On the Kalman-Yacubovich-Popov lemma and common Lyapunov solutions for matrices with regular inertia

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Abstract

In this paper we extend the classical Lefschetz version of the Kalman-Yacubovich-Popov (KYP) lemma to the case of matrices with general regular inertia. We then use this result to derive an easily verifiable spectral condition for a pair of matrices with the same regular inertia to have a common Lyapunov solution (CLS), extending a recent result on CLS existence for pairs of Hurwitz matrices.

1 Introduction

Classical Lyapunov theory provides a strong method for checking the asymptotic stability of linear time-invariant (LTI) systems of the form $\dot{x} = Ax$, $A \in \mathbb{R}^{n\times n}$ without explicitly calculating the eigenvalues of A [1, 2]. The result is that, the zero state of $\dot{x} = Ax$ is asymptotically stable if and only if the solution of the Lyapunov equation

$$A^T P + PA = -Q, \tag{1}$$

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is a symmetric positive definite matrix P for all $Q = Q^T > 0$. Here, the matrix $P = P^T > 0$ is called a *Lyapunov solution* for A. Also, the asymptotic stability of $\dot{x} = Ax$ implies that all the eigenvalues of A have strictly negative real parts, where such matrices are said to be *Hurwitz* .

In recent years many publications [3, 4, 5, 6, 7, 8, 9] have appeared that deal with the existence of common quadratic Lyapunov functions (CQLFs) for families of stable LTI dynamical systems. In an earlier publication, CQLF existence problem has been investigated in conjunction with the stability of LTI systems with uncertain parameters in [10]. Formally, for the case of a pair of systems, the CQLF existence problem amounts to determining necessary and sufficient conditions for the existence of a positive definite symmetric matrix $P = P^T > 0$, $P \in \mathbb{R}^{n\times n}$ that simultaneously satisfies the matrix inequalities

$$A_1^T P + PA_1 < 0 \quad , \qquad A_2^T P + PA_2 < 0 \tag{2}$$

where all eigenvalues of the given matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$ lie in the open left half of the complex plane, that is A_1, A_2 are Hurwitz. When there exists a $P = P^T > 0$ satisfying the above inequalities, then the scalar function $V(x) = x^T Px$ is said to be a common quadratic Lyapunov function (CQLF) for the dynamical systems $\Sigma_{A_i} : \dot{x} = A_i x \ \ i \in \{1,2\}$, and the matrix P is a common Lyapunov solution (CLS) for the Lyapunov inequalities (2). In a slight abuse of notation, we shall often refer to such a P as a CLS for the matrices A_1, A_2. The existence of CQLFs is of considerable importance in a number of engineering problems [11] and consequently the CQLF existence problem has assumed a pivotal role in research on stability theory.

It is generally accepted that determining the existence of a CQLF for a finite set of LTI systems is very difficult to solve analytically. However, in certain situations as in the case of switching between two LTI systems, elegant conditions for the existence of a CQLF may be obtained when restrictions are placed on the matrices A_1 and A_2. Recently, one such result was obtained for the case where A_1 and A_2 are Hurwitz and $rank(A_1 - A_2) = 1$; in this case a CQLF exists for Σ_{A_1} and Σ_{A_2} if and only if the matrix product $A_1 A_2$ does not have any real negative eigenvalues. Furthermore, it has been shown recently in [3] that this result can be seen as a time-domain version of the Kalman-Yacubovich-Popov (KYP) lemma which was introduced by Kalman in [12].

Our primary aim in this paper is to extend this result on CLS existence to the case where the matrices A_1 and A_2 are no longer Hurwitz, but rather have regular inertia [2]. Note that the general problem of CLS existence for matrices with regular inertia has been considered by various authors before [13, 5, 6, 8, 14], and, in particular, results linking CLS existence to the inertia of so-called *convex invertible cones* of matrices have been established for the cases of Hermitian and triangular matrices in $\mathbb{R}^{n\times n}$ and for matrix pairs in $\mathbb{R}^{2\times 2}$. In this paper, we shall extend the KYP lemma from classical stability theory to matrices with regular inertia and show that, in analogy with the classical case of Hurwitz matrices [15], this extension leads to elegant conditions for CLS existence for matrices with regular inertia also.

The main contribution of this paper is the derivation of a simple algebraic condition that is equivalent to CLS existence for a significant class of pairs of matrices in companion form and with the same regular inertia, as stated in the following Theorem (Theorem 3.2 in Section 3.2).

Theorem: *Let A, $A-gh^T$ be two matrices in $\mathbb{R}^{n\times n}$ in companion form and with the same regular inertia, $In(A)=In(A-gh^T)=(n_+,n_-,0)$, where g,h are vectors in $\mathbb{R}^n$. Further, assume that for any pair of eigenvalues, λ_i,λ_j, of A, $\mathrm{Re}(\lambda_i+\lambda_j)\neq 0$. Then, the following statements are equivalent:*

(i) *There exists a symmetric matrix $P=P^T$ in $\mathbb{R}^{n\times n}$ with $In(P)=In(-A)=In(-(A-gh^T))$, and positive definite matrices $Q_1>0$, $Q_2>0$ such that*

$$A^TP+PA=-Q_1$$
$$(A-gh^T)^TP+P(A-gh^T)=-Q_2.$$

(ii) *The matrix rays*

$$\sigma_{\gamma[0,\infty)}(A,A-gh^T)=\{A+\gamma(A-gh^T):\gamma\in[0,\infty)\}$$

and

$$\sigma_{\gamma[0,\infty)}(A^{-1},A-gh^T)=\{A^{-1}+\gamma(A-gh^T):\gamma\in[0,\infty)\}$$

have constant regular inertia.

(iii) *The matrix $A(A-gh^T)$ has no real negative eigenvalues.*

(iv) $1+\mathrm{Re}\{h^T(j\omega I_n-A)^{-1}g\}>0, \quad \forall\omega\in\mathbb{R}.$

2 Mathematical Preliminaries

2.1 Definitions and technical lemmas

In this section we present a number of basic definitions and results that are needed in our later derivations.

Throughout this paper $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers respectively. We denote n-dimensional real Euclidean space by $\mathbb{R}^n$ and the space of $n \times n$ matrices with real entries by $\mathbb{R}^{n\times n}$. Also, we adopt the convention that vectors in $\mathbb{R}^n$ are assumed to be column vectors. For a vector x in $\mathbb{R}^n$, x_i denotes the i^{th} component of x and for A in $\mathbb{R}^{n\times n}$, we denote the entry in the (i, j) position by a_{ij}. I_n denotes the $n \times n$ identity matrix and j is used throughout to denote the complex number satisfying $j^2 = -1$.

<u>Companion matrices:</u>

We say that a matrix $A \in \mathbb{R}^{n\times n}$ is in *companion form* [16, 17] if

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix}, \tag{3}$$

where $a_0, \dots, a_{n-1}$ are real numbers. It is straightforward to verify that if A is in the form (3), then the characteristic polynomial of A is

$$\det(zI_n - A) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0.$$

<u>Matrix Inertia:</u>

The inertia of a matrix $A \in \mathbb{R}^{n\times n}$ is the ordered triple

$$In(A) = (i_+(A), i_-(A), i_0(A)) \tag{4}$$

where $i_+(A), i_-(A), i_0(A)$ are the number of eigenvalues of A in the open right half plane, the open left half plane, and on the imaginary axis, respectively. We say that A has regular inertia if $i_0(A) = 0$.

The Matrix Ray $\sigma_{\gamma[0,\infty)}[A_1, A_2]$:

Later in the paper, we shall refer to the *matrix ray* $\sigma_{\gamma[0,\infty)}[A_1, A_2]$. Formally, this is the parameterized family of matrices of the form

$$\sigma_{\gamma[0,\infty)}[A_1, A_2] = \{A_1 + \gamma A_2 : \gamma \in [0,\infty)\}. \tag{5}$$

We shall say that $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is *non-singular* if $A_1 + \gamma A_2$ is non-singular for all $\gamma \geq 0$; otherwise it is said to be *singular*. It is trivial to show that singularity of the matrix ray $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is equivalent to the matrix product $A_1^{-1}A_2$ having a negative real eigenvalue if A_1 and A_2 are non-singular. Also, we say that $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ has constant inertia if there are fixed non-negative integers n_+, n_-, n_0 such that $In(A_1 + \gamma A_2) = (n_+, n_-, n_0)$ for all $\gamma \geq 0$.

Technical lemmas:

We next record some basic technical facts that shall be used in proving the principal results of this paper.

Lemma 2.1 *Suppose that $A \in \mathbb{R}^{n\times n}$ and is nonsingular. Then*

$$det(\omega^2 I_n + A^2) > 0, \tag{6}$$

for all $\omega \in \mathbb{R}$.

Proof: As the matrix A has real entries and is nonsingular, it follows that for any $\omega \in \mathbb{R}$,

$$\det(\omega^2 I_n + A^2) = |\det(j\omega I_n + A)|^2 > 0. \tag{7}$$

$\square$

Lemma 2.2 *[12] Let $A \in \mathbb{R}^{n\times n}$ and $A - gh^T \in \mathbb{R}^{n\times n}$ be in companion form, where $h, g \in \mathbb{R}^n$ with $g = [0, ...0, 1]^T$. Then we can write*

$$1 + Re\{h^T(j\omega I_n - A)^{-1}g\} = 1 - h^T A(\omega^2 I_n + A^2)^{-1}g$$

The next lemma records the simple fact that any symmetric matrix P which satisfies the Lyapunov inequality for a given matrix A also satisfies the Lyapunov inequality for its inverse, A^{-1}.

Lemma 2.3 *[5] Let $A \in \mathbb{R}^{n\times n}$ have regular inertia. Then for any symmetric $P = P^T$ in $\mathbb{R}^{n\times n}$ with $In(P) = In(-A)$,*

$$A^T P + PA < 0 \tag{8}$$

if and only if

$$(A^{-1})^T P + P(A^{-1}) < 0.$$

Proof: This follows immediately from the observation that

$$(A^{-1})^T P + PA^{-1} = (A^{-1})^T (A^T P + PA)A^{-1}. \tag{9}$$

Also it is evident from the proof that the lemma is valid for all nonsingular A. □

Lemma 2.4 *[18] Let $A, A - gh^T$ be Hurwitz matrices in $\mathbb{R}^{n\times n}$, where $g, h^T \in \mathbb{R}^n$. Then for any complex number s,*

$$1 + h^T (sI - A)^{-1} g = \frac{det(sI - (A - gh^T))}{det(sI - A)}. \tag{10}$$

2.2 The Circle Criterion and Common Lyapunov solutions

One of the most fundamental results on the stability of dynamical systems in the engineering literature is the Circle Criterion. The relevance of the Circle Criterion [15] in our present context stems from the fact that it provides a necessary and sufficient condition for two fixed Hurwitz matrices in companion form to have a common Lyapunov solution. Formally, if A, $A - gh^T$ are two Hurwitz matrices in $\mathbb{R}^{n\times n}$ in companion form, where h, g are vectors in $\mathbb{R}^n$, then they have a CLS if and only if the rational function

$$1 + h^T (zI - A)^{-1} g \tag{11}$$

is strictly positive real (SPR), meaning that

$$1 + \mathrm{Re}\{h^T (j\omega I - A)^{-1} g\} > 0 \tag{12}$$

for all ω in $\mathbb{R}$. Moreover, it follows from Meyer's extension of the KYP Lemma [19] that the condition (12) is sufficient for CQLF existence for Hurwitz matrices A, $A - gh^T$, differing by rank one, but not necessarily in companion form.

Recently in [3, 20], it has been established that the frequency domain condition (12) is equivalent to a simple condition on the eigenvalues of the matrix product $A(A-gh^T)$. This equivalence was first demonstrated in [3] for matrices in companion form and then extended to the case of a general pair of Hurwitz matrices A_1, A_2 with $\mathrm{rank}(A_2 - A_1) = 1$ in [20]. We state the most general form of the result here.

Theorem 2.1 *Let A, $A-gh^T$ be Hurwitz matrices in $\mathbb{R}^{n\times n}$, where $g,h \in \mathbb{R}^n$. Then*

$$1+\mathrm{Re}\{h^T(j\omega I-A)^{-1}g\} > 0 \textit{ for all } \omega \in \mathbb{R}$$

if and only if the matrix product $A(A-gh^T)$ has no negative real eigenvalues.

Proof: Without loss of generality, we may assume that gh^T is in one of the following Jordan canonical forms

$$\text{(i)} \begin{pmatrix} c & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 \\ \vdots & & & \\ 0 & \dots & \dots & 0 \end{pmatrix},$$

$$\text{(ii)} \begin{pmatrix} 0 & \dots & \dots & 0 \\ 1 & \dots & \dots & 0 \\ \vdots & & & \\ 0 & \dots & \dots & 0 \end{pmatrix}. \tag{13}$$

As A and $A-gh^T$ are both Hurwitz, their determinants will have the same sign, so it follows that the product $A(A-gh^T)$ has no negative real eigenvalues if and only if, for all $\lambda > 0$

$$det(\lambda I + (A-gh^T)A) = det(\lambda I + A^2 - gh^T A) > 0$$

If gh^T is in Jordan form then it follows that the expressions

$$det(\lambda I + A^2 - gh^T A)$$

and

$$Re\{det(\lambda I + A^2 - gh^T A - \sqrt{\lambda}jgh^T)\},$$

are identical. Thus, writing $\lambda = \omega^2$ we have that for all real ω

$$Re\{det(\omega^2 I + A^2 - gh^T A - j\omega gh^T)\} > 0. \tag{14}$$

It now follows, after a short calculation (see [3],[21]) that for all $\omega \in \mathbb{R}$

$$Re\{\frac{det(j\omega I - (A - gh^T))}{det(j\omega I - A)}\} > 0. \tag{15}$$

Making use of Lemma 2.4 It follows that for all real ω

$$1 + Re\{h^T (j\omega I - A)^{-1} g\} > 0$$

as claimed. $\square$

Combining the result of Theorem 2.1 with Meyer's extension of the KYP Lemma [19], yields the following spectral condition for CLS existence for Hurwitz matrices differing by rank one.

Theorem 2.2 *[20] Let A, $A - gh^T$ be two Hurwitz matrices in $\mathbb{R}^{n\times n}$ where g, h are vectors in $\mathbb{R}^n$. A necessary and sufficient condition for the existence of a common Lyapunov solution for the matrices A, $A - gh^T$ is that the matrix product $A(A - gh^T)$ does not have any negative real eigenvalues.*

The principal contribution of the present paper is to extend Theorem 2.2 to the case of pairs of matrices with the same regular inertia. First of all, we recall some fundamental facts on the existence of solutions to the Lyapunov inequality for a single matrix with regular inertia. The first part of Theorem 2.3 below is usually referred to as the *General Inertia Theorem* [2], while the second part follows from general results on the existence of solutions to the Sylvester equation $AX + XB = C$ (For instance, see Theorem 4.4.6 in [2]). While the General Inertia Theorem has been established for matrices with complex entries, we state it here for real matrices as we only consider the CLS existence problem for real matrices in this paper.

Theorem 2.3 *General Inertia Theorem [2]*
Let $A \in \mathbb{R}^{n\times n}$ be given. Then there exists a symmetric matrix $P = P^T$ in $\mathbb{R}^{n\times n}$ such that

$$A^T P + PA < 0 \tag{16}$$

if and only if A has regular inertia. In this case, $In(P) = In(-A)$.

Furthermore, if $\lambda_i + \lambda_j \neq 0$ *for all eigenvalues* λ_i, λ_j *of A, then for every* $Q = Q^T < 0$ *in* $\mathbb{R}^{n\times n}$*, there is a unique* $P = P^T$ *with* $In(P) = In(-A)$ *and* $A^T P + PA = Q < 0$.

3 Main results

The two main contributions of this paper are described in the current section. First of all, in Theorem 3.1 we extend the classical Lefschetz [22] version of the Kalman-Yacubovich-Popov (KYP) lemma to the case of matrices with regular inertia and in companion form. Historically, the KYP lemma has played a key role in stability theory and has led to a number of important results on Lyapunov function existence for dynamical systems including the *Circle Criterion* [15] and the *Popov Criterion* [23, 24]. We shall see below that the extension of the KYP lemma to the case of matrices with regular inertia also has implications for the existence of common Lyapunov solutions in this more general context. In particular, in Theorem 3.2 we derive a simple algebraic condition that is equivalent to CLS existence for a significant class of pairs of matrices in companion form, and with the same regular inertia.

3.1 The KYP Lemma for matrices with regular inertia

The classical KYP lemma considered the existence of constrained solutions to the Lyapunov inequality for Hurwitz matrices. More formally, the following question, which we shall address below for matrices with regular inertia, was considered.

Given, A$\in \mathbb{R}^{n\times n}$ Hurwitz, a real constant $\tau > 0$, and a positive definite matrix $D = D^T > 0$, determine conditions for the existence of a vector $q \in \mathbb{R}^n$, a real number $\varepsilon > 0$ and a positive definite matrix $P = P^T > 0 \in \mathbb{R}^{n\times n}$ such that

$$A^T P + PA = -qq^T - \varepsilon D \qquad (17)$$

$$Pg - h = \sqrt{\tau} q. \qquad (18)$$

Before we proceed, we prove the following technical lemma which shall be needed later in this subsection.

Lemma 3.1 *Let $A \in \mathbb{R}^{n\times n}$ to be a nonsingular matrix such that for all pairs λ_i, λ_j of eigenvalues of A, $\mathrm{Re}(\lambda_i + \lambda_j) \neq 0$. Further suppose that g, h are column vectors in $\mathbb{R}^n$ such that for any h, the matrices A ,and $A - gh^T$ can simultaneously be transformed to companion forms using similarity transformations. Then*

$$\mathrm{Re}\{h^T(j\omega I_n - A)^{-1}g\} = 0 \textit{ for all } \omega \in \mathbb{R} \tag{19}$$

implies that $h = 0$.

Proof: Without loss of generality, we can assume that A is in companion form and that $g = (0, \ldots, 1)^T$. We shall argue by contradiction. Assume now that (19) holds and that $h = (h_0, \ldots, h_{n-1})^T$ is non-zero, and consider the rational function $R(z) = h^T(zI_n - A)^{-1}g$. Then we can write

$$R(z) = \frac{h_0 + h_1 z + \cdots + h_{n-1}z^{n-1}}{\det(zI_n - A)}, \tag{20}$$

and moreover, under our assumptions the following facts must hold:

(i) $R(z)$ is not uniformly zero;

(ii) $R(z)$ has at least one pole and any such pole must be an eigenvalue of A;

(iii) $R(z)$ takes strictly imaginary values on the imaginary axis.

From (iii), it follows that the function $R_1(z) = jR(jz)$ takes real values for real z, and hence that R_1 is a real rational function. Thus, the poles of R_1 must be real, or else occur in complex conjugate pairs. Moreover, if λ is any pole of R_1, then $j\lambda$ is a pole of the original function R. From this it follows that R must either have a pole on the imaginary axis or else that there are two poles, λ_i, λ_j of R with $\mathrm{Re}(\lambda_i + \lambda_j) = 0$. Remembering that any pole of R must be an eigenvalue of A, this is a contradiction. Thus h must be zero as claimed. □

Comments:

(i) The proof given above is based on an argument presented in Chapter 8 of [22], where it was shown that for a Hurwitz matrix $A \in \mathbb{R}^{n\times n}$ in companion form, and $g = (0, \ldots, 1)^T$,

$$\mathrm{Re}\{h^T(j\omega I_n - A)^{-1}g\} = 0 \text{ for all } \omega \in \mathbb{R}$$

implies that $h = 0$. This is not in general true for a companion matrix A with regular inertia as can be seen from the simple example

$$A = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}, g = h = (0,1)^T.$$

Clearly, the additional assumption made in Lemma 3.1 , that $\mathrm{Re}(\lambda_i + \lambda_j)$ is non-zero, is automatically satisfied if A is Hurwitz.

(ii) The assumption, that $\mathrm{Re}(\lambda_i + \lambda_j)$ is non-zero, for all eigenvalues λ_i, λ_j of A is satisfied generically. More precisely, given any $A \in \mathbb{R}^{n\times n}$ in companion form with regular inertia which does not satisfy the assumption, and $\varepsilon > 0$, there exists a matrix $A' \in \mathbb{R}^{n\times n}$ in companion form with the same inertia as A such that $\|A - A'\| < \varepsilon$ and $\mathrm{Re}(\lambda_i + \lambda_j)$ is non-zero for all eigenvalues λ_i, λ_j of A'. (Here $\|.\|$ can be any matrix norm on $\mathbb{R}^{n\times n}$.)

(iii) It is important to note that if $\mathrm{Re}(\lambda_i + \lambda_j)$ is non-zero for all eigenvalues λ_i, λ_j of A, then it follows from the last part of Theorem 2.3 that for any negative definite matrix $Q = Q^T < 0$ in $\mathbb{R}^{n\times n}$, there is a unique symmetric $P = P^T$ with $In(P) = In(-A)$ such that $A^T P + PA = Q < 0$. We shall make use of this fact in the proof of Theorem 3.1 below.

We are now in a position to state the principal result of this subsection which is an extension of the classical KYP lemma to the case of matrices with regular inertia.

Theorem 3.1 *Let $A \in \mathbb{R}^{n\times n}$ be a companion matrix with regular inertia such that $\mathrm{Re}(\lambda_i + \lambda_j) \neq 0$ for all $\lambda_i, \lambda_j \in \sigma(A)$, and let $g, h \in \mathbb{R}^n$ be vectors such that $A - gh^T$ is also in companion form. Moreover, let $D = D^T > 0$ in $\mathbb{R}^{n\times n}$ and $\tau > 0$ in $\mathbb{R}$ be given. Then the following two statements are equivalent:*

(i) There exists a symmetric matrix $P = P^T$ in $\mathbb{R}^{n\times n}$ with $In(P) = In(-A)$, a vector $q \in \mathbb{R}^n$ and a scalar $\varepsilon > 0$ such that

$$A^T P + PA = -qq^T - \varepsilon D \tag{21}$$

$$Pg - h = \sqrt{\tau} q. \tag{22}$$

(ii) $\tau + 2Re\{h^T (j\omega I_n - A)^{-1} g\} > 0$ for all $\omega \in \mathbb{R}$.

Proof: For convenience, throughout the proof we shall use the notation $A_{j\omega}$ to denote $(j\omega I_n - A)$ and $m_{j\omega}$ shall denote the complex vector-valued function $A_{j\omega}^{-1}g$. It is then straightforward to check that for any $P = P^T$ in $\mathbb{R}^{n\times n}$,

$$A_{j\omega}^* P + PA_{j\omega} = -(A^T P + PA), \tag{23}$$

Moreover, multiplying the left and right hand sides of (23) by $g^T(A_{j\omega}^{-1})^*$ and $A_{j\omega}^{-1}g$ respectively, we see that

$$g^T P m_{j\omega} + m_{j\omega}^* P g = -m_{j\omega}^*(A^T P + PA)m_{j\omega}. \tag{24}$$

<u>*(i) ⇒ (ii):*</u>

Suppose that the equations (21), and (22) hold. It follows immediately from (21) and (24) that

$$m_{j\omega}^* P g + g^T P m_{j\omega} \;=\; m_{j\omega}^* q q^T m_{j\omega} + \varepsilon m_{j\omega}^* D m_{j\omega}. \tag{25}$$

In (25) we can replace the Pg term using (22) and arrange to get

$$m_{j\omega}^* h + h^T m_{j\omega} + \sqrt{\tau}(m_{j\omega}^* q + q^T m_{j\omega}) = m_{j\omega}^* q q^T m_{j\omega} + \varepsilon m_{j\omega}^* D m_{j\omega}$$

or equivalently,

$$2Re\{h^T m_{j\omega}\} = m_{j\omega}^* q q^T m_{j\omega} - 2\sqrt{\tau} Re\{q^T m_{j\omega}\} + \varepsilon m_{j\omega}^* D m_{j\omega}. \tag{26}$$

It now follows that

$$2Re\{h^T m_{j\omega}\} = |q^T m_{j\omega} - \sqrt{\tau}|^2 - \tau + \varepsilon m_{j\omega}^* D m_{j\omega}, \tag{27}$$

and hence, as D is positive definite and A has regular inertia,

$$\tau + 2Re\{h^T m_{j\omega}\} > 0 \tag{28}$$

for all $\omega \in \mathbb{R}$.

<u>*(ii) ⇒ (i):*</u>

Without loss of generality, we can assume that A is in companion form, and $g = (0,0,\ldots,1)^T$. In this case, it can be verified by direct calculation [12, 16] that for any vector $f = (f_0,\ldots,f_{n-1})^T$ in $\mathbb{R}^n$,

$$f^T(zI_n - A)^{-1} g = \frac{f_0 + f_1 z + \ldots + f_{n-1} z^{n-1}}{det(zI_n - A)}, \tag{29}$$

for $z \in \mathbb{C}$.

For convenience, we shall use $\kappa(\omega)$ and $\pi(\omega)$ to denote

$$\kappa(\omega) = 2\mathrm{Re}\{h^T m_{j\omega}\}, \qquad \pi(\omega) = m_{j\omega}^* D m_{j\omega}, \tag{30}$$

for $\omega \in \mathbb{R}$. Then:

(i) $\tau + \kappa(\omega) > 0$ for all $\omega \in \mathbb{R}$, and $\tau + \kappa(\omega) \to \tau$ as $|\omega| \to \infty$;

(ii) $\pi(\omega) > 0$ for all $\omega \in \mathbb{R}$ and $\pi(\omega) \to 0$ as $|\omega| \to \infty$.

It follows from (i) there exists a positive constant $m_\kappa > 0$ such that $\tau + \kappa(\omega) > m_\kappa$ for all $\omega \in \mathbb{R}$. Also, (ii) implies that there is some constant $M_\pi > 0$ such that $\pi(\omega) < M_\pi$ for all $\omega \in \mathbb{R}$. If we now choose $\varepsilon > 0$ with $\varepsilon < \frac{m_\kappa}{M_\tau}$ then it follows that for all $\omega \in \mathbb{R}$,

$$\tau + 2\mathrm{Re}\{h^T m_{j\omega}\} - \varepsilon m_{j\omega}^* D m_{j\omega} > 0. \tag{31}$$

It can be verified by calculation that the left hand side of (31) can be written in the form:

$$\begin{aligned} &\tau + m_{j\omega}^* h + h^T m_{j\omega} - \varepsilon m_{j\omega}^* D m_{j\omega} \\ &\qquad = \frac{\eta(\omega)}{\det(\omega^2 I_n + A^2)} \end{aligned} \tag{32}$$

where $\eta(.)$ is a polynomial with the following properties.

(i) $\eta(.)$ is a polynomial of degree $2n$ with real coefficients and leading coefficient τ. Thus, any non-real zeroes of $\eta(.)$ occur as complex conjugate pairs.

(ii) Only the even coefficients of η are non-zero. Thus, for any zero z_0 of $\eta(.)$, $-z_0$ is also a zero with the same multiplicity as z_0.

(iii) $\eta(\omega) > 0$ for all $\omega \in \mathbb{R}$. Thus, for any real zero, ω_0, of $\eta(.)$, ω_0 and $-\omega_0$ have the same *even* multiplicity.

It follows from the above considerations that there exists a polynomial $\theta(.)$ of degree n with real coefficients, and leading coefficient $\sqrt{\tau}$, such that

$$\eta(\omega) = \theta(j\omega)\theta(-j\omega), \tag{33}$$

for all $\omega \in \mathbb{R}$. Now, if we define $\psi(z) = \det(zI_n - A)$, then, as the leading coefficient of θ is $\sqrt{\tau}$,

$$\sqrt{\tau} - \frac{\theta(z)}{\psi(z)} = \frac{\nu(z)}{\psi(z)} \tag{34}$$

where $\nu(z) = q_0 + q_1 z + \cdots + q_{n-1}z^{n-1}$ is a polynomial of degree at most $n-1$. Thus, from (29)

$$\frac{\nu(z)}{\psi(z)} = q^T(zI_n - A)^{-1}h \tag{35}$$

where $q = (q_0, \ldots, q_{n-1})^T$.

For this vector q, it follows from Theorem 2.3 that there exists a symmetric matrix $P = P^T$ with $In(P) = In(-A)$ such that

$$A^T P + PA = -qq^T - \varepsilon D. \tag{36}$$

Moreover, combining (32), (35) and (34), we see that

$$\tau + m^*_{j\omega}h + h^T m_{j\omega} - \varepsilon m^*_{j\omega} D m_{j\omega} = |\sqrt{\tau} - q^T m_{j\omega} h|^2 \tag{37}$$

It now follows immediately that

$$\begin{aligned} m^*_{j\omega}h \quad &+ \quad h^T m_{j\omega} - \varepsilon m^*_{j\omega} D m_{j\omega} \\ &= \quad (-m^*_{j\omega}q + \sqrt{\tau})(-q^T m_{j\omega} + \sqrt{\tau}) - \tau \\ &= \quad m^*_{j\omega}qq^T m_{j\omega} - \sqrt{\tau}(q^T m_{j\omega} + m^*_{j\omega}q). \end{aligned}$$

We can now use (24) and (36) to obtain

$$\begin{aligned} m^*_{j\omega}h + h^T m_{j\omega} \quad &- \quad \varepsilon m^*_{j\omega} D m_{j\omega} = m^*_{j\omega}Pg + g^T P m_{j\omega} \\ &- \quad \varepsilon m^*_{j\omega} D m_{j\omega} - \sqrt{\tau}(q^T m_{j\omega} + m^*_{j\omega}q). \end{aligned} \tag{38}$$

After suitably rearranging the equations above we see that

$$\begin{aligned} m^*_{j\omega}Pg + g^T P m_{j\omega} \quad &- \quad m^*_{j\omega}h - h^T m_{j\omega} \\ &- \quad \sqrt{\tau}q^T m_{j\omega} - \sqrt{\tau} m^*_{j\omega} q = 0 \end{aligned} \tag{39}$$

and hence,

$$m_{j\omega}^*(Pg - h - \sqrt{\tau}q) + (Pg - h - \sqrt{\tau}q)^T m_{j\omega} = 0$$
$$\Rightarrow \qquad 2Re\{(Pg - h - \sqrt{\tau}q)^T m_{j\omega}\} = 0. \tag{40}$$

As (40) holds for any real value of ω, it now follows from Lemma 3.1 that $Pg - h = \sqrt{\tau}q$. This completes the proof of the theorem. □

3.2 Common Lyapunov solutions and the KYP Lemma

We shall now show how Theorem 3.1 can be used to obtain simple algebraic conditions for CLS existence for a significant class of pairs of matrices with the same regular inertia in $\mathbb{R}^{n\times n}$.

Theorem 3.2 *Let A, $A - gh^T$ be two matrices in $\mathbb{R}^{n\times n}$ in companion form and with the same regular inertia, $In(A) = In(A - gh^T) = (n_+, n_-, 0)$, where g, h are vectors in $\mathbb{R}^n$. Further, assume that for any pair of eigenvalues, λ_i, λ_j, of A, $\mathrm{Re}(\lambda_i + \lambda_j) \neq 0$. Then, the following statements are equivalent:*

(i) There exists a symmetric matrix $P = P^T$ in $\mathbb{R}^{n\times n}$ with $In(P) = In(-A) = In(-(A - gh^T))$, and positive definite matrices $Q_1 > 0$, $Q_2 > 0$ such that

$$A^T P + PA = -Q_1$$
$$(A - gh^T)^T P + P(A - gh^T) = -Q_2. \tag{41}$$

(ii) The matrix rays $\sigma_{\gamma[0,\infty)}(A, A - gh^T)$ and $\sigma_{\gamma[0,\infty)}(A^{-1}, A - gh^T)$ have the same regular inertia.

(iii) The matrix $A(A - gh^T)$ has no real negative eigenvalues.

(iv) $1 + \mathrm{Re}\{h^T(j\omega I_n - A)^{-1}g\} > 0, \quad \forall \omega \in \mathbb{R}$.

Proof: **(i) ⇒ (ii)**:
Suppose that there is a symmetric $P = P^T$ satisfying (41). From Lemma 2.3 we know that P also satisfies

$$((A - gh^T)^T)^{-1} P + P(A - gh^T)^{-1} < 0 \tag{42}$$

Hence for all $\gamma \in [0, \infty)$

$$\begin{aligned} (A+\gamma(A-gh^T))^T P + P(A+\gamma(A-gh^T)) &< 0 \quad (43) \\ (A+\gamma(A-gh^T)^{-1})^T P + P(A+\gamma(A-gh^T)^{-1}) &< 0 \quad (44) \end{aligned}$$

It now follows immediately from Theorem 2.3 that (ii) is true.

(ii) $\Rightarrow$ (iii):
Assume that (ii) is true. Then, $A^{-1}+\gamma(A-gh^T)$ has regular inertia for all $\gamma > 0$. In particular, $A^{-1}+\gamma(A-gh^T)$ is non-singular for all $\gamma > 0$. It follows immediately that the matrix product $A(A-gh^T)$ has no negative real eigenvalues.

(iii) $\Rightarrow$ (iv):
Assume that $A(A-gh^T)$ has no real negative eigenvalues. As $A, A-gh^T$ have the same regular inertia, it follows that

$$det(\omega^2 I_n + (A-gh^T)A) > 0 \quad (45)$$

for all $\omega \in \mathbb{R}$. This implies that

$$\begin{aligned} det(\omega^2 I_n + (A-gh^T)A) &> 0 \\ \Rightarrow det(I_n\omega^2 + A^2 - gh^T A) &> 0 \end{aligned}$$

and hence

$$det(\omega^2 I_n + A^2) det(I_n - (\omega^2 I_n + A^2)^{-1} gh^T A) > 0.$$

In this last relation we know that $det(\omega^2 I_n + A^2) > 0$ from Lemma 2.1. Thus we can conclude that

$$det(I_n - (\omega^2 I_n + A^2)^{-1} gh^T A) > 0 \quad (46)$$

for all $\omega \in \mathbb{R}$. Now making use of the identity $det(I_n - AB) = det(I_m - BA)$, (where $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times n}$) we can express the last inequality as follows;

$$det(1 - h^T A(\omega^2 I_n + A^2)^{-1} g) > 0. \quad (47)$$

Notice that the argument in the last relation is a scalar, and hence that

$$1 - h^T A(\omega^2 I_n + A^2)^{-1} g = T(\omega^2) > 0. \quad (48)$$

Now comparing this last equation with the result of Lemma 2.2, we see that

$$T(\omega^2) = 1 + Re\{h^T(j\omega I_n - A)^{-1}g\} > 0 \tag{49}$$

which proves (iv).

(iv) $\Rightarrow$ (i):
Finally, assume that (iv) is true. Choose some positive definite $D = D^T > 0$ in $\mathbb{R}^{n\times n}$. Then it follows from Theorem 3.1 (with $\tau = 2$) that there exists a symmetric $P = P^T$ with $In(P) = In(-A)$ and a vector q such that

$$A^T P + PA = -qq^T - \varepsilon D \tag{50}$$
$$Pg - h = \sqrt{2}q. \tag{51}$$

It can be verified by direct computation that this P is a common Lyapunov solution for A, $A - gh^T$. This completes the proof of the theorem. $\square$

Comments:

(i) Note that, in the language of [6], the above result establishes that for any pair of matrices $A, A - gh^T$ in $\mathbb{R}^{n\times n}$ satisfying the hypotheses of the theorem, CLS existence is equivalent to the convex invertible cone generated by $A, A - gh^T$ having constant regular inertia.

(ii) It is sufficient that either one of A, or $A - gh^T$ satisfy the spectral assumption that $\mathrm{Re}(\lambda_i + \lambda_j) \neq 0$ for any pair of eigenvalues λ_i, λ_j of the matrix.

4 Further results on common Lyapunov solutions for matrices with regular inertia

In this section, we shall show that the principal result of [4] on CLS existence for pairs of Hurwitz matrices extends naturally to the more general case of matrices with the same regular inertia. Note that in [20] this result was shown to provide a unifying framework for two of the most significant classes for which conditions for CLS existence are known; namely the class of Hurwitz matrices in $\mathbb{R}^{2\times 2}$ and the class of Hurwitz matrices in $\mathbb{R}^{n\times n}$ in companion form.

The main result established in [4] was concerned with a pair of Hurwitz matrices A_1, A_2 in $\mathbb{R}^{n\times n}$ with no common Lyapunov solution, but for which there exists some common solution $P = P^T \geq 0$ to the *weak* Lyapunov inequalities

$$A_i^T P + PA_i \;\; = \;\; -Q_i \leq 0 \tag{52}$$

where Q_i has rank $n-1$ for $i = 1,2$. It was shown that in these circumstances at least one of the matrix products $A_1A_2, A_1A_2^{-1}$ must have a negative real eigenvalue. Equivalently, one of the matrix rays $\sigma_{\gamma[0,\infty)}[A_1^{-1}, A_2]$, $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is singular. In the later paper [20], it was shown how this fact can be used to derive necessary and sufficient conditions for common Lyapunov solution existence for pairs of Hurwitz matrices in $\mathbb{R}^{2\times 2}$ as well as for pairs of Hurwitz matrices in companion form in $\mathbb{R}^{n\times n}$. Thus, the above fact connects the known conditions for common Lyapunov solution existence for 2×2 matrices and the classical SISO Circle Criterion for matrices in companion form. In Theorem 4.1, we show that this same fact holds when we relax the assumption that the matrices are Hurwitz and only require that they have the same regular inertia. First of all, we recall the following preliminary result from [4] which is needed for the proof of the theorem.

Lemma 4.1 *Let x, y, u, v be non-zero vectors in $\mathbb{R}^n$. Suppose that there is some $k > 0$ such that for all symmetric matrices $P \in Sym(n, \mathbb{R})$*

$$x^T P y = -k u^T P v.$$

Then either

$$x = \alpha u \text{ for some real scalar } \alpha, \text{ and } y = -(\frac{k}{\alpha})v$$

or

$$x = \beta v \text{ for some real scalar } \beta \text{ and } y = -(\frac{k}{\beta})u.$$

Theorem 4.1 *Let A_1, A_2 be two matrices in $\mathbb{R}^{n\times n}$ with the same regular inertia. Suppose that A_1 and A_2 have no strong common Lyapunov solution and, furthermore, that there exists some symmetric $P = P^T$ in $\mathbb{R}^{n\times n}$ such that*

$$A_i^T P + PA_i = -Q_i \leq 0 \quad \text{for } i = 1,2 \tag{53}$$

where $\mathrm{rank}(Q_1) = \mathrm{rank}(Q_2) = n-1$. *Then at least one of the matrix products $A_1A_2^{-1}$, A_1A_2 has a negative real eigenvalue and, equivalently, at least one of the matrix rays $\sigma_{\gamma[0,\infty)}[A_1, A_2]$, $\sigma_{\gamma[0,\infty)}[A_1^{-1}, A_2]$ is singular.*

Proof: As Q_1 and Q_2 have rank $n-1$, there are unique vectors x_1, x_2 in $\mathbb{R}^n$ such that $x_i^T x_i = 1$, $Q_i x_i = 0$ for $i = 1,2$. There are two major steps involved in the proof of this result.

Step 1:

We shall show by contradiction that, under the hypotheses of the theorem, there cannot exist any symmetric H in $\mathbb{R}^{n\times n}$ such that $x_i^T HA_i x_i < 0$ for $i = 1,2$.

Now, suppose that there is some $H = H^T$ in $\mathbb{R}^{n\times n}$ with $x_1^T HA_1 x_1 < 0$, $x_2^T HA_2 x_2 < 0$. For $i = 1,2$, let Ω_i denote the cone

$$\Omega_i = \{x \in \mathbb{R}^n : x^T x = 1, x^T HA_i x \geq 0\}.$$

We shall now show that there exists a positive constant $C_1 > 0$ such that $A_1^T(P + \delta H) + (P + \delta H)A_1 < 0$ provided $0 < \delta < C_1$. If Ω_1 is empty, then any positive C_1 will have this property. Henceforth we shall assume that Ω_1 is non-empty.

Note that the set Ω_1 is closed and bounded and hence compact. Moreover, x_1 or any scalar multiple of x_1 is not in Ω_1 and hence $x^T PA_1 x < 0$ for all $x \in \Omega_1$. Let M_1 denote the maximum value of $x^T PA_1 x$ on Ω_1 and m_1 denote the maximum value of $x^T HA_1 x$ on Ω_1. Then $M_1 < 0$ and if we set $C_1 = |M1|/(m_1 + 1)$ it follows by considering the cases $x \in \Omega_1$, $x \notin \Omega_1$ separately that for all vectors x of Euclidean norm 1,

$$x^T((P + \delta H)A_1)x < 0,$$

and hence $A_1^T(P + \delta H) + (P + \delta H)A_1 < 0$ provided $0 < \delta < C_1$.

An identical argument can be used to show that there exists some constant $C_2 > 0$ such that $A_2^T(P + \delta H) + (P + \delta H)A_2 < 0$ provided $0 < \delta < C_2$. Now choose any δ with $0 < \delta < \min\{C_1, C_2\}$ and it follows that

$$A_i^T(P + \delta H) + (P + \delta H)A_i < 0$$

for $i = 1,2$. The General Inertia Theorem 2.3 now implies that $In(P) = In(-A_1) = In(-A_2)$ and that P is a common Lyapunov solution for A_1, A_2 contradicting the assumptions of the theorem.

Step 2:

As there is no symmetric H in $\mathbb{R}^{n\times n}$ with $x_1^T HA_1 x_1 < 0$, $x_2^T HA_2 x_2 < 0$, it follows that the null spaces of the two linear functionals (defined on the space of symmetric matrices in $\mathbb{R}^{n\times n}$) $H \to x_1^T HA_1 x_1$, $H \to x_2^T HA_2 x_2$ coincide, and that there must be some constant $k > 0$ such that

$$x_1^T H A_1 x_1 = -k x_2^T H A_2 x_2 \tag{54}$$

for all $H = H^T$ in $\mathbb{R}^{n\times n}$.

Now Lemma 4.1 implies that either $x_1 = \alpha x_2$ with $A_1 x_1 = -(\frac{k}{\alpha})A_2 x_2$ for some real α, or $x_1 = \beta A_2 x_2$ and $A_1 x_1 = -(\frac{k}{\beta})x_2$ for some real β. Consider the former situation to begin with. Then we have

$$\begin{aligned} A_1(\alpha x_2) &= -(\frac{k}{\alpha})A_2 x_2 \\ \implies (A_1 + (\frac{k}{\alpha^2})A_2)x_2 &= 0 \end{aligned}$$

and thus the matrix ray $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is singular and the matrix $A_1 A_2^{-1}$ has a negative real eigenvalue. A similar argument shows that in the latter case, $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ is singular and the matrix product $A_1 A_2$ has a negative real eigenvalue. This completes the proof of Theorem 4.1. □

5 Concluding remarks

In this paper we derived a verifiable spectral condition for common Lyapunov solution (CLS) existence for pairs of matrices in $\mathbb{R}^{n\times n}$ in companion form with the same regular inertia; thereby extending a recent result for pairs of Hurwitz matrices in [3]. We have also shown that the principal result of [4] extends directly to the regular inertia case.

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