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ZERO DIVISOR GRAPHS OF POSETS

A Thesis submitted to the
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in Mathematics

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Abstract

In 1988, Beck [10] introduced the notion of coloring of a commutative ring R. Let G be a simple graph whose vertices are the elements of R and two vertices x and y are adjacent if $xy = 0$. The graph G is known as the zero divisor graph of R. He conjectured that, the chromatic number $\chi(G)$ of G is same as the clique number $\omega(G)$ of G. In 1993, Anderson and Naseer [1] gave an example of a commutative local ring R with 32 elements for which $\chi(G) > \omega(G)$.

Further, this concept of zero divisor graphs is well studied in algebraic structures such as rings, semigroups; see Anderson et. al. [1, 2], F. DeMeyer et. al. [14, 15], LaGrange [31, 32], Redmond [53, 54], and in ordered structure such as lattices, meet-semilattices, posets and qosets; see Alizadeh et. al. [9], Estaji and Khashyarmanesh [17], Halaš and Länger [21], Joshi et. al. [27, 28, 29], Lu and Wu [37], Nimbhorkar et. al. [48, 49, 68].

In this Thesis, we deal with the basic properties such as connectivity, diameter, girth (gr), eccentricity (e), radius (r), center, cut-set, clique number (ω), chromatic number (χ), domination number (γ) etc. of the

zero divisor graph of a poset and its complement. This Thesis contains four chapters.

In the first Chapter, we relate lattice properties of a distributive lattice L with graph properties of the corresponding zero divisor graph, $G_{\{0\}}(L)$. Cycles in $G_{\{0\}}(L)$ are investigated. Also an algebraic and a topological characterization is given for the graph $G_{\{0\}}(L)$ to be triangulated or hyper-triangulated. Further, we study edge chromatic number χ' of $G_{\{0\}}(L)$.

In Chapter two, we study the zero divisor graph of a Boolean poset. We determine the diameter, radius, center, eccentricity and domination number of the zero divisor graph of a Boolean poset.

It is easy to observe that the non-isomorphic posets may have isomorphic zero divisor graph. In view of this, the following problem is worth to study.

Problem 2: Find the class $\mathcal{P}$ of posets for which $G(P) \cong G(Q)$ if and only if $P \cong Q$ for $P, Q \in \mathcal{P}$.

We partially answer this problem by proving that the class of Boolean posets is contained in $\mathcal{P}$.

One of the main problems in the theory of zero divisor graphs is the realization of zero divisor graphs. LaGrange [32] characterized the graphs which are realizable as zero divisor graphs of Boolean rings while Lu and Wu [37] considered this problem for general posets. In this Chapter, we also considered the realization problem for Boolean posets.

Chapter three deals with $(G_I(L))^c$, the complement of the zero-divisor graph $G_I(L)$ with respect to a semiprime ideal I of a bounded lattice L. We have obtained necessary and sufficient conditions for $(G_I(L))^c$ to be connected and also determined its diameter, radius, center and girth. Further, we have calculated the vertex chromatic number of $(G_I(L))^c$, where $L = \mathbf{2}^n$. Also a form of Beck's conjecture is proved for $G_I(L)$ when $\omega((G_I(L))^c) < \infty$. In the last section of this Chapter, we study the cut-sets in $(G_I(L))^c$.

The fourth Chapter is devoted for the study of matrices over lattices. The concept of matrices over Boolean algebra was first studied by Luce [38]. He proved that the set of matrices over a Boolean algebra forms a lattice ordered semigroup with zero. Rutherford [55] studied the eigenvalue problem for Boolean matrices. Further, this concept was generalized for more general class of distributive lattices by Tan in [60, 62], see also [30, 40, 61]. Redmond [53] introduced the concept of zero divisor graphs over a non-commutative ring. Among non-commutative rings, matrix rings have received special attention in [11, 13, 42]. Motivated with the work of Redmond [53] about zero divisor graphs of non-commutative rings, we have studied the zero divisor graphs of matrices over lattices.

In this Chapter, we study the basic properties such as connectivity, diameter and girth of the zero divisor graph $\Gamma(M_n(L))$ of $n \times n$ matrices over a lattice L with 0. Further, we consider the zero divisor graph $\Gamma(M_2(C_n))$ of 2×2 matrices over an n-element chain C_n. We have determined the domination number of $\Gamma(M_2(C_n))$. Also we have shown that Beck's Conjecture is true for $\Gamma(M_2(C_n))$. Further, we have proved that $\Gamma(M_2(C_n))$ is a hyper-triangulated graph.

Lemmas, Theorems, Remarks and Definitions are numbered consequently in each chapter without making distinction between them. Figures are sectionally numbered. References and Index are given at the end of the Thesis. References are listed alphabetically and yearwise. The end of the proof is indicated by the symbol $\square$.

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