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The Global Defect Index

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Abstract. We show how far the local defect index determines the behaviour of an ordered medium in the vicinity of a defect.

1. Introduction

A rough model for an ordered medium may be constructed by a manifold M encoding the positions of the particles in space and a map $M \to V$ from M to the so-called **order parameter space** V or, more generally, a section $\sigma : M \to E$ in some fibre bundle E over M with typical fibre V, which describes additional degrees of freedom. We are interested in the consequences imposed on this situation merely by topology, i.e. by continuity assumptions on σ only. In general a bundle $E \to M$ does not admit a section on all of M but only on the complement $M \setminus \widetilde{\Delta}$ of some defect $\widetilde{\Delta} \subset M$. Even if the bundle is trivial there may occur defects $\widetilde{\Delta}$ which can not be removed by changing σ in the vicinity of $\widetilde{\Delta}$ only.

In a variety of examples the defect set is a submanifold (see e.g. [4]). In this case the section is called **regularly defected**. In the present investigation regularity will be tacitly assumed. An arc component $\Delta \subset \widetilde{\Delta}$ of the defect then has a well defined normal bundle $N \to \Delta$, and the behaviour of σ in the vicinity of this defect component is described by the restriction of σ to the sphere bundle SN of N, i.e. by a bundle map $\sigma : SN \to E|_\Delta$.

Definition 1. *The* **local defect index** *of a regularly defected cross section at* $p \in \widetilde{\Delta}$ *is the homotopy class* $\iota_p(\sigma) := [\sigma_p] \in [SN_p, E_p]$, *where* $\sigma_p : SN_p \to E_p$ *denotes the restriction of* σ *to the fibres over* $p \in \widetilde{\Delta}$.

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A regularly defected cross section is called **topologically stable***, if for every arc component* $\Delta \subset \widetilde{\Delta}$ *the local defect index* $\iota_p(\sigma)$ *at some (hence every) point* $p \in \Delta$ *is nontrivial.*

The main objective of this work is to show that the local defect index does not in general suffice to determine the global behaviour of a defect mapping along the defect component Δ. This is more precisely described by the fibre homotopy class of σ over Δ, which we will refer to as the **global defect index** of σ. Recall that two mappings $\sigma_0, \sigma_1 : SN \to E|_\Delta$ over Δ are called **fibre homotopic**, if there is a homotopy H between them consisting of mappings H_t which commute with the projections of the two bundles. By $[SN, E]_\Delta$ we denote the set of fibre homotopy classes of mappings $SN \to E|_\Delta$ over Δ and by $[SN, E]_\Delta^\alpha$ the set of fibre homotopy classes of maps $\sigma : SN \to E|_\Delta$ over Δ, whose local defect index at $p \in \Delta$ equals a given $\alpha \in [SN_p, E_p]$.

In the examples we have in mind we may assume that the normal bundle as well as E are trivial. Nontrivial bundles are treated in [1]. There is a long exact sequence involving the Whitehead product (Theorem 1), from which the set $[SN, E]_\Delta^\alpha$ can be computed by dividing out the action of the fundamental group of the mapping space $\mathrm{Map}_\alpha(S^{n-1}, V)$ of the fibres, see (1). As examples we explicitly treat nematics (Proposition 1), the superfluid dipolefree A-phase ^{3}He (Proposition 2), and (in Proposition 3) the case where V is an H-space — a Lie group for instance. The latter appears in the theory of the superfluid dipole locked A-phase of ^{3}He where $V = SO(3)$. In the case of nematics (see also [3]) there are 4 different types of global defect indices sharing the only nontrivial local defect index. In the other cases above even infinitely many global defect indices with the same local defect index occur.

Single unknotted ring defects in $\mathbb{R}^3$ were considered in [9]. The configurations with only one unknotted ring defect are described by the set $[\mathbb{R}^3 \setminus S^1, V] = [S^2 \vee S^1, V] = \pi_2(V) \times \pi_1(V)/\theta$ where θ is the action of $\pi_1(V)$. Our treatment admits other defects but identifies configurations which are homotopic near the defect component Δ. Thus we are interested in the set $[S^1 \times S^1, V]$, which we compute from the long exact sequence of Theorem 1. After dividing out the action of $\pi_1(V)$, this is related to $[S^2 \vee S^1, V]$ by an exact sequence

$$\pi_2(V) \to [S^2 \vee S^1, V]_{bp} \to [S^1 \times S^1, V]_{bp} \to \pi_1(V) \ ,$$

where $[\cdot, \cdot]_{bp}$ denotes homotopy classes, relative basepoint.

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2. The Whitehead-Sequence

Let $SN := \Delta \times S^{n-1}$ and $E := \Delta \times V$ be trivial bundles. The Exponential Law (see e.g. [2], p. 438) gives us a canonical bijection

$$[\Delta \times S^{n-1}, \Delta \times V]_\Delta \to [\Delta, \mathrm{Map}(S^{n-1}, V)]$$

between the set of fibre homotopy classes of mappings $SN \to E$ and a set of ordinary homotopy classes. Here we denote by $\mathrm{Map}(X, Y)$ the mapping space equipped with the compact-open topology. We will sometimes write $\mathrm{Map}_\alpha(X,Y)$ instead of α for the arc-component of any (hence every) representative of a class $\alpha \in [X,Y]$.

Let us take a closer look at the case $\Delta = S^m$. If we consider a fixed local defect index α as an element in $[S^{n-1}, V]$ we now know that after choosing basepoints we have a canonical bijection

$$[S^m \times S^{n-1}, S^m \times V]^\alpha_{S^m} \approx \pi_m(\mathrm{Map}_\alpha(S^{n-1}, V))/\pi_1(\mathrm{Map}_\alpha(S^{n-1}, V)) \ , \tag{1}$$

where the right-hand side is the quotient of the canonical action of the fundamental group $\pi_1(\mathrm{Map}_\alpha(S^{n-1}, V))$ on the higher homotopy group of this space. In order to calculate the set $[S^m \times S^{n-1}, S^m \times V]^\alpha_{S^m}$ we therefore have to determine the m^{th} homotopy group of the mapping space $\mathrm{Map}_\alpha(S^{n-1}, V)$ and the action of its fundamental group. The first part was done by G. W. Whitehead in [12] and we will summarize his results:

Let $s_0 \in S^{n-1}$ and $v_0 \in V$ be the basepoints and let F_α denote the subspace $\mathrm{Map}_\alpha((S^{n-1}, s_0), (V, v_0))$ of basepoint preserving mappings homotopic to $\alpha \in \pi_m(V, v_0)$. Then the evaluation map

$$\begin{array}{rcl} \tau_\alpha : G_\alpha := \mathrm{Map}_\alpha(S^{n-1}, V) & \to & V \\ f & \mapsto & f(s_0) \end{array}$$

is a Hurewicz-fibration with fibre F_α and so induces the long exact homotopy sequence

$$\cdots \to \pi_{m+1}(V) \xrightarrow{\partial} \pi_m(F_\alpha) \to \pi_m(G_\alpha) \to \pi_m(V) \to \cdots \ . \tag{2}$$

The homeomorphism $S^m \wedge S^{n-1} \cong S^{m+n-1}$ induces an isomorphism

$$\varphi_\alpha : \pi_m(F_\alpha) \cong \pi_{m+n-1}(V) \ . \tag{3}$$

The composition

$$\pi_{m+1}(V) \xrightarrow{\partial} \pi_m(F_\alpha) \xrightarrow{\varphi_\alpha} \pi_{m+n-1}(V)$$

is the Whitehead product from the left with α ([12]). Inserting this into the sequence (2) we obtain the following result.

Theorem 1 (G.W.Whitehead). *If $\rho_\alpha : \pi_{m+1}(V) \to \pi_{m+n-1}(V)$, $\beta \mapsto [\alpha, \beta]$ denotes the Whitehead product with $\alpha \in \pi_{n-1}(V)$, we have the following long exact sequence:*

$$\cdots \to \pi_{m+1}(V) \xrightarrow{\rho_\alpha} \pi_{m+n-1}(V) \to \pi_m(\mathrm{Map}_\alpha(S^{n-1}, V)) \to \pi_m(V) \to \cdots \ .$$

3. Applications

We now want to give some applications of Theorem 1 of physical importance. For this we consider regularly defected cross sections of the trivial bundle $S^3 \times V \to S^3$ with a defect set $\widetilde{\Delta} := \Delta_1 \cup \ldots \cup \Delta_r \cup p_1 \cup \ldots \cup p_s$ consisting of connected closed 1-dimensional submanifolds $\Delta_i \subset S^3$ and points $p_j \in S^3$. Such a regularly defected cross section is just a continous mapping $S^3 \setminus \widetilde{\Delta} \to V$ to the order parameter space V. The physical interpretation is that this mapping defines an ordering of the considered medium, continuous everywhere except at the defect set $\widetilde{\Delta}$.

In order to study the behaviour of such a mapping at a defect component $\Delta \cong S^1$ we consider the induced mapping $\sigma : SN \to \Delta \times V$ from the sphere normal bundle $SN \to \Delta$ of $\Delta \subset S^3$ and its fibre homotopy class, as we have done before. If we require its orientability then the bundle $SN \to \Delta$ is automatically trivial and hence we can restrict ourselves to the case $SN = S^1 \times S^1 \to S^1$. Thus the desired homotopy classes will be elements of $[S^1 \times S^1, V]$. We shall denote by $[S^1 \times S^1, V]^*$ the subset of all those classes whose restriction to the fibre $S^1 \times 1$ is nontrivial.

3.1. Nematics. Consider the case $V := \mathbb{RP}^2$. This is the order parameter space for nematic liquid crystals and in this situation we have the following result that can also be found in [3] where a different proof is given:

Proposition 1. *We have*

$$\#[S^1 \times S^1, \mathbb{RP}^2]^* = 4 \ . \tag{4}$$

Proof. From $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$ we know that $[S^1, \mathbb{RP}^2]$ has two elements and therefore we have $[S^1 \times S^1, \mathbb{RP}^2]^* = [S^1 \times S^1, \mathbb{RP}^2]^\alpha \approx \pi_1(\mathrm{Map}_\alpha(S^1, \mathbb{RP}^2))/\sim$, where $\alpha \in [S^1, \mathbb{RP}^2]$ denotes the nontrivial element and the right-hand side of the equation denotes the quotient of the conjugation operation of the fundamental group on itself. But the group $\pi_1(\mathrm{Map}_\alpha(S^1, \mathbb{RP}^2))$ may be found in the exact sequence of Theorem 1

$$\pi_2(\mathbb{RP}^2) \xrightarrow{\rho^2_\alpha} \pi_2(\mathbb{RP}^2) \to \pi_1(\mathrm{Map}_\alpha(S^1, \mathbb{RP}^2)) \to \pi_1(\mathbb{RP}^2) \xrightarrow{\rho^1_\alpha} \pi_1(\mathbb{RP}^2) \to \ . \tag{5}$$

The action of $\pi_1(\mathbb{RP}^2)$ on $\pi_2(\mathbb{RP}^2)$ is nontrivial. We have $\rho^2_\alpha(\beta) = \alpha \cdot \beta - \beta = -2\beta$ for every $\beta \in \pi_2(\mathbb{RP}^2)$. At the right end of the diagram the groups are abelian. Hence the Whitehead product ρ^1_α is trivial.

Thus (5) yields the exact sequence

$$\mathbb{Z} \xrightarrow{\cdot(-2)} \mathbb{Z} \to \pi_1(\mathrm{Map}_\alpha(S^1, \mathbb{RP}^2)) \to \mathbb{Z}_2 \to 0 \ ,$$

which gives immediately that $\#\pi_1(\mathrm{Map}_\alpha(S^1, \mathbb{RP}^2) = 4$. As every group with four elements is abelian, the number of elements does not change when passing to free homotopy classes and the assertion is proved. □

These four homotopy classes can be explicitly described as follows (see [3]). For any $\zeta \in S^1$ we denote by $[\zeta] \in \mathbb{RP}^2$ its image under the mapping $S^1 \hookrightarrow S^2 \xrightarrow{\pi} \mathbb{RP}^2$. For $k = 0, \ldots, 3$ let

$$\psi_k : S^1 \times S^1 \to \mathbb{RP}^2$$

be the mapping induced by

$$[0,1] \times [0,1] \to \mathbb{RP}^2 \ , \ (t,s) \mapsto [\mathrm{e}^{\pi \mathrm{i}(t+ks)}] \ .$$

We claim that

$$[S^1 \times S^1, \mathbb{RP}^2]^* = \{[\psi_k] \mid k = 0, \dots, 3\} \ .$$

Proof. When restricted to $S^1 \times 1$ each ψ_k represents the nontrivial class $\alpha \in \pi_1(\mathbb{RP}^2)$, so that $[\psi_k] \in [S^1 \times S^1, \mathbb{RP}^2]^*$ for all $k = 0, \dots, 3$. Clearly

$$[\psi_0|_{1 \times S^1}] = [\psi_2|_{1 \times S^1}] = 0 \in \pi_1(\mathbb{RP}^2) \ ,$$

and

$$[\psi_1|_{1 \times S^1}] = [\psi_3|_{1 \times S^1}] = \alpha \in \pi_1(\mathbb{RP}^2) \ .$$

Considered as elements of $\pi_1(\mathrm{Map}_\alpha(S^1, \mathbb{RP}^2))$ the ψ_k satisfy $[\psi_3] = [\psi_2] + [\psi_1]$ and $[\psi_1] = [\psi_0] + [\psi_1]$. It suffices therefore to prove that $[\psi_0] \neq [\psi_2]$.

A straightforward calculation shows that the homomorphism

$$\pi_2(\mathbb{RP}^2) \xrightarrow{\iota} \pi_1(\mathrm{Map}_\alpha(S^1, \mathbb{RP}^2))$$

maps 0 to $[\psi_0]$ and the generator of $\pi_2(\mathbb{RP}^2)$ to $[\psi_2]$. From the exactness of the sequence (5) we infer $[\psi_0] \neq [\psi_2]$. $\square$

3.2. Superfluid dipolefree A-phase ^{3}He. Here we have to consider the order parameter space $V := S^2 \times_{\mathbb{Z}_2} SO(3)$, where the generator of $\mathbb{Z}_2$ acts on S^2 by reversing the sign and on $SO(3)$ via

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \mapsto \begin{pmatrix} -a_1 & -b_1 & c_1 \\ -a_2 & -b_2 & c_2 \\ -a_3 & -b_3 & c_3 \end{pmatrix}$$

(see [5-7]).

The first two homotopy groups of V are (see [8]) :

$$\pi_1(V) = \mathbb{Z}_4 \quad \text{and} \quad \pi_2(V) = \mathbb{Z} \ .$$

As $\pi_1(V)$ is abelian, we can consider the local defect index $\alpha \in [S^1, V]$ as an element of $\pi_1(V)$. We have the fibration

$$p : V = S^2 \times_{\mathbb{Z}_2} SO(3) \to \mathbb{RP}^2$$

associated to the $\mathbb{Z}_2$-principal fibration $S^2 \to \mathbb{RP}^2$. Its homotopy sequence yields that the induced homomorphism

(i) $p_* : \pi_2(V) \to \pi_2(\mathbb{RP}^2)$ is injective and
(ii) $p_* : \pi_1(V) \to \pi_1(\mathbb{RP}^2)$ is surjective.

Since the operation of the fundamental group on the higher homotopy groups is natural we have the following equation for the action of a generator $\iota \in \pi_1(V) \cong \mathbb{Z}_4$ on an arbitrary $\beta \in \pi_2(V) \cong \mathbb{Z}$:

$$p_*(\iota \cdot \beta) = p_*(\iota) \cdot p_*(\beta) = (-1) \cdot p_*(\beta) = -p_*(\beta) = p_*(-\beta) \ .$$

From (i) we deduce that $\iota \cdot \beta = -\beta$ and therefore we have calculated the operation of the fundamental group of V on $\pi_2(V)$ as follows:

$$\begin{array}{ccccc} \pi_1(V) & \times & \pi_2(V) & \to & \pi_2(V) \\ (\iota^k & , & \beta) & \mapsto & (-1)^k\beta \end{array}$$

and so we are able to prove the following

Proposition 2. *Let $V := S^2 \times_{\mathbb{Z}_2} SO(3)$ and let ι be the generator of $\pi_1(V) \cong \mathbb{Z}_4$. If we denote by $[S^1 \times S^1, V]^{\iota^k}$ the set of homotopy classes of mappings whose restrictions to $S^1 \times 1$ equal ι^k , $k \in \mathbb{Z}_4$ then we have the following two cases:*

(i) For $k = 1, 3$ mod 4 we have $\#[S^1 \times S^1, V]^{\iota^k} = 8$,
(ii) for $k = 0, 2$ mod 4 we have $\#[S^1 \times S^1, V]^{\iota^k} = \infty$.

Proof. For the local defect index $\alpha := \iota^k$ we have the exact sequence

$$\begin{array}{ccccccccc} \pi_2(V) & \to & \pi_2(V) & \to & \pi_1(\mathrm{Map}_\alpha(S^1, V)) & \to & \pi_1(V) & \to & \pi_1(V) \\ \beta & \mapsto & \alpha \cdot \beta - \beta & & & & \beta & \mapsto & \alpha \cdot \beta - \beta \end{array}$$

which becomes

$$\mathbb{Z} \xrightarrow{\rho} \mathbb{Z} \to \pi_1(\mathrm{Map}_\alpha(S^1, V)) \to \mathbb{Z}_4 \xrightarrow{0} \mathbb{Z}_4 \ ,$$

where $\rho = (-2)$ in case (i) and $\rho = 0$ in case (ii). In the first case it follows that $\pi_1(\mathrm{Map}_\alpha(S^1, V))$ is abelian and the set $[S^1 \times S^1, V]^{\iota^k} = \pi_1(\mathrm{Map}_\alpha(S^1, V))$ has 8 elements. In the second case it must have infinitely many conjugacy classes and the proposition is proved. □

3.3. H-Space as Fibre. We now assume $\Delta = S^m$ and that the fibre V of E is an H-Space. Recall that on an H-space all the Whitehead products vanish. In particular the action of the fundamental group on the higher homotopy groups is trivial, so that we may regard the local defect index as an element in $\pi_{n-1}(V)$. The following is immediate from Theorem 1.

Proposition 3. *If V is an H-space, then for every $\alpha \in \pi_{n-1}(V)$ we have the equation*

$$\#\pi_m(\mathrm{Map}_\alpha(S^{n-1}, V)) = \#\pi_{m+n-1}(V) \cdot \#\pi_m(V) \ .$$

As a concrete example let $V = S^3$ and assume $n = 4$, such that SN also has S^3 as its fibre. For the number of possible fibre homotopy classes with local defect index $1 \in \pi_3(S^3)$ we have:

Proposition 4. *For every $m \in \mathbb{N}$ we have*

$$\#[S^m \times S^3, S^m \times S^3]^1_{S^m} = \#\pi_m(S^3) \cdot \#\pi_{m+3}(S^3) \ .$$

Especially for $m \neq 3$ we get that

$$\#[S^m \times S^3, S^m \times S^3]^1_{S^m} < \infty \ .$$

Proof. From Proposition 3 we know that

$$\#\pi_m(\mathrm{Map}_1(S^3,S^3)) = \#\pi_m(S^3)\cdot\#\pi_{m+3}(S^3)\ .$$

There is a 1-1 correspondence

$$[S^m\times S^3, S^m\times S^3]^1_{S^m} \approx \pi_m(\mathrm{Map}_1(S^3,S^3))/\pi_1(\mathrm{Map}_1(S^3,S^3))\ .$$

Hence the first assertion follows from the fact that $\mathrm{Map}_1(S^3,S^3)$ is also an H-space and thus the action of its fundamental group on the m^{th} homotopy group is trivial. From [11] we know that the groups $\pi_m(S^3)$ are all finite except for $m=3$ and therefore the second assertion is proved as well. □

With the help of Table A.3.6 in [10] we obtain the following list:

m	1	2	3	4	5	6	7	8	...
$\#[S^m\times S^3, S^m\times S^3]^1_{S^m}$	2	2	∞	4	4	36	30	4	$<\infty$

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