2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009
ThA06.1

Stabilization and Performance over a Gaussian Communication Channel for a Plant with Time Delay

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***Abstract*— Two problems that have received much attention are those of finding the minimum channel signal to noise ratio compatible with closed loop stability, and of finding the optimal performance, in terms of disturbance attenuation, for a channel with specified signal to noise ratio. In this paper, we study these problems for the case in which the plant has relative degree greater than one, and thus introduces a delay greater than one time step. We show that, unlike the relative degree one case, for the problem of stabilization linear time varying control and communication strategies may have advantages over linear time invariant strategies. We derive a lower bound on optimal disturbance response at a fixed terminal time. If the encoder has access to the state of the plant, then this bound is achievable using linear time varying communication and control.**

I. Introduction

Recent years have seen much interest in the limitations imposed on a feedback system by the presence of a communication channel in the feedback path (e.g., [1]–[3], [5], [11]–[14], [17], [18]). The goal of the stabilization problem is to determine the minimal channel capacity required to stabilize an open loop unstable plant. The solution to this problem is known for noise-free data rate limited channels [14] and additive Gaussian noise channels [2]. A more difficult problem is that of determining the optimal performance, in terms of the disturbance response, that is achievable for a channel with given capacity. A lower bound on the mean square value of the state vector of an unstable system stabilized over a data rate limited channel is given in [14], and an analogous bound for an unstable system stabilized over a Gaussian channel is given in [8]. Except in special cases, such as that of a first order system, one would not expect this lower bound to be tight, thus complicating the problem of finding the best *achievable* performance. An alternate problem statement, considered in [6], [7], is to consider the finite horizon problem of minimizing the mean square value of the output of a discrete time system at a specified terminal time. A lower bound is derived in [6] on the best possible performance for arbitrary nonlinear causal communication and control strategies. Moreover, under the assumption that the plant has relative degree one, this lower bound is achievable using strategies that are linear and time-varying [7].

For data rate limited channels, it is shown in [14] that time delay does not affect the data rate required for stabilization. However, the authors of [15] show that time delay does worsen the performance, in terms of the mean square response of the system state to a Gaussian disturbance. For Gaussian channels, it is shown in [2] that if the relative degree of the plant is greater than one (implying a time delay of more than one step), then the channel capacity required to stabilize with linear time invariant (LTI) control is strictly greater than if the relative degree were equal to one.

In the present paper we suppose that the plant has relative degree greater than one, and study the impact of the resulting time delay on the problems of stabilization and performance of a feedback system. The remainder of the paper is outlined as follows. In Section II we define notation and discuss preliminaries. In Section III we consider the problem of finding the minimal signal to noise ratio required to stabilize an unstable plant. For minimum phase plants with relative degree one, this minimal value is known and is achievable with LTI communication and control. We show that linear time varying strategies may prove advantageous when the plant has relative degree greater than one. The problem of performance, in terms of disturbance response, for a system that has been stabilized is considered in Section IV. A lower bound that holds for general causal encoding and decoding schemes is derived, and shown to guarantee a peak in the disturbance response whose size increases with the length of the delay. In Section V we consider only the finite horizon performance problem of minimizing the mean square value of the system output at a fixed terminal time. We provide a lower bound on the disturbance response, and show that this bound is achievable provided that the encoder has access to the state of the plant. Conclusions and directions for further research are provided in Section VI.

II. Notation and Preliminaries

We use upper case letters to denote random variables, lower case letters to denote realizations of these random variables, subscripts to denote elements of a sequence, and superscripts to denote subsequences, e.g., $x^k \triangleq \{x_0, x_1, \ldots, x_k\}$. Denote the expected value of the random variable X by $\mathcal{E}\{X\}$. Given two random variables X and Y, denote the conditional expectation of X given that $Y = y$ by $\mathcal{E}_y\{X\} = \mathcal{E}\{X|Y = y\}$, and the associated random variable by $\mathcal{E}_Y\{X\}$. Given X and $Y = y$, it is well known (cf. [10, p. 504]) that the conditional expectation $\mathcal{E}_y\{X\}$ minimizes the variance of the mean square estimation error with respect to all other functions $g(Y)$: $\mathcal{E}\{(X - \mathcal{E}_Y\{X\})^2\} \leq \mathcal{E}\{(X - g(Y))^2\}$.

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Consider the linear system, or "plant"

$$x_{k+1} = Ax_k + Bu_k + Bd_k, \tag{1}$$
$$y_k = Cx_k, \tag{2}$$

with state $x_k \in \mathbb{R}^n$, control $u_k \in \mathbb{R}$, process disturbance $d_k \in \mathbb{R}$, and output $y_k \in \mathbb{R}$. Assume that x_0 and d_k are realizations of zero mean Gaussian random variables X_0 and D_k, where X_0, D_0, D_1, ... are mutually independent, that X_0 has covariance $\Sigma_{0|-1}$, and that D_k is stationary with variance σ_d^2.

Assume that the relative degree of the plant is equal to $\tau \geq 1$, and thus that its transfer function may be factored as

$$G(z) = G_0(z)z^{-(\tau-1)}. \tag{3}$$

Let (A_0, B_0, C_0) be a state space realization of G_0 with state vector $\xi_k \in \mathbb{R}^m$, where $m+\tau-1 = n$. Then the assumption that $G(z)$ has relative degree τ implies that $C_0 B_0 \neq 0$. The state vector and state equations thus have the form

$$x_k^T = \left[\ \xi_k^T \ \middle|\ (u_{k-(\tau-1)} + d_{k-(\tau-1)}), \ldots, (u_{k-1} + d_{k-1})\ \right],$$

and

$$A = \left[\begin{array}{c|cccc} A_0 & B_0 & 0 & \ldots & 0 \\ \hline 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 \end{array}\right], \quad B = \left[\begin{array}{c} 0 \\ \hline 0 \\ \vdots \\ 0 \\ 1 \end{array}\right], \tag{4}$$

$$C = \left[\ C_0 \ \middle|\ 0 \quad 0 \quad \ldots \quad 0\ \right]. \tag{5}$$

Note that

$$CA^\ell B = \begin{cases} 0, & \ell < \tau - 1, \\ C_0 A_0^{\ell+1-\tau} B_0, & \ell \geq \tau - 1. \end{cases} \tag{6}$$

The control input is computed based on measurements of the plant output received from a Gaussian communication channel $r_k = s_k + n_k$, where the channel noise n_k is a realization of an independent identically distributed Gaussian random process with zero mean and variance σ_n^2. The channel noise is assumed to be independent of the initial state and process disturbance. Assume also that the channel input s_k must satisfy the instantaneous power constraint $\mathcal{E}\{S_k^2\} \leq \mathcal{P}$.

We shall be interested in communication and control strategies in which the channel input depends on the sequence of plant states, $s_k = f_k(x^k)$, and the control input depends on the sequence of channel outputs, $u_k = g_k(r^k)$. Note that the encoder f_k and the decoder g_k are potentially nonlinear and time varying.

Denote the conditional expectations of the plant state X_{k+1} given the channel output histories $R^{k-1} = r^{k-1}$ and $R^k = r^k$ by $\hat{x}_{k|k-1} = \mathcal{E}_{r^{k-1}}\{X_k\}$ and $\hat{x}_{k|k} = \mathcal{E}_{r^k}\{X_k\}$, respectively, and the associated state estimation errors by $\tilde{x}_{k|k-1} = x_k - \hat{x}_{k|k-1}$ and $\tilde{x}_{k|k} = x_k - \hat{x}_{k|k}$. Similarly, denote conditional estimates of the system output by $\hat{y}_{k|k-1}$ and $\hat{y}_{k|k}$, and the conditional output estimation errors by $\tilde{y}_{k|k-1}$ and $\tilde{y}_{k|k}$. The variance of $\tilde{y}_{k|k-1}$ is thus given by $\mathcal{E}\{\tilde{Y}_{k|k-1}^2\} = \mathcal{E}\{(Y_k - \mathcal{E}_{R^{k-1}}\{Y_k\})^2\}$, and a similar expression holds for $\mathcal{E}\{\tilde{Y}_{k|k}^2\}$.

III. Stabilization

It was shown in [2] that if the encoder has access to the states of the plant, then the system (1)-(2) may be stabilized using LTI communication and control strategies provided that the channel signal to noise ratio (SNR) satisfies the lower bound

$$\mathcal{P}/\sigma_n^2 > \prod_{i=1}^{N_\phi} |\phi_i|^2 - 1, \tag{7}$$

where $\{\phi_i : i = 1, \ldots, N_\phi\}$ are the unstable eigenvalues of A: $|\phi_i| \geq 1$. Hence the capacity of the channel, given by $\mathcal{C} = (1/2)\log(1 + \mathcal{P}/\sigma_n^2)$, must satisfy

$$\mathcal{C} > \sum_{i=1}^{N_\phi} \log|\phi_i|. \tag{8}$$

If the encoder has access only to the plant output, then the minimal SNR required for stabilization with LTI communication and control increases with the relative degree, and thus the time delay, of the plant. We illustrate with a special case.

Lemma 1: Assume that

$$G(z) = \frac{z^{-(\tau-1)}}{z - \phi}, \tag{9}$$

where $|\phi| > 1$ and $\tau \geq 1$. Then stabilization with LTI communication and control is possible if and only if the SNR satisfies

$$\mathcal{P}/\sigma_n^2 > (\phi^2 - 1)\phi^{2(\tau-1)}. \tag{10}$$

Furthermore, use of the unity encoder $s_k = y_k$ and the decoder/control law

$$u_k = (1/\phi - \phi)\left(\phi^{\tau-1} r_k + \phi^{\tau-2} u_{k-\tau+1} + \ldots + \phi u_{k-2} + u_{k-1}\right)$$

results in (10) being satisfied with equality asymptotically: $\lim_{k\to\infty} \mathcal{E}\{Y_k^2\} = (\phi^2 - 1)\phi^{2(\tau-1)}\sigma_n^2$.

Proof: Follows from [2, Theorem III.2] or [16, Theorem 2]. ■

It was shown in [8], [9] that if the plant is minimum phase and has relative degree one, then nonlinear time-varying communication and control strategies do not allow stabilization with a smaller channel SNR than that achievable with LTI strategies. For plants that have relative degree greater than one, it turns out that nonlinear, time-varying strategies may have advantages.

Example 1: Consider the plant (9). Suppose that the channel input is given by

$$s_k = \begin{cases} y_k, & k = \ell\tau,\ \ell = 0, 1, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

Suppose also that the control input is given by

$$u_k = \begin{cases} -(\phi^\tau - \phi^{-\tau})r_k, & k = \ell\tau,\ \ell = 0,1,\dots \\ 0, & \text{otherwise.} \end{cases}$$

Then it may be shown that $y_{(\ell+1)\tau} = \phi^\tau y_{\ell\tau} + u_{\ell\tau}$, and $\lim_{\ell\to\infty} \mathcal{E}\{Y_{\ell\tau}^2\} = (\phi^{2\tau} - 1)\sigma_n^2$. The *peak* SNR is thus $\phi^{2\tau} - 1$, which is greater than value (10) achieved with the LTI strategy in Lemma 1. However, the *average* SNR is $(\phi^{2\tau} - 1)/\tau$, which is smaller than (10). ■

The problem of determining the minimal value of SNR for which a plant with relative degree greater than one may be stabilized remains to be solved. Unlike the relative degree one case, the optimal communication and control strategies may be nonlinear and time varying.

IV. Stabilization and Performance over an Infinite Horizon

The authors of [8], [9] derived a lower bound on the mean square value of the state vector, showing that the response to a disturbance would become unbounded if the channel capacity is allowed to approach the minimum required for stabilization. However, this bound does not depend on the relative degree, or time delay, of the plant. Motivated by analogous results on data rate limited channels [15], we now extend the results of [8], [9] to include the effect of delay.

It follows from the decomposition (4)-(5) that the state equations of the plant may be written as

$$\begin{aligned} \xi_{k+1} &= A_0\xi_k + B_0 u_{k+1-\tau} + B_0 d_{k+1-\tau} \\ y_k &= C_0 \xi_k. \end{aligned} \tag{11}$$

We shall assume, with no loss of generality, that all eigenvalues of A_0 are unstable, and thus that $m = N_\phi$. (If this assumption is not satisfied, then the state equations may be further decomposed into stable and unstable subsystems, as described in [8], [9].) Iterating (11) yields

$$\xi_{k+\tau} = A_0^\tau \xi_k + \nu_k + \delta_k,$$

where

$$\nu_k \triangleq \sum_{j=0}^{\tau-1} A_0^j B_0 u_{k-j}, \quad \delta_k \triangleq \sum_{j=0}^{\tau-1} A_0^j B_0 d_{k-j}.$$

Note that ν_k is a function of the channel output sequence r^k, and is thus known at the decoder at time k. Furthermore, δ_k is an m dimensional zero mean Gaussian random vector with constant covariance

$$\mathcal{E}\{\Delta_k \Delta_k^T\} = \sigma_d^2 \sum_{j=0}^{\tau-1} A_0^{\tau-1-j} B_0 B_0^T A_0^{(\tau-1-j)T}.$$

Since Δ_k is Gaussian, the *entropy power* [4] of Δ_k satisfies

$$N(\Delta) = \sigma_d^2 \det{}^{1/m}\left(\sum_{j=0}^{\tau-1} A_0^{\tau-1-j} B_0 B_0^T A_0^{(\tau-1-j)T}\right). \tag{12}$$

Under the assumptions that (A_0, B_0) is reachable, and that $\tau \ge m$, it follows that $N(\Delta) > 0$.

Define the average conditional entropy power [9] of the state ξ_k given the channel output sequence r^ℓ, $\ell \le k$, by $n_{k|\ell} = \mathcal{E}\{N_{R^\ell}(\Xi_k)\}$, where $N_{r^\ell}(\Xi_k)$ is the conditional entropy power of the random variable Ξ_k given that $R^\ell = r^\ell$. Lemmas III.2 and III.3 of [9] may be modified to show that

$$n_{k|k} \ge \left(\frac{\sigma_n^2}{\mathcal{P} + \sigma_n^2}\right)^{\tau/m} n_{k|k-\tau}, \tag{13}$$

$$n_{k+\tau|k} \ge \left(\prod_{i=1}^m |\phi_i|^2\right)^{\tau/m} n_{k|k} + N(\Delta). \tag{14}$$

The bound (13) provides a lower bound on the possible reduction in entropy power due to τ measurements from a channel with capacity $\mathcal{C} = (1/2)\log(1 + \mathcal{P}/\sigma_n^2)$. The bound (14) provides a lower bound on the increase in entropy power over τ time steps due to the unstable state dynamics and the disturbances arriving in this time interval. Combining (13)-(14) yields the recursion

$$n_{k+\tau|k} \ge \gamma^\tau n_{k|k-\tau} + N(\Delta), \tag{15}$$

where

$$\gamma \triangleq \prod_{i=1}^m |\phi_i|^{2/m} e^{-2\mathcal{C}/m}.$$

It follows from (15) that $n_{k|k-\tau} \ge N(\Delta)(1-\gamma^{\tau k})/(1-\gamma^\tau)$. Using the fact that conditional entropy power is a lower bound on conditional variance [4, p. 255], and thus on the mean square value of the state vector [8], [9], we have

$$\mathcal{E}\{\|\Xi_k\|^2\} \ge mN(\Delta)(1-\gamma^{\tau k})/(1-\gamma^\tau). \tag{16}$$

Assume that the channel capacity satisfies the lower bound (8) required for stabilization. Then $\gamma < 1$ and the mean square value of the state is bounded below by

$$\begin{aligned} \sup_k \mathcal{E}\{\|\Xi_k\|^2\} &\ge mN(\Delta)/(1-\gamma^\tau) \\ &= \frac{mN(\Delta)}{1-\gamma}\frac{1}{1+\gamma+\ldots+\gamma^{\tau-1}}. \end{aligned} \tag{17}$$

Example 2: Consider again the system (9). Assume that $\mathcal{C} > \log|\phi|$. Then $N(\Delta)$ given by (12) simplifies to $N(\Delta) = \sum_{j=0}^{\tau-1}\phi^{2j}$, and (17) reduces to

$$\sup_k \mathcal{E}\{\|\Xi_k\|^2\} \ge \frac{\sigma_d^2}{1-(\phi/e^{\mathcal{C}})^2}\left(\frac{\sum_{j=0}^{\tau-1}\phi^{2j}}{\sum_{j=0}^{\tau-1}(\phi/e^{\mathcal{C}})^{2j}}\right). \tag{18}$$

For $\tau = 1$, the fact that the system is first order implies that the bound is achievable using LTI communication and control strategies [9]. ■

It is clear that the lower bound on disturbance response (18) is an increasing function of the relative degree τ. In fact, for $\tau > 1$, this lower bound may be conservative, and the problem of finding the optimal disturbance response remains to be resolved.

V. Performance at a Terminal Time

We now extend the results of [7], and present communication and control strategies $s_k = f_k(x^k)$ and $u_k = g_k(r^k)$ that minimize the mean square value of the system output at the terminal time $k = N + \tau$. The τ-step delay in the plant implies that $y_{N+\tau}$ can depend *only* on the channel output sequence r^N, and thus that we need to choose s_k and u_k only for times $k = 0, \ldots, N$. The optimal value of this cost function is thus

$$J^*_{N+\tau} = \inf_{\substack{f_k, g_k \\ k=0,\ldots,N}} \mathcal{E}\{Y^2_{N+\tau}\}. \quad (19)$$

The fact that the conditional expectation minimizes the mean square estimation error implies that

$$\mathcal{E}\{Y^2_{k+\tau}\} \geq \mathcal{E}\{\tilde{Y}^2_{k+\tau|k}\}. \quad (20)$$

By choosing the control at time N appropriately, the lower bound (20) may be achieved with equality at time $N + \tau$.

Lemma 2: Assume that the system (1)-(2) has relative degree τ. Then the control input

$$u_N = -(CA^{\tau-1}B)^{-1} CA^\tau \hat{x}_{N|N} \quad (21)$$

yields

$$\hat{y}_{N+\tau|N} = 0, \qquad y_{N+\tau} = \tilde{y}_{N+\tau|N}. \quad (22)$$

Hence

$$\mathcal{E}\{y^2_{N+\tau}\} = \mathcal{E}\{(CA^\tau \tilde{x}_{N|N})^2\} + (CA^{\tau-1}B)^2 \sigma_d^2. \quad (23)$$

Proof: Iterating (1) yields

$$x_{k+\tau} = A^\tau x_k + \sum_{j=0}^{\tau-1} A^{\tau-1-j} B(u_{k+j} + d_{k+j}).$$

Left multiplying by C and invoking the assumption of relative degree τ implies that

$$y_{k+\tau} = CA^\tau x_k + CA^{\tau-1}B(u_k + d_k), \quad (24)$$
$$\hat{y}_{k+\tau|k} = CA^\tau \hat{x}_{k|k} + CA^{\tau-1}Bu_k.$$

Setting $k = N$ and applying the control (21) yields (22). Hence $y_{N+\tau} = CA^\tau \tilde{x}_{N|N} + CA^{\tau-1}Bd_N$, and independence of $\tilde{x}_{N|N}$ and d_N implies (23). ■

The result of Lemma 2 shows how to choose the control signal at time N. It remains to choose the channel input at times $k = 0, \ldots, N$ and the control signal at times $k = 0, \ldots, N-1$. By Lemma 2, this problem reduces to one of minimizing the variance of the estimation error $\tilde{y}_{N+\tau|N}$.

We must estimate $y_{N+\tau}$ which, under the assumption on relative degree, is given by

$$y_{N+\tau} = CA^{N+\tau} x_0 + \sum_{j=0}^{N} CA^{N+\tau-1-j} B(d_j + u_j).$$

Since the control signal is known at the decoder, and the disturbance at time N will not affect the plant output until time $N + \tau$, the task of the encoder is to use the channel input sequence s^N to communicate a "message" that is a function of the primitive random variables x_0 and d^{N-1}:

$$m(x_0, d^{N-1}) = CA^{N+\tau} x_0 + \sum_{j=0}^{N-1} CA^{N+\tau-1-j} Bd_j. \quad (25)$$

A. Encoder has access to additional information.

We suppose *temporarily* that the encoder has access to additional information: the channel output and the control input. Access to the state and the control input allows the primitive random variables to be computed. In particular, the disturbance d_k may be computed at time $k+1$:

$$d_k = (CA^{\tau-1}B)^{-1} \left(CA^{\tau-1} x_{k+1} - CA^\tau x_k\right) - u_k.$$

The ability to compute the primitive random variables, in turn, enables the encoder to obtain an estimate of (25) at each time step as the state of a discrete integrator with initial condition $m_0 = CA^{N+\tau} x_0$ and input sequence $v_k = CA^{N+\tau-1-k} Bd_k$ with variance $\sigma_k^2 = (CA^{N+\tau-1-k}B)^2 \sigma_d^2$:

$$m_{k+1} = m_k + v_k. \quad (26)$$

Note that the integrator state (26) at time k provides an estimate of the message $m(x_0, d^{N-1})$ given the primitive random variables available at that time: $m_k = \mathcal{E}_{x_0, d^{k-1}}\{m(X_0, D^{N-1})\}$.

The assumption that the encoder also has access to the channel output allows a Kalman filter for the purpose of estimating m_k to be implemented using the feedback path around the channel, as shown in Figure 1.

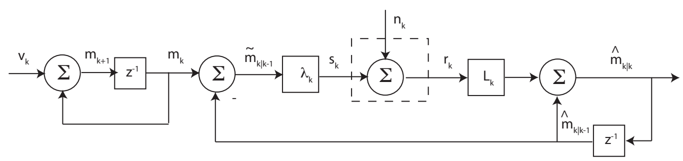

Fig. 1. Communicating the output of a discrete integrator over a channel with feedback.

As described in [6], the state estimate evolves according to

$$\hat{m}_{k|k} = \hat{m}_{k|k-1} + L_k r_k, \qquad \hat{m}_{k+1|k} = \hat{m}_{k|k}$$

where the estimator gain L_k and $\mathcal{M}_{k|k-1} = \mathcal{E}\{\tilde{M}^2_{k|k-1}\}$ satisfy

$$L_k = \frac{1}{\lambda_k} \frac{\mathcal{P}}{\mathcal{P} + \sigma_n^2}, \quad \mathcal{M}_{k+1|k} = \mathcal{M}_{k|k-1} \frac{\sigma_n^2}{\mathcal{P} + \sigma_n^2} + \sigma_k^2,$$

with initial condition $\mathcal{M}_{0|-1} \triangleq CA^{N+\tau} \Sigma_{0|-1} A^{(N+\tau)T} C^T$, and λ_k is adjusted so that $\lambda_k^2 \mathcal{M}_{k|k-1} = \mathcal{P}$.

We have that $\tilde{y}_{N+\tau|N} = \tilde{m}_{N|N} + CA^{\tau-1}Bd_N$. Hence, $\mathcal{E}\{\tilde{Y}^2_{N+\tau|N}\} = \mathcal{E}\{\tilde{M}^2_{N|N}\} + (CA^{\tau-1}B)^2 \sigma_d^2$, and iterating $\mathcal{M}_{k+1|k}$ yields

$$\mathcal{E}\{\tilde{Y}^2_{N+\tau|N}\} = \mathcal{M}_{0|-1} \left(\frac{\sigma_n^2}{\mathcal{P} + \sigma_n^2}\right)^{N+1} + \sigma_d^2 \sum_{j=0}^{N} (C_0 A_0^j B_0)^2 \left(\frac{\sigma_n^2}{\mathcal{P} + \sigma_n^2}\right)^j, \quad (27)$$

where the identity (6) has been applied to the terms in the summation. Note that the variance of the estimation error depends on the relative degree τ only through the response to the initial condition.

For the case $\tau = 1$, it is shown in [6] that the use of more general, potentially nonlinear, encoding and decoding schemes cannot reduce the variance of the estimation error beyond that achievable with the linear schemes depicted in Figure 1. It is straightforward to show that the results of [6] may be extended to yield the same conclusions for arbitrary values of τ.

Example 3: Consider the system (1)-(2) factored as in (3)-(5), with

$$A_0 = \begin{bmatrix} 1.1 & 1 \\ 0 & 1.2 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad (28)$$

$\sigma_d^2 = 1$, $\mathcal{P} = 10$, and $\sigma_n^2 = 5$. Assume that the state ξ_0 has covariance $\Sigma_{0|-1}^{\xi} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and that the remaining entries of the covariance matrix $\Sigma_{0|-1}$ are equal to zero. Plots of the minimum achievable estimation error (27) vs. time N for $\tau = 1, 3, 5$ are given in Figure 2. Note that the only difference for the three cases is the response to the initial condition, which worsens as τ increases. As expected from (27), once the response to the initial state decays to zero, the estimation error variance is independent of τ. ■

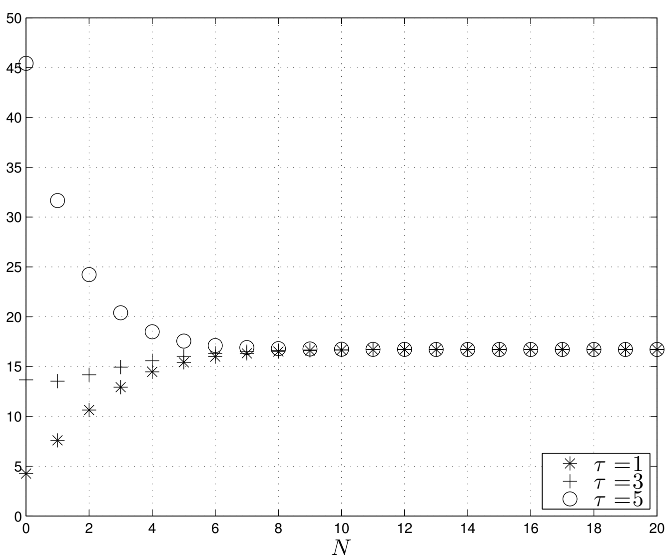

Fig. 2. Optimal estimation error (27) $\mathcal{E}\{\tilde{Y}_{N+\tau|N}^2\}$ vs. N for $\tau = 1, 3, 5$.

It is of interest to determine the asymptotic behavior of the estimation error variance (27) for large values of N. As noted in [7], if the channel SNR satisfies $\mathcal{P}/\sigma_n^2 > \rho^2(A) - 1$, where $\rho(A)$ denotes the spectral radius of A, then

$$\lim_{N\to\infty} \mathcal{E}\{\tilde{Y}_{N+\tau|N}^2\} = \sigma_d^2 \sum_{j=0}^{\infty} (C_0 A_0^j B_0)^2 \left(\frac{\sigma_n^2}{\mathcal{P} + \sigma_n^2}\right)^j . \quad (29)$$

The fact that the limit (29) is finite for SNRs that may be incompatible with closed loop stability is not a contradiction, since the optimal error variance (27) may only be achieved at a given *finite* value of N.

Example 4: Consider again the system (9) discussed in Lemma 1 and Examples 1-2, and assume that $\mathcal{C} > \log|\phi|$. Then (29) simplifies to

$$\lim_{N\to\infty} \mathcal{E}\{\tilde{Y}_{N+\tau|N}^2\} = \frac{\sigma_d^2}{1 - (\phi/e^{\mathcal{C}})^2}. \quad (30)$$

Note that this value is smaller than the lower bound (18), but may only be achieved at a specified terminal time. ■

B. Encoder has access to the plant state only.

We now remove the assumption that the encoder is allowed access to the channel output and the plant input. Specifically, we suppose that the channel input is given by

$$s_k = \lambda_k z_k, \quad z_k = H_k x_k, \quad H_k = CA^{N+\tau-k}, \quad (31)$$

$k = 0, \ldots, N$. It follows from (24) that $z_{N+1} = y_{N+\tau}$.

Proposition 1: Assume that the channel input is given by (31), where λ_k is adjusted so that $\lambda_k^2 \mathcal{E}\{S_k^2\} = \mathcal{P}$. Assume also that we apply the feedback control

$$u_k = -F_k \hat{x}_{k|k}, \qquad F_k = (H_{k+1}B)^{-1} H_{k+1} A, \quad (32)$$

$k = 0, \ldots, N$. The estimate $\hat{x}_{k|k}$ is given by

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k-1} + Bu_k + AL_k(r_k - \lambda_k H_k \hat{x}_{k|k-1}),$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k(r_k - \lambda_k H_k \hat{x}_{k|k-1}),$$

where

$$L_k = \lambda_k \Sigma_{k|k-1} H_k^T / (\mathcal{P} + \sigma_n^2), \quad (33)$$

and $\Sigma_{k|k-1}$ is the solution to the Riccati difference equation

$$\Sigma_{k+1|k} = A\Sigma_{k|k-1}A^T - \frac{A\Sigma_{k|k-1}H_k^T H_k \Sigma_{k|k-1} A^T}{H_k \Sigma_{k|k-1} H_k^T} \frac{\mathcal{P}}{\mathcal{P} + \sigma_n^2} + \sigma_d^2 BB^T. \quad (34)$$

Then $z_k = \tilde{z}_{k|k-1}$ and, at time $k = N+1$, $\mathcal{E}\{Y_{N+\tau}^2\} = \mathcal{E}\{\tilde{Y}_{N+\tau|N}^2\}$, where $\mathcal{E}\{\tilde{Y}_{N+\tau|N}^2\}$ is given by (27).

Proof: With the control (32) applied, it is straightforward to show that $\hat{z}_{k+1|k} = 0$, and thus $z_k = \tilde{z}_{k|k-1}$. In particular, $y_{N+\tau} = z_{N+1} = \tilde{z}_{N+1|N}$, and thus $\mathcal{E}\{Y_{N+\tau}^2\} = CA^{\tau-1}\Sigma_{N+1|N}A^{(\tau-1)T}C^T$, where $\Sigma_{N+\tau|N}$ is given by (34) for $k = N$. Substituting (34) into the expression for $\mathcal{E}\{Y_{N+\tau}^2\}$ and simplifying yields

$$\mathcal{E}\{Y_{N+\tau}^2\} = CA^{\tau}\Sigma_{N|N-1}A^{\tau T}C^T \frac{\sigma_n^2}{\mathcal{P} + \sigma_n^2} + \sigma_d^2 (CA^{\tau-1}B)^2.$$

Applying a similar decomposition to $CA^{\tau}\Sigma_{N|N-1}A^{\tau T}C^T$ and repeating shows that $\mathcal{E}\{Y_{N+1}^2\}$ is given by (27). ■

The control (32) only sets $y_{k+\tau} = \tilde{y}_{k+\tau|k}$ for $k = N$. We now state a general formula for the mean square value of the output at earlier times.

Corollary 3: Assume that the hypotheses of Proposition 1 are satisfied. Then

$$\mathcal{E}\{Y_{k+\tau}^2\} = CA^{\tau-1}\Sigma_{k+1|k}(CA^{\tau-1})^T + CA^{\tau-1}(A - BF_k)\Gamma_{k|k}(A - BF_k)^T(CA^{\tau-1})^T, \quad (35)$$

where F_k is defined by (32), and $\Gamma_{k|k} \triangleq \mathcal{E}\{\hat{X}_{k|k}\hat{X}_{k|k}^T\}$ satisfies the recursion

$$\Gamma_{k+1|k+1} = (A - BF_k)\Gamma_{k|k}(A - BF_k)^T + \frac{\lambda_k^2\Sigma_{k|k-1}H_k^T H_k\Sigma_{k|k-1}}{\mathcal{P} + \sigma_n^2}, \quad (36)$$

with $\Gamma_{0|0} = \lambda_0^2\Sigma_{0|-1}H_0^T H_0\Sigma_{0|-1}/(\mathcal{P} + \sigma_n^2)$. ■

Example 5: Consider the system (28) studied in Example 3. Suppose we apply the communication and control sequences (31) and (32) that minimize the cost (19) for $N = 20$ and $\tau = 3$ (hence the terminal time is equal to 23). The transient value of the estimation error variance $\mathcal{E}\{\tilde{Y}_{k+3|k}^2\}$ and the mean square value of the output $\mathcal{E}\{Y_{k+3}^2\}$ are plotted in Figure 3. Note that $\mathcal{E}\{Y_{k+3}^2\} \geq \mathcal{E}\{\tilde{Y}_{k+3|k}^2\}$, $\forall k$, as we expect from (20), and that equality is achieved at the terminal time $N + 3 = 23$. Also plotted in Figure 3 is the variance of the optimal estimation error at time $k + 3$, as depicted in Figure 2. As predicted, the value of $\mathcal{E}\{\tilde{Y}_{k+3|k}^2\}$, and thus that of $\mathcal{E}\{Y_{k+1}^2\}$, is equal to this lower bound at the terminal time. ■

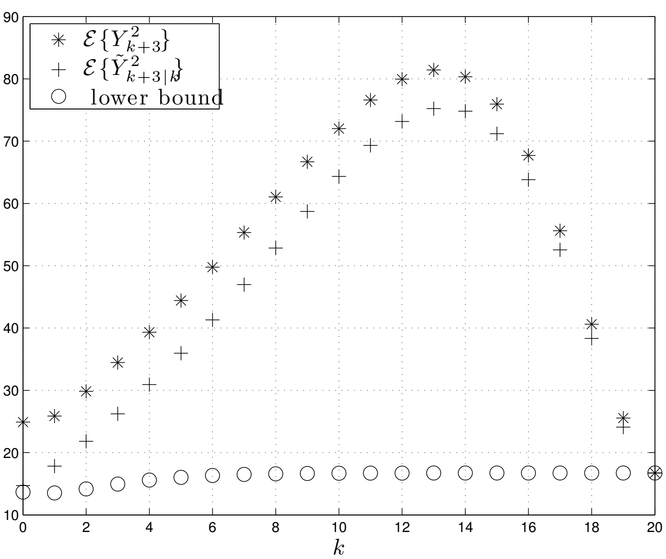

Fig. 3. Plots of $\mathcal{E}\{Y_{k+3}^2\}$ and $\mathcal{E}\{\tilde{Y}_{k+3|k}^2\}$ vs. k for terminal time $N+3 = 23$. Also plotted is the lower bound on $\mathcal{E}\{\tilde{Y}_{k+3|k}^2\}$ from Figure 2.

VI. Conclusions and Future Directions

In this paper we have discussed the impact that time delay due to the relative degree of the plant has upon the problems of stabilization and performance. Under the assumption that the encoder has access to the states of the plant, the optimal value of the terminal cost function (19) is given by (27), and is achieved using the linear time-varying communication and control strategies of Proposition 1. It remains to determine the optimal value of the cost (19) in the case that the encoder has access only to the sequence of plant outputs. In fact, the lower bound derived in Section V-B is optimistic, in that it assumes each disturbance may be computed one time step after it occurs. An approach that takes into account the fact that disturbances do not reach the output for τ time steps and that removes the assumption that the encoder has access to the plant state will be reported elsewhere.

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