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Coding Theory: An approach through metrics which respect support, and other issues

Teoria de Códigos: Uma abordagem usando métricas que respeitam suporte e outros problemas

Campinas

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Supervisor: Marcelo Firer

Este exemplar corresponde à versão final da Tese defendida pelo aluno Roberto Assis Machado e orientada pelo Prof. Dr. Marcelo Firer.

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Resumo

Esta tese estuda métricas definidas por peso e respeitam o suporte dos vetores (TS-métricas) no contexto de teoria de códigos. Sua principal preocupação, considerando famílias específicas de métricas, é explorar e entender alguns resultados "estruturais"das métricas, a saber: descrever o grupo de isometrias lineares e, estabelecer condições para a validade da Identidade MacWilliams (uma relação entre a distribuição de peso de um código e a distribuição de peso - possivelmente de um peso modificado - do código dual) e da propriedade de extensão MacWilliams (quando isometrias lineares entre códigos lineares podem ser estendidas para isometrias lineares em todo o espaço). Esses resultados são primeiro explorados para duas famílias de TS-métricas: as métricas combinatoriais de Gabidulin e as métricas de posets-bloco-rotulados, uma nova família de TS-métricas introduzida neste trabalho. Além disso, é apresentada uma abordagem sistemática ao espaço de todas as TS-métricas, trabalhando em três frentes. Primeiro mostramos que toda TS-métrica pode ser obtida por meio de rotulagem das arestas do cubo de Hamming. Em seguida, introduzimos um operador condicional entre TS-métricas e mostramos que toda TS-métrica pode ser obtida como uma sequência de somas condicionais de métricas poset ou métricas combinatoriais. Ainda considerando as TS-métricas, introduzimos um conceito de dualidade de métrica que generaliza o conceito existente em todas as instâncias conhecidas nas quais vale uma identidade de MacWilliams. Finalmente, como um assunto a parte, apresentamos alguns resultados relevantes em relação à representação de dígrafos.

Palavras-chave: Métricas que respeitam suporte, Identidade de MacWilliams, soma condicional, dualidade de métricas.

Abstract

This thesis concerns about metrics defined by weights that respect vector support (TS-metrics) in the context of code theory. The main goal, considering specific families of metrics, is to explore and understand some “structural” metrics results, namely: to describe the group of linear isometries and to establish conditions for the validity of the MacWilliams Identity (a relationship between the weight distribution of a code and the weight distribution - possibly of a modified weight - of its dual code) and the MacWilliams extension property (when linear isometries between linear codes can be extended to linear isometries in the whole space). These results are explored for two TS-metric families: Gabidulin’s combinatorial metrics and labeled-poset-block metrics, a new family of TS-metrics which is introduced in this work. Also, we present a systematic approach to the space of all TS-metrics, working in three different ways: First, we show that every TS-metric can be obtained by labeling the edges of the Hamming cube. Next, we introduce a conditional operator between TS-metrics and show that every TS-metric can be obtained as a sequence of conditional sums between poset or combinatorial metrics. Still considering TS-metrics, we introduced a concept of metric duality that generalizes the existing ones in all known instances in which a MacWilliams identity holds. Finally, as a separate subject, we present some relevant results regarding the representation of digraphs.

Keywords: Metrics which respect support, MacWilliams’ Identity, conditional sum, metrics duality.

Contents

Introduction

In coding theory, there are two main sources of decoding criteria: a probabilistic (Maximum Likelihood Decoding - MLD) and a metric (Minimum Distance Decoding - MDD). While the first one focuses on the properties of the channel and is the optimal criterion (in terms of minimizing the error probability of the encoding-transmission-decoding process), the last is an approximation to the true value (MLD) that may help in the implementation of decoding algorithms.

The most usual and important channel is the binary symmetric channel, for which the MLD criterion matches the MDD criterion determined by the Hamming metric. This means that given any code $C \subset V$ and any received message $x \in V$, then the MLD and MDD criteria generate the same set of codewords, i.e., $\arg\max_{c\in C} P(x|c) = \arg\min_{c\in C} d_H(x,c)$, for all $x \in V$.

Besides this crucial role, of being matched to the binary symmetric channel, the Hamming metric has two properties that can be stated in a more general setting:

P1 *Weight condition:* The metric d_H is determined by the Hamming weight wt_H, i.e., $d_H(u,v) = \mathrm{wt}_H(u-v)$.

P2 *Support condition:* If the vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are such that $\mathrm{supp}(u) \subset \mathrm{supp}(v)$, then $\mathrm{wt}_H(u) \leq \mathrm{wt}_H(v)$, where $\mathrm{supp}(u) = \{i \in \{1, \ldots, n\} : u_i \neq 0\}$. In this case we say that the metric/weight *respects support*.

The first of these properties makes metrics suitable for working with linear codes. It ensures that the minimal distance equals the minimal weight and it is a necessary and sufficient condition for performing MDD through the syndrome decoding algorithm. We should remark that a metric d is defined by a weight if, and only if, it is invariant by translations, in the sense that $d(u+x, v+x) = d(u,v)$, for all $u, v, x \in \mathbb{F}_q^n$. The second property makes it meaningful in the context of coding theory, in the sense that, considering binary codes, no extra error can improve the result.

Metrics satisfying both of these properties are called *TS-metrics*, where **T** refers to translation invariance and **S** to the support condition.

Many of the metrics that have been explored in the literature in the context of Coding Theory satisfy both properties (see, for example, a recent survey of Gabidulin (GABIDULIN, 2012)). We cite, for example, the combinatorial metrics (GABIDULIN, 1973) and the poset metrics (BRUALDI; GRAVES; LAWRENCE, 1995), as much as its generalizations: poset-block metrics (ALVES; PANEK; FIRER, 2008), digraph metrics (ETZION; FIRER; MACHADO, 2017), weighted-posets (HYUN; KIM; PARK, 2017), etc.

This thesis studies TS-metrics in the context of coding theory. First of all, we consider two families of such metrics, namely the *combinatorial* and the *labeled-poset-block* metrics. These are two large families of TS-metrics, with very small intersection. The first family was introduced in 1973 and the second one is firstly introduced in this work. Neither is studied in the literature, so we start exploring some "*structural*" results of the metrics, namely: to describe the group of linear isometries and to establish conditions for the validity of the MacWilliams Identity and the MacWilliams extension property.

The long term goal is to contribute to a better understanding of the space of all TS-metrics. We do it by exploring the role of two distinguished families of such metrics: Gabidulin's combinatorial metrics and the poset metrics. These two families are roughly disjoint (the intersection of the two families contains only the Hamming metric) and despite of being large families of metrics (in the sense that the number of different metrics grows exponentially with n) they are not sufficient to describe all possible TS-metrics, not even by considering the equivalence classes of metrics[1]. However, any TS-metric can be approximated by combinatorial and poset metrics, in the following sense: we define an operation in the space of TS-metrics, the *conditional sum of metrics*, which allows to obtain any TS-metric as a conditional sum of combinatorial and poset metrics. In some sense, the treatment of different families of TS-metrics can be unified by labeling the edges of the Hamming cube and, in this very generic setting, we can give a useful description of the group of linear isometries. Finally, we explore the concept of *dual weight and metric*, which allows us to produce Mac Williams' type identities in a very generic setting. We show that this concept coincides with the usual concept of dual poset metric (or dual graph metric) in every instance where a MacWilliams Identity is known to hold.

Besides this systematic study of TS-metrics, we present some results in a different direction, concerning the coloring of digraphs. These results were obtained while visiting Prof. Olgica Milenkovic in 2018.

The work is organized as follows:

Chapter 1: Here we study Gabidulin's combinatorial metrics, an important and unexplored instance of TS-metrics. First, we describe the group of linear isometries of combinatorial metrics and characterize those that admit a MacWilliams-type identity. Considering the binary case, we classify the metrics satisfying the MacWilliams extension property (for disconnected coverings) and, for connected coverings, we give a necessary condition for the extension property hold.

Chapter 2: We introduce the labeled-poset-block metrics, a new family of TS-metrics, a generalization of the digraph metrics. We give a full description of the

[1] We say that two metrics on $\mathbb{F}_q^n$ are *equivalent* if they determine the same MDD criterion, in every possible instance. A precise definition is introduced in Chapter 3.

group of linear isometries and determine sufficient conditions to ensure the existence of a MacWilliams' identity and for the validity of the MacWilliams extension property.

Chapter 3: Here we start to look at the set of all TS-metrics. We consider it as a space with a structure determined by an equivalence relation which arises naturally in the context of Coding Theory. We show how these metrics can be obtained from the edge-labeled Hamming cube and, based on this representation, we could obtain a description for the group of linear isometries (for $q > 2$). Next, we introduce the concept of conditional sum of metrics and determine what are the conditions that, out of two metrics respecting the support, the conditional sum give rise to a new such metric. Considering these operators, we show that poset and combinatorial metrics can be considered as generators of the space of TS-metrics, in the sense that any equivalence class contains a representative that can be obtained by a finite conditional sum of poset and combinatorial metrics.

Chapter 4: In this chapter, we reap important results in coding theory for TS-metrics. We introduce a very broad definition of duality of a weight (and hence of a TS-metric) that allows us to understand the concept of a MacWilliams' Identity in a very general sense. Then, we derive an algorithmic result to decide if a MacWilliams' Identity exists and, in the positive case, the identity depends only on the set of metric-spheres, and not on particular codes. The MacWilliams identity for many of the known cases of TS-metrics (Hamming metric, hierarchical posets, combinatorial metrics, and so on), are particular instances of this result, so we should call it a generalized MacWilliams identity.

Chapter 5: This chapter is independent of the previous ones. Consider a directed graph (digraph). To each vertex it is assigned a set of colors, under the following conditions: two vertices are connected if and only if they share at least one color and the head vertex has a strictly smaller color set than the tail. We seek to estimate the smallest possible color set that can explain the observed digraph topology. To address this problem, we introduce the new notion of a directed intersection representation of a digraph and show that it is well-defined for all directed acyclic graphs (DAGs). We then proceed to introduce the directed intersection number (DIN), the smallest number of colors needed to represent a DAG. Our main results are upper and lower bounds on the DIN of DAGs.

1 Combinatorial metrics

The family of combinatorial metrics, an instance of TS-metrics, was introduced in 1973 by E. M. Gabidulin (GABIDULIN, 1973) and it attends some usefulness condition: "The b-burst metric can be considered as a combinatorial metric" (GABIDULIN, 2012). The study of combinatorial metrics rested nearly untouched since its introduction and just recently, after they were mentioned in a survey of 2012 (GABIDULIN, 2012), the interest in these metrics has returned. In order to determine the manageability of such metrics, it is necessary to explore the details of their geometry. This is the direction we work here.

Some subfamilies of combinatorial metrics have been widely explored in the literature, for example, the block metrics in (FENG; XU; HICKERNELL, 2006) and the translational metrics in (MOHAMED; BOSSERT, 2015). In the general setting of combinatorial metrics, we can find very few papers. As one of the few exceptions, we cite the work (BOSSERT; SIDORENKO, 1996) concerning Singleton-type bounds.

The structural coding properties like the MacWilliams' Identities and the MacWilliams' Extension property had not been yet explored in this context. In this chapter we characterize the combinatorial metrics having a MacWilliams-type Identity and we describe the group of linear isometries of such metrics. Concerning the extension property, we have two different cases. If the covering is not connected, we give necessary and sufficient conditions for a combinatorial metric to satisfy the extension property. For the connected case, we give a necessary condition, which we believe is also a sufficient one.

This chapter is organized as follows. In Section 1.1 we present the definition of the combinatorial metric and some basic properties of these metrics. In Section 1.2 we characterize the combinatorial metrics that admits a MacWilliams' identity. In Section 1.3 we describe the group of linear isometries of a space endowed with a combinatorial metric (over an alphabet with $q > 2$ elements). In Section 1.4 we give necessary and sufficient conditions for a disconnected covering to determine a metric which satisfies an extension property of isometries, similar to the MacWilliams Extension Theorem. For the connected case we present a necessary condition.

1.1 Preliminaries

Let $\mathbb{F}_q^n$ be the n-dimensional vector space over the field $\mathbb{F}_q$, $[n] := \{1, \dots, n\}$ and $\mathcal{P}_n := \{A : A \subset [n]\}$ the power set of $[n]$. We say that a family $\mathcal{A} \subset \mathcal{P}_n$ is a *covering* of a set $X \subset [n]$ if $X \subset \cup_{A \in \mathcal{A}} A$. If $\mathcal{F}$ is a covering of $[n]$, then the *$\mathcal{F}$-combinatorial weight* of $u = (u_1, \ldots, u_n) \in \mathbb{F}_q^n$ is the integer-valued map $\mathrm{wt}_{\mathcal{F}}$ defined by

$$\mathrm{wt}_{\mathcal{F}}(u) = \min\{|\mathcal{A}| \ : \ \mathcal{A} \subset \mathcal{F} \text{ and } \mathcal{A} \text{ is a covering of } \mathrm{supp}(u)\},$$

where $\mathrm{supp}(u) := \{i \in [n] : u_i \neq 0\}$ is the *support* of u. Each element $A \in \mathcal{F}$ is called a *basic set* of the covering.

As showed in (GABIDULIN, 1973), the function $d_{\mathcal{F}} : \mathbb{F}_q^n \times \mathbb{F}_q^n \to \mathbb{N}$ defined by

$$d_{\mathcal{F}}(u, v) = \mathrm{wt}_{\mathcal{F}}(u - v)$$

satisfies the axioms of a metric and is called an *$\mathcal{F}$-combinatorial metric.*

Example 1.1.1 (Block Metrics, (FENG; XU; HICKERNELL, 2006))**.** *Suppose that $\mathcal{F}$ is a partition of $[n]$, that is, the basic sets are pairwise disjoint. In this case, the $\mathcal{F}$-combinatorial metric is also called a* block metric*. The particular case, in which every basic set is a singleton ($\mathcal{F} = \{\{1\}, \{2\}, \ldots, \{n\}\}$), determines the classical Hamming metric.*

Example 1.1.2 (b-burst Metric, (Bridwell; Wolf, 1970))**.** *Given an integer b, define $[b] + i := \{1 + i, 2 + i, \ldots, b + i\}$. The family*

$$\mathcal{F}_{[b]} = \{[b], [b] + 1, [b] + 2, \ldots, [b] + (n - b)\}$$

is a covering of $[n]$, where we assume that $b \leq n$. The metric determined by $\mathcal{F}_{[b]}$ is called the b-burst metric*.*

Consider the coverings $\mathcal{F}_1 = \{[n]\}$ and $\mathcal{F}_2 = \{[n], B\}$ where $B \subset [n]$ is any nonempty subset. Both coverings determine the same metric: for every $x, y \in \mathbb{F}_q^n$,

$$d_{\mathcal{F}_1}(u, v) = d_{\mathcal{F}_2}(u, v) = \begin{cases} 0 \text{ if } u = v \\ 1 \text{ if } u \neq v \end{cases}.$$

In order to eliminate multiplicity (different coverings determining the same metric), we will define the *redundancy* of basic sets: given a covering $\mathcal{F}$, we say that $A \in \mathcal{F}$ is *$\mathcal{F}$-redundant* (or just *redundant*) if there is $B \in \mathcal{F}$, with $A \subset B$ and $A \neq B$. We denote by $\overline{\mathcal{F}}$ the family of all redundant basic sets. In the sequence we will present two propositions whose proofs follow straightforward from the previous definitions.

Proposition 1.1.3. *Given a covering $\mathcal{F}$ of $[n]$, the set $\mathcal{F}_2 = \mathcal{F} \setminus \overline{\mathcal{F}}$ is also a covering of $[n]$ and the metrics $d_{\mathcal{F}}$ and $d_{\mathcal{F}_2}$ are equal.*

Proposition 1.1.4. *Two different coverings with no redundancy determine different metrics.*

Due to Proposition 1.1.3, we may (and will) assume that $\mathcal{F}$ has no redundancy. We end this section with a definition which will be used many times later.

Definition 1.1.5. *A covering $\mathcal{F}$ is called a k-*partition *if it is a partition of $[n]$ ($A \cap B = \emptyset$, for all $A, B \in \mathcal{F}, A \neq B$) and every $A \in \mathcal{F}$ has constant cardinality $k = |A|$. In this case, the $\mathcal{F}$-combinatorial metric is called an $(\mathcal{F}, k)$-combinatorial metric.*

We remark that different k-partitions define different metrics. However, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are two k-partitions of $[n]$, there is a permutation $\sigma \in S_n$ that transforms one partition into the other, in the sense that $A \in \mathcal{F}_1 \iff \sigma(A) \in \mathcal{F}_2$. It follows that the action of σ on $\mathbb{F}_q^n$, $\sigma(x_1, x_2, \ldots, x_n) = (x_{\sigma(1)}, x_{\sigma(2)}\ , \ldots x_{\sigma(n)})$ defines an isometry between the two metric spaces. For this reason, while considering a k-partition, we need not to specify a particular one.

1.2 MacWilliams' Identities

The classical MacWilliams identity, presented in (MACWILLIAMS, 1963), is a remarkable result in coding theory that relates, in the case of the Hamming metric, the weight enumerators of a code and of its dual code. When another metric is in place, to establish such relations may not be possible, as we can see in the counterexamples for the Lee metric constructed in (SHI; SHIROMOTO; SOLÉ, 2015) and in the classification of poset-block metrics admitting a MacWilliams-type identity presented in (Pinheiro; Firer, 2012).

Despite the fact that MacWilliams identity had not been studied in the context of combinatorial metrics, the case of block metrics is included in the cases studied in (Pinheiro; Firer, 2012) and this is a special instance of combinatorial metrics: those determined by partitions. We now classify all combinatorial metrics which admits a MacWilliams type identity. We start with some basic definitions.

As usual, the dual of a linear code $\mathcal{C} \subset \mathbb{F}_q^n$ is the linear code $\mathcal{C}^{\perp} = \{u \in \mathbb{F}_q^n \ : u \cdot c = \sum_{i=1}^{n} u_i c_i = 0$ for every $c \in \mathcal{C}\}$.

The $\mathcal{F}$- *weight enumerator* of a code $\mathcal{C}$ is the polynomial

$$W_{\mathcal{C}}^{\mathcal{F}}(x) = \sum_{c \in \mathcal{C}} x^{\mathrm{wt}_{\mathcal{F}}(c)} = \sum_{i=0}^{D} A_i x^i,$$

where $A_i = |\{c \in \mathcal{C} \ : \ \mathrm{wt}_{\mathcal{F}}(c) = i\}|$. When no confusion may arise, we write $W_{\mathcal{C}}(x)$, omitting the index $\mathcal{F}$.

Definition 1.2.1. *A combinatorial metric $d_{\mathcal{F}}$ admits a MacWilliams-type identity if the $\mathcal{F}$-weight enumerator of a code determines the $\mathcal{F}$-weight enumerator of its dual, i.e., if $W_{\mathcal{C}_1}(x) = W_{\mathcal{C}_2}(x)$ then $W_{\mathcal{C}_1^{\perp}}(x) = W_{\mathcal{C}_2^{\perp}}(x)$.*

Restating the results of (FENG; XU; HICKERNELL, 2006) in terms of combinatorial metrics, we have the following:

Proposition 1.2.2. *(FENG; XU; HICKERNELL, 2006) Suppose $\mathcal{F}$ is a partition of $[n]$. The combinatorial metric $d_{\mathcal{F}}$ admits a MacWilliams-type identity if, and only if, $d_{\mathcal{F}}$ is an $(\mathcal{F}, k)$-combinatorial metric, for some $k \in \mathbb{N}$.*

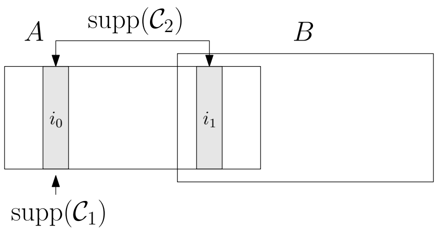
Our goal is to prove that these are all the combinatorial metrics admitting a MacWilliams identity.

Proposition 1.2.3. *Let $d_{\mathcal{F}}$ be a combinatorial metric. If $d_{\mathcal{F}}$ satisfies a MacWilliams-type identity then $\mathcal{F}$ is a partition and $d_{\mathcal{F}}$ is an $(\mathcal{F}, k)$-combinatorial metric for some k.*

Proof. If $\mathcal{F}$ is a partition of $[n]$, the result follows from Proposition 1.2.2.

Suppose $\mathcal{F}$ is not a partition (in particular, $d_{\mathcal{F}}$ is not an $(\mathcal{F}, k)$-combinatorial metric). This means there are $A, B \in \mathcal{F}$ such that $A \cap B \neq \emptyset$ and $A \neq B$. We shall prove that $d_{\mathcal{F}}$ does not satisfy a MacWilliams-type identity.

Let $i_1 \in A \cap B$. Assuming that $\mathcal{F}$ has no redundancy we find that there is $i_0 \in A \setminus B$. We denote by e_i the vector in $\mathbb{F}_q^n$ having the i-th coordinate equals to 1 and $\text{supp}(e_i) = \{i\}$. Consider the unidimensional codes over $\mathbb{F}_q$ with length n given by $\mathcal{C}_1 = span\{e_{i_0}\}$ and $\mathcal{C}_2 = span\{e_{i_0} + e_{i_1}\}$.



By direct computations we conclude that

$$W_{\mathcal{C}_1}(x) = W_{\mathcal{C}_2}(x) = 1 + (q-1)x.$$

Given $c = (c_1, \ldots, c_n) \in \mathcal{C}_1^{\perp}$, we get that $c - c_{i_1}e_{i_0} \in \mathcal{C}_2^{\perp}$, since $c_{i_0} = 0$, and hence we get a linear map $T : \mathcal{C}_1^{\perp} \to \mathcal{C}_2^{\perp}$ by setting $T(c) = c - c_{i_1}e_{i_0}$. By construction, T is an injection, hence a bijection, because $\dim(\mathcal{C}_1) = \dim(\mathcal{C}_2)$.

Claim: The map T preserves weight ($\text{wt}_{\mathcal{F}}(c) = \text{wt}_{\mathcal{F}}(T(c))$ for every $c \in \mathcal{C}_1^{\perp}$) if, and only if, $W_{\mathcal{C}_1^{\perp}}(x) = W_{\mathcal{C}_2^{\perp}}(x)$.

Proof of the claim: If T preserves weight, then, by definition, $W_{\mathcal{C}_1^{\perp}}(x) = W_{\mathcal{C}_2^{\perp}}(x)$.

Suppose now that $W_{\mathcal{C}_1^{\perp}}(x) = W_{\mathcal{C}_2^{\perp}}(x)$. Since $\text{supp}(c) \subset \text{supp}(T(c))$, it follows that $\text{wt}_{\mathcal{F}}(c) \leq \text{wt}_{\mathcal{F}}(T(c))$. On the other hand, consider $\mathcal{A} \subset \mathcal{F}$ to be a covering of $\text{supp}(c)$

such that $|\mathcal{A}| = \mathrm{wt}_{\mathcal{F}}(c)$. Since $\mathrm{supp}(T(c)) \setminus \mathrm{supp}(c) \subset \{i_0\}$, it follows that $\mathrm{supp}(T(c)) \subset \mathcal{A} \cup \{C\}$ for every $C \in \mathcal{F}$ with $i_0 \in C$. Hence, $\mathrm{wt}_{\mathcal{F}}(T(c)) \leq |\mathcal{A}|+1 = \mathrm{wt}_{\mathcal{F}}(c)+1$. Therefore,

$$\mathrm{wt}_{\mathcal{F}}(c) \leq \mathrm{wt}_{\mathcal{F}}(T(c)) \leq \mathrm{wt}_{\mathcal{F}}(c) + 1.$$

Let $D = \max\{\mathrm{wt}_{\mathcal{F}}(c) : c \in \mathcal{C}_1^{\perp}\} = \max\{\mathrm{wt}_{\mathcal{F}}(c) : c \in \mathcal{C}_2^{\perp}\}$. From the inequalities in (1.1) we find that every codeword of $\mathcal{C}_1^{\perp}$ with $\mathcal{F}$-weight D is mapped into a codeword of $\mathcal{C}_2^{\perp}$ with the same weight D. Considering now codewords of $\mathcal{C}_1^{\perp}$ with $\mathcal{F}$-weight $D-1$, the inequalities obtained in (1.1) ensure that they are mapped either into codewords with weight $D-1$ or D. Since T is a bijection and we are assuming that $W_{\mathcal{C}_1^{\perp}}(x) = W_{\mathcal{C}_2^{\perp}}(x)$, all the codewords of $\mathcal{C}_2^{\perp}$ with weight D are image of codewords of $\mathcal{C}_1^{\perp}$ having weight D, hence, if $c \in \mathcal{C}_1^{\perp}$ and $\mathrm{wt}_{\mathcal{F}}(c) = D-1$, then $\mathrm{wt}_{\mathcal{F}}(T(c)) = D-1$. We can move on by induction: assuming that T preserves every weight greater than K, with the same argument we conclude that T maps a codeword $c \in \mathcal{C}_1^{\perp}$ with $\mathrm{wt}_{\mathcal{F}}(c) = K$ into a codeword $T(c)$ with $\mathrm{wt}_{\mathcal{F}}(T(c)) = K$. Hence, the claim is proved.

Returning to the main proof, we shall prove that $d_{\mathcal{F}}$ does not satisfy a MacWilliams-type identity. We recall that $\mathcal{C}_1 = span\{e_{i_0}\}$ and $A, B \in \mathcal{F}$ were chosen to satisfy both $A \cap B \neq \emptyset$ and $i_0 \in A \setminus B$. We define $c := \sum_{i \in B} e_i$, so that $\mathrm{supp}(c) = B$ and $\mathrm{wt}_{\mathcal{F}}(c) = 1$. Furthermore, $c \in \mathcal{C}_1^{\perp}$ because $i_0 \notin B$. Since $\{i_0\} \cup B = \mathrm{supp}(T(c))$ and $\mathcal{F}$ has no redundancy, it follows that $\mathrm{wt}_{\mathcal{F}}(T(c)) > 1$. Thus, T does not preserve weights and, from our claim, it follows that $W_{\mathcal{C}_1^{\perp}}(x) \neq W_{\mathcal{C}_2^{\perp}}(x)$, that is, the combinatorial metric $d_{\mathcal{F}}$ does not satisfy a MacWilliams-type identity.

$\square$

The following theorem is an unified restatement of Propositions 1.2.2 and 1.2.3.

Theorem 1.2.4. *A combinatorial metric $d_{\mathcal{F}}$ admits a MacWilliams-type Identity if, and only if, $d_{\mathcal{F}}$ is an $(\mathcal{F}, k)$-combinatorial metric.*

1.3 Linear $d_{\mathcal{F}}$-isometries

In the context of coding theory, the linear group of isometries has been characterized considering many different metrics (see for example (PANEK et al., 2008; ETZION; FIRER; MACHADO, 2017)) and it has been used as a relevant tool to prove coding-related results. We aim to describe the group of linear isometries of the n-dimensional space $\mathbb{F}_q^n$ when endowed with a combinatorial metric.

We denote by $GL\left(n,\mathcal{F}\right)_q$ the group of linear isometries of $\left(\mathbb{F}_q^n, d_{\mathcal{F}}\right)$, i.e.,

$$GL\left(n,\mathcal{F}\right)_q = \{T : \mathbb{F}_q^n \to \mathbb{F}_q^n \ : T \text{ is linear and } d_{\mathcal{F}}\left(u,v\right) = d_{\mathcal{F}}\left(T\left(u\right), T\left(v\right)\right) \ \forall\ u,v \in \mathbb{F}_q^n\}.$$

Let S_n be the group of permutations of $[n]$. Given $\phi \in S_n$, considering the action of S_n on the coordinates of the elements of $\mathbb{F}_q^n$, we define a map $T_\phi : \mathbb{F}_q^n \to \mathbb{F}_q^n$ as $T_\phi\left((u_1, u_2, \ldots, u_n)\right) = (u_{\phi(1)}, u_{\phi(2)}, \ldots, u_{\phi(n)})$.

To determine whether T_ϕ is a $d_{\mathcal{F}}$-isometry or not, we need the following definition.

Definition 1.3.1. *Let $\mathcal{F}$ be a covering of $[n]$. We say that a permutation $\phi : [n] \to [n]$* preserves $\mathcal{F}$ *if it maps a basic set into a basic set, i.e., $\phi(A) \in \mathcal{F}$ for every $A \in \mathcal{F}$. We denote $G := G^{\mathcal{F}} = \{T_\phi : \phi$ preserves $\mathcal{F}\}$.*

The following proposition determines a relation between permutations preserving coverings and linear $d_{\mathcal{F}}$-isometries.

Proposition 1.3.2. *Given a permutation $\phi \in S_n$, the linear map T_ϕ is a $d_{\mathcal{F}}$-isometry if, and only if, $\phi \in G^{\mathcal{F}}$.*

Proof. Consider $u = (u_1, \ldots, u_n) \in \mathbb{F}_q^n$ with $\mathrm{wt}_{\mathcal{F}}(u) = k$ and let $A_1, \ldots, A_k \in \mathcal{F}$ be a family of basic sets such that $\mathrm{supp}(u) \subset A_1 \cup \cdots \cup A_k$. Given $\phi \in S_n$, from the definition of T_ϕ we have that

$$T_\phi(u) = T_\phi\left(\sum_{i=1}^{n} u_i e_i\right) = \sum_{i=1}^{n} u_i T_\phi(e_i) = \sum_{i=1}^{n} u_i e_{\phi(i)}$$

so that $\mathrm{supp}(T_\phi(u)) = \phi(\mathrm{supp}(u))$.

Suppose that ϕ preserves $\mathcal{F}$. It follows that $\phi(\mathrm{supp}(u)) \subset \phi(A_1) \cup \cdots \cup \phi(A_k)$ and this means that $\mathrm{wt}_{\mathcal{F}}(T_\phi(u)) \le \mathrm{wt}_{\mathcal{F}}(u)$. The equality follows straightforward by using the same construction for the inverse permutation ϕ^{-1}.

On the other hand, if T_ϕ is an isometry, we wish to show that $\phi(A) \in \mathcal{F}$, for every $A \in \mathcal{F}$. Let $u := \sum_{i \in A} e_i$. Since T_ϕ is an isometry and $\mathrm{wt}_{\mathcal{F}}(u) = 1$, there is a basic set $B \in \mathcal{F}$ such that $\mathrm{supp}(T_\phi(u)) = \phi(A) \subset B$. By the same reasoning, it follows that $\phi^{-1}(B)$ is contained in a basic set C and, by construction, $A \subset \phi^{-1}(B) \subset C$. Assuming that $\mathcal{F}$ has no redundancy, we get that $A = \phi^{-1}(B) = C$. Equivalently, $\phi(A) = B$ is a basic set, i.e., ϕ preserves $\mathcal{F}$.

$\square$

Let $\mathcal{F}$ be a covering of $[n]$ and denote $\mathcal{F}^i := \{A \in \mathcal{F} : i \in A\}$. It determines an equivalence relation $\sim_{\mathcal{F}}$ on $[n]$ by the following rule:

$$i \sim_{\mathcal{F}} j \iff \mathcal{F}^i = \mathcal{F}^j,$$

in other words, any basic set containing i also contains j and vice versa.

The next table illustrates the equivalence classes, considering some particular coverings of $\{1, 2, 3, 4, 5\}$.

$\mathcal{F}$	$\mathcal{F}^i$ (list of elements)	Equivalence classes
$\{1, 2, 3\}, \{3, 4, 5\}$	$\mathcal{F}^1 = \mathcal{F}^2 : \{1, 2, 3\}$; $\mathcal{F}^3 : \{1, 2, 3\}, \{3, 4, 5\}$; $\mathcal{F}^4 = \mathcal{F}^5 : \{3, 4, 5\}$	$\{1, 2\}, \{3\}, \{4, 5\}$
$\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}$	$\mathcal{F}^1 : \{1, 2, 3\}; \mathcal{F}^2 : \{1, 2, 3\}, \{2, 3, 4\}$; $\mathcal{F}^3 : \{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}$; $\mathcal{F}^4 : \{2, 3, 4\}, \{3, 4, 5\}; \mathcal{F}^5 : \{3, 4, 5\}$	$\{1\},\{2\},\{3\},\{4\},\{5\}$
$\{1, 2, 3, 4\}, \{2, 3, 4, 5\}$	$\mathcal{F}^1 : \{1, 2, 3, 4\}$; $\mathcal{F}^2 = \mathcal{F}^3 = \mathcal{F}^4 : \{1, 2, 3, 4\}, \{2, 3, 4, 5\}$; $\mathcal{F}^5 : \{2, 3, 4, 5\}$	$\{1\}, \{2, 3, 4\}, \{5\}$

We denote by $\mathcal{H}_{\mathcal{F}} = \{H_1, \ldots, H_s\}$ the set of equivalence classes. We can express $[n] = \bigsqcup_{i=1}^{s} H_i$, where the union is disjoint.

We stress that if an element of an equivalence class H_i belongs to a basic set $A_j \in \mathcal{F}$, then the entire class H_i is contained in A_j. Assuming that $\mathcal{F} = \{A_1, \ldots, A_r\}$, let $M = M(\mathcal{F}; \mathcal{H}_{\mathcal{F}})$ be the $s \times r$ incidence matrix defined as follows

$$m_{ij} = \begin{cases} 1, \text{ if } H_i \subset A_j \\ 0, \text{ otherwise.} \end{cases}.$$

Let v^k be the k-th row of M. We say that H_i *dominates* H_j if $\operatorname{supp}(v^j) \subset \operatorname{supp}(v^i)$. This defines an order relation on the rows of the incidence matrix. We say that H_i is a *head* in a family of equivalence classes if it is a maximal element in the family. We may assume, without loss of generality, that the classes are ordered in a natural way, i.e., given two distinct classes H_i and H_j, if H_i dominates H_j then $i > j$.

For each subset $X \subset [n]$ there is a minimum set $\mathcal{H}_{\mathcal{F}}(X) = \{H_{i_1}, \ldots, H_{i_k}\}$ of equivalence classes of $\mathcal{F}$ such that $X \subset H_{i_1} \cup \cdots \cup H_{i_k}$. The *minimum set header* of X is defined by

$$\widehat{X} = \{i \in X : i \in H_j \text{ for some head } H_j \in \mathcal{H}_{\mathcal{F}}(X)\}.$$

Given $u = (u_1, \ldots, u_n) \in \mathbb{F}_q^n$ the *cleared out* form of u is the vector $\widetilde{u} = (\widetilde{u}_1, \ldots, \widetilde{u}_n)$ where $\widetilde{u}_i = u_i$ if $i \in \widehat{\operatorname{supp}(u)}$ and $\widetilde{u}_i = 0$ otherwise, i.e., we maintain the coordinates corresponding to heads and reset the others.

An $n \times n$-matrix $B = (b_{xy})$ with coefficients in $\mathbb{F}_q$ is said to *respect* M if for every block (submatrix) $B_{ij} = (b_{xy})_{x \in H_i, y \in H_j}$, the following conditions hold:

1. Each block $B_{ii} = (b_{xy})_{x,y \in H_i}$ is an invertible matrix;

2. If $i \neq j$, then $B_{ij} \neq 0$ implies that H_i dominates H_j.

We denote by K_M as the set of all matrices respecting M. We stress that due to the natural ordering in the classes, condition 2 implies that $B_{ij} = 0$ for every $i < j$. Furthermore, we will assume, without loss of generality, that $H_i = \{N_{i-1} + 1, \dots, N_i\}$ where $N_i = \sum_{j=1}^{i} |H_j|$ and $N_0 = 0$. Therefore, in this configuration, B will always be a block lower triangular matrix having B_{ii} in its diagonal.

Lemma 1.3.3. *If $B \in K_M$, then $B^{-1} \in K_M$.*

Proof. We stress that B is an invertible matrix since it is a block triangular matrix in which all the blocks in its diagonal, namely, the blocks $B_{ii} = (b_{xy})_{x,y \in H_i}$, are invertible matrices. Also, due to the triangular shape of matrix B, it is straightforward to conclude that $(B^{-1})_{ii}$ are invertible matrices for every i.

Suppose $(B^{-1})_{ij} \neq 0$ for some $i \neq j$, then $B_{ii}(B^{-1})_{ij} \neq 0$. Since $BB^{-1} = I_n$, we have that $\sum_{k=i}^{j} B_{ik}(B^{-1})_{kj} = 0$ for every $i \neq j$. Hence, there is $k_0 > i$ such that $B_{ik_0}(B^{-1})_{k_0 j} \neq 0$ and so, both B_{ik_0} and $(B^{-1})_{k_0 j}$ are non null matrices. In this way, by repeating the steps for $(B^{-1})_{k_0 j}$, we conclude that $B_{ij} \neq 0$, i.e., that H_i dominates H_j. $\square$

Proposition 1.3.4. *An $n \times n$ matrix B respecting M is a linear $d_{\mathcal{F}}$-isometry, i.e., $K_M \subset GL\left(n, \mathcal{F}\right)_q$.*

Proof. If $\mathrm{wt}_{\mathcal{F}}(u) = k$ for a vector $u \in \mathbb{F}_q^n$, then there are $A_1, \dots, A_k \in \mathcal{F}$ covering the support of u, i.e., $\mathrm{supp}(u) \subset A_1 \cup \dots \cup A_k$. Since B respects M, every covering of $\mathrm{supp}(u)$ also covers $\mathrm{supp}(Bu)$, that is, $\mathrm{wt}_{\mathcal{F}}(Bu) \leq \mathrm{wt}_{\mathcal{F}}(u)$. By Lemma 3.2.11, $B \in K_M$ implies that $B^{-1} \in K_M$ and so we have that $\mathrm{wt}_{\mathcal{F}}(u) = \mathrm{wt}_{\mathcal{F}}(Bu)$. $\square$

The previous proposition ensures that for every vector $u \in \mathbb{F}_q^n$ there is a linear $d_{\mathcal{F}}$-isometry $S \in K_M$ such that $S(u)$ is the cleared out of u, i.e., $S(u) = \widetilde{u}$.

From here on, in this section, we need to assume that $q > 2$.

Lemma 1.3.5. *Let $T \in GL(n, \mathcal{F})_q$ with $q > 2$. Given $e_i \in \mathbb{F}_q^n$, the support of $\widetilde{T(e_i)}$ is contained in a single equivalence class of $\mathcal{F}$.*

Proof. Without loss of generality, let us assume that $\operatorname{supp}(\widetilde{T(e_i)}) \subset H_1 \cup H_2$. In order to be explicit, let $S_i \in K_M$ be a linear $d_{\mathcal{F}}$-isometry such that $S_i(T(e_i)) = \widetilde{T(e_i)}$ and, further, suppose that it decomposes as

$$\widetilde{T(e_i)} = (S_i(T(e_i))_1 + (S_i(T(e_i))_2,$$

with $\operatorname{supp}((S_i(T(e_i))_j) \subset H_j$ for every $j \in \{1,2\}$. We denote $R_i = S_i T$ and $R_i^{-1} = T^{-1}S_i^{-1}$.

By definition, $\operatorname{supp}(\widetilde{T(e_i)})$ contains only heads, and this means that neither H_1 dominates H_2 nor H_2 dominates H_1. It follows that there are equivalence classes $K_1^1, \ldots, K_s^1$ and $K_1^2, \ldots, K_r^2$ such that:

1. The sets $H_1 \cup \left(K_1^1 \cup \cdots \cup K_s^1\right)$ and $H_2 \cup \left(K_1^2 \cup \cdots \cup K_r^2\right)$ are each contained in a single basic set.
2. Neither $H_1 \cup H_2 \cup \left(K_1^1 \cup \cdots \cup K_s^1\right)$ nor $H_1 \cup H_2 \cup \left(K_1^2 \cup \cdots \cup K_r^2\right)$ is contained in a single basic set.

This means there are vectors $u, v \in \mathbb{F}_q^n$ with $\operatorname{supp}(u) \subset K_1^1 \cup \cdots \cup K_s^1$ and $\operatorname{supp}(v) \subset K_1^2 \cup \cdots \cup K_r^2$ such that

$$\mathrm{wt}_{\mathcal{F}}(R_i(e_i)_1 + u) = \mathrm{wt}_{\mathcal{F}}(R_i(e_i)_2 + v) = 1$$

and

$$\mathrm{wt}_{\mathcal{F}}(R_i(e_i) + u) = \mathrm{wt}_{\mathcal{F}}(R_i(e_i) + v) = 2.$$

We now claim that $i \notin \operatorname{supp}(R_i^{-1}(v))$. If this was not the case, we would have that $\operatorname{supp}(e_i + R_i^{-1}(v)) \subset \operatorname{supp}(R_i^{-1}(v))$ and applying R_i to both the vectors we get $\operatorname{supp}(R_i(e_i) + v) \subset \operatorname{supp}(v)$ and this implies $\mathrm{wt}_{\mathcal{F}}(R_i(e_i) + v) \leq \mathrm{wt}_{\mathcal{F}}(v)$, contradicting the fact that $\mathrm{wt}_{\mathcal{F}}(R_i(e_i) + v) = 2$ and $\mathrm{wt}_{\mathcal{F}}(v) = 1$. The same reasoning shows that $i \notin \operatorname{supp}(R_i^{-1}(u))$.

Since we are assuming that $q > 2$, there are $\alpha, \gamma \in \mathbb{F}_q^*$ such that $1+\alpha$ and $1+\gamma$ are not zero. Thus,

$$\mathrm{wt}_{\mathcal{F}}((1+\alpha)R_i(e_i)_1 + R_i(e_i)_2 + u) = \mathrm{wt}_{\mathcal{F}}(R_i(e_i)_1 + (1+\gamma)R_i(e_i)_2 + v) = 2.$$

In a similar manner we can prove that $i \notin \operatorname{supp}(R_i^{-1}(R_i(e_i)_1) + u)$ and $i \notin \operatorname{supp}(R_i^{-1}(R_i(e_i)_2) + v)$. But this means that the i-th coordinates of both vectors $R_i^{-1}(R_i(e_i))_1$ and $R_i^{-1}(R_i(e_i))_2$ are equal to zero, a contradiction since, by definition, $R_i^{-1}(R_i(e_i)) = e_i$.

$\square$

The previous Lemma ensures the existence of a class $H_i \in \mathcal{H}_{\mathcal{F}}$ that contains $\operatorname{supp}(\widetilde{T(e_i)})$. In the next lemma we prove that this class does not depend only on i, but also on T and on the class containing i.

Lemma 1.3.6. *Let $T \in GL(n, \mathcal{F})_q$ with $q > 2$. Given a class $H \in \mathcal{H}_{\mathcal{F}}$, there is $H' \in \mathcal{H}_{\mathcal{F}}$ such that, for every $i \in H$,* $\mathrm{supp}(\widetilde{T(e_i)}) \subset H'$.

Proof. Suppose that $\mathrm{supp}(\widetilde{T(e_i)}) \subset H_1$ and $\mathrm{supp}(\widetilde{T(e_j)}) \subset H_2$ with $H_1 \neq H_2$, for $i, j \in H$. A similar construction of the previous lemma allows us to prove that $T(\widetilde{e_i + e_j})$ is contained in a unique equivalence class and this implies that either H_1 dominates H_2 or H_2 dominates H_1. Let us assume that H_1 dominates H_2. This means there is a basic set A containing H_2 but not H_1. Take a vetor $v \in \mathbb{F}_q^n$ with $\mathrm{supp}(v) \subset A$ such that $\mathrm{wt}_{\mathcal{F}}(T(e_i) + v) = 2$ and $\mathrm{wt}_{\mathcal{F}}(T(e_j) + v) = 1$. Furthermore, since we are assuming $q > 2$ there are $\alpha, \beta \in \mathbb{F}_q$, such that $i \in \mathrm{supp}(\alpha e_i + T^{-1}(v))$ and $j \in \mathrm{supp}(\beta e_j + T^{-1}(v))$ which contradicts the assumption that T is a $d_{\mathcal{F}}$-isometry.

□

Note that it is straightforward that K_M is a normal subgroup of $GL(n, \mathcal{F})_q$ because $\mathrm{supp}(T_\phi \circ \widetilde{T \circ T_\phi^{-1}}(u)) = \mathrm{supp}(\widetilde{T(u)})$, but we still does not have a semidirect product since $G \cap K_M \neq \{I\}$. In order to obtain two subgroups whose intersection is the identity, we shall restrict the permutation part to those that only permutes the equivalence classes in $\mathcal{H}_{\mathcal{F}} = \{H_1, \ldots, H_s\}$. We recall that we labeled the equivalence classes as $H_i = \{N_{i-1} + 1, \ldots, N_i\}$ where $N_i = \sum_{j=1}^{i} |H_j|$ and $N_0 = 0$. It follows that, given $\phi \in S_s$ (where $s = |\mathcal{H}_{\mathcal{F}}|$), it induces a permutation on $[n]$ by setting $\phi_{\mathcal{F}}(N_i + j) = N_{\phi(i)} + j$. We now consider the subgroup of $G_{\mathcal{F}} \subseteq G$ determined by such permutations, that is:

$$G_{\mathcal{F}} := \{T_{\phi_{\mathcal{F}}} : \phi \in S_s\}.$$

Hence, we clearly have a semidirect product, i.e.:

Lemma 1.3.7. *For $q > 2$, the group $G_{\mathcal{F}} K_M$ is the semidirect product $G_{\mathcal{F}} \ltimes K_M$.*

Theorem 1.3.8. *Given a covering $\mathcal{F}$, we have that $G_{\mathcal{F}} \ltimes K_M \subseteq GL(n, \mathcal{F})_q$. Equality holds for $q > 2$. For $q = 2$ equality may or not hold, depending of $\mathcal{F}$.*

Proof. Propositions 5 and 6 ensures that $G_{\mathcal{F}} \ltimes K_M \subseteq GL(n, \mathcal{F})_q$. Lemmas 1.3.5 and 1.3.6 ensure that for each equivalence class H_i of $\mathcal{F}$, there is $S_i \in K_M$ such that $S_i(\widetilde{T(e_j)} = \widetilde{T(e_j)})$ for $j \in H_i$ and $S_i(\widetilde{T(e_k)}) = \widetilde{T(e_k)}$ for every $k \notin H_i$. It follows that, given $T \in GL(n, \mathcal{F})_q$, the composition $S_1 S_2 \cdots S_n T$ determines a permutation of basic sets of $\mathcal{F}$, that is, $S_1 S_2 \cdots S_n T \in G$. It is clear that G can be written as the product $G = G_{\mathcal{F}} \cdot G_1$ where $G_1 := \{I\} \cup (G \setminus G_{\mathcal{F}})$. Furthermore, $G_1 \subset K_M$, hence there is $R \in G_1 \subset K_M$ such that $R S_1 S_2 \cdots S_n T \in G_{\mathcal{F}}$. □

1.4 MacWilliams' Extension Property

To define equivalence among linear codes, there are two distinct approaches, a local one and a global one. For the Hamming metric, F. J. MacWilliams, in her thesis (see (MACWILLIAMS, 1962)), proved that the two approaches are equivalent. To be more precise, we need some definitions.

Definition 1.4.1. *(Local Equivalence) Two linear codes* $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{F}_q^n$ *are* locally $\mathcal{F}$-equivalent *if there exists a linear map* $t : \mathcal{C}_1 \to \mathcal{C}_2$ *which preserves the* $\mathcal{F}$*-weight, in the sense that* $\mathrm{wt}_{\mathcal{F}}(u) = \mathrm{wt}_{\mathcal{F}}(t(u))$*, for all* $u \in \mathcal{C}_1$*.*

Definition 1.4.2. *(Global Equivalence) Two linear codes* $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{F}_q^n$ *are* globally $\mathcal{F}$-equivalent*, or just* $\mathcal{F}$-equivalent*, if there is a* $d_{\mathcal{F}}$*-isometry* $T \in GL(n, \mathcal{F})_q$ *such that* $T(\mathcal{C}_1) = \mathcal{C}_2$*.*

The MacWilliams extension theorem states that every Hamming weight preserving linear map $t : \mathcal{C}_1 \to \mathcal{C}_2$ can be extended to a monomial map. Hence, if $\mathcal{F}$ induces the Hamming metric, then two codes are locally $\mathcal{F}$-equivalent if, and only if, they are $\mathcal{F}$-equivalent. Naturally, in the case of combinatorial metrics, global equivalence implies local equivalence, but the opposite is not always true as we shall see.

Definition 1.4.3. *(MacWilliams' Extension Property - MEP) An* $\mathcal{F}$*-combinatorial metric satisfies the MacWilliams' Extension Property if for any linear codes* $\mathcal{C}_1$ *and* $\mathcal{C}_2$*, every* $\mathcal{F}$*-weight preserving linear map* $t : \mathcal{C}_1 \to \mathcal{C}_2$ *can be extended to a* $d_{\mathcal{F}}$*-isometry, i.e., there is* $T \in GL(n, \mathcal{F})_q$ *such that* $T(c) = t(c)$*, for every* $c \in \mathcal{C}_1$*.*

Proposition 1.4.4. *If there are* $A, B \in \mathcal{F}$ *such that* $|A| \neq |B|$*, then* $d_{\mathcal{F}}$ *does not satisfy MEP.*

Proof. Suppose there are $A, B \in \mathcal{F}$ such that $|A| > |B|$. Define $C \subset [n]$ such that $C \subset A$, $A \cap B \subset C$ and $|C| = |B|$. Let $\sigma : C \to B$ be a bijective map such that $\sigma(i) = i$ for every $i \in B \cap A$. Hence, a local linear map t can be defined by putting $t(e_i) = e_{\sigma(i)}$ for every $i \in C$. By construction, t is an $\mathcal{F}$-weight preserving linear map. Given $i_0 \in A \setminus C$, if T is a linear extension of t and $\alpha_i \in \mathbb{F}_q$ for every $i \in C$, then

$$T\left(e_{i_0} + \sum_{j \in C} \alpha_j e_j\right) = T(e_{i_0}) + \sum_{j \in C} \alpha_j e_{\sigma(j)} = T(e_{i_0}) + \sum_{j \in B} \alpha_{\sigma^{-1}(j)} e_j.$$

We now consider two different cases:

- If $supp(T(e_{i_0})) \cap B = \emptyset$, considering $\alpha_i = 1$ for every $i \in C$, since $\mathcal{F}$ has no redundancy, $\mathrm{wt}_{\mathcal{F}}(e_{i_0} + \sum_{j \in C} \alpha_j e_j) > 1$. Therefore, T is not a $d_{\mathcal{F}}$-isometry.

- If $supp(T(e_{i_0})) \cap B \neq \emptyset$, considering $\alpha_{\sigma^{-1}(i)} = 0$ for every $i \in supp(T(e_{i_0}))$ and $\alpha_i = 1$ otherwise, by the same reasoning T is not a $d_{\mathcal{F}}$-isometry.

$\square$

In order to characterize the disconnected coverings satisfying the MacWilliams extension property, we need the definition of connected components.

Definition 1.4.5. *A family (or covering) $\mathcal{F}$ is said to be* connected *if there are no $\mathcal{A}, \mathcal{B} \subset \mathcal{F}$ such that $\mathcal{A} \cup \mathcal{B} = \mathcal{F}$ with $A \cap B = \emptyset$, for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. A maximal connected subset $\mathcal{A} \subset \mathcal{F}$ is called a* connected component *of $\mathcal{F}$. If $\mathcal{F}$ is not connected, we say that it is disconnected.*

Proposition 1.4.6. *Suppose $\mathcal{F}$ has two connected components. A combinatorial metric $d_{\mathcal{F}}$ satisfies MEP if, and only if, $\mathcal{F}$ is a k-partition.*

Proof. If we suppose that $d_{\mathcal{F}}$ satisfies MEP, Proposition 1.4.4 ensures that $|A| = |B|$, for every $A, B \in \mathcal{F}$. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the two connected components of $\mathcal{F}$. We claim that $|\mathcal{A}_1| = |\mathcal{A}_2| = 1$. Indeed, if $|\mathcal{A}_1| > 1$, there exist $A, B \in \mathcal{A}_1$ such that $A \cap B \neq \emptyset$. Let us define

$$u = \sum_{i \in A} e_i, \quad v = \sum_{i \in B \setminus A} e_i \quad \text{and} \quad w = \sum_{i \in A \cap B} e_i.$$

Consider a basic set $C \in \mathcal{A}_2$ and let $j_0 \in C$ and $i_0 \in A \setminus B$. The linear map $t : span\{u, v\} \to span\{e_{i_0}, e_{j_0}\}$ defined by $t(u) = e_{j_0}$ and $t(v) = e_{i_0}$ is a $\mathcal{F}$-weight-preserving map. Suppose T is a $d_{\mathcal{F}}$-isometry which extends t. Thus,

$$T\left(\sum_{i \in B} e_i\right) = T(v) + T(w) = e_{i_0} + T(w).$$

Since $\mathrm{wt}_{\mathcal{F}}\left(T(\sum_{i \in B} e_i)\right) = \mathrm{wt}_{\mathcal{F}}\left(\sum_{i \in B} e_i\right) = 1$, it follows that $\mathrm{supp}\,(T(w)) \subset \bigcup_{A \in \mathcal{A}_1} A$. On the other hand, for $u + w = \sum_{i \in A \setminus B} e_i$ we get that

$$T\left(\sum_{i \in A \setminus B} e_i\right) = T(u) - T(w) = e_{j_0} - T(w).$$

Since $\mathrm{wt}_{\mathcal{F}}(T(u - w)) = 1$, it follows that $\mathrm{supp}\,(T(w)) \subset \bigcup_{A \in \mathcal{A}_2} A$. Hence, T is not a $d_{\mathcal{F}}$-isometry.

For the other implication, suppose that $\mathcal{F}$ is a k-partition, since there are only two connected components, $k = n/2$: $\mathcal{F} = \{A_1, A_2\}$, with $A_1 \cap A_2 = \emptyset$ and $|A_1| = |A_2| = n/2$. Let $\mathcal{C}_1, \mathcal{C}_2$ be linear codes and $t : \mathcal{C}_1 \to \mathcal{C}_2$ be a $\mathcal{F}$-weight preserving linear map.

Note that every $u \in \mathbb{F}_q^n$ can be uniquely decomposed as $u = u^1 + u^2$, where $\mathrm{supp}(u^1) \subset A_1$ and $\mathrm{supp}(u^2) \subset A_2$.

Given $x \in \mathcal{C}_1$ with $x^1 \neq 0$ and $x^2 = 0$, then $t(x) = t(x^1)$ and $\mathrm{supp}(t(x)) = \mathrm{supp}(t(x^1))$ is contained in either A_1 or A_2, because t is an $\mathcal{F}$-weight preserving linear map. Let us assume that $\mathrm{supp}(t(x)) \subset A_1$. Moreover, we have to remark the following:

- For every codeword $y = y^1 \in \mathcal{C}_1$, $\mathrm{supp}(t(y)) = \mathrm{supp}(t(y^1)) \subset A_1$ (otherwise $\mathrm{wt}_{\mathcal{F}}(x + y) = 1 \neq \mathrm{wt}_{\mathcal{F}}(t(x + y)) = 2$);
- For every codeword $y = y^2 \in \mathcal{C}_1$, $\mathrm{supp}(t(y^2)) \subset A_2$, (otherwise we would have $\mathrm{wt}_{\mathcal{F}}(x + y) = 2$ and $\mathrm{wt}_{\mathcal{F}}(t(x + y)) = 1$);
- A similar situation holds for the previous items if we consider $x = x^1 + x^2$ with $x^1 = 0$ and $x^2 \neq 0$;
- If $x = x^1 + x^2$ with both $x^1 \neq 0$ and $x^2 \neq 0$, then it holds for $t(x) = t(x)^1 + t(x)^2$, i.e., both $t(x)^1 \neq 0$ and $t(x)^2 \neq 0$.

From the above discussion it follows that the map t induces a permutation $\sigma : [2] \to [2]$ of the basic sets A_1 and A_2, in the sense that, $\mathrm{supp}(t(x^i)) \subset A_{\sigma(i)}$, for every $x \in \mathcal{C}_1$ and $i \in [2]$.

Let $\beta = \{c_1, \ldots, c_k\}$ be a basis of $\mathcal{C}_1$ and $t(\beta) = \{t(c_1), \ldots, t(c_k)\}$ be the corresponding basis of $\mathcal{C}_2$. We remark that $t(c_i)^{\sigma(j)}$ depends only on c_i^j, i.e., if $c_i \neq c_{i_0}$ and $c_i^j = c_i^j$, then $t(c_i)^{\sigma(j)} = t(c_{i_0})^{\sigma(j)}$.

Hence, if $\beta_i = \{c^i_{r_1}, \ldots, c^i_{r_{j_i}}\}$ is a linearly independent set, then $S(\beta_i) := \{t(c_{r_1})^{\sigma(i)}, \ldots, t(c_{r_{j_i}})^{\sigma(i)}\}$ is also a linearly independent set. We can extend β_i to a linearly independent set $\beta_i' = \beta_i \cup \{v_1^i, \ldots, v_{s_i}^i\}$, where $\mathrm{supp}(v_j^i) \subset A_i$ and $|\beta_i'| = n/2$. Analogously, $S(\beta_i)$ can be extended to a linearly independent set $S(\beta_i)' = S(\beta_i) \cup \{w_1^i, \ldots, w_{s_i}^i\}$ where $\mathrm{supp}(w_j^i) \subset A_{\sigma(i)}$ and $|S(\beta_i)'| = n/2$. Therefore, $\beta_1' \cup \beta_2'$ and $S(\beta_1)' \cup S(\beta_2)'$ are both basis of $\mathbb{F}_q^n$.

The linear map $T : \mathbb{F}_q^n \to \mathbb{F}_q^n$ defined as

$$T(u) = \begin{cases} t(c_{r_s})^{\sigma(i)}, & \text{if } u = c^i_{r_s} \in \beta_i \\ w_j^{\sigma(i)}, & \text{if } u = v_j^i \end{cases} \tag{1.2}$$

is an extension of t, indeed, if $c \in \mathcal{C}_1$ is a codeword, then $c = \sum_{l=1}^{k} \alpha_l c_l$ with $\alpha_l \in \mathbb{F}_q$ and

$$T(c) = \sum_{l=1}^{k} \alpha_l T(c_l) = \sum_{l=1}^{k} \alpha_l [T(c_l^1) + T(c_l^2)].$$

Since $c_l^1 = \sum_{i=1}^{j_1} \gamma_i c_{r_i}^1$ and $c_l^2 = \sum_{i=1}^{j_2} \lambda_i c_{r_i}^2$ where $\gamma_i, \lambda_i \in \mathbb{F}_q$, we get that $T(c_l^1) = \sum_{i=1}^{j_1} \gamma_i T(c_{r_i}^1)$ and $T(c_l^2) = \sum_{i=1}^{j_2} \lambda_i T(c_{r_i}^2)$. By the definition of T, it follows that

$$T(c) = \sum_{l=1}^{k} \alpha_l \left[\sum_{i=1}^{j_1} \gamma_i t(c_{r_i})^{\sigma(1)} + \sum_{i=1}^{j_2} \lambda_i t(c_{r_i})^{\sigma(2)} \right]$$

and then,

$$T(c) = \sum_{l=1}^{k} \alpha_l \left[t(c_l)^{\sigma(1)} + t(c_l)^{\sigma(2)} \right] = \sum_{l=1}^{k} \alpha_l t(c_l) = t\left(\sum_{l=1}^{k} \alpha_l c_l \right) = t(c),$$

i.e., T is an extension of t.

To conclude, it is enough to prove that T is an $d_{\mathcal{F}}$-isometry. Given $u \in \mathbb{F}_q^n$, we need to consider separately whether $\mathrm{wt}_{\mathcal{F}}(u) = 1$ or $\mathrm{wt}_{\mathcal{F}}(u) = 2$.

If $\mathrm{wt}_{\mathcal{F}}(u) = 1$ we have that $\mathrm{supp}(u) \subset A_i$ for some $i \in [2]$, or equivalently, $u \in span(\beta_i')$ and from Expression 1.2 we get that $T(u) \in span(\beta'_{\sigma(i)})$, so that $\mathrm{wt}_{\mathcal{F}}(T(u)) = 1$.

If $\mathrm{wt}_{\mathcal{F}}(u) = 2$ we have that $\mathrm{supp}(u) \cap A_i \neq \emptyset$ for both $i = 1$ and $i = 2$, or equivalently, $\{u\} \cap span(\beta_i') \neq \emptyset$. Again, from Expression 1.2 we get that $\{T(u)\} \cap span(\beta'_{\sigma(i)}) \neq \emptyset$ for both $i = 1$ and $i = 2$, so that $\mathrm{wt}_{\mathcal{F}}(T(u)) = 2$.

Hence, T preserves $\mathcal{F}$-weight, i.e., it is a $d_{\mathcal{F}}$-isometry.

□

From here up to the end of this chapter, we assume $q = 2$, so that every code will be binary. Furthermore, we will write $GL(n, \mathcal{F})$ instead of $GL(n, \mathcal{F})_2$.

Proposition 1.4.7. *If $\mathcal{F}$ has 3 or more connected components, then the $\mathcal{F}$-combinatorial metric satisfies MEP if, and only if, it is the Hamming metric.*

Proof. It is well known that the Hamming metric satisfies MEP (MacWilliams' Extension Property). For the opposite direction, we may assume, for simplicity, that $\mathcal{F}$ has exactly 3 connected components: $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$. By supposing $d_{\mathcal{F}}$ is not the Hamming metric, Proposition 1.4.4 ensures that $|A|$ is constant and $|A| > 1$, for every $A \in \mathcal{F}$. We consider basic sets $A \in \mathcal{A}_1$, $B \in \mathcal{A}_2$ and $C \in \mathcal{A}_3$ and elements $a_0, a_1 \in A$, $b_0, b_1 \in B$ and $c \in C$ where $a_0 \neq a_1$ and $b_0 \neq b_1$. Define

$$t(e_{a_0} + e_{b_0}) = e_{a_1} + e_c \quad \text{and} \quad t(e_{a_1} + e_{b_1}) = e_{a_1} + e_{b_1}.$$

By construction, t is an $\mathcal{F}$-weight preserving linear map. For any linear extension T of t, we claim that T is not a $d_{\mathcal{F}}$-isometry. Indeed, since

$$T(e_{a_0}) = e_{a_1} + e_c + T(e_{b_0}) \tag{1.3}$$

and

$$T(e_{a_1} + e_{b_1} + e_{b_0}) = e_{a_1} + e_{b_1} + T(e_{b_0}), \tag{1.4}$$

Equation (1.3) ensures that T is a $d_{\mathcal{F}}$-isometry if either $T(e_{b_0}) = e_{a_1}$ or $T(e_{b_0}) = e_c$, but in both the cases we get a contradiction by (1.4). Therefore, $T \notin GL(n, \mathcal{F})$. □

Summarizing the previous results, we have the following theorem:

Theorem 1.4.8. *Let $d_{\mathcal{F}}$ be a combinatorial metric over $\mathbb{F}_2^n$. If $\mathcal{F}$ has l connected components (with $l \geq 2$), $d_{\mathcal{F}}$ satisfies MEP if, and only if, either $l = 2$ and $\mathcal{F}$ is a k-partition or $l > 2$ and $d_{\mathcal{F}}$ is the Hamming metric.*

To complete the characterization of the combinatorial metrics over $\mathbb{F}_2^n$ satisfying the MEP, all is left is the case when the covering $\mathcal{F}$ is connected, which is addressed in the next theorem.

Theorem 1.4.9. *Let $d_{\mathcal{F}}$ be a combinatorial metric over $\mathbb{F}_2^n$. If $\mathcal{F}$ is connected and the metric $d_{\mathcal{F}}$ satisfies MEP then $\mathcal{F} = \mathcal{F}_{n,k} := \{A \subset [n] : |A| = k\}$, for some $1 < k \leq n$.*

Proof. Suppose $|B| = k$ for some $B \in \mathcal{F}$. By Proposition 1.4.4, $|B| = k$ for every $B \in \mathcal{F}$. We have to show that every set $J \subset [n]$ with $|J| = k$ is an element of $\mathcal{F}$. Let $J = \{i_1, \ldots, i_k\} \subset [n]$. Hence, in order to show that J is an element of $\mathcal{F}$, it is sufficient to show that $\mathrm{wt}_{\mathcal{F}}(e_{i_1} + \cdots + e_{i_k}) = 1$ because in this case $J \subset B$ for some $B \in \mathcal{F}$ and since $|B| = |J| = k$, it follows that $J = B$.

Suppose first that $\mathrm{wt}_{\mathcal{F}}(e_{i_1} + \cdots + e_{i_k}) = 2$. Let A_1 and A_2 two basic sets such that $\{i_1, \ldots, i_k\} \subset A_1 \cup A_2$. Consider $B_1 = J \cap (A_1 \setminus A_2)$ and $B_2 = J \setminus B_1$, so $B_1 \cup B_2 = J$, furthermore, $B_1 \subset A_1$ and $B_2 \subset A_2$.

Case 1: $A_1 \cap A_2 \neq \emptyset$: Consider $i_0 \in A_1 \cap A_2$ and let

$$u = \sum_{i \in B_1} e_i \quad \text{and} \quad v = \sum_{i \in B_2} e_i.$$

Note that $i_0 \notin \mathrm{supp}(u)$ since, by construction, $\mathrm{supp}(u) = B_1 \subset A_1 \setminus A_2$ and $i_0 \in A_1 \cap A_2$. Furthermore, $\mathrm{supp}(v) \neq \{i_0\}$, since neither $B_1 \subset B_2$ nor $B_2 \subset B_1$. Hence, if we consider the linear map defined by

$$t(0) = 0, \quad t(e_{i_0} + v) = e_{i_0}, \quad t(v) = u \quad \text{and} \quad t(e_{i_0}) = u + e_{i_0},$$

it is clearly a local linear isometry (we stress that at this point it is important to consider $\mathbb{F}_q$ to be the binary field). Suppose that t can be extended to a global linear isometry T. Since

$$T(u + v) = T(u) + u$$

and $\mathrm{wt}(u+v)=2$ because $\mathrm{supp}(u+v)=J$, it follows that $\mathrm{wt}(T(u)+u)=2$. Note that

$$T(u+e_{i_0})=T(u)+u+e_{i_0}.$$

We now need to consider two different instances:

- Suppose $i_0 \notin \mathrm{supp}(T(u))$. Since $i_0 \notin \mathrm{supp}(u)$, we get that

  $$\mathrm{supp}(T(u)+u+e_{i_0})=\mathrm{supp}(T(u)+u)\cup\{i_0\}.$$

  But $\mathrm{wt}_{\mathcal{F}}(T(u)+u)=2$, hence $\mathrm{wt}_{\mathcal{F}}(T(u)+u+e_{i_0})\geq 2$, a contradiction, since $\mathrm{wt}_{\mathcal{F}}(u+e_{i_0})=1$.

- Suppose now $i_0 \in \mathrm{supp}(T(u))$. Since

  $$T(u+e_{i_0}+v)=T(u)+e_{i_0},$$

  we get that $\mathrm{supp}(T(u)+e_{i_0})\subset\mathrm{supp}(T(u))$, which is also a contradiction because $\mathrm{wt}_{\mathcal{F}}(u+e_{i_0}+v)=2$. Therefore, $\mathrm{wt}_{\mathcal{F}}(e_{i_1}+\cdots+e_{i_k},0)\neq 2$.

Case 2: $A_1\cap A_2=\emptyset$: If there are basic sets $D_1,D_2\in\mathcal{F}$ such that $B_1\subset D_1$, $B_2\subset D_2$ and $D_1\cap D_2\neq\emptyset$, we can apply the Case 1 for $A_1=D_1$ and $A_2=D_2$. The remaining case is when $D_1\cap D_2=\emptyset$ for every basic sets $D_1,D_2\in\mathcal{F}$ such that $B_1\subset D_1$ and $B_2\subset D_2$.

Let $\mathcal{D}=\{D_1,\ldots,D_r\}$ be a connected family of basic sets satisfying $B_1\subset D_1$ and $B_2\subset D_r$. Suppose $\mathcal{D}$ is a minimal family, in the following sense: We assume that $D_i\cap D_{i+1}\neq\emptyset$ and removing any D_i we lose this property. We let $|B_1|=l$, so that $|B_2|=k-l$. Let i_0 be the minimal index such that there is a set of $k-l$ elements $B_2'=\{s_1,\ldots,s_{k-l}\}\subset D_{i_0}$ with $\mathrm{wt}_{\mathcal{F}}(e_{j_1}+\cdots+e_{j_l}+e_{s_1}+\cdots+e_{s_{k-l}})=2$. If $i_0=2$, since $D_1\cap D_2\neq\emptyset$, we can apply case 1 in $A_1=D_1$ and $A_2=D_2$ and obtain a contradiction. If $i_0>2$, considering $B_2''=\{r_1,\ldots,r_{k-l}\}$ such that $B_2''\subset D_{i_0-1}$ and $B_2''\cap D_{i_0}\neq\emptyset$, by the minimality of i_0 there is a basic set D such that $B_1\subset D$ and $B_2''\subset D$, which is a contradiction since $\{D,D_{i_0},\ldots,B_r\}$ is a connected family of basic sets satisfying $B_1\subset D$, $B_2\subset D_r$ with $r-i_0+1<r$ elements, contradicting the minimality of $\mathcal{D}$.

Therefore, by cases 1 and 2, it follows that $\mathrm{wt}_{\mathcal{F}}(e_{i_1}+\cdots+e_{i_k})\neq 2$.

Suppose now that $\mathrm{wt}_{\mathcal{F}}(e_{i_1}+\cdots+e_{i_k})>2$, then there is a subset $\{j_1,\ldots,j_l\}\subset J$ such that $\mathrm{wt}_{\mathcal{F}}(e_{j_1}+\cdots+e_{j_l})=2$ and, using the same previous argument we get a contradiction. So, $\mathrm{wt}_{\mathcal{F}}(e_{i_1}+\cdots+e_{i_k})=1$ and, consequently, $J\in\mathcal{F}$.

$\square$

We define the $\mathcal{F}$-*degree* of a vertex $i\in[n]$ as the cardinality of $\mathcal{F}(i)=\{A\in\mathcal{F}: i\in A\}$. We denote it by degree$(i)$. We note that $\mathcal{F}$-degrees are meaningful only if we assume that the covering $\mathcal{F}$ has no redundancy.

Corollary 1.4.10. *Suppose that the $\mathcal{F}$-degre is not constant. Then, $(\mathbb{F}_2^n, d_{\mathcal{F}})$ does not satisfy the MEP.*

We conjecture that the reciprocal of Theorem 1.4.9 also holds, that is, a metric $d_{\mathcal{F}}$ determined by $\mathcal{F}_{n,k}$ satisfies the MEP, for any $1 \leq k \leq n$. The extremal cases are known: for $k = 1$ we have the Hamming metric and for $k = n$ we have the 0-1 metric ($d_{\mathcal{F}}(u, v) = 1$, for all $u \neq v$) and the MEP is satisfied since every bijection is an isometry. We were able to show, using brute force, that the result holds for small values of n and k, but a proof for the general case is not available and it stays as a conjecture.

2 Labeled-poset-block metrics

This chapter is devoted to introduce a family of TS-metrics which is a generalization for the digraph metrics which arises naturally from the reduced canonical form for directed graphs presented in (ETZION; FIRER; MACHADO, 2017). This canonical form makes a contraction of each maximal cycle into a unique vertex and, then, such vertex is labeled by the number of vertices contained in the original cycle. If we allow this labeling to assume different values, such extension also generalizes the poset-block metrics by labeling every maximal cycle with 1. The goal in this chapter, is to present an structured and more generic family of metrics that may be controlled in the sense that all the coding results obtained in (ETZION; FIRER; MACHADO, 2017) can be extended to this family. Despite the generality of this approach, we also produce a description for the group of linear isometries and determine conditions for a MacWilliams' identity and extension property to be available.

2.1 Posets, blocks, labels and metrics: basic definitions

Let $P = ([m], \preceq_P)$ be a partially ordered set (abbreviated as poset), where $\preceq_p$ is a partial order over $[m] := \{1, \ldots, m\}$. An *ideal* in $P = ([m], \preceq_P)$ is a subset $I \subseteq [m]$ such that, if $b \in I$ and $a \preceq_P b$, then $a \in I$. The set of all ideals in P is denoted by $\mathcal{I}(P)$. Given $A \subseteq [m]$, we denote by $\langle A \rangle_P$ the smallest ideal of P containing A and call it the *ideal generated by* A. An element a of a set $A \subseteq [m]$ is called a *maximal* element of A if $a \preceq_P b$ for some $b \in A$ implies $b = a$. The set of all maximal elements of A is denoted by $\mathcal{M}_P(A)$. Note that if $I \subseteq [m]$ is an ideal, then $\mathcal{M}_P(I)$ is the minimal set that generates I, i.e., $\langle \mathcal{M}_P(I) \rangle_P = I$.

Given two posets P and Q over $[m]$, a *poset isomorphism* is a bijection $\phi : [m] \to [m]$ such that $i \preceq_P j \iff \phi(i) \preceq_Q \phi(j)$. When $P = Q$, ϕ is called a P*-automorphism*. The set of all P-automorphisms is a group denoted by $Aut(P)$.

A *chain* in a poset P is a subset $X \subseteq [m]$ such that any two elements $a, b \in X$ are comparable, in the sense that $a \preceq_P b$ or $b \preceq_P a$. We remark that any (finite) chain has a unique maximal element. The *height* $h(a)$ of an element $a \in P$ is the cardinality of a largest chain having a as the maximal element. The *height* $h(P)$ *of the poset* is the maximal height of its elements, i.e., $h(P) = \max\{h(a) : a \in [m]\}$. The *i-th level* Γ_i^P of a poset P is the set of all elements with height i, i.e., $\Gamma_i^P = \{a \in [m] : h(a) = i\}$. A poset P is *hierarchical* if elements at different levels are always comparable, i.e., $a \in \Gamma_i^P$ and $b \in \Gamma_j^P$ implies $a \prec_P b$ for any $1 \leq i < j \leq h(P)$.

Let us consider a map $\pi : [n] \to [m]$ with $n \geq m$ (called a *block map*). A vector $u \in \mathbb{F}_q^n$ may be written as $u = (u_1, \ldots, u_m)$, where $u_i \in \mathbb{F}_q^{k_i}$, with $k_i = |\pi^{-1}(i)|$, with $n = k_1 + k_2 + \cdots + k_m$. The π-*support* is defined as

$$\mathrm{supp}_\pi(u) = \{i \in [m] : u_i \neq 0\}.$$

Given a block function $\pi : [n] \to [m]$, a poset $P = ([m], \preceq_P)$ and a *label function* $L : [m] \to \mathbb{N}$, the (P, π, L)-*weight* of u is defined as

$$\mathrm{wt}_{(P,\pi,L)}(u) = \sum_{i \in \langle \mathrm{supp}_\pi(u) \rangle_P} L(i).$$

For $u, v \in \mathbb{F}_q^n$, we define the *labeled-poset-block distance* by:

$$d_{(P,\pi,L)}(u, v) = \mathrm{wt}_{(P,\pi,L)}(u - v).$$

Proposition 2.1.1. *If the label function L assumes only positive values, then $d_{(P,\pi,L)}(u, v)$ determines a metric over $\mathbb{F}_q^n$.*

Proof. The proof follows straight from the definitions. $\square$

Remark 2.1.2. *If $\pi : [n] \to [n]$ is the identity map and $L(i) = 1$, for every $i \in [n]$, then the $d_{(P,\pi,L)}$ is a* poset metric.

2.2 (P, π, L)-linear isometries

Let $GL(P, \pi, L)_q$ be the group of linear isometries of the space $\mathbb{F}_q^n$ endowed with a (P, π, L)-metric. Our goal in this section is to give a description for $GL(P, \pi, L)_q$.

To be more precise,

$$\begin{aligned} GL(P, \pi, L)_q = \{T : \mathbb{F}_q^n \to \mathbb{F}_q^n \ : \ T \text{ is linear}, \\ d_{(P,\pi,L)}(u, v) = d_{(P,\pi,L)}(T(u), T(v)), \forall\, u, v \in \mathbb{F}_q^n\} \\ = \{T : \mathbb{F}_q^n \to \mathbb{F}_q^n \ : \ T \text{ is linear}, \\ \mathrm{wt}_{(P,\pi,L)}(u) = \mathrm{wt}_{(P,\pi,L)}(T(u)), \forall\, u \in \mathbb{F}_q^n\} \end{aligned}$$

Similarly to what happens in the case of posets, $GL(P, \pi, L)_q$ can be described as the semi-direct product of two subgroups. We start presenting one of them, which is a subgroup of the permutation group $[m]$ that preserves the involved structures: the order structure P, the block map π and the label function L.

Definition 2.2.1. *A map $\phi : [m] \to [m]$ is a (P, π, L)-automorphism if it is a P-*automorphism *with $L(i) = L(\phi(i))$ and $k_i = k_{\phi(i)}$, for every $i \in [m]$. We denote by $Aut(P, \pi, L)$ the set of all (P, π, L)-automorphisms.*

We remark that $Aut(P,\pi,L)$ is a group. The following proposition follows straight from the definition of $d_{(P,\pi,L)}$.

Proposition 2.2.2. *Let ϕ be a (P,π,L)-automorphism. The linear map $T_\phi : \mathbb{F}_q^n \to \mathbb{F}_q^n$ defined by $T_\phi(e_{ij}) = e_{\phi(i)j}$ is an isometry. Moreover, the map $\varphi : Aut(P,\pi,L) \to GL(P,\pi,L)$ that associates $\phi \mapsto T_\phi$ is an injective homomorphism of groups.*

Proof. For every $u = \sum_{i,j} u_{ij}e_{ij} \in \mathbb{F}_q^n$, the π-support of $T_\phi(u)$ is given by

$$\begin{aligned} \operatorname{supp}_\pi(T_\phi(u)) &= \operatorname{supp}_\pi(T_\phi(\sum_{i,j} u_{ij}e_{ij})) \\ &= \{\phi(i) \in [m] : u_{ij} \neq 0 \text{ for some } j\} \\ &= \{\phi(i) \in [m] : i \in \operatorname{supp}_\pi(u)\}. \end{aligned}$$

Since ϕ is a (P,π,L)-automorphism, we have that $\sum_{i\in I} L(i) = \sum_{i\in I} L(\phi(i))$ for every ideal I. Thus, $\mathrm{wt}_{(P,\pi,L)}(u) = \mathrm{wt}_{(P,\pi,L)}(T_\phi(u))$.

The fact that φ an injective group homomorphism is straightforward and follows from the definitions. $\square$

We denote by $\mathcal{A} := \{T_\phi \in GL(P,\pi,L); \phi \in Aut(P,\pi,L)\}$ the subgroup of isometries *induced by (P,π,L)-automorphisms.*

Proposition 2.2.3. *Let $T : \mathbb{F}_q^n \to \mathbb{F}_q^n$ be a linear isomorphism satisfying the following condition: for every $u_i \in \mathbb{F}_q^{k_i}\setminus\{0\}$, there are $u_i' \in \mathbb{F}_q^{k_i}$ and $v_i \in \mathbb{F}_q^n$ with $\operatorname{supp}_\pi(v_i) \subset \langle i\rangle_P\setminus\{i\}$ such that $T(u_i) = u_i' + v_i$. Then, $T \in GL(P,\pi,L)$.*

Proof. First, note that $\operatorname{supp}_\pi(T(\mathbb{F}_q^{k_i})) \subseteq \langle i\rangle_P$. By the defining property of T, given $u = u_1 + \cdots + u_m \in \mathbb{F}_q^n$ we have that $T(u) = (u_1'+v_1)+\cdots+(u_m'+v_m) = \sum_i (u_i'+v_1^i+\cdots+v_m^i)$, where $v_l = v_l^1 + \cdots + v_l^m$ is written in a canonical decomposition of $\mathbb{F}_q^n = \mathbb{F}_q^{k_1} \oplus \cdots \oplus \mathbb{F}_q^{k_m}$. Note that, if T preserves maximal elements then T also preserves the (P,π,L)-weight. So that, given $u \in \mathbb{F}_q^n$, let $\mathcal{M}(\operatorname{supp}_\pi(u))$ be the set of maximal coordinates of u. It is simple to check if $i \in \mathcal{M}(\operatorname{supp}_\pi(u))$ then $v_j^i = 0$, for every $j \in [m]$. This implies that $i \in \mathcal{M}(\operatorname{supp}_\pi(T(u)))$.

On the other hand, if $j \in \mathcal{M}(\operatorname{supp}_\pi(T(u)))$ then $(u_j' + v_1^j + \cdots + v_m^j) \neq 0$. If $v_l^j \neq 0$, there is then $l \neq j$ such that $l \in \operatorname{supp}_\pi(u)$ and $j \preceq_P l$. This implies that $j \in \mathcal{M}(\operatorname{supp}_\pi(u))$. So that, T preserves the (P,π,L)-weight. $\square$

Let $\mathcal{N}$ be the set of (P,π,L)-linear isometries defined in the previous proposition. Actually, $\mathcal{N}$ is a subgroup of $GL(P,\pi,L)_q$. To prove it, we need two lemmas.

Lemma 2.2.4. *Consider $T \in GL(P,\pi,L)_q$ and $u_i \in \mathbb{F}_q^{k_i} \subset \mathbb{F}_q^n$. If $j \in \mathrm{supp}_\pi(T(u_i))$, then $\sum\limits_{k\in\langle j\rangle_P} L(k) \leq \sum\limits_{k\in\langle i\rangle_P} L(k)$.*

Proof. For every $j \in \mathrm{supp}_\pi(T(u_i))$ we have that $\sum\limits_{k\in\langle j\rangle_P} L(k) \leq \sum\limits_{k\in\langle \mathrm{supp}_\pi(T(u_i))\rangle_P} L(k)$. Since T is a (P,π,L)-isometry, it follows that

$$\sum_{k\in\langle j\rangle_P} L(k) \leq \sum_{k\in\langle \mathrm{supp}_\pi(T(u_i))\rangle_P} L(k) = \sum_{k\in\langle \mathrm{supp}_\pi(u_i)\rangle_P} L(k).$$

□

Lemma 2.2.5. *If $T \in GL(P,\pi,L)_q$ and $u_i \in \mathbb{F}_q^{k_i}$, then $\langle \mathrm{supp}_\pi(T(u_i))\rangle_P$ is a prime ideal.*

Proof. Suppose that $\langle \mathrm{supp}_\pi(T(u_i))\rangle_P$ is not a prime ideal and let $i_1,\ldots,i_r$ be the maximal elements of $\langle \mathrm{supp}_\pi(T(u_i))\rangle_P$. Hence, $T(u_i) = u_{i_1} + \cdots + u_{i_r} + u_{i_{r+1}} + \cdots + u_{i_s}$, where $\mathrm{supp}_\pi(u_{r+j}) \cap \mathcal{M}(\mathrm{supp}_\pi(T(u_i))) = \emptyset$, for every $j \in [s]$. It is clear that $\mathrm{wt}_{(P,\pi,L)}(T^{-1}(u_{i_j})) < \mathrm{wt}_{(P,\pi,L)}(u_i)$, for every $j \in [r+s]$. Furthermore, note that there is $k \in \{1,\ldots,r+s\}$ such that $i \in \mathrm{supp}_\pi(T^{-1}(u_{i_k}))$. From previous lemma, we have that $\mathrm{wt}_{(P,\pi,L)}(u_i)) \leq \mathrm{wt}_{(P,\pi,L)}(u_{i_k})$ which contradicts $\mathrm{wt}_{(P,\pi,L)}(T^{-1}(u_{i_j})) < \mathrm{wt}_{(P,\pi,L)}(u_i)$. Thus, $\langle \mathrm{supp}_\pi(T(u_i))\rangle_P$ is a prime ideal. □

Since $\langle \mathrm{supp}_\pi(T(u_i))\rangle_P$ is a prime ideal, we can define a map $\phi_T : P \to P$ where $\phi_T(i)$ is the unique maximal element in $(\langle \mathrm{supp}_\pi(T(u_i))\rangle_P)$, where $0 \neq u_i \in \mathbb{F}_q^{k_i}$.

Theorem 2.2.6. *Let $T \in GL(P,\pi,L)_q$ and consider the map ϕ_T. Then,*

1. *ϕ_T is a (P,π,L)-automorphism;*
2. *$\kappa : GL(P,\pi,L)_q \to Aut(P,\pi,L)$ defined by $T \mapsto \phi_T$ is a group homomorphism with kernel equal to $\mathcal{N}$. Thus, $\mathcal{N}$ is normal subgroup of $GL(P,\pi,L)_q$;*
3. *$\kappa \circ \varphi(\phi) = \phi$.*

Proof. 1. To show that ϕ_T preserves the poset order, if $i_0 \prec i_1$ then $\sum\limits_{k\in\langle i_0\rangle_P} L(k) < \sum\limits_{k\in\langle i_1\rangle_P} L(k)$. Since T is a (P,π,L)-linear isometry, we have that

$$\sum_{k\in\langle i\rangle_P} L(k) = \sum_{k\in\langle \phi_T(i)\rangle_P} L(k),$$

and so, $\sum\limits_{k\in\langle \phi_T(i_0)\rangle_P} L(k) < \sum\limits_{k\in\langle \phi_T(i_1)\rangle_P} L(k)$. We stress that $\langle \phi_T(i_0)\rangle_P \subseteq \langle \phi_T(i_1)\rangle_P$, since $\mathrm{supp}_\pi(T(\mathbb{F}_q^{k_i})) \subseteq \langle \phi_T(i)\rangle_P$. Therefore, we have that $\phi_T(i_0) \prec \phi_T(i_1)$ and ϕ_T is a (P,π,L)-homomorphism.

To conclude, since we are considering only finite posets, it is enough to show the injectivity of ϕ_T to conclude that it is an automorphism. If $\phi_T(i) = \phi_T(j)$ then $\langle \text{supp}_\pi(T(u_i))\rangle_P = \langle \text{supp}_\pi(T(u_j))\rangle_P$ for u_i and u_j under the conditions of ϕ_T. Since $T \in GL(P,\pi,L)_q$ it follows that $\sum_{k\in\langle\{i,j\}\rangle_P} L(k) = \sum_{k\in\langle \text{supp}(T(u_i+u_j))\rangle_P} L(k)$. Furthermore, since $\langle \text{supp}(T(u_i+u_j))\rangle_P \subseteq \langle \text{supp}(T(u_i))\rangle_P \cup \langle \text{supp}(T(u_j))\rangle_P = \langle \text{supp}(T(u_j))\rangle_P$. Thus, $i = j$ and ϕ_T is injective, hence an automorphism of the poset.

2. Let $T, S \in GL(P,\pi,L)_q$ be such that $\phi_T(i) = j$ and $\phi_S(j) = l$. Given $0 \neq u \in \mathbb{F}_q^{k_i}$ we have that $T(u) = u_j + v_j$, where $0 \neq u_j \in \mathbb{F}_q^{k_j}$ and $\text{supp}_\pi(u_j) \subset \langle j\rangle_P \setminus \{j\}$ and $S(u_j) = u_l + v_l$ under analogous conditions.

   Thus, $ST(u) = S(u_j+v_j) = u_l + v_l + S(v_j)$. Since $\text{supp}_\pi(v_l + S(v_j)) \subset \langle l\rangle_P \setminus \{l\}$ and $u_l \in \mathbb{F}_q^{k_l}$, we have that $\mathcal{M}(\langle \text{supp}(T(u_i+u_j))\rangle_P) = l$, and so, $\phi_{ST}(i) = l = \phi_S(\phi_T(i))$. Therefore, κ is a group homomorphism.

   Furthermore, given $\phi \in Aut(P,\pi,L)$, we have that $\kappa(T_\phi) = \phi$. It shows that κ is a surjective homomorphism.

   Since $\kappa(T) = id$ for each $T \in \mathcal{N}$, then $\mathcal{N} \subseteq ker(\kappa)$. Conversely, if $T \in ker(\kappa)$ then $\text{supp}_\pi(T(\mathbb{F}_q^{k_i})) \subseteq \langle i\rangle_P$ for every $i \in [m]$. Since $T \in GL(P,\pi,L)_q$, for $0 \neq u_i \in \mathbb{F}_q^{k_i}$, there is a nonzero $u_i' \in \mathbb{F}_q^{k_i}$ and $v_i \in \mathbb{F}_q^n$ with $\text{supp}_\pi(v_i) \subseteq \langle i\rangle_P \setminus \{i\}$ such that $T(v_i) = v_i' + u_i$; hence $ker(\phi) = \mathcal{N}$.

3. Follows straight from the definitions.

□

Corollary 2.2.7. *$\mathcal{N}$ is a normal subgroup of $GL(P,\pi,L)_q$.*

Proof. It follows from the fact that $\mathcal{N} = \ker(\phi_T)$, proved in item 2 of Theorem 2.2.6. □

Now we can characterize the group

Theorem 2.2.8. *Every linear isometry S can be written in a unique way as a product $S = F \circ T_\phi$, where $F \in \mathcal{N}$ and $\phi \in Aut(P,\pi,L)$. Furthermore, $GL(P,\pi,L)_q$ is the semi-direct product $GL(P,\pi,L)_q = \mathcal{N} \rtimes \mathcal{A}$.*

2.3 (P,π,L)-Canonical Decomposition of linear codes for hierarchical posets of directed cycles

Two linear codes $\mathcal{C}, \mathcal{C}' \subseteq \mathbb{F}_q^n$ are (P,π,L)-*equivalent* if there is $T \in GL(P,\pi,L)_q$ such that $T(\mathcal{C}) = \mathcal{C}'$.

A decomposition $\mathcal{C} = \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_{h(P)}$ of a code $\mathcal{C}$ is called (P, π, L)-*canonical decomposition* if $\mathrm{supp}_\pi(\mathcal{C}_i) \subseteq \Gamma_i^P$. Working with such decompositions simplifies the computation of all metric invariants of a code. Naturally, not every code admits a (P, π, L)-*canonical decomposition*, but it may be equivalent to a code that has such a decomposition.

Definition 2.3.1. *Let $P = ([m], \preceq)$ be a poset with $h(P)$ levels. We say that a linear code $\mathcal{C} \subseteq \mathbb{F}_q^n$* admits a (P, π, L)-canonical decomposition *if it is (P, π, L)-equivalent to a linear code $\tilde{\mathcal{C}} = \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_{h(P)}$, where* $\mathrm{supp}_\pi(\mathcal{C}_i) \subseteq \Gamma_i^P$.

As we shall see, the hierarchical posets have a crucial role in finding canonical decompositions.

Lemma 2.3.2. *Let $d_{(P,\pi,L)}$ be a metric where poset P is hierarchical with $h(P)$ levels. Let $\mathcal{C} \subset \mathbb{F}_q^n$ be a linear code with* $\mathrm{supp}_\pi(\mathcal{C}) \subset \Gamma_i^P$*, for some $i \in [h(P)]$ and consider $u \in \mathbb{F}_q^n$ such that* $\mathcal{M}(\mathrm{supp}_\pi(u)) \subset \Gamma_i^P$ *and $\tilde{u} \notin \mathcal{C}$, where $\tilde{u}$ is such that $\tilde{u}_i = u_i$ if $i \in$* $\mathcal{M}(\mathrm{supp}_\pi(u))$ *and $\tilde{u}_i = 0$ otherwise. Then, $\mathcal{C} \oplus span\{u\}$ and $\mathcal{C} \oplus span\{\tilde{u}\}$ are (P, π, L)-equivalent codes.*

Proof. Since $\tilde{u} \notin \mathcal{C}$, then $\alpha = \{\tilde{u}, v_1, \dots, v_k\}$ is a basis of $\mathcal{C} \oplus span\{\tilde{u}\}$, where $\{v_1, \dots, v_k\}$ is a basis of $\mathcal{C}$. Extending α, by the canonical vectors, to a basis $\beta = \{\tilde{u}, v_1, \dots, v_k, e_{j_1}, \dots, e_{j_r}\}$ of $\mathbb{F}_q^n$, we construct a linear map $T : \mathbb{F}_q^n \to \mathbb{F}_q^n$ by setting $T(u) = \tilde{u}$, $T(v_i) = v_i$ and $T(e_{j_k}) = e_{j_k}$ that satisfies the desired conditions of the lemma. It is clear that $\Delta = \{u, v_1, \dots, v_k, e_{j_1}, \dots, e_{j_r}\}$ is also a basis of $\mathbb{F}_q^n$.

It is enough to show that T preserves the (P, π, L)-weight. Since P is a hierarchical poset, given $v \in \mathbb{F}_q^n$, we have that $\mathcal{M}(\mathrm{supp}_\pi(u)) \subset \Gamma_j^P$, for some $j \in [h(P)]$. The vector v can be decomposed as $v = w + \lambda u \in \mathbb{F}_q^n$ with $\lambda \in \mathbb{F}_q$ and $w \in span\{\Delta \backslash \{u\}\}$.

If $\lambda = 0$, then $T(w) = w$ and $\mathrm{wt}_{(P,\pi,L)}(w) = \mathrm{wt}_{(P,\pi,L)}(T(w))$.

If $\lambda \neq 0$ and $j \in [h(P)]$ is such that $\mathcal{M}(\mathrm{supp}_\pi(v)) \subset \Gamma_j^P$, then $i \leq j$. If $i < j$, then $\mathrm{wt}_{(P,\pi,L)}(T(v)) = \mathrm{wt}_{(P,\pi,L)}(v)$, since

$$\mathcal{M}(\mathrm{supp}_\pi(T(v)) = \mathcal{M}(\mathrm{supp}_\pi(w + \lambda\tilde{u})) = \mathcal{M}(\mathrm{supp}_\pi(w)) = \mathcal{M}(\mathrm{supp}_\pi(v)).$$

If $i = j$, then

$$\mathcal{M}(\mathrm{supp}_\pi(T(v))) = \mathcal{M}(\mathrm{supp}_\pi(w + \lambda\tilde{u})) = \mathcal{M}(\mathrm{supp}_\pi(w + \lambda u))$$

which implies that $\mathrm{wt}_{(P,\pi,L)}(v) = \mathrm{wt}_{(P,\pi,L)}(T(v))$. □

Remark 2.3.3. *Note that the linear (P, π, L)-isometry T constructed in the proof of previous lemma satisfies* $\mathrm{supp}_\pi(T(\mathcal{C} \oplus \langle u \rangle_P)) \subset \Gamma_i^P$ *and $T(v) = v$, whenever* $\mathrm{supp}_\pi(v) \subset [n] \backslash \Gamma_i^P$*. These properties will play a crucial role to construct a (P, π, L)-equivalent code in a (P, π, L)-canonical form of a code.*

The next theorem is a generalization of the P-canonical decomposition for poset metrics, determined in (FELIX; FIRER, 2012).

Given a poset P and $u \in \mathbb{F}_q^n$, the i-th P-*projection* $\widehat{u}^{P,i} \in \mathbb{F}_q^n$ is defined by $\widehat{u}_j^{P,i} = u_j$ if $j \in \Gamma_i^P$ and $\widehat{u}_j^{P,i} = 0$ otherwise.

Theorem 2.3.4. *The poset P is hierarchical if, and only if, any linear code $\mathcal{D}$ admits a (P, π, L)-canonical decomposition.*

Proof. First, suppose that P is a hierarchical poset with $h(P)$ levels. If $dim(\mathcal{D}) = 1$, it is enough to use Lemma 2.3.2, considering $\mathcal{D} = \mathcal{C} \oplus span\{u\}$ where $\mathcal{C} = \{0\}$ and $0 \neq u \in \mathcal{D}$.

Suppose the result holds for linear codes with dimension smaller then k and let $\mathcal{D} = span\{v_1\} \oplus span\{v_2, \dots, v_k\}$ be a k-dimensional code. The induction hypothesis ensures that, for $\mathcal{D}' = span\{v_2, \dots, v_k\}$, there is a linear isometry T' such that $T'(\mathcal{D}') = \oplus_{i=1}^{l} D_i'$ and $\mathrm{supp}_\pi(D_i') \subset \Gamma_i^P$. Since $T'(v_1) \notin T'(\mathcal{D}')$, then $span\{T'(v_1)\} \cap T'(\mathcal{D}') = \{0\}$ and there exists a level i such that $\widehat{T'(v_1)}^{P,i} \notin \mathcal{D}_i'$.

Denote by i_0 the maximal level with this property and let $u = (u_1, u_2, \dots, u_m)$ be defined by

$$u_i = \begin{cases} T'(v_i), & \text{if} \quad i \in \Gamma_j^P \text{ and } j \leq i_0 \\ 0, & \quad\text{otherwise.} \end{cases}$$

Thus we find that $T'(\mathcal{D}) = span\{u\} \oplus T'(\mathcal{D}')$. Considering $\mathcal{C} = \mathcal{D}_{i_0}'$, Lemma 2.3.2 ensures there is $T \in GL(P, \pi, L)_q$ such that $T(\mathcal{D}_i') = \mathcal{D}_i'$ for $i \neq i_0$ and $\mathrm{supp}_\pi(T(span\{u\} \oplus \mathcal{D}'_{i_0})) = \mathrm{supp}_\pi(span\{\tilde{u}\} \oplus \mathcal{D}'_{i_0}) \subset \Gamma_{i_0}^P$. Therefore, if $\mathcal{D}_i = \mathcal{D}_i'$ for $i \neq i_0$ and $\mathcal{D}_{i_0} = span\{\tilde{u}\} \oplus \mathcal{D}'_{i_0}$, then $\tilde{\mathcal{D}} = T(T'(\mathcal{D})) = \oplus_{i=1}^{l} \mathcal{D}'_i$ is a linear code (P, π, L)-equivalent to $\mathcal{D}$.

On the other hand, suppose that P is not hierarchical and let $i \in [h(P)]$ be the lowest level of P for which P fails to be hierarchical, i.e., there are $a \in \Gamma_i^P$ and $b \in \Gamma_{i+1}^P$ such that $a \npreceq b$. Consider $j \in \pi^{-1}(a)$ and $k \in \pi^{-1}(b)$. The linear code $\mathcal{C} = span\{e_j + e_k\}$ cannot be (P, π, L)-equivalent to a canonically decomposed code $\tilde{\mathcal{C}}$. Indeed, Theorem 2.2.8 insures that any linear (P, π, L)-isometry $T \in GL(P, \pi, L)$ induces an automorphism $\phi_T : [m] \to [m]$. Thus, the ideals $\langle \mathrm{supp}_\pi(T(e_j)) \rangle_P$ and $\langle \mathrm{supp}_\pi(T(e_k)) \rangle_P$ are generated by $\phi_T(\pi(j)) \in [m]$ and $\phi_T(\pi(k)) \in [m]$, respectively. Moreover, since $\pi(j) \npreceq \pi(k)$, we have that $\phi_T(\pi(j)) \npreceq \phi_T(\pi(k))$. It follows that $\mathcal{M}(\mathrm{supp}_\pi(T(span\{e_j + e_k\}))) \supset \{\phi_T(\pi(j)), \phi_T(\pi(k))\}$ is not contained in a single level. Since $dim(\mathcal{C}) = 1$ and $T \in GL(P, \pi, L)$ is taken arbitrarily, we show that $\mathcal{C}$ does not admit a (P, π, L)-canonical decomposition. $\square$

Remark 2.3.5. *In the proof of the previous theorem, we constructed a map T, considering a basis $\beta = \{u_1, \dots, u_k\}$ of $\mathcal{D}$. For the purpose of this Theorem, the choice of the basis is immaterial. For future purpose (Lemma 2.5.6), it is worth to note that the choice may be done in such a way that the linear isometry T (which maps $\mathcal{D}$ into its canonical* *decomposition), restricted to β is defined by $T(u_i) = \tilde{u}_i$. It follows that if P is hierarchical, given a code $\mathcal{C}$ it is possible to find a basis $\beta = \{u_1, u_2, \ldots, u_k\}$ such that, when considering only the maximal components of each u_i, we get a basis $\tilde{\beta} = \{\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_k\}$ such that* $\mathrm{supp}(\tilde{u}_i)$ *is contained in a single level of P and $\tilde{\beta}$ generates a code $\mathcal{D}$ that is a canonical decomposition of $\mathcal{C}$.*

2.4 MacWilliams' Identity and Extension Property

The existence of a (P, π, L)-canonical decomposition is a very useful tool, allowing to simplify the computation of many metric invariants (minimal distance, packing and covering radius) and also to determine conditions which ensure the validity of important results in coding theory, such as the MacWilliams' Extension Property. Just as an example, we show how it allows to determine a type of MacWilliams' Identity for linear codes.

Definition 2.4.1. *A (P, π, L)-structure satisfies the unique decomposition property if, for $1 \leq i \leq h(P)$, given $S, S' \subseteq \Gamma_i^P$ such that*

$$\sum_{a \in S} L(a) = \sum_{b \in S'} L(b),$$

there is a bijection $g : S \to S'$ such that $L(a) = L(g(a))$ and $|\pi^{-1}(a)| = |\pi^{-1}(g(a))|$ for all $a \in S$.

The (P, π, L)-*weight enumerator* of a linear code $\mathcal{C}$ is the polynomial

$$W_{\mathcal{C}}^{(P,\pi,L)}(X) = \sum_{i=0}^{n} A_i^{(P,\pi,L)}(\mathcal{C}) X^i$$

where $A_i^{(P,\pi,L)}(\mathcal{C}) = |\{c \in \mathcal{C} : \mathrm{wt}_{(P,\pi,L)}(c) = i\}|$.

As we know, given a poset $P([n], \preceq_P)$ its *dual* is the poset $P^{\perp}([n], \preceq_{P^{\perp}})$ defined by the opposite relations

$$i \preceq_P j \in E \iff j \preceq_{P^{\perp}} i.$$

It is easy to see that a set $J \in \mathcal{I}(P)$ (i.e., J is an ideal in P) if, and only if, its complement $J^c \in \mathcal{I}(P^{\perp})$.

Definition 2.4.2. *(The MacWilliams Identity) A (P, π, L)-weight* admits *a MacWilliams Identity if for every linear code $\mathcal{C} \subseteq \mathbb{F}_q^n$, the (P, π, L)-weight enumerator $W_{\mathcal{C}}^{(P,\pi,L)}(X)$ of $\mathcal{C}$ determines the $(P^{\perp}, \pi, L)$-weight enumerator $W_{\mathcal{C}^{\perp}}^{(P^{\perp},\pi,L)}$ of the dual code $\mathcal{C}^{\perp}$.*

Theorem 2.4.3. *Consider a (P, π, L)-weight where P is a hierarchical poset. The (P, π, L)-weight admits the MacWilliams Identity if, and only if, it satisfies the unique decomposition property .*

Proof. This proof can be obtained in a similar way presented by Etzion in (ETZION; FIRER; MACHADO, 2017), details is provided in Appendix 2.5 at the end of this chapter. □

The MacWilliams' Extension Property is defined in the same way we did in Chapter 1, when considering combinatorial metrics.

Definition 2.4.4. *(*The MacWilliams Extension Property*) We say that* $(\mathbb{F}_q^n, d_{(P,\pi,L)})$ satisfies the MacWilliams Extension Property *if for any pair of linear codes* $\mathcal{C}$ *and* $\mathcal{C}'$ *and any linear map* $t : \mathcal{C} \to \mathcal{C}'$ *preserving the* (P, π, L)*-weight, there is a* (P, π, L)*-isometry* $T \in GL(P, \pi, L)_q$ *such that* $T|_{\mathcal{C}} = t$*.*

Differently of the poset metrics, the sufficient conditions on the triple (P, π, L) that ensures the existence of a MacWilliams Identity (assuming P being a hierarchical poset and L admits the unique decomposition property) are not enough to characterize those metric spaces $(\mathbb{F}_q^n, d_{(P,\pi,L)})$ that satisfy the MacWilliams Extension Property. Indeed, let $n = 6, m = 3$. Consider the poset P to be the anti-chain poset (only the trivial relations) on $[3]$ and consider the block map

$$\pi(1) = \pi(2) = 1, \pi(3) = \pi(4) = 2 \text{ and } \pi(5) = \pi(6) = 3,$$

and $L : [3] \to \mathbb{Z}$ be constant equal to 2. Then, from Theorem 2.4.3, we have that the (P, π, L)-weight admits the MacWilliams Identity. However, it does not satisfy the extension property. Indeed, consider the linear codes $\mathcal{C}_1 = \{000000, 100010, 101000, 001010\}$ and $\mathcal{C}_2 = \{000000, 010010, 110011, 100001\}$. The linear map $t : \mathcal{C}_1 \to \mathcal{C}_2$ defined by $t(100010) = 010010$ and $t(101000) = 110011$ preserves the (P, π, L)-weight but any possible linear extension does not preserve the (P, π, L)-weight of the vector 100000; that is, t cannot be extended to $T \in GL(P, \pi, L)$. An actual proof that it does not satisfy the extension property will follow from Theorem 2.4.8, but the key point is that $3 = |\mathrm{supp}_\pi(\mathcal{C}_1)| \neq |\mathrm{supp}_\pi(\mathcal{C}_2)| = 2$.

Such a situation can be avoided by assuming an additional constraint.

Definition 2.4.5. *(*Condition Ω*) We say that the* (P, π, L)*-structure satisfies the condition* Ω *if*

$$|\{i \in [m] : |\pi^{-1}(i)| \geq 2 \text{ and } L(i) = k\}| \leq 2,$$

for every $k \in L([m])$*.*

From here on, we assume that the field $\mathbb{F}_q$ is binary, i.e., $q = 2$.

We have proved that every code has a canonical decomposition for a (P, π, L)-weight when P is a hierarchical poset. Looking for a condition to ensure the extension property, we start with the simplest hierarchical poset: Let us assume P is an antichain, a poset with only one level. Given a linear code $\mathcal{C} \subset \mathbb{F}_2^n$, we define

$$I_j(\mathcal{C}) := \{k \in \operatorname{supp}_\pi(\mathcal{C}); L(k) = j\}.$$

When considering two codes $\mathcal{C}_1$ and $\mathcal{C}_2$, we shall write $I_j(\mathcal{C}_i) = I_j^i$.

Lemma 2.4.6. *Let P be an anti-chain and (P, π, L) a labeled-poset-block satisfying the UDP and the Ω condition. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be linear codes in $(\mathbb{F}_2^n, d_{(P,\pi,L)})$ and $t : \mathcal{C}_1 \to \mathcal{C}_2$ a linear map preserving the (P, π, L)-weight. Then, $|I_j^1| = |I_j^2|$, for all $j \leq r = \max\{L(k) : k \in [m]\}$.*

Proof. Let us decompose the support of the codes as $\operatorname{supp}_\pi(\mathcal{C}_i) = I_1^i \cup I_2^i \cup \cdots \cup I_r^i$. It is important to stress that some of the I_j^i may be empty; however, once $I_j^1 = \emptyset$, we must have $I_j^2 = \emptyset$ as well. Indeed, suppose that $I_j^1 \neq \emptyset$. This means there is $u \in \mathcal{C}_1$ such that $\operatorname{supp}_\pi(u) \cap I_j^1 \neq \emptyset$. But since P is assumed to be an anti-chain, we have that

$$\mathrm{wt}_{(P,\pi,L)}(u) = \sum_{i \in \operatorname{supp}_\pi(u)} L(i) = \sum_{i \in \operatorname{supp}_\pi(t(u))} L(i) = \mathrm{wt}_{(P,\pi,L)}(t(u))$$

and the UDP ensures that $\operatorname{supp}_\pi(t(u)) \cap I_j^2$ is also nonempty.

Let us consider $2 \leq j \leq r$. Suppose that $|I_j^1| \leq |I_j^2|$, since $|I_j^i| \leq 2$ (the Ω condition), then the possible values for $(|I_j^1|, |I_j^2|)$ are $(0,0), (0,1), (0,2), (1,1), (1,2)$ and $(2,2)$. The cases $(0,1)$ and $(0,2)$ cannot occur, since, as we just saw, $I_j^1 = \emptyset$ *iff* $I_j^2 = \emptyset$. We need to discard the case $(|I_j^1|, |I_j^2|) = (1,2)$. Let $u \in \mathcal{C}_1$ be a vector such that $\operatorname{supp}_\pi(u) = I_j^1$. The UDP ensures there is no $v \in \mathcal{C}_2$ such that $\operatorname{supp}_\pi(v) = I_j^2$. However, there must be $v, w \in \mathcal{C}_2$ such that $\operatorname{supp}_\pi(v) \cup \operatorname{supp}_\pi(w) = I_j^2$ and then we have that either $\operatorname{supp}_\pi(v) = I_j^2$, or $\operatorname{supp}_\pi(w) = I_j^2$, or $\operatorname{supp}_\pi(v + w) = I_j^2$, a contradiction.

Now, we need to prove that $I_1^1 = I_1^2$. We consider a set $X = \{u_1, u_2, \ldots, u_s\}$ such that $\bigcup_{i=1}^{s} (\operatorname{supp}_\pi(u_i) \cap I_1^1) = I_1^1$, with s minimal. A careful use of the inclusion-exclusion principle and the UDP ensures that

$$\left| \bigcup_{i=1}^{s} (\operatorname{supp}_\pi(u_i) \cap I_1^1) \right| = \left| \bigcup_{i=1}^{s} (\operatorname{supp}_\pi(t(u_i)) \cap I_1^2) \right| \leq |I_1^2|$$

and we get that $|I_1^1| \leq |I_1^2|$. A similar reasoning for the inverse t^{-1} ensures that $|I_1^1| = |I_1^2|$. □

Proposition 2.4.7. *Let P be an anti-chain poset. The metric space $(\mathbb{F}_2^n, d_{(P,\pi,L)})$ satisfies the MacWilliams Extension Property if, and only if, L satisfies both the UDP and the Condition Ω.*

Proof. First of all, we shall prove that the two stated conditions are sufficient. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two linear codes and let $t : \mathcal{C}_1 \to \mathcal{C}_2$ be a linear map that preserves the (P, π, L)-weight. Given $u, v \in \mathcal{C}_1$, the UDP ensures the existence of the bijections $g_1 : \mathrm{supp}_\pi(u) \to \mathrm{supp}_\pi(t(u))$ and $g_2 : \mathrm{supp}_\pi(v) \to \mathrm{supp}_\pi(t(v))$. We claim that it is possible to choose g_1 and g_2 in such a way that, if $i \in \mathrm{supp}_\pi(u) \cap \mathrm{supp}_\pi(v)$, then $g_1(i) = g_2(i)$. Indeed, suppose that $g_1(i) \neq g_2(i)$ and consider the linear codes $\mathcal{C} = span\{u, v\}$ and $\mathcal{C}' = span\{t(u), t(v)\}$. The only obstructions to g_1 and g_2 to satisfy this condition would be if either $|I_{L(i)}(\mathcal{C})| < |I_{L(i)}(\mathcal{C}')|$ or $|I_{L(i)}(span\{u\})| < |I_{L(i)}(span\{t(u)\})|$, contradicting Lemma 2.4.6. Hence, there is a bijection $\phi_t : \mathrm{supp}_\pi(\mathcal{C}_1) \to \mathrm{supp}_\pi(\mathcal{C}_2)$ such that, for every $u \in \mathcal{C}_1$ the restriction map $\phi_t : \mathrm{supp}_\pi(u) \to \mathrm{supp}_\pi(t(u))$ is a bijection preserving the L-weight. Since L satisfies the UDP and

$$\sum_{i \in [m] \setminus \mathrm{supp}_\pi(\mathcal{C}_1)} L(i) = \sum_{i \in [m] \setminus \mathrm{supp}_\pi(\mathcal{C}_2)} L(i),$$

then ϕ_t can be extended to $\varphi : [m] \to [m]$.

Given $u \in \mathbb{F}_2^n$, we write $u = u_1 + \cdots + u_m$, where $\mathrm{supp}_\pi(u_i) \subset \{i\} \subset [m]$. It follows that $\mathcal{C}_j \subset \mathcal{C}_{j1} \oplus \cdots \oplus \mathcal{C}_{jm}$, where $\mathcal{C}_{ji} = \{u_i; u \in \mathcal{C}_j\}$, for $j = 1, 2$.

Let $\beta_j = \{u_{j1}, \ldots, u_{jk_j}\}$ be a basis of $\mathcal{C}_{1j}$ and $\alpha_{\varphi(j)} = \{t(u)_{\varphi(j)1}, \ldots, t(u)_{\varphi(j)k_j}\}$ be a basis of $\mathcal{C}_{2\varphi(j)}$. We consider W_j to be the subspace of $\mathbb{F}_2^n$ isomorphic to $\mathbb{F}_2^{L(i)}$ such that $\mathrm{supp}_\pi(e_{jk}) = v_j$ and we extend the basis β_j of $\mathcal{C}_{1j}$ to a basis $\beta'_j = \{x_{j1}, \ldots, x_{jk_j}, e_{j1}, \ldots, e_{jr_j}\}$ of W_j. It follows that, $\mathbb{F}_2^n = \bigoplus_{i=1}^{m} W_j$.

In the same way, $\alpha'_{\varphi(j)} = \{t(x)_{\varphi(j)1}, \ldots, t(x)_{\varphi(j)k_j}, f_{\varphi(j)1}, \ldots, f_{\varphi(j)r_j}\}$ is a basis of $W_{\varphi(j)}$, an extension of the basis $\alpha_{\varphi(j)}$ of $\mathcal{C}_{2\varphi(j)}$.

Let $T : \mathbb{F}_2^n \to \mathbb{F}_2^n$ be the linear map defined by $T(x_{ij}) = t(x)_{\varphi(i)j}$ and $T(e_{ij}) = f_{\varphi(i)j}$. By construction, T is a linear (P, π, L)-isometry that extends t.

Now we prove that the two stated conditions are necessary.

<u>UDP</u>: Consider the sets $S, S' \subset [m]$ and the linear codes $\mathcal{C}_1 = span\{u\}$ and $\mathcal{C}_2 = span\{v\}$, where $u = \sum_{i \in S} e_i$ and $v = \sum_{i \in S'} e_i$.

If $\sum_{i \in S} L(i) = \sum_{i \in S'} L(i)$ we have that $\mathrm{wt}_{(P,\pi,L)}(u) = \mathrm{wt}_{(P,\pi,L)}(v)$ hence the linear map $t : \mathcal{C}_1 \to \mathcal{C}_2$ defined by $t(u) = v$ preserves the (P, π, L)-weight. Suppose that t can be extended to $T \in GL(P, \pi, L)$. Then, Lemma 2.3.2 ensures that T defines a (P, π, L)-automorphism ϕ such that $\phi(S) = S'$, but his means that $L(i) = L(\phi(i))$, for all $i \in S$, i.e., the UDP is satisfied.

<u>Condition Ω</u>: Suppose $|\{i \in [m] : |\pi^{-1}(i)| \geq 2 \text{ and } L(i) = l\}| > 2$, for some $l \in L([m])$. Let $i_1, i_2, i_3 \in [m]$ be such that $L(i_j) = l$ and $|\pi^{-1}(i_j)| \geq 2$ for $j \in \{1, 2, 3\}$. For $j = 1, 2, 3$, let e_{a_j} and e_{b_j} be two different vetors in $\mathbb{F}_2^n$ such that $\mathrm{supp}_\pi(e_{a_j}) = \mathrm{supp}_\pi(e_{b_j}) =$

i_j, which existence is ensured by the fact that $|\pi^{-1}(i_j)| \geq 2$. Let us define the linear codes $\mathcal{C}_1 = span\{e_{a_1} + e_{a_2}, e_{a_1} + e_{a_3}\}, \mathcal{C}_2 = span\{e_{a_1} + e_{a_3}, e_{b_1} + e_{b_3}\} \subset \mathbb{F}_2^n$. Since every the codewords has (P, π, L)-weight is equal to $2l$, any linear isomorphism t between the codes preserves the (P, π, L)-weight. However, $3 = |I_l^1|$ and $|I_l^2| = 2$ so t can not be extended to an isometry $T \in GL(P, \pi, L)$. □

Theorem 2.4.8. *Let $P = ([m], \preceq_P)$ be a hierarchical poset. Then, $(\mathbb{F}_2^n, d_{(P,\pi,L)})$ satisfies the MacWilliams Extension Property if, and only if, L satisfies the UDP and for each $1 \leq i \leq h(P)$, the restriction of L to the level Γ_i^P satisfies Condition Ω.*

Proof. Since we are assuming P to be hierarchical, the canonical decomposition (Theorem 2.3.4) allows us to assume every linear code is in the canonical decomposition. Consider $\mathcal{C} = \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_{h(P)}$ and $\mathcal{C}' = \mathcal{C}'_1 \oplus \cdots \oplus \mathcal{C}'_{h(P)}$.

Let us assume that UDP and the Condition Ω both hold. Let $t : \mathcal{C}_1 \to \mathcal{C}_2$ be a local isometry. Given $u \in \mathcal{C}_i$, we have $t(u) = t_i(u) + F_i(u)$, where $F_i : \mathcal{C}_i \to \sum_{j<i} \mathcal{C}'_j$ and $t_i : \mathcal{C}_i \to \mathcal{C}'_i$ are both linear maps. Then, it is easy to verify that $t_i : \mathcal{C}_i \to \mathcal{C}'_i$ is also a linear isometry. Since $\text{supp}(\mathcal{C}_i), \text{supp}(\mathcal{C}'_i) \subset \Gamma_i^P$, we can consider $\mathcal{C}_i, \mathcal{C}'_i \subset \mathbb{F}_2^{n_i}$ to be equipped with the metric $((\Gamma_i^P, \preceq_P), \pi, L)$ on $\mathbb{F}_2^{n_i}$, where $(\Gamma_i^P, \preceq_P)$ denotes de subposet induced by P on level Γ_i^P, an antichain poset. The previous proposition ensures that each t_i admits an extension T_i to $\mathbb{F}_2^{n_i}$, it means that, there is a linear $((\Gamma_i^P, \preceq_P), \pi, L)$-isometry $T_i \in GL((\Gamma_i^P, \preceq_P), \pi, L)$ of $\mathbb{F}_2^{n_i}$ into itself and $T_i|_{\mathcal{C}_i} = t_i$. The linear map $T : \mathbb{F}_2^n \to \mathbb{F}_2^n$ defined by $T(u_1 + \cdots + u_{h(P)}) = (T_1 + F_1)(u_1) + \cdots + (T_{h(P)} + F_{h(P)})(u_{h(P)})$ is a (P, π, L)-isometry, since P is hierarchical and each $T_i + F_i$ is a $d_{(P,\pi,L)}$ isometry restricted to the vectors with support on level Γ_i^P or bellow, with at least one element in Γ_i^P.

Suppose that the UDP and the Condition Ω are holds until the level Γ_{j-1}^P of P but one of them fails on level Γ_j^P. Since P is assumed to be hierarchical, it is possible to construct codes $\mathcal{C}_1$ and $\mathcal{C}_2$ as in Proposition 2.4.7, just taking the care to consider the generator vectors u, v (in case the UDP does not hold) and the vectors e_{a_i}, e_{b_i} (in case condition Ω does not hold) to have the support on the j-level. Then, proceeding as in the proof of Proposition 2.4.7, we construct a local isometry that can not be extended to a global one. □

2.5 Proof for the MacWilliams Identity

Although the proof for MacWilliams' Identity in the context of the labeled-poset-block metrics follows similarly as in the case considering digraph metrics, presented by Etzion et. al (ETZION; FIRER; MACHADO, 2017), here we present every detail. We start by presenting some crucial facts regarding additive characters and, then we follow by adapting MacWilliams' original approach. An *additive character* χ of $\mathbb{F}_q$ is a nontrivial homomorphism of the additive group $\mathbb{F}_q$ into the multiplicative group of complex numbers with 1-norm. The next lemma is well known.

Lemma 2.5.1. *Let $\mathcal{C} \subset \mathbb{F}_q^n$ be a linear code and let χ of $\mathbb{F}_q$ be an additive character. Then,*

$$\sum_{u \in \mathcal{C}} \chi(u \cdot v) = \begin{cases} |\mathcal{C}|, & \text{if} \quad v \in \mathcal{C}^\perp \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.5.2. *Given a (P, π, L)-weight and linear code $\mathcal{C} \subset \mathbb{F}_q^n$, then*

$$A_i^{(P,\pi,L)}(\mathcal{C}) = \frac{1}{|\mathcal{C}^\perp|} \sum_{1 \leq j \leq n} \sum_{u \in \mathcal{C}^\perp \cap \overline{S}_j} \sum_{v \in S_i} \chi(u \cdot v),$$

where S_i and $\overline{S}_j$ are the spheres of radii i and j considering the weights induced by the triples (P, π, L) and $(P^\perp, \pi, L)$, respectively, i.e., $S_i = \{u \in \mathbb{F}_q^n : \mathrm{wt}_{(P,\pi,L)}(u) = i\}$ and $\overline{S}_j = \{u \in \mathbb{F}_q^n : \mathrm{wt}_{(P^\perp,\pi,L)}(u) = j\}$.

Proof. Since $A_i^{(P,\pi,L)} = |\mathcal{C} \cap S_i|$, it follows that $A_i^{(P,\pi,L)} = \sum_{u \in \mathcal{C} \cap S_i} 1$. Lemma 2.5.1 implies that

$$\begin{aligned} A_i^{(P,\pi,L)}(\mathcal{C}) &= \sum_{v \in S_i} \frac{1}{|\mathcal{C}^\perp|} \sum_{u \in \mathcal{C}^\perp} \chi(u \cdot v) \\ &= \frac{1}{|\mathcal{C}^\perp|} \sum_{u \in \mathcal{C}^\perp} \sum_{v \in S_i} \chi(u \cdot v) \\ &= \frac{1}{|\mathcal{C}^\perp|} \sum_{1 \leq j \leq n} \sum_{u \in \mathcal{C}^\perp \cap \overline{S}_j} \sum_{v \in S_i} \chi(u \cdot v). \end{aligned}$$

□

Lemma 2.5.3. *A (P, π, L)-weight admits the MacWilliams Identity if, given $i, j \in \{\mathrm{wt}_P(u) : u \in \mathbb{F}_q^n\}$, and $u, u' \in S_i$, then*

$$\sum_{v \in \overline{S}_j} \chi(u \cdot v) = \sum_{v \in \overline{S}_j} \chi(u' \cdot v).$$

Proof. Let $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{F}_q^n$ be linear codes such that $W_{\mathcal{C}_1}^{(P,\pi,L)}(X) = W_{\mathcal{C}_2}^{(P,\pi,L)}(X)$. By hypotheses, $\sum_{v \in \overline{S}_j} \chi(u \cdot v)$ depends only on the choice of i and j, so that we can consider $p_{ij} := \sum_{v \in \overline{S}_j} \chi(u \cdot v)$. From Lemma 2.5.2,

$$\begin{aligned} A_j^{(P^\perp,\pi,L)}(\mathcal{C}_k^\perp) &= \frac{1}{|\mathcal{C}_k|} \sum_{1 \leq i \leq n} \sum_{u \in \mathcal{C}_k \cap S_i} p_{ij} \\ &= \frac{1}{|\mathcal{C}_k|} \sum_{1 \leq i \leq n} A_i^{(P,\pi,L)}(\mathcal{C}_k) p_{ij}, \text{ for } k \in \{1, 2\}. \end{aligned}$$

Since $A_i^{(P,\pi,L)}(\mathcal{C}_1) = A_i^{(P,\pi,L)}(\mathcal{C}_2)$, we have that $W_{\mathcal{C}_1^{\perp}}^{(P^{\perp},\pi,L)}(X) = W_{\mathcal{C}_2^{\perp}}^{(P^{\perp},\pi,L)}(X)$. $\square$

Lemma 2.5.4. *Let (P,π,L)-weight be such that P is an anti-chain poset, i.e., $h(P)=1$. Given $I \in \mathcal{I}(P)$, let $S_I = \{u \in \mathbb{F}_q^n : \operatorname{supp}_\pi(u) = I\}$. Then, for $u \in S_I$ we have that,*

$$\sum_{v\in S_{J^c}} \chi(u\cdot v) = (-1)^{|I\cap J^c|} \prod_{i\in I^c\cap J^c} (q^{k_i}-1) \prod_{j\in J^c} (q^{k_j}-1)^{|J^c|-1}.$$

Proof. Given $u \in \mathbb{F}_q^n$, the vector $\overline{u}^i \in \mathbb{F}_q^n$ is defined by $\overline{u}^i_j = u_j$ if $\pi(j) = i$ and $\overline{u}^i_j = 0$ otherwise. Thus,

$$\begin{aligned}
\sum_{v\in S_{J^c}} \chi(u\cdot v) &= \sum_{v\in S_{J^c}} \chi\left(\sum_{i=1}^{m} \overline{u}^i\cdot\overline{v}^i\right)\\
&= \sum_{v\in S_{J^c}} \prod_{i\in J^c} \chi\left(\overline{u}^i\cdot\overline{v}^i\right) \prod_{i\in J} \chi\left(\overline{u}^i\cdot\overline{v}^i\right)\\
&= \prod_{i\in J^c} \sum_{v\in S_{J^c}} \chi\left(\overline{u}^i\cdot\overline{v}^i\right),
\end{aligned}$$

where in the first equality we consider the fact that χ is a group homomorphism and we separate the product in the J and J^c parts and the second follows from the fact that $\overline{u}^i\cdot\overline{v}^i = 0$ for $i \in J$ and $\chi(0) = 1$. We note that, given $\overline{v}^1 + \cdots + \overline{v}^m \in S_{J^c}$ we have that $\overline{v}^i \in \mathbb{F}_q^{k_i}\backslash\{0\} \subset \mathbb{F}_q^n$ if $i \in J^c$ and $\overline{v}^i = 0$, otherwise. It follows that

$$\begin{aligned}
\sum_{v\in S_{J^c}} \chi\left(\overline{u}^i\cdot\overline{v}^i\right) &= \sum_{\overline{v}^1+\cdots+\overline{v}^m\in S_{J^c}} \chi\left(\overline{u}^i\cdot\overline{v}^i\right)\\
&= \prod_{j\in J^c\backslash\{i\}} (q^{k_j}-1) \sum_{\overline{v}^i\in\mathbb{F}_q^{k_i}\backslash\{0\}\subset\mathbb{F}_q^n} \chi\left(\overline{u}^i\cdot\overline{v}^i\right).
\end{aligned}$$

Furthermore, from Lemma 2.5.1, we have that $\sum_{v\in\mathbb{F}_q^{k_i}\backslash\{0\}} \chi(v\cdot u) = q^{k_i}-1$ if $u=0$ and equals -1, otherwise. Hence,

$$\begin{aligned}
&\sum_{v\in S_{J^c}} \chi(u\cdot v) =\\
&\quad= (-1)^{|\operatorname{supp}_\pi(u)\cap J^c|} \prod_{i\in J^c\cap([m]\backslash\operatorname{supp}_\pi(u))} (q^{k_i}-1) \prod_{j\in J^c} (q^{k_j}-1)^{|J^c|-1}\\
&\quad= (-1)^{|I\cap J^c|} \prod_{i\in I^c\cap J^c} (q^{k_i}-1) \prod_{j\in J^c} (q^{k_j}-1)^{|J^c|-1}.
\end{aligned}$$

where the second equality follows from the fact $\operatorname{supp}_\pi(u)\cap J^c = I\cap J^c$, since $h(P)=1$. $\square$

In the next theorem, we will use a variation of the previous lemma, considering not only the ideal J^c in $P^\perp$, but the family $\overline{J^c}$ of all ideals K^c in $P^\perp$ such that $\sum_{i\in J^c} L(i) =$

$\sum_{i\in K^c} L(i)$. Given $I, J \in \mathcal{I}(P)$ and $u \in S_I$, then

$$\begin{aligned}\sum_{v\in S_{\overline{J^c}}} \chi(u\cdot v) &= \sum_{K^c\in\overline{J^c}} \sum_{v\in S_{K^c}} \chi(u\cdot v)\\ &= \sum_{K^c\in\overline{J^c}} (-1)^{|I\cap K^c|} \prod_{i\in I^c\cap K^c} (q^{k_i}-1) \prod_{j\in K^c} (q^{k_j}-1)^{|K^c|-1}.\end{aligned}$$

Theorem 2.5.5. *Let (P,π,L)-weight be such that P is an anti-chain. Then,* $\mathrm{wt}_{(P,\pi,L)}$ *admits the MacWilliams Identity if, and only if, (P,π,L)-structure satisfies the UDP.*

Proof. It is clear to see that

$$\sum_{i\in I} L(i) = \sum_{i\in J} L(i) \iff \sum_{i\in I^c} L(i) = \sum_{i\in J^c} L(i).$$

Given $u, u' \in S_r$, consider the ideals $I_1 = \mathrm{supp}_\pi(u)$ and $I_2 = \mathrm{supp}_\pi(u')$. This means that $\sum_{i\in I_1} L(i) = \sum_{i\in I_2} L(i) = r$. If the (P,π,L)-structure satisfies the UDP, then there is a bijection $g : I_1 \to I_2$ such that $L(i) = L(g(i))$ and $|\pi^{-1}(i)| = |\pi^{-1}(g(i))|$. Furthermore, since $\sum_{i\in I_1^c} L(i) = \sum_{i\in I_2^c} L(i)$, then g may be extended to $[m]$. It implies that there is a (P,π,L)-automorphism $\varphi \in Aut(P,\pi,L)$ such that $\varphi(I_1) = I_2$.

Given $J \in \mathcal{I}(P)$, let $r = \sum_{i\in J} L(i)$. Lemma 2.5.4 ensures that

$$\sum_{v\in S_r} \chi(u\cdot v) = \sum_{v\in S_r} \chi(u'\cdot v).$$

It follows straightforward from Lemma 2.5.3 that $\mathrm{wt}_{(P,\pi,L)}$ admits the MacWilliams Identity.

On the other hand, let $S, S' \subset [m]$ be minimal sets such that: *(i)* $\sum_{i\in S} L(i) = \sum_{i\in S'} L(i)$; *(ii)* there is no bijection $g : S \to S'$ such that $L(i) = L(g(i))$ and $|\pi^{-1}(i)| = |\pi^{-1}(g(i))|$. Consider the linear codes $\mathcal{C}_1 = span\{u\}$ and $\mathcal{C}_2 = span\{v\}$, where $u = \sum_{i\in\pi^{-1}(S)} e_i$ and $v = \sum_{i\in\pi^{-1}(S')} e_i$. By construction, the (P,π,L)-weight enumerators of $\mathcal{C}_1$ and $\mathcal{C}_2$ are equal. Furthermore, by the minimality of S and S', $\mathrm{supp}_\pi(\mathcal{C}_1) \cap \mathrm{supp}_\pi(\mathcal{C}_2) = \emptyset$ and either $L(i) \neq L(j)$ or $|\pi^{-1}(i)| = |\pi^{-1}(j)|$, for any $i \in S$ and $j \in S'$. It follows that there exists $i_0 \in S \cup S'$ such that $L(i_0) \leq L(j)$ or $|\pi^{-1}(i_0)| \leq |\pi^{-1}(j)|$, for each $j \in S \cup S'$. Suppose $L(i_0) \leq L(j)$ and $i_0 \in S$, then there is $w \in \mathbb{F}_q^n \setminus \mathcal{C}_1^\perp$ such that $\mathrm{supp}_\pi(w) = \{i_0\}$, which implies $A_{L(i_0)}^{(P,\pi,L)}(\mathcal{C}_1^\perp) < A_{L(i_0)}^{(P,\pi,L)}(\mathcal{C}_2^\perp)$.

Now, suppose that given any $S, S' \subset [m]$ such that $\sum_{i\in S} L(i) = \sum_{i\in S'} L(i)$, there exists $g : S \to S'$ such that $L(i) = L(g(i))$. It follows that $|\pi^{-1}(i)| \neq |\pi^{-1}(g(i))|$. Let $i_0 \in [m]$ be such that $L(i_0) \leq L(i)$, for every $i \in S$. In addition, without loss of generality assume that $|\pi^{-1}(i_0)| < |\pi^{-1}(g(i_0))|$. The minimality constraint on S and S' implies $S = \{i_0\}$ and $S' = \{g(i_0)\}$. Since any $w \in \mathbb{F}_q^n$ with $\mathrm{supp}_\pi(w) = \{g(i_0)\}$ belongs to $\mathcal{C}_1^\perp$, it follows that $A_{L(i_0)}^{(P,\pi,L)}(\mathcal{C}_2^\perp) < A_{L(i_0)}^{(P,\pi,L)}(\mathcal{C}_1^\perp)$. □

Lemma 2.5.6. *Let* $\mathrm{wt}_{(P,\pi,L)}$ *be such that* P *is a hierarchical poset with* l *levels. Let* $\mathcal{C}$ *be a linear code and* $\mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_l$ *its* (P,π,L)*-canonical decomposition. Let* $\mathcal{D}_i = \{y \in \mathcal{C}_i^\perp : \mathrm{supp}_\pi(v) \subset \Gamma_i^P\}$ *and* $\mathcal{D} = \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_l$*. Then,* $\mathcal{C}^\perp$ *and* $\mathcal{D}$ *are* (P,π,L)*-equivalent.*

Proof. Let $\alpha = \{u_1, u_2, \ldots, u_r\}$ be a basis of $\mathcal{C}$ such that $\{\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_r\}$ generates $\mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_l$, as ensured by Remark 2.3.5. Since P is assumed to be hierarchical, so is $P^\perp$ and it follows, again by Remark 2.3.5, that $\mathcal{C}^\perp$ admits a basis $\beta = \{v_1, v_2, \ldots, v_{n-r}\}$ such that $\tilde{\beta} = \{\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_{n-r}\}$ generates a code $(P^\perp,\pi,L)$-equivalent to $\mathcal{C}^\perp$, that is a $(P^\perp,\pi,L)$-canonical decomposition of $\mathcal{C}^\perp$. We claim that this code is already $\mathcal{D}$. Indeed, to conclude that, we just need to prove $\tilde{v}_j \cdot \tilde{v}_k = 0$, for all $1 \le j \le r$ and $1 \le k \le n-r$. Actually, since all those vectors have the support in a single level, it is enough to prove $\tilde{v}_j \cdot \tilde{v}_k = 0$ for $\tilde{u}_j \in \tilde{\alpha}$ and $\tilde{v}_k \in \tilde{\beta}$ such that $\mathrm{supp}_\pi(\tilde{u}_j), \mathrm{supp}_\pi(\tilde{v}_k) \subset \Gamma_i^P$. Note that, if $\mathrm{supp}_\pi(\tilde{u}_j) \subset \Gamma_i^P$, then $\mathrm{supp}_\pi(u_j) \subset \Gamma_1^P \cup \Gamma_2^P \cup \cdots \cup \Gamma_i^P$. Analogously, if $\mathrm{supp}_\pi(\tilde{v}_k) \subset \Gamma_i^P$, then $\mathrm{supp}_\pi(v_k) \subset \Gamma_i^P \cup \Gamma_{i+1}^P \cup \cdots \cup \Gamma_l^P$, since the metric is induced by $(P^\perp,\pi,L)$. This implies that $u_j \cdot v_k = \tilde{u}_j \cdot \tilde{v}_k$ and since $u_j \cdot v_k = 0$, it follows that $\tilde{u}_j \cdot \tilde{v}_k = 0$.

□

Theorem 2.5.7. *Let* $\mathrm{wt}_{(P,\pi,L)}$ *be such that* P *is a hierarchical poset. Then,* $\mathrm{wt}_{(P,\pi,L)}$ *admits a MacWilliams' Identity if, and only if,* (P,π,L) *satisfies the UDP.*

Proof. Suppose that the P has m_i elements in the i-th level. From Lemma 2.5.6, we may assume, without loss of generality, that a linear code $\mathcal{C} \subset \mathbb{F}_q^n$ is already in a (P,π,L)-canonical form $\mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_{h(P)}$ and its dual is equivalent to $\mathcal{C}^\perp = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \cdots \oplus \mathcal{D}_{h(P)}$, where $\mathcal{D}_i$ is defined as in Lemma 2.5.6.

Furthermore, since the (P,π,L)-structure satisfies the Unique Decomposition Property, the (Γ_i^P,π,L) (where Γ_i^P denotes the poset induced by P on its i-th) also satisfies the Unique Decomposition Property and so, Theorem 2.4.3 ensures that $W_{\mathcal{C}_i}^{(\Gamma_i^P,\pi,L)}(X)$ determines $W_{\mathcal{D}_i}^{(\Gamma_i^P,\pi,L)}(X)$. Since $W_{\mathcal{C}}^{(P,\pi,L)}(X)$ can be expressed in terms of the $W_{\mathcal{C}_i}^{(\Gamma_i^P,\pi,L)}(X)$'s (and similarly for $W_{\mathcal{C}^\perp}^{(\Gamma_i^P,\pi,L)}(X)$), we conclude the proof.

□

3 Weights which respect support

In the literature, matching between channels and metrics (that is, the coincidence of the maximum likelihood decoding and nearest neighbor decoding) is an underestimated subject of study. Despite the large number of channels that are studied and the large number of metrics described in the literature in the context of Coding Theory (see, for example, [Chapter 16] in (DEZA; DEZA, 2009), and a recent survey of Gabidulin (GABIDULIN, 2012)), there are a few examples of classical metrics and channels which are proved to be matched.

Although matching channels and metrics is not widely studied, there are some advances establishing whenever a discrete memoryless channel admits a metric, (QURESHI, 2019), (D'OLIVEIRA; FIRER, 2019). From another perspective, we can find in the literature large families of metrics satisfying the basic decoding conditions described in the Introduction. We cite, for example, the poset metrics of Brualdi (BRUALDI; GRAVES; LAWRENCE, 1995), Gabidulin's combinatorial metrics (GABIDULIN, 1973), poset-block metrics (ALVES; PANEK; FIRER, 2008) and digraph metrics (ETZION; FIRER; MACHADO, 2017).

All these generalize the Hamming metric and they represent very large families of metrics over a vector space $\mathbb{F}_q^n$ (large in the sense that each of these families grows exponentially with n). Nevertheless, they are not sufficient to determine all the MDD criteria (which is the rule for the MDD decoding) satisfying the support condition. Example 3.0.1 illustrates such an affirmation for the smallest possible case, $n = 2$.

Before introducing the example, we should remark that different metrics may determine the same MMD, and in this case, we should consider such metrics to be equivalent. To be more precise: two metrics d_1 and d_2 over a space V are *decoding-equivalent* if given any code $\mathcal{C} \subset \mathbb{F}_q^n$ and any received message $u \in \mathbb{F}_q^n$, the MDDs determined by both metrics generate the same set of codewords, i.e.,

$$\arg\min_{c\in\mathcal{C}} d_1(u,c) = \arg\min_{c\in\mathcal{C}} d_2(u,c), \text{ for any } \mathcal{C} \subset \mathbb{F}_q^n, u \in \mathbb{F}_q^n. \tag{3.1}$$

It is not difficult to prove that for metrics defined by weights, being equivalent means that, when ordering the vectors in $\mathbb{F}_q^n$ according to the two different weights we get the same ordering (see Definition 3.1.3 for details).

Example 3.0.1. *Let us consider the space $\mathbb{F}_2^2 = \{00, 10, 01, 11\}$. In this case we have 4 non decoding-equivalent criteria. In the table bellow we present these criteria and check each that can be determined by a metric in one of the large families we have mentioned:*

poset wt_P*, poset-blocks* wt_{PB}*, combinatorial* wt_C *and digraph* wt_D*. It is worth to note that only the first one can be determined by any of these families of metrics.*

Criterion	wt_H	wt_P	wt_{PB}	wt_C	wt_D
$\mathrm{wt}(10) = \mathrm{wt}(01) < \mathrm{wt}(11)$	✓	✓	✓	✓	✓
$\mathrm{wt}(10) = \mathrm{wt}(01) = \mathrm{wt}(11)$			✓	✓	✓
$\mathrm{wt}(10) < \mathrm{wt}(01) = \mathrm{wt}(11)$		✓	✓		✓
$\mathrm{wt}(10) < \mathrm{wt}(01) < \mathrm{wt}(11)$					

Table 1 – Decoding criteria which respect support in $\mathbb{F}_2^2$

We stress that different metrics can be decoding-equivalent. In fact, even though the second criterion in the table ($\mathrm{wt}(u)$ *is constant for* $u \neq 0$*) may be determined by a poset-block* wt_{PB} *and also by a digraph* wt_D *weight, simple computations shows that* $\mathrm{wt}_{PB}(u) = 1$ *and* $\mathrm{wt}_D(u) = 2$*, for* $u \neq 0$*. More important, we note that the last decoding criterion can not be determined by any metric belonging to one of these families.*

In this chapter we aim to reduce the gap between the known and studied TS-metrics and the family of all possible TS-metrics. To do so, we first introduce a systematic approach to the space of all TS-metrics, by labeling the edges of the Hamming cube. Then we introduce a conditional operator on metrics, which allows us to obtain new TS-metrics out of given one. We show that any TS metric can be obtained, by a sequence of conditional sums, out of the poset, digraph and combinatorial metrics. For this reason, this chapter moves one step forward in a long term goal to develop an “approximation theory” of metrics in the context of coding theory.

3.1 S-weights

In this chapter, we are concerned not with particular metrics, but to decoding criteria, that is, we shall look into the decoding-equivalence classes of metrics as defined in 3.1. A simpler equivalence relation can be stated considering only the metrics, or weights, as we shall do in Definition 3.1.3.

Before we do it, we shall establish a condition on weights to ensure that it determines a distance satisfying the TS conditions.

Definition 3.1.1. *A function* $\mathrm{wt} : \mathbb{F}_q^n \to \mathbb{Z}$ *is a weight that respects support if the following holds:*

1. $\mathrm{wt}(u) \geq 0$ *and equality implies* $u = 0$*;*

2. *If* $\mathrm{supp}(u) \subset \mathrm{supp}(v)$*, then* $\mathrm{wt}(u) \leq \mathrm{wt}(v)$*, where* $\mathrm{supp}(u) = \{i \in [n] : u_i \neq 0\}$.

Remark 3.1.2. *From here on, we will consider only weights respecting support of vectors, hence we will call it just* ***weight****. In addition, it is worth to stress that if* $\mathrm{supp}(u) = \mathrm{supp}(v)$*, then* $\mathrm{wt}(u) = \mathrm{wt}(v)$.

A weight determines a semi-metric, by defining $d(u, v) = \mathrm{wt}(u - v)$, and two weights determine the same semi-metric if, and only if, they are equal. In order to guarantee that $d(u, v) = \mathrm{wt}(u - v)$ is a metric it is required the triangle inequality which we ignore in this work due to the fact that every semi-metric d (on a finite space) can be rescaled to a decoding-equivalent metric d' as follows:

$$d'(u,v) = \begin{cases} 0, & \text{if} \quad u = v, \\ d(u,v) + \max\limits_{x,y \in \mathbb{F}_q^n} d(x,y), & \text{if} \quad u \neq v. \end{cases}$$

So, to understand the space of all TS-metrics, it is enough to study the space of all weights up to the following equivalence:

Definition 3.1.3. *We say that two weights* wt_1 *and* wt_2 *are* equivalent *(denoted by* $\mathrm{wt}_1 \sim \mathrm{wt}_2$*) if*

$$\mathrm{wt}_1(u) < \mathrm{wt}_1(v) \iff \mathrm{wt}_2(u) < \mathrm{wt}_2(v), \forall u, v \in \mathbb{F}_q^n.$$

It is not difficult to see that two weights are equivalent if, and only if, they are decoding-equivalent (see (D'OLIVEIRA; FIRER, 2019) for details).

3.2 Weights respecting support

In order to explore properties of S-weights, our approach is to construct general S-weights in a way it can inherit the knowledge accumulated about poset, digraph and combinatorial metrics. For this purpose, we shall represent S-weights by labeling the edges of the Hamming cube. This approach allows us to obtain a description for the group of linear isometries which is a fundamental tool in the context of coding theory. Indeed, being in the same orbit of this group is the definition of code equivalence and the structure of these groups is used to determine whenever the MacWilliams' extension property is satisfied and so forth.

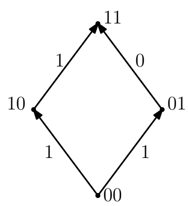
We start by considering the Hamming cube $\mathcal{H}^n$ as a directed graph, where $\mathbb{F}_2^n$ is the set of vertices, and $[u, v]$ is an (directed) arc if, and only if, $d_H(u, v) = 1$ and $\mathrm{wt}_H(u) < \mathrm{wt}_H(v)$. In this case, v is called the *head* and u the *tail* of the arc. A *trail* τ in $\mathcal{H}^n$ with *initial vertex* u and *final vertex* v is a sequence $[u, u_1], [u_1, u_2], \ldots, [u_r, v]$ of consecutive arcs.

To every arc is assigned a non-negative integer $\delta([u,v])$, called the *label* of the arc. The pair $(\mathcal{H}^n, \delta)$ is called a *δ-labeled Hamming cube*. We call it as labeled to avoid confusion with weight.

Remark 3.2.1. *It is important to stress that, by simplicity and some abuse of notation, we consider $u \in \mathbb{F}_q^n$ and, we associate the vector $u = (u_1, \ldots, u_n)$ to a vertex $u' = (u_1', \ldots, u_n')$ in the Hamming cube defined as follows: $u_i' = 1$, if $u_i \neq 0$, and $u_i' = 0$, if $u_i = 0$. We shall abuse of the notation and omit the prime symbol $(')$, identifying u and u' according to the context.*

It is simple to see that given an S-weight $\mathrm{wt} : \mathbb{F}_q^n \to \mathbb{N}$, by setting $\delta([u,v]) = \mathrm{wt}(v) - \mathrm{wt}(u)$ we have that $\mathrm{wt}(w) = \sum_{[u,v]\in\tau} \delta([u,v])$ where τ is any *trail* from the null vector to w. By simplicity, we denote $\delta(\tau) = \sum_{[u,v]\in\tau} \delta([u,v])$ whenever no confusion may arise. This shows that every S-weight can be represented by a δ-labeled Hamming cube. From here on, except if explicitly stated, we assume that every trail has 0 as its initial vertex.

Not every $(\mathcal{H}^n, \delta)$ determines an S-weight. For this to happen, the δ function should avoid the situation depicted in the following example. Consider the δ-Hamming cube below.



This δ function does not induce an S-weight because the sum of weights on the left and right trails from 00 to 11 are different, i.e., $\delta([00,10]) + \delta([10,11]) \neq \delta([00,01]) + \delta([01,11])$. To avoid this we need to ensure that the label of a trail depends exclusively on its end point, not on the trail itself.

Proposition 3.2.2. *The map $\mathrm{wt}_\delta(w) = \delta(\tau)$, where τ is a trail from 0 to w, is an S-weight if, and only if, it does not depends on the trail ($\delta(\tau) = \delta(\tau')$, for any trails τ, τ' in $\mathcal{H}^n$ starting at u and ending at v) and $\delta([0, e_i]) > 0$, for every $i \in \{1, \ldots, n\}$.*

Proof. The proof follows directly from the definitions. □

We are concerned exclusively with labels which determine a weight, so, from here on, we shall assume that a label δ satisifies the conditions of Proposition 3.2.2. Also, we say that δ and δ' are equivalent if $\mathrm{wt}_\delta \sim \mathrm{wt}_{\delta'}$. It is, a priori, difficult to determine if a given labeled graph determines a metric belonging to a specific known family of metrics. In the case when $\max_{u \in \mathcal{H}^n} \mathrm{wt}_\delta(u) \le 2$ and $\delta([u,v]) \in \{0,1\}$ we can show that wt_δ is a combinatorial metric, introduced by Gabidulin in (GABIDULIN, 1973).

Example 3.2.3. *Combinatorial weight: Let δ be determined by an S-weight* wt *as before and suppose $\max_{u \in \mathbb{F}_q^n} \mathrm{wt}(u) \le 2$. In this particular case,* wt *determines a combinatorial metric if, and only if, $\delta([u,v]) \in \{0,1\}$. The non trivial side is proved as follows: Since we are assuming that the δ-Hamming cube satisfies the conditions required in Lemma 3.2.2, it follows that $\delta([0,e_i]) = 1$, for every $i \in [n]$. Let $\mathcal{P}([n])$ be the power set of $[n]$. Consider $M = \{\tau : \tau$ is maximal with property $\delta(\tau) = 1\}$ and $N = \{u \in \mathbb{F}_2^n :$ u is the ending vertex for some $\tau \in M\}$. It follows that δ is determined by the combinatorial weight induced by the covering $\mathcal{F} = \{\mathrm{supp}(u) : u \in N\}$.*

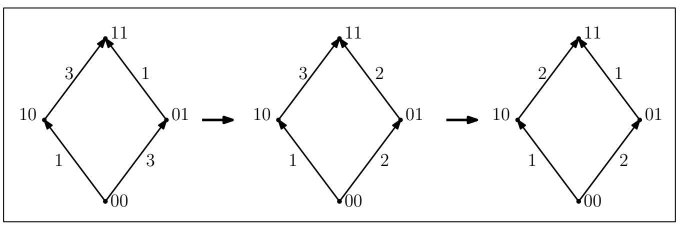
Poset weight: The constraint $\max_{u \in \mathbb{F}_q^n} \mathrm{wt}(u) \le 2$ is sufficient to guarantee that no poset metric can be equivalent to this S-weight wt*, for $n > 2$ since $\mathrm{wt}_P(11\cdots 1) = n$, for any poset weight. But on the other side, if wt_P is a poset weight then the δ-Hamming cube, determined by wt_P, has the following property: $\delta(\tau_u) < n$ (τ_v denotes a trail from the null vector to v) implies there is $i \notin \mathrm{supp}(u)$ such that $\delta(\tau_{u+e_i}) = \delta(u) + 1$.*

As already remarked, different labels can give rise to equivalent weights and metrics. We now show how we can give a standard form of a δ-labeled Hamming cube which represents all the labels determining the same equivalence class.

Definition 3.2.4. *Let δ be obtained from an S-weight* wt*. We say that δ is in a standard form if, given a trail τ with $\delta(\tau) = k > 1$ there is a trail τ' (not depending on τ, but on k) such that $\delta(\tau') = k - 1$.*

Definition 3.2.5. *We say that an S-weight* wt admits a standard form *if it is equivalent to an S-weight which determines a label δ in a standard form.*

Example 3.2.6. *Consider the figure bellow. The δ-Hamming cube on the left is not in a standard form since $\delta([00,01]) = 3$ and there is no trail τ with $\delta(\tau) = 2$. In the middle, we assign the value $\delta([00,01]) = 2$, and, since $\delta([00,10]) + \delta([10,11]) = 4$, Lemma 3.2.2 imposes $\delta([01,11]) = 2$ to get a weight which defines an S-weight. Now, on the right side, we repeat the procedure for the trail $\tau = \{[00,01],[01,11]\}$ which has $\delta(\tau) = 4$ while there is no trail τ' with $\delta(\tau) = 3$.*



We remark that since the values of δ decrease at every step, the algorithm will have a stopping point.

Proposition 3.2.7. *Every S-weight admits a unique standard form. Two labels are equivalent if, and only if, they admit the same standard form.*

Proof. The proof is a simple extension of the steps we detail in Example 3.2.6. First we consider the ordered list $\gamma(\mathrm{wt}) = (0 = r_1 = \mathrm{wt}(0), r_2, \ldots, r_l)$ of values assumed by the S-weight wt. Note that, given vectors $u, v \in \mathbb{F}_q^n$ such that $\mathrm{wt}(u) = r_i$ and $\mathrm{wt}(v) = r_{i+1}$, then any trail τ from u to v contains an arc (x, y) such that $\delta((x, y)) = r_{i+1} - r_i$.

We shall start with maximum values r_l and r_{l-1}, and if $r_l - r_{l-1} > 1$, then any arc (x, y) contained in trails between vector $u, v \in \mathbb{F}_q^n$ such that $\mathrm{wt}(u) = r_{l-1}$ and $\mathrm{wt}(v) = r_l$ is relabeled as 1.

So that, we follow this procedure in pairs from the maximum to minimum values r_i's. Since, every step reduces the sum of all differences $r_i - r_{i-1}$, it follows that the algorithm stops, and by construction, the weight induced by the standard form of Hamming cube is equivalent to the input one. $\square$

The previous proposition allows us to consider labels only in the standard form.

3.2.1 Group of linear isometries

Now, we turn our attention to describe the group of linear isometries in a space $\mathbb{F}_q^n$ endowed with a metric determined by an S-weight wt, i.e.,

$$GL(n, q, \mathrm{wt}) = \{T : \mathbb{F}_q^n \to \mathbb{F}_q^n : T \text{ is linear}, \mathrm{wt}(u) = \mathrm{wt}(T(u))\}.$$

The group $GL(n, q, \mathrm{wt})$ will be described in Theorem 3.2.15. As we shall see, this description is a generalization of the labeled-poset-block case presented in Theorem 2.2.8 which, on its side, is another link on a chain of results that started at Rosenbloom-Tsfasman metric (LEE, 2003) and passing through (CHO; KIM, 2006), (PANEK et al., 2008), (ALVES; PANEK; FIRER, 2008) and (ETZION; FIRER; MACHADO, 2017)). On this context of TS-metrics, this description is the last possible link.

Before we reach it, we need to establish some definitions and auxiliary propositions.

Let $\phi \in S_n$ and denote by T_ϕ the induced map on $\mathbb{F}_q^n$, defined by $T_\phi(u_1, \ldots, u_n) = (u_{\phi(1)}, \ldots, u_{\phi(n)})$. We say that ϕ *preserves* δ if $\delta([u, v]) = \delta([T_\phi(u), T_\phi(v)])$ for every arc $[u, v]$. Let $Aut(\mathcal{H}, \delta)$ be the group of automorphisms of the Hamming cube $\mathcal{H}$ which preserve δ.

Proposition 3.2.8. *Let $\phi \in S_n$. The linear map $T_\phi : \mathbb{F}_q^n \to \mathbb{F}_q^n$ is an isometry if, and only if, $T_\phi \in Aut(\mathcal{H}, \delta)$.*

Proof. Given a trail τ, it follows that $\phi(\tau)$ is also a trail, since ϕ is an automorphism of $\mathcal{H}$. Since the weight does not depend on the trail and from the fact that ϕ respects δ, then $\mathrm{wt}(u) = \delta(\tau_u) = \delta(\phi(\tau_u)) = \mathrm{wt}(T(u))$, for any trail τ_u from the zero vector to u.

If $T_\phi \notin Aut(\mathcal{H}, \delta)$, it is clear that there is at least an arc $[u, v]$ in which $\delta([u, v]) \neq \delta([T_\phi(u), T_\phi(v)])$. Let

$$B = \{u \in \mathbb{F}_2^n : \exists [u, v] \text{ such that } \delta([u, v]) \neq \delta([T_\phi(u), T_\phi(v)])\}$$

and consider $u_0 \in \arg\min_{u \in B} |\mathrm{supp}(u)|$. Since $u_0 \in B$, there is a v_0 such that $\delta([u_0, v_0]) \neq \delta([T_\phi(u_0), T_\phi(v_0)])\}$. It follows that $\mathrm{wt}_\delta(v_0) \neq \mathrm{wt}(T(v_0))$. □

Let $\alpha = \{e_1, e_2, \ldots, e_n\}$ be the usual basis of $\mathbb{F}_q^n$. Given $i, j \in [n]$, consider the linear map $T_{ij} : \mathbb{F}_q^n \to \mathbb{F}_q^n$ defined by $T_{ij}(e_i) = e_i + e_j$ and $T_{ij}(e_k) = e_k$, for $k \neq i$. Denote

$$A_i = \{i\} \cup \{j \in [n] : T_{ij}, T_{ji} \in GL(n, q, \mathrm{wt})\}.$$

It is immediate to check that either $A_i = A_j$ or $A_i \cap A_j = \emptyset$, this implies in the existence of an equivalence relation $\sim_{\mathrm{wt}}$ on $[n]$ defined as follows:

$$i \sim_{\mathrm{wt}} j \iff A_i = A_j.$$

Each A_i is an equivalence class but, to avoid working with multi sets, we denote by $\mathcal{S}_{\mathrm{wt}} = \{H_1, \ldots, H_s\}$ the set of equivalence classes, so that $[n] = \bigsqcup_{i=1}^{n} H_i$. Naturally, given A_i, there is a unique $j \in [s]$ such that $A_i = H_j$.

An equivalence class H_i *dominates* H_j if for any two vectors $u, v \in \mathbb{F}_q^n$ with $\mathrm{supp}(u) \subset H_i$ and $\mathrm{supp}(v) \subset H_j$ we have that $\mathrm{wt}(u + w) + \mathrm{wt}(u + v + w)$ for any vector $w \in \mathbb{F}_q^n$ such that $\mathrm{supp}(w) \cap \mathrm{supp}(u) = \emptyset$.

We say that an equivelence class H_i is a *head* in a family of equivalence classes if it is a maximal element in the family. We may assume, without loss of generality, that the classes are ordered in a topological order, i.e., given two distinct classes H_i and H_j, if H_j dominates H_i then $i < j$.

For each subset $X \subset [n]$ there is a minimum set $\mathcal{S}_{\mathrm{wt}}(X) = \{H_{i_1}, \ldots, H_{i_k}\}$ of equivalence classes such that $X \subset H_{i_1} \cup \cdots \cup H_{i_k}$. The *minimum set header* of X is defined by

$$\widehat{X} = \{i \in X : i \in H_j \text{ for some head } H_j \in \mathcal{S}_{\mathrm{wt}}(X)\}.$$

Definition 3.2.9. *The* cleared out *form of $u = (u_1, \ldots, u_n) \in \mathbb{F}_q^n$ is the vector $\widetilde{u} = (\widetilde{u}_1, \ldots, \widetilde{u}_n)$ where $\widetilde{u}_i = u_i$ if $i \in \widehat{\mathrm{supp}(u)}$ and $\widetilde{u}_i = 0$ otherwise, i.e., we maintain the coordinates contained in the head classes of $\mathcal{S}_{\mathrm{wt}}(\mathrm{supp}(u))$ and set the others do be 0.*

Definition 3.2.10. *Given a weight* wt *over* $\mathbb{F}_q^n$*, we say that a map* T preserves domination of classes *(with respect to* wt*) if:*

(1) T *is a linear map;*

(2) For all $i \in [n]$*,* $i \in \mathrm{supp}(T(e_i))$*;*

(3) If $j \in \mathrm{supp}(T(e_i))$*,* $i \in H_k$*, and* $j \in H_l$*, then either* $H_k = H_l$ *or* H_k *dominates* H_l*.*

We denote by $N(\mathcal{H}^n, \delta)$ *the set of all maps preserving domination.*

We shall prove that $N(\mathcal{H}, \delta)$ is a group. Since trivially it is closed under composition and the identity is in $N(\mathcal{H}, \delta)$, to prove that it is a group, all is needed is the following Lemma:

Lemma 3.2.11. *If* $T \in N(\mathcal{H}, \delta)$*, then* $T^{-1} \in N(\mathcal{H}, \delta)$*.*

Proof. We stress that T is an one-to-one map and the minimum set header of $T(u)$ is the image under T of the cleared-out form of u (the minimum set header of $\mathrm{supp}(u)$), that is, $\widetilde{T(u)} = T(\tilde{u})$, so that $T^{-1} \in N(\mathcal{H}, \delta)$. □

Proposition 3.2.12. *Any map* $T \in N(\mathcal{H}, \delta)$ *is a linear isometry.*

Proof. Given $i \in \mathrm{supp}(T(u))$, there is k such that $i \in H_k$. If $i \in \mathrm{supp}(u)$, then, due to the support condition, it follows $\mathrm{wt}(T(u)) \leq \mathrm{wt}(u)$. If $i \notin \mathrm{supp}(u)$, condition (3) in the definition of $N(\mathcal{H}^n, \mathrm{wt})$ ensures that there is $j \in \mathrm{supp}(u)$ such that $j \in H_l$ with either $l = k$ or H_l dominating H_k. This means that every class containing elements in the support of $T(u)$ either contains an element in the support of u or is dominated by such a class. For both the cases, it follows that $\mathrm{wt}(T(u)) \leq \mathrm{wt}(u)$. The previous lemma ensures that $T^{-1} \in N(\mathcal{H}^n, \mathrm{wt})$. Applying the same reasoning for T^{-1} we get that $\mathrm{wt}(u) \leq \mathrm{wt}(T(u))$, hence T is an isometry. □

The next two results will be needed to prove Theorem 3.2.15, by showing that given $T \in GL(n, q, \mathrm{wt})$ there is $T' \in Aut(\mathcal{H}, \delta)$ such that $T' \circ T \in N(\mathcal{H}, \delta)$.

Proposition 3.2.13. *Suppose* $q > 2$*. For any* $T \in GL(n, q, \mathrm{wt})$ *and* $e_i \in \mathbb{F}_q^n$*, there is* $\widetilde{T} \in N(\mathcal{H}, \delta)$ *such that* $\mathrm{supp}(\widetilde{T}(T(e_i)) \subset H$*, for some equivalence class* $H \in \mathcal{S}_{\mathrm{wt}}$*.*

Proof. Let $T \in GL(n, q, \mathrm{wt})$. Given $e_i \in \mathbb{F}_q^n$ suppose that $\mathrm{supp}(\widetilde{T(e_i)}) \subset H_j \cup H_k$, where $\widetilde{\{j, k\}}$ is minimal with this property. We denote by $\widetilde{T}$ a linear isometry such that $\widetilde{T}(e_i) = \widetilde{T(e_i)}$, where $\tilde{u}$ is the cleared out form of u. We remark that there are many such maps. This implies there are $u, v, w \in \mathbb{F}_q^n$ such that

1- $\widetilde{T}(T)(e_i) = u + v$;

2- $\mathrm{supp}(u) \subset H_j$;

3- $\mathrm{supp}(v) \subset H_k$;

4- $\mathrm{supp}(w) \cap (\mathrm{supp}(u) \cup \mathrm{supp}(v)) = \emptyset$;

5- $\delta((u + w, u + v + w)) \neq 0$;

6- $\delta((v + w, u + v + w)) \neq 0$.

This implies that $\mathrm{wt}(u + w) < \mathrm{wt}(u + v + w)$ and $\mathrm{wt}(v + w) < \mathrm{wt}(u + v + w)$.

We claim that $i \notin \mathrm{supp}(\widetilde{T}^{-1}(w))$, otherwise we would have that $\mathrm{supp}(e_i + \widetilde{T}^{-1}T^{-1}(w)) \subset \mathrm{supp}(\widetilde{T}^{-1}(w))$ which is a contradiction since $\mathrm{wt}(w) \leq \mathrm{wt}(u + w) < \mathrm{wt}(u + v + w)$.

Since we are assuming $q > 2$ there are $\alpha, \beta \in \mathbb{F}_q^*$, such that $1 + \alpha$ and $1 + \beta$ are not zero. Thus,

$$\mathrm{wt}(u + v + w) = \mathrm{wt}((1 + \alpha)u + v + w) = \mathrm{wt}(u + (1 + \beta)v + w).$$

In a similar way we can prove that

$$i \notin \mathrm{supp}\left(\left(\widetilde{T}^{-1}\right)(u + w)\right) \text{ and } i \notin \mathrm{supp}\left(\left(\widetilde{T}^{-1}\right)(v + w)\right).$$

But this means that the i-th coordinate of both the vectors $\left(\widetilde{T}^{-1}T^{-1}\right)(u)$ and $\left(\widetilde{T}^{-1}T^{-1}\right)(v)$ equal zero, which is a contradiction since, by definition, $\left(\widetilde{T}^{-1}\right)(u + v) = e_i$.

We considered the case where we needed two minimum head sets to cover $\mathrm{supp}(\widetilde{T(e_i)})$. This case may be used as the inductive step in case we have $\mathrm{supp}(\widetilde{T(e_i)}) \subset H_{i_1} \cup H_{i_2} \cup \cdots \cup H_{i_k}$.

□

Lemma 3.2.14. *If $T \in GL(n, q, \mathrm{wt})$, then given $i \in [n]$ the linear map $T_i : \mathbb{F}_q^n \to \mathbb{F}_q^n$ defined as*

$$T_i(e_j) = \begin{cases} \widetilde{T(e_j)}, & \text{if} \quad i = j, \\ T(e_j), & \text{if} \quad j \in [n] \backslash \{i\}. \end{cases}$$

is a linear isometry.

Proof. For any $u = \sum_{j=1}^{n} u_j e_j$, $T_i(u) = T_i(\sum_{j=1}^{n} u_j e_j) = T_i(u_i e_i) + T(\sum_{j \neq i} u_j e_j)$. Since there is $\widetilde{T} \in N(\mathcal{H}, \delta)$ such that $\widetilde{T}(T(e_i)) = e_k$, it follows that for any $v \in \mathbb{F}_q^n$ such that $v_k = 0$, $\delta(u + e_k, u + T(e_i)) = 0$.

So, given $v \in \mathbb{F}_q^n$ with $v_k = 0$, $T_i(v) = T(v)$. If $v_k \neq 0$, it follows that $\delta(T(v - v_k e_k) + e_k, T(v - v_k e_k) + T(e_i)) = 0$, which proves that T_i is a linear isometry. $\square$

Hence, we clearly have a semi-direct product, i.e.:

Theorem 3.2.15. *Let* wt *be an* S*-weight over* $\mathbb{F}_q^n$ *with* $q > 2$*. Then, the group* $GL(n, q, \mathrm{wt})$ *of all linear isometries of* $\mathbb{F}_q^n$ *is the semi-direct product*

$$GL(n, q, \mathrm{wt}) = Aut(\mathcal{H}, \delta) \ltimes N(\delta, \mathrm{wt}),$$

where $Aut(\mathcal{H}, \delta)$ *and* $N(\mathcal{H}^n, \mathrm{wt})$ *are described in Proposition 3.2.8 and Definition 3.2.10, respectively.*

Proof. First of all, it is clear that $Aut(\mathcal{H}, \delta) \cap N(\mathcal{H}, \delta) = \{Id\}$. Moreover, consider the map that associates to $T \in GL(n, q, \mathrm{wt})$ to a permutation of coordinates through the map $T_1 \circ T_2 \circ \cdots \circ T_n \circ T$, where the T_i's are constructed as in Lemma 3.2.14. It follows that $N(\mathcal{H}, \delta)$ is the kernel of this map, hence it is a normal subgroup of $GL(n, q, \mathrm{wt})$. It follows that $N(\mathcal{H}, \delta)$ is normal in the product $Aut(\mathcal{H}, \delta)N(\mathcal{H}, \delta)$ so that $Aut(\mathcal{H}, \delta)N(\mathcal{H}, \delta) = Aut(\mathcal{H}, \delta) \ltimes N(\mathcal{H}, \delta) = \{Id\}$. Propositions 3.2.8 and 3.2.12 ensures that $Aut(\mathcal{H}, \delta) \ltimes N(\delta) \subseteq GL(n, q, \mathrm{wt})$.

For $q > 2$, It follows from Lemma 3.2.14 that given $T \in GL(n, q, \mathrm{wt})$, there is a sequence of $T_1, \ldots, T_n \in N(\mathcal{H}, \delta)$ such that $T_1 \circ T_2 \circ \cdots \circ T_n \circ T \in Aut(\delta)$ which proves the equality, i.e., $GL(n, q, \mathrm{wt}) = Aut(\delta) \ltimes N(\delta)$. $\square$

3.3 Conditional sums

In the previous section we showed how weights are related to δ-Hamming cubes in the most general setting. Such weights, in full generality, are not yet studied, but there are families of weights that are reasonably understood. These families can be used to either approximate general weights or, alternatively, to be considered as bricks with which we can construct new weights. As already remarked, the most studied such metrics are the families of poset (and its generalizations, such as poset-block metrics (ALVES; PANEK; FIRER, 2008), digraph (ETZION; FIRER; MACHADO, 2017) and weighted digraph (HYUN; KIM; PARK, 2017)) and combinatorial metrics. It is worth to note that these two families are virtually complementary, in the sense that the Hamming metric is the unique metric that belongs to both of them.

So, we are left with two fundamental questions to be addressed in this section:

1. (Proposition 3.3.8) How can we obtain a new S-weight out from two given S-weights?

2. (Theorem 3.3.13) How large is the family of S-weights that can be constructed from a combination of poset and combinatorial weights?

We start by presenting a simple conditional sum which permits to obtain new S-weights out of given ones.

Proposition 3.3.1. *Let* wt_1 *and* wt_2 *be S-weights. Then, the k-sum*

$$(\mathrm{wt}_1 \oplus_k \mathrm{wt}_2)(u) = \begin{cases} \mathrm{wt}_1(u), & \textit{if } \mathrm{wt}_1(u) < k, \\ \mathrm{wt}_1(u) + \mathrm{wt}_2(u), & \textit{if } \mathrm{wt}_1(u) \geq k. \end{cases}$$

is a S-weight.

Proof. The proof follows straightforward from the definitions. □

We remark that for $k = 0, 1$, $\mathrm{wt}_1 \oplus_k \mathrm{wt}_2$ is the usual sum $\mathrm{wt}_1 + \mathrm{wt}_2$. The previous proposition implies that the set of all S-weights endowed with directed sum or k-sum is a magma, i.e., the set of all weights is closed under "$\oplus_k$".

Definition 3.3.2. *Let* $C : \mathbb{F}_q^n \to \{\texttt{true}, \texttt{false}\}$ *be a binary map. We say that* C respects support *if* $C(u) = \texttt{true}$ *implies* $C(v) = \texttt{true}$ *for every v such that* $\mathrm{supp}(u) \subset \mathrm{supp}(v)$.

Definition 3.3.3. *Let* wt_1 *and* wt_2*. Given a condition* $C : \mathbb{F}_q^n \to \{\texttt{true}, \texttt{false}\}$*, the* conditional sum $\mathrm{wt}_1 \oplus_C \mathrm{wt}_2$ *is defined as*

$$(\mathrm{wt}_1 \oplus_C \mathrm{wt}_2)(u) = \begin{cases} \mathrm{wt}_1(u), & \textit{if } C(u) = \texttt{false}, \\ \mathrm{wt}_1(u) + \mathrm{wt}_2(u), & \textit{if } C(u) = \texttt{true}. \end{cases}$$

Proposition 3.3.4. *The conditional sum* $\mathrm{wt}_1 \oplus_C \mathrm{wt}_2$ *of two weights is a weight if the condition C respects support.*

Proof. The proof follows straightforward from the definitions. □

Remark 3.3.5. *We stress that the conditional sum $\oplus_k$ may be replaced by a similar sum in which we consider different binary maps* respecting the support *of vectors $u \in \mathbb{F}_q^n$. For instance, given two weights* wt_1 *and* wt_2*, let us consider the (H,k)-binary map (in which $(H,k)(u) = \texttt{true}$, if $\mathrm{wt}_H(u) \geq k$, and $(H,k)(u) = \texttt{false}$, otherwise) defined as follows*

$$(\mathrm{wt}_1 \oplus_{(H,k)} \mathrm{wt}_2)(u) = \begin{cases} \mathrm{wt}_1(u), & \textit{if } \mathrm{wt}_H(u) < k, \\ \mathrm{wt}_1(u) + \mathrm{wt}_2(u), & \textit{if } \mathrm{wt}_H(u) \geq k. \end{cases}$$

Example 3.3.6. *As we saw on Table 1, the last criterion is not determined by any poset or combinatorial metric. Now we show how we can obtain it by considering a conditional sum. Let P be the poset over* $[2]$ *in which* $1 \preceq_P 2$*. The poset weight* wt_P *satisfies the criterion* $\mathrm{wt}(00) < \mathrm{wt}(10) < \mathrm{wt}(01) = \mathrm{wt}(11)$*. The $(H,1)$-conditional sum* $\mathrm{wt}_P \oplus_{(H,1)} \mathrm{wt}_H$ *(where*

wt_H *is the Hamming weight) satisfies the criterion* $\mathrm{wt}(00) < \mathrm{wt}(10) < \mathrm{wt}(01) < \mathrm{wt}(11)$, *the last row of Table 1.*

We should remark that operating with conditional sums leads to a lot of redundancies, in the sense that operating with weights may not lead to a new weight. It happens, for example, for $\mathrm{wt} \oplus_0 \mathrm{wt}$.

We wish to know under what conditions we have that wt_1, wt_2 and $\mathrm{wt}_1 \oplus_C \mathrm{wt}_2$ are all equivalent. We start with the following lemma:

Lemma 3.3.7. *Let* wt_1 *and* wt_2 *be equivalent weights. Suppose that* $\mathrm{wt}_1 \oplus_C \mathrm{wt}_2$ *is also equivalent to* wt_1 *and* wt_2 *for a given condition* C*. Then,*

1. *If* $\mathrm{wt}_1(u) = \mathrm{wt}_1(v)$ $(u \neq v)$ *and* $C(u) = \texttt{true}$, *then* $C(v) = \texttt{true}$.
2. *If* $\mathrm{wt}_1(u) < \mathrm{wt}_1(v)$ *and* $C(u) = \texttt{true}$, *then either* $C(v) = \texttt{true}$ *or* $\mathrm{wt}_1(u) + \mathrm{wt}_2(u) < \mathrm{wt}_1(v)$.

Proof. The proof follows straightforward from definitions of conditional sum and from the fact that wt_1 is equivalent to $\mathrm{wt}_1 \oplus_C \mathrm{wt}_2$. □

The second part of the previous Lemma ensures that, if $\mathrm{wt}_1 \sim \mathrm{wt}_2 \sim \mathrm{wt}_1 \oplus_C \mathrm{wt}_2$ we can choose all of them to be equal, and assume all the values $0, 1, \dots, \max\limits_{u \in \mathbb{F}_q^n} \mathrm{wt}_1(u)$. We can prove the following result:

Proposition 3.3.8. *Let* wt *be a* S*-weight such that* $\sigma(\mathrm{wt}) = \{0, 1, \dots, \max\limits_{u \in \mathbb{F}_q^n} \mathrm{wt}(u)\}$*. If* $\mathrm{wt} \sim \mathrm{wt} \oplus_C \mathrm{wt}$, *then* $\oplus_C = \oplus_k$, *for some* $k \in \mathbb{N}$.

Proof. Let $u \in \mathbb{F}_q^n$ be such that $C(u) = \mathit{true}$. If $v \in \mathbb{F}_q^n$ with $\mathrm{wt}(v) = \mathrm{wt}(u)$, then from statement 1 of Lemma 3.3.7 implies that $C(v) = \mathit{true}$. If $\mathrm{wt}(u) < \mathrm{wt}(v)$, statement 2 of Lemma 3.3.7 ensures that $C(v) = \mathtt{\mathit{true}}$ or $\mathrm{wt}(u) + \mathrm{wt}(u) < \mathrm{wt}(v)$. Since we are assuming that $\sigma(\mathrm{wt}) = \{0, 1, \dots, \max\limits_{u \in \mathbb{F}_q^n} \mathrm{wt}(u)\}$, it follows that the situation wherein $2\mathrm{wt}(u) < \mathrm{wt}(v)$.

So that, in both cases it is enough to choose $k = \min\{\mathrm{wt}(u) : u \in \mathbb{F}_q^n \text{ satisfies } C\}$. □

Definition 3.3.9. *Let* wt_1 *and* wt_2 *be two* S*-weights. We say that* wt_2 *is a* refinement *of* wt_1 *if* $\mathrm{wt}_1(u) < \mathrm{wt}_1(v)$ *implies* $\mathrm{wt}_2(u) < \mathrm{wt}_2(v)$, *for every* $u, v \in \mathbb{F}_2^n$.

Definition 3.3.10. *Let* wt *be a refinement of* wt'*. Let* $\mathcal{B} = \{B_0, B_1, \dots, B_k\}$ *be a partition of* $\mathbb{F}_2^n$ *such that*

1) wt' *is refinement of* wt *restricted to* B_j*;*

2) $\mathrm{wt}(u) < \mathrm{wt}(v)$ *for* $u \in B_i$*,* $v \in B_j$ *and* $i < j$*.*

Let $\mathcal{B} = \{B_0, B_1, \ldots, B_{\kappa(\mathrm{wt},\mathrm{wt}')}\}$ *be a partition with minimum number of parts satisfying these properties. It is called a* partition of $(\mathbb{F}_2^n, \mathrm{wt})$ induced by wt'*. The number* $\kappa(\mathrm{wt}, \mathrm{wt}')$ *is called* incompatibility degree *of the weights. We shall see that such a partition exist when* wt *is a refinement of* wt'*.*

Proposition 3.3.11. *If* wt *is a refinement of* wt'*, then there is a partition induced by* wt'*.*

Proof. From Definition 3.3.9, if $\mathrm{wt}'(u) < \mathrm{wt}'(v)$ then $\mathrm{wt}(u) < \mathrm{wt}(v)$. Consider the list of vectors $u \in \mathbb{F}_2^n$ with ordering induced by wt, i.e., $\mathrm{wt}(u_i) < \mathrm{wt}(u_j)$ implies $i < j$. Since wt is a refinement of wt' it follows that the ordering induced by wt is also an ordering induced by wt'. Further,

$$B_k = \{u_i : \mathrm{wt}(u_i)\mathbf{R}\mathrm{wt}(u_j) \iff \mathrm{wt}'(u_i)\mathbf{R}\mathrm{wt}'(u_j), \forall\, i_{k-1} < i, j \leq i_k, \text{ where } \mathbf{R} \in \{=, <\}\}.$$

Let B_0 be the set with the first vectors in the list until the first order fail, i.e., while $\mathrm{wt}(u_{i_0}) < \mathrm{wt}(u_{i_0+1})$, $\mathrm{wt}'(u_{i_0}) = \mathrm{wt}'(u_{i_0+1})$. By following inductively such a procedure for every B_i, we produce a partition $\mathcal{B}$ respecting both properties in Definition 3.3.9. By construction, it follows that this is minimum partition. □

Remark 3.3.12. *It follows straighforward that* $\kappa(\mathrm{wt}, \mathrm{wt}') = 0$*, if and only if,* wt *is equivalent to* wt'*.*

Theorem 3.3.13. *Every* S*-weight can be realized as a finite sequence of conditional sums of poset and combinatorial weights.*

Proof. Consider the covering $\mathcal{F} = \{[n]\}$. The weight $\mathrm{wt}_{\mathcal{F}}$ satisfies

$$\mathrm{wt}_{\mathcal{F}}(u) = \begin{cases} 0, & \text{if } \ u = 0, \\ 1, \text{ if} & \text{otherwise.} \end{cases}$$

Let wt be a weight. It is clear that any S-weight is a refinement of the $\mathrm{wt}_{\mathcal{F}}$ and that $\max\limits_{u \in B_i} \mathrm{wt}_{\mathcal{F}}(u) = \min\limits_{u \in B_{i+1}} \mathrm{wt}_{\mathcal{F}}(u)$. Let $\mathcal{B} = \{B_1, \ldots, B_{\kappa(\mathrm{wt}, \mathrm{wt}_{\mathcal{F}})}\}$ be a partition of $(\mathbb{F}_2^n, \mathrm{wt})$ induced by $\mathrm{wt}_{\mathcal{F}}$. Define the condition $C_i : \mathbb{F}_2^n \to \{\texttt{true}, \texttt{false}\}$ as `true`, for every $u \in B_i \cup B_{i+1} \cup \cdots \cup B_\kappa$ and `false`, otherwise. Since each conditional sum $\oplus_{C_i}$ keeps any relation among vectors $u, v \in B_i \cup B_{i+1} \cup \cdots \cup B_\kappa$ and fix the relation between vectors $u \in B_{i-1}$ and $v \in B_i$, it follows that the weight $\mathrm{wt}_{\mathcal{F}} \oplus_{C_1} \cdots \oplus_{C_\kappa} \mathrm{wt}_{\mathcal{F}}$ is equivalent to wt. □

Remark 3.3.14. *The construction presented in the previous theorem is universal but not efficient. The number of summands may be extremely large. Actually, the maximum value for* $\kappa(\mathrm{wt}, \mathrm{wt}_{\mathcal{F}})$ *is* $2^n - 1$ *when* wt *has maximum spectrum*[1]*, i.e., when* $u \neq v$ *implies*

[1] The spectrum σ of an S-weight $\mathrm{wt} : \mathbb{F}_q^n \to \mathbb{Z}$ is given by $\sigma(\mathrm{wt}) = \{\mathrm{wt}(u) : u \in \mathbb{F}_q^n\}$.

$\mathrm{wt}_{\mathcal{F}}(u) \neq \mathrm{wt}_{\mathcal{F}}(v)$. *In the next section we shall produce bounds for the number of summands that are needed in this process.*

3.3.1 Lower and upper bounds: an algorithmical approach

In this section, we turn our attention to discuss how the new S-weights can be obtained from *basic* ones (poset and combinatorial weights), in order to find an approach which may enable us to recycle some properties from the basic S-weighs which many metric invariants for codes and properties in this metric spaces are studied in the literature of Coding Theory: (BRUALDI; GRAVES; LAWRENCE, 1995), (FELIX; FIRER, 2012), (GABIDULIN, 1973),(PANEK et al., 2008), (Machado; Pinheiro; Firer, 2017), (PINHEIRO; MACHADO; FIRER, 2019).

We start with a simple result that gives us an lower bound for the number of conditional summands of basic S-weights that are necessary to be equivalent to a given S-weight.

Theorem 3.3.15. *Let* wt *be an* S*-weight with spectrum equals to* k*. It is necessary at least* $\left\lceil \frac{k}{n} \right\rceil$ *summands to produce an equivalent weight* wt' *of* wt*, where* wt' *is obtained by conditional sums of poset and combinatorial weights.*

Proof. Since a poset or combinatorial weight can assume value at most n different values, it follows that any $\left\lceil \frac{k}{n} \right\rceil - 2$ conditional sum of such a weights

$$\left(\mathrm{wt}_1 \oplus_{C_1} \mathrm{wt}_2 \oplus_{C_2} \cdots \oplus_{C_{\lceil \frac{k}{n} \rceil - 2}} \mathrm{wt}_{\lceil \frac{k}{n} \rceil - 1}\right)(u) \leq n\left(\left\lceil \frac{k}{n} \right\rceil - 1\right) < k.$$

Therefore, $\sigma\left(\mathrm{wt}_1 \oplus_{C_1} \mathrm{wt}_2 \oplus_{C_2} \cdots \oplus_{C_{\lceil \frac{k}{n} \rceil - 2}} \mathrm{wt}_{\lceil \frac{k}{n} \rceil - 1}\right) < k$. □

Theorem 3.3.16 (Upper bound)**.** *If* wt *is a refinement of* wt_1*, then it is necessary at most* $\kappa(\mathrm{wt}, \mathrm{wt}_1)$ *conditional sums* $(\mathrm{wt}_1 \oplus_{C_1} \cdots \oplus_{C_\kappa} \mathrm{wt}_{\kappa+1})$ *to obtain an* S*-weight equivalent to* wt*.*

Proof. Let $\mathcal{B} = \{B_0, \ldots, B_\kappa\}$ be the partition of $(\mathbb{F}_2^n, \mathrm{wt})$ induced by wt_1. We remark that $\max\limits_{u \in B_i} \mathrm{wt}_1(u) = \min\limits_{u \in B_{i+1}} \mathrm{wt}_1(u)$. Let condition $C_i : \mathbb{F}_2^n \to \{\texttt{true}, \texttt{false}\}$ be defined as $\texttt{true}$ for every $u \in B_i \cup B_{i+1} \cup \cdots \cup B_\kappa$ and $\texttt{false}$, otherwise. Let $\mathrm{wt}_2 = \mathrm{wt}_3 = \cdots = \mathrm{wt}_{\kappa+1}$ be defined by $\mathrm{wt}_i(u) = 1$ for $u \neq 0, i \geq 2$. It follows that, $\mathrm{wt}_1 \oplus_{C_1} \cdots \oplus_{C_\kappa} \mathrm{wt}_{\kappa+1}$ is an S-weight equivalent to wt. □

Corollary 3.3.17 (Recursive upper bound)**.** *If* wt *is a refinement of* $\mathrm{wt}^j := \mathrm{wt}_1 \oplus_{C_1} \oplus \cdots \oplus_{C_{j-1}} \mathrm{wt}_j$*, then it is necessary at most* $j + \kappa(\mathrm{wt}, \mathrm{wt}^j)$ *conditional sums* $(\mathrm{wt}^j \oplus_{C_j} \mathrm{wt}_{j+1} \cdots \oplus_{C_\kappa} \mathrm{wt}_{\kappa+1})$ *to obtain an* S*-weight equivalent to* wt*.*

Although it is intuitive, we illustrate that the recursive upper bound given in Corollary 3.3.16 can perform better than the bound given in Theorem 3.3.16.

Example 3.3.18. *We start this example by considering an S-weight* wt *over* $\mathbb{F}_2^3$ *that has the following weight ordering:*

$$0 = \mathrm{wt}(000) < \mathrm{wt}(100) < \mathrm{wt}(010) = \mathrm{wt}(001) < \mathrm{wt}(110) < \mathrm{wt}(101) < \mathrm{wt}(011) = \mathrm{wt}(111).$$

Consider the poset weights wt_P *and* $\mathrm{wt}_{P'}$ *in which* $P = ([3], \{1 \prec_P 2, 1 \prec_P 3\})$ *and* $P' = ([3], \{1 \prec_{P'} 2 \prec_{P'} 3\})$*. It follows that* $\kappa(\mathrm{wt}, \mathrm{wt}_P) = 2$ *while* $\kappa(\mathrm{wt}, \mathrm{wt}_P \oplus_C \mathrm{wt}_{P'}) = 0$*, where* $C(u) = \texttt{true}$ *if, and only, if* $u \in \{110, 101, 011, 111\}$*.*

By combining both Theorem 3.3.13 and Corollary 3.3.17 we can describe an algorithm to find a conditional sum of poset and combinatorial weights that is equivalent to a given S-weight. The hard part in this algorithm is the first step where it is necessary to find a poset or combinatorial weight that fits better in $(\mathbb{F}_2^n, \mathrm{wt})$.

Given an S-weight wt follows that next steps:

Step 1 - Find the poset or combinatorial weight wt_1 that is refined by wt and minimizes $\kappa(\mathrm{wt}, \mathrm{wt}_1)$;

Step 2 - Construct the partition $\mathcal{B}_1 = \{B_0, \dots, B_\kappa\}$ of $(\mathbb{F}_2^n, \mathrm{wt})$ induced by wt_1;

Step 3 - Let $C_i : \mathbb{F}_2^n \to \{\texttt{true}, \texttt{false}\}$ be defined as $\texttt{true}$ for every $u \in B_i \cup B_{i+1} \cup \cdots \cup B_\kappa$ and $\texttt{false}$, otherwise.

Step 4 - Return to Step 1 replacing wt_1 by wt_i in which wt is a refinement of wt_i restricted to $\mathcal{B}\backslash\{B_0\}$ and $\kappa(\mathrm{wt}, \mathrm{wt}_1)$ by $\kappa(\mathrm{wt}, \mathrm{wt}_1 \oplus_{C_1} \cdots \oplus_{C_{i-1}} \mathrm{wt}_i)$.

Example 3.3.19. *Consider a S-weight* wt *over* $\mathbb{F}_2^3$ *as described in Example 3.3.18, i.e., with the weight ordering* $0 = \mathrm{wt}(000) < \mathrm{wt}(100) < \mathrm{wt}(010) = \mathrm{wt}(001) < \mathrm{wt}(110) < \mathrm{wt}(101) < \mathrm{wt}(011) = \mathrm{wt}(111)$*. Simple computations show that PC-weights in which* wt *is a refinement of them are induced by the poset* $P = ([3], \{1 \prec_P 2, 1 \prec_P 3\})$*, or by the coverings* $\mathcal{F}_1 = \{\{1, 2, 3\}\}$*,* $\mathcal{F}_2 = \{\{1, 2\}, \{3\}\}$*. Note that the partitions of* $(\mathbb{F}_2^3, \mathrm{wt})$ *induced by* wt_P*,* $\mathrm{wt}_{\mathcal{F}_1}$ *and* $\mathrm{wt}_{\mathcal{F}_2}$ *are given by*

$$\mathcal{B}_P = \{\{000, 100, 010, 001\}, \{110\}, \{101, 011, 111\}\},$$

$$\mathcal{B}_{\mathcal{F}_1} = \{\{000, 100\}, \{010, 001\}, \{110\}, \{101\}, \{011, 111\}\} \text{ and}$$

$$\mathcal{B}_{\mathcal{F}_2} = \{\{000, 100\}, \{010, 001\}, \{110, 101\}, \{011, 111\}\}, \text{ respectively.}$$

It follows that $\kappa(\mathrm{wt}, \mathrm{wt}_P)$ *is the minimum for any PC-weight. By setting* $C_1(u) = \texttt{true} \iff u \in \mathcal{B}_P\backslash\{B_0\}$*, weight induced by* $P' = ([3], \{1 \prec_{P'} 2 \prec_{P'} 3\})$ *or* $\mathcal{F}_2$ *is such that* wt *is equivalent to both* $\mathrm{wt}_P \oplus_{C_1} \mathrm{wt}_{P'}$ *and* $\mathrm{wt}_P \oplus_{C_1} \mathrm{wt}_{\mathcal{F}_2}$*.*

Note that this new procedure of looking for the poset or combinatorial weight reduces a lot the number of summands compared to that obtained in Theorem 3.3.13. In this example, while Theorem 3.3.13 needs 4 *summands, Corollary 3.3.17 needs only* 2.

4 Weight Duality

When considering $\mathbb{F}_q^n$ endowed with a metric d determined by a weight wt, the *weight distribution* of a linear code $\mathcal{C} \subset \mathbb{F}_q^n$ counts how many codewords there are with a given weight. It is an important invariant in coding theory since it enables to bound the error probability of a code. However, determining the weight distribution is a hard task, since $\mathcal{C}$ has q^k elements, each with n coordinates, and the parameters n and k are typically very large. However, desirable codes have large rate k/n and very often $n-k$ is small. The *dual code* $\mathcal{C}^\perp$, defined by the equation $\mathcal{C}^\perp = \{u \in \mathbb{F}_q^n \mid \langle u, c\rangle = 0, \forall c \in \mathcal{C}\}$ is an $n-k$-dimensional linear code and the weight distributions of $\mathcal{C}$ and $\mathcal{C}^\perp$ are related by the remarkable MacWilliams Identity, a fundamental result in coding theory, proved by J. MacWilliams in 1961, considering the Hamming metric.

When looking for a MacWilliams identity for poset metrics, (KIM; OH, 2005) realized that not only the dual code is involved, but also a metric duality should be considered. The duality of the metric could not be recognized for the Hamming case since, as a particular case of poset metric, the Hamming metric is self-dual. The duality of the metric showed up to be crucial for the MacWilliams Identity for many of the metrics to which it was proved to hold, as, for example, in the case of poset-blocks (Pinheiro; Firer, 2012) and digraph metrics (ETZION; FIRER; MACHADO, 2017).

In this work we adopt a general approach which embraces all the previous ones for which the orbits of a vector under the group of linear isometries depend only on the weight. First of all we construct the dual weight $\mathrm{wt}^\perp$ of a given weight wt (over a finite set), by considering the spheres centered at the origin. With this in hand, we are able to determine a necessary and sufficient condition for the existence of a MacWilliams' Identity.

As mentioned, we shall prove that, for the poset and combinatorial metrics that admit a MacWilliams Identiy, namely the hierachical posets and the k-partitions, respectively, this duality coincides with the usual definitions of dual of a hierachical poset and the self-dual combinatorial metric.

4.1 The dual of a weight function

In this section we turn our attention to develop a generic definition of the dual weight based only on the partition of $\mathbb{F}_q^n$ into spheres. The first discussion concerning the weight duality was raised in the context of poset weights in (KIM; OH, 2005). The authors generalized MacWilliams' identity for poset metrics, and classified the subfamily of poset metrics whose admit such an identity. To do so, they related the P-weight enumerator of a linear code and the $P^{\perp}$-weight enumerator of its dual code, where $P^{\perp}$ denotes the dual poset.

Given a weight wt, the *ordered spectrum of* Λ_{wt} of wt is defined as $\Lambda_{\mathrm{wt}} := (r_1, r_2, \ldots, r_{\sigma(\mathrm{wt})})$, where $\{\mathrm{wt}(u) : u \in \mathbb{F}_q^n\} = \{r_1, r_2, \ldots, r_{\sigma(\mathrm{wt})}\}$ is the set of values assumed by the weight wt, $0 = r_1 < r_2 < \cdots < r_{\sigma(\mathrm{wt})}$ and $\sigma(\mathrm{wt}) = |\{\mathrm{wt}(u) : u \in \mathbb{F}_q^n\}|$. Denote by e_j the *basic vector* such that $\mathrm{supp}(e_j) = \{j\}$ and the j-th coordinate is 1. Let $S_i = \{u \in \mathbb{F}_q^n : \mathrm{wt}(u) = i\}$ be the sphere of radius i centered at the 0 vector and let $\{S_i : i \in \Lambda_{\mathrm{wt}}\}$ be the set of spheres. The *array of spheres* of the weight wt is the ordered list $\mathcal{S}_{\mathrm{wt}} = (S_{r_1}, S_{r_2}, \ldots, S_{r_{\sigma(\mathrm{wt})}})$. Since we are considering only metrics defined by weights, it follows (Property 2, see Definition 3.1.1) that if $e_j \in \widehat{S_i}$, then any $\lambda e_j \in S_i$, for any $\lambda \in \mathbb{F}_q \setminus \{0\}$. So, for every sphere S_i we consider the subset $\widehat{S_i} \subset S_i$ consisting of multiples of the basic vectors: $\widehat{S_i} := \{\lambda e_j : e_j \in S_i \text{ and } \lambda \in \mathbb{F}_q \setminus \{0\}\}$.

Definition 4.1.1. *We say that S_i is a* basic sphere *if there is a basic vector $e_j \in S_i$, i.e., $\widehat{S_i} \neq \emptyset$. The* array of basic spheres, *denoted by $\mathcal{S}_{\mathrm{wt}}^B$, is the sub-list of $\mathcal{S}_{\mathrm{wt}}$ obtained by removing the non-basic spheres.*

The set $\mathcal{S}_{\mathrm{wt}}^B$ of all basic spheres induces a sub-list Λ_{wt}^B of Λ_{wt} in which $S_i \in \mathcal{S}_{\mathrm{wt}}^B$ if, and only if, $i \in \Lambda_{\mathrm{wt}}^B$.

Let wt be a weight and consider array of spheres $\mathcal{S}_{\mathrm{wt}} = (S_{r_1}, S_{r_2}, \ldots, S_{r_{\sigma(\mathrm{wt})}})$ and $\mathcal{S}_{\mathrm{wt}}^B = (S_{r_{i_1}}, S_{r_{i_2}}, \ldots, S_{r_{i_\alpha}})$.

Consider an arbitrary ordering $(u_1, u_2, \ldots, u_l)$ of $\bigcup\limits_{i \in \Lambda_{\mathrm{wt}}^B} S_i$, the set of basic spheres. The dual $\mathcal{S}_{\mathrm{wt}}^{\perp} = (S_{r_1}^{\perp}, S_{r_2}^{\perp}, \ldots, S_{r^{\perp}_{\sigma(\mathrm{wt})}})$ is obtained by the procedure as follows:

> **Step 1 -** Consider the family $\mathcal{V}_{\mathrm{wt}^{\perp}}^B = (S_{r_{i_1}}^{\perp}, \ldots, S_{r_{i_\alpha}}^{\perp})$ obtained from $\mathcal{S}_{\mathrm{wt}}^B$ with the reverse ordering, i.e., $S_{r_{i_j}}^{\perp} = S_{r_{(i_\alpha - i_j + 1)}}$.

In the next step, we will replace the non-basic vectors contained in basic spheres.

Step 2 - For $u := u_1 \in S^{\perp}_{r_{i_1}} \cup \cdots \cup S^{\perp}_{r_{i_\alpha}}$, there is r_{i_j} such that $u \in S^{\perp}_{r_{i_j}}$. Let $\eta(u) \geq j$ be the lowest integer in which $\mathrm{supp}(u) \subset \mathrm{supp}(\widehat{S}^{\perp}_{r_{i_1}} \cup \cdots \cup \widehat{S}^{\perp}_{r_{i_{\eta(u)}}})$. Then, u is moved to $S^{\perp}_{r_{i_{\eta(u)}}}$. That is, we set $S^{\perp}_{r_{i_j}} := S^{\perp}_{r_{i_j}} \setminus \{u\}$ and $S^{\perp}_{r_{i_{\eta(u)}}} := S^{\perp}_{r_{i_{\eta(u)}}} \cup \{u\}$.
Repeat it for all the elements $u_2, u_3, \ldots, u_l$.

Remark 4.1.2. *Since $e_k \in S^{\perp}_{r_{i_j}}$ implies $k \in \mathrm{supp}(\widehat{S}^{\perp}_{r_{i_j}})$, it follows that any basic vector e_k stays at the original basic sphere, so that, the algorithm does not change any $\widehat{S}^{\perp}_{r_i}$, that is, at any round of Step 2, Λ^B_{wt} is unchanged. Moreover, the spheres S_i may be modified at the rounds of Step 2, but the sets $\widehat{S}_i$ are always the same.*

Now we re-arrange the family $\mathcal{V}^B_{\mathrm{wt}^\perp}$ determined in Step 1. We will need to insert the non-basic spheres (which up to this point rested untouched) into the ordered list of basic spheres, the output of Step 2.

Step 3 - Consider the sub-string $\mathcal{S}_{\mathrm{wt}} \setminus \mathcal{S}^B_{\mathrm{wt}} = (S_{r_1}, S_{r_{j_1}}, S_{r_{j_2}}, \ldots, S_{r_{j_\beta}})$, be an ordered family of length $\beta + 1$. The zero sphere S_{r_1} is placed at the first position of the string $\mathcal{V}^B_{\mathrm{wt}^\perp}$, i.e., $\mathcal{V}^B_{\mathrm{wt}^\perp} := (S_{r_1}, S^{\perp}_{r_{i_1}}, \ldots, S^{\perp}_{r_{i_\alpha}})$. Fix $\omega(0) = 0$.
For $l = 1, 2, \ldots, \beta$, let

$$\omega(l) = \min \left\{ \varsigma(l) \in [\alpha] : \exists u \in S_{r_{j_l}} \text{ such that } \mathrm{supp}(u) \subset \mathrm{supp}(\bigcup_{k=1}^{\varsigma(l)} \widehat{S}^{\perp}_{r_{i_k}}) \right\}.$$

If $\omega(l) \neq \omega(l-1)$ place $S_{r_{j_l}}$ at the $(\omega(l)+1)$-th coordinate of $\mathcal{V}^B_{\mathrm{wt}^\perp}$;
Else, if $\omega(l) = \omega(l-1)$, then place $S_{r_{j_l}}$ at the $(\omega(l-1)+2)$-th coordinate of $\mathcal{V}^B_{\mathrm{wt}^\perp}$.

In the next step, we just rename the labels of the string $\mathcal{V}^B_{\mathrm{wt}^\perp}$, the output of Step 3.

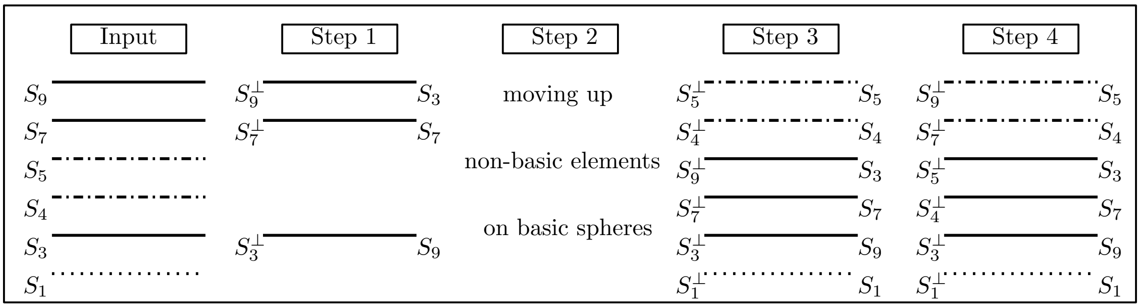
Step 4 - Replace the subscript j of the i-th position S_j of $\mathcal{V}^B_{\mathrm{wt}^\perp}$ by r_i.

The outcome of Step 4 is a string of disjoint subsets $\mathcal{V}^B_{\mathrm{wt}^\perp} = (S^{\perp}_{r_1}, S^{\perp}_{r_{i_1}}, \ldots, S^{\perp}_{r_{\sigma(\mathrm{wt})}})$.

In the next step, we move up the vectors in this string of spheres in a cascade fashion, to ensure that the metric will respect support of vectors.

Step 5 - Move vectors $u \in S^{\perp}_j$ to the sphere in higher position $\chi(u)$, in which there is a vector $v \in S^{\perp}_{\chi(u)}$ such that $\mathrm{supp}(v) \subset \mathrm{supp}(u)$. To be more precise, given $u \in \mathbb{F}_q^n$, let $j(u)$ be such that $u \in S^{\perp}_{j(u)}$. We set $R(u) := \{v \in \mathbb{F}_q^n : \mathrm{supp}(v) \subset \mathrm{supp}(u)\}$. Set $\chi(u) = \max\{r_i : S^{\perp}_{r_i} \cap R(u) \neq \emptyset\}$. We remark that $\chi(u) \geq j(u)$. Now, we move u from $S^{\perp}_{j(u)}$ to $S^{\perp}_{\chi(u)}$ and move on to the next vector. We remark that in case $\chi(u) = j(u)$ nothing is done. The transitivity of the continence of (sets) support ensure that Step 5 does no loop, that is, once u is moved, it will not need to be moved again.

Next we try to illustrate how this algorithm works step-by-step.



Definition 4.1.3. *The* dual weight $\mathrm{wt}^\perp$ *of the weight* wt *is determined by string* $\mathcal{S}_{\mathrm{wt}}^\perp$ *obtained at the end algorithm:*

$$\mathrm{wt}^\perp(u) = i, \text{ for every } u \in S_i^\perp.$$

Example 4.1.4. *Let* wt_P *be a poset weight with* $P = ([3], \{1 \prec_P 3, 2 \prec_P 3\})$ *and let us consider the binary case* $q = 2$*. Then we have*

$$\mathcal{S}_{\mathrm{wt}_P} = (S_0 = \{000\}, S_1 = \{100, 010\}, S_2 = \{110\}, S_3 = \{101, 011, 001, 111\})$$

and

$$\mathcal{S}^B_{\mathrm{wt}_P} = (S_1 = \{100, 010\}, S_3 = \{101, 011, 001, 111\}).$$

Step 1: *We set*

$$\mathcal{V}^B_{\mathrm{wt}^\perp} = (S_1^\perp = \{101, 011, 001, 111\}, S_3^\perp = \{100, 010\}).$$

Step 2: *We move the (non-basic) vectors* 101, 011 *and* 111 *to* $S_3^\perp$ *and get*

$$\mathcal{V}^B_{\mathrm{wt}^\perp} = (S_1^\perp = \{001\}, S_3^\perp = \{100, 010, 101, 011, 111\}).$$

Step 3: *We place the non-basic spheres* S_0 *and* S_2 *into the ordered string that came out of the previous step and get:*

$$\mathcal{V}^B_{\mathrm{wt}^\perp} = (S_0^\perp = \{000\}, S_1^\perp = \{001\}, S_3^\perp = \{100, 010, 101, 011, 111\}, S_2^\perp = \{110\}).$$

Step 4: *We just relabel the ordered string that came out of the previous step and get:*

$$\mathcal{V}^B_{\mathrm{wt}^\perp} = (S_0^\perp = \{000\}, S_1^\perp = \{001\}, S_2^\perp = \{100, 010, 101, 011, 111\}, S_3^\perp = \{110\}).$$

Step 5: *We note that* $\mathrm{supp}(110) \subset \mathrm{supp}(111)$*. This means that it is necessary to move* 111 *to* $S_3^\perp$ *and we get the outcome*

$$\mathcal{S}^\perp_{\mathrm{wt}_P} = (S_0^\perp = \{000\}, S_1^\perp = \{001\}, S_2^\perp = \{100, 010, 101, 011\}, S_3^\perp = \{110, 111\}).$$

We remark that the dual poset $\mathrm{wt}_{P^\perp}$ *is defined by* $P^\perp = ([3], \{3 \prec_{P^\perp} 1, 3 \prec_{P^\perp} 2\})$ *and it is straightforward to check that* $\mathcal{S}_{\mathrm{wt}_{P^\perp}} = \mathcal{S}_{\mathrm{wt}_P^\perp}$.

Theorem 4.1.5. *If* $\mathrm{wt} : \mathbb{F}_q^n \to \mathbb{R}$ *is a weight, then its dual* $\mathrm{wt}^\perp$ *is also a weight.*

Proof. Since $\Lambda_{\mathrm{wt}^\perp} := \mathcal{V}^B_{\mathrm{wt}^\perp}$ is equal to Λ_{wt} and $S_1^\perp = S_1$ is placed at the first coordinate of $\Lambda_{\mathrm{wt}^\perp} := \mathcal{V}^B_{\mathrm{wt}^\perp}$, it follows that $\mathrm{wt}^\perp$ satisfies Property 1. Steps 2 and 3 of the algorithm ensure that Property 2 holds for $\mathrm{wt}^\perp$. Indeed, suppose that $\mathrm{supp}(u) \subset \mathrm{supp}(v)$, for $u \in S_{r_i}^\perp$ and $v \in S_{r_j}^\perp$. If $S_{r_i}^\perp$ and $S_{r_j}^\perp$ are basic spheres, the minimalities of $\eta(u)$ and $\eta(v)$, in Step 2, combined with $\mathrm{supp}(u) \subset \mathrm{supp}(v)$ imply that $\eta(u) \le \eta(v)$, i.e., $\mathrm{wt}^\perp(u) \le \mathrm{wt}^\perp(v)$. If $S_{r_j}^\perp$ is a non-basic sphere, $r_i \le \omega(j) = r_j$. So that, $\mathrm{wt}^\perp(u) \le \mathrm{wt}^\perp(v)$ as required. □

4.2 MacWilliams-type Identity

The wt-*weight enumerator* of a code $\mathcal{C}$ is the polynomial

$$W_{\mathcal{C}}^{\mathrm{wt}}(X) = \sum_i A_i^{\mathrm{wt}}(\mathcal{C}) X^i$$

where $A_i^{\mathrm{wt}}(\mathcal{C}) = |\{c \in \mathcal{C} : \mathrm{wt}(c) = i\}|$.

Definition 4.2.1. *A weight* wt *admits a MacWilliams-type identity if the* wt*-weight enumerator of a linear code determines the* $\mathrm{wt}^\perp$*-weight enumerator of its dual, i.e., given two linear codes* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *such that* $W_{\mathcal{C}_1}^{\mathrm{wt}}(X) = W_{\mathcal{C}_2}^{\mathrm{wt}}(X)$ *then* $W_{\mathcal{C}_1^\perp}^{\mathrm{wt}^\perp}(X) = W_{\mathcal{C}_2^\perp}^{\mathrm{wt}^\perp}(X)$.

Lemma 4.2.2. *Let* $\mathcal{C} \subset \mathbb{F}_q^n$ *be a linear code and let* χ *of* $\mathbb{F}_q$ *be an additive character. Then,* $\sum_{x \in \mathcal{C}} \chi(u \cdot v) = |\mathcal{C}|$ *if* $v \in \mathcal{C}^\perp$ *and* 0 *otherwise.*

Lemma 4.2.3. *Let* wt *be a weight. Given a linear code* $\mathcal{C}$ *of* $\mathbb{F}_q^n$*, then*

$$A_i^{\mathrm{wt}}(\mathcal{C}) = \frac{1}{|\mathcal{C}^\perp|} \sum_{1 \le j \le n} \sum_{x \in \mathcal{C}^\perp \cap S_j^\perp} \sum_{y \in S_i} \chi(x \cdot y),$$

where S_i *and* $S_j^\perp$ *are the spheres of radii* i *and* j *considering the metric induced by the weights* wt *and* $\mathrm{wt}^\perp$*, respectively, i.e.,* $S_i = \{x \in \mathbb{F}_q^n : \mathrm{wt}(x) = i\}$ *and* $S_j^\perp = \{x \in \mathbb{F}_q^n : \mathrm{wt}^\perp(x) = j\}$.

Proof. First, note that $A_i^{\mathrm{wt}}(\mathcal{C}) = |\mathcal{C} \cap S_i| = \sum_{u \in \mathcal{C} \cap S_i} 1$. Lemma 4.2.2 implies that

$$\begin{aligned} A_i^{\mathrm{wt}}(\mathcal{C}) &= \sum_{v \in S_i} \frac{1}{|\mathcal{C}^\perp|} \sum_{u \in \mathcal{C}^\perp} \chi(u \cdot v) \\ &= \frac{1}{|\mathcal{C}^\perp|} \sum_{u \in \mathcal{C}^\perp} \sum_{v \in S_i} \chi(u \cdot v) \\ &= \frac{1}{|\mathcal{C}^\perp|} \sum_{1 \le j \le n} \sum_{u \in \mathcal{C}^\perp \cap S_j^\perp} \sum_{v \in S_i} \chi(u \cdot v). \end{aligned}$$

□

Theorem 4.2.4. *A weight* wt *admits the MacWilliams Identity if, given $i, j \in [n]$, and $u, u' \in S_i$, then*

$$\sum_{v \in S_j^{\perp}} \chi(u \cdot v) = \sum_{v \in S_j^{\perp}} \chi(u' \cdot v).$$

Proof. Let $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{F}_q^n$ be linear codes such that $W_{\mathcal{C}_1}^{\mathrm{wt}}(X) = W_{\mathcal{C}_2}^{\mathrm{wt}}(X)$. Since wt satisfies the first condition we have that $p_{ij} := \sum_{v \in S_j^{\perp}} \chi(u \cdot v)$ does not depend on the choice of $u \in S_i$. Therefore, $W_{\mathcal{C}_1^{\perp}}^{\mathrm{wt}^{\perp}}(X) = W_{\mathcal{C}_2^{\perp}}^{\mathrm{wt}^{\perp}}(X)$, since, from Lemma 4.2.3,

$$\begin{aligned} A_j^{\mathrm{wt}^{\perp}}(\mathcal{C}_k^{\perp}) &= \frac{1}{|\mathcal{C}_k|} \sum_{1 \le i \le n} \sum_{u \in \mathcal{C}_k \cap S_i} p_{ij} \\ &= \frac{1}{|\mathcal{C}_k|} \sum_{1 \le i \le n} A_i^{\mathrm{wt}}(\mathcal{C}_k) p_{ij}, \text{ for } k \in \{1, 2\}. \end{aligned}$$

□

4.3 Relation to the usual MacWilliams' Identity

In this section we shall prove that the MacWilliams-type Identity stated in Theorem 4.2.4 is a generalization of the MacWilliams Identity already known for combinatorial metrics (as it appears in Chapter 1) or the poset metrics (as it appears in (KIM; OH, 2005)). To establish it we will show that a combinatorial weight is self dual and that the dual weight determined by a hierarchical poset is the weight determined by the dual poset.

Proposition 4.3.1. *Any combinatorial weight is self-dual, that is,* $\mathrm{wt}_{\mathcal{F}}^{\perp} = \mathrm{wt}_{\mathcal{F}}$, *for any covering* $\mathcal{F}$ *of* $[n]$.

Proof. First note that $\mathrm{wt}_{\mathcal{F}}(e_i) = 1$, for any covering $\mathcal{F}$ and any basic vector $e_i \in \mathbb{F}_q^n$. This implies the set of basic spheres $\mathcal{S}_{\mathrm{wt}_{\mathcal{F}}}^{B}$ is composed only by the radius 1 sphere S_1. Due to this fact: 1) Step 1 does not change any order; 2) Step 2 does not move any vector to another basic sphere; 3) Every non-basic sphere is placed at original coordinate position; 4) There is nothing to relabel, since at this point the order is exactly like the one of the input weight $\mathrm{wt}_{\mathcal{F}}$; 5) Since up to Step 4, the spheres are exactly as defined by $\mathrm{wt}_{\mathcal{F}}$, Step 5 does not change any vector from the original sphere. □

Proposition 4.3.2. *Let P be an hierarchical poset and* wt_P *the corresponding weight. Then, the dual weight coincides with the weight defined by the dual poset, i.e.,* $\mathrm{wt}_P^{\perp} = \mathrm{wt}_{P^{\perp}}$.

Proof. Let $N_i = |\Gamma_i^P|$, for each level Γ_i^P of P and denote by S_i^P and $(S_i^P)^\perp$ the sphere of radius i concerning the P-weight and dual of the P-weight, respectively. It follows that

$$
\begin{aligned}
S_1^P &= \{\lambda e_i : i \in \Gamma_1^P \text{ and } \lambda \in \mathbb{F}_q \setminus \{0\}\} \\
S_2^P &= \{\lambda_1 e_i + \lambda_2 e_j : i, j \in \Gamma_1^P, i \neq j \text{ and } \lambda_1, \lambda_2 \in \mathbb{F}_q \setminus \{0\}\} \\
&\vdots \qquad\qquad\qquad \vdots \\
S_{N_1}^P &= \{\lambda_1 e_{i_1} + \lambda_2 e_{i_2} + \cdots + \lambda_{N_1} e_{i_{N_1}} : \{i_1, i_2, \ldots, i_{N_1}\} = \Gamma_1^P \text{ and } \lambda_1, \lambda_2, \ldots, \lambda_{N_1} \in \mathbb{F}_q \setminus \{0\}\} \\
S_{N_1+1}^P &= \{\lambda e_i + u : i \in \Gamma_2^P, \lambda \in \mathbb{F}_q \setminus \{0\} \text{ and } \mathrm{supp}(u) \subset \Gamma_1^P\} \\
&\vdots \qquad\qquad\qquad \vdots \\
S_{N_1+\cdots+N_{h(P)}}^P &= \{\lambda_1 e_{i_1} + \cdots + \lambda_{N_{h(P)}} e_{i_{N_{h(P)}}} + u : \text{ with} \\
&\qquad \{i_1, \ldots, i_{N_{h(P)}}\} = \Gamma_{h(P)}^P, \lambda_1, \ldots, \lambda_{N_{h(P)}} \in \mathbb{F}_q \setminus \{0\} \text{ and } \mathrm{supp}(u) \cap \Gamma_{h(P)}^P = \emptyset\}.
\end{aligned}
$$

Step 1 and Step 4 of the algorithm ensure that $S_1^{P^\perp} = \{\lambda e_i : i \in \Gamma_{h(P)}^P \text{ and } \lambda \in \mathbb{F}_q \setminus \{0\}\} \subset (S_1^P)^\perp$. In addition, Step 2 ensures that $u \in (S_1^P)^\perp$ if, and only if, $|\mathrm{supp}(u)| = 1$. This implies that $S_1^{P^\perp} = (S_1^P)^\perp$.

Step 3 and Step 4 ensure that $S_2^{P^\perp} = \{\lambda_1 e_i + \lambda_2 e_j : i, j \in \Gamma_{h(P)}^P \text{ and } \lambda_1, \lambda_2 \in \mathbb{F}_q \setminus \{0\}\} \subset (S_2^P)^\perp$. Further, Steps 3 and 5 ensure that any vector $\lambda_1 e_i + \lambda_2 e_j + u \notin (S_2^P)^\perp$, for $\{i, j\} \subset \Gamma_{h(P)}^P = \Gamma_1^{P^\perp}$ and $u \neq 0$ satisfying $\mathrm{supp}(u) \cap \{i, j\} = \emptyset$.

By using repetitively the procedure as before, we show that every $S_i^{P^\perp} = (S_i^P)^\perp$, so that $\mathrm{wt}_P^\perp = \mathrm{wt}_{P^\perp}$. □

Corollary 4.3.3. *If* wt *is either a combinatorial or poset weight admitting a MacWilliams Identity, then the identity coincides with the MacWilliams-type Identity of Theorem 4.2.4.*

The identification between the dual weight and the weight determined by the dual poset established in Proposition 4.3.2 is not valid for a general poset, as we can see in the following tiny example.

Example 4.3.4. *Let* wt_P *be a poset weight with* $P = ([3], \{1 \prec_P 3\})$*. Then*

$$\mathcal{S}_{\mathrm{wt}_P} = (S_0 = \{000\}, S_1 = \{100, 010\}, S_2 = \{110, 001, 101\}, S_3 = \{011, 111\}).$$

The dual poset $P^\perp = ([3], \{3 \prec_{P^\perp} 1\})$ *has array of spheres*

$$\mathcal{S}_{\mathrm{wt}_{P^\perp}} = (S_0 = \{000\}, S_1 = \{001, 010\}, S_2 = \{011, 100, 101\}, S_3 = \{110, 111\}).$$

Now, the dual weight $\mathrm{wt}_P^\perp$ *has array of spheres*

$$\mathcal{S}_{\mathrm{wt}_P}^\perp = (S_0 = \{000\}, S_1 = \{001\}, S_2 = \{010, 110, 100, 101\}, S_3 = \{011, 111\})$$

so that $\mathcal{S}_{\mathrm{wt}_{P^\perp}} \neq \mathcal{S}_{\mathrm{wt}_P}^\perp$.

To conclude, we should remark that it is a work-in-progress to prove that this concept of weight duality coincides with the one established for digraphs (as in (ETZION; FIRER; MACHADO, 2017)) or the labeled-poset block (as in Chapter 2), in the instances where a MacWilliams' Identity holds. There are some missing details, but it seems to be the case.

It may happen that the concept of weight duality still need to be refined in order to encompass all the instances of known weights respecting support. We conjecture that it may be refined in order to coincide with a weight that is *equivalent* (but not equal) to the ones established in the particular families of weights respecting support known in the literature (posets, poset-blocks, digraphs and labeled-poset-blocks).

5 Representation of directed acyclic graphs

5.1 Introduction

[1] In the WWW network, a number of pages are devoted to *topic or item disambiguation*; in disambiguation pages, a number of identical names of designators are used to describe different entities which are further clarified and narrowed down in context via links to more specific pages. For example, typing the word “Michael Jordan” into a search engine such as Google produces a Wikipedia page which lists sportists, actors, scientists and other persons bearing this name. From this web page, one can choose to follow a link to any one of the items sharing the same two keywords, “Michael” and “Jordan”. Most of the specific pages do not link back to the disambiguation page: For example, following the link to “Michael Jordan (footballer)” does not allow for returning to the disambiguation page, and may hence be viewed as a directed link. Furthermore, disambiguation pages tend to have little content, usually in the form of lists, while the pages that link to it tend to have significantly more information about one of the individuals.

Motivated by such directed networks of webpages, we consider the following problem, illustrated by a small-scale directed graph depicted in Figure 1. Assume that the vertices A, B, C, D correspond to four web-pages that contain different collections of topics, files or networks, represented by color-coded rectangles (For example, each color may correspond to a different person bearing the same name). Two web-pages are linked to each other if they have at least one topic in common (e.g., the same name or some other shared feature). For a directed graph, in addition to the shared content assumption one needs to provide an explanation for the direction of the links, i.e., which vertex in the arc represents the tail and which vertex in the arc represents the head. In the context of the above described web-page linkages, it is reasonable to assume that a webpage links to another terminal webpage if the latter covers more topics, i.e., contains additional information compared to the source page. In Figure 1, the link between web-pages A and B is directed from A to B, since B lists three topics, while A lists only two. This give rise to two generative constraints for the existence of a directed edge: Shared information content and content size dominance. This is a natural generative assumption, which has been exploited in a similar form in a number of data mining contexts (TSOURAKAKIS, 2015; DAU; MILENKOVIC, 2017).

Often, one is only presented with the directed graph topology of a directed graphs and asked to determine the latent vertex content leading to the observed topology.

[1] Initial findings on the topic were presented at ISIT 2019.

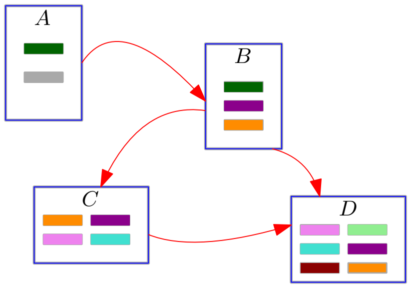
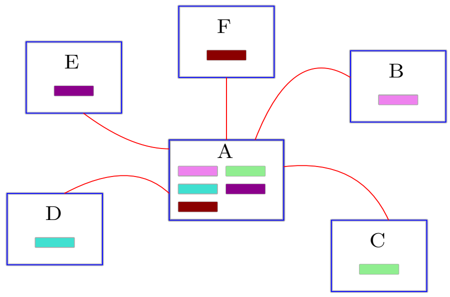


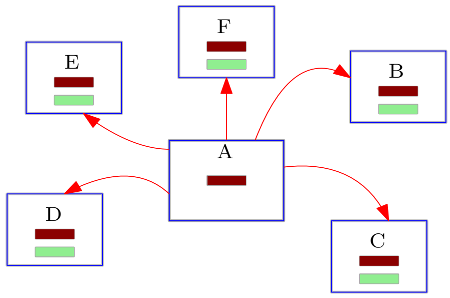
Figure 1 – An information storage network such as the World Wide Web. Each vertex contains a list of color-coded topics or files, representing its information content (e.g., vertex B contains a green, purple and orange topic). Vertices A and B are connected through an arc (A,B) since they share the green-colored topic and A lists two, while B lists three files.

A problem of particular interest is to determine the *smallest* topic/information content that explains the observed digraph. This question may be formally described as follows. Let $D = (V, A)$ be a directed graph with vertex set V and arc set A, and assume that each vertex $v \in V$ is associated with a nonempty subset $\varphi(v)$ of a finite ground set $\mathcal{C}$, called the *color set*, such that $(u, v) \in A$ if and only if $|\varphi(u) \cap \varphi(v)| \geq 1$ and $|\varphi(u)| < |\varphi(v)|$ (i.e., two vertices share an arc if their color sets intersect and the color set of the tail is strictly smaller than the color set of the head). If such a representation is possible, we refer to it as a *directed intersection representation*. The question of interest is to determine the smallest cardinality of the ground set $\mathcal{C}$ which allows for a directed intersection representation of a digraph D with $|V| = n$ vertices, henceforth termed the directed intersection number of D. Clearly, not all digraphs allow for such a representation. For example, a directed triangle $D\,(V, A)$ with $V = \{1, 2, 3\}$ and $A = \{(1, 2), (2, 3), (3, 1)\}$ does not admit a directed intersecting representation, as such a representation would require $|\varphi(1)| < |\varphi(2)| < |\varphi(3)| < |\varphi(1)|$, which is impossible. The same is true of every digraph that contains cycles, but as we subsequently show, every directed acyclic graph (DAG) admits a directed intersection representation. We focus on connected DAGs, although our results apply to disconnected graphs with either no or some small modifications.

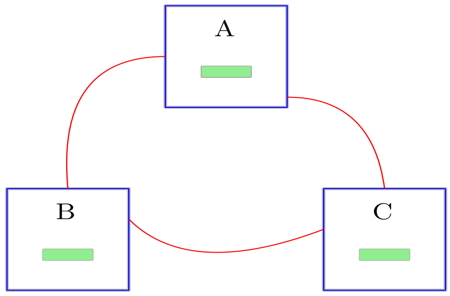
The problem of finding directed intersection representations of digraphs is closely associated with the intersection representation problem for undirected graphs. Intersection representations are of interest in many applications such as keyword conflict resolution, traffic phasing, latent feature discovery and competition graph analysis (PULLMAN, 1983; ROBERTS, 1985; DAU; MILENKOVIC; PULEO, 2017). Formally, the vertices $v \in V$ of a graph $G(V, E)$ are associated with subsets $\varphi(v)$ of a ground set $\mathcal{C}$ so that $(u, v) \in E$ if and only if $|\varphi(u) \cap \varphi(v)| \geq 1$. The intersection number (IN) of the graph $G = (V, E)$ is the smallest size of the ground set $\mathcal{C}$ that allows for an intersection representation, and it is well-defined for all graphs. Finding the intersection number of a graph is equivalent to finding the edge clique cover number, as proved by Erdós, Goodman and Posa in (ERDöS;



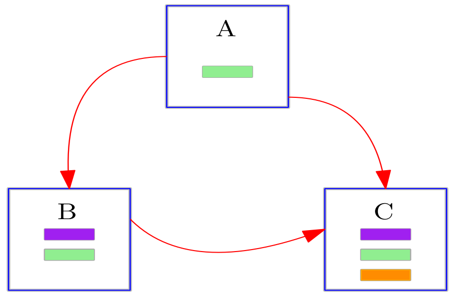
(a) The intersection number of a star graph is equal to $|E| = n - 1$ (e.g., 5).



(b) The DIN of any star digraph is 2.



(c) The intersection number of a complete graph is 1.



(d) The DIN of a “complete” DAG on three vertices is exactly 3.

Figure 2 – A comparison of the intersection numbers and DINs of the star and complete graph/DAG.

GOODMAN; PóSA, 1966); determining the edge clique cover number is NP-hard, as shown by Orlin (ORLIN, 1977). The intersection number of an undirected graph may differ vastly from the DIN of some of its directed counterparts, whenever the latter exists. This is illustrated by two examples in Figure 2.

The paper is organized as follows. Section 5.2 contains a constructive proof that all DAGs have a finite directed intersection representation and algorithmically identifies representations using a suboptimal number of colors. As a consequence, the constructive algorithm establishes a bound on the DIN of arbitrary DAGs with a prescribed number of vertices. In the same section, we inductively prove an improved upper bound which is $\frac{5\,n^2}{8} - \frac{3\,n}{4} + 1$. In Section 5.3 we introduce the notion of DIN-extremal DAGs and describe constructions of acyclic digraphs with DINs equal to

$$\frac{n^2}{2} + \lfloor \frac{n^2}{16} - \frac{n}{4} + \frac{1}{4} \rfloor - 1.$$

5.2 Representations of Directed Acyclic Graphs

We use the notation and terminology described below. Whenever clear from the context, we omit the argument n.

The in-degree of a vertex v is the number of arcs for which v is the head, while the out-degree is the number of arcs for which v is the tail. The set of in-neighbors of v is the set of vertices sharing an arc with v as the head, and is denoted by $N^-(v)$. The set of out-neighbors $N^+(v)$ is defined similarly.

For a given acyclic digraph $D(V, A)$, let $\Gamma : V \rightarrow \mathbb{N}$ be a mapping that assigns to each vertex $v \in V$ the length of the longest directed path that terminates at v. The map Γ induces a partition of the vertex set V into *levels* $(V_0, \ldots, V_\ell)$, such that $V_i = \{v \in V : \Gamma(v) = i\}$. We refer to V_i, $i = 1, \ldots, \ell$ as the *longest path decomposition* of V and the graph G. Clearly, there is no arc between any pair of vertices u and v at the same level V_i, $i = 1, \ldots, \ell$, as this would violate the longest path partitioning assumption. Note that although the longest path problem is NP-hard for general graphs, it is linear time for DAGs. Finding the longest path in this case can be accomplished via topological sorting (BATTISTA; TAMASSIA, 1988).

Lemma 5.2.1. *Every DAG $D(V, A)$ on n vertices admits a directed intersection representation. Moreover, $DIN(n) \leq \frac{5}{8}n^2 - \frac{1}{4}n$.*

Proof. We prove the existence claim and upper bound by describing a constructive color assignment algorithm.

Step 1: We order the vertices of the digraph as $V = (v_1, v_2, \ldots, v_n)$ so that if $(v_i, v_j) \in A$, then $i < j$. One such possible ordering is henceforth referred to as a left-to-right order, and it clearly well-defined as the digraph is acyclic. We then construct the longest path decomposition and order the vertices in the graph starting from the first level and proceeding to the last level. The order of vertices inside each level is irrelevant.

Step 2: We group vertices into pairs in order of their labels, i.e., (v_{2i-1}, v_{2i}), for $1 \leq i \leq \frac{n}{2}$, and then assign to each vertex v_i, $i = 1, \ldots, n$, a color set distinct from the color set of all other vertices. The sizes of the color sets equal $\frac{n}{2} - \lceil \frac{i}{2} \rceil$.

Remark 5.2.2. *In this step we used exactly*

$$2 \cdot \left(\frac{n}{2} - 1 + \frac{n}{2} - 2 + \ldots + 1 \right) = 2 \cdot \frac{1 + \frac{n}{2} - 1}{2} \cdot \left(\frac{n}{2} - 1 \right) = \frac{n^2}{4} - \frac{n}{2} \tag{5.1}$$

distinct colors. Those colors are going to be reused to accomodate for arcs between pairs.

Step 3: For each $1 \leq i \leq n - 2$, we assign common colors for arcs from v_i to vertices belonging to pairs that follow the pair in which v_i lies. More precisely:

- If $(v_i, v_{2j-1}) \notin A$ and $(v_i, v_{2j}) \notin A$ for some j such that $2 \cdot \lceil \frac{i}{2} \rceil < 2j - 1 \leq n - 1$, then we do nothing and move to the next step.

• If $(v_i, v_{2j-1}) \in A$ and $(v_i, v_{2j}) \notin A$ for some j such that $2 \cdot \lceil \frac{i}{2} \rceil < 2j-1 \leq n-1$, then we copy one color from $\varphi(v_i)$ not previously used in Step 3 and place it into the color set of v_{2j-1}, $\varphi(v_{2j-1})$.

• If $(v_i, v_{2j-1}) \notin A$ and $(v_i, v_{2j}) \in A$ for some j such that $2 \cdot \lceil \frac{i}{2} \rceil < 2j-1 \leq n-1$, then we copy one color from $\varphi(v_i)$ not previously used in Step 3 and place it into the color set of v_{2j}, $\varphi(v_{2j})$.

• If $(v_i, v_{2j-1}) \in A$ and $(v_i, v_{2j}) \in A$ for some j such that $2 \cdot \lceil \frac{i}{2} \rceil < 2j-1 \leq n-1$, then we copy one color from $\varphi(v_i)$ not previously used in Step 3 and place it into both $\varphi(v_{2j-1})$ and $\varphi(v_{2j})$.

Remark 5.2.3. *Since each vertex v_i has a color set $\varphi(v_i)$ with $\frac{n}{2} - \lceil \frac{i}{2} \rceil$ colors, and there are $\frac{n}{2} - \lceil \frac{i}{2} \rceil$ pairs following the pair that vertex v_i is located in the previously fixed left-to-right ordering, we will never run out of colors during the above color assignment process.*

The color sets obtained after the previously described procedure are denoted by φ'.

Step 4: To the color sets of each pair of vertices (v_{2i-1}, v_{2i}), we add at most $3i$ new colors. The augmented color sets, denoted by φ'', satisfy 1) if $v_{2i-1}v_{2i}$ is an arc, then $|\varphi''(v_{2i-1})| = \frac{n}{2} + 2i - 2$ and $|\varphi''(v_{2i})| = \frac{n}{2} + 2i - 1$; 2) if $v_{2i-1}v_{2i}$ is not an arc, then $|\varphi''(v_{2i-1})| = |\varphi''(v_{2i})| = \frac{n}{2} + 2i - 1$.

In **Step 4** we add at most

$$\frac{n}{2} + 2i - 1 - 1 - \left(\frac{n}{2} - i\right) = 3i - 2$$

colors to the color set of v_{2i-1} and at most

$$\frac{n}{2} + 2i - 1 - \left(\frac{n}{2} - i\right) = 3i - 1$$

colors to the color set of v_{2i} to reach the desired color-set sizes. Note that some colors may be reused so that at this step, at most $3i - 1$ new colors are actually needed for a pair (v_{2i-1}, v_{2i}). Note that in **Step 3**, for each pair (v_{2i-1}, v_{2i}), we added in total at most $2i - 2$ colors to both $\varphi'(v_{2i-1})$ and $\varphi'(v_{2i})$. Since $3i - 2 > 2i - 2$, we added at least one color in common for the pair (v_{2i-1}, v_{2i}) so that the intersection condition is satisfied when $v_{2i-1}v_{2i}$ is an arc.

Thus, the number of colors used so far is at most

$$(3 \cdot 1 - 1) + (3 \cdot 2 - 1) + \ldots + \left(3 \cdot \frac{n}{2} - 1\right) = 3 \cdot \left(1 + 2 + \ldots + \frac{n}{2}\right) - \frac{n}{2}$$

$$= 3 \cdot \frac{1 + \frac{n}{2}}{2} \cdot \frac{n}{2} - \frac{n}{2} = \frac{3}{8}n^2 + \frac{n}{4}. \tag{5.2}$$

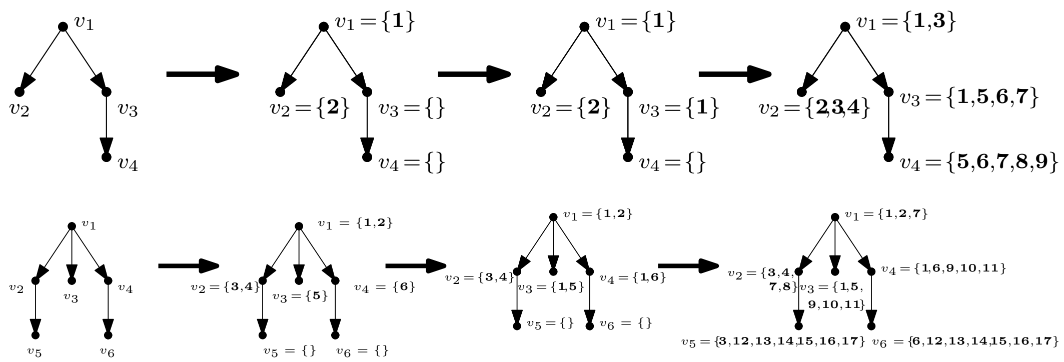


Figure 3 – Directed intersection representations for two rooted trees with four and six vertices, respectively. The representations were obtained by using a vertex partition according to the longest terminal path and the constructive algorithm of Lemma 5.2.1.

Next, we claim that φ'' is a valid representation that uses at most $\frac{5}{8}n^2 - \frac{n}{4}$ colors. From (5.1) and (5.2), we know that we used at most

$$\frac{n^2}{4} - \frac{n}{2} + \frac{3}{8}n^2 + \frac{n}{4} = \frac{5}{8}n^2 - \frac{n}{4}$$

colors.

The size condition obviously holds since $|\varphi''(v_i)| = \frac{n}{2} + i - 1$ and $(v_i, v_j \in A$ implies $|\varphi(v_i)| < |\varphi(v_j)|$. The intersection condition also holds since for each (v_i, v_j) with $i < j$, one has

- If $(v_i, v_j) \in A$, then

1) If (v_i, v_j) is a pair, then $\varphi''(v_i)$ and $\varphi''(v_j)$ have by the previous procedure at least one color in common.

2) If (v_i, v_j) is not a pair, then we added a color for this arc in **Step 3**.

- If $(v_i, v_j) \notin A$, then

1) If (v_i, v_j) is a pair, then by previous procedure $|\varphi''(v_i)| = |\varphi''(v_j)|$.

2) If (v_i, v_j) is not a pair, then $\varphi''(v_i)$ and $\varphi''(v_j)$ have no color in common based on **Step 2** and **Step 3**. □

On the example of the directed rooted tree shown in Figure 3, we see that more careful book-keeping and repeating of the colors used at the different levels allows one to reduce the cardinality of the representation set $\mathcal{C}$ compared to the one guaranteed by the construction of Lemma 5.2.1. If the vertices of the tree on the top figure are labeled according to the preorder traversal of the tree (MORRIS, 2017) as v_1, v_2, v_3, and v_4, the longest terminal path vertex partition equals $V_0 = \{v_1\}$, $V_1 = \{v_2, v_3\}$, $V_2 = \{v_4\}$. Using this decomposition and Lemma 5.2.1, we arrive at a bound for the DIN equal to 9. It is straightforward to see the actual DIN of the tree equals 5. Similarly, the algorithm of

Lemma 5.2.1 assigns 17 distinct colors to the vertices of the tree depicted at the bottom of the figure, while the actual DIN of the tree equals 6. Nevertheless, as we will see in the next section, a color assignment akin to the one described in Lemma 5.2.1 is needed to handle a number of Hamiltonian DAGs.

The algorithm described in the proof of Lemma 5.2.1 established that every DAG has a directed intersection representation and introduced an algorithmic upper bound on the DIN number of any DAG on n vertices with a leading term $\frac{5}{8}n^2$. An improved upper bound may be obtained using (nonconstructive) inductive arguments, as described in our main result, Theorem 5.2.4, and its proof. For simplicity, we only present the proof for even n.

Theorem 5.2.4. *Let $D = (V, A)$ be an acyclic digraph on n vertices. If n is even, then*

$$DIN(D) \leq \frac{5n^2}{8} - \frac{3n}{4} + 1.$$

Proof. We prove a stronger statement which asserts that for a left-to-right ordering of the vertices V of an arbitrary acyclic digraph D, there exists a representation φ such that

(a) $|\varphi(v_1)| = \frac{n}{2}$, $|\varphi(v_2)| \geq \frac{n}{2}$, and $|\varphi(v_i)| \geq \frac{n}{2} + 1$ for $3 \leq i \leq n$.

(b) For each pair (v_{2i-1}, v_{2i}), if $(v_{2i-1}, v_{2i}) \in A$ then $|\varphi(v_{2i-1})| = |\varphi(v_{2i})| - 1$, and if $(v_{2i-1}, v_{2i}) \notin A$ then $|\varphi(v_{2i-1})| = |\varphi(v_{2i})|$ for $1 \leq i \leq \frac{n}{2}$.

(c) $\cup_{i=1}^{n} \varphi(v_i)$ contains at most $\frac{5n^2}{8} - \frac{3n}{4} + 1$ colors.

The base case $n = 2$ is straightforward, as a connected DAG contains only one arc. In this case, we use $\{1\}$ for the head and $\{1, 2\}$ for the tail, and this representation clearly satisfies (a), (b), and (c).

We hence assume $n \geq 4$ and delete the arc (v_1, v_2) from D to obtain a new digraph D'; the ordering $(v_3, \ldots, v_n)$ is still a left-to-right ordering of D'. Thus, by the induction hypothesis, D' has a representation φ' satisfying

1) $|\varphi'(v_3)| = \frac{n}{2} - 1$, $|\varphi'(v_4)| \geq \frac{n}{2} - 1$, and $|\varphi'(v_i)| \geq \frac{n}{2}$ for $5 \leq i \leq n$;

2) For each pair of vertices (v_{2i-1}, v_{2i}), if $(v_{2i-1}, v_{2i}) \in A$, then $|\varphi(v_{2i-1})| = |\varphi(v_{2i})| - 1$, and if $(v_{2i-1}, v_{2i}) \notin A$, then $|\varphi(v_{2i-1})| = |\varphi(v_{2i})|$ for $2 \leq i \leq \frac{n}{2}$, and

3) The representation φ' uses at most

$$\frac{5(n-2)^2}{8} - \frac{3(n-2)}{4} + 1 = \frac{5n^2}{8} - \frac{3n}{4} + 1 - \left(\frac{5}{2}n - 4\right) \tag{5.3}$$

colors.

We initialize our procedure by letting $\varphi = \varphi'$.

Case 1: $(v_1, v_2) \notin A$.

Step 1: Assign to v_1 a set of $\dfrac{n}{2}-1$ new colors, say $\{\alpha_1, \ldots, \alpha_{\frac{n}{2}-1}\}$. Let $\varphi(v_1) = \{\alpha_1, \ldots, \alpha_{\frac{n}{2}-1}\}$. Assign to v_2 a set of $\dfrac{n}{2}-1$ new colors, say $\{\beta_1, \ldots, \beta_{\frac{n}{2}-1}\}$, all of which are distinct from the colors in $\{\alpha_1, \ldots, \alpha_{\frac{n}{2}-1}\}$. Let $\varphi(v_2) = \{\beta_1, \ldots, \beta_{\frac{n}{2}-1}\}$.

Step 2: Add the same color γ to both $\varphi(v_1)$ and $\varphi(v_2)$.

Step 3: For arcs including v_1, and for each $2 \leq i \leq \dfrac{n}{2}$, we perform the following procedure:

- If $(v_1, v_{2i-1}) \in A$ and $(v_1, v_{2i}) \in A$, then we copy a color from $\varphi(v_1)$ (say, α_{i-1}) to both $\varphi(v_{2i-1})$ and $\varphi(v_{2i})$.
- If $(v_1, v_{2i-1}) \in A$ and $(v_1, v_{2i}) \notin A$, then we copy a color from $\varphi(v_1)$ (say, α_{i-1}) to $\varphi(v_{2i-1})$.
- If $(v_1, v_{2i-1}) \notin A$ and $(v_1, v_{2i}) \in A$, then we copy a color from $\varphi(v_1)$ (say, α_{i-1}) to $\varphi(v_{2i})$.
- If $(v_1, v_{2i-1}) \notin A$ and $(v_1, v_{2i}) \notin A$, then we do nothing.

Step 4: For arcs including v_2, and for each $2 \leq i \leq \dfrac{n}{2}$, we perform the following procedure:

- If $(v_2, v_{2i-1}) \in A$ and $(v_2, v_{2i}) \in A$, then we copy a color from $\varphi(v_2)$ (say, β_{i-1}) to both $\varphi(v_{2i-1})$ and $\varphi(v_{2i})$.
- If $(v_2, v_{2i-1}) \in A$ and $(v_2, v_{2i}) \notin A$, then we copy a color from $\varphi(v_2)$ (say, β_{i-1}) to $\varphi(v_{2i-1})$.
- If $(v_2, v_{2i-1}) \notin A$ and $(v_2, v_{2i}) \in A$, then we copy a color from $\varphi(v_2)$ (say, β_{i-1}) to $\varphi(v_{2i})$.
- If $(v_2, v_{2i-1}) \notin A$ and $(v_2, v_{2i}) \notin A$, then we do nothing.

Next, assume that the DAG representation φ is as constructed above.

Step 5: For each $2 \leq i \leq \dfrac{n}{2}$, we add colors to both $\varphi(v_{2i-1})$ and $\varphi(v_{2i})$ so that the new representation φ satisfies

$$|\varphi(v_j)| - |\varphi'(v_j)| = 3.$$

In the process, we reuse colors to minimize the number of newly added colors. Since the procedures in Step 3 and Step 4 increase the color set of each vertex by at most 2, one may need to add as many as 3 new colors to a vertex representation (Note that we actually only need the difference to be 2, but for consistency with respect to Case 2 we set the value to 3). As an example, assume that we added $j \in \{0, 1, 2\}$ colors to $\varphi(v_{2i-1})$ and $k \in \{0, 1, 2\}$ colors to $\varphi(v_{2i})$ in Step 3 and Step 4. Then, we need to add $\max\{3-j, 3-k\}$ colors to obtain the desired representation, which for $j = 0$ or $k = 0$ results in 3 new colors. This is repeated for each pair, with at most 3 distinct added colors.

Claim 5.2.5. The representation φ includes at most $\dfrac{5}{2}n - 4$ new colors.

Proof. We used

$$\frac{n}{2} - 1 + \frac{n}{2} - 1 + 1 = n - 1$$

colors in Step 1 and Step 2. We used at most $3 \cdot (\dfrac{n}{2} - 1)$ in Step 5. Therefore, we used at most

$$n - 1 + \frac{3}{2}n - 3 = \frac{5}{2}n - 4$$

new colors in total. □

Claim 5.2.6. The color assignments φ constitute a valid representation satisfying conditions **(a)**, **(b)**, and **(c)**.

Proof. **(i):** For a pair of vertices (u, w) such that $u \in V - \{v_1, v_2\}$ and $w \in V - \{v_1, v_2\}$, we consider the following cases

1) If $(u, w) \in A$, then since φ' constituted a valid representation, we have that a) the intersection condition holds for φ because the two vertices still have representations with a color in common, and b) the size condition holds since we added three colors to both the color sets of u and w.

2) If $(u, w) \notin A$, and if u, w belong to different pairs, then since φ' is a valid representation and we added distinct colors to different pairs of vertices in Step 5, φ is a valid representation. This claim holds since if the vertices u and w have no color in common in φ', then they still have no color in common after different colors are added in Step 5. Furthermore, if the representation sets of the vertices had the same size before we added three colors to each color set, the sizes will remain the same. If u, w belong to the same pair, their color set sizes were the same in φ' and they stay the same after colors are added in Step 5. Hence, φ is still valid.

Similarly, for a pair of vertices (u, w) such that $u \in \{v_1, v_2\}$ and $w \in V - \{v_1, v_2\}$, we consider the following cases.

1) If $(u, w) \in A$, then the intersection condition holds for φ because we added a common color to the color sets of u and w in Step 3 or Step 4. Furthermore, the size condition holds since

$$|\varphi(w)| = |\varphi'(w)| + 3 \geq \frac{n}{2} - 1 + 3 > \frac{n}{2} = |\varphi(u)|.$$

Therefore, φ is a valid representation.

2) If $(u, w) \notin A$, then φ is valid since we did not add any common color to the color sets of the two vertices, and the set $\varphi'(u)$ was obtained by augmenting it with distinct colors.

Recall that under Case 1, $(v_1, v_2) \notin A$ and $|\varphi(v_1)| = |\varphi(v_2)|$. Hence, φ is a valid representation.

In addition, we have

(a): $|\varphi(v_1)| = |\varphi(v_2)| = \frac{n}{2}$ and $|\varphi(v_i)| \geq \frac{n}{2} - 1 + 3 \geq \frac{n}{2} + 1$, for $3 \leq i \leq n$.

(b): For each pair (v_{2i-1}, v_{2i}), if $(v_{2i-1}, v_{2i}) \in A$, then $|\varphi'(v_{2i-1})| = |\varphi'(v_{2i})| - 1$. Thus,

$$|\varphi(v_{2i-1})| = |\varphi'(v_{2i-1})| + 3 = |\varphi'(v_{2i})| - 1 + 3 = |\varphi(v_{2i})| - 1.$$

If $(v_{2i-1}, v_{2i}) \notin A$, where $2 \leq i \leq \frac{n}{2}$, then $|\varphi'(v_{2i-1})| = |\varphi'(v_{2i})|$. Thus,

$$|\varphi(v_{2i-1})| = |\varphi'(v_{2i-1})| + 3 = |\varphi'(v_{2i})| + 3 = |\varphi(v_{2i})|.$$

These properties also hold for $i = 1$, as previously established.

(c): By Claim 5.2.5, we used at most $\frac{5}{2}n - 4$ new colors. □

Case 2: $(v_1, v_2) \in A$.

Step 1: This step follows along the same lines as Step 1 of Case 1.

Step 2: Add a common color γ to both $\varphi(v_1)$ and $\varphi(v_2)$ to satisfy the intersection constraint, and add a new color δ to $\varphi(v_2)$ to satisfy the size constraint.

Step 3: This step follows along the same lines as Step 3 of Case 1.

Step 4: This step follows along the same lines as Step 4 of Case 1.

Step 5: This step follows along the same lines as Step 4 of Case 1.

Using the same counting arguments as before, it can be shown that the above steps introduce $\frac{5}{2}n - 3$ new colors (see the claim below).

Claim 5.2.7. We used at most $2.5n - 3$ new colors.

Claim 5.2.8. One can remove (save) one color from the given representation.

Proof. **Case 1:** $(v_2, v_3) \in A$.

Case 1.1: $(v_2, v_4) \in A$. Then $\beta_1 \in \varphi(v_3) \cap \varphi(v_4)$ and we can save one color for the pair (v_3, v_4) in Step 5 as only two colors suffice.

Case 1.2: $(v_2, v_4) \notin A$.

Case 1.2.1: $(v_1, v_3) \in A$. If $(v_1, v_4) \in A$, then $\alpha_1 \in \varphi(v_3) \cap \varphi(v_4)$ and we can save one color introduced in Step 5. If $(v_1, v_4) \notin A$, then $\beta_1 \in \varphi(v_3)$ and $\alpha_1 \in \varphi(v_3)$. We replace $\beta_1 \in \varphi(v_3)$ by δ and replace $\beta_1 \in \varphi(v_2)$ by α_1 and remove β_1. This saves one color.

Case 1.2.2: $(v_1, v_4) \in A$. Since $\beta_1 \in \varphi(v_3)$ and $\alpha_1 \in \varphi(v_4)$, we can discard one color used in Step 5.

Case 1.2.3: $(v_1, v_3) \notin A$ and $(v_1, v_4) \notin A$. Then α_1 is unused and we can thus replace α_1 in $\varphi(v_1)$ by δ to save one color.

Case 2: $(v_2, v_3) \notin A$.

Case 2.1: $(v_2, v_4) \in A$. Then $\beta_1 \in \varphi(v_4)$. If $(v_1, v_3) \in A$, then $\alpha_1 \in \varphi(v_3)$ and we can save a color in Step 5. Thus, we may assume that $(v_1, v_3) \notin A$. In this case, if $(v_1, v_4) \in A$, then $\alpha_1 \in \varphi(v_4)$ and we replace $\alpha_1 \in \varphi(v_4)$ by a color we used in Step 5 for v_3 (recall that in Step 5, we added three new colors to $\varphi(v_3)$ and only reused one of them in $\varphi(v_4)$; hence, there are two colors remaining). In addition, we replace $\alpha_1 \in \varphi(v_1)$ by β_1 to save one color. Thus, we may assume $(v_1, v_4) \notin A$. Then, α_1 is not used in the second pair and we may replace $\alpha_1 \in \varphi(v_1)$ by δ to save one color.

Case 2.2: $(v_2, v_4) \notin A$.

Case 2.2.1: If $(v_1, v_3) \in A$ and $(v_1, v_4) \in A$, then $\alpha_1 \in \varphi(v_3) \cap \varphi(v_4)$ and we saved a color in Step 5.

Case 2.2.2: If $(v_1, v_3) \notin A$ and $(v_1, v_4) \notin A$, then we may replace $\beta_1 \in \varphi(v_2)$ by α_1 to save one color.

Case 2.2.3: If $(v_1, v_3) \in A$ and $(v_1, v_4) \notin A$ or $(v_1, v_3) \notin A$ and $(v_1, v_4) \in A$, then we modify Step 5 by requiring that the color sets be augmented by two rather than three colors. This allows us to save at least one color. □

Claim 5.2.9. The representation φ is valid and it satisfies conditions **(a)**, **(b)**, and **(c)**.

Proof. We separately consider two cases.

- For Case 2.2.3,

For a pair of vertices (u, w) such that $u \in V - \{v_1, v_2\}$ and $w \in V - \{v_1, v_2\}$, we consider the following cases.

1) If $(u, w) \in A$, then since φ' constituted a valid representation we have that a) the intersection condition holds for φ because the two vertices still have a representation with a color in common, and b) the size condition holds since we added two colors to both the color set of u and w.

2) If $(u, w) \notin A$, and if u, w belong to different pairs, then since φ' is a valid representation and we added distinct colors to different pairs in Step 5, φ is a valid representation. This claim holds since if the vertices u and w have no color in common in φ', then they still have no color in common after different colors are added in Step 5. Furthermore, if the color set representations of two vertices had the same size, then since we added two colors to both color sets, the color sets of the vertices will still have the same size. If u, w belong to the same pair, then their color size were the same in φ' and remain the same after colors are added in Step 5. Hence, φ is a valid representation.

Similarly, for a pair of vertices (u, w) such that $u \in \{v_1, v_2\}$ and $w \in V - \{v_1, v_2, v_3, v_4\}$, we consider the following cases.

1) If $(u, w) \in A$, then a) the intersection condition holds for φ because we added one common color in Step 3 or Step 4, and b) the size condition holds since

$$|\varphi(w)| = |\varphi'(w)| + 2 \geq \frac{n}{2} + 2 > \frac{n}{2} + 1 \geq |\varphi(u)|.$$

Therefore, φ is a valid representation.

2) If $(u, w) \notin A$, then φ is valid since

$$\varphi(w) \geq \frac{n}{2} + 2 > \frac{n}{2} + 1 \geq \varphi(u)$$

and we did not add a common color for the two vertices, and $\varphi'(u)$ was obtained by adding distinct colors to $\varphi(u)$.

For (v_1, v_3), when $(v_1, v_3) \in A$ we added α_1 to $\varphi(v_3)$ so that

$$|\varphi(v_3)| = \frac{n}{2} + 1 > \frac{n}{2} = |\varphi(v_1)|.$$

When $(v_1, v_3) \notin A$ we added distinct colors to $\varphi(v_1)$ and $\varphi(v_3)$. Thus, φ is valid.

For (v_1, v_4), when $(v_1, v_4) \in A$ we added α_1 to $\varphi(v_4)$ so that

$$|\varphi(v_4)| = \frac{n}{2} + 1 > \frac{n}{2} = |\varphi(v_1)|.$$

When $v_1 v_4 \notin A$ we added distinct colors to $\varphi(v_1)$ and $\varphi(v_4)$. Thus, φ is valid.

For (v_2, v_3), we added distinct colors to $\varphi(v_2)$ and $\varphi(v_3)$. Thus, φ is valid.

For (v_2, v_4), we added distinct colors to $\varphi(v_2)$ and $\varphi(v_4)$. Thus, φ is valid.

For (v_1, v_2), since $(v_1, v_2) \in A$, $\gamma \in \varphi(v_1) \cap \varphi(v_2)$, and $|\varphi(v_1)| = |\varphi(v_2)| - 1$ we have that φ is valid.

To verify that conditions **(a)**, **(b)** and **(c)** are satisfied, observe that:

(a): $|\varphi(v_1)| = |\varphi(v_2)| - 1 = \frac{n}{2}$ and $|\varphi(v_i)| \geq \frac{n}{2} + 1$ for $3 \leq i \leq n$.

(b): For each pair (v_{2i-1}, v_{2i}), if $(v_{2i-1}, v_{2i}) \in A$ then $|\varphi'(v_{2i-1})| = |\varphi'(v_{2i})| - 1$. Thus,

$$|\varphi(v_{2i-1})| = |\varphi'(v_{2i-1})| + 2 = |\varphi'(v_{2i})| - 1 + 2 = |\varphi(v_{2i})| - 1.$$

This claim is also true for $i = 1$, which we already showed.

If $(v_{2i-1}, v_{2i}) \notin A$, where $2 \leq i \leq \frac{n}{2}$, then $|\varphi'(v_{2i-1})| = |\varphi'(v_{2i})|$. Thus,

$$|\varphi(v_{2i-1})| = |\varphi'(v_{2i-1})| + 2 = |\varphi'(v_{2i})| + 2 = |\varphi(v_{2i})|.$$

(c): By Claim 5.2.7 and Claim 5.2.8, we used at most $2.5n - 4$ new colors.

• For the other cases,

For a pair of vertices (u, w) such that $u \in V - \{v_1, v_2\}$ and $w \in V - \{v_1, v_2\}$, we consider the following cases.

If $(u, w) \in A$ then since φ' was valid 1) the intersection condition still holds for φ because they still have color in common and 2) the size condition still hold since we added three colors to each of the color set of u and w.

If $(u, w) \notin A$, and the two vertices are in different pairs then since φ' was valid and we added distinct colors to different pairs in Step 5, we have that φ is valid because if u and v have no color in common in φ' then they still have no color in common after we added different colors in Step 5; if they had the same size in φ' then since we added three colors to each color set their sizes remain the same. If the two vertices are in the same pair then their color size was the same in φ' and it stays the same after adding colors in Step 5. Hence, φ is still valid.

For a pair of vertices (u, w) such that $u \in \{v_1, v_2\}$ and $w \in V - \{v_1, v_2\}$, we consider the following cases.

If $(u, w) \in A$ then 1) the intersection condition holds for φ because we added a common color in Step 3 or Step 4 to the color sets of u and w and 2) the size condition hold since

$$|\varphi(w)| = |\varphi'(w)| + 3 \geq \frac{n}{2} - 1 + 3 > \frac{n}{2} + 1 \geq |\varphi(u)|.$$

Therefore, φ is valid.

If $(u, w) \notin A$ then φ is valid since we did not add any common color for them and u uses distinct colors from φ'.

For (v_1, v_2), since $(v_1, v_2) \in A$, $\gamma \in \varphi(v_1) \cap \varphi(v_2)$, and $|\varphi(v_1)| = |\varphi(v_2)| - 1$ we have that φ is valid.

To verify that conditions **(a)**, **(b)** and **(c)** are satisfied, observe that:

(a): $|\varphi(v_1)| = |\varphi(v_2)| - 1 = \frac{n}{2}$ and $|\varphi(v_i)| \geq \frac{n}{2} - 1 + 3 \geq \frac{n}{2} + 1$ for $3 \leq i \leq n$.

(b): For each pair (v_{2i-1}, v_{2i}), if $(v_{2i-1}, v_{2i}) \in A$ then $|\varphi'(v_{2i-1})| = |\varphi'(v_{2i})| - 1$. Thus,

$$|\varphi(v_{2i-1})| = |\varphi'(v_{2i-1})| + 3 = |\varphi'(v_{2i})| - 1 + 3 = |\varphi(v_{2i})| - 1.$$

This claim is also true for $i = 1$, which we already showed.

If $(v_{2i-1}, v_{2i}) \notin A$, where $2 \leq i \leq \frac{n}{2}$, then $|\varphi'(v_{2i-1})| = |\varphi'(v_{2i})|$. Thus,

$$|\varphi(v_{2i-1})| = |\varphi'(v_{2i-1})| + 3 = |\varphi'(v_{2i})| + 3 = |\varphi(v_{2i})|.$$

(c): By Claim 5.2.7 and Claim 5.2.8, we used at most $2.5n - 4$ new colors. This proves the claim. □

This completes the proof of the theorem. □

5.3 Extremal DIN Digraphs and Lower Bounds

The derivations in the previous section proved that for any DAG D on n vertices, one has

$$DIN(D) \leq \frac{5n^2}{8} - \frac{3n}{4} + 1. \tag{5.4}$$

In comparison, the intersection number of any graph on n vertices is upper bounded by $\frac{n^2}{4}$ (ERDöS; GOODMAN; PóSA, 1966). Furthermore, the existence of undirected graphs that meet the bound $\frac{n^2}{4}$ can be established by observing that the intersection number of a graph is equivalent to its edge-clique cover number and by invoking Mantel's theorem (MANTEL, 1907) which asserts that any triangle-free graph on n vertices can have at most $\frac{n^2}{4}$ edges. The extremal graphs with respect to the intersection number are the well-known Turan graphs $T(n, 2)$ (TURÁN, 1954).

Consequently, the following question is of interest in the context of directed intersection representations: Do there exist DAGs that meet the upper bound in (5.4) and which DIN values are actually achievable? To this end, we introduce the notion of *DIN-extremal* DAGs: A DAG on n vertices is said to be DIN-extremal if it has the largest DIN among all DAGs with the same number of vertices.

Directed path DAGs, e.g., directed acyclic graphs $D(V, A)$ with $V = \{1, 2, \ldots, n\}$ and $A = \{(1, 2), (2, 3), (3, 4), \ldots, (n-1, n)\}$ have DINs that scale as $\frac{n^2}{4}$. The following result formalizes this observation.

Proposition 5.3.1. *Let $D(V, A)$ be a directed path on n vertices. If n is even, then $DIN(D) = \frac{n^2+2n}{4}$; if n is odd, then $DIN(D) = \frac{n^2+2n+1}{4}$.*

The proof of the result is straightforward and hence omitted.

Figure 4 provides examples of DIN-extremal DAGs for $n \leq 7$ vertices. These graphs were obtained by combining computer simulations and proof techniques used in establishing the upper bound of (5.4). Direct verification for large n through exhaustive search is prohibitively complex, as the number of connected/disconnected DAGs with n vertices follows a "fast growing" recurrence (ROBINSON, 1977). For example, even for $n = 6$, there exist 5984 different unlabeled DAGs. Note that all listed extremal DAGs are Hamiltonian, e.g., they contain a directed path visiting each of the n vertices exactly once. As such, the digraphs have a unique topological order induced by the directed path, and for the decomposition described on page 5 one has $|V_i| = 1$ for all $i \in [n]$. Note that the bound in (5.4) for $n = 2, 3, 4, 5, 6, 7$ equals $2, 4, 8, 12, 19, 26$, respectively. Hence, the upper bound in (4) is loose for $n \geq 6$.

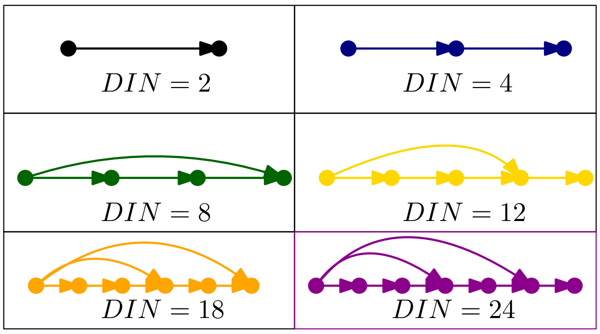


Figure 4 – Examples of DIN-extremal graphs for $n \leq 7$.

For all $n \leq 7$ the extremal digraphs are what we refer to as *source arc-paths*, illustrated in Figure 5 a),b). A source arc-path on n vertices has the following arc set

$$A = \{(v_1, v_{2k}) : k \in [\lfloor n/2 \rfloor]\} \cup \{(v_k, v_{k+1}) : k \in [n-1]\}.$$

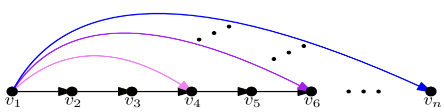
It is straightforward to prove the following result.

Proposition 5.3.2. *The DIN of a source arc-path on n vertices is equal to $\lfloor \frac{n^2}{2} \rfloor = \lfloor \frac{4n^2}{8} \rfloor$. Hence, the DIN of source arc-paths is by $\frac{n^2}{8}$ smaller than the leading term of the upper bound* (5.4)*.*

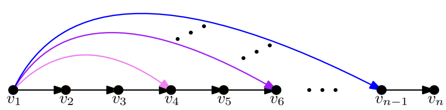
Proof. A *directed triangle* in a digraph $D = (V, A)$ is a collection of three vertices $\{v_i, v_j, v_k\}$ such that $(v_i, v_j) \in A$, $(v_j, v_k) \in A$, and $(v_i, v_k) \in A$. Since a source arc-path avoids directed triangles and every vertex has a color set of different size than another (due to the presence of the directed Hamiltonian path), every color may be used at most twice. We need $\frac{n}{2}$ colors for $\varphi(v_1)$ to represent the arcs $v_1 v_{2i}$, where $1 \leq i \leq \frac{n}{2}$. Since the size of the color sets φ increases along the directed path, vertex v_j in the natural ordering has $\varphi(v_j) \geq \frac{n}{2} + j - 1$. Furthermore, $(v_{2i}, v_{2j}) \notin A$ for a source arc-path, for all $1 \leq i < j \leq \frac{n}{2}$. Thus, $\varphi(v_{2i}) \cap \varphi(v_{2j}) = \emptyset$, $1 \leq i < j \leq \frac{n}{2}$. This implies the number of colors needed is

$$\geq \frac{n}{2} + 1 + \frac{n}{2} + 3 + \cdots + \frac{n}{2} + n - 1 = \frac{n}{2} \cdot \frac{n}{2} + \frac{(1 + n - 1)(\frac{n}{2})}{2} = \frac{n^2}{2}.$$

To show that the above lower bound is met, we exhibit the following representation φ with $\frac{n}{2}$ colors:



(a) Source arc-path, n even.



(b) Source arc-path, n odd.

1) $\varphi(v_1) = \{c_1, \ldots, c_{\frac{n}{2}}\}$, $\varphi(v_2) = \{c_1, f_1, g_{1,1}, \ldots, g_{\frac{n}{2}-1,1}\}$.
2) For $2 \leq i \leq \dfrac{n}{2} - 1$,

$$\varphi(v_{2i}) = \{c_i, d_i, f_i, g_{1,i}, \ldots, g_{\frac{n}{2}+2i-4,i}\},$$

$$\varphi(v_n) = \{c_{\frac{n}{2}}, d_{\frac{n}{2}}, g_{1,\frac{n}{2}}, \ldots, g_{\frac{n}{2}+n-3,\frac{n}{2}}\}.$$

3) For $2 \leq i \leq \dfrac{n}{2} - 1$,

$$\varphi(v_{2i-1}) = \{d_i, f_{i-1}, g_{1,i}, \ldots, g_{\frac{n}{2}+2i-4,i}\}.$$

$$\varphi(v_{n-1}) = \{f_{\frac{n}{2}-1}, d_{\frac{n}{2}}, g_{1,\frac{n}{2}}, \ldots, g_{\frac{n}{2}+n-4,\frac{n}{2}}\}.$$

$\square$

For $n \geq 8$, there exist DAGs with DINs that exceed those of source arc-paths which are obtained by adding carefully selected additional arcs. For even integers n, the DIN of such graphs equals

$$\frac{n^2}{2} + \lfloor \frac{n^2}{16} - \frac{n}{4} + \frac{1}{4} \rfloor - 1.$$

A digraph with the above DIN has a vertex set $V = \{v_1, \ldots, v_n\}$ and arcs constructed as follows:

Step 1: Initialize the arc set as $A = \emptyset$.

Step 2: Add to A arcs of a source-arc-path, i.e.,

$$A = A \cup \{(v_1, v_{2i}) : i \in [\frac{n}{2}]\} \cup \{(v_j, v_{j+1}) : j \in [n-1]\}.$$

Step 3: Add arcs with tails and heads in the set $\{v_3, v_5, \ldots, v_{n-1}\}$ according to the following rules:

Step 3.1: If $\dfrac{n-2}{2}$ is even, then let

$$X = \{v_3, v_5, \ldots, v_{\frac{n}{2}}\} \text{ and } Y = \{v_{\frac{n}{2}+2}, \ldots, v_{n-1}\}.$$

Add all arcs between X and Y except for $(v_{\frac{n}{2}}, v_{\frac{n}{2}+2})$.

Step 3.2: If $\dfrac{n-2}{2}$ is odd, then let

$$X = \{v_3, v_5, \ldots, v_{\frac{n}{2}+1}\} \text{ and } Y = \{v_{\frac{n}{2}+3}, \ldots, v_{n-1}\}.$$

Add all arcs between X and Y except for $(v_{\frac{n}{2}+1}, v_{\frac{n}{2}+3})$.

The above described digraphs have no directed triangles and their number of arcs equals

$$\lfloor \frac{(\frac{n}{2}-1)^2}{4} \rfloor - 1 = \lfloor \frac{n^2}{16} - \frac{n}{4} + \frac{1}{4} \rfloor - 1.$$

We start with the following lower bound on the DIN number of the augmented source-arc-path graphs.

Proposition 5.3.3. *The DIN of the above family of graphs is at least*

$$\frac{n^2}{2} + \lfloor \frac{n^2}{16} - \frac{n}{4} + \frac{1}{4} \rfloor - 1.$$

Proof. Due to the presence of the arc of a source-arc-path, v_1 requires at least $\frac{n}{2}$ colors. Furthermore, since the graph is Hamiltonian, the size of the color sets increases along the path. Based on the previous two observations, one can see that v_i requires at least $\frac{n}{2} + i - 1$ colors for all $i \in [n]$.

Since there are no arcs in the digraph induced by the vertex set $\{v_2, v_4, \ldots, v_n\}$ with even labels, the color sets of these vertices have to be mutually disjoint. Thus, the number of colors needed to color vertices with even indices is at least

$$\frac{n}{2} + 1 + \frac{n}{2} + 3 + \ldots + \frac{n}{2} + n - 1 = \frac{n^2}{2}.$$

Since the digraphs avoid directed triangles and every pair of vertices has a different color set sizes, we require one additional color to represent each of the arcs added in Step 3. Due to the absence of directed triangle, we need at least $\lfloor \frac{n^2}{16} - \frac{n}{4} + \frac{1}{4} \rfloor - 1$ colors. Furthermore, the color sets used for the two previously described vertex sets are disjoint. Thus, the number of colors required is at least

$$\frac{n^2}{2} + \lfloor \frac{n^2}{16} - \frac{n}{4} + \frac{1}{4} \rfloor - 1.$$

□

To show that the above number of colors suffices to represent the digraphs under consideration, we provide next a representation φ using $\frac{n^2}{2} + \lfloor \frac{n^2}{16} - \frac{n}{4} + \frac{1}{4} \rfloor - 1$ colors.

We start by exhibiting a representation φ' of the source-arc-path that uses $\frac{n^2}{2}$ colors and then change the color assignments accordingly:

1) Set $\varphi(v_1) = \{c_1, \ldots, c_{\frac{n}{2}}\}$ and $\varphi(v_2) = \{c_1, f_1, g_{1,1}, \ldots, g_{\frac{n}{2}-1,1}\}$.

2) For $2 \le i \le \frac{n}{2} - 1$, set

$$\varphi(v_{2i}) = \{c_i, d_i, f_i, g_{1,i}, \ldots, g_{\frac{n}{2}+2i-4,i}\},$$

and

$$\varphi(v_n) = \{c_{\frac{n}{2}}, d_{\frac{n}{2}}, g_{1,\frac{n}{2}}, \ldots, g_{\frac{n}{2}+n-3,\frac{n}{2}}\}.$$

3) For $2 \leq i \leq \frac{n}{2} - 1$, set

$$\varphi(v_{2i-1}) = \{d_i, f_{i-1}, g_{1,i}, \ldots, g_{\frac{n}{2}+2i-4,i}\},$$

and

$$\varphi(v_{n-1}) = \{f_{\frac{n}{2}-1}, d_{\frac{n}{2}}, g_{1,\frac{n}{2}}, \ldots, g_{\frac{n}{2}+n-4,\frac{n}{2}}\}.$$

Let $m := \lfloor \frac{n^2}{16} - \frac{n}{4} + \frac{1}{4} \rfloor - 1$.

1') Set $\Gamma_{2i-1} = \{g_{1,i}, \ldots, g_{\frac{n}{2}+2i-4,i}\}$.

2') Order the m arcs in the graph induced by $\{v_3, v_5, \ldots, v_{n-1}\}$ in an arbitrary fashion, say $\{e_1, \ldots, e_m\}$. Set a counter variable to $k = 1$.

3') For $e_k = (v_{2i-1}, v_{2j-1})$, assign a previously unused color h_k to both $\varphi(v_{2i-1})$ and $\varphi(v_{2j-1})$. Pick one color g' from Γ_{2i-1} and a color g'' from Γ_{2j-1} not previously used in the procedure. Set

$$\varphi(v_{2i-1}) = \varphi(v_{2i-1}) \cup h_k - g', \qquad \text{and} \qquad \Gamma_{2i-1} = \Gamma_{2i-1} - g',$$

$$\varphi(v_{2j-1}) = \varphi(v_{2j-1}) \cup h_k - g'', \qquad \text{and} \qquad \Gamma_{2j-1} = \Gamma_{2j-1} - g''.$$

Let $k = k + 1$. If $k \leq m$, go to Step 3'), otherwise stop.

4') Since each v_{2i-1} has degree at most $\frac{n}{4}$ on the digraph induced by $\{v_3, \ldots, v_{n-1}\}$ and at step $k = 1$ we had $|\Gamma_{2i-1}| = \frac{n}{2} + 2i - 4$, we do not run out of colors to replace. This follows since when we choose g' from Γ_{2i-1} we always have $\geq \frac{n}{2} + 2i - 4 - \frac{n}{4}$ colors available.

5') Since g', g'' were used twice in φ' and deleted only once in the processing steps (and thus remain in the union of the colors), each iteration of the procedure in 3) introduces exactly one new color (e.g., h_k) to φ. Therefore, the number of colors used is

$$\frac{n^2}{2} + m = \frac{n^2}{2} + \lfloor \frac{n^2}{16} - \frac{n}{4} + \frac{1}{4} \rfloor - 1.$$

This completes the construction of digraphs on n vertices with DIN values $\frac{n^2}{2} + \lfloor \frac{n^2}{16} - \frac{n}{4} + \frac{1}{4} \rfloor - 1$.

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Appendix - Index