Optimal Initial Public Offering design with aftermarket trading.*

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November 4, 2009

*We are particularly grateful to Marco Pagano and Uday Rajan for their most helpful comments on this paper. We also thank Roberto Burguet, Alejandro Manelli, Pablo Bekker and Peter Hammond for their valuable input. We thank seminar and conference participants at Vigo University, INRA Montpellier, University College Dublin, NUI Maynooth, WP Carey Business School Arizona State University, 82nd Western Economic Association and at 9th SAET Conference.

Abstract

We characterize the optimal pricing and allocation of shares in the presence of distinct adverse selection problems. Some investors have private information at the time of the IPO and sell their shares in the after-market upon facing liquidity needs. Others learn their private interest in the after-market, and sell their shares strategically. The optimal mechanism trades-off informational rents and rents to strategic traders. Flipping facilitates truthful information revelation. When liquidity needs are likely, it is optimal to allocate all shares to investors informed at the IPO stage. Otherwise, some shares are allocated to those who trade strategically in the after-market.

I Introduction

Most of the literature considering the optimal allocation and pricing of shares in initial public offerings (hereafter IPO) ignores trading in the after-market and its impact on the value investors assigned to the shares they bid for. Yet, incorporating after-market trading in the IPO analysis is critical when there is residual uncertainty about the value of the stock that some traders can learn and use. Indeed, if an investor has access to some private information in the after-market, he can use it strategically and make a profit at the expense of uninformed investors. Each investor's prospect of winning or losing money in the secondary market is likely to affect his willingness to pay for the shares and consequently the offer price.

It is not always feasible to get rid of all residual uncertainty, and thus of some potential strategic trading, despite best intentions. Chen and Wilhelm (2008) argue that underwriters are often unable to capture all the relevant information during the road show. It may be that information arrives sequentially or else, that all investors have not been able to process it all during the time frame of the IPO. Therefore, whether it is desirable or not, strategic trading in the after-market is often unavoidable and should be taken into account when characterizing an optimal IPO design.

Evidence of strategic trading in the aftermarket is empirically documented in Krigman et al. (1999), Minnigoulov (2001), and more recently in Boehmer et al. (2006). According to these articles, flipping is a significant predictor of future stock performance.[1] They establish that some of the after-market trading is based on information that has not been revealed during the IPO stage.

Ellul and Pagano (2006), Busaba and Chang (2005), and Chen and Wilhelm (2008) show that incorporating after-market trading leads to interesting new results. Focusing on under-pricing, Ellul and Pagano (2006) proposes a new rationale for the well documented under-pricing phenomenon.[2] Interestingly, these authors show that the amount of under-pricing is correlated with liquidity risk. They consider a setting with two types of private information, one affecting the primary market (aka pre-market) and the other arising in the after-market. In their paper underpricing is required to compensate investors who buy shares in an IPO and potentially liquidate these in the after-market where some traders have superior information.[3] Busaba and Chang (2005) compares price discovery under book-building and fixed price offering. Under book-building, private information is revealed during the road show in exchange for some informational rents. By opposition, price discovery takes place in the after-market under fixed price offering. While several seminal papers have established that book-building leads to a lower cost of price discovery, Busaba and Chang (2005) shows that incorporating after-market trading leads to new conclusions.[4] First, they show

[1] Flipping is defined as quickly selling some of the shares acquired during the IPO.

[2] The under-pricing is defined as the (often positive) difference between the price resulting from the first day of trading and the IPO offer price.

[3] An additional rationale for under-pricing can be found in Ellis et al. (2000) and Boehmer and Fishe (2000) who establish a link between the underwriter's trading profits in the after-market and the IPO underpricing.

[4] The seminal papers in question are Benveniste and Wilhelm (1990), Spatt and Srivastava

that introducing after-market trading increases the required under-pricing under both methods. More importantly they establish that, unless entry is restricted under book-building, fixed price offering targeting uninformed investors minimizes underpricing. Finally, Chen and Wilhelm (2008) argues that, in a context where relevant information arrives sequentially, it is optimal to resort to discriminatory pricing to sell the shares. However, regulatory rules require that IPO shares must be sold at a uniform price. These authors then show how a strategic allocation of the shares to institutional investors, together with price stabilization practices from underwriters during the after-market, enables to overcome the uniform pricing constraint so as to reach a more efficient outcome.

Our paper proposes a complementary analysis to this new strand of research. It characterizes the optimal pricing and allocation of shares in the presence of distinct adverse selection problems, one affecting the pre-market and the other the after-market. We consider that the underwriter faces two types of investors. As in Benveniste and Spindt (1989), some have private information about the company going public that reflects their interest for the shares at the time of the IPO (these are called type 1 investors). By opposition, others learn their private interest for the stock in the after-market, that is after the shares have been allocated and priced (called type 2 investors).[5] The realized value of the share reflects the interest of all the investors.

(1991), Benveniste and Busaba (1997), and Biais and Faugeron-Crouzet (2001).

[5] As in Chen and Wilhelm (2008), we consider that the after-market stage does not refer solely to the first day of trading.

The discrepancy in assessing relevant information is motivated by stylized facts described and analyzed in several papers. On the one hand, the relevant financial information is typically complex and diverse and, on the other hand, investors are not identical in their ability to process this information. The former is supported by Hirst and Hopkins (1998) suggesting that the time and mental attention needed for processing financial information is non-trivial.[6] The latter is motivated in Peng (2005) where information capacity constraint agents do not learn their information simultaneously. This is even more likely to arise in an IPO where the time frame is too narrow to incorporate all the relevant information, as highlighted by Chen and Wilhelm (2008). Finally, Libby et al. (2002) provides an interesting survey supporting our assumption as it describes experimental analysis on financial information processing.

We model the after-market as a competitive dealer market. Because the investors acquire their information in different periods, they trade for different reasons in the after-market. Following Ellul and Pagano (2006), we consider that type 1 investors, who have revealed their interest at the IPO stage, sell if they face liquidity needs. The other type of investors sell or keep their stock depending on their information.

Due to the distinct adverse selection problems we show that two sources of rents emerge: the usual informational rents and the rents from trading the asset strategically. The optimal solution and how much money has to be "left on the

[6] Hirshleifer and Teoh (2003) shows that even the format, in which financial information is presented, matters.

table" depends on how many investors of either type the underwriter faces. The underwriter can eliminate all rents by allocating all the shares to type 2 investors provided they can exhaust the issue. When this is not the case, the optimal allocation and price result from trading-off both types of rents. We then establish the following results.

First, we show that the level of informational rents decreases as type 1 investors are more likely to face liquidity needs. Because the dealers infer information from the offer price, which reflects the expressions of interest, down-playing interest results in a lower bid price leading highly interested type 1 investors to incur a self-inflected punishment when they report a low interest for the shares. Thus, as liquidity needs become more likely, the incentive to misreport high interest weakens and consequently informational rents become negligible.[7] Interestingly, this may explain the finding in Aggarwal (2000) according to which penalty bids, punishing flipping, are seldom implemented. Allowing flipping can, according to our finding, facilitate truthful information revelation.

Second, as in Benveniste and Spindt (1989), priority is given to investors reporting high interest at the IPO stage. When these do not exhaust the issue, the allocation of the remaining shares between type 2 investors and type 1 investors with low interest, is decided as follows. When liquidity needs are likely to arise, the remaining shares are sold to type 1 investors reporting low interest. By opposition, type 2 investors receive some shares when type 1 investors are

[7] This contrasts with Busaba and Chang (2005) where investors benefit from lying as they transfer their informational advantage to the after-market.

long-term holders i.e. when they are unlikely to face liquidity needs. The intuition for this result is the following. Allocating shares to type 2 investors is both beneficial and detrimental. On the one hand, it facilitates incentive compatibility as it mitigates type 1 investors access to a share when reporting low interest. On the other hand, it generates rents from strategic trading and decreases type 1 investors' willingness to pay for the stock. Indeed, since type 1 investors face an informational disadvantage in the after-market, their overall willingness to pay for the shares decreases if part of the shares is allocated to type 2 investors. The optimal allocation is such that type 2 investors are given shares when the informational rents are significant, that is when liquidity needs are unlikely. How unlikely must liquidity needs be depends on how relevant is type 2 investors' information relative to that of type 1 and how likely it is to face a type 1 investor with high interest.

Third, we show that when there are not enough type 1 investors to exhaust the issue (and not enough type 2 either), the underwriter is inclined to sell shares to type 2 investors for a wider range of the parameters. In this situation the underwriter is unable to eliminate the informational rents as well as the rents from informed trading. Yet, to minimize the rents from strategic trading the best is to favour type 2 investors as these rents decrease the more informed trading takes place in the after-market.

The paper unfolds as follows. The next section presents the model. The third section depicts the outcome of the after-market. The fourth section characterizes the optimal IPO design. Finally, we conclude in section 5.

II The Model

The model is built upon Benveniste and Spindt (1989). A firm contracts an underwriter to allocate Q shares to I investors. We abstract from any agency problem between the firm and the underwriter and assume that the objective is to maximize the revenue. To keep matters simple we assume that each investor bids to buy at most 1 share and that we have $Q < I$.

The investors differ in their ability to process financial information. As a result they learn their interest for the stock at different periods. We consider that n $(0 \leq n \leq I)$ investors learn their interest for the stock during the road show. These are labelled type 1 investors. By opposition, we call type 2 investors those who learn their interest for the share in the after-market, once the shares have been allocated.

We assume that all investors have a high or low interest for the shares. Let $\eta^i \in \{-\eta, +\eta\}$ denote type 1 investor i's interest $(i \leq n)$. Each η^i is an i.i.d. realization of a random variable $\widetilde{\eta} \in \{-\eta, +\eta\}$ such that $\widetilde{\eta} = +\eta$ with probability $q \in [0, 1]$. Let $\varepsilon^j \in \{-\varepsilon, +\varepsilon\}$ denote type 2 investor j's interest $(j > n)$. Each ε^j is the realization of an i.i.d. random variable $\widetilde{\varepsilon} \in \{-\varepsilon, +\varepsilon\}$. We consider that the expectation of $\widetilde{\varepsilon}$ is equal to 0. Therefore, $\widetilde{\varepsilon} = +\varepsilon$ with probability $\frac{1}{2}$.

We assume that the value of a share is given by

$$V_k = v + \eta_k + \sum_{j \in (I-n)} \varepsilon^j,$$

where $v > 0$ is a commonly known variable and where $\eta_k = \sum_{i=1,..,n} \eta^i$. Notice that

$$E_{\varepsilon}\left(V_{k+1}-V_{k}\right)=2\eta,$$

where E_{ε} denotes the expectation over ε.

To make the analogy with Benveniste and Spindt (1989) note that type 1 investors are similar to their so-called regular investors and type 2 are the occasional investors. In their paper, the residual uncertainty captured by a parameter λ is of no significance. In this paper, the residual uncertainty, measured by the last term of V_k, can be used strategically.

The timing for the IPO is as follows. Initially, each type 1 investor learns his interest for the stock. Then, the underwriter announces the offer price and allocation rule for each possible messages sent by type 1 investors. Finally, aware of the mechanism, type 1 investors report their interest and an allocation and offer price are implemented. Note that the underwriter is not restricted to allocating the shares to the type 1 investors. However, she must select an offer price that each investor accepts once information is updated.

We model the after-market as a competitive dealer market. Type 1 investors sell their share if they face liquidity needs which arise with probability z. Type 2 investors learn their information and trade strategically. Given the market structure, it is optimal for them to sell when they have low interest. We do not allow for short selling.

We characterize the revenue maximizing offer price and allocation for each possible value of n. As this is a sequential game, we solve by backwards induction to find a sub-game perfect Nash equilibrium. Solving for the after-market first allows us to calculate the willingness to pay for a share for both types of investors.

III The aftermarket and the investors' expected revenue.

A The after-market.

The after-market is modeled as a dealership market as in Glosten and Milgrom (1985).

Let state k refer to the state of nature where k type 1 investors reported high interest i.e. reported $+\eta$. Let p_k^b refer to the bid price when the dealer infers, from the offer price, that state k has occurred.[8] Let $x_k(s)$ denote the probability that a type 1 investor who revealed interest $s \in \{+\eta, -\eta\}$ in state k gets a share. Finally, let $\widehat{x}_k$ denote the probability with which a type 2 investor gets a share. Full allocation of the shares implies that $\forall n$ and $\forall k \leq n$

$$kx_k(+\eta) + (n-k)x_k(-\eta) + (I-n)\widehat{x}_k = Q. \tag{1}$$

Therefore, the proportion of type 1 investors present in the after-market is given by:

$$\alpha_k = \frac{kx_k(+\eta) + (n-k)x_k(-\eta)}{Q}.$$

[8] This is in sharp contrast with Pagano and Ellul (2006) where the state of Nature becomes public information.

The proportion of informed, and therefore strategic, investors is given by

$$\beta_k = \frac{(I-n)\widehat{x}_k}{Q}.$$

Given that type 1 investors trade if they face liquidity needs, while type 2 investors trade strategically, the overall probability of facing a sell order is given by

$$\alpha_k z + \beta_k \frac{1}{2} = \alpha_k z + (1-\alpha_k)\,\frac{1}{2}.$$

(When there is no strategic trading this probability is equal to z. When there is only strategic trading it is 1/2.)

The competitive bid price, in state k, is then given by

$$\begin{aligned} p_k^b &= \frac{\alpha_k z}{\alpha_k z + \beta_k \frac{1}{2}}\,(v+\eta_k) + \frac{\beta_k \frac{1}{2}}{\alpha_k z + \beta_k \frac{1}{2}}\,(v+\eta_k-\varepsilon) \\ &\Rightarrow \; p_k^b = v+\eta_k - \varepsilon \frac{\beta_k \frac{1}{2}}{\alpha_k z + \beta_k \frac{1}{2}}. \end{aligned}$$

B The investors' expected revenue.

The unconditional probability of state k is given by

$$\pi_k = \begin{bmatrix} n \\ k \end{bmatrix} q^k (1-q)^{n-k}.$$

From the vantage point of a type 1 investor with low interest state k occurs with probability

$$\pi'_k = \begin{bmatrix} n-1 \\ k \end{bmatrix} q^k (1-q)^{n-1-k}.$$

Finally, state k occurs with probability π'_{k-1} for a type 1 investor with high interest.

Let u_k denote the value of a share to a type 1 investor. It is given by

$$u_k = zp_k^b + (1-z)(v + \eta_k),$$

which simplifies to

$$u_k = v + \eta_k - \frac{z\varepsilon\beta_k}{\beta_k + 2z\alpha_k}. \tag{2}$$

Indeed, with probability z, the investor will sell the share in the after-market and get p_k^b for it, while he keeps the share with probability $(1-z)$.

Let $U(\eta^i, s)$ denote the expected revenue to a type 1 investor when he reports $s \in \{-\eta, +\eta\}$ while his true interest is $\eta^i \in \{-\eta, +\eta\}$. A type 1 investor honestly reporting high interest receives:

$$U(+\eta, +\eta) = \sum_{k=0}^{n-1} \pi'_k x_{k+1}(+\eta). \left[u_{k+1} - p_{k+1}^o\right], \tag{3}$$

while a type 1 investor honestly reporting low interest gets

$$U(-\eta, -\eta) = \sum_{k=0}^{n-1} \pi'_k . x_k(-\eta). \left[u_k - p_k^o\right]. \tag{4}$$

Finally, let $\widehat{U}$ denote the expected revenue to a type 2 investor. We have:

$$\widehat{U} = \sum_{k=0}^{n} \pi_k \widehat{x}_k \left[\widehat{u}_k - p_k^o\right],$$

where $\widehat{u}_k$ denotes a type 2 investor's willingness to pay for a share in state k, given that he will sell the share in the after-market if his interest is low and keep it if his interest is high. More precisely we have

$$\widehat{u}_k = \frac{1}{2}p_k^b + \frac{1}{2}(v + \eta_k + \varepsilon),$$

which simplifies to

$$\widehat{u}_k = v + \eta_k + \frac{\varepsilon z\alpha_k}{\beta_k + 2z\alpha_k}. \tag{5}$$

IV The pre-market.

The underwriter selects the offer price in each state and the allocation of shares so as to maximize the expected revenue

$$Max\ Q \sum_{k=0}^{n} \pi_k p_k^o$$

subject to (i), (ii), (iii) and (iv) described below.

(i) Incentive compatibility requires that an investor truthfully reports his interest. As is typically the case (and despite after-market trading), the only relevant constraint concerns type 1 investors with high interest. We must have

$$U(+\eta, +\eta) \geq U(+\eta, -\eta),$$

where $U(+\eta, +\eta)$ is given by (3) and

$$U(+\eta, -\eta) = \sum_{k=0}^{n-1} \pi'_k . x_k(-\eta). \left[z p_k^b + (1-z) v_{k+1} - p_k^o \right]. \tag{6}$$

(ii) Ex-post voluntary participation states that all investors must accept to pay the offer price once it is posted. Therefore we must have

$$p_k^o \leq u_k \text{ and } p_k^o \leq \widehat{u}_k,$$

(iii) Full allocation of the shares as defined by (1).

(iv) Finally, we must have $\widehat{x}_k$, $x_k(+\eta)$ and $x_k(-\eta)$ all between 0 and 1. Thus, rationing may have to take place in some instances.

Lemma 1: *To guarantee incentive compatibility, the rents to a type 1 investor reporting high interest must be such that*

$$U(+\eta, +\eta) \geq U(-\eta, -\eta) + 2\eta(1-z) \sum_{k=0}^{n-1} \pi'_k . x_k(-\eta).$$

Proof. See Appendix 1. ■

Interestingly one can see that the cost of implementing truthful revelation depends on z. Indeed, reporting low interest decreases the bid price that the investor receives if he sells the share in the aftermarket. Thus, the more likely a type 1 investor is to sell a share in the aftermarket, the weaker his incentive to misreport high interest.

Lemma 2: *The expected revenue of the seller can be written as*

$$\begin{aligned} W_n &= Q\sum_{k=0}^{n}\pi_k\left(v+\eta_k\right)-\left(I-n\right)\widehat{U}-nqU(+\eta,+\eta) \\ &\quad -n(1-q)U(-\eta,-\eta). \end{aligned} \tag{7}$$

Proof. See Appendix 1. ■

By opposition to Benveniste and Spindt (1989) the "money left on the table" represents not only the informational rents issued to type 1 investors, but also the rents type 2 investors extract from trading strategically.

First we present the optimal allocation and prices for the specific cases $n = 0$ and $n = I$.

Lemma 3:

- *When there are no type 1 investors* $(n = 0)$*, the optimal offer price is* $p^o_{n=0} = v$ *and we have* $W_0 = Qv$.
- *When there are no type 2 investors* $(n = I)$*, the optimal offer price is* $p^o_k = v + \eta_k$ *for all* $k = 0, ..., I - 1$ *and* p^o_I *is such that the incentive constraint binds. The allocation of shares gives priority to those reporting*

high interest, $x_k(+\eta) = \min\left\{1, \frac{Q}{k}\right\}$, *and the remaining shares are allocated to the ones reporting low interest,* $x_k(-\eta) = \min\left(0, \dfrac{Q-k}{I-k}\right)$. *The underwriter's expected revenue is given by*

$$W_I = Q\sum_{k=0}^{I}\pi_k\left(v+\eta_k\right) - 2\frac{q}{1-q}\eta\left(1-z\right)\sum_{k=0}^{Q-1}\pi_k\left(Q-k\right).$$

Proof. The case $n = 0$ is straightforward. When $n = I$ the objective function depends solely on $x_k(-\eta)$ $k = 0, ..., I-1$ which determines the level of informational rents. The larger $x_k(-\eta)$, the lower is the revenue. (See Appendix 2 for more details.) ■

Generally, the optimal mechanism results from trading off informational rents and rents gathered by type 2 investors. Whether or not the underwriter can eliminate part or both of these rents depends on how many type 2 investors she faces.

Lemma 4: *When there are enough type 2 investors to absorb the issue* $(I - n \geq Q)$, *it is possible to extract the whole surplus. This can be done either by allocating all the shares to type 2 investors or, when there are also enough type 1 investors to absorb the issue, one can allocate all the shares to the type 1 investors in state* n *at a price* $p_n^o = u_n$ *and allocate all the shares to the type 2 investors when at least one type 1 investor reports low interest.*

At the solution, type 1 investors have no rents no matter what they reveal so incentive compatibility holds weakly. Thus, the above mechanism would not prevail if information acquisition were costly. Costly information acquisition is a complex topic in itself and is analyzed in details in Sherman (2000) and Sherman and Titman (2003). We now consider the complementary situations.

A Enough type 1 investors and too few type 2 investors to exhaust the issue.

Consider any $n > max\{Q, I-Q\}$.[9] In proposition 1 we characterize the optimal offer prices and describe intuitively the allocation rule. Below, we formally characterize the optimal allocation rule both, mathematically and graphically.

Proposition 1: *For any $n > max\{Q, I-Q\}$, the offer price is such that $p_k^o = u_k$ for any $k = 0, ..., n-1$ and p_n^o is such that the incentive constraint binds:*

$$u_n - p_n^o = \frac{2\eta(1-z)}{\pi'_{n-1} x_n(+\eta)} \sum_{k=0}^{n-1} \pi'_k . x_k(-\eta).$$

The optimal allocation rule is such that priority is given to type 1 investors reporting high interest. The remaining shares are allocated to type 2 investors when z is sufficiently low. As z increases, a non-decreasing proportion of these shares are sold to type 1 investors reporting low interest. For z high enough, all shares are sold to type 1 investors.

Proof. See Appendix 2. ■

We now describe precisely the optimal allocation. Type 1 investors reporting high interest get served systematically. Thus, for all $k \geq Q$ we have $x_k(-\eta) = \widehat{x}_k = 0$ and $x_k(+\eta) = \frac{Q}{k}$. For all $k < Q$, we have $x_k(+\eta) = 1$ while $\widehat{x}_k$ and

[9] These two possibilities result from considering both $I > 2Q$ and $I = 2Q$.

$x_k(-\eta)$ solve:

$$(n-k)\, x_k(-\eta) + (I-n)\, \widehat{x}_k = Q - k.$$

The optimal $\widehat{x}_k$ is characterized below (see Appendix 2 for details). Let us introduce the variable χ defined as $\chi = \dfrac{(1-q)\,\varepsilon}{\eta q}$.

- For $0 < \chi < 2$ there exist a unique z_1 and $z_2(k)$ with $\frac{1}{2} < z_2(k) < z_1 < 1$ such that

$$\widehat{x}_k = \begin{cases} \min\left\{1, \frac{Q-k}{I-n}\right\} \text{ for } z \leq z_2(k), \\ x^* \text{ for any } z \in [z_2(k), z_1]\,, \\ 0 \text{ for } z \geq z_1, \end{cases}$$

  where the interior solution solves

$$[(I-n)\, x^*\, (1-2z) + 2zQ]^2 = \frac{z^2 Q^2 \chi}{(1-z)}. \tag{8}$$

  The solution $\widehat{x}_k$ is continuous.

- For $\chi = 2$ we have

$$\widehat{x}_k = \begin{cases} \min\left\{1, \frac{Q-k}{I-n}\right\} \text{ for } z < \frac{1}{2}, \\ 0 \text{ for } z > \frac{1}{2}. \end{cases}$$

  At $z = \frac{1}{2}$ any $\widehat{x}_k \in \left[0, \min\left\{1, \frac{Q-k}{I-n}\right\}\right]$ leads to the same payoff.

- For $\chi > 2$, there exists a unique $z_3(k) \in \left]0, \frac{1}{2}\right[$ such that $\widehat{x}_k = \min\left\{1, \frac{Q-k}{I-n}\right\}$ when $z \leq z_3(k)$ and $\widehat{x}_k = 0$, for all k for $z \geq z_3(k)$.[10]

The graphs below helps visualize the optimal allocation rule. They represent the optimal $\widehat{x}_k$ for each possible χ.

[10] At $z = z_3(k)$, the solution is discontinous. Both $\widehat{x}_k = 0$ and $\widehat{x}_k = \min\left\{1, \frac{Q-k}{I-n}\right\}$, lead to the same payoff.

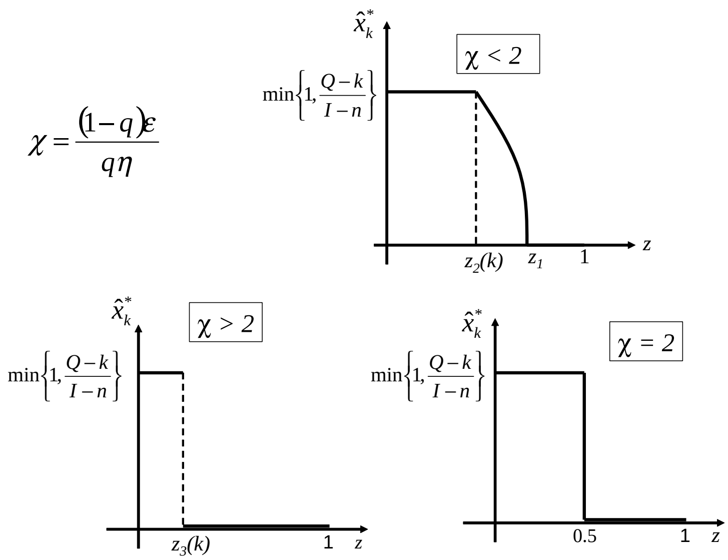

Figure 1: Optimal values for $\widehat{x}_k$ when there are enough type 1 investors to exhaust the issue but not enough type 2 investors.

The rationale for the offer price is straightforward. When there are not enough type 2 investors to absorb the issue, the offer price is bounded above by type 1 investors' willingness to pay, u_k. In order to eliminate the rents of those reporting low interest, it is optimal to set $p_k^o = u_k$ for all $k = 0, ..., n-1$. Finally p_n^o is set so as to guarantee incentive compatibility.

The optimal allocation is decided as follows. Because both, the informational rents and the rents from informed trading are decreasing as $x_k(+\eta)$ increases, it is best to favour type 1 investors reporting high interest. When type 1 investors reporting high interest cannot exhaust the issue, the remaining shares must either go to type 1 investors reporting low interest or to type 2 investors. When z is high, informational rents are less of a burden and it is best to serve only type 1 investors to save on rents generated by informed trading. By opposition, when z is low, in order to decrease the significant informational rents, it is best to favour type 2 investors at the expense of type 1 investors with low interest.

The extent to which one wants to use type 2 investors depends on χ. More precisely there are two factors that can affect χ: how important is ε relative to η and on q. Indeed, considering (7), one can see that a higher ε relative to η increases the rents from informed trading while a high q gives more weight to informational rents. Therefore, the underwriter will prefer to stay away from type 2 investors when ε/η is high or/and when q is low. Because an increase in ε/η increases χ and a decrease in q increases χ as well, we find that the larger χ the narrower the range of z over which type 2 investors are being allocated shares.

The total revenue, for the general case ($\varepsilon > 0$) can be written as follows

$$\begin{aligned} W_n &= Q\sum_{k=0}^{n} \pi_k\left(v+\eta_k\right) - 2\eta\left(1-z\right)\frac{q}{1-q}\sum_{k=0}^{Q-1}\pi_k\left(Q-k\right) \\ &\quad - \left(I-n\right)\sum_{k=0}^{Q-1}\pi_k\widehat{x}_k\left[\frac{z\varepsilon Q}{\left(1-2z\right)\left(I-n\right)\widehat{x}_k+2zQ} - \frac{2\eta q}{\left(1-q\right)}\left(1-z\right)\right]. \end{aligned} \tag{9}$$

One must subtract the informational rents and the rents from informed trading to the expected value of the share. Note that by allocating the shares to type 2 investors, one saves on the informational rents as the last term of the above expression shows.

Corollary 1 : *When there is no residual uncertainty that can be strategically used, i.e. when $\varepsilon = 0$ and therefore $\chi = 0$, it is optimal to allocate the remaining shares to type 2 investors for any z in $[0,1]$, so that $\widehat{x}_k = \min\left\{1, \frac{Q-k}{I-n}\right\}$ $\forall z$. The total revenue is then given by*

$$W_n = Qv + nQ\eta\,(2q-1) - 2\eta(1-z)\frac{q}{1-q}\sum_{k=0}^{k'-1}\pi_k\,[k'-k]\,, \tag{10}$$

with $k' = Q - I + n$.

In all states of nature where type 1 investors reporting low interest do not get shares (all k such that $x_k(-\eta) = 0$) informational rents are equal to zero. Therefore, these rents are positive only when there are not enough type 2 investors and type 1 investors reporting high interest to exhaust the issue ($k < k'$). This is represented by the third term in (10). Finally, we can link the result in Corollary 1 to Bennouri and Falconieri (2006) who also show that allocating some items to uninformed bidders can raise the revenue by decreasing informational rents.

B Too few type 1 and too few type 2 investors to exhaust the issue.

The main difference between this case and the preceding one is that $\widehat{x}_k$ can no longer be set equal to 0. Indeed, since the number of type 1 investors is too low to absorb the issue, type 2 investors are now guaranteed to get some shares in all states of nature and we have $\widehat{x}_k \geq \dfrac{Q-n}{I-n}$.

Proposition 2: *When* $I < 2Q$ and $I - Q < n < Q$, *the offer price* $p_k^o = u_k$ *for any* $k = 0, ..., n-1$ *and* p_n^o *is such that the incentive constraint binds. The optimal allocation of shares is similar to that describe in Proposition 1 except that* $\widehat{x}_k$ *is now bounded below by* $\dfrac{Q-n}{I-n}$, *and the different equilibria depend on whether* χ *is greater, equal or less than* $2\dfrac{Q}{n}$.

Proof. see Appendix 3. ■

We describe below the optimal allocation rule.

- For $0 < \chi < 2\dfrac{Q}{n}$, there exist unique $z_5(k)$ and z_4 with $\frac{1}{2} < z_5(k) < z_4$, such that

$$\widehat{x}_k = \begin{cases} \min\left\{1, \frac{Q-k}{I-n}\right\} \text{ for } z \leq z_5(k), \\ x^* \text{ for any } z \in [z_5(k), z_4], \\ \dfrac{Q-n}{I-n} \text{ for } z \geq z_4, \end{cases}$$

  where the interior solution solves (8). The solution $\widehat{x}_k$ is continuous.

- For $\chi = 2\dfrac{Q}{n}$,

$$\widehat{x}_k = \begin{cases} \min\left\{1, \frac{Q-k}{I-n}\right\} \text{ for } z < \frac{1}{2}, \\ \dfrac{Q-n}{I-n} \text{ for } z > \frac{1}{2}. \end{cases}$$

  At $z = \frac{1}{2}$ we have a continuum of solutions as any $\widehat{x}_k$ such that $\dfrac{Q-n}{I-n} \leq \widehat{x}_k \leq \min\left\{1, \frac{Q-k}{I-n}\right\}$ leads to the same payoff.

- For $\chi > 2\dfrac{Q}{n}$, there exists a unique $z_6(k)$ such that $\widehat{x}_k = \min\left\{1, \frac{Q-k}{I-n}\right\}$ when $z \leq z_6(k)$ and $\widehat{x}_k = \dfrac{Q-n}{I-n}$ for $z \geq z_6(k)$.

At the solution we have

$$
\begin{aligned}
W_n &= Q\sum_{k=0}^{n}\pi_k\left(v+\eta_k\right)-\pi_n\frac{z\varepsilon Q\left(Q-n\right)}{\left(1-2z\right)\left(Q-n\right)+2zQ} \\
&\quad -2\eta\frac{q}{1-q}\left(1-z\right)\sum_{k=0}^{n-1}\left(Q-k\right)\pi_k \qquad (11)\\
&\quad -\left(I-n\right)\frac{Q}{n}\sum_{k=0}^{n-1}\pi_k\widehat{x}_k\left[\frac{z\varepsilon n}{\left(1-2z\right)\left(I-n\right)\widehat{x}_k+2zQ}-2\eta\frac{q}{1-q}\left(1-z\right)\right].
\end{aligned}
$$

The figure below, as the one before, helps to visualize the optimal allocation.

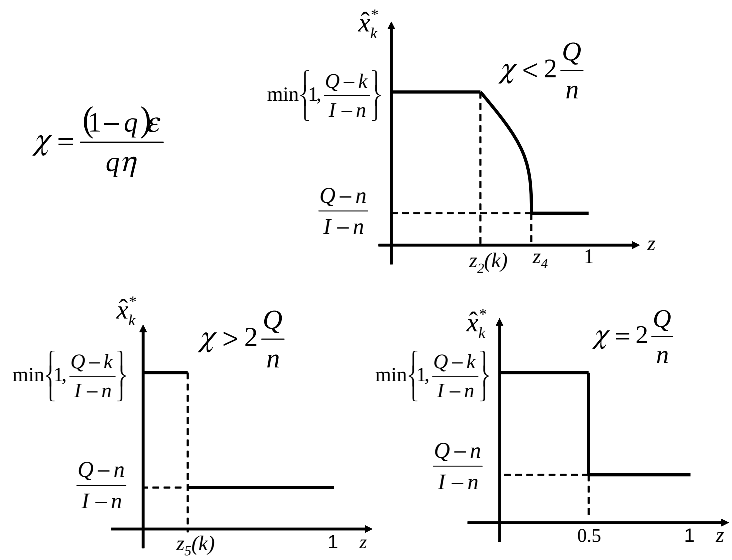

Figure 2: Optimal value for$\widehat{x}_k$, when there are not enough type 1 and not enough type 2 to exhaust the issue (but enough of both).

Corollary 2: *When there are not enough type 1 investors to exhaust the issue, the underwriter relies on type 2 investors for a wider range of parameters.*

The above corollary highlights the fact that χ must now be greater than $2\dfrac{Q}{n}$ for the optimal allocation rule to change. Since $\dfrac{Q}{n} > 1$, type 2 investors are used for a wider range of χ. As there are too few of each type of investors, it is impossible to eliminate the rents from strategic trading. Yet, to minimize these rents the best is to favour type 2 investors. Strategic rent decrease the more informed trading takes place in the after-market.

The following corollary gives the optimal allocation rule and the total revenue when there is no residual uncertainty.

Corollary 3 : *When $\varepsilon = 0$ and therefore $\chi = 0$, it is optimal to allocate the remaining shares to type 2 investors for any z* in $[0, 1]$, *so that* $\widehat{x}_k = \min\left\{1, \frac{Q-k}{I-n}\right\} \forall z$. *The total revenue is then given by*

$$\begin{aligned} W_n &= Qv + nQ\eta\,(2q-1) - 2\eta(1-z)\frac{q}{1-q}\sum_{k=0}^{k'-1}\pi_k\left[Q\left(2-\frac{I}{n}\right)-k\right] \qquad (12) \\ &\quad +2\eta(1-z)\frac{q}{1-q}\left(\frac{Q}{n}-1\right)\sum_{k=k'}^{n-1}\pi_k\left[Q-k\right]. \end{aligned}$$

This corollary follows the same intuition as corollary 1. One difference is that we must always give part of the shares to type 2 investors as type 1 investors are not numerous enough. This does not generate any cost as $\varepsilon = 0$. Instead, it allows to decrease the amount of informational rents. The savings made by allocating shares to type 2 investors outweight the cost when k is high enough (so that type 1 investors reporting low are not served). This is depicted by the last term of (12) which was systematically equal to 0 in the previous situation.

V On the optimal number of type 1 investors.

According to Lemma 4, it would seem that a greater number of type 2 investors is optimal as it allows to eliminate both types of rents. This would indeed be true if type 1 investors had a perfectly correlated information and where the value of the shares did not depend on the number of investors reporting high interest. In our setting, the value of the shares is sensitive to the investors' interest. Indeed, the larger the number of type 1 investors revealing high interest at the IPO stage, the higher is the value of the share. As a consequence, having more type 1 investors can pay off. Ideally it would be beneficial to have a large number of both, type 1 and type 2 investors. However, we have considered a slightly different situation whereby the set of interested investors is fixed and some will acquire their information faster than the others. Within that setting we ask to what extent does the underwriter benefit from "generating" more type 1 investors by facilitating information acquisition?

Unfortunately, due to the complexity of the solution, we cannot reach closed form solutions in terms of the optimal number of type 1 versus type 2 investors. However we present now the results we can establish.

Corollary 4: *Given (7) when $q \leq \frac{1}{2}$ it is optimal to have $n = 0$. For $q > \frac{1}{2}$ and $z = 1$, it is optimal to have $n = I$.*

When $q \leq \frac{1}{2}$ type 1 investors are more likely to reveal low interest and therefore depreciate the value of the shares. Moreover, by gathering a large number of such investors, the underwriter will end up paying more informational rents. When $z = 1$, type 1 investors have no incentive to misreport their information. Therefore, provided $q > \frac{1}{2}$, having $n = I$ maximizes the revenue. By continuity, for z close enough to 1, $n = I$ is also optimal. These results apply whether or not there is residual uncertainty, i.e. whether or not the underwriter has to pay rents from informed trading.

When $\varepsilon = 0$ and $z \neq 1$, there are no rents from informed trading. The overall effect of n on both expressions (10) and (12) is not straightforward. On the one hand a higher n increases the first term of W_n provided $q > \frac{1}{2}$. In a model similar to Benveniste and Spindt (1989), a greater number of investors means a lower probability of accessing a share (whatever the signal reported) and thus lower informational rents. In other words, more competition increases the revenue. In our model, the impact of a higher n is not so clear as more type 1 investors also means less type 2 investors to rely on. This has two implications. First, as the underwriter cannot rely as much on type 2 investors informational rents are paid more often (i.e. for more value of k). Second, in every k where informational rents are paid, the amount paid is larger. This is so because the probability of getting a share when reporting low interest is potentially higher as n increases. Finally, in the case where there are too few of each type of investors to exhaust the issue, a higher n also affects the last positive term of (12).

Below are the graphs from the following example.

Example 1 *The values for the parameters are as follows:* $I = 10$, $Q = 4$, $\varepsilon = 2.5$, $\eta = 1$, $q = 0.51$.

In that case, when z is small the underwriter derives higher total revenue with $n = I - Q + 1$ type 1 investors than if he were to face $n = I$ type 1 investors. When z is large the opposite happens.

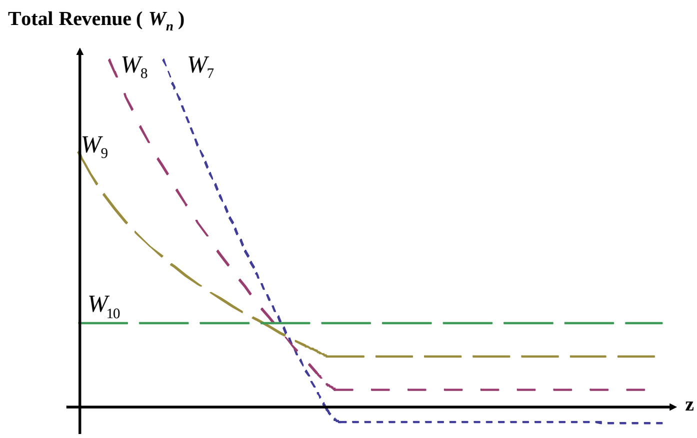

Figure 5: Total revenue as a function of n and z. The different curves represent the underwriter's total revenue W_n for different values of n.

VI Conclusion

In this paper we have analyzed optimal IPO design incorporating after-market trading which allows us to consider distinct adverse selection problems. Indeed, we consider that, due to the nature of financial information and due to the small time frame of an IPO, investors differ in the information they can access and use at the time of the IPO. Some learn their interest for the share at the time of the IPO while others only learn their interest in the after-market, once the shares have been allocated. Our main findings are summarized below.

The underwriter faces two types of rents: the traditional informational rents and the rents associated with strategic trading in the after-market. We find that the level of informational rents decreases as the investors who know and report their information at the IPO stage are more likely to face liquidity needs. In other words, allowing the investors who are informed at the time of the IPO to flip their share can facilitate truthful information report. Indeed investors who lie about their true interest and report a low instead of a high interest depress the bid price in the after-market.

When the investors who learn their information in the after-market are numerous enough to exhaust the issue, the underwriter eliminates all rents by giving them all the shares. When this is not the case, the optimal allocation of shares results from trading-off both types of rents. As in Benveniste and Spindt (1989), priority is given to investors truthfully reporting high interest. If these are not numerous enough to exhaust the issue, the remaining shares either go to those reporting low interest or to the investors who learn their interest in the after-market. When liquidity needs are likely to arise, because informational rents become negligible, the remaining shares are sold to those reporting low interest. By opposition, in the presence of long-term holders who require higher informational rents, it is optimal to sell the remaining shares to investors able to flip their shares strategically. The extent to which generating strategic trading is best depends on the relative impact of each type of investors' interest on the future value of the stock and on how likely it is to face a highly interested investor.

We believe that several empirical and theoretical work could follow from this analysis. As expressed, in our setting, it is difficult to say anything about the optimal balance between both types of investors. We have established the benefits and costs associated with each of these. Maybe one could consider this issue more precisely simplifying the setting. Also, one could set as endogenous the choice of each investors in terms of information gathering and revelation. Possibly, some investors may try and grab a bit of both rents. Empirically, it may be interesting to try and disentangle the rents that are left over. It has been argued, in several papers, that informational rents on their own do not explain the amount of money that is being "left on the table". Would rents from strategic trading fill the gap? Finally, it can be interesting to find some empirical evidence of using flipping as a tool to induce truthful revelation.

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VIII Appendix

- **Appendix 1**

  Proof of Lemma 1:

The incentive compatibility constraint states that investors with high interest must reveal it truthfully. Let $v_k = v + \eta_k$. The expected profit that would result from reporting low interest while true interest is high is given by (6). Misreporting interest changes the offer price which also affects the price on the aftermarket. Given (4), (6) can be re-written as:

$$U(+\eta,-\eta) = U(-\eta,-\eta) + 2\eta(1-z)\sum_{k=0}^{n-1}\pi'_k.x_k(-\eta).$$

The IC constraint is such that

$$U(+\eta,+\eta) \geq U(-\eta,-\eta) + 2\eta(1-z)\sum_{k=0}^{n-1}\pi'_k.x_k(-\eta). \tag{13}$$

This concludes the proof for Lemma 1.

Proof of Lemma 2:

Given the full allocation of the shares, we can rewrite the objective of the seller as

$$Max\sum_{k=0}^{n}\pi_k\left[kx_k(+\eta)+(n-k)x_k(-\eta)+(I-n)\widehat{x}_k\right]p_k^o.$$

Note that $x_n(-\eta)=0$ and that $x_0(+\eta)=0$ and finally that

$$\pi_k = \frac{n\,(1-q)}{n-k}\pi'_k \text{ and } \pi_{k+1} = \frac{nq}{k+1}\pi'_k. \tag{14}$$

Given all this we have

$$\sum_{k=0}^{n}\pi_k k x_k(+\eta)p_k^o = nq\sum_{k=0}^{n-1}\pi'_k u_{k+1}x_{k+1}\left(+\eta\right) - nqU\left(+\eta,+\eta\right),$$

and

$$\sum_{k=0}^{n}\pi_k(n-k)x_k(-\eta)p_k^o = n\,(1-q)\sum_{k=0}^{n-1}\pi'_k u_k x_k\left(-\eta\right) - n\,(1-q)\,U\left(-\eta,-\eta\right).$$

We can rewrite the objective function as

$$\begin{aligned} W_n \quad = \quad & nq\sum_{k=0}^{n-1}\pi'_k u_{k+1}x_{k+1}\left(+\eta\right) + n\,(1-q)\sum_{k=0}^{n-1}\pi'_k u_k x_k\left(-\eta\right) \qquad (15) \\ & + \sum_{k=0}^{n}\pi_k(I-n)\widehat{x}_k p_k^o - n\,(1-q)\,U(-\eta,-\eta) - nqU\left(+\eta,+\eta\right). \end{aligned}$$

Note that

$$nq\sum_{k=0}^{n-1}\pi'_k u_{k+1}x_{k+1}\left(+\eta\right)=\sum_{k=0}^{n}k\pi_k x_k\left(+\eta\right)u_k,$$

and

$$n\left(1-q\right)\sum_{k=0}^{n-1}\pi'_k u_k x_k\left(-\eta\right)=\sum_{k=0}^{n}(n-k)\pi_k x_k\left(-\eta\right)u_k,$$

and finally that

$$\sum_{k=0}^{n}\pi_k(I-n)\widehat{x}_k p_k^o=\sum_{k=0}^{n}\pi_k(I-n)\widehat{x}_k\left(p_k^o-u_k\right)+\sum_{k=0}^{n}\pi_k(I-n)\widehat{x}_k u_k$$

Substituting the above in (15) we obtain

$$\begin{aligned}W_n &= \sum_{k=0}^{n}\pi_k(I-n)\widehat{x}_k\left(p_k^o-u_k\right)-n\left(1-q\right)U(-\eta,-\eta)-nqU\left(+\eta,+\eta\right)\\ &\quad+\sum_{k=0}^{n}\pi_k u_k\left[kx_k\left(+\eta\right)+\left(n-k\right)x_k(-\eta)+(I-n)\widehat{x}_k\right].\end{aligned}$$

Given that

$$kx_k\left(+\eta\right)+\left(n-k\right)x_k(-\eta)+(I-n)\widehat{x}_k=Q,$$

we have

$$\begin{aligned}W_n &= Q\sum_{k=0}^{n}\pi_k u_k+\sum_{k=0}^{n}\pi_k(I-n)\widehat{x}_k\left(p_k^o-u_k\right)\\ &\quad -n\left(1-q\right)U(-\eta,-\eta)-nqU\left(+\eta,+\eta\right).\end{aligned}$$

Considering the definition of u_k, and the fact that

$$\widehat{u}_k-u_k=\frac{z\varepsilon}{\beta_k+2z\alpha_k}$$

we have

$$\begin{aligned}W_n &= Q\sum_{k=0}^{n}\pi_k(v+\eta_k)-\sum_{k=0}^{n}\pi_k(I-n)\widehat{x}_k\left(\widehat{u}_k-p_k^o\right)-nqU\left(+\eta,+\eta\right)\\ &\quad -n\left(1-q\right)U(-\eta,-\eta),\end{aligned}$$

which proves Lemma 2.

- **Appendix 2.**

Since $(I-n)<Q$, even in state 0, type 1 investors must get some of the shares to exhaust the issue. Therefore $x_0(-\eta)>0$ and the only relevant ex-post incentive constraint is given by $p_k^o \leq u_k$. To guarantee that $U(-\eta,-\eta)=0$ we set $p_k^o = u_k$ for $k=0,..,n-1$ and set p_n^o as described in Proposition 1, so that the IC constraint binds. Incorporating this in the revenue function we get

$$\begin{aligned} W_n &= Q\sum_{k=0}^{n}\pi_k\left(v+\eta_k\right)-nq2\eta\left(1-z\right)\sum_{k=0}^{n-1}\pi_k' x_k(-\eta) \\ &\quad -\left(I-n\right)\sum_{k=0}^{n-1}\pi_k\widehat{x}_k\frac{z\varepsilon}{\beta_k(1-2z)+2z}-(I-n)\pi_n\widehat{x}_n\left(\widehat{u}_n-p_n^o\right). \end{aligned} \tag{16}$$

It is obviously optimal to set $x_k(+\eta)$ as high as possible. Therefore, for any $k\geq Q$ all the shares go to those reporting high interest (and as a result $\widehat{x}_n=0$).

When $k\leq Q-1$, we must determine the optimal $x_k(-\eta)$ and $\widehat{x}_k$ knowing that full allocation requires that

$$(n-k)\,x_k(-\eta)+(I-n)\,\widehat{x}_k=Q-k.$$

Incorporating the results given above in terms of $x_k(+\eta)$ and p_k^o we can rewrite the objective function as:

$$W_n=Q\sum_{k=0}^{n}\pi_k\left(v+\eta_k\right)-2\eta\frac{q}{1-q}\left(1-z\right)\sum_{k=0}^{Q-1}\pi_k\left(Q-k\right)+(I-n)\sum_{k=0}^{Q-1}\pi_k f\left(\widehat{x}_k\right),$$

where

$$f\left(\widehat{x}_k\right)=2\eta\frac{q}{1-q}\left(1-z\right)\widehat{x}_k-\frac{z\varepsilon Q\widehat{x}_k}{\left(I-n\right)\widehat{x}_k\left(1-2z\right)+2zQ}.$$

We have $\dfrac{dW_n}{d\widehat{x}_k} = (I-n)\pi_k \dfrac{df}{d\widehat{x}_k}$ for $k = 0, ..., Q-1$, and

$$\frac{df}{d\widehat{x}_k} = 2\left[\eta\left(1-z\right)\frac{q}{\left(1-q\right)} - \frac{z^2Q^2\varepsilon}{\left(\left(I-n\right)\widehat{x}_k\left(1-2z\right)+2zQ\right)^2}\right]$$

and

$$\frac{d^2f}{d\widehat{x}_k^2} = 4\left(I-n\right)z^2Q^2\varepsilon\frac{\left(1-2z\right)}{\left(\left(I-n\right)\widehat{x}_k\left(1-2z\right)+2zQ\right)^3}.$$

Note that for $z = \frac{1}{2}$, the objective function is linear in $\widehat{x}_k$. We have $\dfrac{df}{d\widehat{x}_k} = 2\left[\dfrac{2-\chi}{4\eta q(1-q)}\right]$ so that the objective function is independent from the value of $\widehat{x}_k$ for $\chi = 2$.

For any $z > \frac{1}{2}$, the objective function is concave. We have

$$\left.\frac{dW_n}{d\widehat{x}_k}\right|_{\widehat{x}_k=0} \geq 0 \Leftrightarrow z \leq z_1$$

and

$$\left.\frac{dW_n}{d\widehat{x}_k}\right|_{\widehat{x}_k=\min\left\{1,\frac{Q-k}{I-n}\right\}} \leq 0 \Leftrightarrow z \geq z_2(k)$$

where z_1 and $z_2(k)$ are the unique solutions to[11]

$$\left(1-z_1\right) = \frac{\chi}{4}, \tag{17}$$

$$1-z_2 = \begin{cases} \dfrac{z_2^2Q^2\chi}{\left(\left(I-n\right)\left(1-2z_2\right)+2z_2Q\right)^2} \text{ for } k \in \left[0,k'\right], \\ \dfrac{z_2^2Q^2\chi}{\left(Q-k\left(1-2z_2\right)\right)^2} \text{ for } k \in \left[k',Q-1\right], \end{cases} \tag{18}$$

where $k' \in]0, Q-1[$ is defined such that $k' = Q-(I-n)$. For any $k' < k < Q$ few shares remain to be allocated so that type 2 investors have to be rationed.

[11] While it is technically possible to have $z_1 < 0$ for $\chi > 4$, this variable plays no role for any $\chi > 2$ and therefore we need not worry about this possibility.

Note that $z_2(k)$ is continuous, with $\dfrac{dz_2}{dk} = 0$ for $k \in [0, k']$, and we have (using the implicit function theorem) $\dfrac{dz_2}{dk} > 0$ for $k \in [k', Q-1]$ when $z_2(k) > \frac{1}{2}$.

Finally, for $z < \frac{1}{2}$, the objective function is convex in $\widehat{x}_k$. We need to examine its values at $\widehat{x}_k = 0$ and $\widehat{x}_k = \min\left\{1, \frac{Q-k}{I-n}\right\}$. Let $z_3(k)$ be defined such that

$$W_n|_{\widehat{x}_k=0} = W_n|_{\widehat{x}_k=\min\left\{1,\frac{Q-k}{I-n}\right\}}.$$

More precisely $z_3(k)$ is the unique solution to

$$2(1-z_3) = \begin{cases} \dfrac{z_3\chi Q}{(I-n)+2z_3(Q-I+n)} \text{ for } k \in [0,k'], \\ \dfrac{z_3\chi Q}{Q-k(1-2z_3)} \text{ for } k \in [k', Q-1]. \end{cases}$$

Note that $z_3(k)$ is continuous, with $\dfrac{dz_3}{dk} = 0$ for $k \in [0, k']$, and we have (using the implicit function theorem) $\dfrac{dz_3}{dk} < 0$ for $k \in [k', Q-1]$ when $z_3(k) > \frac{1}{2}$.

Setting $\widehat{x}_k = \min\left\{1, \frac{Q-k}{I-n}\right\}$ is optimal when $z \leq z_3(k)$ (provided $z_3 \leq \frac{1}{2}$). When $z > z_3(k)$ it is optimal to set $\widehat{x}_k = 0$.

The optimal solution is then easily determined given the above. There are 3 possibilities.

- For any $\chi > 2$, we have $z_1 < z_3(k) < z_2(k) < \frac{1}{2}$. In that case, for any $z \geq \frac{1}{2}$ the function is concave and decreasing at $\widehat{x}_k = 0$, for all $k \leq Q-1$, therefore, it is optimal to set $\widehat{x}_k = 0$, for all $k \leq Q-1$. When $z < \frac{1}{2}$, the function is convex and it is optimal to set $\widehat{x}_k = \min\left\{1, \frac{Q-k}{I-n}\right\}$, for all $k \leq Q-1$ when $z \leq z_3(k)$ and $\widehat{x}_k = 0$, for all $k \leq Q-1$ for $z \geq z_3(k)$.

- For $\chi = 2$, we have $z_1 = z_2(k) = z_3(k) = \frac{1}{2}$. The optimal allocation is such that

$$\widehat{x}_k = \begin{cases} \min\left\{1, \frac{Q-k}{I-n}\right\} \text{ for all } k = 0, .., Q-1 \text{ for } z < \frac{1}{2}, \\ 0 \text{ for all } k = 0, .., Q-1 \text{ for } z > \frac{1}{2}. \end{cases}$$

At $z = \frac{1}{2}$ we have a continuum of solutions as any value for $\widehat{x}_k$ leads to the same payoff.

- $0 < \chi < 2$ we have $\frac{1}{2} < z_2(k) < z_3(k) < z_1$. The optimal allocation in that case is such that

$$\widehat{x}_k = \begin{cases} \min\left\{1, \frac{Q-k}{I-n}\right\} \text{ for all } k = 0, .., Q-1 \text{ for } z \leq z_2(k), \\ x^* \text{ for any } z \in [z_2(k), z_1], \\ 0 \text{ for all } k = 0, .., Q-1 \text{ for } z \geq z_1, \end{cases}$$

where the interior solution solves

$$((I-n)\, x^* (1-2z) + 2zQ)^2 = \frac{z^2 Q^2 \chi}{(1-z)}.$$

One can easily check that at $z = z_2(k)$ we have $x^* = 1$ if $k \leq k'$ and $x^* = \frac{Q-k}{I-n}$ if $k > k'$, while $x^* = 0$ at $z = z_1$.

- **Appendix 3.**

Starting with (16), the main difference is that now we cannot have $\widehat{x}_n = 0$. It is still clear that it is optimal to set $x_k(+\eta) = 1$ and therefore

$$(n-k)\, x_k(-\eta) + (I-n)\widehat{x}_k = (Q-k)\,.$$

Taking this into account we can re-write the objective function as

$$\begin{aligned} W_n &= Q\sum_{k=0}^{n}\pi_k\left(v+\eta_k\right)-\pi_n\frac{z\varepsilon Q\left(Q-n\right)}{\left(1-2z\right)\left(Q-n\right)+2zQ} \\ &\quad -2\eta\frac{q}{1-q}\left(1-z\right)\frac{Q}{n}\sum_{k=0}^{n-1}\left(Q-k\right)\pi_k+\left(I-n\right)\frac{Q}{n}\sum_{k=0}^{n-1}\pi_k g(\widehat{x}_k), \end{aligned}$$

where

$$g(\widehat{x}_k)=2\eta\frac{q}{1-q}\left(1-z\right)\widehat{x}_k-\frac{z\varepsilon n\widehat{x}_k}{\left(I-n\right)\widehat{x}_k\left(1-2z\right)+2zQ}.$$

We have $\dfrac{dW_n}{d\widehat{x}_k}=\left(I-n\right)\dfrac{Q}{n}\pi_k\dfrac{dg}{d\widehat{x}_k}$ where

$$\frac{dg}{d\widehat{x}_k}=2\eta\left(1-z\right)\frac{q}{\left(1-q\right)}-\frac{2Qn\varepsilon z^2}{\left(\left(I-n\right)\widehat{x}_k\left(1-2z\right)+2zQ\right)^2}$$

and

$$\frac{d^2g}{d\widehat{x}_k^2}=4\left(I-n\right)nQ\varepsilon z^2\frac{\left(1-2z\right)}{\left(\left(I-n\right)\widehat{x}_k\left(1-2z\right)+2zQ\right)^3}.$$

Evaluating the derivative at $\widehat{x}_k=\dfrac{Q-n}{I-n}$ we have

$$\left.\frac{dW_n}{d\widehat{x}_k}\right|_{\widehat{x}_k=\frac{Q-n}{I-n}}\geq 0\Leftrightarrow z\leq z_4,$$

where z_4 is the unique solution to

$$\left(1-z_4\right)-\frac{z_4^2Q\chi n}{\left(Q-n\left(1-2z_4\right)\right)^2}=0. \tag{19}$$

We have

$$\left.\frac{dW_n}{d\widehat{x}_k}\right|_{\widehat{x}_k=\min\left\{1,\frac{Q-k}{I-n}\right\}}\leq 0\Leftrightarrow z\geq z_5(k),$$

where

$$1-z_5=\begin{cases}\dfrac{z_5^2Q\chi n}{\left(\left(I-n\right)\left(1-2z_5\right)+2z_5Q\right)^2} & \text{for } k\in\left[0,k'\right], \\ \dfrac{z_5^2Q\chi n}{\left(Q-k\left(1-2z_5\right)\right)^2} & \text{for } k\in\left[k',Q-1\right],\end{cases} \tag{20}$$

Note that $z_5(k)$ is continuous, with $\dfrac{dz_5}{dk} = 0$ for $k \in [0, k']$, and we have (using the implicit function theorem) $\dfrac{dz_5}{dk} > 0$ for $k \in [k', Q-1]$ when $z_5(k) > \frac{1}{2}$.

Finally, comparing the values of W_n at $\widehat{x}_k = \dfrac{Q-n}{I-n}$ and $\widehat{x}_k = \min\left\{1, \frac{Q-k}{I-n}\right\}$ shows that

$$W_n|_{\widehat{x}_k = \frac{Q-n}{I-n}} \geq W_n|_{\widehat{x}_k = \min\left\{1, \frac{Q-k}{I-n}\right\}} \Leftrightarrow z \geq z_6(k),$$

where z_6 is the unique solution to

$$1 - z_6 = \begin{cases} \dfrac{z_6^2 Q \chi n}{\left((I-n)\left(1-2z_6\right) + 2z_6 Q\right)\left(Q - n\left(1-2z_6\right)\right)} & \text{for } k \in [0, k'], \\ \dfrac{z_6^2 Q \chi n}{\left(Q - k(1-2z_6)\right)\left(Q - n\left(1-2z_6\right)\right)} & \text{for } k \in [k', Q-1]. \end{cases} \tag{21}$$

Note that $z_6(k)$ is continuous, with $\dfrac{dz_6}{dk} = 0$ for $k \in [0, k']$, and we have (using the implicit function theorem) $\dfrac{dz_6}{dk} < 0$ for $k \in [k', Q-1]$ when $z_6(k) > \frac{1}{2}$.

One can verify that $z_2(k) = z_4 = z_5(k) = \frac{1}{2}$ for $\chi = 2\dfrac{Q}{n}$. For any $\chi > 2\dfrac{Q}{n}$, we have $z_4 < z_5(k) < z_2(k) < \frac{1}{2}$. And finally for any $0 < \chi < 2\dfrac{Q}{n}$, we have $\frac{1}{2} < z_2(k) < z_5(k) < z_4$. Following the same logic as that used for proposition 1, one can easily prove proposition 2.

- Appendix 4.

We detail here the example provided in subsection 4.3.

The values of the parameters we are considering are given by: $I = 10$, $Q = 4$, $\varepsilon = 2.5$, $\eta = 1$, $q = 0.51$.

The underwriter issues 4 shares. In total there are 10 investors. As we are looking at case 1 (enough type 1 investors and too few type 2 investors to exhaust the issue), we must have that the number of investors at the IPO is strictly greater than 6. Given the values of ε, η and q, we get that $\chi > 2$.

Given the parameter values and for $n = 7$, we obtain that[12]

$$z_3\left(3\right) = 0.4249,\ \ z_3\left(2\right) = 0.4387,\ \ z_3 = z_3\left(1\right) = 0.4478.$$

The probability for type 2 investors of getting a share is shown on the graph below as a function of z:

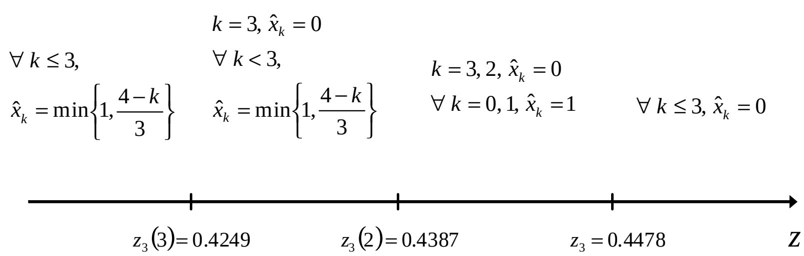
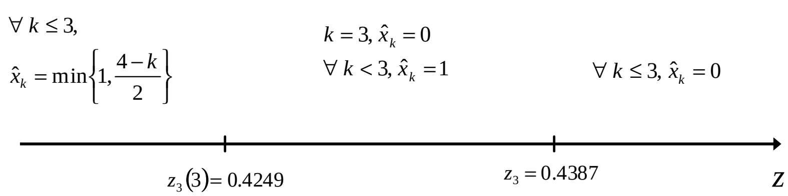


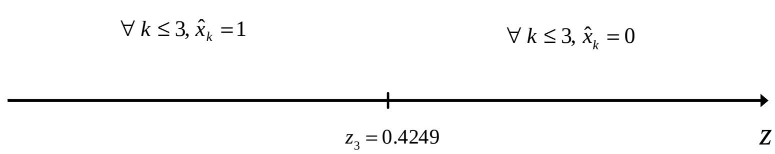
Figure 3: Probability for type 2 of getting a share when $n = 7$. The figures are based on $I = 10$, $Q = 4$, $\varepsilon = 2.5$, $\eta = 1$, $q = 0.51$.

Repeating the same exercise for $n = 8$ and $n = 9$, we obtain

$n = 8$



$n = 9$



[12] The values of z vary with n.

Figure 4: Probability for type 2 of getting a share when $n = 8$ and $n = 9$. The figures are based on $I = 10$, $Q = 4$, $\varepsilon = 2.5$, $\eta = 1$, $q = 0.51$.

We then compare, given $\widehat{x}_k$, the total revenue with different number of investors at the IPO where the total revenue is given by (9).