$ is compact, it follows from continuity that by choosing $\varepsilon > 0$ sufficiently small, we can guarantee that $\mathbf{v}_\varepsilon := \mathbf{v} + \varepsilon\mathbf{1} \succ \mathbf{0}$ satisfies $\mathbf{v}_\varepsilon^\mathsf{T}\mathbf{A}_j\mathbf{x}_j < 0$ for all $\mathbf{x}_j \in \mathscr{C}_j \cap \{\mathbf{x} \succeq \mathbf{0} : \|\mathbf{x}\|_1 = 1\}$ and all $j = 1, \ldots, N$, where $\mathbf{1}$ is the vector of all ones.

Finally, it is easy to see that $\mathbf{v}_\varepsilon^\mathsf{T}\mathbf{A}_j\mathbf{x}_j < 0$ is true even without the norm requirement on $\mathbf{x}_j$. This completes the proof of the theorem. □

A very practical way of partitioning the state space would be to partition it using *simplicial cones* $\mathscr{C}_j$. These are cones generated by non-negative, non-singular generating matrices $\mathbf{Q}_j \in \mathbb{R}^{n\times n}$:

$$\mathscr{C}_j := \left\{ \mathbf{x} \,\middle|\, \mathbf{x} = \textstyle\sum_{i=1}^n \alpha_i \mathbf{q}_j^{(i)}, \alpha_i \geq 0, i = 1, \ldots, n \right\} \tag{5}$$

where $j = 1, \ldots, N$ and $\mathbf{q}_j^{(i)}$ denotes the ith column of $\mathbf{Q}_j$. In that case, we may include the cone generating matrices into the second statement of Theorem 1 to reword it slightly to:

[...]
2. There are vectors $\mathbf{w}_j \succeq \mathbf{0}$ not all zero s. t. $\sum_{j=1}^N \mathbf{B}_j\mathbf{w}_j \succeq \mathbf{0}$, with $\mathbf{B}_j := \mathbf{A}_j\mathbf{Q}_j$.

This new statement 2 can now be easily tested by running a feasibility check on a suitably defined linear program, see [14] for more details.

3.2 Switching in entire positive orthant

An important special case of the previous results is when the $\mathbf{Q}_j$ matrices are the identity matrix, namely when we seek a common linear co-positive Lyapunov function for a finite set of linear positive systems. For that, some additional notation is required: Let the set containing all possible mappings $\sigma : \{1, \ldots, n\} \to \{1, \ldots, N\}$ be called $\mathscr{S}_{n,N}$, for positive integers n and N. Given N matrices $\mathbf{A}_j$, these mappings will then be used to construct matrices $\mathbf{A}_\sigma(\mathbf{A}_1, \ldots, \mathbf{A}_N)$ in the following way:

$$\mathbf{A}_\sigma(\mathbf{A}_1, \ldots, \mathbf{A}_N) := \left[\mathbf{a}_{\sigma(1)}^{(1)}\ \mathbf{a}_{\sigma(2)}^{(2)}\ \ldots\ \mathbf{a}_{\sigma(n)}^{(n)}\right] \tag{6}$$

where $\mathbf{a}_j^{(i)}$ denotes the ith column of $\mathbf{A}_j$. In other words, the ith column $\mathbf{a}_\sigma^{(i)}$ of $\mathbf{A}_\sigma$ is the ith column of one of the $\mathbf{A}_1, \ldots, \mathbf{A}_N$ matrices, depending on the mapping $\sigma \in \mathscr{S}_{n,N}$ chosen. We then have the following condition:

Theorem 2. *Given N Hurwitz and Metzler matrices $\mathbf{A}_1, \ldots, \mathbf{A}_N \in \mathbb{R}^{n\times n}$, the following statements are equivalent:*

1. *There is a vector $\mathbf{v} \in \mathbb{R}_+^n$ such that $\mathbf{v}^\mathsf{T}\mathbf{A}_j \prec \mathbf{0}$ for all $j = 1, \ldots, N$.*
2. *$\mathbf{A}_\sigma(\mathbf{A}_1, \ldots, \mathbf{A}_N)$ is Hurwitz for all $\sigma \in \mathscr{S}_{n,N}$.*

Proof. Given in [14]. □

Remark. Since the submission of [14] it has come to our attention that this result may also be deduced from the more general results on **P**-matrix sets given in [19].

Theorem 2 states that N positive LTI systems have a common linear co-positive Lyapunov function $V(\mathbf{x}) = \mathbf{v}^\mathsf{T}\mathbf{x}$ if and only if the $\mathbf{A}_\sigma(\mathbf{A}_1,\ldots,\mathbf{A}_N)$ matrices are Hurwitz matrices, for all $\sigma \in \mathscr{S}_{n,N}$. In that case, the switched system formed by these subsystems is uniformly asymptotically stable under arbitrary switching.

Finally, note that when the $\mathbf{A}_j\mathbf{Q}_j$ in Theorem 1 (or its reworded version) are Metzler and Hurwitz, then the Hurwitz condition of Theorem 2 can also be used to give a solution to the state dependent switching problem.

4 Solution to motivating examples

We shall now use these results to answer the problems posed in Section 2.

1) Classes of switched time-delay systems

We can show stability under arbitrary switching and delays if two conditions are met: (a) there is a matrix $\tilde{\mathbf{A}}_\mathrm{d}$ such that $\big(\mathbf{A}_\mathrm{d}(t) - \tilde{\mathbf{A}}_\mathrm{d}\big) \preceq \mathbf{0}$ for all t, i. e. there is a matrix $\tilde{\mathbf{A}}_\mathrm{d}$ that is entry-wise greater or equal than $\mathbf{A}_{\mathrm{d}i}$ for all $i = 1,\ldots,M$; (b) for all $\sigma \in \mathscr{S}_{n,N}$ the matrices $\mathbf{A}_\sigma\big(\mathbf{A}_1+\tilde{\mathbf{A}}_\mathrm{d}\,,\ \ldots\ ,\ \mathbf{A}_N+\tilde{\mathbf{A}}_\mathrm{d}\big)$ are Hurwitz. This can be seen by noting that (b) guarantees (by applying Theorem 2) the existence of a vector $\mathbf{v} \succ \mathbf{0}$ such that $\mathbf{v}^\mathsf{T}\big[\mathbf{A}(t) + \tilde{\mathbf{A}}_\mathrm{d}\big] \prec \mathbf{0}$. Then, consider the following Lyapunov-Krasovskii functional, [7, 13].

$$V(\psi) = \mathbf{v}^\mathsf{T}\psi(0) + \mathbf{v}^\mathsf{T}\tilde{\mathbf{A}}_\mathrm{d}\int_{-\tau}^{0}\psi(\theta)\mathrm{d}\theta$$

for some $\mathbf{v} \succ \mathbf{0}$. Clearly $V(\psi) \geq \mathbf{v}^\mathsf{T}\psi(0) \geq a\|\psi(0)\|_\infty$ with $a = \min_i\{v_i\} > 0$ and $\|\cdot\|_\infty$ being the maximum modulus norm.

Next, define $\mathbf{x}_t := \{\,\mathbf{x}(t+\theta) \mid \theta \in [-\tau,0]\,\}$ as the trajectory segment of the states in the interval $[t-\tau\,, t]$. Then, if condition (a) is met, the directional derivative of the above functional along the solutions of (1) will be

$$\begin{aligned}
\dot{V}(\mathbf{x}_t) &= \mathbf{v}^\mathsf{T}\dot{\mathbf{x}}(t) + \mathbf{v}^\mathsf{T}\tilde{\mathbf{A}}_\mathrm{d}\big[\mathbf{x}(t) - \mathbf{x}(t-\tau)\big] \\
&= \mathbf{v}^\mathsf{T}\big[\mathbf{A}(t)\mathbf{x}(t) + \mathbf{A}_\mathrm{d}(t)\mathbf{x}(t-\tau)\big] + \mathbf{v}^\mathsf{T}\tilde{\mathbf{A}}_\mathrm{d}\big[\mathbf{x}(t) - \mathbf{x}(t-\tau)\big] \\
&= \underbrace{\mathbf{v}^\mathsf{T}\big[\mathbf{A}(t) + \tilde{\mathbf{A}}_\mathrm{d}\big]}\mathbf{x}(t) + \mathbf{v}^\mathsf{T}\underbrace{\big[\mathbf{A}_\mathrm{d}(t) - \tilde{\mathbf{A}}_\mathrm{d}\big]}_{\preceq 0}\mathbf{x}(t-\tau) \\
&\leq \quad -\mathbf{p}^\mathsf{T}\mathbf{x}(t) \\
&\leq \quad -\beta\|\mathbf{x}(t)\|_\infty
\end{aligned}$$

where $\beta = \min_i\{p_i\} > 0$. It then follows (see for instance [7]) that the switched system is uniformly asymptotically stable.

2) Switched positive systems with multiplicative noise
Through Theorem 2 we know that if $\mathbf{A}_\sigma(\mathbf{A}_1,\ldots,\mathbf{A}_N)$ is a Hurwitz matrix for all $\sigma \in \mathscr{S}_{n,N}$, then there exists a common LCLF for the system. In that case, since $\mathbf{D}(t)\mathbf{x} \succ \mathbf{0}$, the system will be stable for any $\mathbf{D}(t)$.

3) Robustness of switched pos. systems with channel dep. multiplicative noise
Again, our principal result can be used to give conditions such that this system is stable. A sufficient requirement for asymptotic stability here would be that $\mathbf{A}_\sigma\left(\mathbf{B}_1,\ldots,\mathbf{B}_q\right)$ is a Metzler and Hurwitz matrix for all $\sigma \in \mathscr{S}_{n,q}$, where $q = N^{(n+1)}$ and $\mathbf{B}_1,\ldots,\mathbf{B}_q$ are all the matrices of the form $\left[\mathbf{A}_{i_0} + \mathbf{C}_{i_1}^{[1]} + \cdots + \mathbf{C}_{i_N}^{[n]}\right]$ with $i_0,\ldots,i_N \in \{1,\ldots,N\}$.

Further, by exploiting simple properties of Metzler matrices, we will also get the robust stability of the related system:

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \left[\mathbf{C}^{[1]}(t)\mathbf{D}^{[1]}(t) + \cdots + \mathbf{C}^{[n]}(t)\mathbf{D}^{[n]}(t)\right]\mathbf{x}$$

where the $\mathbf{D}^{[i]}(t)$ are non-negative diagonal matrices whose diagonal entries are strictly positive, but with entries bounded less than one. This latter result is important as it can be used to model uncertain communication channel characteristics.

4) Numerical example
With $\mathbf{A},\mathbf{B},\mathbf{C}$ given as in (2), it turns out that all $\mathbf{A}_\sigma(\mathbf{A},\mathbf{B},\mathbf{C})$ are Hurwitz matrices, for any $\sigma \in \mathscr{S}_{3,3}$; hence a switched linear positive system with these matrices will be uniformly asymptotically stable under arbitrary switching. If, however, the (3,1)-element of $\mathbf{C}$ is changed from 8 to 14 — note that after the change $\mathbf{C}$ is still a Metzler and Hurwitz matrix — then the matrix $\mathbf{A}_{(3,2,3)} = \left[\mathbf{c}^{(1)}\ \mathbf{b}^{(2)}\ \mathbf{c}^{(3)}\right]$ will have an eigenvalue $\lambda \simeq 1.7$ which violates the Hurwitz condition.

5 Conclusion

In this paper, after presenting a few motivating examples, we have reviewed necessary and sufficient conditions for the existence of a certain type of Lyapunov function for switched linear positive systems. We then illustrated and commented on the implications of our results. Future work will consider switched positive systems with time delay, and we suspect that the results reviewed here will be of great value in this future study.

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