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On blow-up loci of instantons in higher dimensions

Sobre lugares de blow-up de instantons em dimensões superiores

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Resumo

Estudamos aspectos analíticos fundamentais de conexões Yang-Mills em dimensões superiores ou iguais a quatro, seguindo o célebre artigo de G. Tian nos *Annals of Mathematics* (2000). O objetivo central é fornecer uma apresentação auto-suficiente à análise fundamental de Tian sobre o fenômeno de não-compacidade dos espaços de moduli de conexões Yang-Mills com energia L^2 uniformemente limitada. Isto culmina em uma relação notável entre teoria de calibres e geometria calibrada: sob hipóteses adequadas, verifica-se que os lugares de 'blow-up' de sequências de conexões Yang-Mills definem ciclos calibrados.

Palavras-chave: Yang-Mills, Teoria de; Geometria calibrada; Grupos de holonomia; Conexões (Matemática); Blow-up locus.

Abstract

We study fundamental analytical aspects of Yang-Mills connections in dimensions higher than or equal to four, following the famous work by G. Tian in the *Annals of Mathematics* (2000). The main objective is to provide a self-contained introduction to the fundamental analysis of Tian on the noncompactness phenomenon of moduli spaces of Yang-Mills connections with uniformly bounded L^2-energy. This culminates in a remarkable relation between gauge theory and calibrated geometry: under suitable conditions, one verifies that the 'blow-up' sets of sequences of Yang-Mills connections define calibrated cycles.

Keywords: Yang-Mills theory; Calibrated geometry; Holonomy groups; Connections (Mathematics); Blow-up locus.

List of symbols

$\mathbb{N}$: natural numbers, with the convention that $0 \notin \mathbb{N}$.

$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

$\mathbb{Z}$: integers.

$\mathbb{R}$: real numbers.

$\mathbb{R}_+$: strictly positive real numbers.

$\mathbb{C}$: complex numbers.

$\mathbb{K}$: scalar field $= \mathbb{R}$ or $\mathbb{C}$.

G: compact Lie group.

$\mathfrak{g} := \mathrm{Lie}(G)$.

$\langle \cdot, \cdot \rangle_{\mathfrak{g}}$: Ad_G-invariant inner product on $\mathfrak{g}$.

M: smooth (connected) $n-$manifold (without boundary, unless otherwise stated).

$C^\infty(M, \mathbb{K})$: space of smooth functions $M \to \mathbb{K}$.

$C^\infty(M) := C^\infty(M, \mathbb{R})$.

$\Gamma(F)$: space of *smooth* sections of a smooth fiber bundle F.

$\Omega^k(M) := \Gamma(\Lambda^k T^* M)$: space of smooth $k-$forms on M.

g: Riemannian metric or a gauge transformation.

$\mathrm{d}V_g$: Riemannian volume $n-$form.

$* : \Lambda^k T^* M \to \Lambda^{n-k} T^* M$: Hodge star duality operator on $k-$forms.

μ_g: natural Radon measure associated to $\mathrm{d}V_g$.

d_g: Riemannian distance function on (M, g).

$B_r(p) \equiv B_r(p;g)$: *open* d_g-ball of radius $r > 0$ and center p.

$\overline{B}_r(p) \equiv \overline{B}_r(p;g)$: closed d_g-ball of radius $r > 0$ and center p.

$\mathrm{inj}_g(p)$: injectivity radius of (M,g) at p.

$\mathrm{inj}_g(M) \coloneqq \inf\{p \in M : \mathrm{inj}_g(p)\}$.

d: exterior derivative or trivial connection.

d^*: formal L^2-adjoint of the exterior derivative.

D^g: Levi-Civita connection.

R^g: Riemann curvature $(1,3)-$tensor.

Rm^g: Riemann curvature $(0,4)-$tensor.

Ric^g: Ricci transformation.

$\mathrm{div}^g X \coloneqq \mathrm{trace}(Z \mapsto D^g_Z X)$.

$\Delta_g \coloneqq \mathrm{dd}^* + \mathrm{d}^*\mathrm{d} : \Omega^k(M) \to \Omega^k(M)$: (positive definite) Hodge-de Rham Laplacian on $k-$forms.

$\Delta_g^- \coloneqq -\mathrm{d}^*\mathrm{d} : C^\infty(M) \to C^\infty(M)$: (negative definite) Laplace-Beltrami operator on functions.

Z^{2m}: (almost) complex manifold of real dimension $2m$.

$T^{1,0}Z$: holomorphic tangent bundle of Z.

$T^{0,1}Z$: anti-holomorphic tangent bundle of Z.

$\Lambda^{p,q}T^*Z_{\mathbb{C}} \coloneqq \Lambda^p\left(T^{1,0}Z\right)^* \otimes_{\mathbb{C}} \Lambda^q\left(T^{0,1}Z\right)^*$.

(Z,ω): Kähler manifold with Kähler form ω.

(Z,ω,Υ): Calabi-Yau manifold with Kähler form ω and holomorphic volume form Υ.

(Y^7,ϕ): G_2-manifold with G_2-structure $3-$form ϕ.

(X^8,Φ): Spin(7)$-$manifold with Spin(7)$-$structure $4-$form Φ.

E: smooth (complex or real) vector bundle (over M, Z, Y or X) with structure group G.

$\mathcal{T}^r_s(E) \coloneqq \Gamma\left(\left(\bigotimes^r E\right) \otimes \left(\bigotimes^s E^*\right)\right)$: space of smooth $(r,s)-$tensor fields on E.

$\Omega^k(M,E) \coloneqq \Gamma(\Lambda^k T^*M \otimes E)$.

$\Omega^{p,q}(Z,E) \coloneqq \Gamma(\Lambda^{p,q}T^*Z_{\mathbb{C}} \otimes E)$.

$\Omega^k(Z,E) = \bigoplus_{p+q=k} \Omega^{p,q}(Z,E)$.

∇: connection on a vector bundle.

F_∇: curvature of ∇.

d_∇: exterior covariant derivative induced by ∇.

d_∇^*: formal L^2-adjoint of d_∇.

$\Delta_\nabla \coloneqq \mathrm{d}_\nabla \mathrm{d}_\nabla^* + \mathrm{d}_\nabla^* \mathrm{d}_\nabla$: generalized Hodge-de Rham Laplacian.

$L^p(M,E)$: Lebesgue space of L^p-sections of $E \to M$.

$\|\cdot\|_p$: Lebesgue L^p-norm.

$W^{k,p}(M,E)$: Sobolev space of $W^{k,p}-$sections of $E \to M$.

$\|\cdot\|_{k,p}$: Sobolev $W^{k,p}-$norm.

$\mathfrak{g}_E$: adjoint bundle of a $G-$bundle E.

$\mathfrak{A}(E)$: space of smooth $G-$connections on E.

$\mathfrak{A}^{k,p}(E)$: Sobolev space of $G-$connections on E.

$\mathcal{G}(E)$: space of smooth gauge transformations of E.

$\mathcal{G}^{k,p}(E)$: Sobolev space of $W^{k,p}$ gauge transformations of E.

μ: measure on a topological space.

$\operatorname{spt}(\mu) \coloneqq X \setminus \bigcup\{U \subseteq X : U \text{ is open and } \mu(U) = 0\}$: support of a measure μ on X.

χ_A: indicator function of A.

$(f\mu)(E) \coloneqq \int_E f \mathrm{d}\mu$.

$\mu \lfloor A \coloneqq \chi_A \mu$.

$\mu_i \rightharpoonup \mu$: μ_i weakly* converges to μ as Radon measures.

$\mu \ll \nu$: the measure μ is absolutely continuous with respect to the measure ν.

$\mathscr{H}^s$: $s-$dimensional Hausdorff measure.

$\Theta^{*s}(\mu,x)$: upper $s-$dimensional density of μ at x.

${\Theta_*}^s(\mu, x)$: lower $s-$dimensional density of μ at x.

$\Theta^s(\mu, x)$: $s-$dimensional density of μ at x.

$\mathscr{D}^k(M)$: Fréchet space of smooth $k-$forms on M with compact support.

$\mathscr{D}_k(M) \coloneqq \left(\mathscr{D}^k(M)\right)'$: space of $k-$currents on M.

$T_i \rightharpoonup T$: T_i weakly* converges to T as $k-$currents.

$\partial T \coloneqq T \circ \mathrm{d}$: boundary of a current T.

$\|T\|$: total variational measure of the current T.

$\mathbf{M}(T)$: mass of the current T.

$\mathbf{M}_k(M)$: space of finite mass $k-$currents.

$\mathbf{M}_{k,\mathrm{loc}}(M)$: space of locally finite mass $k-$currents.

$\mathbf{N}_k(M)$: space of normal $k-$currents.

$\mathbf{N}_{k,\mathrm{loc}}(M)$: space of locally normal $k-$currents.

$\mathbf{R}_k(M)$: space of integer rectifiable $k-$currents.

$\mathbf{R}_{k,\mathrm{loc}}(M)$: space of locally integer rectifiable $k-$currents.

$\mathbf{I}_k(M)$: space of integral $k-$currents.

$\mathbf{I}_{k,\mathrm{loc}}(M)$: space of locally integral $k-$currents.

$\mathcal{Z}_k(M) \coloneqq \{T \in \mathbf{I}_k(M) : \partial T = 0\}$: $k-$cycles.

$\mathcal{B}_k(M) \coloneqq \{\partial S : S \in \mathbf{I}_{k+1}(M)\} \subseteq \mathcal{Z}_k(M)$: $k-$boundaries.

Contents

Introduction

The advent of the Yang-Mills theory in the mid-70s had a strong influence on the development of mathematical areas such as differential geometry and topology over the last quarter of the twentieth century [Don05]. In particular, outstanding results on topology of $4-$manifolds derive from the study of moduli spaces of so-called anti-self-dual (ASD) instantons, special first order solutions of the Yang-Mills equation in four dimensions.

Using analytical works of Taubes [Tau82] and Uhlenbeck [Uhl82b, Uhl82a], Donaldson [Don83] was able to show that certain intersection forms could not be realized by compact, simply-connected smooth $4-$manifolds. One year earlier, Freedman [Fre82] had classified all compact, simply-connected *topological* $4-$manifolds, so that Donaldson's result automatically gave several examples of previously unknown nonsmoothable $4-$manifolds. Later, Taubes [Tau87] proved a generalization of Donaldson's theorem for oriented asymptotically periodic $4-$manifolds. This implied the existence of *uncountably* many *exotic* smooth structures on $\mathbb{R}^4$, i.e. the existence of an uncountable family of diffeomorphism classes of oriented $4-$manifolds homeomorphic to $\mathbb{R}^4$ (see also the earlier work of Gompf [Gom85]). Ultimately, Donaldson has extended his work significantly and produced deep new invariants distinguishing smooth $4-$manifolds with the same intersection form [Don90] (cf. [FU84, DK90]).

In the late-90s, the hugely influential work by Donaldson-Thomas [DT98] provided profound insights on the possibility of extending the familiar constructions in lower dimensional gauge theory to higher dimensional situations, in the presence of appropriate special geometric structures. A generalization of the notion of instanton to the higher dimensional setting was first considered by physicists in [CDFN83]; see also [BKS98, Car98]. While the classification of differentiable structures is much better understood in dimensions larger than four [Sco05], it is expected, optimistically, that the study of instantons in higher dimensions would allow one to define invariants of, for instance, manifolds of restricted holonomy such as Calabi-Yau, G_2 and $\mathrm{Spin}(7)$ manifolds.

In order to carry out the program outlined by Donaldson-Thomas rigorously, one would like to have higher-dimensional analogues of the compactness results of Uhlenbeck. In fact, a major issue in the study of Yang-Mills connections is the potential failure of compactness: previous results of Uhlenbeck [Uhl], Price [Pri83], and Nakajima [Nak88] imply that given any sequence $\{\nabla_i\}$ of Yang-Mills connections with uniformly bounded L^2-energy, $\|F_{\nabla_i}\|^2_{L^2} \leq \Lambda$, on a $G-$bundle $E \to M$, there exists a closed subset $S \subseteq M$, the 'blow-up set' of $\{\nabla_i\}$, with Hausdorff codimension at least 4, such that, up to gauge transformations, a subsequence of ∇_i converges to a Yang-Mills connection in $C^\infty_{\text{loc}}-$topology outside S.

In his celebrated paper [Tia00], Tian started this analytical programme by proving foundational regularity results concerning blow-up loci of general sequences of Yang-Mills connections, notably showing that these possess natural geometric structure. Supported primarily by Price's monotonicity formula [Pri83] and a curvature estimate due to Uhlenbeck and Nakajima [Nak88], Tian's analysis is similar to Lin's work on the analogous compactness problem for harmonic maps [Lin99].

Tian's paper begins introducing a very general type of anti-self-duality equation, to be studied for connections on $G-$bundles over an oriented Riemannian manifold (M, g) endowed with a closed $(n-4)-$form Ξ. A connection ∇ on a $G-$bundle E over M is called a $\Xi-$anti-self-dual instanton if

$$*(\Xi \wedge F_\nabla) = -F_\nabla.$$

This is a first order equation, depending on Ξ, which implies the second order Yang-Mills equation $\mathrm{d}^*_\nabla F_\nabla = 0$. Moreover, when M is a closed manifold, each $\Xi-$ASD instanton has an a priori L^2-energy bound, depending only on E, M and Ξ.

For suitable choices of Ξ these $\Xi-$ASD equations include the familiar ASD equations in 4−dimensions, the Hermitian Yang-Mills equations, and the higher-dimensional equations of Donaldson-Thomas: G_2- and $\mathrm{Spin}(7)-$instantons (in particular, complex ASD instantons).

In this setting, Tian's major breakthrough was the discovery of a specific relationship between gauge theory and calibrated geometry: when Ξ is a calibration, under general conditions he proves that the blow-up set of a sequence of $\Xi-$ASD instantons defines a $\Xi-$*calibrated cycle* [Tia00, Theorem 4.3.2]. In particular, this implies that the blow-up set is volume-minimizing [HL82]. Furthermore, by known regularity results in geometric measure theory [Alm84], it follows that the blow-up set is the closure of a smooth $\Xi-$calibrated submanifold.

The present work aims to provide a comprehensive treatment of Tian's work, specifically Chapters 1 − 4 of [Tia00], culminating with its foundational result relating gauge theory and calibrated geometry. A compilation of the main results presented in this dissertation (concerning Tian's work) is as follows:

Theorem A (Uhlenbeck, Price, Nakajima, Tian)**.** *Let (M, g) be a connected, compact and oriented Riemannian $n-$manifold, with $n \geq 4$. Let E be a $G-$bundle over M, where G is a compact Lie group, and let $\{\nabla_i\}$ be a sequence of smooth Yang-Mills connections on E with $\mathcal{YM}(\nabla_i) := \|F_{\nabla_i}\|^2_{L^2}$ uniformly bounded. Then after passing to a subsequence the following holds:*

(i) There exist a closed subset $S \subseteq M$ such that $\mathscr{H}^{n-4}(S) < \infty$, a smooth Yang-Mills connection ∇ on $E|_{M\setminus S}$, and a sequence of gauge transformations $\{g_i\} \subseteq \mathcal{G}(E|_{M\setminus S})$, such that $g_i^* \nabla_i$ converges to ∇ in $C^\infty_{loc}-$topology on $M \setminus S$.

(ii) The current[1] $c_2(\nabla) \in \mathscr{D}_{n-4}(M)$ defined by

$$c_2(\nabla)(\varphi) := \frac{1}{8\pi^2} \int_M \varphi \wedge (\mathrm{tr}(F_\nabla \wedge F_\nabla) - \mathrm{tr}(F_\nabla) \wedge \mathrm{tr}(F_\nabla)), \quad \forall \varphi \in \mathscr{D}^{n-4}(M),$$

is closed.

(ii) There exist a constant $\varepsilon_0 > 0$, depending only on the geometry of (M, g), n and G, and an upper semi-continuous $\mathscr{H}^{n-4}-$integrable function $\Theta : S \to [\varepsilon_0, \infty[$ such that, as Radon measures,

$$\mu_i := |F_{\nabla_i}|^2 \mathrm{d}V_g \rightharpoonup \mu = |F_\nabla|^2 \mathrm{d}V_g + \Theta \mathscr{H}^{n-4} \lfloor S.$$

(iii) S decomposes as $S = \Gamma \cup \mathrm{sing}(\nabla)$ with

$$\Gamma := \mathrm{spt}(\Theta \mathscr{H}^{n-4} \lfloor S) \quad \text{and}$$

$$\mathrm{sing}(\nabla) := \{x \in M : \Theta^{*n-4}(\mu_\nabla, x) > 0\};$$

Γ is countably $\mathscr{H}^{n-4}-$rectifiable, i.e. at $\mathscr{H}^{n-4}-$a.e. $x \in \Gamma$, it has a well-defined tangent space $T_x\Gamma$, and $\mathrm{sing}(\nabla)$ is a $\mathscr{H}^{n-4}-$negligible closed set[2].

(iv) For $\mathscr{H}^{n-4}-$a.e. smooth point $x \in \Gamma$, i.e. for $\mathscr{H}^{n-4}-$a.e. $x \in \Gamma$ such that $T_x\Gamma$ is well-defined and $x \notin \mathrm{sing}(\nabla)$, there exists a non-flat connection $I(x)$ on $T_x\Gamma^\perp$ satisfying $\mathcal{YM}(I(x)) \leq \Theta(x)$ whose pull-back to T_xM is a non-flat Yang-Mills connection $B(x)$ realized as the limit of a blowing up of the sequence $\{\nabla_i\}$ around x.

Theorem B (Tian)**.** *Assume the situation of Theorem A. If, furthermore, $\{\nabla_i\}$ is a sequence of $\Xi-$anti-self-dual instantons[3], with respect to some closed $(n-4)-$form Ξ on M, then ∇ is a $\Xi-$anti-self-dual instanton on $M \setminus S$ and the following holds:*

---

[1] For definitions concerning currents, see Section A.6 of Appendix A.

[2] Tao-Tian [TT04] further shows that in case ∇ is *stationary*, e.g. when ∇ is a $\Xi-$anti-self-dual instanton (see Section 4.5), then ∇ extends, modulo gauge transformations, to a smooth Yang-Mills connection on a $G-$bundle $\tilde{E}$ over $M \setminus \mathrm{sing}(\nabla)$ which is isomorphic to E over $M \setminus S$.

[3] cf. Definition 2.3.1.

(i) For $\mathscr{H}^{n-4}-$a.e. smooth point $x \in \Gamma$, the $(n-4)-$form $\Xi_x := \Xi|_{T_xM}$ restricts to one of the volume forms induced by g on $T_x\Gamma$, and $I(x)$ is a non-trivial anti-self-dual instanton on $(T_x\Gamma^{\perp}, g|_{T_x\Gamma^{\perp}})$ with respect to the orientation given by $*\Xi_x|_{T_x\Gamma^{\perp}}$. Equivalently, $B(x)$ is a Ξ_x-anti-self-dual instanton on $(T_xM, g|_{T_xM})$.

(ii) Suppose $G \subseteq U(r)$. The current $c_2(\Gamma,\Theta) \in \mathscr{D}_{n-4}(M)$ defined by

$$c_2(\Gamma,\Theta)(\varphi) := \frac{1}{8\pi^2}\int_M \langle\varphi, \Xi|_\Gamma\rangle \Theta \mathrm{d}\left(\mathscr{H}^{n-4}\lfloor\Gamma\right), \quad \forall\varphi \in \mathscr{D}^{n-4}(M),$$

is a closed integral current. Moreover, the instanton charge density is conserved:

$$c_2(\nabla_i) \rightharpoonup c_2(\nabla) + c_2(\Gamma,\Theta). \tag{B1}$$

In particular, if Ξ is a calibration then $c_2(\Gamma,\Theta)$ is a $\Xi-$calibrated cycle.

Overview of the chapters. I begin Chapter 1 introducing some terminology on $G-$bundles, followed by a general discussion on connections, curvatures and some other related differential operators, including a classical Bochner-Weitzenböck formula [BLJ81] for the generalized Hodge-de Rham Laplacian on $\mathfrak{g}_E-$valued $2-$forms, and a review on Sobolev spaces of connections (Section 1.1). We also provide some material on holonomy groups and basic Chern-Weil theory (Sections 1.2 and 1.3). Next, we explain the variational formulation of the weak and strong Yang-Mills equation over Riemannian manifolds, comment some of its symmetries and give a brief discussion on gauge fixing (Section 1.4). We end the chapter with a brief recap on basic aspects of $4-$dimensional gauge theory, introducing (anti-)self-dual instantons and giving two well-known interpretations of this notion: one topological, via Chern-Weil theory, and one geometrical, in the context of complex geometry (Section 1.5).

In Chapter 2 we introduce the basic language of calibrated geometry (cf. [HL82]) and the notion of instanton in higher dimensions (cf. [CDFN83], [DT98], [Tia00] et al.). Since these notions naturally arise in the context of manifolds of special (or reduced) holonomy, we begin with a brief discussion on Riemannian holonomy groups, stating the so-called Berger's classification theorem and giving short descriptions of the special geometries associated to the groups $\mathrm{U}(m)$ (Kähler), $\mathrm{SU}(m)$ (Calabi-Yau), and the exceptional cases G_2 and $\mathrm{Spin}(7)$ (Section 2.1). Next, aside from a brief review on minimal submanifolds, we introduce calibrations and calibrated submanifolds, proving some of their basic properties and studying the classical examples on special holonomy manifolds (Section 2.2). Then, we present two known approaches for the generalization of the familiar $4-$dimensional notion of instanton, focusing on the approach employed in [Tia00], first explored by physicists [CDFN83, BKS98], based on the presence of a closed $(n-4)-$form Ξ on the base manifold M^n (Section 2.3). The second approach, originally introduced by Carrion in [Car98], is based on the presence of an $N(H)-$structure on M^n, for some closed Lie subgroup $H \subseteq \mathrm{SO}(n)$, where $N(H)$ denotes the normalizer of H in $\mathrm{SO}(n)$. These two points of view are shown to coincide in cases of interest, and various analogies between such instantons and calibrated submanifolds are pointed out.

Chapter 3 provides the analytical backbone of the main results to be developed in Chapter 4. Aside from a review on Uhlenbeck's compactness theorems [Uhl82b, Uhl82a] (Section 3.1), we study two core results in the analysis of Yang-Mills fields in higher dimensions: a monotonicity formula due to Price [Pri83] (Section 3.2), and a local pointwise estimate on Yang-Mills fields with sufficiently small normalized L^2-norm over small balls, due to Uhlenbeck and Nakajima [Uhl82b, Nak88] (Section 3.3). In particular, these results are used to show by standard methods that sequences of Yang-Mills connections with uniformly bounded L^2-energy are $C^\infty_{\mathrm{loc}}-$convergent away from a blow-up set of Hausdorff codimension at least 4, where the normalized L^2-energy of the sequence concentrates (Section 3.4). We then finish the chapter studying some properties of a class of singular Yang-Mills connections, called *admissible* Yang-Mills connections (cf. [Tia00, §2.3]), for which we are able to define the first two terms of the Chern character in the sense of currents (Section 3.5). At this stage, parts (i) and (ii) of Theorem A are already proved.

Finally, following closely Tian's work [Tia00], in Chapter 4 we study the structure of blow-up loci of sequences of Yang-Mills connections with uniformly bounded L^2-energy. The main results of this chapter corresponds to Theorem A (iii)$-$(iv) and Theorem B. Fixing such a sequence $\{\nabla_i\}$ with limit connection ∇, and blow-up set S, we start showing that S decomposes into two closed pieces, one involving energy loss (Γ) and one involving the formation of singularities ($\mathrm{sing}(\nabla)$), where the later is readily shown to be an $\mathscr{H}^{n-4}-$negligiable set (Section 4.1). Next, we show a first regularity result: Γ is (countably) $\mathscr{H}^{n-4}-$*rectifiable*, i.e. at $\mathscr{H}^{n-4}-$a.e. $x \in \Gamma$, the approximate $(n-4)-$dimensional tangent space $T_x\Gamma$ of Γ exists (Section 4.2). Then we move to the analysis of the behavior of ∇_i, for i sufficiently large, near a smooth point $x \in \Gamma$, i.e. a point $x \notin \mathrm{sing}(\nabla)$ at which $T_x\Gamma$ is well-defined. Using blow-up analysis techniques that goes back to Lin's work [Lin99], we show that, for $\mathscr{H}^{n-4}-$a.e. smooth point $x \in \Gamma$, we can find a non-flat connection $I(x)$ on $T_x\Gamma^\perp$ satisfying the energy inequality $\mathcal{YM}(I(x)) \leq \Theta(\mu, x)$ and whose pull-back to T_xM is the limit of a blowing up of the sequence $\{\nabla_i\}$ around x (Section 4.3).

Then we turn to the case in which $\{\nabla_i\}$ is a sequence of $\Xi-$anti-self-dual instantons (Section 4.4). We show that, at $\mathscr{H}^{n-4}-$a.e. $x \in \Gamma$, the approximate tangent space $T_x\Gamma$ is calibrated by Ξ and an ASD instanton 'bubbles off' transversely; $I(x)$ is a (non-flat) ASD instanton. Finally, we conclude the proof of Theorem B by showing that, for $G \subseteq \mathrm{U}(r)$, the natural $(n-4)-$current $c_2(\Gamma, \Theta)$ defined by the triple $(\Gamma, \Xi, \frac{1}{8\pi^2}\Theta)$ is a closed integral current satisfying (B1). We end the chapter introducing *stationary* admissible Yang-Mills connections and stating a Tian's theorem that the blow-up locus of a general sequence of Yang-Mills connections with uniformly bounded L^2-energy defines a minimal cycle if, and only if, the weak limit connection is stationary (Section 4.5).

Chapter 1

Geometry and gauge theory

We start Section 1.1 reviewing the basic terminology on $G-$bundles. Then we recall some important aspects of connections and curvatures on $G-$bundles and introduce important differential operators associated to connections on $G-$bundles over Riemannian manifolds. At the end, we include two special topics that will be needed in Chapter 3: a Bochner-Weitzenböck formula and Sobolev spaces of connections. Next, in Section 1.2, we review some topics concerning holonomy groups of connections on real vector bundles, including the so-called holonomy principle and the Ambrose-Singer theorem. As for the Section 1.3, we give a quick exposition on the basic Chern-Weil representation of characteristic classes. Then, in Section 1.4, we review some variational aspects of the Yang-Mills equation, deriving it as the Euler-Lagrange equation for the Yang-Mills functional. We finish the section with a brief discussion on gauge fixing. Finally, in Section 1.5, we recall the $4-$dimensional notion of (A)SD instantons, as special first order solutions of the corresponding Yang-Mills equation, and provide two well-known interpretations of this notion; one topological, via Chern-Weil theory, and one geometrical, in the context of complex geometry.

1.1 Connections and curvature

We will work with connections exclusively from the point of view of vector bundles. There are many standard references for the topics we review in this section. In particular, we recommend [DK90, §2.1] and [FU84, §2]. Perhaps only the last two topics, concerning a Bochner-Weitzenböck formula and Sobolev spaces of connections, deserve more specific references, which are pointed out in the text.

$G-$**bundles.** Let $\pi : E \to M$ be a $\mathbb{K}-$vector bundle of rank r and structure group $G \subseteq \mathrm{GL}(r, \mathbb{K})$; from now on, we will say simply that E is a $G-$*bundle*. This means that E admits a bundle atlas $\{(U_\alpha, \varphi_\alpha)\}$, of local trivializations

$$\varphi_\alpha = (\pi, \phi_\alpha) : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^r,$$

whose associated transition functions $\{g_{\alpha\beta}\}$ take values in G: for all α, β with $U_\alpha \cap U_\beta \neq \emptyset$, we can write

$$\begin{aligned} \varphi_\alpha \circ \varphi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{K}^r &\to (U_\alpha \cap U_\beta) \times \mathbb{K}^r \\ (x, v) &\mapsto (x, g_{\alpha\beta}(x)v), \end{aligned}$$

where $g_{\alpha\beta}(x) = \phi_\alpha \circ (\phi_\beta|_{E_x})^{-1} \in G$, for each $x \in U_{\alpha\beta} \coloneqq U_\alpha \cap U_\beta$. This type of atlas is also known as a $G-$*atlas* for E. A local trivialization

$$\varphi = (\pi, \phi) : \pi^{-1}(U) \to U \times \mathbb{K}^r$$

is compatible with such a $G-$atlas when $\phi \circ (\phi_\alpha|_{E_x})^{-1} \in G$, for any α with $U \cap U_\alpha \neq \emptyset$ and $x \in U \cap U_\alpha$; in this case, φ is called a $G-$*trivialization*.

Examples of structure group G that we will be interested include $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$. In fact, more generally, we will be interested in the groups $\mathrm{U}(r)$ and $\mathrm{SU}(r)$, when $\mathbb{K} = \mathbb{C}$, and $\mathrm{SO}(r)$, $r \geq 3$, when $\mathbb{K} = \mathbb{R}$. In any case, we will suppose G to be a *compact* Lie group. In particular, the Lie algebra $\mathfrak{g}$ of G admits some Ad_G-invariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$. In fact, in Yang-Mills theory it is common to take G to be a compact *semi-simple* Lie group, in which case we have a canonical choice of Ad_G-invariant inner product on $\mathfrak{g}$: minus the (negative definite) *Cartan-Killing form*[1] of $\mathfrak{g}$. However, because of our interest in the unitary groups $\mathrm{U}(r)$ (e.g. when working with Hermitian-Yang-Mills connections), in general we will not suppose semi-simplicity for G. Instead, since G is compact, we may suppose[2] $G \subseteq \mathrm{O}(r)$ or $\mathrm{U}(r)$, according to the respective cases $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and fix once and for all the Ad_G-invariant inner product on $\mathfrak{g}$ to be the one induced by the canonical trace inner product:

$$\langle X, Y \rangle_{\mathfrak{g}} \coloneqq -\mathrm{tr}(XY), \quad \forall X, Y \in \mathfrak{g}. \tag{1.1.1}$$

It is worth noting that minus the Cartan-Killing form of $\mathfrak{su}(r)$ (resp. $\mathfrak{so}(r)$) differs from the above choice of inner product by the constant factor $2r$ (resp. $r-2$).

[1]If $K_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denotes the Cartan-Killing form of $\mathfrak{g}$, i.e. the (symmetric) bilinear form on $\mathfrak{g}$ given by $K_{\mathfrak{g}}(X, Y) \coloneqq \mathrm{tr}\,(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$, for each $X, Y \in \mathfrak{g}$, then the compactness of G implies $K_{\mathfrak{g}}$ is a negative semi-definite bilinear form; in general, $K_{\mathfrak{g}}$ is non-degenerate if, and only if, $\mathfrak{g}$ is a semi-simple Lie algebra.

[2]Every continuous representation $\rho : G \to \mathrm{Aut}(V)$ of a compact Lie group G into a finite-dimensional $\mathbb{K}-$vector space V is *unitary*, i.e. the $G-$module V admits a $\rho(G)-$invariant inner product.

Remark 1.1.1. The condition of G being contained in $\mathrm{U}(r)$ (resp. $\mathrm{O}(r)$) implies that our bundle $E \to M$ is endowed with a Hermitian (resp. Euclidean) metric h, i.e. a smooth assignment of a Hermitian (resp. Euclidean) inner product h_x on E_x, for each $x \in M$. Indeed, given $x \in M$ choose U_α such that $x \in U_\alpha$ and define

$$h_x := (\phi_\alpha|_{E_x})^* \, h_0,$$

where h_0 is the canonical Hermitian (resp. Euclidean) inner product on $\mathbb{C}^r$ (resp. $\mathbb{R}^r$). To see this is well-defined, note that whenever $x \in U_\alpha \cap U_\beta$ we have $g_{\alpha\beta}(x) = \phi_\alpha \circ (\phi_\beta|_{E_x})^{-1} \in G \subseteq \mathrm{U}(r)$ (resp. $\mathrm{O}(r)$), so that

$$\begin{aligned} (\phi_\beta|_{E_x})^* \, h_0 &= \left(\phi_\beta|_{E_x} \circ (\phi_\alpha|_{E_x})^{-1} \circ \phi_\alpha|_{E_x}\right)^* h_0 \\ &= (g_{\beta\alpha}(x) \circ \phi_\alpha|_{E_x})^* \, h_0 \\ &= (\phi_\alpha|_{E_x})^* \, h_0. \end{aligned}$$

One may readly check that $h : x \mapsto h_x$ is a smooth assignment, i.e. for each pair of smooth local sections $s, t \in \Gamma(E|_U)$, where $U \subseteq M$ is any open subset, the map $h(s,t) : x \mapsto h_x(s,t)$ is a smooth $\mathbb{K}-$valued function on U. Moreover, since each $\phi_\alpha|_{E_x}$ is a $\mathbb{K}-$linear isomorphism, it is clear that each h_x is a Hermitian (resp. Euclidean) inner product on E_x.

Conversely, if we start with a complex (resp. real) vector bundle $E \to M$ endowed with a Hermitian (resp. Euclidean) metric h, then the usual Gram–Schmidt process ensures the existence of local orthonormal frames for E, which are just another way to speak of $\mathrm{U}(r)-$ (resp. $\mathrm{O}(r)-$) local trivializations for E. In particular, a $\mathrm{U}(r)-$bundle $E \to M$ is just a complex vector bundle of rank r over M endowed with a Hermitian metric h; such bundles are also known as *unitary* (or *Hermitian*) vector bundles.

By a similar reasoning, we see that an $\mathrm{SU}(r)-$bundle E, for example, is just a $\mathrm{U}(r)-$bundle endowed with a fixed trivialization τ on its top exterior power $\Lambda^r E^*$ (i.e., a smooth section $\tau \in \Gamma(\Lambda^r E^*)$, assigning for each $x \in M$ an orientation $0 \neq \tau(x) \in \Lambda^r E_x^*$ on the fiber E_x); a local $\mathrm{SU}(r)-$trivialization is just one for which the associated local frame is orthonormal and oriented. ♢

We denote by $\mathrm{Aut}_G(E)$ the **bundle of $G-$automorphisms** of E, i.e. $\mathrm{Aut}_G(E)$ is the bundle of groups over M whose fiber at a point $x \in M$ consists of all $g \in \mathrm{GL}(E_x)$ acting as $G-$isomorphisms on E_x, that is, all $g \in \mathrm{GL}(E_x)$ whose matrix representation with respect to some (therefore any) local $G-$trivialization of E lies in $G \subseteq \mathrm{GL}(r, \mathbb{K})$. The space of smooth sections of $\mathrm{Aut}_G(E)$ is denoted by $\mathcal{G}(E)$, and is called the group of **gauge transformations** of E. We note that $\mathcal{G}(E)$ is endowed with a natural group structure given by pointwise composition. Alternatively, $\mathcal{G}(E)$ is naturally identified (as a group) with the set of all $G-$bundle automorphisms $g : E \to E$ (i.e. diffeomorphisms $g : E \to E$ covering the identity map $\mathbb{1}_M : M \to M$ such that, for each $x \in M$, the restriction $g_x := g|_{E_x} : E_x \to E_x$ lies in $\mathrm{Aut}_G(E)_x$) with the group structure given by the composition of maps.

Another important bundle in this setting is the **adjoint bundle** $\mathfrak{g}_E$ of E, the *real* vector subbundle of $\mathrm{End}(E) = E^* \otimes E$ whose fiber at a point $x \in M$ consists of all those endomorphisms $T : E_x \to E_x$ whose matrix representation with respect to a local $G-$trivialization of E lies in the (real) Lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}(r, \mathbb{K})$. Alternatively, if $\{g_{\alpha\beta} : U_{\alpha\beta} \to G\}$ is a family of transition functions for E then $\mathfrak{g}_E$ is the real vector bundle given by the transition functions

$$\mathrm{Ad}(g_{\alpha\beta}) : U_{\alpha\beta} \to \mathrm{GL}(\mathfrak{g}),$$

where $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ denotes the canonical adjoint action of G on $\mathfrak{g}$.

Now, since G is a compact Lie group, the Lie algebra $\mathfrak{g}$ is *reductive*, meaning that its Levi decomposition[3] has the form

$$\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}(\mathfrak{g}), \tag{1.1.2}$$

where $\mathfrak{s}$ is a semi-simple ideal of $\mathfrak{g}$ and $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$. (In particular, if $Z(G)$ denotes the center of G, the compact Lie group $G/Z(G)$ is semi-simple, with Lie algebra $\mathfrak{s}$.) Furthermore, since $G \subseteq \mathrm{GL}(r, \mathbb{K})$, we have explicitly:

$$\mathfrak{s} = \mathfrak{g} \cap \mathfrak{sl}(r, \mathbb{K}) \quad \text{and} \quad \mathfrak{z}(\mathfrak{g}) = \mathfrak{g} \cap \mathfrak{l},$$

where $\mathfrak{l} \subseteq \mathfrak{gl}(r, \mathbb{K})$ denotes the Lie algebra of scalar matrices. Every element $X \in \mathfrak{g}$ decomposes accordingly as

$$X = \left(X - \frac{1}{r}\mathrm{tr}(X)\mathbb{1}\right) \oplus \left(\frac{1}{r}\mathrm{tr}(X)\mathbb{1}\right) \in \mathfrak{s} \oplus \mathfrak{z}(\mathfrak{g}).$$

The $\mathfrak{s}-$component (or *trace-free component*) of X will be denoted by X^0.

It follows from the above decomposition (1.1.2) that the adjoint bundle $\mathfrak{g}_E$ splits as

$$\mathfrak{g}_E = \mathfrak{g}_E^{(0)} \oplus \underline{\mathfrak{z}(\mathfrak{g})}, \tag{1.1.3}$$

where $\mathfrak{g}_E^{(0)}$ is the adjoint bundle of $E \times_G G/Z(G)$, the trace-free endomorphisms in $\mathfrak{g}_E$, and $\underline{\mathfrak{z}(\mathfrak{g})}$ is the trivial bundle with fibre equal to $\mathfrak{z}(\mathfrak{g})$.

Connections. We now recall the definition of a connection on E from the covariant derivative (Koszul) point of view.

[3] If $\mathfrak{r}(\mathfrak{g})$ denotes the (solvable) radical of a finite-dimensional (real) Lie algebra $\mathfrak{g}$, then $\mathfrak{g}$ is the semi-direct product of $\mathfrak{r}(\mathfrak{g})$ and a (necessarily semi-simple) subalgebra $\mathfrak{s}$.

Definition 1.1.2. A smooth **connection** (or *covariant derivative*) ∇ on E is a $\mathbb{K}-$linear map

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

satisfying the *Leibniz rule*

$$\nabla(fs) = \mathrm{d}f \otimes s + f\nabla s, \quad \text{for each } f \in C^\infty(M), \text{ and } s \in \Gamma(E). \tag{1.1.4}$$

Remark 1.1.3. Suppose E is a *complex* vector bundle, i.e. suppose $\mathbb{K} = \mathbb{C}$, so that $\Gamma(E)$ has a natural structure of $C^\infty(M, \mathbb{C})-$module. In this situation, we can give a natural $C^\infty(M, \mathbb{C})-$module structure on $\Gamma(T^*M \otimes E)$ in such a way that it is identified with $\Gamma(T^*M_\mathbb{C} \otimes_{C^\infty(M,\mathbb{C})} E)$ canonically. Thus, for example, when M is a manifold endowed with an almost complex structure J, one may want to look at ∇ as an operator on $\Gamma(E)$ taking values in $\Gamma(T^*M_\mathbb{C} \otimes_{C^\infty(M,\mathbb{C})} E)$ instead of $\Gamma(T^*M \otimes E)$ (e.g. we will do so in paragraph 1.5). In this case, it makes sense to write $\mathrm{d}f \otimes s = \mathrm{d}f_1 \otimes s + \mathrm{d}f_2 \otimes (\mathbf{i}s)$ for each $f = f_1 + \mathbf{i}f_2 \in C^\infty(M, \mathbb{C})$, with $f_i \in C^\infty(M)$, so that the Leibniz rule (1.1.4) holds more generally for *complex*-valued smooth functions. ♢

In what follows we list some important properties of connections:

(i) *The difference of connections is a tensor.* It follows from the Leibniz rule (1.1.4) that the difference $\nabla - \nabla'$ of connections is an *algebraic operator* (i.e. $C^\infty(M)-$linear), therefore defines an element $A \in \Omega^1(M, \mathrm{End}(E))$ such that

$$(\nabla - \nabla')\, s = As, \quad \text{for each } s \in \Gamma(E) \equiv \Omega^0(M, E).$$

Here A acts algebraically on the space of sections $\Omega^0(M, E)$ via the natural contraction map[4]

$$\Omega^0(M, E) \times \Omega^1(M, \mathrm{End}(E)) \to \Omega^1(M, E).$$

Conversely, given a covariant derivative ∇ on E and an element $A \in \Omega^1(M, \mathrm{End}(E))$, the operator $\nabla' := \nabla + A : \Omega^0(M, E) \to \Omega^1(M, E)$ is clearly linear and verifies the Leibniz rule, thus defines a covariant derivative on E. This means that, if nonempty[5], the space of covariant derivatives on $E \to M$ is an affine space modeled on $\Omega^1(M, \mathrm{End}(E))$.

(ii) *Connections are local operators.* Another consequence of Definition 1.1.2 is that a connection ∇ is a *local operator*, that is to say, it *decreases support*: if $s \in \Gamma(E)$

[4] e.g. $(s, \omega \otimes T) \mapsto \omega \otimes (Ts)$, for each $\omega \in \Omega^1(M)$, $T \in \Gamma(\mathrm{End}(E))$ and $s \in \Omega^0(M, E)$.

[5] We will not include a proof here of the existence of connections on a given vector bundle. We just note that we suppose our manifolds to be paracompact, so connections do exist and the proof is a standard application of partitions of unit.

vanishes on some open subset $U \subseteq M$ then ∇s also vanishes on U. (By linearity, this is also equivalent to say that the value of ∇s at x depends only on those of s in an arbitrarily small neighborhood of x.) Indeed, let $x \in U$ and pick a cutoff function $\rho \in C^\infty(M)$ supported in U and identically 1 in a neighborhood $V \Subset U$ of x, i.e. with $\text{supp}(\rho) \subseteq U$ and $\rho|_V \equiv 1$ for some open $V \ni x$, $\overline{V} \subset U$. Then, $\rho s \equiv 0$ so that, by linearity,

$$\nabla(\rho s) \equiv 0.$$

On the other hand, as $s(x) = 0$ and $\rho(x) = 1$, by the Leibniz rule

$$\begin{aligned}\nabla(\rho s)(x) &= (\mathrm{d}\rho)(x) \otimes s(x) + \rho(x)(\nabla s)(x) \\ &= (\nabla s)(x).\end{aligned}$$

Therefore, $(\nabla s)(x) = 0$ proving the claim.

This shows in particular that if ∇ is a connection on E, then ∇ restricts to a connection on $E|_U$, provided that $U \subseteq M$ is an open subset. Thus if $\{U_\alpha\}$ is an open cover of M, then ∇ is completely determined by the induced connections ∇_α on each of the restricted bundles $E|_{U_\alpha}$.

(iii) *Connections are covariant objects.* Connections can be pulled back by means of a smooth map $f : M' \to M$. Let us briefly recall some facts about induced bundles $f^*E \to M'$. In terms of transition functions, if $\{g_{\alpha\beta}\}$ is a family of transition functions for the $G-$bundle E subordinated to an open cover $\{U_\alpha\}$ of M, then f^*E is determined by the transition functions $\{f^*g_{\alpha\beta}\}$ subordinated to the open cover $\{f^{-1}(U_\alpha)\}$ of M'. In particular, f^*E is also a $G-$bundle. A concrete description of the total space of the induced bundle is

$$f^*E = \{(x', v) \in M' \times E : f(x') = \pi(v)\}.$$

Whenever $U \subseteq M$ is an open subset, each $s \in \Gamma(E|_U)$ induces $f^*s \in \Gamma(f^*E|_{f^{-1}(U)})$ defined by

$$(f^*s)(x') := s(f(x')), \quad \forall x' \in f^{-1}(U).$$

If $\{e_1, \dots, e_r\}$ is a local frame for E over U, then it is quite easy to see $\{f^*e_1, \dots, f^*e_r\}$ is a local frame for f^*E over $f^{-1}(U)$.

Given a smooth connection ∇ on E and a $G-$atlas $\{(U_\alpha, \varphi_\alpha)\}$ for E, the *pull-back connection* $f^*\nabla$ on f^*E is defined in such a way that in each induced local frame $\{f^*e_i^\alpha : i = 1, \dots, r\}$ we have

$$(f^*\nabla)(f^*e_i^\alpha) := f^*(\nabla e_i^\alpha), \tag{1.1.5}$$

where in the RHS of the last equation f^* acts as the natural extension of $f^*(\omega\otimes s) := (f^*\omega)\otimes(f^*s)$, for $\omega\in\Omega^1(U_\alpha)$ and $s\in\Gamma(E|_{U_\alpha})$. (This means that if $\{A_\alpha\}$ is the collection of 'gauge potentials' associated to ∇, then $\{f^*A_\alpha\}$ is the collection of 'gauge potentials' associated to $f^*\nabla$ on the induced local trivializations for f^*E – see the next paragraph.)

Local description of connections. Combining properties (i) and (ii) above, we get the following local description of connections. Consider an atlas $\{(U_\alpha,\varphi_\alpha)\}$ of local trivializations for E. Then, we may write

$$\nabla_\alpha = \mathrm{d} + A_\alpha, \tag{1.1.6}$$

where d is the *trivial product connection* on $U_\alpha\times\mathbb{K}^r$, which takes a section $s=(s_1,\dots,s_r)$ to[6] $\mathrm{d}s=(\mathrm{d}s_1,\dots,\mathrm{d}s_r)$, and[7] $A_\alpha\in\Omega^1\left(U_\alpha,\mathfrak{gl}(r,\mathbb{K})\right)$. The meaning of the above equality is that, identifying (via φ_α) local sections of $E|_{U_\alpha}$ with (column) vector-valued functions, the induced covariant derivative on $E|_{U_\alpha}$ acts on sections as the sum $\mathrm{d}+A_\alpha$. The matrix A_α of local 1−forms is called the *connection matrix* or *gauge potential* of ∇ with respect to $(U_\alpha,\varphi_\alpha)$.

In an overlap $U_\alpha\cap U_\beta\neq\emptyset$, a straightforward computation shows that the gauge potentials A_α and A_β are related by

$$A_\alpha = g_{\alpha\beta}A_\beta g_{\alpha\beta}^{-1} + g_{\alpha\beta}\mathrm{d}g_{\alpha\beta}. \tag{1.1.7}$$

$G-$**connections.** A connection ∇ on E is called a $G-$**connection** if its associated gauge potentials A_α with respect to local $G-$trivializations $(U_\alpha,\varphi_\alpha)$ of E lie in $\Omega^1\left(U_\alpha,\mathfrak{g}\right)$. For example, if $G=\mathrm{U}(r)$ and $\langle\cdot,\cdot\rangle$ is the associated metric on E, then the condition for ∇ to be a $G-$connection may be rephrased globally as:

$$\mathrm{d}\langle s_1,s_2\rangle = \langle\nabla s_1,s_2\rangle + \langle s_1,\nabla s_2\rangle,\quad \forall s_1,s_2\in\Gamma(E).$$

Here $\langle\cdot,\cdot\rangle$ is naturally extended so that

$$\langle\omega\otimes s,t\rangle = \langle s,\omega\otimes t\rangle = \langle s,t\rangle\omega,$$

whenever $\omega\in\Omega^1(M)$ and $s,t\in\Gamma(E)$.

We see that $G-$connections differ by an element in $\Omega^1(M,\mathfrak{g}_E)$ rather than just $\Omega^1(M,\mathrm{End}(E))$, so that the **space of smooth** $G-$**connections** on E, hereafter denoted by $\mathfrak{U}(E)$, is an (infinite-dimensional) affine space modeled on $\Omega^1(M,\mathfrak{g}_E)$. Thus, when we fix a smooth reference $G-$connection ∇_0,

$$\mathfrak{U}(E) = \{\nabla_0 + A : A\in\Omega^1(M,\mathfrak{g}_E)\}.$$

[6] By a slight abuse of notation, we also denote by d the exterior derivative operator.

[7] In this notation, $\mathfrak{gl}(r,\mathbb{K})$ is though as the trivial bundle $U_\alpha\times\mathfrak{gl}(r,\mathbb{K})$ over U_α.

We use this affine structure to endow $\mathfrak{U}(E)$ with the $C^\infty_{\text{loc}}-$topology coming from the model $\Omega^1(M, \mathfrak{g}_E)$. By definition, a sequence $\{\nabla_i\} \subseteq \mathfrak{U}(E)$ converges to $\nabla \in \mathfrak{U}(E)$ if, and only if, $\{\nabla_i - \nabla\} \subseteq \Omega^1(M, \mathfrak{g}_E)$ converges to zero in $C^\infty_{\text{loc}}-$topology[8] on M.

☞ **Convention:** *Unless otherwise stated, from now on we drop the prefix $G-$ and assume we are dealing only with $G-$objects. Thus, for example, 'a connection on E' will actually mean 'a $G-$connection on E', a 'local trivialization for E' will actually mean a 'a local $G-$trivialization of E', and so on.*

Now we turn attention to some important differential operators induced by a connection $\nabla \in \mathfrak{U}(E)$ on $E \to M$. We start noting that ∇ induces a collection of exterior differential operators

$$\mathrm{d}_\nabla : \Omega^k(M, E) \to \Omega^{k+1}(M, E), \quad k \geq 0,$$

uniquely determined by the following properties (see [MT97, p. 170, Lemma 17.6]):

(i) d_∇ is $\mathbb{K}-$linear, for each $k \geq 0$;

(i) $\mathrm{d}_\nabla = \nabla$ on $\Omega^0(M, E)$;

(ii) $\mathrm{d}_\nabla(\omega \wedge \xi) = \mathrm{d}\omega \wedge \xi + (-1)^k \omega \wedge \mathrm{d}_\nabla \xi$, for each $\omega \in \Omega^k(M)$ and $\xi \in \Omega^l(M, E)$.

Here $\wedge : \Omega^k(M) \times \Omega^l(M, E) \to \Omega^{k+l}(M, E)$ is the naturally extended wedge product acting trivially on the $E-$component.

The curvature of a connection. We note that the composition

$$\mathrm{d}_\nabla \circ \mathrm{d}_\nabla : \Omega^0(M, E) \to \Omega^2(M, E)$$

is $C^\infty(M)-$linear: indeed, for $f \in C^\infty(M)$ and $s \in \Gamma(E)$ we have

$$\begin{aligned} \mathrm{d}_\nabla \circ \mathrm{d}_\nabla(fs) &= \mathrm{d}_\nabla(\mathrm{d}f \otimes s + f\nabla s) \\ &= \mathrm{d}^2 f \otimes s - \mathrm{d}f \wedge \nabla s + \mathrm{d}f \wedge \nabla s + f\mathrm{d}_\nabla \circ \mathrm{d}_\nabla s \\ &= f\mathrm{d}_\nabla \circ \mathrm{d}_\nabla s. \end{aligned}$$

Hence, there exists a unique section $F_\nabla \in \Omega^2(M, \mathrm{End}(E))$, called the **curvature** of ∇, such that

$$F_\nabla s = \mathrm{d}_\nabla \circ \mathrm{d}_\nabla s, \quad \forall s \in \Gamma(E).$$

[8]See the first paragraph of Section A.6 for a construction of such topology in a simplified context.

Local description of the curvature. Consider an atlas of local trivializations $(U_\alpha, \varphi_\alpha)$ for E. Let again A_α be the associated gauge potentials of ∇ and let $F_\alpha := [F_\nabla]_\alpha$ be the local matrix representations of F_∇. Then, a local computation gives the following Cartan's formula:

$$F_\alpha = \mathrm{d}A_\alpha + A_\alpha \wedge A_\alpha, \tag{1.1.8}$$

where $A_\alpha \wedge A_\alpha$ is the matrix of local 2−forms

$$(A_\alpha \wedge A_\alpha)^i_j := \sum_k (A_\alpha)^i_k \wedge (A_\alpha)^k_j, \quad 1 \leq i, j \leq r.$$

Here, ξ^i_j denotes the components of ξ with respect to the local frame induced on $\mathrm{End}(E)$; more intrinsically, there is a natural extension of the wedge product to $\Omega^\bullet(M, \mathrm{End}(E)) = \bigoplus_{k\geq 0} \Omega^k(M, \mathrm{End}(E))$ such that

$$(\omega \otimes T) \wedge (\eta \otimes S) = (\omega \wedge \eta) \otimes (T \circ S), \quad \text{for each } \omega, \eta \in \Omega^\bullet(M) \text{ and } S, T \in \Gamma(\mathrm{End}(E)).$$

Using (1.1.8) and (1.1.7), one further shows that

$$F_\alpha = g_{\alpha\beta} F_\beta g_{\alpha\beta}^{-1} = \mathrm{Ad}(g_{\alpha\beta}) F_\beta, \quad \text{on } U_{\alpha\beta} \neq \emptyset. \tag{1.1.9}$$

Shrinking U_α if necessary, we can also consider local coordinates $(x^1, \ldots, x^n)$ and write

$$A_\alpha = A_{\alpha,i} \otimes \mathrm{d}x^i, \quad \text{for } A_{\alpha,i} \in \mathfrak{g}, \tag{1.1.10}$$

and

$$F_\alpha = \frac{1}{2} F_{\alpha,ij} \otimes \mathrm{d}x^i \wedge \mathrm{d}x^j, \quad \text{for } F_{\alpha,ij} \in \mathfrak{gl}(r, \mathbb{K}). \tag{1.1.11}$$

It then follows from (1.1.8) that

$$F_{\alpha,ij} = \partial_i A_{\alpha,j} - \partial_j A_{\alpha,i} + [A_{\alpha,i}, A_{\alpha,j}], \tag{1.1.12}$$

where $[\cdot,\cdot]$ is the commutator of $\mathfrak{g} \subseteq \mathfrak{gl}(r, \mathbb{K})$. In particular, we have $F_{\alpha,ij} \in \mathfrak{g}$ for each $i, j = 1, \ldots, n$. Thus, the curvature F_∇ lies actually in $\Omega^2(M, \mathfrak{g}_E)$.

Now we note that ∇ induces a connection on $\mathrm{End}(E)$, still denoted by ∇, in the following manner: for $T \in \Gamma(\mathrm{End}(E))$, we put

$$(\nabla T)(s) := \nabla(Ts) - T(\nabla s), \quad \text{for each } s \in \Gamma(E), \tag{1.1.13}$$

where $T(\nabla s)$ denotes the action of the endomorphism T on the E component of ∇s. This connection in fact reduces to a connection on $\mathfrak{g}_E \subseteq \mathrm{End}(E)$, since ∇ is a G−connection. As we have done before, this induces operators

$$\mathrm{d}_\nabla : \Omega^k(M, \mathfrak{g}_E) \to \Omega^{k+1}(M, \mathfrak{g}_E), \quad k \geq 0.$$

If $\xi \in \Omega^p(M, \mathfrak{g}_E)$ and $\xi_\alpha := [\xi]_\alpha$ is the local representation via a local trivialization $(U_\alpha, \varphi_\alpha)$, then one can show that

$$[\mathrm{d}_\nabla \xi]_\alpha = \mathrm{d}\xi_\alpha + [A_\alpha, \xi_\alpha], \tag{1.1.14}$$

where $[\omega \otimes T, \eta \otimes S] := (\omega \wedge \eta) \otimes [T, S]_{\mathfrak{g}}$ is the *graded commutator*; more generally, if $\eta \in \Omega^q(M, \mathfrak{g}_E)$,

$$[\xi, \eta] := \xi \wedge \eta - (-1)^{pq} \eta \wedge \xi. \tag{1.1.15}$$

Lemma 1.1.4 (Bianchi identity)**.** *A smooth connection $\nabla \in \mathfrak{U}(E)$ satisfies*

$$\mathrm{d}_\nabla F_\nabla = 0. \tag{1.1.16}$$

Proof. It suffices to check the identity in a local trivialization. By (1.1.14) and Cartan's formula (1.1.8), we have

$$\begin{aligned} [\mathrm{d}_\nabla F_\nabla]_\alpha &= \mathrm{d}F_\alpha + [A_\alpha, F_\alpha] \\ &= \mathrm{d}(\mathrm{d}A_\alpha + A_\alpha \wedge A_\alpha) + [A_\alpha, \mathrm{d}A_\alpha + A_\alpha \wedge A_\alpha]. \end{aligned}$$

Now, by the Leibniz rule and (1.1.15),

$$\mathrm{d}(\mathrm{d}A_\alpha + A_\alpha \wedge A_\alpha) = \mathrm{d}A_\alpha \wedge A_\alpha - A_\alpha \wedge \mathrm{d}A_\alpha$$

and

$$\begin{aligned} [A_\alpha, \mathrm{d}A_\alpha + A_\alpha \wedge A_\alpha] &= A_\alpha \wedge \mathrm{d}A_\alpha - \mathrm{d}A_\alpha \wedge A_\alpha + A_\alpha \wedge A_\alpha \wedge A_\alpha - A_\alpha \wedge A_\alpha \wedge A_\alpha \\ &= A_\alpha \wedge \mathrm{d}A_\alpha - \mathrm{d}A_\alpha \wedge A_\alpha. \end{aligned}$$

Summing these equations we get the desired result. ■

Gauge equivalence. We now recall the concept of gauge equivalence for connections on E. Firstly, we note the existence of a canonical action of $\mathcal{G}(E)$ on $\Gamma(E)$: a gauge transformation $g \in \mathcal{G}(E)$ acts on a section $s \in \Gamma(E)$ giving rise to the new section $gs \in \Gamma(E)$ defined by

$$(gs)(x) := g_x(s(x)), \quad \forall x \in M.$$

This extends naturally to an action of $\mathcal{G}(E)$ on $\Omega^k(M, E)$ by acting trivially on the form part. That said, we can define the following 'pullback' action[9] of $\mathcal{G}(E)$ on the space of smooth $G-$connections $\mathfrak{U}(E)$: an element $g \in \mathcal{G}(E)$ acts on $\nabla \in \mathfrak{U}(E)$ by

$$g^*\nabla := g^{-1} \circ \nabla \circ g,$$

[9]Here, the use of the terminology 'pullback' is just a reference to the fact that the described action is a *right* action. Some authors defines the action of $\mathcal{G}(E)$ on $\mathfrak{U}(E)$ to be the corresponding left action, where g acts on ∇ by 'pushforward': $g \cdot \nabla := g \circ \nabla \circ g^{-1}$.

i.e. $g^*\nabla : \Omega^0(M,E) \to \Omega^1(M,E)$ is the map given by

$$(g^*\nabla)(s) \coloneqq g^{-1}\nabla(gs), \quad \forall s \in \Omega^0(M,E).$$

This defines indeed a $G-$connection on E. First of all, $g^*\nabla$ is clearly a $\mathbb{K}-$linear map. To check the Leibniz rule, let $f \in C^\infty(M)$ and $s \in \Omega^0(M,E)$; since the actions of $C^\infty(M)$ and $\mathcal{G}(E)$ on $\Omega^k(M,E)$ commutes, we have

$$\begin{aligned}(g^*\nabla)(fs) &= g^{-1}(\nabla g(fs)) = g^{-1}(\nabla f(gs)) \\ &= g^{-1}(\mathrm{d}f \otimes (gs) + f\nabla(gs)) = \mathrm{d}f \otimes s + f(g^{-1}\nabla(gs)) \\ &= \mathrm{d}f \otimes s + f(g^*\nabla)s.\end{aligned}$$

Moreover, a straightforward calculation shows that, if (U,φ) is a local $(G-)$trivialization for E and we let A and g^*A be the associated gauge potentials of ∇ and $g^*\nabla$ respectively, then we obtain the following transformation law:

$$g^*A = g^{-1}Ag + g^{-1}\mathrm{d}g = \mathrm{Ad}(g^{-1})A + g^*\theta_{MC}, \tag{1.1.17}$$

where $g, g^{-1} : U \to G$ are seen here as sections of $U \times G \simeq \mathrm{Aut}(E|_U)$ via φ, and θ_{MC} is the Maurer-Cartan form[10] of G. This shows that $g^*\nabla$ is in fact a $G-$connection. Finally, one checks directly from the definition that $g^*(h^*\nabla) = (h \circ g)^*\nabla$ for each $g \in \mathcal{G}(E)$ and $\nabla \in \mathfrak{U}(E)$, characterizing a right action.

We say that two connections $\nabla, \nabla' \in \mathfrak{U}(E)$ are **gauge equivalent** if they live in the same orbit of the above action of $\mathcal{G}(E)$ on $\mathfrak{U}(E)$, i.e. if there exists $g \in \mathcal{G}(E)$ such that $\nabla' = g^*\nabla$. It is immediate that

$$F_{g^*\nabla} = g^{-1}F_\nabla g, \quad \forall g \in \mathcal{G}(E), \nabla \in \mathfrak{U}(E). \tag{1.1.18}$$

We now introduce some other important differential operators induced by connections on *real* vector bundles defined over (oriented) *Riemannian* manifolds.

Let (M,g) be an oriented manifold endowed with a Riemannian metric g, and let $F \to M$ be a real vector bundle over M endowed with a metric $\langle \cdot, \cdot \rangle$. Recall that g distinguishes a unique torsion-free $\mathrm{O}(n)-$connection D^g on TM, the so-called *Levi-Civita* connection associated to g. In particular, taking the tensor product with D^g, a connection $\nabla \in \mathfrak{U}(F)$ on $F \to M$ induces connections

$$\nabla : \Omega^k(M,F) \to \Gamma(T^*M \otimes \Lambda^k T^*M \otimes F), \quad \text{for each } k \geq 0. \tag{1.1.19}$$

[10] $\theta_{MC} \in \Omega^1(G,\mathfrak{g})$ is the unique left invariant $1-$form on G with values in $\mathfrak{g}$ such that $(\theta_{MC})_1 : \mathfrak{g} \to \mathfrak{g}$ is the identity map.

Recalling the definition of the exterior differential operator d_∇ on $\Omega^k(M,F)$, we see that $\mathrm{d}_\nabla = \wedge \circ \nabla$, where $\wedge : \Lambda^1 \otimes \Lambda^k \to \Lambda^{k+1}$ is the natural map.

The metric g naturally induces metrics on every tensor bundle of M. In particular, we get (Euclidean) metrics on every exterior power $\Lambda^k T^*M$. Taking the tensor product of these with the metric $\langle \cdot, \cdot \rangle$ on F gives rise to (Euclidean) metrics, still denoted by $\langle \cdot, \cdot \rangle$, on the bundles $\Lambda^k T^*M \otimes F$. One readily checks that the induced connections defined in the above paragraph are compatible with the respective induced metrics.

Let $\mathrm{d}V_g$ be the Riemannian volume $n-$form on (M,g) determined by the orientation on M together with g. Then the Hodge star operator, $* : \Lambda^k T^*M \to \Lambda^{n-k}T^*M$, isomorphically interchanges forms of complementary degree by the relation $\alpha \wedge *\beta = (\alpha, \beta)_g \mathrm{d}V_g$, where $\alpha, \beta \in \Lambda^k T^*M$ and $(\cdot,\cdot)_g$ denotes the natural metric on $\Lambda^k T^*M$ induced by g. Given any vector bundle $W \to M$, we define $* : \Omega^k(M,W) \to \Omega^{n-k}(M,W)$ as acting trivially on the W part: $*(\alpha \otimes s) := (*\alpha) \otimes s$.

If $\xi, \eta \in \Omega^k(M,F)$, at least one of which has compact support, we define their L^2-**inner product** by

$$\langle \xi, \eta \rangle_{L^2} := \int_M \langle \xi, \eta \rangle \mathrm{d}V_g.$$

This gives rise to formal L^2-*adjoint* operators for ∇ and d_∇:

$$\nabla^* : \Gamma(T^*M \otimes \Lambda^k T^*M \otimes F) \to \Omega^k(M,F),$$
$$\mathrm{d}_\nabla^* : \Omega^{k+1}(M,E) \to \Omega^k(M,E).$$

For example, d_∇^* is characterized by the equation

$$\langle \mathrm{d}_\nabla \xi, \eta \rangle_{L^2} = \langle \xi, \mathrm{d}_\nabla^* \eta \rangle_{L^2},$$

which is valid for forms ξ, η at least one of which has compact support. Furthermore, using Stokes' theorem, one can show that

$$\mathrm{d}_\nabla^* = (-1)^{n(k+1)+1} * \mathrm{d}_\nabla *, \quad \text{on } \Omega^k(M,F).$$

Mimicking definitions in the context of Riemannian geometry, we get the following two important second order operators acting on $\Omega^k(M,F)$:

- the **generalized Hodge-de Rham Laplacian**

$$\Delta_\nabla := \mathrm{d}_\nabla \mathrm{d}_\nabla^* + \mathrm{d}_\nabla^* \mathrm{d}_\nabla : \Omega^k(M,F) \to \Omega^k(M,F),$$

- the **covariant** (or *rough*) **Laplacian**

$$\nabla^* \nabla : \Omega^k(M,F) \to \Omega^k(M,F).$$

For example, one can show that in any orthonormal local frame $(e_1,\dots,e_n)$ of TM we have

$$\nabla^*\nabla\xi = -\sum_{j=1}^{n}\nabla^2(e_j,e_j)\xi,$$

where $\nabla^2(X,Y) \coloneqq \nabla_X\nabla_Y - \nabla_{D^g_X Y}$ is the invariantly defined *Hessian operator*.

A Bochner-Weitzenböck Formula. (cf. [BLJ81, pp. 199-200]) Consider now our $\mathbb{K}-$vector bundle E over M with compact structure group G. Fix $\nabla \in \mathfrak{U}(E)$ a smooth connection on E. By Ad−invariance, the inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ naturally induces a metric[11] on the real vector bundle $\mathfrak{g}_E$. Then, letting $\mathfrak{g}_E$ play the role of F in the discussion of the previous paragraph, and considering the induced connection, still denoted by ∇, on $\mathfrak{g}_E$ (see 1.1.13), we have the associated operators Δ_∇ and $\nabla^*\nabla$ acting on $\mathfrak{g}_E-$valued $k-$forms.

It turns out that these operators have the same principal symbol and their difference is a zero order (algebraic) operator, i.e. $C^\infty(M)-$linear. We will present now (without proof) the formula that gives the precise difference between these operators on the space $\Omega^2(M,\mathfrak{g}_E)$. Such a formula is often called a **Böchner-Weitzenböck formula**.

Fix an orthonormal local frame $\{e_1,\dots,e_n\}$ of TM. Recall that the *Ricci transformation* $\mathrm{Ric}^g : T_xM \to T_xM$ is given by

$$\mathrm{Ric}^g(X) = \sum_{j=1}^{n} R^g(X,e_j)e_j,$$

where R^g stands for the *Riemann curvature tensor* associated to g. We extend the Ricci transformation to act on 2−forms by putting:

$$(\mathrm{Ric}^g \wedge I)(X,Y) \coloneqq \mathrm{Ric}^g(X)\wedge Y + X\wedge \mathrm{Ric}^g(Y), \quad \forall X,Y \in \mathfrak{X}(M).$$

Now define a zero-order operator $\mathscr{F}_\nabla : \Omega^2(M,\mathfrak{g}_E) \to \Omega^2(M,\mathfrak{g}_E)$ given by

$$\mathscr{F}_\nabla(\xi)(X,Y) \coloneqq \sum_{j=1}^{n}\left\{[F_\nabla(e_j,X),\xi(e_j,Y)] - [F_\nabla(e_j,Y),\xi(e_j,X)]\right\}, \quad \forall X,Y \in \mathfrak{X}(M).$$

For each $\xi \in \Omega^2(M,\mathfrak{g}_E)$, we write

$$(\xi\circ \mathrm{Ric}^g \wedge I)(X,Y) \coloneqq \xi(\mathrm{Ric}^g(X),Y) + \xi(X,\mathrm{Ric}^g(Y)), \quad \text{and}$$
$$(\xi\circ 2R^g)(X,Y) \coloneqq \sum_{j=1}^{n}\xi(e_j,R^g(X,Y)e_j), \quad \forall X,Y \in \mathfrak{X}(M).$$

[11]In a local trivialization the elements of $\mathfrak{g}_E$ are represented by matrices in $\mathfrak{g}$. Moreover, any two such representations differ by the adjoint action of G on $\mathfrak{g}$. Thus there is well-defined induced metric $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}_E$ such that, for all $T,S \in \Omega^0(\mathfrak{g}_E)$, we have locally

$$\langle T,S\rangle|_{U_\alpha} = \langle [T]_\alpha,[S]_\alpha\rangle_{\mathfrak{g}}.$$

The following formula can be found in [BLJ81, Theorem 3.10, p. 200].

Theorem 1.1.5 (Bochner-Weitzenböck formula)**.** *For any* $\xi \in \Omega^2(M, \mathfrak{g}_E)$ *we have*

$$\Delta_\nabla \xi = \nabla^* \nabla \xi + \xi \circ (\mathrm{Ric}^g \wedge I + 2R^g) + \mathscr{F}_\nabla(\xi).$$

Sobolev spaces of connections. (cf. [Weh04, Appendix A and Appendix B]) In this paragraph, we suppose M is a compact manifold. Let $E \to M$ be a $G-$bundle over M. Given $k \in \mathbb{N}$ and $1 \leq p < \infty$, we want to introduce the Sobolev space $\mathfrak{U}^{k,p}(E)$ of $W^{k,p}$ connections on E.

We start noting that the metric on the adjoint bundle $\mathfrak{g}_E$ (determined by the Ad_G-invariant inner product on $\mathfrak{g}$), combined with the metric induced by g on T^*M, gives rise to a natural (tensor product) metric on the tensor bundle $T^*M \otimes \mathfrak{g}_E$. If we fix a smooth connection $\nabla_0 \in \mathfrak{U}(E)$, this induces (by tensoring with the Levi-Civita connection) a naturally associated compatible connection on $T^*M \otimes \mathfrak{g}_E$. Thus, we can speak of the Sobolev spaces $W^{k,p}(M, T^*M \otimes \mathfrak{g}_E)$, for each $1 \leq p < \infty$ and $k \in \mathbb{N}_0$ (cf. Section B.2 of Appendix B). In this context, we can define the **Sobolev space of** $W^{k,p}$ **connections** on E by

$$\mathfrak{U}^{k,p}(E) \coloneqq \{\nabla_0 + A : A \in W^{k,p}(M, T^*M \otimes \mathfrak{g}_E)\}.$$

By the compactness of M, we know from Theorem B.2.9 that $W^{k,p}(M, T^*M \otimes \mathfrak{g}_E)$ does not depend on the choices of metrics and compatible connections on the involved bundles. Moreover, since any two smooth reference connections $\nabla_0, \nabla_0' \in \mathfrak{U}(E)$ differ by an element of $\Omega^1(M, \mathfrak{g}_E)$ and, by compactness of M, there is a bounded inclusion $\Omega^1(M, \mathfrak{g}_E) \hookrightarrow W^{k,p}(M, T^*M \otimes \mathfrak{g}_E)$, we see that $\mathfrak{U}^{k,p}(E)$ is well-defined.

We topologize $\mathfrak{U}^{k,p}(E)$ using its affine structure: by definition, a sequence $\{\nabla_i\} \subseteq \mathfrak{U}^{k,p}(E)$ converges to $\nabla \in \mathfrak{U}^{k,p}(E)$ if, and only if, $\|\nabla_i - \nabla\|_{p,k} \to 0$ as $i \to \infty$.

We know that a smooth gauge transformation $g \in \mathcal{G}(E)$ acts on a smooth connection $\nabla = \nabla_0 + A \in \mathfrak{U}(E)$ by

$$g^*\nabla = g^{-1} \circ \nabla \circ g = \nabla_0 + g^{-1}\nabla_0 g + g^{-1}Ag.$$

Thus, naturally enough, one expects the relevant group of gauge transformations in the context of $W^{k,p}-$connections is[12]

$$\mathcal{G}^{k+1,p}(E) \coloneqq W^{k+1,p}(M, \mathrm{Aut}(E)).$$

In fact, using the Sobolev embedding theorem one can prove the following [Weh04, Lemma A.5, p. 175 & Lemma A.6, p. 176]:

[12]The heuristic is that we need one more derivative of regularity on g in order to $g^*A = g^{-1}\nabla_0 g + g^{-1}Ag$ lie in $W^{k,p}$ whenever $A \in W^{k,p}$.

Proposition 1.1.6. *Let $k \in \mathbb{N}_0$ and let $1 \leq p < \infty$ be such that $(k+1)p > n$. Then, $\mathcal{G}^{k+1,p}(E) \subseteq C^0(M, \mathrm{Aut}_G(E))$ and $\mathcal{G}^{k+1,p}(E)$ is a topological group with respect to composition. Moreover, the pullback action $\mathcal{G}^{k+1,p}(E) \times \mathfrak{U}^{k,p}(E) \to \mathfrak{U}^{k,p}(E)$ is a continuous map. In particular, for $p > \frac{n}{2}$, $\mathcal{G}^{2,p}(E)$ acts continuously in $\mathfrak{U}^{1,p}(E)$.*

Given a smooth connection $\nabla = \nabla_0 + A \in \mathfrak{U}(E)$, its curvature (or field) is just

$$F_\nabla = \nabla^2 = F_{\nabla_0} + \mathrm{d}_{\nabla_0} A + [A, A] \in \Omega^2(M, \mathfrak{g}_E).$$

More generally, we have [Uhl82a, Lemma 1.1]:

Lemma 1.1.7. *Let $1 < p < \infty$ be such that $2p \geq n$. Then, the curvature map $\nabla \mapsto F_\nabla$ on $\mathfrak{U}(E)$ extends to a quadratic map*

$$\mathfrak{U}^{1,p}(E) \to L^p(M, \Lambda^2 T^* M \otimes \mathfrak{g}_E).$$

Sketch of proof. We know that $F_\nabla = F_{\nabla_0} + \nabla_0 A + [A, A]$, with $F_{\nabla_0} \in C^\infty$ and $\nabla_0 A \in L^p$. By the Sobolev embedding (Theorem B.2.10), we have $W^{1,p} \subseteq L^q$ for $\frac{1}{q} \geq \frac{1}{p} - \frac{1}{n}$; in this case, by Hölder's inequality, the quadratic term $A \mapsto [A, A]$ lies in $L^{q/2}$. In order to obtain $L^{q/2} \subseteq L^p$, the Sobolev embedding requires $\frac{1}{p} \geq \frac{2}{q} \geq \frac{2}{p} - \frac{2}{n}$, i.e. $2p \geq n$. ∎

1.2 Holonomy groups

Fix $E \to M$ a real vector bundle of rank r endowed with a smooth connection ∇. We now give a brief review on the basics of holonomy groups, fixing terminology and notations that we will need for the Chapter 2. The main references for this section are [Joy07, Joy04] and [CS12].

Parallel transport. Let $\gamma : [0, 1] \to M$ be a smooth path from $x = \gamma(0)$ to $y = \gamma(1)$. A section $s \in \Gamma(E)$ is said to be ∇**−parallel along** γ when the composition $s \circ \gamma \in \Gamma(\gamma^* E)$ is $\gamma^*\nabla$−parallel[13], i.e. when

$$(\gamma^*\nabla)(s \circ \gamma) = 0. \tag{1.2.1}$$

Since the closed interval $[0, 1]$ is contractible, the induced bundle $\gamma^* E \to [0, 1]$ is trivial, i.e. it admits a global frame $\{E_i\}$ over $[0, 1]$. Writing $s \circ \gamma = x^j E_j$ and letting $A = (A^i_j)$ denote the gauge potential of $\gamma^*\nabla$ with respect to $\{E_i\}$, we see equation (1.2.1) translates into the following typical linear ODE:

$$\dot{x} + Ax = 0,$$

[13]This pull-back notation is not entirely rigorous since $[0, 1]$ is a manifold with boundary. Now, the smoothness of γ means there exists a smooth path $\tilde{\gamma} :]-\varepsilon, 1+\varepsilon[\to M$, for some $\varepsilon > 0$, such that $\tilde{\gamma}|_{[0,1]} = \gamma$. So when we talk about $\gamma^*\nabla$ we are in fact looking at $\tilde{\gamma}^*\nabla$ restricted to $[0, 1]$ (one can show this is independent of the extension $\tilde{\gamma}$).

where $x = (x^1, \ldots, x^n) : [0,1] \to \mathbb{R}^r$. Thus, invoking the well-known existence and uniqueness theorem for ODE's, one shows that, given an element $v \in E_x$, there exists a unique $\nabla-$parallel section $s_{\gamma,v}$ along γ satisfying the initial condition $s_{\gamma,v}(x) = v$. Moreover, by linearity of the equation, the solution depends linearly on the initial condition. This allows us to define the linear map

$$\begin{aligned} P_\gamma : E_x &\to E_y \\ v &\mapsto s_{\gamma,v}(y), \end{aligned}$$

called the **parallel transport along** γ with respect to ∇. This map is invertible, with inverse given by $P_{\gamma^{-1}}$, where $\gamma^{-1}(t) := \gamma(1-t)$ for each $t \in [0,1]$.

We can also define P_γ for a (continuos) *piecewise* smooth path γ simply as the composition of the parallel transport maps along its smooth pieces (in the appropriate order). One can show this is well-defined by the uniqueness part of the above cited ODE theorem.

In particular, if α is a (piecewise) smooth path starting at $\alpha(0) = y$, then $P_\alpha \circ P_\gamma = P_{\alpha\cdot\gamma}$, where $\alpha \cdot \gamma$ is the concatenation of γ and α:

$$\alpha \cdot \gamma(t) := \begin{cases} \gamma(2t), \text{ if } t \in [0, 1/2], \\ \alpha(2t-1) \text{ if } t \in [1/2, 1]. \end{cases}$$

The holonomy principle. We can now recall the definition of the holonomy group of ∇.

Definition 1.2.1 (Holonomy group)**.** Given $x \in M$, the subgroup of $\mathrm{GL}(E_x)$ given by

$$\mathrm{Hol}_x(\nabla) := \{P_\gamma : \gamma \text{ is a piecewise smooth loop based at } x\}$$

is called the **holonomy group** of ∇ at x.

Lemma 1.2.2. *If $x, y \in M$ are connected by a piecewise smooth path $\gamma : [0,1] \to M$, $\gamma(0) = x$ and $\gamma(1) = y$, then*

$$\mathrm{Hol}_y(\nabla) = P_\gamma \cdot \mathrm{Hol}_x(\nabla) \cdot P_{\gamma^{-1}}.$$

It is a simple fact that if $\gamma : [0,1] \to M$ is a *continuous* path connecting $x = \gamma(0)$ and $y = \gamma(1)$, then there exists a *smooth* path $\tilde{\gamma} : [0,1] \to M$ connecting $x = \tilde{\gamma}(0)$ and $y = \tilde{\gamma}(1)$. In fact, we can take $\tilde{\gamma}$ in the same homotopy class of γ with fixed end points (see e.g. [Kos07, p. 8, Theorem 2.5]). Thus, the above lemma gives us a precise relation between the holonomy groups of ∇ at any two points lying in the same connected component of M.

If M is connected we conclude the holonomy group $\mathrm{Hol}_x(\nabla)$ is independent of the base point x in the following sense. A choice of basis on E_x induces an identification

$\mathrm{GL}(E_x) \simeq \mathrm{GL}(r, \mathbb{R})$ and, therefore, a corresponding faithful representation $\mathrm{Hol}_x(\nabla) \hookrightarrow \mathrm{GL}(r, \mathbb{R})$. From linear algebra, we know that a different choice of basis in E_x will change this identification by a conjugation in $\mathrm{GL}(r, \mathbb{R})$. Thus, up to equivalence, there is a well-defined faithful representation of $\mathrm{Hol}_x(\nabla)$ on the typical fiber $\mathbb{R}^r$ of E, called the *holonomy representation*. In this language, the above lemma shows that $\mathrm{Hol}_x(\nabla)$ and $\mathrm{Hol}_y(\nabla)$ have the same holonomy representation. In other words, when regarded as a subgroup of $\mathrm{GL}(r, \mathbb{R})$ defined up to conjugation, the holonomy group is independent of the choice of base point.

☞ **Convention:** *From now on, we assume M is a* connected *manifold and when we write* $\mathrm{Hol}(\nabla)$ *(omitting the base point), we are implicitly regarding the holonomy group of* ∇ *as a subgroup of* $\mathrm{GL}(r, \mathbb{R})$ *defined up to conjugation.*

One outcome of the above discussion is that the holonomy group is a *global invariant* of the connection. The next result shows that $\mathrm{Hol}(\nabla)$ 'controls' the existence of $\nabla-$parallel sections on the tensor powers of[14] E [Joy07, Proposition 2.5.2, p. 33].

Theorem 1.2.3 (Holonomy principle)**.** *Let $E \to M$ be a vector bundle over a connected smooth manifold M and write $\mathcal{T}^r_s(E) := (\bigotimes^r E) \otimes (\bigotimes^s E^*)$. Fix a base point $x \in M$, so that $\mathrm{Hol}_x(\nabla)$ acts on E_x, and so on $\mathcal{T}^r_s(E_x)$. Then, any $(r,s)-$tensor $t_x \in \mathcal{T}^r_s(E_x)$ that is invariant under $\mathrm{Hol}_x(\nabla)$ is the value at x of a $(r,s)-$tensor field $t \in \Gamma(\mathcal{T}^r_s(E))$ which is $\nabla-$parallel ($\nabla t = 0$). Conversely, any parallel tensor field $t \in \Gamma(\mathcal{T}^r_s(E))$ is fixed in the fiber over x by the action of $\mathrm{Hol}_x(\nabla)$.*

Corollary 1.2.4. *If $G \subseteq \mathrm{GL}(E_x)$ is the subgroup which fixes $t|_x$ for all parallel tensors t on M, then $\mathrm{Hol}_x(\nabla) \subseteq G$.*

The following result shows that the holonomy group $\mathrm{Hol}(\nabla)$ is a connected Lie group when M is simply-connected [Joy07, Proposition 2.2.4, p. 26].

Proposition 1.2.5. *Suppose M is simply-connected and ∇ is a connection on a real vector bundle $E \to M$. Then $\mathrm{Hol}(\nabla)$ is a connected Lie subgroup of $\mathrm{GL}(r, \mathbb{R})$.*

This leads us to consider the notion of *restricted* holonomy groups.

Definition 1.2.6 (Restricted holonomy group)**.** Given $x \in M$, we define the **restricted holonomy group** of ∇ at x to be the subgroup

$$\mathrm{Hol}^0_x(\nabla) := \{P_\gamma : \gamma \text{ is a null-homotopic piecewise smooth loop based at } x\} \subseteq \mathrm{Hol}_x(\nabla).$$

[14] The statement in Joyce's book is given for $E = TM$ but the proof is clearly valid for any vector bundle $E \to M$

As for the case of the holonomy group, we can regard $\mathrm{Hol}^0_x(\nabla)$ as a subgroup of $\mathrm{GL}(r, \mathbb{R})$ defined up to conjugation, so that we can omit the base point x and write $\mathrm{Hol}^0(\nabla)$. The next proposition gathers some properties of $\mathrm{Hol}^0(\nabla)$ [Joy07, Proposition 2.2.6, p. 27].

Proposition 1.2.7. $\mathrm{Hol}^0(\nabla)$ *is the connected component of* $\mathrm{Hol}(\nabla)$ *containing the identity and a Lie subgroup of* $\mathrm{GL}(r, \mathbb{R})$*. Moreover, if* M *is simply-connected then* $\mathrm{Hol}^0(\nabla) = \mathrm{Hol}(\nabla)$*.*

The Ambrose-Singer theorem. With the above proposition in mind, we can make the following

Definition 1.2.8 (Holonomy algebra)**.** The **holonomy algebra** $\mathfrak{hol}_x(\nabla)$ of ∇ at $x \in M$ is the Lie algebra of $\mathrm{Hol}^0_x(\nabla)$.

Up to the adjoint action of $\mathrm{GL}(r, \mathbb{R})$, we can also speak of the holonomy algebra $\mathfrak{hol}(\nabla)$ as a Lie subalgebra of $\mathfrak{gl}(r, \mathbb{R})$. The next proposition shows that the holonomy algebra $\mathfrak{hol}(\nabla)$ constrains F_∇ (cf. [Joy07, p. 30, Proposition 2.4.1]).

Proposition 1.2.9. *For each* $x \in M$*, the curvature* $F_\nabla|_x$ *of* ∇ *at* x *lies in* $\Lambda^2 T^*_x M \otimes \mathfrak{hol}_x(\nabla)$*.*

In fact, a result due to W. Ambrose and I. M. Singer shows that $\mathfrak{hol}(\nabla)$ is *determined* by F_∇ (cf. [Joy07, p. 31, Theorem 2.4.3. (a)]):

Theorem 1.2.10 (Ambrose-Singer)**.** *Suppose* M *is a connected manifold,* $E \to M$ *is a vector bundle over* M*, and* ∇ *is a smooth connection on* E*. Then, for each* $x \in M$*, the holonomy algebra* $\mathfrak{hol}_x(\nabla)$ *is the Lie subalgebra of* $\mathrm{End}(E_x)$ *which, as a vector space, is spanned by all elements of* $\mathrm{End}(E_x)$ *of the form*

$$P_\gamma^{-1}[(F_\nabla)(v, w)]P_\gamma,$$

where $\gamma : [0, 1] \to M$ *varies on the collection of all piece-wise smooth paths starting at* $\gamma(0) = x$*, and* $v, w \in T_{\gamma(1)}M$*.*

Remark 1.2.11 (Flat connections)**.** It immediately follows from the Ambrose-Singer theorem that if ∇ is a *flat* connection, i.e. if $F_\nabla = 0$, then the restricted holonomy group $\mathrm{Hol}^0_x(\nabla)$ is trivial for each $x \in M$. This implies that for a flat connection ∇ the parallel transport $P_\gamma : E_x \to E_y$ depends only upon the homotopy class (with fixed end-points) of the path γ between x and y. In fact, if γ is homotopic to another path $\tilde{\gamma}$, which without loss of generality we can assume to be piece-wise smooth[15], then the concatenation $\tilde{\gamma}^{-1} \cdot \gamma$

[15] See the first paragraph succeeding Lemma 1.2.2.

is a null-homotopic (piece-wise smooth) loop based at x. Thus, from the triviality of $\mathrm{Hol}^0_x(\nabla)$, we get that $\mathbb{1}_{T_xM} = P_{\tilde{\gamma}^{-1}\cdot\gamma} = P_{\tilde{\gamma}}^{-1} \circ P_\gamma$, i.e. $P_\gamma = P_{\tilde{\gamma}}$, as we wanted.

In particular, for each fixed base point $x \in M$, each flat connection on a $G-$bundle induces a holonomy representation $\pi_1(M,x) \to \mathrm{Aut}(E_x) = G$. Ultimately, this leads to the well-known one-to-one correspondence between gauge-equivalence classes of flat $G-$connections over M and conjugacy classes of representations $\pi_1(M) \to G$ (cf. [DK90, pp. 49-50]). ♢

1.3 Chern-Weil approach to characteristic classes

Here we give a brief account on the Chern-Weil polynomials representing characteristic classes. This section is based on [Mil74, Appendix C].

Fundamental lemma of Chern-Weil theory. Let $E \to M$ be a $\mathbb{K}-$vector bundle of rank r over M and $\{(U_\alpha, \varphi_\alpha)\}$ an atlas of local trivializations for E with associated transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{K}).$$

If ∇ is an arbitrary connection on $E \to M$, we know its curvature F_∇ together with the local description of E leads to local curvature matrices of $2-$forms $F_\alpha := [F_\nabla]_\alpha \in \Omega^2(U_\alpha, \mathfrak{gl}(r,\mathbb{K}))$ on M. The wedge product combined with matrix multiplication then allow us to form powers of these matrices. In particular, we can evaluate a polynomial function $P : \mathfrak{gl}(r,\mathbb{K}) \to \mathbb{K}$ on F_α giving rise to a sum of exterior forms of even degree on U_α. Now recall from (1.1.9) that in the overlaps $U_\alpha \cap U_\beta \neq \emptyset$ the F_α and F_β are related by the adjoint action of $\mathrm{GL}(r,\mathbb{K})$: in fact,

$$F_\alpha = g_{\alpha\beta} F_\beta g_{\alpha\beta}^{-1}, \quad \text{on } U_\alpha \cap U_\beta.$$

So if P is a $\mathrm{GL}(r,\mathbb{K})-$**invariant** polynomial, i.e.

$$P(gXg^{-1}) = P(X), \quad \forall g \in \mathrm{GL}(r,\mathbb{K}),$$

then we can associate a *globally defined* element $P(F_\nabla) \in \Omega^{\mathrm{even}}_{\mathbb{K}}(M) := \bigoplus_{k\geq 0} \Omega^{2k}_{\mathbb{K}}(M)$ putting

$$P(F_\nabla)|_{U_\alpha} := P(F_\alpha), \quad \text{for each } \alpha.$$

Of course, in general, the $P(F_\nabla)$ will be a sum of exterior forms of various even degrees. But, if we suppose further that P is a **homogeneous** polynomial, i.e. a sum of monomials of a fixed degree, say of degree m, then $P(F_\nabla) \in \Omega^{2m}_{\mathbb{K}}(M)$. Also note that we could replace the hypothesis on P being invariant to the more flexible one of P being a sum of invariant homogeneous polynomials of increasing degrees, since we know that $Q(F_\nabla) = 0$ whenever $2\deg(Q) > \dim M$.

An important point about $P(F_\nabla)$ is that it is well behaved with respect to induced bundles. This means that if M' is some smooth manifold and $f : M' \to M$ is a smooth map, then

$$P(f^*\nabla) = f^*P(\nabla).$$

This is a direct consequence of the definition of $f^*\nabla$: if $(U_\alpha, \varphi_\alpha)$ is a local trivialization of E, then it follows from (1.1.5) and (1.1.8) that

$$[F_{f^*\nabla}]_\alpha = f^*[F_\nabla]_\alpha,$$

where $[F_{f^*\nabla}]_\alpha$ denotes the local description of $F_{f^*\nabla}$ with respect to the induced local trivialization.

It turns out that $P(F_\nabla)$ in fact defines a cohomology class on M which is independent of the initially choosen connection ∇ on E. This is the content of the next result, which is the core of Chern-Weil theory [Mil74, pp. 296-298]:

Proposition 1.3.1 (Fundamental Lemma of Chern-Weil theory)**.** *Let P be a homogeneous $\mathrm{GL}(r,\mathbb{K})-$invariant polynomial of degree m, and ∇ a connection on a $\mathbb{K}-$vector bundle $E \to M$ of rank r. Then:*

(i) $P(F_\nabla)$ is a closed *$2m-$form; therefore defines an element $[P(F_\nabla)] \in H^{2m}_{dR}(M,\mathbb{K})$;*

(ii) The cohomology class $[P(F_\nabla)]$ is independent of the choosen connection *∇, i.e. if ∇_0 and ∇_1 are connections on E then $P(F_{\nabla_0}) - P(F_{\nabla_1})$ is an* exact *form.*

Proof. (i) Let $P'(M) : \mathfrak{gl}(r,\mathbb{K}) \to \mathbb{K}$ be the derivative of P at an element $M \in \mathfrak{gl}(r,\mathbb{K})$. If $X \in \mathfrak{gl}(r,\mathbb{K})$ and $g :]-\varepsilon,\varepsilon[\to \mathrm{GL}(r,\mathbb{K})$ is given by $g(t) = e^{tX}$, then

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}P(gMg^{-1})|_{t=0} \quad \text{(by the invariance of } P) \\ &= P'(M)(XM - MX), \end{aligned} \tag{1.3.1}$$

where in the last equation we used the fact that $(gMg^{-1})'(0) = XM - MX$. Now take $M = [F_\nabla]_\alpha \equiv F_\alpha$ and $X = A_\alpha$, where $\nabla|_{U_\alpha} = \mathrm{d} + A_\alpha$ on the local trivialization φ_α. Then, operating with $\wedge$ in place of the usual multiplication, equation (1.3.1) reads:

$$P'(F_\alpha) \wedge [A_\alpha, F_\alpha] = 0. \tag{1.3.2}$$

On the other hand, the chain rule gives

$$\mathrm{d}(P(F_\alpha)) = P'(F_\alpha) \wedge \mathrm{d}F_\alpha. \tag{1.3.3}$$

Now, the Bianchi identity (1.1.16) says that $\mathrm{d}F_\alpha + [A_\alpha, F_\alpha] = 0$; therefore,

$$\mathrm{d}(P(F_\alpha)) = -P'(F_\alpha) \wedge [A_\alpha, F_\alpha] = 0,$$

as we claimed.

(ii) Suppose ∇_0 and ∇_1 are two different connections on E. Write $p : M \times \mathbb{R} \to M$ for the canonical projection. Consider the induced connections $\nabla'_l := p^*\nabla_l$, $l = 0, 1$, on $p^*E \to M$ and the convex combination connection

$$\nabla := t\nabla'_1 + (1-t)\nabla'_0,$$

where $t : M \times \mathbb{R} \to \mathbb{R}$ is the natural projection. Note that if $i_l : M \to M \times \mathbb{R}$ denotes the function $x \mapsto (x, l)$, $l = 0, 1$, then we can identify $i_l^*\nabla$ with ∇_l, $l = 0, 1$, as connections on E. Being a smooth map, it follows that

$$i_l^*(P(F_\nabla)) = P(F_{\nabla_l}).$$

Now, the maps i_0 and i_1 are clearly homotopic, so they induce the same map in cohomology. In particular,

$$P(F_{\nabla_0}) = i_0^*(P(F_\nabla)) = i_1^*(P(F_\nabla)) = P(F(\nabla_1)).$$

■

In summary, each invariant homogeneous polynomial P on $\mathfrak{gl}(r, \mathbb{K})$ determines a *characteristic cohomology class* $c_P(E) = [P(F_\nabla)]$ in $H^*_{dR}(M, \mathbb{K})$ depending only on the isomorphism class of the vector bundle E, and such that if $f : M' \to M$ is smooth then $c_P(f^*E) = f^*c_P(E)$, where the left hand-side represents the cohomology class of the pull-back bundle f^*E and the right hand-side is the image of the cohomology class associated to E under the pull-back map induced by f in cohomology.

Chern classes. Let $\mathbb{K} = \mathbb{C}$. For each $X \in \mathfrak{gl}(r, \mathbb{C})$ and $1 \leq k \leq r$, write $\sigma_k(X)$ for the $k-$th elementary symmetric polyomial function on the eigenvalues of X, so that

$$\det(\mathbb{1} + tX) = 1 + t\sigma_1(X) + \ldots + t^r\sigma_r(X).$$

More explicitly, if $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$ are the eigenvalues of $X \in \mathfrak{gl}(r, \mathbb{C})$, then

$$\sigma_k(X) = \sum_{1 \leq i_1 < \ldots < i_k \leq r} \lambda_{i_1} \ldots \lambda_{i_k},$$

for each $1 \leq k \leq r$. In particular, $\sigma_1(X) = \mathrm{tr}\, X$ and $\sigma_r(X) = \det X$.

It is well known that every symmetric polynomial $P : \mathfrak{gl}(r, \mathbb{C}) \to \mathbb{C}$ has a unique representation as a polynomial in these elementary functions $\sigma_1, \ldots, \sigma_r$. From this, one can prove [Mil74, p. 299, Lemma 6]:

Proposition 1.3.2. *The ring of* $\mathrm{GL}(r, \mathbb{C})-$*invariant polynomials is* $\mathbb{C}[\sigma_1, \ldots, \sigma_r]$*, i.e. every invariant polynomial on* $\mathfrak{gl}(r, \mathbb{C})$ *can be expressed as a polynomial function of* $\sigma_1, \ldots, \sigma_r$*.*

Definition 1.3.3 (Chern classes and character in de Rham cohomology)**.** Let $E \to M$ be a complex vector bundle of rank r and let ∇ be a smooth connection on E. For $1 \leq k \leq r$,

1. the k**-th Chern class** of E is the element:
$$c_k(E) := \left(\frac{-1}{2\pi\mathbf{i}}\right)^k [\sigma_k(F_\nabla)] \in H^{2k}_{dR}(M, \mathbb{C}).$$
2. the k**-th Chern character** of E is the element:
$$\mathrm{ch}_k(E) := \frac{(-1)^k}{(2\pi\mathbf{i})^k k!} [\mathrm{tr}(F_\nabla \wedge \ldots \wedge F_\nabla)] \in H^{2k}_{dR}(M, \mathbb{C}).$$

For example, the first two Chern classes are given by:
$$c_1(E) = \frac{\mathbf{i}}{2\pi}[\mathrm{tr}(F_\nabla)] \tag{1.3.4}$$
and
$$c_2(E) = \frac{-1}{8\pi^2}[\mathrm{tr}(F_\nabla) \wedge \mathrm{tr}(F_\nabla) - \mathrm{tr}(F_\nabla \wedge F_\nabla)]. \tag{1.3.5}$$

1.4 Yang-Mills equation on Riemannian manifolds

In this section we review the variational formulation of the (weak/strong) Yang-Mills equation on a Riemannian $n-$manifold by means of the Yang-Mills energy functional, and point out some of its basic symmetries. The references for this section are [Weh04, Appendix A, pp. 172-173, and Chapter 9, pp. 141-142] and [Uhl82b, §1].

Yang-Mills functional. Let (M, g) be an oriented Riemannian $n-$manifold and let $E \to M$ be a $G-$bundle over M. Denote by $\langle \cdot, \cdot \rangle$ the natural tensor product metric on $\Lambda^2 T^*M \otimes \mathfrak{g}_E$ induced by g and the Ad_G-invariant inner product (1.1.1) on $\mathfrak{g}$. Then, for each $\xi, \zeta \in \Omega^2(M, \mathfrak{g}_E)$, we have
$$\langle \xi, \zeta \rangle \mathrm{d}V_g = \langle \xi \wedge *\zeta \rangle_{\mathfrak{g}} = -\mathrm{tr}(\xi \wedge *\zeta),$$
where $\langle \xi \wedge *\zeta \rangle_{\mathfrak{g}}$ represents the contraction of $\xi \wedge *\zeta$ by the induced invariant metric on $\mathfrak{g}_E$.

If $|\cdot|$ denotes the induced pointwise norm on sections of $\Lambda^2 T^*M \otimes \mathfrak{g}_E$, then for each $\nabla \in \mathfrak{U}(E)$ we get a function $|F_\nabla| : M \to \mathbb{R}$. By the Ad_G-invariance of $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ and the transformation behaviour (1.1.18), it follows that
$$|F_{g^*\nabla}| = |F_\nabla|, \quad \text{for each } g \in \mathcal{G}(E) \text{ and } \nabla \in \mathfrak{U}(E). \tag{1.4.1}$$
In other words, the function $\nabla \mapsto |F_\nabla|$ is invariant under the action of $\mathcal{G}(E)$ (*gauge invariant*).

Definition 1.4.1. The **Yang-Mills functional**

$$\mathcal{YM} : \mathfrak{U}(E) \to [0, \infty]$$

associates to each connection $\nabla \in \mathfrak{U}(E)$ its L^2-**energy**:

$$\mathcal{YM}(\nabla) := \|F_\nabla\|^2_{L^2} = \int_M |F_\nabla|^2 \mathrm{d}V_g = -\int_M \mathrm{tr}(F_\nabla \wedge *F_\nabla).$$

Of course, in general, the L^2-energy of a smooth connection ∇ may be infinite, i.e. F_∇ may not be L^2-integrable, if the base manifold M is not compact. Moreover, in such case there also are no natural Sobolev spaces of connections. On the other hand, if the base manifold M is compact, then $\mathcal{YM}$ is clearly finite on the whole space of smooth connections $\mathfrak{U}(E)$. Moreover, one can prove that in such case $\mathcal{YM}$ extends to $\mathfrak{U}^{1,p}(E)$, for each $2 \leq p < \infty$ such that[16] $p \geq \frac{4n}{4+n}$.

It follows directly from (1.4.1) that $\mathcal{YM}$ is a gauge invariant functional on $\mathfrak{U}(E)$, i.e.

$$\mathcal{YM}(g^*\nabla) = \mathcal{YM}(\nabla), \quad \text{for each } g \in \mathcal{G}(E) \text{ and } \nabla \in \mathfrak{U}(E). \tag{1.4.2}$$

Moreover, we note that $\mathcal{YM}$ is *conformally invariant* if, and only if, $n = 4$. Indeed, if we scale g by some positive smooth function f on M, then the pointwise inner product on $2-$forms scales by f^{-2}, while the Riemannian volume $n-$form scales by $f^{n/2}$. Thus, an integral $\int_M |F_\nabla|^2 \mathrm{d}V_g$ transforms to $\int_M f^{\frac{n}{2}-2} |F_\nabla|^2 \mathrm{d}V_g$, so that $\mathcal{YM}$ stays invariant precisely when $n = 4$.

Yang-Mills equation. The following proposition gives the first variational formula of $\mathcal{YM}$ with respect to compactly supported variations.

Proposition 1.4.2. *Let $\nabla \in \mathfrak{U}(E)$ with $\mathcal{YM}(\nabla) < \infty$. If $\{\nabla_t\}_{t \in]-\varepsilon,\varepsilon[}$ is a compactly supported smooth variation of ∇, then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{YM}(\nabla_t)|_{t=0} = 2 \left\langle \mathrm{d}^*_\nabla F_\nabla, B \right\rangle_{L^2},$$

where[17]

$$B := \frac{\mathrm{d}}{\mathrm{d}t} \nabla_t \Big|_{t=0} \in \Gamma_0(T^*M \otimes \mathfrak{g}_E).$$

In particular, ∇ is a critical point of $\mathcal{YM}$ with respect to compactly supported smooth variations if, and only if, ∇ satisfy the (strong) **Yang-Mills equation**[18]*:*

$$\mathrm{d}^*_\nabla F_\nabla = 0. \tag{1.4.3}$$

[16] The latter condition comes from the Sobolev embedding $W^{1,p} \hookrightarrow L^4$ (cf. Theorem B.2.10) which ensures $[A, A]$ and hence $F_\nabla = F_{\nabla_0} + \mathrm{d}_{\nabla_0} A + [A, A]$ lies in L^2, whenever $\nabla = \nabla_0 + A \in \mathfrak{U}^{1,p}(E)$.

[17] Here and in what follows, $\Gamma_0(\cdot)$ denotes the subset of $\Gamma(\cdot)$ consisting of all compactly supported elements.

[18] If M is a compact manifold with (possibly empty) boundary ∂M, the Yang-Mills equation becomes

Proof. Recall that a smooth variation of ∇ is just a smooth path $t \mapsto \nabla_t$ on $\mathfrak{U}(E)$ starting at $\nabla_0 = \nabla$. We say that a smooth variation $\{\nabla_t\}$ is compactly supported provided there exists a precompact open subset $U \Subset M$ such that, writing $\nabla_t = \nabla + A_t$, where $A_t \in \Omega^1(M, \mathfrak{g}_E)$, then each A_t has compact support contained in $\overline{U}$. The statement that ∇ is a critical point of $\mathcal{YM}$ with respect to compactly supported variations means simply that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{YM}(\nabla_t)|_{t=0} = 0$ for all such variations.

By the affine space structure of $\mathfrak{U}(E)$, we may restrict ourselves to variations of the form $\nabla_t = \nabla + tB$, where $B \in \Gamma_0(T^*M \otimes \mathfrak{g}_E)$. In this case, locally, we have:

$$\begin{aligned} [F_{\nabla_t}]_\alpha &= \mathrm{d}(A_\alpha + tB) + (A_\alpha + tB) \wedge (A_\alpha + tB) \\ &= F_\alpha + t(\mathrm{d}B + B \wedge A_\alpha + A_\alpha \wedge B) + t^2(B \wedge B) \\ &= [F_\nabla + t(\mathrm{d}_\nabla B) + t^2(B \wedge B)]_\alpha. \end{aligned}$$

Hence, globally,

$$F_{\nabla_t} = F_\nabla + t(\mathrm{d}_\nabla B) + t^2(B \wedge B).$$

In particular,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{YM}(\nabla_t)|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\langle F_{\nabla_t}, F_{\nabla_t}\rangle_{L^2}|_{t=0} \\ &= 2\left\langle \frac{\mathrm{d}}{\mathrm{d}t}F_{\nabla_t}|_{t=0}, F_\nabla \right\rangle_{L^2} \\ &= 2\langle \mathrm{d}_\nabla B, F_\nabla\rangle_{L^2} \\ &= 2\langle B, \mathrm{d}_\nabla^* F_\nabla\rangle_{L^2}, \end{aligned}$$

where in the second equality we use[19] $\mathcal{YM}(\nabla) < \infty$, and the last equality follows from the definition of d_∇^* as a formal L^2-adjoint of d_∇ (provided the compact support of B does not intersect the boundary of M, in case M has nonempty boundary). The result follows. ■

Remark 1.4.3. Since $\mathrm{d}_\nabla^* = \pm * \mathrm{d}_\nabla *$, the Yang-Mills equation can be rewritten as

$$\mathrm{d}_\nabla * F_\nabla = 0.$$

Also, note that this equation does *not* depend on the choice of orientation on M; indeed, the only thing in the equation depending on the choice of orientation is the $*-$operator, and this later only changes by a minus sign under a change of orientation, clearly not affecting the equation. ♢

the system:

$$\begin{cases} \mathrm{d}_\nabla^* F_\nabla = 0 \\ *F_\nabla|_{\partial M} = 0. \end{cases}$$

[19]Indeed, we need this to ensure that F_{∇_t} is L^2-integrable so we can apply the standard theorem of differentiation under the integral sign, e.g. as in [Fol13, Theorem 2.27].

Definition 1.4.4 (Yang-Mills connections)**.** A smooth connection $\nabla \in \mathfrak{U}(E)$ is called a **Yang-Mills connection** when ∇ satisfies the Yang-Mills equation (1.4.3). In this case, the associated curvature tensor of ∇ is called a **Yang-Mills field**.

Another important class of connections is given by the *weak* Yang-Mills connections, which are critical points of the Yang-Mills action on appropriate Sobolev spaces.

Definition 1.4.5 (Weak Yang-Mills connections)**.** Suppose M is compact. Let $1 \leq p < \infty$ be such that $p > \frac{n}{2}$, and in case $n = 2$ assume in addition $p \geq \frac{4}{3}$. A connection $\nabla \in \mathfrak{U}^{1,p}(E)$ is called a **weak Yang-Mills connection** when ∇ satisfies the **weak Yang-Mills equation**:

$$\int_M \langle F_\nabla, \mathrm{d}_\nabla B \rangle \mathrm{d}V_g = 0, \quad \forall B \in \Gamma_0(T^*M \otimes \mathfrak{g}_E). \tag{1.4.4}$$

Remark 1.4.6. The Yang-Mills functional need not be defined nor finite on weak Yang-Mills connections. The conditions imposed on p, coming from Sobolev embedding (Theorem B.2.10), are made to ensure the weak Yang-Mills equation makes sense for those connections and, at the same time, for the equation to be preserved by the action of $\mathcal{G}^{2,p}(E)$ (this later condition forces the strict inequality $p > \frac{n}{2}$). Moreover, it is not a priori clear if weak Yang-Mills connections satisfy the strong Yang-Mills equation, although this is true for sufficiently regular connections (see, for instance, [Weh04, Lemma 9.3, p. 142]). ♢

We note that both the weak and strong Yang-Mills equations are invariant under gauge transformations. For the weak equation this is more subtle and we refer the reader to [Weh04, Lemma 9.2, p. 142]. For the strong equation, in the light of Proposition 1.4.2, one can deduce this fact from the invariance (1.4.2) of the Yang-Mills functional on $\mathfrak{U}(E)$. Alternatively, one can check directly that, for each $\nabla \in \mathfrak{U}(E)$ and $g \in \mathcal{G}(E)$ we have

$$\mathrm{d}_{g^*\nabla} * F_{g^*\nabla} = g^{-1}(\mathrm{d}_\nabla * F_\nabla)g.$$

In particular, the solutions of the Yang-Mills equation, as either Yang-Mills connections or fields, are an invariant space under gauge transformations. This *gauge freedom* turns out to be the main difficulty in treating the regularity theory of these equations.

Gauge fixing. (cf. [Uhl82b, §1]) Lets take a closer look to the Yang-Mills equation (1.4.3) in a local gauge $(U_\alpha, \varphi_\alpha)$ of the $G-$bundle E. Suppose we have coordinates $(x^1, \ldots, x^n)$ on U_α; write $\nabla_\alpha = \mathrm{d} + A_\alpha$, $F_\alpha = \mathrm{d}A_\alpha + A_\alpha \wedge A_\alpha$ and recall the local expressions (1.1.10) and (1.1.11). For simplicity, assume also that $(g_{ij}) = (\delta_{ij})$, i.e. g is *flat* on U_α. Then,

$$\mathrm{d}^*_\nabla F_\nabla = \mathrm{d}^* F_\alpha - *[A_\alpha, *F_\alpha]$$

and

$$\mathrm{d}^* F_\alpha = -\sum_{i,j} \frac{\partial F_{\alpha,ij}}{\partial x^i} \otimes \mathrm{d}x^j.$$

Hence

$$\mathrm{d}^*_\nabla F_\nabla = -\sum_{i,j} \left(\frac{\partial F_{\alpha,ij}}{\partial x^i} + [A_{\alpha,i}, F_{\alpha,ij}] \right) \otimes \mathrm{d}x^j.$$

Therefore, on the gauge $(U_\alpha, \varphi_\alpha)$ the equation (1.4.3) reads

$$\sum_i \frac{\partial F_{\alpha,ij}}{\partial x^i} + [A_{\alpha,i}, F_{\alpha,ij}] = 0, \quad \forall j = 1, \ldots, n.$$

Of course, for a general metric g a similar (more complicated) equation holds. Now, if the gauge group G is an abelian Lie group (and therefore all brackets on $\mathfrak{g}$ are zero; in particular, $F_\alpha = \mathrm{d}A_\alpha$), the Bianchi identity and Yang-Mills equation for ∇_α reduces, respectively, to $\mathrm{d}F_\alpha = \mathrm{d}^2 A_\alpha = 0$ and $\mathrm{d}^* F_\alpha = 0$. The system $\mathrm{d}F_\alpha = 0$, $\mathrm{d}^* F_\alpha = 0$ then forms an **elliptic system** for F_α. This is the basic linear model for the regularity theory.

In the non-abelian case, a non-smooth gauge transformation g can turn a smooth field F_α into a discontinuous one $gF_\alpha g^{-1}$. Thus, the choice of a 'good' gauge is much more important to the non-linear theory. The linearized Yang-Mills equations written for A_α are $\mathrm{d}^*\mathrm{d}A_\alpha = 0$. By the last paragraph, this is exactly the Yang-Mills equation if G is abelian. This single system for A_α is *not* elliptic and, as in the Hodge theory for exact forms on manifolds, one usually adds a second equation such as $\mathrm{d}^* A_\alpha = 0$ to remedy the situation. For the abelian case, this involves solving the linear equation $\mathrm{d}^*(A_\alpha + \mathrm{d}u) = \mathrm{d}^*\tilde{A}_\alpha = 0$ for $u : U_\alpha \to \mathfrak{g}$. Here $\tilde{A}_\alpha \coloneqq g^* A_\alpha = g^{-1} A_\alpha g + g^{-1}\mathrm{d}g$, where $g = e^u \in C^\infty(U_\alpha, G) \simeq \mathcal{G}(E|_{U_\alpha})$.

The equation $\mathrm{d}^* A_\alpha = 0$ can also be added to the non-linear theory as a method of *choosing a good gauge*. In general, to find such a gauge we need to solve the non-linear elliptic equation: $\mathrm{d}^*(g^{-1} A_\alpha g + g^{-1}\mathrm{d}g) = 0$ for $g \in C^\infty(U_\alpha, G)$. Such a gauge is often called a *Coulomb gauge*. In the seminal works [Uhl82b, Theorems 2.7 and 2.8] and [Uhl82a, Theorem 1.3], K. Uhlenbeck solves the general problem of constructing Coulomb gauges over model domains of interest under, respectively, L^∞ and $L^{n/2}-$boundedness hypothesis on the curvature norm. In Section 3.1 of Chapter 3 we will see more about the latter (cf. Theorem 3.1.2).

One can also study the theory of the Yang-Mills equation on Lorentzian manifolds, as it originally come from Physics: a generalization of Maxwell's equations on the Minkowski space-time $\mathbb{R}^{3+1}$. The resulting equation is of weakly hyperbolic type and turns out to be very hard to study.

1.5 Instantons in four-dimensions

We now recall the familiar 4−dimensional notion of instanton and give some well known interpretations of this notion. The main references are [DK90, §2.1.3-2.1.5] and [FU84, pp. 36-37] (see also [Sco05, Chapter 9, pp. 351-354]).

Let (M, g) be an oriented Riemannian 4−manifold. A special feature of this setting is that the Hodge star operator on 2−forms,

$$* : \Lambda^2 T^*M \to \Lambda^2 T^*M,$$

is an involutive[20] self-adjoint[21] automorphism. Hence, $*|_{\Lambda^2 T^*M}$ has eigenvalues ± 1 and splitts $\Lambda^2 T^*M$ orthogonally into the corresponding eigenbundles $\Lambda^2_\pm T^*M$:

$$\Lambda^2 T^*M = \Lambda^2_+ T^*M \oplus \Lambda^2_- T^*M, \tag{1.5.1}$$

where $\Lambda^2_\pm T^*M := \{\omega \in \Lambda^2 T^*M : *\omega = \pm\omega\}$. Fiberwise, this phenomenon corresponds to the exceptional Lie algebra isomorphism

$$\mathfrak{so}(4) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3),$$

which at the level of Lie groups reads

$$\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2).$$

Indeed, as SO(4)−modules, one has $\Lambda^2(\mathbb{R}^4)^* \simeq \mathfrak{so}(4)$, and such isomorphism maps the $*$−eigenspaces Λ^2_+ and Λ^2_- onto the two 3−dimensional commuting ideals in $\mathfrak{so}(4)$, isomorphic to $\mathfrak{so}(3)$. We note that the SO(4)−modules Λ^2_+ and Λ^2_- are both irreducible and 3−dimensional, but not SO(4)−isomorphic (see [Bes08, §1.123-1.125, p. 50]).

Defining $\Omega^2_\pm(M) := \Gamma(\Lambda^2_\pm T^*M)$, we get an L^2−orthogonal decomposition of $\Omega^2(M)$ as

$$\Omega^2(M) = \Omega^2_+(M) \oplus \Omega^2_-(M).$$

Correspondingly, every $\omega \in \Omega^2(M)$ can be written as $\omega = \omega^+ \oplus \omega^-$, with

$$\omega^\pm := \frac{\omega \pm *\omega}{2} \in \Omega^2_\pm(M).$$

Definition 1.5.1. A 2−form $\omega \in \Omega^2(M)$ is called **anti-self-dual** (resp. self-dual) if $\omega^+ = 0$ (resp. $\omega^- = 0$).

---

[20] In general, $*^2 = (-1)^{k(4-k)}\mathbb{1}$ when acting on $\Lambda^k T^*M$ (see e.g. [Pet06, Lemma 26, p. 203]).

[21] with respect to the natural metric $(\cdot,\cdot)_g$ on $\Lambda^2 T^*M$ induced by g; here recall the defining property of $*$ as the unique bundle isomorphism $\Lambda^k T^*M \simeq \Lambda^{n-k} T^*M$ such that $\alpha \wedge *\beta = (\alpha, \beta)_g \mathrm{d}V_g$ for all $\alpha, \beta \in \Lambda^k T^*M$.

Terminology: From now on we will abbreviate (anti-)self-dual as (A)SD.

Remark 1.5.2. Note that a change of orientation on M changes the Hodge star operator $*$ by a sign and thus reverses the roles of $\Lambda^2_+T^*M$ and $\Lambda^2_-T^*M$. Moreover, as the action of the Hodge star on 2−forms of a 4−manifold is conformally invariant, the (A)SD condition is conformally invariant. ◊

Given a $G-$bundle E over M, the bundle splitting (1.5.1) immediately extends to $\mathfrak{g}_E-$valued 2−forms, resulting into the L^2-orthogonal decomposition

$$\Omega^2(M,\mathfrak{g}_E) = \Omega^2_+(M,\mathfrak{g}_E) \oplus \Omega^2_-(M,\mathfrak{g}_E),$$

where $\Omega^2_\pm(M,\mathfrak{g}_E) := \Gamma(\Lambda^2_\pm T^*M \otimes \mathfrak{g}_E)$. For $\nabla \in \mathfrak{U}(E)$, we write

$$F_\nabla = F^+_\nabla \oplus F^-_\nabla \in \Omega^2_+(M,\mathfrak{g}_E) \oplus \Omega^2_-(M,\mathfrak{g}_E).$$

This give rise to a very important class of solutions for the Yang-Mills equation in four dimensions.

Definition 1.5.3. Let (M,g) be an oriented Riemannian 4−manifold and let E be a $G-$bundle over M, where G is a compact semi-simple Lie group[22]. A smooth connection $\nabla \in \mathfrak{U}(E)$ is called an **ASD** (resp. SD) **instanton** when $F^+_\nabla = 0$ (resp. $F^-_\nabla = 0$).

A few observations are in order.

- The (A)SD equation $*F_\nabla = \pm F_\nabla$ is both *gauge invariant* and *conformally invariant*. For the gauge invariance, note that if $\nabla \in \mathfrak{U}(E)$ is an (A)SD instanton and $g \in \mathcal{G}(E)$, then

  $$*F_{g^*\nabla} = *(g^{-1}F_\nabla g) = g^{-1}(*F_\nabla)g = \pm F_{g^*\nabla}.$$

  As for the conformal invariance, it follows from the conformal invariance of the Hodge star $*$ on 2−forms in four dimensions.

- *Every (A)SD instanton is a Yang-Mills connection.* Indeed, if $\nabla \in \mathfrak{U}(E)$ then

  $$*F_\nabla = \pm F_\nabla \quad \Rightarrow \quad \mathrm{d}_\nabla(*F_\nabla) = \pm \mathrm{d}_\nabla F_\nabla = 0,$$

  by the Bianchi identity (1.1.16). Also notice that the (A)SD equation $F^\pm_\nabla = 0$ is a (nonlinear, unless G is abelian) *first-order* p.d.e. on the connection, while the Yang-Mills equation $\mathrm{d}^*_\nabla F_\nabla = 0$ is a (nonlinear, unless G is abelian) *second-order* p.d.e. on the connection. One moral is that (A)SD instantons provide a fertile source

[22]In Chapter 2 we will extend this definition allowing G to be any compact Lie group; see Definition 2.3.1 (ii) and the subsequent discussion.

of examples of Yang-Mills connections. Nonetheless, one can construct examples of Yang-Mills connections which are neither SD nor ASD. For instance, Sibner, Sibner and Uhlenbeck [SSU89] were the first to give such examples when $M = S^4$ and $G = \mathrm{SU}(2)$; two years later, Sadun and Segert [SS91] showed that non-self-dual Yang-Mills connections exist on all $\mathrm{SU}(2)-$bundles over S^4 with second Chern number[23] not equal to ± 1. See also [Wan91] for examples on $M = S^3 \times S^1$ (and $S^2 \times S^2$) with group $G = \mathrm{SU}(2)$.

Topological energy bounds from Chern-Weil theory. Suppose M is a closed oriented Riemannian 4−manifold and let $E \to M$ be an $\mathrm{SU}(r)-$bundle over M. In what follows, we will show that $\mathcal{YM} : \mathfrak{U}(E) \to \mathbb{R}$ is bounded below by a number depending only on the topology of E. Furthermore, the sign of such lower bound obstructs the existence of either SD or ASD instantons on E, which are shown to be the *absolute minima* of $\mathcal{YM}$.

Given $\nabla \in \mathfrak{U}(E)$, by the basic Chern-Weil theory developed in Section 1.3, we know that the topological characteristic class $c_2(E)$ is represented by

$$\begin{aligned} c_2(E) &= -\frac{1}{8\pi^2}[\mathrm{tr}(F_\nabla) \wedge \mathrm{tr}(F_\nabla) - \mathrm{tr}(F_\nabla \wedge F_\nabla)] \\ &= \frac{1}{8\pi^2}[\mathrm{tr}(F_\nabla \wedge F_\nabla)]. \quad (\text{since } F_\nabla \in \Omega^2(M, \mathfrak{su}_E) \text{ is trace-free}) \end{aligned}$$

Define the **topological charge** $\kappa(E)$ of E by pairing $c_2(E)$ with the fundamental class $[M]$:

$$\kappa(E) := \langle c_2(E), [M] \rangle = \frac{1}{8\pi^2} \int_M \mathrm{tr}(F_\nabla \wedge F_\nabla).$$

Write

$$F_\nabla = F_\nabla^+ \oplus F_\nabla^- \in \Omega^2_+(M, \mathfrak{g}_E) \oplus \Omega^2_-(M, \mathfrak{g}_E).$$

As this decomposition is L^2-orthogonal, it folows that

$$\begin{aligned} 8\pi^2\kappa(E) &= -\langle F_\nabla, *F_\nabla \rangle_{L^2} \\ &= -\langle F_\nabla^+ + F_\nabla^-, F_\nabla^+ - F_\nabla^- \rangle_{L^2} \\ &= -\|F_\nabla^+\|^2_{L^2} + \|F_\nabla^-\|^2_{L^2}. \end{aligned}$$

On the other hand,

$$\mathcal{YM}(\nabla) = \|F_\nabla\|^2_{L^2} = \|F_\nabla^+\|^2_{L^2} + \|F_\nabla^-\|^2_{L^2}.$$

Thus we get two identities:

$$\mathcal{YM}(\nabla) = 2\|F_\nabla^\pm\|^2_{L^2} \pm 8\pi^2\kappa(E).$$

In particular, $\mathcal{YM}(\nabla) \geq 8\pi^2|\kappa(E)|$. Thus:

[23] The second Chern number of a complex vector bundle E over an oriented compact 4−manifold M is the integer $C_2(E)$ given by the natural pairing $\langle c_2(E), [M] \rangle$.

- if $\kappa(E) = 0$ then the absolute minima for $\mathcal{YM}$ are precisely the (A)SD *flat* connections;
- if $\kappa(E) > 0$ then E does not admit SD instantons and
  ∇ is an **absolute minima** for $\mathcal{YM}$ $\iff$ $\mathcal{YM}(\nabla) = 8\pi^2\kappa(E)$ $\iff$ ∇ is an **ASD** instanton;
- if $\kappa(E) < 0$ then E does not admit ASD instantons and
  ∇ is an absolute minima for $\mathcal{YM}$ $\iff$ $\mathcal{YM}(\nabla) = -8\pi^2\kappa(E)$ $\iff$ ∇ is an SD instanton.

ASD instantons on $\mathbb{R}^4$. (cf. [Jar05, §2.3],[Mar11, §3.2],[Nab11, §6.3]) Consider the Euclidean space $M = \mathbb{R}^4$, with the standard structure of oriented Riemannian manifold, and let E be a (necessarily) trivial $G-$bundle over M, where G is a compact semi-simple Lie group. Then, letting $x^1, \dots, x^n$ denote Euclidean coordinates, any connection $\nabla \in \mathfrak{U}(E)$ can be written globally as $\nabla = \mathrm{d} + A$, for some

$$A = \sum_{i=1}^{4} A_i \otimes \mathrm{d}x^i, \quad A_i : \mathbb{R}^4 \to \mathfrak{g}.$$

Furthermore, we can write (cf. 1.1.12)

$$F_\nabla = \frac{1}{2} \sum F_{ij} \otimes \mathrm{d}x^i \wedge \mathrm{d}x^j,$$

with

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j].$$

In this context, we have explicitly:

$$\nabla \text{ is an ASD instanton} \quad \iff \quad \begin{cases} F_{12} + F_{34} & = 0 \\ F_{13} + F_{42} & = 0 \\ F_{14} + F_{23} & = 0 \end{cases} . \tag{1.5.2}$$

The first non-trivial explicit examples of ASD instantons on $\mathbb{R}^4$, with finite L^2-energy and gauge group $G = \mathrm{SU}(2) \simeq \mathrm{Sp}(1)$, was given in the classical paper [BPST75]. The simplest solution, called the *basic instanton*, has the potential given by

$$A(x) := \frac{1}{|x|^2 + 1} \mathrm{Im}(q\mathrm{d}\overline{q}),$$

where q is the quaternion $x^1 + x^2\mathbf{i} + x^3\mathbf{j} + x^4\mathbf{k}$, while $\mathrm{Im}(q\mathrm{d}\overline{q})$ denotes the imaginary part of the product quaternion $q\mathrm{d}\overline{q}$. Here we are regarding $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as a basis of the Lie algebra $\mathfrak{su}(2) \simeq \mathrm{Im}(\mathbb{H})$. Computing the curvature of $\nabla = \mathrm{d} + A$, one gets

$$F_\nabla(x) := \frac{1}{(|x|^2 + 1)^2} \mathrm{d}q \wedge \mathrm{d}\overline{q}.$$

Note that the action density function

$$|F_\nabla|^2(x) = \frac{48}{(|x|^2+1)^4}$$

has a bell-shaped profile centered at the origin and decaying like r^{-8}. Furthermore, one can show that ∇ has *topological charge* 1, i.e.

$$\kappa(\nabla) := \frac{1}{8\pi^2}\int_M \mathrm{tr}(F_\nabla \wedge F_\nabla) = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \frac{48}{(|x|^2+1)^4}\mathrm{d}x = \frac{1}{8\pi^2}\mathrm{Vol}(S^3)\int_0^\infty \frac{48r^3}{(r^2+1)^4}\mathrm{d}r = 1.$$

More generally, given $x_0 \in \mathbb{R}^4$ and $\lambda \in \mathbb{R}_+$, if $t_{x_0,\lambda} : \mathbb{R}^4 \to \mathbb{R}^4$ denotes the isometry given by

$$t_{x_0,\lambda}(x) := \lambda^{-1}(x - x_0), \quad \forall x \in \mathbb{R}^4,$$

then the pull-back connection $\nabla_{x_0,\lambda} := t^*_{x_0,\lambda}\nabla$ is still an ASD instanton; more explicitly, letting x_0 correspond to the quaternion q_0, we can write

$$A_{x_0,\lambda}(x) = \frac{1}{|x-x_0|^2+\lambda^2}\mathrm{Im}\left((q-q_0)\mathrm{d}\overline{q}\right)$$

and

$$F_{\nabla_{x_0,\lambda}}(x) = \frac{\lambda^2}{(|x-x_0|^2+\lambda^2)^2}\mathrm{d}q \wedge \mathrm{d}\overline{q}.$$

The action density function

$$|F_{\nabla_{x_0,\lambda}}|^2(x) = \frac{48\lambda^4}{(|x-x_0|^2+\lambda^2)^4}$$

has a bell-shaped profile centered at x_0 and one still has

$$\kappa(\nabla_{x_0,\lambda}) = \frac{1}{8\pi^2}\mathcal{YM}(\nabla_{x_0,\lambda}) = \frac{1}{8\pi^2}\mathrm{Vol}(S^3)\int_0^\infty \frac{48\lambda^4 r^3}{(r^2+\lambda^2)^4}\mathrm{d}r = 1. \quad \text{(indep. of } x_0 \text{ and } \lambda)$$

On the other hand, for fixed x_0,

$$\sup_{x\in\mathbb{R}^4} |F_{\nabla_{x_0,\lambda}}|^2(x) = |F_{\nabla_{x_0,\lambda}}|^2(x_0) = \lambda^{-4}48 \xrightarrow{\lambda\downarrow 0} \infty.$$

Thus, as $\lambda \downarrow 0$, the action density function $|F_{\nabla_{x_0,\lambda}}|^2$ concentrates more and more at x_0. We shall refer to x_0 as the *center* and λ as the *scale* of the potential $A_{x_0,\lambda}$.

Instantons of topological charge k, also called *pseudoparticles*, can be obtained by “superimposing” k basic instantons, via the so-called *’t Hooft Ansatz*. Given $y_i \in \mathbb{R}^4$ and $\lambda_i \in \mathbb{R}_+$, $i = 1, \dots, k$, consider the positive harmonic function $\rho : \mathbb{R}^4 \to \mathbb{R}$ given by

$$\rho(x) := 1 + \sum_{i=1}^k \frac{\lambda_i^2}{|x-y_i|^2}.$$

Then the connection given by the potential $A = A_\mu \otimes \mathrm{d}x^\mu$, with

$$A_\mu := \mathbf{i}\sum_\nu \sigma_{\mu\nu}\frac{\partial}{\partial x^\nu}\ln(\rho),$$

is an ASD instanton; here, $\sigma_{\mu\nu}$, $\mu, \nu = 1, 2, 3, 4$, are the skew-symmetric matrices given by:

$$\sigma_{jl} \coloneqq \frac{1}{4\mathbf{i}}[\sigma_j, \sigma_l], \quad \sigma_{j4} \coloneqq \frac{1}{2}\sigma_j, \quad j, l = 1, 2, 3,$$

where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices. This is physically interpreted as a configuration with k instantons, where λ_i are constants that corresponds to the *size* of the instanton at the point y_i.

In a certain sense, $\mathrm{SU}(2)-$instantons are also the 'building blocks' for instantons with general structure group. More precisely, let G be a compact semi-simple Lie group, and let $\rho : \mathfrak{su}(2) \to \mathfrak{g}$ be any injective Lie algebra homomorphism. Then, for example,

$$\rho(A_{0,1}) = \frac{1}{|x|^2 + 1}\rho(\mathrm{Im}(q\mathrm{d}\overline{q}))$$

indeed defines a $G-$instanton on $\mathbb{R}^4$. While this guarantees the existence of $G-$instantons on the Euclidean space $\mathbb{R}^4$, observe that this instanton might be reducible (e.g. ρ can simply be the obvious inclusion of $\mathfrak{su}(2)$ into $\mathfrak{su}(r)$ for some $r \geq 3$) and that its charge depends on the choice of representation ρ. Furthermore, it is not clear whether every $G-$instanton can be obtained in this way.

Remark 1.5.4 (ADHM construction)**.** For each $k \in \mathbb{N}$, the so-called 'ADHM construction', due to Atiyah et al. [AHDM78], gives a correspondence between gauge-equivalence classes of ASD instantons ∇ with group $\mathrm{SU}(r)$ and fixed topological charge $\kappa(\nabla) = k$, and equivalence classes of certain systems of finite-dimensional algebraic data, for group $\mathrm{SU}(r)$ and index k [DK90, §3.3]. This gives a complete description of finite-energy ASD instantons over $\mathbb{R}^4$ with gauge group $\mathrm{SU}(r)$. ♢

Holomorphic structures and connections. In this paragraph we recall very briefly the Nirenberg-Newlander integrability theorem relating holomorphic structures and certain types of connections in the context of complex vector bundles over complex manifolds. In particular, this will serve as background material for the final paragraph §1.5 on ASD instantons and holomorphic structures.

Notation: We adopt the following notations in this paragraph and the next:

- Z: complex manifold of complex dimension m, i.e. a smooth $2m-$manifold endowed with an integrable almost complex structure J;
- $E \to Z$: (smooth) *complex* vector bundle over Z;
- $\Omega^{p,q}(Z, E) \coloneqq \Gamma(\Lambda^{p,q}T^*Z_{\mathbb{C}} \otimes E)$: $(p, q)-$forms on Z with values on E;
- $\Omega^k(Z, E) = \bigoplus_{p+q=k} \Omega^{p,q}(Z, E)$: complex $k-$forms on Z with values on E.

Definition 1.5.5. A **holomorphic structure** $\mathcal{E}$ on a complex vector bundle $\pi : E \to Z$ is an additional complex manifold structure on the total space E in such a way that π is a holomorphic map and the bundle admits an atlas of biholomorphic trivializations. We call E endowed with the holomorphic structure $\mathcal{E}$ a **holomorphic vector bundle** and denote it by $\mathcal{E} \to Z$.

Alternatively, a holomorphic vector bundle $\mathcal{E} \to Z$ is a complex vector bundle $E \to Z$ determined by a $\mathrm{GL}(r, \mathbb{C})-$cocycle $\{g_{\alpha\beta}\}$ of *holomorphic* transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{C})$.

Given a holomorphic structure $\mathcal{E}$ on $E \to Z$, we can associate a unique $\mathbb{C}-$linear operator

$$\overline{\partial}_{\mathcal{E}} : \Omega^0(Z, E) \to \Omega^{0,1}(Z, E)$$

such that, for each $f \in C^\infty(Z, \mathbb{C})$ and $s \in \Gamma(E)$, we have

(i) $\overline{\partial}_{\mathcal{E}}(fs) = (\overline{\partial} f) \otimes s + f(\overline{\partial}_{\mathcal{E}} s)$;

(ii) If $U \subseteq Z$ is an open subset, then $\left(\overline{\partial}_{\mathcal{E}} s\right)|_U = 0$ if, and only if, s is a holomorphic map over U.

The construction of $\overline{\partial}_{\mathcal{E}}$ is as follows. By hypothesis, E admits an atlas of local frames $\{(e_{\alpha,1}, \ldots, e_{\alpha,r})\}_\alpha$ whose associated transition functions $\{g_{\alpha\beta}\}$ are *holomorphic* maps. Given $s \in \Gamma(E)$, write

$$s|_{U_\alpha} = \sum_i s^i_\alpha \otimes e_{\alpha,i}, \quad s^i_\alpha \in C^\infty(U_\alpha, \mathbb{C}), i = 1, \ldots, k.$$

and define

$$\left(\overline{\partial}_{\mathcal{E}} s\right)|_{U_\alpha} := \sum_i (\overline{\partial} s^i_\alpha) \otimes e_{\alpha,i}. \tag{1.5.3}$$

This operator clearly satisfies properties (i) and (ii). To see it is well-defined it suffices to note that

$$\overline{\partial}(gv) = (\overline{\partial} g)v + g(\overline{\partial} v) = g(\overline{\partial} v),$$

whenever g is a *holomorphic* change of coordinates and v is the local representation of a section of E.

Of course, such operator $\overline{\partial}_{\mathcal{E}}$ can be extended to give rise to $\mathbb{C}-$linear operators

$$\overline{\partial}_{\mathcal{E}} : \Omega^{p,q}(Z, E) \to \Omega^{p,q+1}(Z, E), \quad \text{for all } p, q \geq 0,$$

such that

$$\overline{\partial}_{\mathcal{E}}(\omega \wedge s) = (\overline{\partial}\omega) \otimes s + (-1)^{p+q}\omega \wedge (\overline{\partial}_{\mathcal{E}} s),$$

whenever $\omega \in \Omega^{p,q}(Z)$ and $s \in \Omega^0(Z, E)$. Since Z is a complex manifold (therefore $\overline{\partial}^2 = 0$), it follows from the definition of $\overline{\partial}_{\mathcal{E}}$ (1.5.3) that $\overline{\partial}^2_{\mathcal{E}} := \overline{\partial}_{\mathcal{E}} \circ \overline{\partial}_{\mathcal{E}} = 0$.

Now let ∇ be a (smooth) connection on the complex vector bundle $E \to Z$. Here we regard ∇ as a map from $\Gamma(E) = \Omega^0(Z, E)$ to $\Omega^1(Z, E)$ (see Remark 1.1.3). Then, the bi-degree splitting of $\Omega^1(Z, E)$ induces a corresponding splitting of ∇ as

$$\nabla = \partial_\nabla \oplus \overline{\partial}_\nabla : \Omega^0(Z, E) \to \Omega^{1,0}(Z, E) \oplus \Omega^{0,1}(Z, E).$$

By the Leibniz rule, the $\mathbb{C}-$linear operator $\overline{\partial}_\nabla : \Omega^0(Z, E) \to \Omega^{0,1}(Z, E)$ automatically satisfies (i):

$$\overline{\partial}_\nabla(fs) = \overline{\partial} f \otimes s + f \overline{\partial}_\nabla s,$$

for each $f \in C^\infty(Z, \mathbb{C})$ and $s \in \Gamma(E)$.

More generally, we introduce the following terminology.

Definition 1.5.6. A $\mathbb{C}-$linear operator $\overline{\partial}_E : \Omega^0(Z, E) \longrightarrow \Omega^{0,1}(Z, E)$ is called a **partial connection** on E if it satisfies the '$\overline{\partial}-$Leibniz rule':

$$\overline{\partial}_E(fs) = \overline{\partial} f \otimes s + f \overline{\partial}_E s,$$

for each $f \in C^\infty(Z, \mathbb{C})$ and $s \in \Gamma(E)$.

Given a holomorphic structure $\mathcal{E}$ on E, it is clear that the induced operator $\overline{\partial}_{\mathcal{E}}$ is a partial connection. The non-trivial question is when a given partial connection $\overline{\partial}_E$ comes from a holomorphic structure $\mathcal{E}$ on E, in the following sense:

Definition 1.5.7 (Integrability)**.** A partial connection $\overline{\partial}_E$ on E is called **integrable** if it equals the partial connection $\overline{\partial}_{\mathcal{E}}$ induced by a holomorphic structure $\mathcal{E}$ on E.

In these terms, we can state the following deep general result (for a proof see [DK90, § 2.2.2]):

Theorem 1.5.8 (Nirenberg-Newlander)**.** *If $\overline{\partial}_E$ is a partial connection on E, then*

$$\overline{\partial}_E \text{ is integrable} \quad \Longleftrightarrow \quad \overline{\partial}_E^2 = 0.$$

Now note that if $\nabla = \partial_\nabla \oplus \overline{\partial}_\nabla$ is a connection on $E \to Z$, then

$$F_\nabla = \partial_\nabla^2 \oplus (\partial_\nabla \overline{\partial}_\nabla + \overline{\partial}_\nabla \partial_\nabla) \oplus \overline{\partial}_\nabla^2.$$

In particular,

$$F_\nabla^{0,2} = 0 \quad \Longleftrightarrow \quad \overline{\partial}_\nabla^2 = 0.$$

Definition 1.5.9 (Compatibility)**.** A connection ∇ on $E \to Z$ is said to be **compatible** with a holomorphic structure $\mathcal{E}$ on E when $\overline{\partial}_\nabla$ is an integrable partial connection with $\overline{\partial}_\nabla = \overline{\partial}_{\mathcal{E}}$.

In conclusion, Theorem 1.5.8 implies the following relation between holomorphic structures and connections:

Corollary 1.5.10. *A connection ∇ on $E \to Z$ is compatible with a holomorphic structure $\mathcal{E}$ on E if, and only if, $F_\nabla^{0,2} = 0$.*

If, furthermore, $E \to Z$ is a *Hermitian* vector bundle and ∇ is a $\mathrm{U}(r)-$connection (*unitary* connection) on E, then ∇ is compatible with a holomorphic structure on E (i.e. $F_\nabla^{0,2} = 0$) if, and only if, $F_\nabla \in \Omega^{1,1}(Z, E)$. Indeed, if ∇ is unitary then $F_\nabla \in \Omega^2(Z, \mathfrak{u}(r)_E)$, hence

$$F_\nabla^{0,2} = -(F_\nabla^{2,0})^*.$$

On Hermitian bundles, a holomorphic structure distinguishes a unique compatible $\mathrm{U}(r)-$connection [DK90, Lemma 2.1.54]:

Proposition 1.5.11. *Suppose E is a $\mathrm{U}(r)-$bundle over Z. Then, a holomorphic structure $\mathcal{E}$ on E induces a unique compatible connection $\nabla \in \mathfrak{U}(E)$; such connection is called the* **Chern connection** *of the holomorphic $\mathrm{U}(r)-$bundle $\mathcal{E} \to Z$.*

ASD instantons and holomorphic structures. To end this chapter, we now recall an important interpretation of the ASD instanton equation in the context of $\mathrm{SU}(r)-$bundles over complex Hermitian surfaces. The references are [DK90, pp. 46-47] and [Sco05, pp. 369-370].

Let Z be a **Hermitian surface**, i.e. a (smooth) $4-$manifold endowed with an integrable complex structure I and a Riemannian metric g with respect to which I is an orthogonal transformation. In particular, Z is a Riemannian $4-$manifold with a preferable orientation fixed by I.

In this context, we have two decompositions of the complexified $2-$forms of Z:

$$\Omega^2(Z) = \Omega^{2,0} \oplus \Omega^{1,1} \oplus \Omega^{0,2} \quad \text{and} \quad \Omega^2(Z) = \Omega^2_+ \oplus \Omega^2_-.$$

Denote by ω the fundamental 2-form of $(1,1)-$type induced by the pair (g, I):

$$\omega(X, Y) := g(IX, Y), \quad \forall X, Y \in \mathfrak{X}(Z).$$

Then ω induces a decomposition of $\Omega^{1,1}(Z)$ as $\Omega^{1,1}(Z) = \Omega_0^{1,1} \oplus \Omega^0 \cdot \omega$, where $\Omega_0^{1,1} := (\Omega^0 \cdot \omega)^\perp \cap \Omega^{1,1}$.

By a straightforward local computation, the relation between the above decompositions is given by [DK90, Lemma 2.1.57]:

Proposition 1.5.12. *Let Z be a Hermitian complex surface as above. Then:*

- $\Omega^2_+ = \Omega^{2,0} \oplus \Omega^0 \cdot \omega \oplus \Omega^{0,2}$.

- $\Omega^2_- = \Omega^{1,1}_0$.

Therefore, we get [DK90, Proposition 2.1.59]:

Theorem 1.5.13. *Let $E \to Z$ be an* $\mathrm{SU}(r)-$*bundle over a Hermitian surface Z. If $\nabla \in \mathfrak{U}(E)$, then*

$$\nabla \text{ is an ASD instanton} \quad \Longleftrightarrow \quad \begin{cases} F^{0,2}_\nabla = 0 \text{ (integrability condition)} \\ \hat{F}_\nabla := F_\nabla \cdot \omega = 0 \end{cases}$$

Combining this result with the discussion of the last paragraph, the conclusion is that, in complex geometry, the ASD instanton condition splits naturally into two pieces, one of which has a simple geometric interpretation as an integrability condition. In particular, this suggests that ASD instantons, rather then SD instantons, are preferable in this setting. This is one of the reasons one chooses to work with ASD instantons rather then SD instantons when doing gauge theory, even in the general context of oriented Riemannian $4-$manifolds. From now on we also follow this convention.

Chapter 2

Instantons in higher dimensions

In the presence of appropriate geometric structures on the base manifold M^n, the familiar 4−dimensional notion of instanton (cf. Section 1.5) can be generalized to the higher dimensional context $n > 4$. In this chapter we present two approaches for such generalization. In chronological order, the first approach, first explored by physicists [CDFN83, BKS98], is based on the presence of an appropriate $(n-4)-$form on M. As for the second approach, originally introduced by Reyés Carrion in [Car98], one needs M to be equiped with an $N(H)-$structure, for some closed Lie subgroup $H \subseteq \mathrm{SO}(n)$, where $N(H)$ denotes the normalizer of H in $\mathrm{SO}(n)$. These two points of view turns out to coincide in cases of interest, namely special holonomy manifolds, and were further popularized by the works of Donaldson and Thomas [DT98], Tian [Tia00] et al.

We start this chapter with a discussion of Berger's classification theorem of Riemannian holonomy groups (Section 2.1). In particular, we give short descriptions of the special geometries associated to the holonomy groups $\mathrm{U}(m)$ (Kähler), $\mathrm{SU}(m)$ (Calabi-Yau), G_2 and $\mathrm{Spin}(7)$. Next, in Section 2.2, we introduce the language of calibrated geometry and its relations with special holonomy manifolds. Then, following the previously mentioned approaches, in Section 2.3 we explain generalizations of the notion of instanton for oriented Riemannian $n-$manifolds, $n \geq 4$, endowed with appropriate geometric structure. In fact, we will be interested in the cases where the holonomy group of g is realized as a normalizer $N(H) \subsetneq \mathrm{SO}(n)$ appearing in Berger's list of special geometries. We pay particular attention to the corresponding notions of instantons associated to the holonomy reductions $\mathrm{SU}(m) = N(\mathrm{U}(m)) \subseteq \mathrm{SO}(2m)$, $\mathrm{G}_2 = N(\mathrm{G}_2) \subseteq \mathrm{SO}(7)$ and $\mathrm{Spin}(7) = N(\mathrm{Spin}(7)) \subseteq \mathrm{SO}(8)$, with emphasis on the last two 'exceptional' cases.

2.1 Riemannian metrics with special holonomy groups

The main references for this section are [Joy07, Joy04] and [Bry86].

Riemannian holonomy groups and Berger's classification. Let (M, g) be a Riemannian $n-$manifold and consider D^g the associated Levi-Civita connection on the real $\mathrm{O}(n)-$bundle $TM \to M$. Recall that D^g is uniquely determined by the following properties (cf. [Joy03, p. 40, Theorem 3.1.1]):

(i) D^g is *torsion-free*, i.e. $D^g_X Y - D^g_Y X = [X, Y]$ for all $X, Y \in \mathfrak{X}(M)$;

(ii) D^g is compatible with g, i.e. $D^g g = 0$.

Write $\mathrm{Hol}_x(g)$ for the holonomy group $\mathrm{Hol}_x(D^g)$ of D^g at x (cf. Section 1.2). Since the subgroup of $\mathrm{GL}(T_xM)$ preserving $g|_{T_xM}$ is $\mathrm{O}(T_xM)$, the metric compatibility (ii) implies, via Theorem 1.2.3, that $\mathrm{Hol}_x(g) \subseteq \mathrm{O}(T_xM)$. In particular, we can regard $\mathrm{Hol}(g) := \mathrm{Hol}(D^g)$ as a subgroup of $O(n)$, well-defined up to conjugation in $O(n)$. By connectedness, the restricted holonomy group $\mathrm{Hol}^0(g) := \mathrm{Hol}^0(D^g)$ is a subgroup of $\mathrm{SO}(n)$, defined up to conjugation (by $\mathrm{O}(n)$), and the holonomy algebra $\mathfrak{hol}(g)$ is a Lie subalgebra of $\mathfrak{so}(n)$, defined up to the adjoint action (by $\mathrm{O}(n)$).

The Riemann curvature tensor $R^g := F_{D^g}$ of g has a number of symmetries besides the obvious skew-symmetry in its first two arguments. To express such symmetries it is convenient to lower the last index of R^g. We define

$$Rm^g(X, Y, Z, W) := g(R^g(X, Y)Z, W), \quad \forall X, Y, Z, W \in \mathfrak{X}(M).$$

We shall refer to both R^g and Rm^g as the *Riemann curvature* of g. In terms of components, with respect to any local frame, the tensor R^g is represented by R^i_{jkl} and the tensor Rm^g is represented by R_{ijkl}. Also, we denote the total covariant derivative $D^g Rm^g$ in components by $R_{ijkl;m}$. With these notations, the following result summarizes important symmetries of Rm^g and $D^g Rm^g$ [Joy07, Theorem 3.1.2].

Proposition 2.1.1. *Let (M, g) be a Riemannian manifold with Riemann curvature R_{ijkl}. Then:*

$$R_{ijkl} = -R_{ijlk} = -R_{jikl} = R_{klij}, \tag{2.1.1}$$

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0, \quad \textit{(algebraic/1}^{st}\textit{ Bianchi identity)} \tag{2.1.2}$$

and $$R_{ijkl;m} + R_{ijlm;k} + R_{ijmk;l} = 0. \quad \textit{(differential/2}^{nd}\textit{ Bianchi identity)} \tag{2.1.3}$$

Remark 2.1.2. The identity (2.1.3) is simply a rephrasing of the Bianchi identity (1.1.16) in this context. ♢

At each point $x \in M$, regarding $\mathfrak{hol}_x(g)$ as a subspace of the anti-symmetric endomorphisms $\mathfrak{so}(T_xM)$ of T_xM, it follows from Proposition 1.2.9 that R^i_{jkl} lies in $\Lambda^2 T^*_x M \otimes \mathfrak{hol}_x(g)$. By equation (2.1.1), we see that R_{ijkl} is an element of $\Lambda^2 T^*_x M \otimes \Lambda^2 T^*_x M$, so that identifying $\mathfrak{so}(T_xM)$ with $\Lambda^2 T_x^* M$ using g, we can also think of R_{ijkl} as an element of $\Lambda^2 T_x^* M \otimes \mathfrak{hol}_x(g)$. Furthermore, using the first Bianchi identity (2.1.2), we get [Joy07, Theorem 3.1.7, p. 43]:

Proposition 2.1.3. *Let (M, g) be a Riemannian manifold with Riemann curvature R_{ijkl}. Then R_{ijkl} lies in the subspace $S^2\mathfrak{hol}_x(g)$ of $\Lambda^2 T_x^* M \otimes \Lambda^2 T_x^* M$ at each point $x \in M$.*

Combining this theorem with the Bianchi identities of Proposition 2.1.1, gives quite strong restrictions on the curvature tensor of a Riemannian metric g with a prescribed holonomy group $\mathrm{Hol}(g)$ [Joy07, p. 43]. These restrictions, together with Ambrose-Singer theorem 1.2.10, are the basis of the (algebraic) classification of Riemannian holonomy groups.

A theorem due to de Rham (see [Joy07, Theorem 3.2.7, p. 47]) shows that if (M, g) is a complete, simply-connected Riemannian manifold, then there exist complete, simply-connected Riemannian manifolds (M_j, g_j) for $j = 1, \ldots, k$, such that the holonomy representation of $\mathrm{Hol}(g_j)$ is irreducible, (M, g) is isometric to the Riemannian product $(M_1 \times \ldots \times M_k, g_1 \times \ldots \times g_k)$, and $\mathrm{Hol}(g) = \mathrm{Hol}(g_1) \times \ldots \times \mathrm{Hol}(g_k)$. Thus, in looking for a classification of the possible holonomy groups of (M^n, g), we are mainly interested in the cases where $\mathrm{Hol}^0(g)$ acts irreducibly on $\mathbb{R}^n$.

In 1955, M. Berger [Ber55] gave a list of all the possible irreducible holonomy groups for Riemannian metrics. We state it here as follows (cf. [Bry86]):

Theorem 2.1.4 (Berger). *Let M be a connected, simply-connected*[1] *$n-$dimensional manifold, and let g be a Riemannian metric on M. Suppose that, for some $x \in M$, $\mathrm{Hol}_x(g)$ acts irreducibly on T_xM. Then either g is a locally symmetric metric or else one of the following holds:*

(i) $\mathrm{Hol}(g) = \mathrm{SO}(n)$,

(ii) $n = 2m$, $m \geq 2$ *and* $\mathrm{Hol}(g) = \mathrm{U}(m)$,

(iii) $n = 2m$, $m \geq 2$ *and* $\mathrm{Hol}(g) = \mathrm{SU}(m)$,

(iv) $n = 4m$, $m \geq 2$ *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)$,

(v) $n = 4m$, $m \geq 2$ *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$,

(vi) $n = 7$ *and* $\mathrm{Hol}(g) = \mathrm{G}_2$,

(vii) $n = 8$ *and* $\mathrm{Hol}(g) = \mathrm{Spin}(7)$.

[1] If $\pi(M) \neq 1$ then the universal cover $(\tilde{M}, \tilde{g})$ of (M, g) has $\mathrm{Hol}(\tilde{g}) = \mathrm{Hol}^0(g)$.

Remark 2.1.5. A Riemmanian metric g on M is called *locally symmetric* if every point $p \in M$ admits an open neighborhood U_p in M, and an involutive isometry $\sigma_p : U_p \to U_p$ with unique fixed point p. For more on this we refer the reader to [Joy07, §3.3]. $\lozenge$

Remark 2.1.6. Later, Simons [Sim62] gave another proof of Theorem 2.1.4. See also the more recent proof by Olmos [Olm05]. $\lozenge$

From now on we shall refer to the list of groups (i)-(vii) as *Berger's list*. A very thorough discussion of Berger's theorem, including discussions of each of the geometries associated to the groups in Berger's list, analogies with the four normed division algebras, and the principles behind Berger's original proof, can be found in Joyce's book [Joy07, §3.4].

It can be shown that the space of Riemannian metrics g on M^n for which $\mathrm{Hol}(g) = \mathrm{SO}(n)$ is both open and dense in the space of Riemannian metrics on M. Thus, one says that $\mathrm{SO}(n)$ is the holonomy group of a *generic* metric on M. The other groups on Berger's list are called **special holonomy groups**. In what follows, we give brief descriptions of metrics with these holonomy groups, except the cases (iv) and (v) which will not be fundamental for our later purposes.

Metrics with holonomy $\mathbf{U}(m) \subseteq \mathbf{O}(2m)$. (cf. [Bry86], [Sal89, Chapter 3] and [Joy07]) Let $\mathbb{C}^m$ have complex coordinates $(z^1, \ldots, z^m)$. The unitary group $\mathrm{U}(m)$ may be defined as the set of complex linear endomorphisms of $\mathbb{C}^m$ preserving the Hermitian form

$$\eta_0 = \sum_{j=1}^{m} \mathrm{d}z^j \otimes \mathrm{d}\overline{z^j}.$$

Defining real coordinates $(x^1, \ldots, x^{2m})$ on $\mathbb{C}^m \simeq \mathbb{R}^{2m}$ by $z^j = x^{2j-1} + \mathbf{i}x^{2j}$, $j = 1, \ldots, m$, we can write

$$\eta_0 = \sum_{j=1}^{m} \mathrm{d}x^j \otimes \mathrm{d}x^j - 2\mathbf{i} \sum_{j=1}^{m} \mathrm{d}x^{2j-1} \wedge \mathrm{d}x^{2j}.$$

The real part $g_0 = \mathrm{Re}(\eta_0)$ is the standard Euclidean inner product on $\mathbb{R}^{2m}$, so that $\mathrm{U}(m)$ acts on $\mathbb{R}^{2m}$ as the subgroup of $\mathrm{O}(2m)$ which fixes the real $2-$form $-2\omega_0 := \mathrm{Im}(\eta_0)$. The group $\mathrm{U}(m)$ also commutes with the real endomorphism I_0 of $\mathbb{R}^{2m}$ such that $I_0 dx^{2j-1} = \mathrm{d}x^{2j}$ and $I_0 \mathrm{d}x^{2j} = -\mathrm{d}x^{2j-1}$ $(j = 1, \ldots, m)$. It can be shown that ω_0 and I_0 are equivalent in the presence of the inner product g_0; for instance, $\omega_0(x, y) = g_0(I_0 x, y)$ for all $x, y \in \mathbb{R}^{2m}$.

It follows from the holonomy principle (Theorem 1.2.3) that a Riemannian metric g on a $2m-$dimensional manifold Z^{2m} has holonomy $\mathrm{Hol}(g) \subseteq \mathrm{U}(m)$ if, and only if, Z admits natural tensors $I \in \mathrm{End}(TZ)$ and $\omega \in \Omega^2(Z)$, parallel with respect to the Levi-Civita connection D^g, such that g, I, and ω can be written in the form g_0, I_0, and ω_0 at each point of Z. A Riemannian metric g on a $2m-$dimensional manifold Z^{2m} with $\mathrm{Hol}(g) \subseteq U(m)$ is called a *Kähler metric*.

A $\mathrm{U}(m)$−structure on a smooth $2m$−manifold Z is specified by a pair (I, ω), where $I \in \mathrm{End}(TZ)$ is an almost complex structure, $I^2 = -\mathbb{1}$, and $\omega \in \Omega^2(Z)$ is a non-degenerate real 2−form such that $g(\cdot,\cdot) := \omega(\cdot, I\,\cdot\,)$ defines a Riemannian metric on Z. A $\mathrm{U}(m)$−structure (I, ω) on Z^{2m} is *torsion free* when both I and ω are D^g−parallel with respect to the induced metric g. A $2m$−dimensional manifold Z^{2m} endowed with a torsion-free $U(m)$−structure (I, ω) is called a *Kähler m−fold* and ω its *Kähler form*. The following is standard:

Proposition 2.1.7. *Let (I, ω) be a $U(m)$−structure on Z^{2m}. Denote by g the natural Riemannian metric induced by (I, ω). Then the following are equivalent:*

(i) g is a Kähler metric.

(ii) (I, ω) is torsion-free.

(iii) I is integrable[2] *and* $\mathrm{d}\omega = 0$.

Note that Kähler m−folds are essentially Riemmanian $2m$−manifolds with holonomy contained in $U(m)$. Henceforth, we will denote a Kähler m−fold by a pair (Z^{2m}, ω), omitting the underlying complex structure I and metric g.

Metrics with holonomy $\mathbf{SU}(m) \subseteq \mathbf{O}(2m)$. (cf. [Bry86], [Joy07, Chapter 7] and [CHNP13, Sec.2]) As above, identify $\mathbb{R}^{2m}$ with $\mathbb{C}^m$ with complex coordinates $(z^1, \ldots, z^m)$ and define a complex m−form Υ_0 on $\mathbb{C}^m$ by

$$\Upsilon_0 := \mathrm{d}z^1 \wedge \ldots \wedge \mathrm{d}z^m.$$

The subgroup of $\mathrm{U}(m) \subseteq O(2m)$ preserving g_0, ω_0 and Υ_0 is $\mathrm{SU}(m)$.

By the holonomy principle, a Riemannian metric g on a $2m$−dimensional manifold Z^{2m} has holonomy $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$ if, and only if, g is a Kähler metric, say, with associated complex structure I and Kähler form ω, and further Z admits a natural D^g−parallel complex $(m, 0)$−form Υ such that g, I, ω and Υ have pointwise models g_0, I_0, ω_0 and Υ_0. A Riemannian metric g on a $2m$−dimensional manifold Z^{2m} with $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$ is called a *Calabi-Yau metric*.

An $\mathrm{SU}(m)$−*structure* on a smooth $2m$−manifold Z is specified by a triple (I, ω, Υ), where (I, ω) defines a $\mathrm{U}(m)$−structure on Z, and Υ is a nowhere vanishing complex $(m, 0)$−form on (Z, I) satisfying

$$\frac{\omega^m}{m!} = \mathbf{i}^{m^2} 2^{-m} \Upsilon \wedge \overline{\Upsilon}. \tag{2.1.4}$$

[2]i.e. I is induced from a complex manifold structure on Z

This is a normalization condition that the natural volume forms induced by ω and Υ are equal, or equivalently that $|\Upsilon|^2 = 2^m$ with respect to the induced metric[3]. An $\mathrm{SU}(m)-$structure (I, ω, Υ) on Z^{2m} is called *torsion-free* when $D^g\Upsilon = D^g\omega = 0$ with respect to its induced metric g. A $2m-$manifold Z^{2m} endowed with a torsion-free $\mathrm{SU}(m)-$structure (I, ω, Υ) is called a *Calabi-Yau* $m-$*fold*.

One can show that given an $\mathrm{SU}(m)-$structure (I, ω, Υ) on Z^{2m}, then $\mathrm{d}\Upsilon = 0$ implies that the complex structure I is integrable and Υ is a holomorphic $(m, 0)-$form. In particular, Υ holomorphically trivializes the canonical bundle $K_Z = \Lambda^m_{\mathbb{C}}(T^{1,0}Z)^*$ of (Z, I). Since the first Chern class $c_1(Z) := c_1(T^{1,0}Z)$ turns out to be a characteristic class of K_Z, namely $-c_1(K_Z)$, it follows that $c_1(Z) = 0$. If also $d\omega = 0$ then Z is a Kähler manifold, so that the induced metric g has $\mathrm{Hol}(g) \subseteq \mathrm{U}(m)$. Furthermore, since Υ is a holomorphic form of constant norm, the later condition forces $D^g\Upsilon = 0$, so that the holonomy of g reduces further to $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$. In particular, an $\mathrm{SU}(m)-$structure (I, ω, Υ) is torsion-free if, and only if, $\mathrm{d}\Upsilon = \mathrm{d}\omega = 0$.

The well-known linear relation between the curvature of the canonical bundle and the Ricci curvature of a Kähler metric implies [Joy07, Proposition 7.1.1, p. 123]:

Proposition 2.1.8. *Suppose* (Z^{2m}, ω) *is a Kähler* $m-$*fold and let* g *be its associated compatible Riemannian metric. Then* $\mathrm{Hol}^0(g) \subseteq \mathrm{SU}(m)$ *if, and only if,* g *is Ricci-flat* $(\mathrm{Ric}^g \equiv 0)$.

Finally, a fundamental result in this context is Yau's solution of the Calabi conjecture [Yau78], which has the following important consequence [Joy07, Theorem 7.1.2, p. 124]:

Theorem 2.1.9. *Let* (Z^{2m}, I) *be a compact complex manifold admitting some Kähler metric and such that* $c_1(Z) = 0$. *Then there is a unique Ricci-flat Kähler metric in the cohomology class of each Kähler form on* Z.

Since a generic Kähler metric on a complex $m-$fold has holonomy $U(m)$, in the light of Proposition 2.1.8 we see the above theorem constructs metrics with special holonomy $\subseteq SU(m)$ on compact complex $m-$folds.

Henceforth, we will denote a Calabi-Yau $m-$fold by a triple (Z, ω, Υ), omitting the underlying complex structure I and metric g.

The exceptional cases $\mathrm{G}_2 \subseteq \mathbf{SO}(7)$ and $\mathrm{Spin}(7) \subseteq \mathbf{SO}(8)$. (cf. [Bry86, Bry87] and [Joy07, Chapter 11]) We start with a definition of the Lie group G_2 due to R. Bryant [Bry86, Bry87].

[3] This implies that $\mathrm{Re}(\Upsilon)$ has comass ≤ 1 (cf. Section 2.2.2).

Definition 2.1.10. Let $(x^1, \ldots, x^7)$ be Euclidean coordinates on $\mathbb{R}^7$. Define a 3−form ϕ_0 on $\mathbb{R}^7$ by

$$\phi_0 := \mathrm{d}x^{123} - \mathrm{d}x^{145} - \mathrm{d}x^{167} - \mathrm{d}x^{246} + \mathrm{d}x^{257} - \mathrm{d}x^{347} - \mathrm{d}x^{356}. \tag{2.1.5}$$

Here we write $\mathrm{d}x^{ij\ldots l}$ as shorthand for $\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \ldots \wedge \mathrm{d}x^l$. The subgroup of $\mathrm{GL}(7, \mathbb{R})$ preserving ϕ_0 under the standard (pull-back) action is the exceptional Lie group G_2:

$$\mathrm{G}_2 := \{g \in \mathrm{GL}(7, \mathbb{R}) : g^*\phi_0 = \phi_0\}.$$

Remark 2.1.11. Our definition of ϕ_0 differs from the one given by Bryant by an orientation-preserving change of coordinates. Our sign conventions follows [SW10, Wal13a].

A useful way to interpret ϕ_0 is to write $\mathbb{R}^7 \simeq \mathbb{R}^3 \oplus \mathbb{R}^4$, letting $\mathbb{R}^3$ have coordinates (x^1, x^2, x^3) and $\mathbb{R}^4$ have coordinates (x^4, x^5, x^6, x^7), with the standard choice of orientations

$$\mathrm{vol}_3 := \mathrm{d}x^{123} \quad \text{and} \quad \mathrm{vol}_4 := \mathrm{d}x^{4567},$$

on $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively. Note that the 2−forms

$$\begin{aligned} \eta_1^+ &:= \mathrm{d}x^{45} + \mathrm{d}x^{67}, \\ \eta_2^+ &:= \mathrm{d}x^{46} + \mathrm{d}x^{75}, \\ \eta_3^+ &:= \mathrm{d}x^{47} + \mathrm{d}x^{56}, \end{aligned}$$

give us an orthogonal basis for the self-dual 2−forms on $\mathbb{R}^4$. With these identifications, we can write

$$\phi_0 = \mathrm{vol}_3 - \mathrm{d}x^1 \wedge \eta_1^+ - \mathrm{d}x^2 \wedge \eta_2^+ - \mathrm{d}x^3 \wedge \eta_3^+.$$

◊

By definition, G_2 is a closed Lie subgroup of $\mathrm{GL}(7, \mathbb{R})$. Moreover, one can check directly that for every $x, y \in \mathbb{R}^7$, we have

$$(x \lrcorner \phi_0) \wedge (y \lrcorner \phi_0) \wedge \phi_0 = 6 g_7(x, y) \mathrm{vol}_7, \tag{2.1.6}$$

where g_7 and vol_7 denotes, respectively, the standard metric and orientation of $\mathbb{R}^7$. In particular, we see that $\mathrm{G}_2 \subseteq \mathrm{SO}(7)$.

The following theorem summarizes some general facts about the Lie group G_2; for a proof see [Bry87, Theorem 1, p. 539].

Theorem 2.1.12. G_2 *is a* 14−*dimensional compact,* 2−*connected, simple Lie group.*

Definition 2.1.13. Let V be a 7−dimensional real vector space. A 3−form $\phi \in \Lambda^3 V^*$ is said to be **positive** if there exists a linear isomorphism $u : V \to \mathbb{R}^7$ so that $\phi = u^*\phi_0$, where $\phi_0 \in \Lambda^3(\mathbb{R}^7)^*$ is given by (2.1.5). The set of positive 3−forms on V is denoted by $\Lambda^3_+ V^*$.

Remark 2.1.14. Note that $\Lambda^3_+ V^* \simeq \mathrm{GL}(7,\mathbb{R})/\mathrm{G}_2$, so that (by dimension count) $\Lambda^3_+ V^*$ is an open subset of $\Lambda^3 V^*$. ♢

Definition 2.1.15. Let Y^7 be a smooth 7−manifold. We let $\Lambda^3_+(T^*Y)$ be the (open) subbundle of $\Lambda^3 T^*Y$ whose fiber over a point $y \in Y$ is $\Lambda^3_+(T^*_y Y)$. We denote by $\Omega^3_+(Y)$ the space of smooth sections of $\Lambda^3_+(T^*Y)$. An element $\phi \in \Omega^3_+(Y)$ is called a **positive 3−form** on Y.

By the holonomy principle (Theorem 1.2.3), it follows that a Riemannian metric g on a (connected) 7−manifold Y has $\mathrm{Hol}(g) \subseteq \mathrm{G}_2$ if, and only if, Y possesses a parallel positive 3−form $\phi \in \Omega^3_+(Y)$.

Note that a positive 3−form on Y^7 is equivalent to a G_2−structure on Y^7. Indeed, denote by $\mathcal{F}$ the frame bundle of Y. Then, given $\phi \in \Omega^3_+(Y)$ we can form

$$\mathcal{P}_\phi := \{u \in \mathcal{F} : u^*\phi_0 = \phi_y, \text{ where } u : T_yY \to \mathbb{R}^7\}.$$

It is easy to see $\mathcal{P}_\phi$ defines a principal subbundle of $\mathcal{F}$ with fibre G_2, i.e. a G_2−structure on Y. Conversely, a G_2−structure $\mathcal{P} \subseteq \mathcal{F}$ on Y determines a unique positive 3−form $\phi \in \Omega^3_+(Y)$ by

$$\phi_y := u_y^*\phi_0,$$

where $u_y \in \mathcal{P}_y$, for all $y \in Y$. (This is well-defined precisely because $\mathcal{P}$ is a principal G_2−subbundle: two frames $u_y, u'_y \in \mathcal{P}_y$ are related as $u'_y = g^{-1} \circ u_y$, for some $g \in \mathrm{G}_2 = \mathrm{Stab}(\phi_0)$.) It is clear that such constructions are inverse of each other. Henceforth we will *not distinguish* between G_2−structures and positive 3−forms on Y^7.

Since $\mathrm{G}_2 \subseteq \mathrm{SO}(7)$, a G_2−structure ϕ on Y^7 determines a Riemannian metric g_ϕ and orientation vol_ϕ on Y. Indeed, these are uniquely determined pointwise by the relation (2.1.6). In particular, ϕ determines a $*$−Hodge operator on $\Lambda^\bullet T^*Y$.

Definition 2.1.16. A G_2−structure ϕ on Y^7 is called **torsion-free** when ϕ is parallel with respect to the induced Levi-Civita connection:

$$D^{g_\phi}\phi = 0. \tag{2.1.7}$$

If ϕ is a torsion-free G_2−structure on Y^7, the pair (Y^7, ϕ) is called a G_2−**manifold**.

Thus, a G_2−manifold (Y^7, ϕ) is essentially a Riemannian 7−manifold (Y^7, g_ϕ) with $\mathrm{Hol}(g_\phi) \subseteq \mathrm{G}_2$.

Remark 2.1.17. The torsion-free condition (2.1.7) turns out to be a very complicated *non-linear* p.d.e. on ϕ. The non-linearity is due to the dependency of the metric g_ϕ itself (hence the Levi-Civita connection) on ϕ. ♢

Example 2.1.18. $(\mathbb{R}^7, \phi_0)$, with ϕ_0 given by (2.1.5), is the model example of G_2-manifold.

The following theorem gives a non-trivial characterization for the torsion-free condition (2.1.7):

Theorem 2.1.19 (Fernández-Gray)**.** *Let Y^7 be a (connected) $7-$manifold and let $\phi \in \Omega^3_+(Y)$. Denote by $*$ the Hodge star operator induced by ϕ on Y. Then the following are equivalent:*

(i) (Y, ϕ) *is a* G_2-*manifold.*

(ii) $\mathrm{d}\phi = 0 = \mathrm{d} * \phi$.

Remark 2.1.20. Again, since $*$ depends on ϕ, $\mathrm{d} * \phi = 0$ is a non-linear condition on ϕ. ♢

For a proof of the above result the reader can consult the original paper [FG82, Theorem 5.2], or see [Sal89, Lemma 11.5, p. 160].

Exploring curvature restrictions imposed by the holonomy group, just as in Theorem 2.1.3, and using some representation theory, one can prove the following (see [Sal89, Proposition 11.8, p. 162]):

Proposition 2.1.21. *If g is a Riemannian metric on a (connected) $7-$manifold Y^7 with* $\mathrm{Hol}(g) \subseteq \mathrm{G}_2$*, then g is Ricci-flat (*$\mathrm{Ric}^g \equiv 0$*).*

Moreover, from the classification of Riemannian holonomy groups (cf. Theorem 2.1.4), one has [Joy07, Theorem 11.1.7, p. 230]:

Theorem 2.1.22. *The only connected* non-trivial *Lie subgroups of* G_2 *which can be the holonomy group of a Riemannian metric on $7-$manifolds are:*

(i) $\mathrm{SU}(2)$*, acting on* $\mathbb{R}^7 \simeq \mathbb{R}^3 \oplus \mathbb{C}^2$ *trivially on* $\mathbb{R}^3$ *and as usual in* $\mathbb{C}^2$*,*

(ii) $\mathrm{SU}(3)$*, action on* $\mathbb{R}^7 \simeq \mathbb{R} \oplus \mathbb{C}^3$ *trivially on* $\mathbb{R}$ *and as usual in* $\mathbb{C}^3$*.*

Thus, if ϕ is torsion-free G_2-*structure on a $7-$manifold, then* $\mathrm{Hol}^0(g_\phi)$ *is one of* $\{1\}$, $\mathrm{SU}(2)$, $\mathrm{SU}(3)$ *or* G_2*.*

This theorem implies that from certain lower dimensional geometries we can obtain G_2-manifolds. More precisely, the inclusions $\mathrm{SU}(2) \subseteq \mathrm{G}_2$ and $\mathrm{SU}(3) \subseteq \mathrm{G}_2$ imply that from each Calabi-Yau $2-$ or $3-$fold we can make a G_2-manifold.

Example 2.1.23 (G_2-manifolds from Calabi-Yau 2−folds)**.** Let (Z^4, ω, Υ) be a Calabi-Yau 2−fold, and let (x^1, x^2, x^3) be coordinates on $\mathbb{R}^3$ or $T^3 := S^1 \times S^1 \times S^1$. Then the 3−form

$$\phi := \mathrm{d}x^{123} - \mathrm{d}x^1 \wedge \omega - \mathrm{d}x^2 \wedge \mathrm{Re}(\Upsilon) - \mathrm{d}x^3 \wedge \mathrm{Im}(\Upsilon)$$

defines a torsion-free G_2-structure on $Y^7 := \mathbb{R}^3 \times Z^4$ or $T^3 \times Z^4$ compatible with the natural product metric and orientation structures.

Example 2.1.24 (G_2-manifolds from Calabi-Yau 3−folds)**.** Let (Z^6, ω, Υ) be a Calabi-Yau 3−fold. Let t be a coordinate on $\mathbb{R}$ or S^1. Then the 3−form

$$\phi := \mathrm{d}t \wedge \omega + \mathrm{Re}(\Upsilon)$$

defines a torsion-free G_2-structure on $Y^7 := \mathbb{R} \times Z^6$ or $S^1 \times Z^6$, compatible with the natural product metric and orientation structures.

Note that the above examples have holonomy strictly contained in G_2. Examples of metrics with holonomy *exactly* G_2 are harder to come by. In fact, for a long period after Berger's classification (Theorem 2.1.4), the exceptional holonomy groups G_2 and Spin(7) remained a mystery [Joy04]. At first, Bryant [Bry87] proved the local existence of such metrics, and constructed some explicit incomplete examples. Then, Bryant-Salamon [BS89] constructed the first examples of *complete* metrics with holonomy (exactly) G_2 and Spin(7) on noncompact manifolds. Later, Joyce [Joy96a, Joy96b] constructed the first examples of metrics with holonomy (exactly) G_2 and Spin(7) on *compact* manifolds; also see [Joy00].

Nowadays, a particularly important method in the construction of compact G_2-manifolds, with full holonomy G_2, is the so-called *twisted connected sum construction.* From a pair of smooth asymptotically cilindrical Calabi-Yau 3−folds $V_\pm$, this construction establishes a non-trivial way to glue the products $S^1 \times V_\pm$, truncated sufficiently far along the noncompact end, producing a compact G_2-manifold $Y^7 := (S^1 \times V_+) \,\tilde{\#}\, (S^1 \times V_-)$ with holomomy exactly G_2. This method was first developed by Kovalev [Kov03], based on an idea of Donaldson. Then the construction was improved by Kovalev-Lee [KL00] and, more recently, corrected and extended significantly by Corti, Haskins, Nordström and Pacini [CHNP15].

The twisted connected sum construction provided a major breakthrough in the study of G_2-manifolds and, along with the orbifold resolution construction of Joyce [Joy00, Sec. 11 e 12], is one of only two methods available, at the time of writing, for production of compact manifolds with holonomy exactly G_2.

We now turn to a brief discussion of the holonomy group Spin(7).

Definition 2.1.25. Let $\mathbb{R}^8 = \mathbb{R} \times \mathbb{R}^7$ have coordinates $(x^0, x^1, \ldots, x^7)$. Define a $4-$form Φ_0 on $\mathbb{R}^8$ by

$$\Phi_0 \coloneqq \mathrm{d}x^0 \wedge \phi_0 + \psi_0, \tag{2.1.8}$$

where ϕ_0 is the $3-$form given by (2.1.5), and $\psi_0 \coloneqq *_7\phi_0$. The $\mathrm{GL}(8, \mathbb{R})-$stabilizer of Φ_0 under the (standard) pull-back action is the Lie group Spin(7):

$$\mathrm{Spin}(7) \coloneqq \{g \in \mathrm{GL}(8, \mathbb{R}) : g^*\Phi_0 = \Phi_0\}.$$

Theorem 2.1.26 ([Bry87, Theorem 4, p. 545])**.** Spin(7) *is a simple, compact and* $1-$*connected Lie group of dimension 21. Furthermore,* Spin(7) *is a subgroup of* SO(8)*.*

Definition 2.1.27. Let W^8 be a $8-$dimensional real vector space. A $4-$form $\Phi \in \Lambda^4 W^*$ is called **definite** if there exists a linear isomorphism $u : W \to \mathbb{R}^8$ such that $\Phi = u^*\Phi_0$. We denote by $\Lambda^4_+(W^*)$ the set of definite $4-$forms on W.

Let X^8 be an $8-$dimensional (smooth) manifold. Define the bundle $\Lambda^4_+(T^*X)$ of definite $4-$forms on X to be the subbundle of $\Lambda^4 T^*X$ whose fiber at $x \in X$ is $\Lambda^4_+(T^*_x X)$. A smooth section $\Phi \in \Gamma(\Lambda^4_+(T^*X))$ is called a definite $4-$form on X. The space of definite $4-$forms on X is denoted by $\Omega^4_+(X)$.

Note that a definete $4-$form $\Phi \in \Omega^4_+(X)$ determines and is determined by a unique $\mathrm{Spin}(7)-$structure on X. Thus, it is customary to call such Φ a $\mathrm{Spin}(7)-$structure on X.

Since $\mathrm{Spin}(7) \subseteq \mathrm{SO}(8)$, a $\mathrm{Spin}(7)-$structure Φ on X determines a unique Riemannian metric g_Φ and orientation vol_Φ on X; in particular, we have an associated $*-$operator acting on $\Lambda^\bullet T^*X$.

Definition 2.1.28. A $\mathrm{Spin}(7)-$structure Φ on a smooth $8-$manifold X is called *torsion free* if $D^{g_\Phi}\Phi = 0$. A pair (X^8, Φ) where Φ is a torsion-free $\mathrm{Spin}(7)-$structure on X^8 is called a $\mathrm{Spin}(7)-$**manifold**.

By the holonomy principle (Theorem 1.2.3), a connected Riemannian $8-$manifold (X^8, g) has $\mathrm{Hol}(g) \subseteq \mathrm{Spin}(7)$ if, and only if, (X, g) has a torsion-free $\mathrm{Spin}(7)-$structure Φ. Thus, a $\mathrm{Spin}(7)-$manifold (X^8, Φ) is essentially a Riemannian $8-$manifold (X^8, g_Φ) with $\mathrm{Hol}(g_\Phi) \subseteq \mathrm{Spin}(7)$.

Example 2.1.29. $(\mathbb{R}^8, \Phi_0)$ where Φ_0 is given by (2.1.8) is the model example.

The next results are analogues of Theorem 2.1.19, Proposition 2.1.21 and Theorem 2.1.22. The proofs of the first two can be found, respectively, in [Sal89, Lemma 12.4, p. 176] and [Sal89, Corollary 12.6, p. 176].

Theorem 2.1.30. *Suppose* Φ *is a* $\mathrm{Spin}(7)-$*structure on a* $8-$*manifold* X^8*. Denote by* $*$ *the induced Hodge star operator on* X*. Then the following are equivalent:*

(i) (X, Φ) *is a* $\mathrm{Spin}(7)-$*manifold.*

(ii) $\mathrm{d}\Phi = 0$.

Proposition 2.1.31. *Let g be a Riemannian metric on a (connected) $8-$manifold X^8 such that $\mathrm{Hol}(g) \subseteq \mathrm{Spin}(7)$. Then g is Ricci-flat.*

From Berger's classification theorem (Theorem 2.1.4), one deduces:

Theorem 2.1.32 ([Joy07, Theorem 11.4.7, p. 241])**.** *The only connected* non-trivial *Lie subgroups of* $\mathrm{Spin}(7)$ *which can be holonomy groups of Riemannian metrics on $8-$manifolds are:*

(i) $\mathrm{SU}(2)$*, acting on* $\mathbb{R}^8 \simeq \mathbb{R}^4 \oplus \mathbb{C}^2$ *trivially on* $\mathbb{R}^4$ *and as usual on* $\mathbb{C}^2$*,*

(ii) $\mathrm{SU}(2) \times \mathrm{SU}(2)$*, acting on* $\mathbb{R}^8 \simeq \mathbb{C}^2 \oplus \mathbb{C}^2$ *in the obvious way,*

(iii) $\mathrm{SU}(3)$*, acting on* $\mathbb{R}^8 \simeq \mathbb{R}^2 \oplus \mathbb{C}^3$ *trivially on* $\mathbb{R}^2$ *and as usual on* $\mathbb{C}^3$*,*

(iv) G_2*, acting on* $\mathbb{R}^8 \simeq \mathbb{R} \oplus \mathbb{R}^7$ *trivially on* $\mathbb{R}$ *and as usual on* $\mathbb{R}^7$*.*

(v) $\mathrm{Sp}(2)$*, acting as usual on* $\mathbb{R}^8 \simeq \mathbb{H}^2$*.*

(vi) $\mathrm{SU}(4)$*, acting as usual on* $\mathbb{R}^8 \simeq \mathbb{C}^4$*.*

Therefore, if Φ is a torsion-free $\mathrm{Spin}(7)-$*structure on an $8-$manifold, then* $\mathrm{Hol}^0(g_\Phi)$ *is one of* $\{1\}$, $\mathrm{SU}(2)$, $\mathrm{SU}(2) \times \mathrm{SU}(2)$, $\mathrm{SU}(3)$, G_2, $\mathrm{Sp}(2)$, $\mathrm{SU}(4)$ *or* $\mathrm{Spin}(7)$.

We give two particularly interesting instances of the use of these inclusions to obtain $\mathrm{Spin}(7)-$manifolds (with holonomy strictly contained in $\mathrm{Spin}(7)$).

Example 2.1.33 ($\mathrm{Spin}(7)-$manifolds from G_2-manifolds)**.** Let (Y^7, ϕ) be a G_2-manifold. Let t be a coordinate on $\mathbb{R}$ or S^1. Then the $4-$form

$$\Phi := \mathrm{d}t \wedge \phi + \psi,$$

where $\psi = *_Y \phi$, defines a torsion-free $\mathrm{Spin}(7)-$structure on $X^8 := \mathbb{R} \times Y$ or $S^1 \times Y$, compatible with the canonical product metric and orientation.

Example 2.1.34 ($\mathrm{Spin}(7)-$manifolds from Calabi-Yau $4-$folds)**.** Let (Z^8, ω, Υ) be a Calabi-Yau $4-$fold. Then the $4-$form

$$\Phi := \frac{1}{2}\omega \wedge \omega + \mathrm{Re}(\Upsilon)$$

defines a torsion-free $\mathrm{Spin}(7)-$structure on Z^8 compatible with its metric and orientation.

2.2 Calibrated Geometry

This section is based on [Joy03, §4.1-4.2], [HL82], and the lecture notes [Lot14] and [Nor12].

2.2.1 Minimal Submanifolds

In this subsection we give a brief recap on the basic definitions concerning minimal submanifolds. We follow the exposition of Joyce's book [Joy03, §4.1] and also Lotay's lecture notes [Lot14]. A classical good reference on this subject is Lawson's lecture notes [Law80].

We start defining what we mean by a submanifold.

Definition 2.2.1 (Submanifold)**.** Let M be smooth manifold. A **submanifold** of M is a one-to-one immersion $\iota : N \hookrightarrow M$, where N is some smooth manifold. When N is oriented, we say that $\iota : N \hookrightarrow M$ is an *oriented submanifold*. Two submanifolds $\iota : N \hookrightarrow M$ and $\iota' : N' \hookrightarrow M$ are **isomorphic** if there exists a diffeomorphism $\varphi : N \to N'$ such that $\iota = \iota' \circ \varphi$.

We regard isomorphic submanifolds as the same. In particular, endowing $\iota(N)$ with the manifold structure of N via ι, we do not distinguish between the submanifolds $\iota : N \hookrightarrow M$ and $\iota(N) \hookrightarrow M$ (i.e. one can think of N as a subset of M whose inclusion map is ι).

Remark 2.2.2. We do not require a submanifold $\iota : N \hookrightarrow M$ to have the induced topology of the ambient manifold M, i.e. $\iota : N \hookrightarrow M$ is not necessarily a topological embedding (a homeomorphism onto its image). Anyway, by the implicit function theorem we know that any point $p \in N$ has an open neighborhood V such that $\iota|_V$ is a topological embedding. Thus, when we are treating local questions, we can suppose N is an embedded submanifold of M. ♢

In order to give the variational approach to minimal submanifolds including noncompact submanifolds, we need the following:

Definition 2.2.3 (Variations with compact support)**.** Let $\iota : N \hookrightarrow M$ be a submanifold and let $S \subseteq N$ be an open subset whose closure in N is compact. A (smooth) **variation** of ι *supported in* S is a smooth map

$$F : N \times]{-1}, 1[\to M$$

such that, writing $\iota_t := F(\cdot, t)$, the following holds:

(i) $\iota_0 = \iota$;

(ii) $\iota_t : N \to M$ is a submanifold, for all t;

(iii) $\iota_t|_{N\setminus S} \equiv \iota|_{N\setminus S}$, for all t.

In this case, $V_F \in \mathfrak{X}(N)$ defined by

$$V_F(p) := F(p,\cdot)_* \frac{\partial}{\partial t}\Big|_{t=0}, \quad \forall p \in N,$$

is called the **variational vector field** associated to the variation $F = \{\iota_t\}$.

Now we can define minimal submanifolds.

Definition 2.2.4 (Minimal submanifolds)**.** Let (M, g) be a Riemannian manifold and let $\iota : N \hookrightarrow M$ be an oriented submanifold of M. Denote by $\mathrm{d}V_{\iota^* g}$ the induced Riemannian volume form on N. Then, for each precompact open set $S \Subset N$, it is well-defined the *volume of* S with respect to $\iota : N \hookrightarrow M$:

$$\mathrm{Vol}(\iota|_S) := \int_S \mathrm{d}V_{\iota^* g} < \infty.$$

We say that $\iota : N \hookrightarrow M$ is a **minimal submanifold** of M when for each precompact open subset[4] $S \Subset N$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Vol}(\iota|_S)\Big|_{t=0} = 0,$$

for all variations $\{\iota_t\}$ of ι supported in S.

Thus, a minimal submanifold of M is just a stationary point, with respect to compactly supported variations, of the natural volume functional on oriented submanifolds of M.

We can also give a p.d.e. approach to minimal submanifolds by means of the *mean curvature vector* of submanifolds.

Definition 2.2.5 (Second fundamental form and the mean curvature vector)**.** Let (M, g) be a Riemannian manifold and let $\iota : N \hookrightarrow M$ be a submanifold of M. Then, the tangent bundle of M restricted to N decomposes orthogonally as

$$\iota^* TM = \iota_* TN \oplus \nu^\iota(N),$$

where $\nu^\iota(N)$, called the **normal bundle** of $\iota : N \hookrightarrow M$, is the vector subbundle of $\iota^* TM$ whose fiber at a point $q \in N$ is the orthogonal complement of $\iota_* T_q N$ in $(\iota^* TM)_q \simeq T_q M$

[4] If $\partial N \neq \emptyset$ one requires S to be in the interior of N.

with respect to g. The **second fundamental form** of $\iota : N \hookrightarrow M$ is the section B^ι of $\left(\bigodot^2 T^*N\right) \otimes \nu^\iota(N)$ such that, for all $X, Y \in \mathfrak{X}(N)$,

$$B^\iota(X,Y) = \pi_{\nu^\iota(N)} \circ \left[D^g_{\iota_* X}(\iota_* Y)\right],$$

where $\pi_{\nu^\iota(N)} : \iota^* TM \to \nu^\iota(N)$ is the orthogonal projection map.

The **mean curvature vector** H^ι of $\iota : N \hookrightarrow M$ is the section of $\nu^\iota(N)$ given by

$$H^\iota := \mathrm{tr}_{\iota^* g} B^\iota.$$

Now a straightforward calculation gives the following characterization of minimal submanifolds [Law80, Theorem 1, p. 7].

Theorem 2.2.6. *$\iota : N \hookrightarrow M$ is a minimal submanifold of (M, g) if, and only if, $H^\iota \equiv 0$.*

Note that, by the definition, B^ι depends nonlinearly on the second derivatives of ι, thus so does H^ι. Therefore, the above theorem implies the minimal submanifold condition can be seen as a (nonlinear) p.d.e. of second order on ι, namely, $H^\iota \equiv 0$.

Example 2.2.7. For immersed curves $\gamma : I \to M$, the zero mean curvature condition $H^\gamma \equiv 0$ is equivalent to the geodesic equation $(\gamma^* D^g)(\dot\gamma) = 0$.

Example 2.2.8. Let $f : U \subseteq \mathbb{R}^k \to \mathbb{R}^{n-k}$ be a smooth map from an open subset U of $\mathbb{R}^k$. Then, the graph $\Gamma(f)$ of f is a submanifold of $\mathbb{R}^n$ by means of the natural inclusion map $\iota : \Gamma(f) \hookrightarrow \mathbb{R}^k \times \mathbb{R}^{n-k}$. One can show $H^\iota = 0$ (i.e. $\iota : \Gamma(f) \hookrightarrow \mathbb{R}^n$ is a minimal submanifold) if, and only if,

$$\mathrm{div}\left(\frac{\mathrm{grad}(f)}{\sqrt{1 + |\mathrm{grad}(f)|^2}}\right) = 0.$$

2.2.2 Calibrated Submanifolds

The notion of calibration was introduced by Harvery-Lawson (1982) in their seminal paper [HL82].

Throughout this section, we let (M, g) be a Riemannian $n-$manifold. For each $x \in M$, we write $\mathrm{Gr}_+(k, T_xM)$ to denote the *Grassmanian of oriented $k-$planes* in T_xM, i.e.

$$\mathrm{Gr}_+(k, T_xM) := \{V \leq T_xM : V \text{ is an oriented } k\text{−subspace of } T_xM\},$$

and we set

$$\mathrm{Gr}_+(k, TM) := \bigcup_{x \in M} \mathrm{Gr}_+(k, T_xM).$$

Elements of $\mathrm{Gr}_+(k, TM)$ are called **oriented tangent k-planes** of M. Note that g induces an inner product $g|_V$ on each $V \in \mathrm{Gr}_+(k, TM)$, which together with the orientation of V gives rise to a preferred volume form vol_V on V. In particular, for each $x \in M$, we get an inclusion

$$\mathrm{Gr}_+(k, T_xM) \hookrightarrow \Lambda^k T_xM$$

mapping each $V \in \mathrm{Gr}_+(k, T_xM)$ into the unit simple $k-$vector $\xi_V := e_1 \wedge \ldots \wedge e_k$, where $\{e_i\}$ is any oriented orthonormal basis of V.

Recall that each $\phi_x \in \Lambda^k T_x^*M$ defines a linear functional $\langle \phi_x, \cdot \rangle : \Lambda^k T_xM \to \mathbb{R}$ by means of the natural pairing

$$\langle \cdot, \cdot \rangle : \Lambda^k T_x^*M \otimes \Lambda^k T_xM \to \mathbb{R},$$

defined on simple elements by

$$\langle \alpha_1 \wedge \ldots \wedge \alpha_k, v_1 \wedge \ldots \wedge v_k \rangle := \det\left(\alpha_i(v_j)\right).$$

If $\phi \in \Omega^k(M)$, the **comass** of ϕ at $x \in M$ is given by

$$\|\phi\|_x^* := \sup\{\langle \phi_x, \xi_V \rangle : V \in \mathrm{Gr}_+(k, T_xM)\}.$$

More generally, if $A \subseteq M$ is any subset, we define the comass of ϕ on A by

$$\|\phi\|_A^* := \sup\{\|\phi\|_x^* : x \in A\}.$$

When $A = M$, we simply write $\|\phi\|^*$ for the comass of ϕ on M.

When $\phi \in \Omega^k(M)$ and $V \in \mathrm{Gr}_+(k, TM)$, the restriction $\phi|_V$ is a scalar multiple $\lambda_V \in \mathbb{R}$ of vol_V by dimension reasons. In case $\lambda_V \leq 1$, we write $\phi|_V \leq \mathrm{vol}_V$. Note that

$$\|\phi\|_x^* = \sup\{\lambda_V : V \in \mathrm{Gr}_+(k, T_xM)\}.$$

In particular, $\|\phi\|^* \leq 1$ if, and only if, $\phi|_V \leq \mathrm{vol}_V$ for all $V \in \mathrm{Gr}_+(k, TM)$.

Definition 2.2.9. A $k-$form $\phi \in \Omega^k(M)$ is called a **calibration** on (M, g) if

(i) (ϕ is closed) $\mathrm{d}\phi = 0$.

(ii) (ϕ has comass ≤ 1) $\|\phi\|^* \leq 1$.

In this case, we define the $\phi-$**Grassmannian** $\mathscr{G}(\phi)$ as the collection of oriented tangent $k-$planes of M where ϕ assumes its maximum, i.e.

$$\mathscr{G}(\phi) := \{V \in \mathrm{Gr}_+(k, TM) : \phi|_V = \mathrm{vol}_V\}.$$

An element $V \in \mathscr{G}(\phi)$ is called a $\phi-$**calibrated** (tangent) $k-$plane.

Remark 2.2.10. Using the Euclidean metric, identify $\Lambda^k\mathbb{R}^n \simeq \Lambda^k(\mathbb{R}^n)^*$ and embed $\mathrm{Gr}_+(k,\mathbb{R}^n) \hookrightarrow \Lambda^k(\mathbb{R}^n)^*$. Next, consider the Hodge star operator, which gives isometries $* : \Lambda^k(\mathbb{R}^n)^* \to \Lambda^{n-k}(\mathbb{R}^n)^*$ and $* : \Lambda^k\mathbb{R}^n \to \Lambda^{n-k}\mathbb{R}^n$. Then, for an oriented $k-$plane $V \in \mathrm{Gr}_+(k,\mathbb{R}^n)$ its Hodge star dual $*V$ is the unique orthogonal oriented $(n-k)-$plane $V^\perp$ such that if $\phi \in \Lambda^k(\mathbb{R}^n)^*$ with $\phi|_V = \alpha\mathrm{vol}_V$ for $\alpha \in \mathbb{R}$ then $*\phi|_{V^\perp} = \alpha\mathrm{vol}_{V^\perp}$.

This implies the following. Let (M,g) be an oriented Riemannian $n-$manifold and let $\phi \in \Omega^k(M)$ be a harmonic form (i.e. $\mathrm{d}\phi = 0 = \mathrm{d}*\phi$). Then, ϕ is a calibration if and only if $*\phi$ is a calibration, and in this case we have further $*\mathscr{G}(\phi) = \mathscr{G}(*\phi)$. ◊

Example 2.2.11. Any $k-$form $\phi \neq 0$ on $\mathbb{R}^n$ with constant coefficients (hence $\mathrm{d}\phi = 0$) can be rescaled so that it becomes a calibration with at least one oriented $k-$plane $V_0 \leq \mathbb{R}^n$ for which $V_0 \in \mathscr{G}(\phi)$. Indeed, since $\mathrm{Gr}_+(k,\mathbb{R}^n)$ is compact, the comass $\kappa := \|\phi\|^*_{\mathbb{R}^n} \neq 0$ of ϕ on $\mathbb{R}^n$ is attained at some oriented $k-$plane $V_0 \leq \mathbb{R}^n$. Thus, whenever $\lambda' \leq \lambda := 1/\kappa$, the $k-$form $\lambda'\phi$ is a calibration in $\mathbb{R}^n$, and in case $\lambda' = \lambda$ we have $\lambda\phi|_{V_0} = \mathrm{vol}_{V_0}$.

Although there are many calibrations, as the above example shows, it can happen that a calibration just admits a few calibrated tangent $k-$planes, i.e. the $\phi-$Grassmannian $\mathscr{G}(\phi)$ may be too 'small'. The interesting calibrations are the ones for which $\mathscr{G}(\phi)$ is 'big' enough so that it distinguishes a meaningful collection of $k-$submanifolds of M whose tangent spaces lie in $\mathscr{G}(\phi)$. This motivates the following.

Definition 2.2.12. Let $\phi \in \Omega^k(M)$ be a calibration on (M,g). If $\iota : N \hookrightarrow M$ is an oriented $k-$dimensional submanifold of M, then N is called a $\phi-$**calibrated submanifold** (or a $\phi-$**submanifold** for short) when $\iota_*TN \subseteq \mathscr{G}(\phi)$ (as bundles), i.e. when

$$\iota^*\phi = \mathrm{d}V_{\iota^*g},$$

where $\mathrm{d}V_{\iota^*g}$ is the Riemannian volume form on N induced by ι^*g and the orientation of N. The collection of $\phi-$submanifolds of M is called the $\phi-$**geometry** of M.

It turns out that having $\phi-$submanifolds greatly restricts the calibrations ϕ one wants to consider. The next lemma gives us a key distinguished property $\phi-$submanifolds satisfy (at least in the compact case).

Proposition 2.2.13. *Let $\phi \in \Omega^k(M)$ be a calibration on (M,g). If $\iota : N \hookrightarrow M$ is a compact $\phi-$submanifold then its volume is the topological invariant given by $\langle[\iota^*\phi],[N]\rangle$, and it is a minimal submanifold, minimizing volume in its homology class.*

Proof. Let $\iota' : N' \hookrightarrow M$ be another compact oriented $k-$submanifold of M such that $\partial N = \partial N'$ and $[N] = [N']$ in $H_k(M,\mathbb{R})$ (i.e. $N - N' = \partial X$, for some $(k+1)-$submanifold X of M). Then,

$$\mathrm{Vol}(\iota) := \int_N \mathrm{d}V_{\iota^*g} = \int_N \iota^*\phi = \int_{N'} (\iota')^*\phi \leq \int_{N'} \mathrm{d}V_{(\iota')^*g} =: \mathrm{Vol}(\iota'),$$

where the second equality follows from the condition of N being a ϕ-submanifold, in the third equality we used the homology condition on N' together with Stokes' theorem and the fact that ϕ is closed, and in the last inequality we used the fact that ϕ has comass ≤ 1.

To see that this implies $\iota : N \hookrightarrow M$ is a minimal submanifold, note that for small t a variation $\iota_t : N \hookrightarrow M$ (cf. Definition 2.2.3) of ι determines the same homology class inside M. Thus, the last inequality shows that

$$\mathrm{Vol}(\iota) \leq \mathrm{Vol}(\iota_t),$$

so that ι is a critical point of the volume functional on compact oriented $k-$submanifolds.

■

Corollary 2.2.14. *There are no compact calibrated submanifolds in a contractible Riemannian manifold* (M, g)*. (e.g.* $(\mathbb{R}^n, g_0)$*)*

Proof. Let $1 \leq k \leq n$. By Poincaré's lemma, if $\phi \in \Omega^k(M)$ is a calibration then there exists $\eta \in \Omega^{k-1}(M)$ such that $\phi = \mathrm{d}\eta$ (indeed, $\mathrm{d}\phi = 0$). Thus, if $\iota : N \hookrightarrow M$ is a compact (without boundary) $\phi-$submanifold, using Stokes' theorem we get

$$0 < \mathrm{Vol}(\iota) = \int_N \iota^* \phi = \int_N \mathrm{d}(\iota^* \eta) = 0. \quad (\Rightarrow\Leftarrow)$$

■

We note that the $\phi-$submanifold condition (cf. Definition 2.2.12) for an oriented compact $k-$dimensional submanifold $\iota : N \hookrightarrow M$ depends upon its tangent spaces; it is a *first* order p.d.e. on the immersion ι. On the other hand, as we have already seen in the previous section (see Theorem 2.2.6), the minimal submanifold condition for such a submanifold turns out to be a *second* order p.d.e. on the immersion ι ($H^\iota \equiv 0$). This suggests, via Proposition 2.2.13, that calibrated geometry is a great source of examples of minimal submanifolds. This fact is quite analogous to that in the realm of gauge theory involving ASD instantons and Yang-Mills connections (cf. Section 1.5). In fact, in the next section we shall extend this analogy with the general notion of $\Xi-$ASD instantons. Furthermore, in this work we shall see a striking *concrete* relation between gauge theory and calibrated geometries in dimensions greater than four (cf. [Tia00]). For this it will be useful to generalize the notion of $\phi-$submanifolds in M to the more general setting of currents in M, which turns out to be the measure-geometric generalization of the notion of submanifolds. In what follows we use some notations and terminology that are introduced in Appendix A (see Section A.6).

Definition 2.2.15 ($\phi-$currents)**.** Let $\phi \in \Omega^k(M)$ be a calibration on (M,g). Then an integral $k-$current $T = (\Gamma, \xi, \Theta) \in \mathbf{I}_k(M)$ (cf. Definition A.6.16) is said to be a $\phi-$**calibrated current** (or $\phi-$**current** for short) if

$$\phi|_{T_x\Gamma} = \xi(x), \quad \text{for } \mathscr{H}^k - \text{a.e. } x \in \Gamma.$$

Definition 2.2.16 (Mass-minimizing currents)**.** A current $T \in \mathbf{I}_{k,\mathrm{loc}}(M)$ is called **mass-minimizing** if

$$\mathbf{M}(S) \leq \mathbf{M}(S')$$

whenever $S, S' \in \mathbf{I}_k(M)$, $\|T\| = \|S\| + \|T - S\|$ (i.e. S is a *piece* of T) and $\partial S = \partial S'$.

We have the following result in parallel with Proposition 2.2.13.

Proposition 2.2.17. *Let $\phi \in \Omega^k(M)$ be a calibration on (M,g). Then any compactly supported $\phi-$calibrated cycle $T \in \mathscr{Z}_k(M) \subseteq \boldsymbol{I}_k(M)$ is mass-minimizing in its homology class.*

Proof. Write $T = (\Gamma, \xi, \Theta)$ and let $T' = (\Gamma', \xi', \Theta') \in \mathscr{Z}_k(M)$ be a compactly supported cycle homologous to T, say $T - T' = \partial R$, where $R \in \mathbf{I}_{k+1}(M)$. Then, unraveling definitions, we have:

$$\begin{aligned}
\mathbf{M}(T) = \int_\Gamma \Theta \mathrm{d}\mathscr{H}^k &= \int_\Gamma \langle \phi, \xi \rangle \Theta \mathrm{d}\mathscr{H}^k \quad (T \text{ is } \phi-\text{calibrated}) \\
&= \int_{\Gamma'} \langle \phi, \xi' \rangle \Theta' \mathrm{d}\mathscr{H}^k + R(\mathrm{d}\phi) \quad (T - T' = \partial R) \\
&\leq \int_{\Gamma'} \Theta' \mathrm{d}\mathscr{H}^k = M(T'). \quad (\phi \text{ has comass} \leq 1)
\end{aligned}$$

■

In this context, it is worth mentioning the following deep interior regularity result due to Almgren [Alm84].

Theorem 2.2.18 (Almgren)**.** *If $T \in \mathbf{I}_{k,loc}(M)$ is mass-minimizing, then $\mathring{T} := \mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$ is a smooth $k-$dimensional minimal submanifold of M except by a singular set $\Sigma \subseteq \mathring{T}$ of Hausdorff dimension at most $k-2$.*

Calibrations and Riemannian holonomy groups. There is a natural method of constructing interesting calibrations ϕ on Riemannian manifolds (M,g) with special holonomy, in such a way that $\mathscr{G}(\phi)$ contains families of calibrated tangent $k-$planes with reasonable large dimension.

Let $H \subseteq \mathrm{SO}(n)$ be a possible holonomy group for a Riemannian metric. Thus H acts on the $k-$forms $\Lambda^k(\mathbb{R}^n)^*$ of $\mathbb{R}^n$. Suppose that $\phi_0 \in \Lambda^k(\mathbb{R}^n)^*$ is a nonzero $H-$invariant

$k-$form on $\mathbb{R}^n$. Modulo rescaling, we can assume that ϕ has comass ≤ 1 and that $\mathscr{G}(\phi_0) \neq \emptyset$, i.e. $\phi_0|_V = \mathrm{vol}_V$ for at least one $k-$plane $V \leq \mathbb{R}^n$ (see Example 2.2.11). Thus, from the $H-$invariance of ϕ_0, if $V \in \mathscr{G}(\phi_0)$ then $h \cdot V \in \mathscr{G}(\phi_0)$ for *every* $h \in H$. This usually means $\mathscr{G}(\phi_0)$ is reasonably big.

Now suppose (M, g) is a connected Riemannian $n-$manifold with $\mathrm{Hol}(g) = H$. Then, by the holonomy principle (Theorem 1.2.3), there exists a global parallel (hence *closed*) $k-$form ϕ on M which is pointwise linearly identified with ϕ_0. It follows that ϕ also has comass ≤ 1 and, therefore, is a *calibration* on M. Moreover, for each $x \in M$, we have $\mathscr{G}(\phi) \cap T_x M \simeq \mathscr{G}(\phi_0)$, so that by the above invariance we may expect the $\phi-$geometry of M is non-trivial.

In what follows, we explore the above procedure for the holonomy groups $\mathrm{U}(m)$, $\mathrm{SU}(m)$, G_2 and $\mathrm{Spin}(7)$, introducing corresponding interesting calibrated geometries.

Complex submanifolds. Let $H = \mathrm{U}(m) \subseteq \mathrm{SO}(2m)$. Then H preserves the standard Kähler $2-$form ω_0 on $\mathbb{R}^{2m}$. The following classical lemma shows that $\omega_0^k/k!$ has comass ≤ 1 for each $1 \leq k \leq m$ [Law80, Proposition 4, p. 34].

Lemma 2.2.19 (Wirtinger's inequality)**.** *Consider $\mathbb{C}^m = \mathbb{R}^{2m}$ with complex coordinates $z^j = x^{2j-1} + ix^{2j}$, $j = 1, \ldots, m$, and let ω_0 be the standard Kähler form*

$$\omega_0 = \frac{i}{2}\sum_{j=1}^{m} \mathrm{d}z^j \wedge \mathrm{d}\overline{z^j} = \sum_{j=1}^{m} \mathrm{d}x^{2j-1} \wedge \mathrm{d}x^{2j}.$$

Then, for each $1 \leq k \leq n$, given any collection of $2k$ unitary *vectors $v_1, \ldots, v_{2k} \in \mathbb{R}^{2m}$, we have*

$$\frac{\omega_0}{k!}(v_1, \ldots, v_{2k}) \leq 1.$$

Corollary 2.2.20. *Let (Z^{2m}, I, ω) be a Kähler $m-$fold. Then, for each $1 \leq k \leq m$, the $2k-$form $\dfrac{\omega^k}{k!}$ is a calibration on Z. Moreover, an oriented real $2k-$submanifold N in Z is calibrated if, and only if, N is a* complex $k-$dimensional submanifold *of (Z^{2m}, I), i.e. $I(T_x N) = T_x N$ for all $x \in N$.*

There are lots of examples in this setting. For instance, the complex projective spaces $\mathbb{CP}^m$ have many complex submanifolds defined as the zero set of a collection of homogeneous polynomials. These are called *complex algebraic varieties*, and are studied in the subject of *complex algebraic geometry*. In truth, it is worth saying that the motivation for the general calibration condition comes from the long-known properties complex submanifolds enjoy as minimal submanifolds of Kähler manifolds. For more details and examples of complex submanifolds in Kähler manifolds we refer the reader to [Law80, Chapter 1, §6].

Special Lagrangians. Let $H = \mathrm{SU}(m) \subseteq \mathrm{SO}(2m)$. Then H preserves not only the standard Kähler form ω_0 but also the holomorphic volume form Υ_0. It turns out that $\mathrm{Re}(\Upsilon_0)$ is a calibration on $\mathbb{C}^m$. In fact, the following holds [HL82, Theorem 1.14, p. 89]:

Lemma 2.2.21. *Consider $\mathbb{C}^m = \mathbb{R}^{2m}$ with complex coordinates $(z^1, \ldots, z^m)$, let ω_0 be the standard Kähler form and let Υ_0 be the holomorphic volume form*

$$\Upsilon_0 \coloneqq \mathrm{d}z^1 \wedge \ldots \wedge \mathrm{d}z^m.$$

Then

$$|\Upsilon(e_1, \ldots, e_m)| \leq 1,$$

for all unit vectors $e_1, \ldots, e_m \in \mathbb{C}^m$ with equality if, and only if, $V = \mathrm{span}_{\mathbb{R}}\{e_1, \ldots, e_n\}$ is a Lagrangian *plane, i.e. $\omega_0|_V \equiv 0$.*

Corollary 2.2.22. *Let $(Z^{2m}, \omega, \Upsilon)$ be a Calabi–Yau $m-$fold. Then $\mathrm{Re}(e^{\mathrm{i}\theta}\Upsilon)$ is a calibration on Z for any $\theta \in \mathbb{R}$.*

Definition 2.2.23. Let $(Z^{2m}, \omega, \Upsilon)$ be a Calabi–Yau $m-$fold, and let L be an oriented real $m-$submanifold of Z. We call L a **special Lagrangian** submanifold (or SL $m-$fold for short) if L is calibrated with respect to $\mathrm{Re}(\Upsilon)$. More generally, if L is calibrated with respect to $\mathrm{Re}(e^{\mathrm{i}\theta}\Upsilon)$, for some real number $\theta \in \mathbb{R}$, then L is called **special Lagrangian with phase** $e^{\mathrm{i}\theta}$.

Remark 2.2.24. Let L be an oriented real $m-$submanifold of $\mathbb{C}^m$. Then it is easy to see that L is a SL $m-$fold with phase $e^{\mathrm{i}\theta}$ if, and only if, $e^{-\mathrm{i}\theta}L$ is a SL $m-$fold. ♢

By Lemma 2.2.21, we see that a $m-$submanifold L of a Calabi-Yau $m-$fold $(Z^{2m}, \omega, \Upsilon)$ admits an orientation making it into a SL $m-$fold (with phase 1) if, and only if, $\omega|_L \equiv 0$ (i.e. L is Lagrangian) and $\mathrm{Im}(\Upsilon)|_L \equiv 0$. More generally, using Remark 2.2.24, it follows that L admits an orientation making it into a special Lagrangian with phase $e^{\mathrm{i}\theta}$ if, and only if, $\omega|_L \equiv 0$ and $(\cos\theta \mathrm{Im}(\Upsilon) - \sin\theta \mathrm{Re}(\Upsilon))\,|_L \equiv 0$.

For more on special Lagrangian geometry, as well as examples, we refer the reader to [Joy07, Chapter 8].

Associative and coassociative submanifolds. Let $H = \mathrm{G}_2 \subseteq \mathrm{SO}(7)$. The next result follows from [HL82, Theorem 1.4, p. 113] and Remark 2.2.10.

Lemma 2.2.25. *The $3-$form ϕ_0 given by (2.1.5) and the $4-$form $\psi_0 = *\phi_0$ are calibrations on $\mathbb{R}^7$.*

Corollary 2.2.26. *Let (Y, g_ϕ) be a G_2-manifold. Then ϕ and $\psi = *\phi$ are calibrations.*

Definition 2.2.27. An oriented $3-$submanifold P (resp. $4-$submanifold Q) of Y is called **associative** (resp. coassociative) if P (resp. Q) is a $\phi-$calibrated (resp. $\psi-$calibrated) submanifold (cf. Definition 2.2.12).

Example 2.2.28. Consider the model G_2-manifold $(Y, \phi) = (\mathbb{R}^7, \phi_0)$ of Example 2.1.18 and consider the natural orthogonal decomposition $\mathbb{R}^7 = \mathbb{R}^3 \oplus \mathbb{R}^4$ as in Remark 2.1.11. Then, from the definition of ϕ_0 (2.1.5), it is easy to see that (with the natural choices of orientation - as prescribed in Remark 2.1.11)

$$P := \mathbb{R}^3 \times \{0\} \subseteq \mathbb{R}^3 \oplus \mathbb{R}^4 = \mathbb{R}^7 \text{ is associative and}$$
$$Q := \{0\} \times \mathbb{R}^4 \subseteq \mathbb{R}^3 \oplus \mathbb{R}^4 = \mathbb{R}^7 \text{ is coassociative.}$$

More generally, if (Z^4, ω, Υ) is a Calabi-Yau $2-$fold and we let $(Y^7 := \mathbb{R}^3 \times Z, \phi)$ be the G_2-manifold of Example 2.1.23, then (with the obvious orientations)

$$P := \mathbb{R}^3 \times \{0\} \subseteq \mathbb{R}^3 \oplus Z = Y \text{ is associative and}$$
$$Q := \{0\} \times Z \subseteq \mathbb{R}^3 \oplus Z = Y \text{ is coassociative.}$$

The following result, which we state without proof, gives us a good source of examples of associative and coassociative submanifolds.

Proposition 2.2.29. *Let (Y, ϕ) be a G_2-manifold with an isometric involution $\sigma \neq \mathbb{1}$. If $\sigma^*\phi = \phi$ (resp. if $\sigma^*\phi = -\phi$), then*

$$\mathrm{Fix}(\phi) := \{p \in M : \sigma(p) = p\}$$

is a closed embedded associative (resp. coassociative) submanifold in Y.

We refer the reader to [Joy07, pp. 268-269] for a proof of the above result as well as for examples of both associative and a coassociative submanifolds arising in this way.

Example 2.2.30. The recent work [CHNP15] gives various concrete examples of associative submanifolds in G_2-manifolds arising from the twisted connected sum construction.

Next we state a reduction result to lower-dimensional calibrated geometries. Its proof follows rather easily from the compatibility of the involved structures.

Proposition 2.2.31. *Let (Z^6, ω, Υ) be a Calabi-Yau $3-$fold and consider the cylindrical G_2-manifold $(Y := \mathbb{R} \times Z, \phi)$ of Example 2.1.24. Then:*

(a) $N := \mathbb{R} \times \Sigma \subseteq Y$ associative (resp. coassociative) if and only if Σ is a complex curve (resp. special Lagrangian $3-$fold with phase $-i$).

(b) $N \subseteq \{x\} \times Z \subseteq Y$ is associative (resp. coassociative) if and only if N is a special Lagrangian $3-$fold (resp. complex surface).

Cayley submanifolds. Let $H = \mathrm{Spin}(7) \subseteq \mathrm{SO}(8)$. Then one can prove [HL82, Theorem 1.24, p. 118]:

Lemma 2.2.32. *The 4−form Φ_0 given by (2.1.8) is a calibration on $\mathbb{R}^8$.*

Corollary 2.2.33. *Suppose (X^8, Φ) is a Spin(7)−manifold. Then Φ is a calibration on (X, g_Φ).*

Definition 2.2.34. An oriented 4−submanifold L of X is called **Cayley** if L is a Φ−submanifold.

We can make examples of Cayley 4−folds from lower dimensional simpler calibrations cf. Examples 2.1.33 and 2.1.34.

Example 2.2.35. Let (Y^7, ϕ) be a G_2−manifold and consider the Spin(7)−manifold $(X^8 := \mathbb{R} \times Y^7, \Phi)$ given by Example 2.1.33. Then it follows that:

(i) L is an associative 3−fold in Y if and only if $\mathbb{R} \times L$ is Cayley in X.

(ii) For each $x \in \mathbb{R}$, L is a coassociative 4−fold in Y if and only if $\{x\} \times L$ is Cayley in X.

Similarly, we have

Example 2.2.36. Let (Z^8, ω, Υ) be a Calabi-Yau 4−fold and consider the Spin(7)−manifold $\left(X^8 := Z^8, \Phi := \frac{1}{2}\omega \wedge \omega + \mathrm{Re}(\Upsilon_0)\right)$ of Example 2.1.34. Then:

(i) L is a holomorphic surface in Z if and only if L is Cayley in X.

(ii) L is a special Lagrangian 4−fold in Z if and only if L is Cayley in X.

2.3 Anti-self-duality in higher dimensions

We present two well established approaches to the notion of instanton in higher dimensions and show these approaches coincide for (connected) Riemannian n−manifolds (M^n, g) whose holonomy group $\mathrm{Hol}(g)$ is one of the following groups: $\mathrm{U}(m)$ $(n = 2m \geq 4)$, G_2 $(n = 7)$ and Spin(7) $(n = 8)$.

Instantons via closed $(n-4)$−forms. This approach was originally explored by physicists in [CDFN83] for flat spaces; see also [BKS98], and further [Tia00, Section 1.2]. Suppose $n \geq 4$ and let (M^n, g) be an oriented Riemannian manifold. Given $\Xi \in \Omega^{n-4}(M)$, we define the following $*$−Hodge-type operator acting on 2−forms:

$$\begin{aligned} *_\Xi : \Lambda^2 T^*M &\to \Lambda^2 T^*M \\ \omega &\mapsto *(\Xi \wedge \omega). \end{aligned}$$

We note that $*_\Xi$ is trace-free, self-adjoint and satisfies $*_\Xi = 0$ if, and only if, $\Xi = 0$. Of course, for a given $G-$bundle E over M, there is also a natural extension of $*_\Xi$ to $\mathfrak{g}_E-$valued $2-$forms by acting trivially on the $\mathfrak{g}_E-$component. This leads to the following generalization of the $4-$dimensional notion of anti-self-duality.

Definition 2.3.1 ($\Xi-$ASD instantons)**.** Let (M, g) be an oriented Riemannian manifold endowed with a closed $(n-4)-$form $\Xi \in \Omega^{n-4}(M)$. Let E be a $G-$bundle over M, where G is a compact Lie group.

(i) Suppose that G is a semi-simple Lie group. In this case, a connection $\nabla \in \mathfrak{U}(E)$ on E is called a $\Xi-$**anti-self-dual instanton** ($\Xi-$ASD instanton) if

$$*(\Xi \wedge F_\nabla) = -F_\nabla. \tag{2.3.1}$$

(ii) With no semi-simplicity hypothesis on G, it is convenient to relax the above definition as follows (cf. [Wal13a, Remark 1.90, p. 30]). Recalling the decomposition (1.1.3), a connection $\nabla \in \mathfrak{U}(E)$ on E is called a $\Xi-$**anti-self-dual instanton** ($\Xi-$ASD instanton) if the $\mathfrak{g}_E^{(0)}-$component of F_∇, instead of F_∇, satisfies (2.3.1) and the $\mathfrak{z}(\mathfrak{g})-$component of F_∇ is a $\mathfrak{z}(\mathfrak{g})-$valued harmonic $2-$form[5].

Remark 2.3.2. While (ii) indeed generalizes (i), we note that $\Xi-$ASD instantons on a $G-$bundle E are essentially equivalent to $\Xi-$ASD intantons on the associated $G/Z(G)-$bundle $E \times_G G/Z(G)$. In other words, we can always reduce to the semi-simple case (i). ◊

Remark 2.3.3. A particular case of interest encompassed by (ii) is $G = \mathrm{U}(r)$. For later purposes, we introduce some terminology here. The quotient group $\mathrm{U}(r)/Z(\mathrm{U}(r)) \simeq \mathrm{U}(r)/\mathrm{U}(1)$, denoted henceforth by $\mathrm{PU}(r)$, is called the **projective unitary group** of rank r. We call E a $\mathbf{PU}(r)-$**bundle** when E is the associated bundle $\tilde{E} \times_{\mathrm{U}(r)} \mathrm{PU}(r)$ of a $\mathrm{U}(r)-$bundle $\tilde{E}$. ◊

In the classical case where M is an oriented Riemannian $4-$manifold, there is a natural choice of 0-form Ξ, namely $\Xi = *\mathrm{d}V_g = 1$, for which $*_\Xi = *$. Of course, the corresponding $\Xi-$anti-self-duality notion is precisely the familiar one developed in Section 1.5.

For generic Ξ, the algebraic equation (2.3.1) is an over-determined system and have no solutions at all (i.e. -1 need not be an eigenvalue of $*_\Xi$; for instance, when $n = 4$, let Ξ be any constant function $\neq 1$). In any case, in analogy with the classical $4-$dimensional

[5]The $\mathfrak{g}_E^{(0)}-$component of F_∇ is simply its trace-free component F_∇^0, and the $\mathfrak{z}(\mathfrak{g})-$component of F_∇ is simply $\frac{1}{r}\mathrm{tr}(F_\nabla) \otimes \mathbb{1}$. Thus, ∇ is a $\Xi-$ASD instanton if, and only if, $*(\Xi \wedge F_\nabla^0) = -F_\nabla^0$ and $\mathrm{tr}(F_\nabla)$ is a harmonic $2-$form.

notion, if $\nabla \in \mathfrak{U}(E)$ is a $\Xi-$ASD instanton then ∇ is automatically a Yang-Mills connection. Indeed, for the case (i) of Definition 2.3.1, this is an immediate consequence of $\mathrm{d}\Xi = 0$ and the Bianchi identity (1.1.16):

$$\mathrm{d}_\nabla * F_\nabla = -\mathrm{d}_\nabla(\Xi \wedge F_\nabla) = -(\underbrace{\mathrm{d}\Xi}_{=0} \wedge F_\nabla + (-1)^{n-4}\Xi \wedge \underbrace{\mathrm{d}_\nabla F_\nabla}_{=0}) = 0.$$

As for the general case (ii) of Definition 2.3.1, we have (compare [Tia00, Lemma 1.2.1]):

Proposition 2.3.4. *Let E be a $G-$bundle, where G is a compact Lie group, and let $\nabla \in \mathfrak{U}(E)$. If ∇ is a $\Xi-$ASD instanton (cf. Definition 2.3.1 (ii)) then ∇ is a Yang-Mills connection. Moreover, if $G = \mathrm{U}(r)$ and M is closed, we have the following a priori L^2-energy bound on ∇:*

$$\|F_\nabla\|^2_{L^2} - \frac{1}{r}|\mathrm{tr}(F_\nabla)|^2_{L^2} = 4\pi^2\left\langle\left(2c_2(E) - \frac{r-1}{r}c_1(E)^2\right) \cup [\Xi], [M]\right\rangle.$$

Proof. First note that, since $\mathrm{d}(\mathrm{tr}(F_\nabla)) = \mathrm{tr}(\mathrm{d}_\nabla F_\nabla)$ and $\nabla \mathbb{1} = 0$, it follows from the Bianchi identity (1.1.16) together with the Leibniz rule that $\mathrm{d}_\nabla F^0_\nabla = 0$. Furthermore, a straightforward computation gives:

$$\begin{aligned}
\mathrm{d}^*_\nabla F_\nabla &= \mathrm{d}^*_\nabla\left(\frac{1}{r}\mathrm{tr}(F_\nabla)\otimes \mathbb{1}\right) \pm *\mathrm{d}_\nabla\left(*F^0_\nabla\right) \\
&= \frac{1}{r}\left(\mathrm{d}^*\mathrm{tr}(F_\nabla)\right)\otimes \mathbb{1} \mp *\mathrm{d}_\nabla\left(\Xi \wedge F^0_\nabla\right) \quad (F^0_\nabla \text{ is } \Xi-\text{ASD}) \\
&= 0 \mp *\left(\mathrm{d}\Xi \wedge F^0_\nabla + (-1)^{n-4}\Xi \wedge \mathrm{d}_\nabla F^0_\nabla\right) \quad (\mathrm{tr}(F_\nabla) \text{ is harmonic}) \\
&= 0. \quad (\mathrm{d}\Xi = 0 \text{ and } \mathrm{d}_\nabla F^0_\nabla = 0)
\end{aligned}$$

This proves that ∇ is Yang-Mills.

Now suppose $G = \mathrm{U}(r)$ and M is a compact manifold without boundary. First, by (1.3.4) and (1.3.5), we have

$$4\pi^2\left(2c_2(E) - \frac{r-1}{r}c_1(E)^2\right) = \mathrm{tr}(F_\nabla \wedge F_\nabla) - \frac{1}{r}\mathrm{tr}(F_\nabla)\wedge \mathrm{tr}(F_\nabla). \tag{2.3.2}$$

Next, note that the decomposition $F_\nabla = F^0_\nabla + \frac{1}{r}\mathrm{tr}(F_\nabla)\otimes \mathbb{1}$ is L^2-orthogonal; indeed,

$$\begin{aligned}
\langle F^0_\nabla, \mathrm{tr}(F_\nabla)\otimes \mathbb{1}\rangle_{L^2} &= -\int_M \mathrm{tr}\left(F^0_\nabla \wedge *(\mathrm{tr}(F_\nabla)\otimes \mathbb{1})\right) \\
&= -\int_M \mathrm{tr}(F^0_\nabla \wedge *\mathrm{tr}(F_\nabla)) \\
&= -\int_M \mathrm{tr}(F^0_\nabla)\wedge *\mathrm{tr}(F_\nabla) \\
&= 0.
\end{aligned}$$

Thus

$$\|F_\nabla\|^2_{L^2} = \|F^0_\nabla\|^2_{L^2} + \left\|\frac{1}{r}\mathrm{tr}(F_\nabla)\otimes \mathbb{1}\right\|^2_{L^2} = \|F^0_\nabla\|^2_{L^2} + \frac{1}{r}|\mathrm{tr}(F_\nabla)|^2_{L^2}. \tag{2.3.3}$$

Now, using F^0_∇ is $\Xi-$ASD gives

$$\|F^0_\nabla\|^2_{L^2} = -\int_M \mathrm{tr}(F^0_\nabla \wedge *F^0_\nabla) = \int_M \mathrm{tr}(F^0_\nabla \wedge F^0_\nabla \wedge \Xi).$$

On the other hand,

$$F^0_\nabla \wedge F^0_\nabla = F_\nabla \wedge F_\nabla - \frac{2}{r}\mathrm{tr}(F_\nabla) \wedge F_\nabla + \frac{1}{r^2}\mathrm{tr}(F_\nabla) \wedge \mathrm{tr}(F_\nabla) \otimes \mathbb{1}.$$

Therefore,

$$\begin{aligned}\|F^0_\nabla\|^2_{L^2} &= \int_M \mathrm{tr}(F_\nabla \wedge F_\nabla \wedge \Xi) - \frac{2}{r}\int_M \mathrm{tr}(F_\nabla) \wedge \mathrm{tr}(F_\nabla) \wedge \Xi + \frac{\mathrm{tr}(\mathbb{1})}{r^2}\int_M \mathrm{tr}(F_\nabla) \wedge \mathrm{tr}(F_\nabla) \wedge \Xi \\ &= \int_M \mathrm{tr}(F_\nabla \wedge F_\nabla \wedge \Xi) - \frac{1}{r}\int_M \mathrm{tr}(F_\nabla) \wedge \mathrm{tr}(F_\nabla) \wedge \Xi\end{aligned}$$

Plugging this last equation in (2.3.3) and comparing with (2.3.2) gives the desired result. ■

Remark 2.3.5. The above result, e.g. with $G = \mathrm{SU}(r)$, should be compared with Proposition 2.2.13. In fact, these results provides various similarities between $\Xi-$ASD instantons and $\Xi-$calibrated submanifolds: both are first order solutions of second order Euler-Lagrange equations. Furthermore, these solutions in fact minimize their respective defining (energy/volume) functionals, attaining topological (energy/volume) lower bounds. ◊

The following simple linear algebra result is the essence of the importance of calibrated submanifolds in the study of gauge theory, via the bubbling phenomena we will study in Chapter 4. In fact, it is the reason why the bubbling locus of a sequence of $\Xi-$ASD instantons is $\Xi-$calibrated (cf. Theorem 4.4.6) and why ASD instantons bubbles off transversely (cf. Theorem 4.3.6).

Proposition 2.3.6 (ASD instantons bubbles off transversely)**.** *Suppose $n > 4$ and consider $(\mathbb{R}^n, g_0)$ with the standard flat metric g_0. Let $\Xi \in \Omega^{n-4}(\mathbb{R}^n)$ be a closed $(n-4)-$form and let $\mathbb{R}^n = \mathbb{R}^{n-4} \oplus \mathbb{R}^4$ be an orthogonal decomposition, with associated projection map $\pi : \mathbb{R}^n \to \mathbb{R}^4$. Let E be a $G-$bundle over $\mathbb{R}^4$ where G is a compact Lie group. If $I \in \mathfrak{U}(E)$ is a* non-flat *connection then the following are equivalent:*

(i) $\nabla := \pi^* I$ is a $\Xi-$ASD instanton.

(ii) There exists an orientation on $\mathbb{R}^{n-4}$ with respect to which it is calibrated[6] by Ξ and I is an ASD instanton on $\mathbb{R}^4$.

[6] Although we do not suppose Ξ has comass ≤ 1, the meaning of the statement is that there exists an orientation on $\mathbb{R}^{n-4}$ with respect to which Ξ is the volume form on $\mathbb{R}^{n-4}$ induced by $g_0|_{\mathbb{R}^{n-4}}$.

Proof. For simplicity, we can assume without loss of generality that G is semi-simple, so that we are in case (i) of Definition 2.3.1. In fact, otherwise we can just work in the associated $G/Z(G)-$bundle (see Remark 2.3.2).

Let $x^1, \ldots, x^n$ be oriented orthonormal coordinates of $\mathbb{R}^n$ such that $x^1, \ldots, x^{n-4}$ are coordinates for $\mathbb{R}^{n-4}$. Put

$$\Phi_{n-4} := \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{n-4} \quad \text{and} \quad \Phi_4 := *\Phi_{n-4} = \mathrm{d}x^{n-3} \wedge \ldots \wedge \mathrm{d}x^n,$$

and write

$$\Xi = \alpha\Phi_{n-4} + \Xi_0,$$

for some $\alpha \in \mathbb{R}$ and $\Xi_0 \in \Omega^{n-4}(\mathbb{R}^n)$ such that $\Xi_0|_{\mathbb{R}^{n-4}} = 0$. Then, it is clear that

$$*(\Xi \wedge F_\nabla) = \alpha * (\Phi_{n-4} \wedge F_\nabla). \tag{2.3.4}$$

(i)$\Rightarrow$(ii): Choose (temporarily) Φ_4 to be the orientation of $\mathbb{R}^4$. Then, from (2.3.4) and the assumption of (i) we get[7]

$$-F_I = \alpha *_{\mathbb{R}^4} F_I.$$

Since $F_I \neq 0$, it follows that $\alpha = \pm 1$ (the possible eigenvalues of $*$ on $\Lambda^2(\mathbb{R}^4)^*$). If $\alpha = 1$, we are done. If $\alpha = -1$, just choose the reverse orientation on $\mathbb{R}^4$ (note that this changes $*_{\mathbb{R}^4}$ by a minus sign).

(ii)$\Rightarrow$(i): We can assume that Φ_{n-4} is positively oriented with respect to the orientation on $\mathbb{R}^{n-4}$ predicted by (ii). For, otherwise, recalling that $n > 4$, we can simply make the coordinate change $(x^1, \ldots, x^{n-4}, x^{n-3}, \ldots, x^n) \mapsto (x^1, \ldots, -x^{n-4}, -x^{n-3}, \ldots, x^n)$.

Thus, by assumption, $\alpha = 1$. Also, we fix the compatible orientation given by Φ_4 on $\mathbb{R}^4$. Then, using (2.3.4) and the hypothesis that F_I is ASD, we get

$$*(\Xi \wedge F_\nabla) = \pi^*(*_{\mathbb{R}^4} F_I) = -\pi^* F_I = -F_\nabla,$$

as we wanted. ∎

Appropriate $(n-4)-$forms and Riemannian holonomy groups. (cf. [ACD02, pp. 8-9] and [Sal89, p. 61]) In this paragraph we show that manifolds with reduced holonomy gives us an appropriate setting in which we can find natural closed $(n-4)-$forms Ξ for which the $\Xi-$ASD criterion (2.3.1) is not void.

Let M be a $n-$manifold, where $n \geq 4$. We will call $\Xi \in \Omega^{n-4}(M)$ an **appropriate** $(n-4)-$**form** on M when the symmetric operator $*_\Xi$ (2.3.1) admits the identically -1 constant function as one of its (necessarily real-valued) eigenvalues.

[7]Recall that $F_\nabla = \pi^* F_I$ and that π is a submersion.

A basic observation is the following. Suppose $H \subseteq \mathrm{SO}(n)$ is a Lie subgroup preserving a nonzero 4−form $\Psi_0 \in \Lambda^4(\mathbb{R}^n)^*$. Then, if M is endowed with an H−structure $\mathcal{P} \subseteq \mathcal{F}(M)$, we automatically get a corresponding well-defined nowhere zero 4−form Φ on M pointwise linearly identified with Ψ_0. Explicitly, $\Phi \in \Omega^4(M)$ is defined by putting

$$\Psi_x \coloneqq (u_x^{-1})^*\Psi_0,$$

for any chosen frame $u_x \in \mathcal{P}_x$, for all $x \in M$. This is well-defined since any two frames $u_x, \tilde{u}_x \in \mathcal{P}_x$ are related by the right multiplication of an element in H: $\tilde{u}_x = h^{-1} \circ u_x$, for some $h \in H$. Thus, the H−invariance of Ψ_0 ensures $(u_x^{-1})^*\Psi_0 = (\tilde{u}_x^{-1})^*\Psi_0$.

In fact, by the same reasoning, since $H \subseteq \mathrm{SO}(n)$, it follows that M has the structure of an oriented Riemannian manifold. In particular, we are able to define $\Xi \coloneqq *\Psi \in \Omega^{n-4}(M)$. Now note that the matrix of $*_\Xi$ with respect to any H−frame $u \in \mathcal{P}$ is constant and equal to the matrix of the operator $*_{\Xi_0}$ acting on $\Lambda^2(\mathbb{R}^n)^*$, where $\Xi_0 \coloneqq *\Psi_0 \in \Lambda^{n-4}(\mathbb{R}^n)^*$. Since $*_{\Xi_0}$ is a nonzero symmetric operator (indeed, $\Xi_0 \neq 0$), it admits a nonzero eigenvalue $0 \neq \lambda \in \mathbb{R}$. Thus, it follows that $-\lambda^{-1}\Xi$ is an appropriate $(n-4)$−form on M.

There are several examples of subgroups $H \subseteq \mathrm{SO}(n)$ admitting nonzero H−invariant 4−forms. Suppose $H \subseteq \mathrm{SO}(n)$ is a closed Lie subgroup with Lie algebra $\mathfrak{h} \subseteq \mathfrak{so}(n) \simeq \Lambda^2(\mathbb{R}^n)^*$. Then, the Killing form $K_\mathfrak{h}$ of $\mathfrak{h}$ can be seen as an H−invariant element of $S^2(\mathfrak{h}) \subseteq S^2\left(\Lambda^2(\mathbb{R}^n)^*\right)$. Thus, we can define a corresponding H−invariant 4−form Ψ_0^H on $\mathbb{R}^n$ by

$$\Psi_0^H \coloneqq \mathrm{alt}(K_\mathfrak{h}) \in \left(\Lambda^4(\mathbb{R}^n)^*\right)^H,$$

where $\mathrm{alt} : S^2\left(\Lambda^2(\mathbb{R}^n)^*\right) \to \Lambda^4(\mathbb{R}^n)^*$ denotes the alternation (or wedging) map. If M is a manifold endowed with an H−structure, we denote by Ψ^H the induced 4−form on M. Exploiting this situation with H being the holonomy group of a Riemannian manifold yields the following result [Sal89, Lemma 5.3, p. 61].

Lemma 2.3.7. *Let (M, g) be a Riemannian manifold with holonomy group $H \subset \mathrm{SO}(n)$. Then, the above procedure defines a nonwhere zero parallel 4−form Ψ^H on M, except possibly when H is the isotropy representation of a symmetric space.*

Ψ^H is often called the *fundamental 4−form* associated to the holonomy reduction. Since Ψ^H is D^g−parallel, it follows that it is a harmonic 4−form, so that $\Xi^H \coloneqq *\Psi^H$ defines a closed $(n-4)$−form on M. As observed earlier, modulo rescaling, Ξ^H defines an appropriate $(n-4)$−form on M.

Now suppose $H \subseteq \mathrm{SO}(n)$ is a simple Lie group, e.g. $H = \mathrm{G}_2 \subseteq \mathrm{SO}(7)$ or $\mathrm{Spin}(7) \subseteq \mathrm{SO}(8)$ (cf. Theorems 2.1.12 and 2.1.26). Since Ξ_0^H is by construction H−invariant, the operator $*_{\Xi_0^H}$ trivially commutes with the action of H, so by Schur's lemma the irreducible representations of H in $\Lambda^2(\mathbb{R}^n)^*$ are eigenspaces for $*_{\Xi_0^H}$. Since H is simple, it follows that the Lie algebra $\mathfrak{h} \subseteq \mathfrak{so}(n) \simeq \Lambda^2(\mathbb{R}^n)^*$ is an eigeinspace for $*_{\Xi_0^H}$.

In the situation of Lemma 2.3.7, it follows that the natural subbundle $\tilde{\mathfrak{h}} \subseteq \Lambda^2 T^*M$ determined by $\mathfrak{h}$ is one of the eigenbundles of the operator $*_{\Xi^H}$. If $0 \neq \lambda =$ const. is the corresponding eigenvalue, then scaling Ξ^H by $-\lambda^{-1}$ furnishes an appropriate closed $(n-4)-$form $\tilde{\Xi}^H$ on M whose -1 eigenbundle is precisely $\tilde{\mathfrak{h}}$. The corresponding $\tilde{\Xi}^H-$anti-self-duality notion is then a natural constraint coming from the holonomy reduction of (M, g), in analogy with the constraints imposed by the holonomy group on the Riemann curvature tensor (cf. Proposition 2.1.3). This leads us to another version of the notion of instanton in higher dimensions, which we briefly describe in the next paragraph.

Instantons via Lie groups. There is a generalized notion of instanton available for any oriented Riemannian $n-$manifold equipped with an $N(H)-$structure [Car98]. In what follows, we sketch the basic idea of this theory.

Let $H \subseteq \mathrm{SO}(n)$ be a closed Lie subgroup. Then we can write

$$\Lambda^2(\mathbb{R}^n)^* \simeq \mathfrak{so}(n) = \mathfrak{h} \oplus \mathfrak{h}^\perp. \tag{2.3.5}$$

Let (M, g) be an oriented Riemannian $n-$manifold; recall that

$$\Lambda^2 T^*M = \mathcal{F}(M) \times_{\mathrm{SO}(n)} \Lambda^2(\mathbb{R}^n)^*.$$

Note that if M has an $H-$structure then the decomposition (2.3.5) readily passes to $2-$forms on M. But it turns out that, in practice, one often has an $N(H)-$structure instead of an $H-$structure. Here $N(H) \supseteq H$ denotes the normalizer of H inside $\mathrm{SO}(n)$. So, suppose (M, g) has an $N(H)-$structure $\mathcal{P} \subseteq \mathcal{F}(M)$. Since H is closed, it follows that $N(H)$ is a closed Lie subgroup of $\mathrm{SO}(n)$. Moreover, one can easily verify that $\mathfrak{h} \subseteq \mathfrak{n}(\mathfrak{h}) =: \mathrm{Lie}(N(H))$ is invariant under the adjoint action of $N(H)$. This means that the lie algebra $\mathfrak{h}$ determines a distinguished subbundle $\tilde{\mathfrak{h}}$ of $\Lambda^2 T^*M$:

$$\tilde{\mathfrak{h}} := \mathcal{P} \times_{N(H)} \mathfrak{h} \subseteq \Lambda^2 T^*M.$$

In particular, if $E \to M$ is a $G-$bundle over M, using the metric g we get an associated orthogonal projection map

$$\pi_{\mathfrak{h}} : \Lambda^2 T^*M \otimes \mathfrak{g}_E \to \tilde{\mathfrak{h}} \otimes \mathfrak{g}_E.$$

This leads to the following generalized notion of instanton.

Definition 2.3.8. Suppose M is an $n-$manifold endowed with an $N(H)-$structure, and let E be a $G-$bundle over M where G is a compact semi-simple Lie group. A connection $\nabla \in \mathfrak{U}(E)$ is called an $H-$**instanton** if

$$\pi_{\mathfrak{h}} F_\nabla = F_\nabla,$$

i.e. if the curvature $2-$forms F^i_j lies in the subspace $\mathfrak{h} \subseteq \Lambda^2$.

n	H	$N(H) \subseteq \mathrm{SO}(n)$
4	$\mathrm{SU}(2)$	$\mathrm{SO}(4)$
$2m > 4$	$\mathrm{SU}(m)$	$\mathrm{U}(m)$
7	G_2	G_2
8	$\mathrm{Spin}(7)$	$\mathrm{Spin}(7)$

Table 2.1: Certain Lie groups $H \subseteq SO(n)$ whose normalizer $N(H)$ in $SO(n)$ is a Lie group appearing in Berger's list.

For instance, in the classical case where M is an oriented Riemannian 4−manifold, i.e. when M comes equiped with an $N(H) = \mathrm{SO}(4)$−structure, where $H = \mathrm{SU}(2)$, then an $\mathrm{SU}(2)$−instanton corresponds to the ordinary notion of (A)SD instanton: $\mathfrak{h} = \mathfrak{su}(2) \simeq \Lambda^2_\pm$.

In looking for manifolds endowed with an $N(H)$−structure, a natural ideia is to consider Riemannian manifolds of reduced holonomy. In particular, we are led to consider each $N(H)$ arrising in Berger's list (2.1.4) of special geometries.

Let (M, g) be an oriented Riemannian n−manifold with $\mathrm{Hol}(g) = N(H) \subsetneq \mathrm{SO}(n)$, where $N(H) \neq \mathrm{SO}(4)$ is one of the groups in Table 2.1 (cf. [Car98, p. 6, Table 1.]). By the construction of the last paragraph, we have a naturally associated parallel 4−form $\Phi^{N(H)}$ arising from the holonomy reduction. It turns out that the corresponding notions of H−instanton and $\tilde{\Xi}^{N(H)}$−ASD instanton induced on auxiliary G−bundles $E \to M$ are coincident, provided $\tilde{\Xi}^{N(H)}$ is an appropriate rescaling of $\Xi^{N(H)}$. In the next sections we briefly study each of these cases.

2.3.1 Hermitian-Yang-Mills connections

We start with the following definition, which can be motivated by the discussion in Section 1.5.

Definition 2.3.9. Let (Z, ω) be a Kähler manifold and let E be an $\mathrm{SU}(r)$− or a $\mathrm{PU}(r)$−bundle over Z. A connection $\nabla \in \mathfrak{U}(E)$ is called **Hermitian-Yang-Mills** (HYM) if

$$F_\nabla^{0,2} = 0 \quad \text{and} \quad \Lambda_\omega F_\nabla = 0. \tag{2.3.6}$$

Here Λ_ω is the dual of the Lefschetz operator $L_\omega := \omega \wedge \cdot$.

Remark 2.3.10. Recalling Definition 2.3.1 (ii) and Remark 2.3.2, one can also work with $\mathrm{U}(r)$−bundles and instead of the second part of (2.3.6) require that $\Lambda_\omega F_\nabla = \lambda \mathbb{1}_E$, for some $\lambda \in \mathbf{i}\mathbb{R}$. In this case, if we suppose (Z, ω) is *compact* (without boundary) Kähler manifold, the value of λ is fixed by topological data as follows. Denote by g the underlying Riemannian metric of (Z, ω); thus,

$$\mathrm{d}V_g = \frac{\omega^m}{m!} \quad \text{and} \quad * \, \omega = \frac{\omega^{m-1}}{(m-1)!}.$$

Recalling (1.3.4), it follows that

$$c_1(E) \wedge *\omega = \lambda r \frac{\omega^m}{m!}.$$

Therefore

$$\lambda = \frac{m\langle c_1(E) \cup [\omega]^{m-1}, [Z]\rangle}{r\langle[\omega], [Z]\rangle}.$$

◊

It follows from Corollary 1.5.10 that a HYM connection $\nabla \in \mathfrak{U}(E)$ induces a holomorphic structure $\mathcal{E}$ on E.

In order to clarify the importance of HYM connections, we now recall the so-called Donaldson-Uhlenbeck-Yau correspondence [Don85][UY86], albeit we stress that it is beyond the scope of this text, and the author's expertise, to provide a thorough discussion of this result; the interested reader is referred to the excellent books [LT95, Kob14].

Suppose that (Z, ω) is a *compact* Kähler manifold and let $\mathcal{E} \to Z$ be a holomorphic vector bundle. For a coherent subsheaf $\mathcal{F} \subseteq \mathcal{E}$, we put[8]

$$\begin{aligned} c_1(\mathcal{F}) &:= c_1(\det \mathcal{F}^{**}), \\ \deg_\omega(\mathcal{F}) &:= \int_Z c_1(\mathcal{F}) \wedge \omega^{m-1}, \\ \mu_\omega(\mathcal{F}) &:= \frac{\deg_\omega(\mathcal{F})}{\operatorname{rank}(\mathcal{F})}. \end{aligned}$$

$\mathcal{E}$ is called:

- **stable** if $\mu_\omega(\mathcal{F}) < \mu_\omega(\mathcal{E})$, for each coherent subsheaf $\mathcal{F} \subseteq \mathcal{E}$ with $0 < \operatorname{rank}(\mathcal{F}) < \operatorname{rank}(\mathcal{E})$.
- **polystable** if $\mathcal{E} = \bigoplus_i \mathcal{E}_i$ where each $\mathcal{E}_i$ is stable and satisfies $\mu_\omega(\mathcal{E}_i) = \mu_\omega(\mathcal{E})$.

In these terms, we can state the following deep result, which was proved by Donaldson [Don85] for complex algebraic surfaces, and proved by Uhlenbeck-Yau [UY86] for compact Kähler manifolds:

Theorem 2.3.11 (Donaldson-Uhlenbeck-Yau)**.** *Let E be an* $\mathrm{SU}(r)-$ *or a* $\mathrm{PU}(r)-$*bundle over a* compact *Kähler manifold (Z, ω). There exists a one-to-one correspondence between gauge equivalence classes of HYM connections on E and isomorphism classes of polystable holomorphic bundles $\mathscr{E}$ whose underlying bundle is E.*

This gives a very general relation between Yang-Mills theory over Kähler manifolds (differential geometry) and Mumford-Takemoto's theory of stability (algebraic geometry).

The following straightforward result realizes HYM connections as a particular instance of $\Xi-$ASD instantons, for suitable choice of Ξ.

[8] $\mathcal{F}^* := \operatorname{Hom}(\mathcal{F}, \mathcal{O}_Z)$, where $\mathcal{O}_Z$ is the structure sheaf of Z.

Lemma 2.3.12. *Let E be an $\mathrm{SU}(r)-$ or a $\mathrm{PU}(r)-$bundle over a Kähler manifold (Z^{2m},ω) of complex dimension $m \geq 2$. Consider the following closed $(2m-4)-$form Ξ on Z:*

$$\Xi := \frac{\omega^{m-2}}{(m-2)!}.$$

Then, a connection $\nabla \in \mathfrak{A}(E)$ is a $\Xi-$ASD instanton if, and only if, ∇ is HYM.

Finally, let $H = \mathrm{SU}(m) \subseteq \mathrm{SO}(2m)$, so that $N(H) = \mathrm{U}(m)$. If ω_0 is the standard Kähler form on $\mathbb{R}^{2m}$, then (cf. [Sal89, Chapter 3])

$$\Lambda^2(\mathbb{R}^{2m})^* = [[\Lambda^{2,0}]] \oplus [\Lambda_0^{1,1}] \oplus \langle \omega_0 \rangle,$$

where $[[\Lambda^{2,0}]] \otimes \mathbb{C} = \Lambda^{2,0} \oplus \Lambda^{0,2}$ and $[\Lambda_0^{1,1}] \otimes \mathbb{C} = \Lambda_0^{1,1} := \ker\left(\Lambda_{\omega_0}|_{\Lambda^{1,1}}\right)$. Furthermore, via the isomorphism $\mathfrak{so}(2m) \simeq \Lambda^2(\mathbb{R}^{2m})^*$, one has

$$\mathfrak{h} = \mathfrak{su}(m) \simeq [\Lambda_0^{1,1}].$$

It follows that, for $\mathrm{SU}(r)-$ or $\mathrm{PU}(r)-$bundles, the notion of $\mathrm{SU}(m)-$instantons (cf. Definition 2.3.8) coincides with the notion of HYM connections.

2.3.2 G_2-instantons

This section is based on [Wal13a, Chapter 1] and [SW10].

We start citing a key result [SW10, Theorem 8.4]:

Proposition 2.3.13. *$\Lambda^2(\mathbb{R}^7)^*$ decomposes orthogonally into*

$$\Lambda^2(\mathbb{R}^7)^* = \Lambda_7^2 \oplus \Lambda_{14}^2,$$

where Λ_7^2 and Λ_{14}^2 are irreducible representations of G_2, with $\dim \Lambda_d^2 = d$, given by

$$\Lambda_7^2 := \{\alpha : *_{\phi_0}\alpha = 2\alpha\} = \{v \lrcorner \phi_0 : v \in \mathbb{R}^7\} \simeq \Lambda_7^1, \quad \textit{and}$$
$$\Lambda_{14}^2 := \{\alpha : *_{\phi_0}\alpha = -\alpha\} = \{\alpha : \alpha \wedge \psi_0 = 0\} \simeq \mathfrak{g}_2 \equiv \mathrm{Lie}(\mathrm{G}_2),$$

where $\psi_0 := *\phi_0$, and the last isomorphism comes from the metric identification $\Lambda^2(\mathbb{R}^7) \simeq \mathfrak{so}(7) \supseteq \mathfrak{g}_2$.

It follows that we have an analogous splitting of $\Lambda^2 T^*Y$ for every almost G_2-manifold (Y^7,ϕ). By slight abuse of notation, we will also denote the corresponding summands by Λ_d^2. Moreover, we can now extend for general compact Lie groups G the notion of G_2-instanton on $G-$bundles given in Definition 2.3.8 as follows.

Definition 2.3.14. Let (Y^7, g_ϕ) be a G_2-manifold and let E be a $G-$bundle over a Y, where G is a compact Lie group. A connection $\nabla \in \mathfrak{U}(E)$ is called a G_2-**instanton** if ∇ is a $\phi-$ASD instanton (cf. Definition 2.3.1).

Remark 2.3.15. In the above situation, suppose further that G is semi-simple. Then, by Proposition 2.3.13, a G_2-instanton $\nabla \in \mathfrak{U}(E)$ is characterized by the following equivalent conditions:

(i) $*(\phi \wedge F_\nabla) = -F_\nabla$;

(ii) $F_\nabla \wedge \psi = 0$;

(iii) $\pi_7(F_\nabla) = 0$, where π_7 denotes the orthogonal projection from $\Lambda^2 T^*Y$ to Λ^2_7;

(iv) The curvature tensor F_∇ lies in the subspace $\mathfrak{g}_2 \otimes (\mathfrak{g}_E)_y \subseteq \Lambda^2 T^*_y Y \otimes (\mathfrak{g}_E)_y$ at each point $y \in Y$. ♢

Example 2.3.16. By Proposition 2.1.3 and Remark 2.3.15 (iv), the Levi-Civita connection D^{g_ϕ} of a connected G_2-manifold (Y^7, g_ϕ) is a G_2-instanton on the tangent bundle TY.

In the next example, we give an extension for the case $n = 7$ and $\Xi = \phi_0$ of Proposition 2.3.6.

Example 2.3.17 (G_2-instantons from ASD instantons)**.** Let (Z^4, ω, Υ) be a Calabi-Yau 2−fold and consider the G_2-manifold (Y^7, ϕ) of Example 2.1.23, where $Y^7 := \mathbb{R}^3 \times Z^4$ (resp. $Y^7 := T^3 \times Z^4$); write $\pi_Z : Y \to Z$ for the natural projection map. The following result relates $\mathbb{R}^3-$invariant (resp. T^3-invariant) G_2-instantons over Y with ASD connections over Z:

Proposition 2.3.18. *Let E be a $G-$bundle over Z, where G is a compact semi-simple Lie group. A connection $I \in \mathfrak{U}(E)$ is an ASD instanton if, and only if, $\nabla := \pi_Z^* I$ is a G_2-instanton.*

Proof. Note that

$$*(\phi \wedge F_\nabla) = *(\mathrm{d}x^{123} \wedge F_\nabla) = \pi_Z^*(*_Z F_I),$$

where in the first equality we used that $\omega \wedge F_\nabla = \mathrm{Re}(\Upsilon) \wedge F_\nabla = \mathrm{Im}(\Upsilon) \wedge F_\nabla = 0$, and on the second equality we used the compatibility of ϕ with the product structures on Y. Since π_Z is a submersion and $F_\nabla = \pi_Z^* F_I$, the result follows. ■

Remark 2.3.19 (G_2−instanton equations in $\mathbb{R}^7$)**.** Consider the model G_2−manifold $(\mathbb{R}^7, \phi_0)$ of Example 2.1.18. Let $\tilde{E}$ be a (necessarily trivial) G−bundle over $\mathbb{R}^7$, where G is a compact semi-simple Lie group, and let $\nabla \in \mathfrak{U}(\tilde{E})$. Letting $x^1, \ldots, x^7$ denote the standard Euclidean coordinates in $\mathbb{R}^7$, we write

$$F_\nabla = \frac{1}{2}\sum F_{ij} \otimes \mathrm{d}x^i \wedge \mathrm{d}x^j, \quad F_{ij} : \mathbb{R}^7 \to \mathfrak{g}.$$

By Remark 2.3.15, ∇ is a G_2−instanton if and only if $F_\nabla \wedge \psi_0 = 0$. Here $\psi_0 := *\phi_0$ can be written as

$$\psi_0 = \mathrm{d}x^{4567} - \mathrm{d}x^{1247} - \mathrm{d}x^{1256} - \mathrm{d}x^{2345} - \mathrm{d}x^{2367} - \mathrm{d}x^{3146} - \mathrm{d}x^{3175}.$$

By straightforward calculations, we get:

$$\begin{aligned} F_\nabla \wedge \psi_0 \quad = \quad & (-F_{16} + F_{25} - F_{34}) \otimes \mathrm{d}x^{123456} + (-F_{17} + F_{24} + F_{35}) \otimes \mathrm{d}x^{123457} + \\ & (-F_{14} - F_{27} + F_{36}) \otimes \mathrm{d}x^{123467} + (-F_{15} - F_{26} - F_{37}) \otimes \mathrm{d}x^{123567} + \\ & (F_{12} - F_{47} - F_{56}) \otimes \mathrm{d}x^{124567} + (F_{13} + F_{46} - F_{57}) \otimes \mathrm{d}x^{134567} + \\ & (F_{23} - F_{45} - F_{67}) \otimes \mathrm{d}x^{234567}. \end{aligned}$$

Hence, ∇ is a G_2−instanton if, and only if,

$$\begin{cases} F_{25} = F_{16} + F_{34}; & F_{12} = F_{47} + F_{56}; \\ F_{17} = F_{24} + F_{35}; & F_{57} = F_{13} + F_{46}; \\ F_{36} = F_{14} + F_{27}; & F_{23} = F_{45} + F_{67}; \\ F_{62} = F_{15} + F_{37}. & \end{cases} \tag{2.3.7}$$

Now write $\mathbb{R}^7 = \mathbb{R}^3 \oplus \mathbb{R}^4$ as in Remark 2.1.11. Denoting by $\pi : \mathbb{R}^7 \to \mathbb{R}^4$ the natural projection, suppose that $\tilde{E} = \pi^* E$ is the pull-back of a G−bundle E over $\mathbb{R}^4$ and that $\nabla = \pi^* I$, for some $I \in \mathfrak{U}(E)$. Then, it follows explicitly from (2.3.7) and (1.5.2) that ∇ is a G_2−instanton if, and only if, I is an ASD instanton. Recalling from Example 2.2.28 that $\mathbb{R}^3 \times \{0\} \subseteq \mathbb{R}^7$ is ϕ_0−calibrated, this gives an explicit instance of Proposition 2.3.6 (for $n = 7$ and $\Xi = \phi_0$). ◊

Example 2.3.20 (G_2−instantons from HYM-connections)**.** Let (Z^6, ω, Υ) be a Calabi-Yau 3−fold and consider the G_2−manifold (Y^7, ϕ) of Example 2.1.24, where $Y^7 := \mathbb{R} \times Z^6$ (resp. $Y^7 := S^1 \times Z^6$); denote by $\pi_Z : Y \to Z$ the natural projection map. The following result relates $\mathbb{R}$−invariant (resp. S^1−invariant) G_2−instantons over Y with HYM connections over Z (cf. [SE15, Proposition 8] or [SEW15, Proposition 3.10]):

Proposition 2.3.21. *Let E be an* $\mathrm{SU}(r)$− *or a* $\mathrm{PU}(r)$−*bundle over Z. A connection $\nabla \in \mathfrak{U}(E)$ is HYM if, and only if, $\pi_Z^* \nabla$ is a* G_2−*instanton.*

Sketch of proof. The main point is to note that, in this context of a Calabi-Yau 3−fold, the HYM condition (2.3.6) is equivalent to

$$F_\nabla \wedge \mathrm{Im}(\Upsilon) = 0 \quad \text{and} \quad F_\nabla \wedge \omega \wedge \omega = 0.$$

Hence, noting that

$$\psi = *(\mathrm{d}t \wedge \omega + \mathrm{Re}(\Upsilon)) = \frac{1}{2}\omega \wedge \omega - \mathrm{d}t \wedge \mathrm{Im}(\Upsilon),$$

and using the fact that $\pi_Z^*\nabla$ is a G_2-instanton precisely when $F_{\pi_Z^*\nabla} \wedge \psi = 0$, the result follows. ■

Remark 2.3.22. This basic result gives a way to obtain G_2-instantons on the halves $Y^7_\pm := S^1 \times Z^6_\pm$ of Kovalev's twisted connected sum construction, by solving the HYM problem on $Z^6_\pm$. This is indeed the essential motivation for the analysis in Sá Earp's works [SE09, SE15]. ♢

Example 2.3.23. At the time of writing, non-trivial examples of G_2-instantons are hard to come by. Until recently, some progress have been made exploiting the constructions of the known examples of G_2-manifolds.

In his PhD thesis [SE09], Sá Earp started the project of studying G_2-instantons over twisted connected sums [Kov03], and made significant progress towards the problem of obtaining a HYM connection on each asymptotically cylindrical Calabi-Yau halve of Kovalev's construction.

Later, on the other hand, Walpuski [Wal13a, Wal13b] presented a general method for constructing G_2-instantons over G_2-manifolds arising from Joyce's generalized Kummer construction [Joy96a, Joy00], providing some concrete examples with structure group SO(3). Moreover, building on Sá Earp's work [SE15], Sá Earp and Walpuski [SEW15] have recently provided a sufficient condition to produce G_2-instantons over G_2-manifolds arising from the twisted connected sum construction [Kov03, CHNP15] (although they not provide examples of the required input).

More recently, Jacob and Walpuski [JW16] proved an analogue of the Donaldson–Uhlenbeck–Yau theorem (cf. Theorem 2.3.11) for asymptotically cylindrical Kähler manifolds, handling reflexive sheaves. This provides examples of (singular) HYM connections over a certain class of complete non-compact Kähler manifolds, generalizing the result of Sá Earp [SE15].

Another interesting construction of G_2-instantons, due to Clarke [Cla14], provides non-trivial examples of G_2-instantons on the trivial $\mathrm{SU}(2)-$bundle over the Bryant-Salamon [BS89] total space of the spinor bundle $\mathbb{S}(S^3)$ of the round 3−sphere S^3.

Topological energy bounds from Chern-Weil theory. Suppose (Y^7, g_ϕ) is a compact G_2-manifold and let E be an $\mathrm{SU}(r)-$bundle over Y. For $\nabla \in \mathfrak{U}(E)$, write $F_\nabla = F_\nabla^7 \oplus F_\nabla^{14}$ according to the decomposition of Λ^2 induced by $*_\phi$. Define the topological number

$$\kappa(E,[\phi]) \coloneqq \langle c_2(E) \cup [\phi], [Y] \rangle .$$

Then we have:

$$\begin{aligned} 8\pi^2\kappa(E,[\phi]) &= \int_Y \mathrm{tr}(F_\nabla \wedge F_\nabla) \wedge \phi \\ &= -\langle F_\nabla, *(F_\nabla \wedge \phi)\rangle \\ &= -\langle F_\nabla^7 + F_\nabla^{14}, 2F_\nabla^7 - F_\nabla^{14}\rangle \\ &= -2\|F_\nabla^7\|_{L^2}^2 + \|F_\nabla^{14}\|_{L^2}^2 \end{aligned}$$

On the other hand, $\mathcal{YM}(\nabla) = \|F_\nabla^7\|_{L^2}^2 + \|F_\nabla^{14}\|_{L^2}^2$. Therefore

$$\mathcal{YM}(\nabla) = 3\|F_\nabla^7\|_{L^2}^2 + 8\pi^2\kappa(E,[\phi]) \quad \text{and} \quad \mathcal{YM}(\nabla) = \frac{1}{2}\left(3\|F_\nabla^{14}\|_{L^2}^2 - 8\pi^2\kappa(E,[\phi])\right).$$

Hence, if $\kappa(E,[\phi]) > 0$ then G_2-instantons are absolute minima of $\mathcal{YM}$ attaining the topological energy bound $8\pi^2\kappa(E,[\phi])$; if $\kappa(E,[\phi]) < 0$ then E does not admit G_2-instantons at all.

2.3.3 Spin(7)−instantons

This section is based on [Wal13a, Chapter 1] and [SW10].

The following result can be found in [SW10, p. 52, Theorem 9.5].

Proposition 2.3.24. $\Lambda^2(\mathbb{R}^8)^*$ *decomposes orthogonally into*

$$\Lambda^2(\mathbb{R}^8)^* = \Lambda^2_7 \oplus \Lambda^2_{21},$$

where Λ^2_7 *and* Λ^2_{21} *are irreducible representations of* $\mathrm{Spin}(7)$*, with dim* $\Lambda^2_d = d$*, given by*

$$\begin{aligned} \Lambda^2_7 &\coloneqq \{\alpha : *_{\Phi_0}\alpha = 3\alpha\} \\ \Lambda^2_{21} &\coloneqq \{\alpha : *_{\Phi_0}\alpha = -\alpha\} \simeq \mathfrak{spin}(7), \end{aligned}$$

where the last isomorphism comes from the metric identification $\Lambda^2(\mathbb{R}^8) \simeq \mathfrak{so}(8) \supseteq \mathfrak{spin}(7)$.

It follows that we have an analogous eigenspace decomposition of $\Lambda^2 T^* X$ with respect to $*_\Phi$ for every almost $\mathrm{Spin}(7)-$manifold (X^8, Φ). By slight abuse of notation, we will also denote the corresponding summands by Λ^2_d.

In the light of the above result, we now extend for general compact Lie groups G the notion of $\mathrm{Spin}(7)-$instanton on $G-$bundles given in Definition 2.3.8.

Definition 2.3.25. Let (X^8, Φ) be a $\mathrm{Spin}(7)-$manifold and let E be a $G-$bundle over X where G is a compact Lie group. A connection $\nabla \in \mathfrak{U}(E)$ is called a $\mathrm{Spin}(7)-$**instanton** if ∇ is a $\Phi-$ASD instanton (cf. Definition 2.3.1).

Remark 2.3.26. In the above situation, suppose further that G is semi-simple. Thus, by Proposition 2.3.24, a connection $\nabla \in \mathfrak{U}(E)$ is a $\mathrm{Spin}(7)-$instanton precisely when one of the following equivalent conditions holds:

(i) $*(\Phi \wedge F_\nabla) = -F_\nabla$;

(ii) $\pi_7(F_\nabla) = 0$, where π_7 denotes the orthogonal projection from $\Lambda^2 T^*X$ to Λ^2_7;

(iii) The curvature tensor F_∇ lies in the subspace $\mathfrak{spin}(7) \otimes (\mathfrak{g}_E)_x \subseteq \Lambda^2 T^*_x X \otimes (\mathfrak{g}_E)_x$ at each $x \in X$. $\diamondsuit$

Example 2.3.27. It follows from Proposition 2.1.3 and Remark 2.3.26 (iii) that if (X^8, g_Φ) is a connected $\mathrm{Spin}(7)-$manifold, then D^{g_Φ} is a $\mathrm{Spin}(7)-$instanton on TX.

Example 2.3.28 ($\mathrm{Spin}(7)-$instantons from G_2-instantons)**.** Let (Y^7, ϕ) be a G_2-manifold and consider the associated $\mathrm{Spin}(7)-$manifold (X^8, Φ) of Example 2.1.33, where $X^8 := \mathbb{R} \times Y^7$ (resp. $X^8 := S^1 \times Y^7$). The following result relates $\mathbb{R}-$invariant (resp. S^1-invariant) $\mathrm{Spin}(7)-$instantons over X with G_2-instantons connections over Y:

Proposition 2.3.29. *Let E be a $G-$bundle over Y, where G is a compact semi-simple Lie group. A connection $\nabla \in \mathfrak{U}(E)$ is a G_2-instanton if, and only if, $\pi_Y^*\nabla$ is a $\mathrm{Spin}(7)-$instanton.*

Proof. By Remark 2.3.26, we know that ∇ is a G_2-instanton $\iff \phi \wedge F_\nabla = - *_Y F_\nabla$ $\iff \psi \wedge F_\nabla = 0$. Thus, noting that

$$\begin{aligned} *(\Phi \wedge F_{\pi_Y^*\nabla}) &= *(\mathrm{d}t \wedge \pi_Y^*(\phi \wedge F_\nabla) + \pi_Y^*(\psi \wedge F_\nabla)) \\ &= \pi_Y^*(*_Y(\phi \wedge F_\nabla)) + *\pi_Y^*(\psi \wedge F_\nabla), \end{aligned}$$

we are done. ■

Example 2.3.30. $\mathrm{Spin}(7)-$instantons was the subject of Lewis' PhD thesis [Lew99]. In particular, he constructs a non-trivial example on a $\mathrm{SU}(2)-$bundle over a particular compact Riemannian $8-$manifold, constructed by Joyce [Joy96a], with holonomy exactly $\mathrm{Spin}(7)$. Recently, a construction for $\mathrm{Spin}(7)-$instantons on $\mathrm{Spin}(7)-$manifolds arising from Joyce's work [Joy99] was given by Tanaka [Tan12]. Moreover, Clarke [Cla14] constructs a (symmetric) $\mathrm{Spin}(7)-$instanton with structure group $\mathrm{SU}(2)$ on the Bryant-Salamon [BS89] negative spinor bundle $\mathbb{S}^-(S^4)$, which is smooth away from the Cayley base (zero section) S^4, and blows up along the later.

Complex ASD instantons. In this brief paragraph, we introduce the notion of complex ASD instanton over Calabi-Yau 4−folds and realize it as a particular instance of the notion of Spin(7)−instanton. Complex ASD instantons and its underlying 'complex gauge theory' was notably studied by R. Thomas in his PhD thesis [Tho97]; see also Donaldson-Thomas [DT98].

Let (Z^8, ω, Υ) be a Calabi Yau 4−fold and consider the following operator:

$$\begin{aligned} *_\Upsilon : \Omega^{0,p}(Z) &\to \Omega^{0,4-p}(Z) \\ \omega &\mapsto \overline{*}(\omega \wedge \Upsilon), \end{aligned}$$

where $\overline{*} : \Lambda^{p,q}T^*Z \to \Lambda^{n-p,n-q}T^*Z$ is the usual anti-linear Hodge star operator on Kähler manifolds. It follows that $*_\Upsilon$ gives an endomorphism

$$*_\Upsilon : \Omega^{0,2} \to \Omega^{0,2}$$

which is self-adjoint and squares to the identity, splitting $\Omega^{0,2}$ orthogonally into *real* subspaces $\Omega^{0,2}_\pm$ corresponding to the eigenvalues ± 1, in complete analogy with the familiar real 4−dimensional case.

Definition 2.3.31. A connection $\nabla \in \mathfrak{A}(E)$ on an $\mathrm{SU}(r)-$ or a $\mathrm{PU}(r)-$bundle E over Z^8 is called a **complex ASD instanton** if

$$*_\Upsilon F^{0,2}_\nabla = -F^{0,2}_\nabla. \tag{2.3.8}$$

We can fit this notion into the context of $\Xi-$ASD instantons as follows. Consider on (Z^8, ω, Υ) the natural Spin(7)−structure Φ of Example 2.1.34. Then we have:

Lemma 2.3.32. *Let E be an $\mathrm{SU}(r)-$ or a $\mathrm{PU}(r)-$bundle over Z^8 and let $\nabla \in \mathfrak{A}(E)$. Then ∇ is a complex ASD instanton if, and only if, ∇ is a* Spin(7)−*instanton with respect to Φ.*

The proof of this lemma is just a matter of unraveling the definitions and taking account of bi-degrees.

Topological energy bounds from Chern-Weil theory. Suppose (X^8, Φ) is a compact Spin(7)−manifold and let E be a $\mathrm{SU}(r)-$bundle over X. Let $\nabla \in \mathfrak{A}(E)$ and write $F_\nabla = F^7_\nabla \oplus F^{21}_\nabla$ according to the decomposition of Λ^2 induced by $*_\Phi$. Define the topological number

$$\kappa(E, [\Phi]) := \langle c_2(E) \cup [\Phi], [X] \rangle .$$

Then

$$\begin{aligned} 8\pi^2\kappa(E,[\Phi]) &= \int_X \mathrm{tr}(F_\nabla \wedge F_\nabla)\wedge \Phi \\ &= -\langle F_\nabla, *(F_\nabla \wedge \Phi)\rangle \\ &= -\langle F_\nabla^7 + F_\nabla^{21}, 3F_\nabla^7 - F_\nabla^{21}\rangle \\ &= -3\|F_\nabla^7\|_{L^2}^2 + \|F_\nabla^{21}\|_{L^2}^2 \end{aligned}$$

On the other hand, $\mathcal{YM}(\nabla) = \|F_\nabla^7\|_{L^2}^2 + \|F_\nabla^{21}\|_{L^2}^2$. Therefore

$$\mathcal{YM}(\nabla) = 4\|F_\nabla^7\|_{L^2}^2 + 8\pi^2\kappa(E,[\Phi]) \quad \text{and} \quad \mathcal{YM}(\nabla) = \frac{1}{3}\left(4\|F_\nabla^{21}\|_{L^2}^2 - 8\pi^2\kappa(E,[\Phi])\right).$$

Hence, if $\kappa(E,[\Phi]) > 0$ then Spin(7)−instantons are the absolute minima of $\mathcal{YM}$, which attains the topological energy bound $8\pi^2\kappa(E,[\Phi])$; if $\kappa(E,[\Phi]) < 0$ then E does not admit Spin(7)−instantons at all.

Chapter 3

Analytical aspects of Yang-Mills connections

In this chapter we shall be interested in the study of analytical results regarding the weak-convergence and regularity theory of Yang-Mills connections in dimensions higher than (or equal to) four, following particularly the seminal works of Uhlenbeck [Uhl82b, Uhl82a], Price [Pri83], Nakajima [Nak88] and Tian [Tia00].

Section 3.1 gives a brief account on weak and strong Uhlenbeck compactness results for connections with uniform L^p-bounds on curvature, where $1 < p < \infty$, $2p > n$ (Theorems 3.1.1 and 3.1.6). The section ends with a consequent compactness result for Yang-Mills connections with locally uniformly bounded curvatures (modulo passing to a subsequence) allowing for noncompact base manifolds. Thenceforth we leave the general setting of arbitrary dimensions and consider only higher dimensional base manifolds.

In Section 3.2 we deduce Price's monotonicity formula for Yang-Mills fields (Theorem 3.2.1), as well as an interesting corollary regarding Yang-Mills connections with finite L^2-energy over the standard (flat) Euclidean space $\mathbb{R}^n$ for $n \geq 5$. Next, in Section 3.3, we derive a local estimate, due to Uhlenbeck and Nakajima, for the $L^\infty-$norm of Yang-Mills fields with sufficiently small normalized L^2-norm on small geodesic balls (Theorem 3.3.1). Then in Section 3.4 we derive a noncompactness phenomenon along sets of Hausdorff codimension at least four, for general sequences of Yang-Mills connections with uniformly L^2-bounded curvatures (Theorem 3.4.4).

Finally, in Section 3.5, we study some properties of a particular class of singular Yang-Mills connections, called *admissible* Yang-Mills connections, arising as weak-limits of sequences of Yang-Mills connections with uniformly L^2-bounded curvatures (cf. Theorem 3.4.4), for which we define generalized first and second Chern classes in the sense of currents (cf. Theorem 3.5.8).

☞ **Convention:** *Throughout this chapter, unless otherwise stated,* (M, g) *denotes a connected, oriented, Riemannian* $n-$*manifold, and* E *denotes a* $G-$*bundle over* M*, where* G *is a compact Lie group.*

3.1 Uhlenbeck's compactness theorems

In the seminal paper [Uhl82a], Uhlenbeck proved the local existence of the so-called Coulomb gauges for connections with local $L^{n/2}-$norm of the curvature sufficiently small. In particular, this enabled her to prove a global weak compactness theorem for arbitrary fields with bounded L^p-norm, for some $p > n/2$.

In this section we make a review of the so-called compactness results of Uhlenbeck, following closely the excellent book [Weh04]. Our exposition will in fact be very brief and sketchy, for the results we recall here need a huge amount of background machinery which is out of the scope of this work to introduce. This section is intended only to collect and organize the main ideas of some important compactness results for future reference. In this way we hope to make the material of this dissertation more complete.

Weak Uhlenbeck compactness. *In this paragraph, unless otherwise stated, we suppose our base manifold* M *is a* compact *manifold with (possibly empty) boundary.* Recall from Proposition 1.1.6 that for $1 < p < \infty$ such that $p > \frac{n}{2}$, the space of $W^{2,p}$ gauge transformations $\mathcal{G}^{2,p}(E)$ forms a topological group (with respect to composition) which acts continuously on the space of $W^{1,p}$ connections $\mathfrak{U}^{1,p}(E)$. In particular, we may consider the topological quotient $\mathcal{M}^p := \mathfrak{U}^{1,p}(E)/\mathcal{G}^{2,p}(E)$. In this context, Uhlenbeck's weak compactness theorem asserts the weak compactness of subsets of the form $\{[\nabla] \in \mathcal{M}^p : \|F_\nabla\|_p \leq \Lambda\}$, for any constant $\Lambda > 0$. Indeed, we can state it as follows (cf. [Uhl82a, Theorem 1.5] and [Weh04, Theorem 7.1, p. 108]):

Theorem 3.1.1 (Uhlenbeck)**.** *Suppose* $1 < p < \infty$ *is such that* $2p > n$*. Let* $\{\nabla_i\} \subseteq \mathfrak{U}^{1,p}(E)$ *be a sequence of connections such that* $\|F_{\nabla_i}\|_p$ *is uniformly bounded. Then, after passing to a subsequence, there exist gauge transformations* $g_i \in \mathcal{G}^{2,p}(E)$ *such that* $g_i^*\nabla_i$ *converges weakly in* $\mathfrak{U}^{1,p}(E)$*.*

The main step in the proof of this weak compactness theorem is to show that 'Coulomb gauges' exist over small trivializing neighborhoods $U \subseteq M$ of E. In a fixed local trivialization $E|_U \simeq U \times \mathbb{K}^r$, note that the spaces $\mathfrak{U}^{1,p}(E|_U)$ and $\mathcal{G}^{2,p}(E|_U)$ are represented, respectively, by $W^{1,p}(U, T^*U \otimes \mathfrak{g})$ and $W^{2,p}(U, G)$. In the following theorem [Weh04, p. 91, Theorem 6.1], for $A \in W^{1,p}(U, T^*U \otimes \mathfrak{g})$ and $g \in W^{2,p}(U, G)$, we use the notations:

$$F_A := \mathrm{d}A + A \wedge A,$$
$$\mathcal{E}^q(A) := \|F_A\|^q_{L^q(U)}, \text{ and}$$
$$g^*A := g^{-1}Ag + g^{-1}\mathrm{d}g.$$

Theorem 3.1.2 (Local Coulomb gauge). *Let $1 < q \leq p < \infty$ be such that $q \geq \frac{n}{2}$, $p > \frac{n}{2}$ and, in case $q < n$, assume in addition $p \leq \frac{nq}{n-q}$. Then there exist constants $\kappa_0 \geq 0$ and $\gamma_0 > 0$ such that the following holds: for each $x \in M$, given any neighborhood $\tilde{U}$ of x in M there exists a smaller neighborhood $U \subseteq \tilde{U}$ of x such that for every $A \in W^{1,p}(U, T^*U \otimes \mathfrak{g})$ with $\mathcal{E}^q(A) \leq \gamma_0$ there exists a gauge transformation $g \in W^{2,p}(U, G)$ such that g^*A is in* Coulomb gauge, *i.e. the following holds:*

(i) $\mathrm{d}^*(g^*A) = 0$.

(ii) $*(g^*A)|_{\partial U} = 0$.

(iii) $\|g^*A\|_{W^{1,q}(U)} \leq \kappa_0 \|F_A\|_{L^q(U)}$.

(iv) $\|g^*A\|_{W^{1,p}(U)} \leq \kappa_0 \|F_A\|_{L^p(U)}$.

Remark 3.1.3. In Wehrheim's book [Weh04, Remark 6.2 (a), p. 91] it is shown further that the above theorem also holds for the case $q = p = \frac{n}{2}$ provided $n \geq 3$. In this way, the above theorem is a generalization of Uhlenbeck's original version [Uhl82a, Theorem 1.3], which corresponds to the case $q = \frac{n}{2}$ and $n \geq p \geq \frac{n}{2}$ of the above result[1]. ♢

The above theorem is proved by first solving the boundary value problem given by (i)-(ii) and then deducing (iii)-(iv) from *a priori* bounds. Such *a priori* bounds are given by the following regularity result.

Theorem 3.1.4 (Regularity for $\mathrm{d} \oplus \mathrm{d}^*$). *Let (M, g) be a compact $n-$manifold with (possibly empty) boundary and let $1 < p < \infty$. Then, there exists a constant $C > 0$ such that for every $A \in W^{1,p}(M, T^*M)$ satisfying $*A|_{\partial M} = 0$ we have*

$$\|A\|_{1,p} \leq C\left(\|\mathrm{d}A\|_p + \|\mathrm{d}^*A\|_p + \|A\|_p\right).$$

Moreover, if in addition $H^1(M, \mathbb{R}) = 0$, then we can drop the $\|A\|_p$ term on the RHS of the above estimate.

[1] "*(...) it seems that in order to obtain a $W^{1,p}-$control in (iv) for $p > n$, one needs small energy for $q > \frac{n}{2}$.*" [Weh04, Remark 6.2 (c), p. 92].

In fact, given the local aspect of Theorem 3.1.2, one actually first reduces the general setting to model cases. Given $x \in M$ and $\delta > 0$, according to weather $x \in \operatorname{int}(M)$ or $x \in \partial M$, we may find an appropriate domain $B \subseteq \mathbb{R}^n$, a constant $\sigma \in]0,1]$ and a chart $\psi_\sigma : B \to M$ centered at x such that[2]

$$\|\sigma^{-2}\psi_\sigma^* g - \mathbb{1}\|_{2,\infty} \leq \delta.$$

Thus, by working in such special local coordinates, it suffices to prove the local Coulomb gauge theorem when $M = B$ is equipped with a smooth metric g satisfying $\|g - \mathbb{1}\|_{2,\infty} \leq \delta$, for some sufficiently small $\delta > 0$, and then examine the effect of rescaling the metric.

A key property of a local Coulomb gauge, explored in the proof of Theorem 3.1.1, is that in such a gauge we can pass the uniform L^p-control on the curvatures F_{∇_i} to a uniform $W^{1,p}-$control on the connections matrices (cf. Theorem 3.1.2 (iv)). Thus, by the reflexiveness of the Sobolev spaces $W^{1,p}$, in each local Coulomb gauge we can extract a weakly $W^{1,p}-$convergent subsequence of the connections ∇_i (as a consequence of the Banach-Alaoglu theorem - see Appendix B). Ultimately, one has to patch together these local gauges in a suitable way to complete the proof of the weak compactness theorem.

For the sake of completeness, we state below a general patching result. First, fix in G the natural bi-invariant Riemannian metric induced by $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ and let d_G denote the Riemannian distance function in G with respect to such metric. Next, let $\Delta_{\exp} > 0$ be the radius of a convex geodesic ball $B_{\Delta_{\exp}}(1_G) \subseteq G$ centered at 1_G, such that the following holds:

1. The exponential map exp restricted to $B_{\Delta_{\exp}}(0) \subseteq \mathfrak{g}$ is a diffeomorphism onto $B_{\Delta_{\exp}}(1_G)$.
2. For all $g, h \in B_{\Delta_{\exp}}(1_G)$ there exists a unique minimal geodesic from g to h and this lies within $B_{\Delta_{\exp}}(1_G)$.

Lemma 3.1.5 ([Weh04, p. 111, Lemma 7.2])**.** *Let M be an $n-$manifold and let $p > \frac{n}{2}$. Suppose $\{U_\alpha\}$ is a locally finite open covering of M by precompact sets U_α, where α runs a countable index set I. Then, there exist open subsets $V_\alpha \subseteq U_\alpha$ still covering of M such that the following holds.*

(i) Let $k \in \mathbb{N}$ and let $g_{\alpha\beta}, h_{\alpha\beta} \in \mathcal{G}^{k+1,p}(U_\alpha \cap U_\beta)$ be two sets of transition functions satisfying the cocycle conditions and

$$d_G(g_{\alpha\beta}, h_{\alpha\beta}) \leq \Delta_{exp}, \quad \forall \alpha, \beta \in I.$$

[2]Here we are identifying $\psi_\sigma^* g$ with its matrix representation in canonical coordinates, and $\mathbb{1}$ denotes the identity matrix of order n.

Then there exist local gauge transformations $h_\alpha \in \mathcal{G}^{k+1,p}(V_\alpha)$ *for all* $\alpha \in I$ *such that, on all intersections* $V_\alpha \cap V_\beta$*, we have*

$$h_\alpha^{-1} h_{\alpha\beta} h_\beta = g_{\alpha\beta}.$$

(ii) *Let the* $h_{\alpha\beta}$ *in (i) run through a sequence* $h^i_{\alpha\beta}$ *of sets of transition functions such that* $g_{\alpha\beta}, h^i_{\alpha\beta} \in \mathcal{G}^{k+1,p}(U_\alpha \cap U_\beta)$ *for all* $k < K$*, where* $2 \leq K \leq \infty$*. Assume that for every* $\alpha, \beta \in I$ *and* $k < K$ *there is a uniform bound on* $\|(h^i_{\alpha\beta})^{-1} \mathrm{d} h^i_{\alpha\beta}\|_{W^{k,p}(U_\alpha \cap U_\beta)}$*.*

Then, the gauge transformations h^i_α *in (i) are constructed in such a way that for every* $\alpha \in I$ *and* $k < K$ *they satisfy* $h^i_\alpha \in \mathcal{G}^{k+1,p}(V_\alpha)$ *and*

$$\sup_{i \in I} \|(h^i_\alpha)^{-1} \mathrm{d} h^i_\alpha\|_{W^{k,p}(V_\alpha)} < \infty.$$

Strong Uhlenbeck compactness. *In this paragraph, unless otherwise stated, we suppose our base manifold* M *is a* compact *manifold with (possibly empty) boundary*[3]*.* Besides the generality and power of the weak compactness theorem, it can be greatly improved when we restrict ourselves to sequences of (weak) *Yang-Mills* connections.

Theorem 3.1.6 (Strong Uhlenbeck compactness)**.** *Let* $1 < p < \infty$ *be such that* $p > \frac{n}{2}$ *and, in case* $n = 2$*, assume in addition* $p \geq \frac{4}{3}$*. Let* $\{\nabla_i\} \subseteq \mathfrak{U}^{1,p}(E)$ *be a sequence of* weak *Yang-Mills connections such that* $\|F_{\nabla_i}\|_p$ *is uniformly bounded. Then, after passing to a subsequence, there exist gauge transformations* $g_i \in \mathcal{G}^{2,p}(E)$ *such that* $\{g_i^* \nabla_i\} \subseteq \mathfrak{U}(E)$ *is a sequence of smooth Yang-Mills connections that converges to a smooth Yang-Mills connection* $\nabla \in \mathfrak{U}(E)$ *in* $C^\infty-$*topology.*

Corollary 3.1.7. *Let* $\{\nabla_i\} \subseteq \mathfrak{U}(E)$ *be a sequence of smooth Yang-Mills connections with uniformly bounded curvatures* $|F_{\nabla_i}| \leq \Lambda$*. Then, after passing to a subsequence, there exist gauge transformations* $g_i \in \mathcal{G}(E)$ *such that* $g_i^* \nabla_i$ *converges to a smooth Yang-Mills connection* $\nabla \in \mathfrak{U}(E)$ *in* $C^\infty-$*topology.*

The key result in the proof of the strong Uhlenbeck compactness is the existence of global relative Coulomb gauges.

Theorem 3.1.8 (Relative Coulomb gauge)**.** *Let* $1 < p \leq q < \infty$ *be such that* $p > \frac{n}{2}$ *and* $\frac{1}{n} > \frac{1}{q} > \frac{1}{p} - \frac{1}{n}$*. Fix a reference connection* $\nabla_0 \in \mathfrak{U}^{1,p}$ *and let a constant* $c_0 > 0$ *be given. Then there exist constants* $\delta > 0$ *and* $C > 0$ *such that the following holds. For every* $\nabla \in \mathfrak{U}^{1,p}$ *with*

$$\|\nabla - \nabla_0\|_q \leq \delta \quad \textit{and} \quad \|\nabla - \nabla_0\|_{1,p} \leq c_0,$$

there exists a gauge transformation $g \in \mathcal{G}^{2,p}(E)$ *such that*

[3] It is important to note that, in such context, we consider the Yang-Mills equation $\mathrm{d}^*_\nabla F_\nabla = 0$ with the boundary condition $*F_\nabla|_{\partial M} = 0$ (see the footnote of number 18 in Chapter 1).

(i) $\mathrm{d}^*_{\nabla_0}(g^*\nabla - \nabla_0) = 0$.

(ii) $\|g^*\nabla - \nabla_0\|_q \leq C\|\nabla - \nabla_0\|_q$.

(iii) $\|g^*\nabla - \nabla_0\|_{1,p} \leq C\|\nabla - \nabla_0\|_{1,p}$.

By iteration of regularity results, one of the consequences of the relative Coulomb gauge theorem is the following:

Theorem 3.1.9 (Regularity of weak Yang-Mills connections)**.** *Let $1 < p < \infty$ be such that $p > \frac{n}{2}$ and, in case $n = 2$, assume in addition $p \geq \frac{4}{3}$. Then, for every weak Yang-Mills connection $\nabla \in \mathfrak{U}^{1,p}(E)$ there exists a gauge transformation $g \in \mathcal{G}^{2,p}(E)$ such that $g^*\nabla$ is a smooth connection.*

Once one proves such results, the strong Uhlenbeck compactness (Theorem 3.1.6) is basically reduced to the weak Uhlenbeck compactness (Theorem 3.1.1) without using a further patching argument[4]. The argument, due to Dietmar Salamon, can be outlined as follows (cf. [Weh04, p. 153]). First, by the weak compactness theorem, after passing to a subsequence, we may find gauge transformations $g_i \in \mathcal{G}^{2,p}(E)$ such that $g_i^*\nabla_i$ converges in the weak $W^{1,p}-$topology to some $\nabla \in \mathfrak{U}^{1,p}(E)$. It can be shown that ∇ also is a weak Yang-Mills connection[5], so that after a gauge transformation we can suppose it is smooth (by Theorem 3.1.9). Moreover, after passing to a further subsequence, we can suppose that $\|\nabla_i - \nabla\|_{1,p}$ is bounded and that, for a suitable $1 < p \leq q < \infty$ such that the Sobolev embedding $W^{1,p} \hookrightarrow L^q$ is compact, the ∇_i converges to ∇ in the L^q-norm. Finally, one puts the connections ∇_i in relative Coulomb gauge with respect ∇ (Theorem 3.1.8). The $C^\infty-$convergence follows from the fact that the Yang-Mills equation together with the relative Coulomb gauge condition form an elliptic system, thus provide uniform bounds on all $W^{k,p}-$norms of the connections, and the compactness then follows from the compact Sobolev embeddings (Theorem B.2.10 (ii)).

A compactness theorem. We finish this section with a compactness result for (smooth) Yang-Mills connections whose proof follows the same line of argument of the proof of the weak Uhlenbeck compactness.

Theorem 3.1.10. *Let $\{\nabla_i\} \subseteq \mathfrak{U}(E)$ be a sequence of smooth Yang-Mills connections with the following property. For each $x \in M$, there exist a neighborhood U of x and a subsequence $\{i'\} \subseteq \{i\}$ such that $|F_{\nabla_{i'}}|$ is uniformly bounded on U. Then, there exist a*

[4]The 'standard' proof of the strong compactness theorem essentially follows the same line of argument of the proof of the weak compactness: one finds local Coulomb gauges in which one has convergent subsequences and then use a patching construction to obtain global gauges (see e.g. [DK90, §4.4.2−4.4.3]).

[5]Here one needs $p > \frac{4}{3}$ in case $n = 2$.

single subsequence $\{i''\} \subseteq \{i\}$, a sequence of smooth gauge transformations $\{g_{i''}\} \subseteq \mathcal{G}(E)$ and a smooth Yang-Mills connection $\nabla \in \mathfrak{U}(E)$ such that the sequence $g^*_{i''}\nabla_{i''}$ converges to ∇ in $C^\infty-$topology on compact subsets of M.

The proof of Theorem 3.1.10 uses the local Coulomb gauge Theorem 3.1.2, elliptic regularity and the following standard patching construction from [DK90, Corollary 4.4.8, p. 160]:

Proposition 3.1.11. *Let $\{\nabla_i\} \subseteq \mathfrak{U}(E)$ be a sequence of smooth Yang-Mills connections with the following property. For each $x \in M$, there exist a neighborhood U of x, a subsequence $\{i'\} \subseteq \{i\}$, and gauge transformations $\{g_{i'}\} \subseteq \mathcal{G}(E|_U)$ such that $g^*_{i'}\nabla_{i'}$ is convergent in $C^\infty-$topology on compact sets in U. Then, there exist a single subsequence $\{i''\}$ and smooth gauge transformations $g_{i''} \in \mathcal{G}(E)$ such that $g^*_{i''}\nabla_{i''}$ converges in $C^\infty-$topology on compact sets over all of M.*

One of the goals of the next two sections is to achieve a compactness theorem for Yang-Mills connections with the more natural assumption of uniform boundedness on the L^2-norm of the curvatures[6]. For this one needs to use *a priori* estimates to bound the pointwise norm of curvature and the convergence will be possible only away from a blow-up set, where the L^2-energy of the sequence concentrates (cf. Section 3.4, Theorem 3.4.4).

3.2 Price's monotonicity formula

☞ **Warning:** *For the rest of this chapter, we assume that $n := \dim M \geq 4$.*

Price's monotonicity formula [Pri83] is a key result in the analysis of Yang-Mills fields in higher dimensions. In particular, it allows normalized L^2-energy estimates on a ball to pass down to any smaller (concentric) ball. Following Tian [Tia00, §2.1], in this section we derive a slightly modified version of Price's original formula that will play a pivotal role in the rest of this work.

A variational formula. We start by deriving a variational formula for the Yang-Mills action along vector fields with compact support (cf. [Tia00, pp. 208-210] and [Pri83, pp. 141-146]).

Let $X \in \mathfrak{X}(M)$ be a vector field with compact support and denote by $\{\phi_t\}$ the *flow* of X, i.e. the induced $1-$parameter family of diffeomorphisms $\phi_t : M \to M$. Note that each ϕ_t restricts to the identity map outside the support of X.

[6]Note that the Yang-Mills equation is the Euler-Lagrange equation of the Yang-Mills L^2-energy functional (cf. Section 1.4). It is in this sense that L^2-energy bounds forms a more 'natural' assumption.

Given a smooth connection $\nabla \in \mathfrak{U}(E)$ with $\mathcal{YM}(\nabla) < \infty$, the flow of X induces a compactly supported variation $\{\nabla_t\}$ of ∇ as follows. Denote by $P_t : E_x \to E_{\phi_t(x)}$ the parallel transport, with respect to ∇, along the path $\{\phi_s(x)\}_{0\le s\le t}$ (cf. Section 1.2). Note that P_t acting on sections of E gives rise to sections of the induced bundle ϕ_t^*E, in such a way that

$$P_t(fs) = (\phi_t^* f)P_t s, \tag{3.2.1}$$

for each $f \in C^\infty(M)$ and $s \in \Gamma(E)$.

For each t, consider the pull-back connection $\phi_t^*\nabla$ on $\phi_t^*E \to M$ (see 1.1) and define:

$$\nabla_t s \coloneqq (P_t)^{-1} (\phi_t^*\nabla)(P_t s), \quad \text{for each } s \in \Gamma(E).$$

It is clear that $\nabla_0 = \nabla$. To verify that each ∇_t indeed defines a connection, note first that linearity follows from the fact that P_t (therefore $(P_t)^{-1}$) and $\phi_t^*\nabla$ are linear maps. Moreover, for every $f \in C^\infty(M)$ and $s \in \Gamma(E)$ we have:

$$\begin{aligned}
\nabla_t(fs) &= (P_t)^{-1}(\phi_t^*\nabla)(P_t(fs)) \\
&= (P_t)^{-1}(\phi_t^*\nabla)((\phi_t^*f)P_t s) \quad \text{(by (3.2.1))} \\
&= (P_t)^{-1}(\mathrm{d}(\phi_t^*f)\otimes(P_t s) + \phi_t^*f(\phi_t^*\nabla)(P_t s)) \quad (\phi_t^*\nabla \text{ is a connection}) \\
&= \phi_{-t}^*(\mathrm{d}(\phi_t^*f))\otimes s + \phi_{-t}^*(\phi_t^*f)(P_t)^{-1}((\phi_t^*\nabla)(P_t s)) \quad \text{(by (3.2.1))} \\
&= \mathrm{d}f\otimes s + f\nabla_t s.
\end{aligned}$$

Now, for each t, we have $F_{\phi_t^*\nabla} = \phi_t^*F_\nabla$, so that the associated curvature F_{∇_t} is given by

$$F_{\nabla_t} = (P_t)^{-1}\circ(\phi_t^*F_\nabla)\circ P_t.$$

Therefore:

$$F_{\nabla_t}(Y,Z) = (P_t)^{-1}\cdot F_\nabla(\mathrm{d}\phi_t(Y),\mathrm{d}\phi_t(Z))\cdot(P_t), \quad \forall Y,Z\in\mathfrak{X}(M). \tag{3.2.2}$$

We now wish to calculate $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{YM}(\nabla_t)\Big|_{t=0}$. Given a local orthonormal frame $\{e_i\}$ of TM, we have

$$\begin{aligned}
|F_{\nabla_t}|^2(x) &= \sum_{i,j}|F_{\nabla_t}(e_i(x),e_j(x))|^2_{\mathfrak{g}} \\
&= \sum_{i,j}|F_\nabla(\mathrm{d}\phi_t(e_i(x)),\mathrm{d}\phi_t(e_j(x)))|^2_{\mathfrak{g}} \quad \text{(by (3.2.2) and Ad-invariance).}
\end{aligned}$$

Thus, we can write[7]

$$\mathcal{YM}(\nabla_t) = \int_M \sum_{i,j}|F_\nabla(\mathrm{d}\phi_t(e_i(x)),\mathrm{d}\phi_t(e_j(x)))|^2_{\mathfrak{g}}\mathrm{d}V_g(x).$$

[7]Note that we can always cover M with open subsets over which TM trivializes by means of orthonormal frames - the tangent bundle of a Riemannian manifold is an $O(n)-$bundle; pick a partition of unity subordinate to such a cover to localize the integrand.

By changing variables, we get

$$\mathcal{YM}(\nabla_t) = \int_M \sum_{i,j} |F_\nabla(\mathrm{d}\phi_t(e_i(\phi_t^{-1}(x))), \mathrm{d}\phi_t(e_j(\phi_t^{-1}(x))))|_{\mathfrak{g}}^2 \mathrm{Jac}(\phi_t^{-1})\mathrm{d}V_g(\phi_t^{-1}(x)).$$

Now note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{d}\phi_t(e_j(\phi_t^{-1}(x)))\Big)\Big|_{t=0} = -[X, e_i](x),$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{Jac}(\phi_t^{-1})(x)\mathrm{d}V_g(\phi_t^{-1}(x))\Big)\Big|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big((\phi_t^{-1})^*\mathrm{d}V_g\Big)(x)\Big|_{t=0} \\ &= (\mathcal{L}_{-X}\mathrm{d}V_g)(x) \\ &= -(\mathrm{div}\ X)(x)\mathrm{d}V_g(x).\end{aligned}$$

Thus, the Leibniz rule and the chain rule give (cf.[Tia00, pp. 209-210]):

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{YM}(\nabla_t)\Big|_{t=0} = -\int_M \left(|F_\nabla|^2 \mathrm{div}\ X + 4\sum_{i,j=1}^n \langle F_\nabla([X, e_i], e_j), F_\nabla(e_i, e_j)\rangle_{\mathfrak{g}}\right)\mathrm{d}V_g.$$

If D denotes the Levi-Civita connection of (M, g), note that we can write

$$\begin{aligned}&\sum_{i,j=1}^n \langle F_\nabla([X, e_i], e_j), F_\nabla(e_i, e_j)\rangle_{\mathfrak{g}} \\ = &-\sum_{i,j=1}^n \Big(\langle F_\nabla(D_{e_i}X, e_j), F_\nabla(e_i, e_j)\rangle_{\mathfrak{g}} - \langle F_\nabla(D_X e_i, e_j), F_\nabla(e_i, e_j)\rangle_{\mathfrak{g}}\Big) \quad (D \text{ is torsion-free}) \\ = &-\sum_{i,j=1}^n \left(\langle F_\nabla(D_{e_i}X, e_j), F_\nabla(e_i, e_j)\rangle_{\mathfrak{g}} - \sum_{k=1}^n g(D_X e_i, e_k)\langle F_\nabla(e_k, e_j), F_\nabla(e_i, e_j)\rangle_{\mathfrak{g}}\right).\end{aligned}$$

Further, since D is compatible with g,

$$\begin{aligned}&\sum_{i,j,k=1}^n g(D_X e_i, e_k)\langle F_\nabla(e_k, e_j), F_\nabla(e_i, e_j)\rangle_{\mathfrak{g}} \\ = &-\sum_{i,j,k=1}^n g(D_X e_k, e_i)\langle F_\nabla(e_k, e_j), F_\nabla(e_i, e_j)\rangle_{\mathfrak{g}} \\ = &-\sum_{i,j,k=1}^n g(D_X e_k, e_i)\langle F_\nabla(e_i, e_j), F_\nabla(e_k, e_j)\rangle_{\mathfrak{g}} \quad (\text{symmetry of } \langle\cdot,\cdot\rangle_{\mathfrak{g}}) \\ = &-\sum_{i,j,k=1}^n g(D_X e_i, e_k)\langle F_\nabla(e_k, e_j), F_\nabla(e_i, e_j)\rangle_{\mathfrak{g}}. \quad (\text{interchanging names of } i \text{ and } k)\end{aligned}$$

So we conclude that

$$\sum_{i,j=1}^n \langle F_\nabla([X, e_i], e_j), F_\nabla(e_i, e_j)\rangle_{\mathfrak{g}} = -\sum_{i,j=1}^n \langle F_\nabla(D_{e_i}X, e_j), F_\nabla(e_i, e_j)\rangle_{\mathfrak{g}}.$$

Summing up the above calculations, we have the following formula for the *first variation of $\mathcal{YM}$ along X*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{YM}(\nabla_t)\Big|_{t=0} = -\int_M \left(|F_\nabla|^2 \mathrm{div}\ X - 4\sum_{i,j=1}^n \langle F_\nabla(D_{e_i}X, e_j), F_\nabla(e_i, e_j)\rangle_{\mathfrak{g}}\right)\mathrm{d}V_g. \quad (3.2.3)$$

If ∇ is a Yang-Mills connection, recalling Proposition 1.4.2, we deduce the variational formula:

$$\int_M \left(|F_\nabla|^2 \text{div } X - 4 \sum_{i,j=1}^{n} \langle F_\nabla(D_{e_i} X, e_j, F_\nabla(e_i, e_j)\rangle \right) \mathrm{d}V_g = 0. \tag{3.2.4}$$

Later we shall see that this stationary condition turns out to be the main ingredient in the proof of Price's monotonicity; see Remark 3.2.2.

A word on notation. Henceforth, we will use the following notations concerning any (connected) Riemannian manifold (M, g):

- d_g: Riemannian distance function on (M, g) (see e.g. [Aub82, §2.1]).
- $B_r(p) \equiv B_r(p; g)$: *open* d_g-ball of radius $r > 0$ and center p.
- $\overline{B}_r(p) \equiv \overline{B}_r(p; g)$: closed d_g-ball of radius $r > 0$ and center p.
- $\text{inj}_g(p)$: injectivity radius of (M, g) at p.
- $\text{inj}_g(M) \coloneqq \inf\{p \in M : \text{inj}_g(p)\}$.
- μ_g: natural Radon measure on M associated to the Riemannian volume $n-$form $\mathrm{d}V_g$, whenever M is oriented[8].

The monotonicity formula. We prove a monotonicity formula for Yang-Mills fields, which is essentially due to Price [Pri83]; its proof follows Price's original arguments with almost no modifications. The minor modification of the original formula (compare Theorem 3.2.1 with [Pri83, Theorem 1', pp. 154-155]) will be important to us in the bubbling analysis of Chapter 4; see the footnote of number 4 in Section 4.3.

The metric g enters into the problem as follows. For each fixed point $p \in M$, we let $0 < r_p < \text{inj}_g(p)$ be a small enough radius with the following properties: there are normal coordinates $x^1, \ldots, x^n$ centered at p in the geodesic ball $B_{r_p}(p)$ such that, for some constant $c(p) \geq 0$, the metric components $g_{ij} \coloneqq g(\partial/\partial x^i, \partial/\partial x^j)$ satisfies the following estimates:

1. $|g_{ij} - \delta_{ij}| \leq c(p)|x|^2$.
2. $|\partial_k g_{ij}| \leq c(p)|x|$.

Note that, from the key properties $g_{ij}(p) = 0$ and $\partial_k g_{ij}(p) = 0$ of normal coordinates, the Taylor expansions of g_{ij} and $\partial_k g_{ij}$ at p show that the constants r_p and $c(p)$ can be chosen depending only on $\text{inj}_g(p)$ and the curvature of g [GJW02]; thus, for instance,

[8] see Example A.2.3 of Appendix A.

when g is flat we can take any $r_p < \mathrm{inj}_g(p)$ and $c(p) = 0$. If M is a *compact* manifold, then it follows from [Heb00, p. 16, Theorem 1.3] that we can choose uniform constants $0 < \delta_0 = \delta_0(M, g) < \mathrm{inj}_g(M)$ and $c_0 = c_0(M, g) \geq 0$ such that the above holds with $r_p = \delta_0$ and $c(p) = c_0$ for *any* point $p \in M$. In such setting, these will *always* be our fixed choices of r_p and $c(p)$.

☞ **Convention:** *In what follows we will always denote by $O(1)$ a quantity bounded by a constant depending only on $n := \dim M$.*

We are now in position to state and prove [Tia00, Theorem 2.1.2, p. 212]:

Theorem 3.2.1 (Price)**.** *Let $p \in M$, and let r_p and $c(p)$ be as above. Then there exists a nonnegative constant $a = a(n, p, g) \geq O(1)c(p)$ such that the following holds. Let $\nabla \in \mathfrak{U}(E)$ be a Yang-Mills connection with finite L^2-energy. Then for all $0 < \sigma < \rho \leq r_p$ we have:*

$$\begin{aligned} & e^{a\rho^2}\rho^{4-n}\int_{B_\rho(p)} |F_\nabla|^2 \,\mathrm{d}V_g - e^{a\sigma^2}\sigma^{4-n}\int_{B_\sigma(p)} |F_\nabla|^2 \,\mathrm{d}V_g \\ \geq \;& 4\int_{B_\rho(p)\setminus B_\sigma(p)} e^{ar^2} r^{4-n}\left|\frac{\partial}{\partial r} \lrcorner F_\nabla\right|^2 \mathrm{d}V_g, \end{aligned}$$

where r denotes the radial distance function on[9] *$B_{r_p}(p)$. Furthermore:*

(i) If $(M, g) = (\mathbb{R}^n, g_0)$, where g_0 denotes the standard flat metric, then we can take $a = 0$ and the above inequality holds for every $\rho \in]0, \infty[$.

(ii) If M is compact, we can choose uniform *constants $a \geq 0$ and $\delta_0 > 0$ so that the above holds for every $0 < \sigma < \rho \leq \delta_0$.*

Proof. Without loss of generality we can suppose $\rho < r_p$; the case $\rho = r_p$ follows by the obvious approximation argument. Let $\xi(r)$ be a C^∞ cut-off function on the interval $[0, r_p]$, and define the cut-off radial vector field

$$X = X_\xi := \xi(r) r \frac{\partial}{\partial r}.$$

(By declaring $X \equiv 0$ outside its compact support, we see that X defines a compactly supported (smooth) vector field on M.) Let $\{e_i\}_{1\leq i\leq n}$ be an orthonormal local frame of TM near p such that $e_1 = \frac{\partial}{\partial r}$. Recalling that the unit radial vector field $\frac{\partial}{\partial r}$ is the velocity of a (radial) geodesic, it follows that

$$D_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r} = 0.$$

[9] Recall the previously introduced normal coordinates in $B_{r_p}(p)$.

Thus

$$D_{\frac{\partial}{\partial r}} X = (\xi r)' \frac{\partial}{\partial r} = (\xi' r + \xi) \frac{\partial}{\partial r}.$$

Moreover, for $i \geq 2$, we have

$$D_{e_i} X = \xi r D_{e_i} \frac{\partial}{\partial r} = \xi D_{r e_i} \frac{\partial}{\partial r} = \xi \sum_{j=2}^{n} b_{ij} e_j,$$

where

$$b_{ij} := g\left(D_{r e_i} \frac{\partial}{\partial r}, e_j \right) \quad (j = 2, \dots, n)$$

satisfies

$$|b_{ij} - \delta_{ij}| = O(1) c(p) r^2.$$

By straightforward computations, we get

$$\begin{aligned} & |F_\nabla|^2 \text{div } X - 4 \sum_{i,j=1}^{n} \langle F_\nabla(D_{e_i} X, e_j), F_\nabla(e_i, e_j) \rangle_{\mathfrak{g}} \\ & = \xi' r |F_\nabla|^2 + (n-4) \xi |F_\nabla|^2 + O(1) c(p) r^2 \xi |F_\nabla|^2 - 4 \xi' r \left| \frac{\partial}{\partial r} \lrcorner F_\nabla \right|^2 . \end{aligned} \tag{3.2.5}$$

We choose, for $\tau \in [\sigma, \rho]$, $\xi(r) = \xi_\tau(r) = \phi\left(\frac{r}{\tau}\right)$ with $\phi = \phi_\varepsilon \in C^\infty([0, \infty[)$, $\varepsilon > 0$ small so that $(1+\varepsilon)\rho < r_p$ (recall that $\rho < r_p$), satisfying: $\phi(t) = 1$ for $t \in [0,1]$, $\phi(t) = 0$ for $t \in [1+\varepsilon, \infty[$, and $\phi'(t) \leq 0$. Then

$$\tau \frac{\partial}{\partial \tau}(\xi_\tau(r)) = -r \xi_\tau'(r). \tag{3.2.6}$$

Noting that $\xi_\tau(r) \neq 0$ precisely when $r \leq (1+\varepsilon)\tau$, it follows from equations (3.2.5), (3.2.6) and the variational formula (3.2.4) that

$$\tau \frac{\partial}{\partial \tau} \int_M \xi_\tau |F_\nabla|^2 \mathrm{d}V_g + \left((4-n) + O(1) c(p) \tau^2 \right) \int_M \xi_\tau |F_\nabla|^2 \mathrm{d}V_g = 4 \tau \frac{\partial}{\partial \tau} \int_M \xi_\tau \left| \frac{\partial}{\partial r} \lrcorner F_\nabla \right|^2 \mathrm{d}V_g.$$

Choosing a nonnegative number $a \geq O(1) c(p)$, and multiplying the above equation by $e^{a\tau^2} \tau^{3-n}$, we get

$$\begin{aligned} \frac{\partial}{\partial \tau} \left(e^{a\tau^2} \tau^{4-n} \int_M \xi_\tau |F_\nabla|^2 \mathrm{d}V_g \right) &= 4 e^{a\tau^2} \tau^{4-n} \frac{\partial}{\partial \tau} \int_M \xi_\tau \left| \frac{\partial}{\partial r} \lrcorner F_\nabla \right|^2 \mathrm{d}V_g \\ &+ (-O(1) c(p) + 2a)\, e^{a\tau^2} \tau^{5-n} \int_M \xi_\tau |F_\nabla|^2 \mathrm{d}V_g. \end{aligned} \tag{3.2.7}$$

Since the second term of the RHS of (3.2.7) is nonnegative (therefore can be dropped), the result follows by integrating over $[\sigma, \rho]$ and letting $\varepsilon \downarrow 0$.

The final assertions (i) and (ii) follows from the discussion preceding the theorem. ■

Remark 3.2.2. In the last section of this chapter we will introduce a special type of singular connections called *admissible* connections (see Definition 3.5.1). In brief, these are smooth connections away from singular sets of Hausdorff codimension at least 4 (cf. Section A.2). In particular, the singular set of such connections are negligible under the integral sign (see Remark 3.5.3). It follows from the above proof that Price's monotonicity formula is also true for any admissible connection ∇ satisfying the variational formula (3.2.4) for every compactly-supported vector field X on M. $\lozenge$

Remark 3.2.3. Following the same arguments of the above proof, Tian [Tia00, Theorem 2.1.1] proves a slightly generalized version of Theorem 3.2.1. He needs such version of the formula to perform a proof of the existence of tangent cone measures of blow-up loci [Tia00, Lemma 3.2.1]. By the direct way we will prove the rectifiability of blow-up loci in Chapter 4, we will not need to provide a separated proof for such existence result; see Theorem 4.2.1 and Remark 4.2.2. $\lozenge$

It follows from Price's monotonicity that the map

$$\rho \mapsto \rho^{4-n} e^{a\rho^2} \int_{B_\rho(p)} |F_\nabla|^2 \mathrm{d}V_g$$

is *non-decreasing* for $\rho \in]0, r_p]$. This will be important in Chapter 4. Moreover, we have the following curious corollary, showing in particular that for $n \geq 5$ every finite-energy Yang-Mills connection over the Euclidean space $\mathbb{R}^n$ is necessarily flat (see [Pri83, Corollary 2, p. 148]).

Corollary 3.2.4. *Let $\nabla \in \mathfrak{A}(E)$ be a Yang-Mills connection with $\mathcal{YM}(\nabla) < \infty$ on a (necessarily trivial) $G-$bundle E over $(\mathbb{R}^n, g_0)$, where g_0 is the standard flat metric. If there is some $x \in \mathbb{R}^n$ such that*[10]

$$\|F_\nabla\|^2_{L^2(B_R(x))} = o(R^{n-4}) \quad \text{as } R \to \infty, \tag{3.2.8}$$

then ∇ is a flat *connection. In particular, if $n \geq 5$ then ∇ is flat.*

Proof. Suppose, by contradiction, that $F_\nabla \neq 0$. Then there exists some $R_0 > 0$ large enough so that

$$\Delta := R_0^{4-n} \|F_\nabla\|^2_{L^2(B_{R_0}(x))} > 0.$$

On the other hand, for each $R \geq R_0$, Theorem 3.2.1 (i) implies that

$$\Delta \leq R^{4-n} \|F_\nabla\|^2_{L^2(B_R(x))}.$$

Thus, making $R \to \infty$ and using the hypothesis (3.2.8) we conclude $\Delta \leq 0$ ($\Rightarrow\Leftarrow$). This proves the main statement.

[10] Here we use the standard little-o notation.

For the final assertion, simply note that the constant function is $o(R^{n-4})$ when $n \geq 5$, and that $\|F_\nabla\|^2_{L^2(B_R(x))} \leq \mathcal{YM}(\nabla) = \text{const.} < \infty$ (by hypothesis) for every $x \in \mathbb{R}^n$ and $R > 0$. ■

Remark 3.2.5. The normalized L^2-norm

$$\rho^{4-n} \int_{B_\rho(p)} |F_\nabla|^2 \mathrm{d}V_g \tag{3.2.9}$$

is also known as the *scaling-invariant* L^2-norm of F_∇. Indeed, scale g by some positive constant $\lambda \in \mathbb{R}_+$ and let $\tilde{g} \coloneqq \lambda g$. It follows easily that $B_{\lambda^{1/2}\rho}(x;\tilde{g}) = B_\rho(x;g)$, for all $x \in M$. Furthermore, the pointwise inner product on $2-$forms scales by λ^{-2}, and the Riemannian volume $n-$forms scales by $\lambda^{n/2}$; thus,

$$(\lambda^{1/2}\rho)^{4-n} \int_{B_{\lambda^{1/2}\rho}(p;\tilde{g})} |F_\nabla|^2_{\tilde{g}} \mathrm{d}V_{\tilde{g}} = \rho^{4-n} \int_{B_\rho(p)} |F_\nabla|^2 \mathrm{d}V_g.$$

In this language, Price's monotonicity formula allows one to pass control on the scaling-invariant L^2-norm of a Yang-Mills field over a small geodesic ball to any other concentric ball of smaller radius. ◊

3.3 $\varepsilon-$regularity theorem

Motivated by Schoen's method [Sch84, Theorem 2.2] in proving the *a priori* pointwise estimate for stationary harmonic maps, Nakajima [Nak88, p. 387, Lemma 3.1] combined Price's monotonicity formula together with an appropriate Bochner-Weitzenböck formula to obtain a local $L^\infty-$estimate for Yang-Mills fields satisfying a smallness condition on their normalized L^2-norm over a sufficiently small geodesic ball. Similar results also appears in earlier works by Uhlenbeck, see e.g. [Uhl82b, Theorem 3.5] and [UY86, Theorem 5.1][11]. The following statement of Nakajima's result, which we will refer to as the *$\varepsilon-$regularity theorem*, is adapted from [Tia00, Theorem 2.2.1, p. 213].

Theorem 3.3.1 (Uhlenbeck-Nakajima)**.** *Let (M,g) be an oriented Riemannian $n-$manifold, with $n \geq 4$, and let E be a $G-$bundle over M where G is a compact Lie group. Given a point $p \in M$, there are constants $\varepsilon_0 > 0$ and $C \geq 0$ such that the following holds. Let $\nabla \in \mathfrak{U}(E)$ be a Yang-Mills connection with finite L^2-energy. Then for any*[12] $0 < \rho \leq r_p$, *if*

$$\varepsilon \coloneqq \rho^{4-n} \int_{B_\rho(p)} |F_\nabla|^2 \mathrm{d}V_g < \varepsilon_0$$

then

$$\sup_{x \in B_{\frac{\rho}{4}}(p)} |F_\nabla|^2(x) \leq C\varepsilon\rho^{-4}.$$

[11] The latter refers the reader to a paper by Uhlenbeck [Uhl] that was never published.

[12] r_p is as in §3.2; see the discussion preceding Theorem 3.2.1.

Furthermore, if M is compact *then we can take the constants ε_0 and C to be independent of the chosen point $p \in M$.*

This theorem is of fundamental importance in compactness theory of Yang-Mills connections in higher dimensions. Following Tian's work [Tia00], in the next section we will provide a key application of such result (cf. Theorem 3.4.4) which implies that a sequence of Yang-Mills connections with uniformly L^2-bounded curvatures may fail to have a $C^\infty_{\text{loc}}-$convergent subsequence modulo gauge transformations. Indeed, the associated curvatures of such sequences of connections satisfy the hypothesis of the above a priori estimate, provided we look at balls outside a suitable subset $S \subseteq M$ of Hausdorff codimension[13] at least 4, where the curvatures 'blows up'. Only away from S we get uniform local bounds on the curvatures, so that we can apply the standard techniques (cf. section 3.1) to extract a $C^\infty-$convergent subsequence.

Still following [Tia00], in the last section of this chapter we also apply Theorem 3.3.1 to defining generalized Chern-Weil forms for admissible Yang-Mills connections, which are kinds of singular connections (cf. Theorem 3.5.8).

The rest of this section is devoted to give a proof of Theorem 3.3.1. Our proof is based on Tian's proof [Tia00, pp. 213-215], which explores the same method of Nakajima's proof [Nak88, Lemma 3.1, pp. 387-388] but fits better in our present notation. We will need the following preliminary lemmas.

Lemma 3.3.2 (Bochner type estimate)**.** *Suppose (M, g) is an oriented Riemannian $n-$manifold, and let E is a $G-$bundle over M. Given $p \in M$ and $0 < r < \text{inj}_g(p)$, there are constants $c, c' > 0$, where c depends at most on n and the supremum bound of the Riemannian curvature R^g on $\overline{B}_r(p)$, and c' depends at most on n and G, such that the following holds. If ∇ is a Yang-Mills connection on E, then*

$$\Delta_g^- |F_\nabla|^2 \geq -c|F_\nabla|^2 - c'|F_\nabla|^3 \quad \text{on } B_r(p), \tag{3.3.1}$$

where $\Delta_g^- := -\mathrm{d}^*\mathrm{d} : C^\infty(M) \to C^\infty(M)$ is the Laplace-Beltrami operator with respect to g.

Proof. We start noting that, for any $\xi \in \Omega^k(M, \mathfrak{g}_E)$, we have

$$\begin{aligned} \Delta_g^- |\xi|^2 &= -\mathrm{d}^*\mathrm{d}\langle \xi, \xi\rangle = -2\mathrm{d}^*\langle \nabla\xi, \xi\rangle \\ &= 2 * \mathrm{d} * \langle \nabla\xi, \xi\rangle = 2 * \mathrm{d}\langle *\nabla\xi, \xi\rangle \\ &= 2 * (\langle \nabla * \nabla\xi, \xi\rangle + \langle *\nabla\xi, \nabla\xi\rangle) = 2\left(|\nabla\xi|^2 - \langle \nabla^*\nabla\xi, \xi\rangle\right). \end{aligned}$$

On the other hand, recalling the Bochner-Weitzenböck formula (1.1.5), for any $\xi \in \Omega^2(M, \mathfrak{g}_E)$ we can write

$$\Delta_\nabla \xi = \nabla^*\nabla\xi + \{\mathrm{R}, \xi\} + \{F_\nabla, \xi\},$$

[13] See Definition A.2.6 for the notion of Hausdorff dimension.

where the brackets $\{,\}$ indicate algebraic multilinear expressions. Combining these facts and using that $\Delta_\nabla F_\nabla = 0$, i.e. ∇ is a Yang-Mills connection, we get:

$$\begin{aligned} 0 = -2\langle \Delta_\nabla F_\nabla, F_\nabla\rangle &= -2\langle \nabla^*\nabla F_\nabla, F_\nabla\rangle + \{\mathrm{R}, F_\nabla, F_\nabla\} + \{F_\nabla, F_\nabla, F_\nabla\} \\ &= \Delta_g^-|F_\nabla|^2 - 2|\nabla F_\nabla|^2 + \{\mathrm{R}, F_\nabla, F_\nabla\} + \{F_\nabla, F_\nabla, F_\nabla\} \\ &\leq \Delta_g^-|F_\nabla|^2 + \{\mathrm{R}, F_\nabla, F_\nabla\} + \{F_\nabla, F_\nabla, F_\nabla\}. \end{aligned}$$

Therefore

$$\Delta_g^-|F_\nabla|^2 \geq -c|F_\nabla|^2 - c'|F_\nabla|^3 \quad \text{on } B_r(p),$$

where $c > 0$ is a constant depending only on n and the supremum bound of the Riemannian curvature R^g on $\overline{B}_r(p)$, and $c' > 0$ is a constant depending only on n and G. ■

The next lemma, which we state without proof, is a standard Harnack-Moser inequality; see e.g. [SC92, Theorem 5.1, p. 425] and [GT01, Theorem 9.20, p. 244].

Lemma 3.3.3 (Harnack-Moser inequality)**.** *Suppose (M, g) is an oriented Riemannian $n-$manifold. Let $p \in M$, $0 < r < 2^{-1}\mathrm{inj}_g(p)$ and $C_0 > 0$ be given. If $u \in C^2(\overline{B}_r(p))$ is* nonnegative *and satisfies*

$$\Delta_g^- u \geq -C_0 u \quad \textit{on } B_r(p),$$

then

$$\sup_{B_{\frac{r}{2}}(p)} u \leq C'(1 + C_0 r^2)^{1+\frac{n}{2}} e^{Cr\sqrt{K}} (r^2 \mu_g(B_r(x)))^{-1} \int_{B_r(p)} u \mathrm{d}V_g,$$

where C and C' are positive constants depending only on n, and $-K \leq 0$ is a lower bound for the Ricci curvature of g on $B_{2r}(p)$.

The rest of this section is dedicated to the proof of Theorem 3.3.1. Since both the normalized L^2-energy of ∇ and the stated bound on F_∇ are not affected by the scaling $g \mapsto \lambda g$, $\rho \mapsto \lambda^{1/2}\rho$, for any constant $\lambda > 0$ (see e.g. Remark 3.2.5), we can suppose $\rho = 1$. So we have

$$\varepsilon := \int_{B_1(p)} |F_\nabla|^2 \mathrm{d}V_g \leq \varepsilon_0, \tag{3.3.2}$$

and we want to prove that for a sufficiently small $\varepsilon_0 > 0$, depending at most on n, G and g near p, we get the estimate

$$\sup_{x \in B_{\frac{1}{4}}(p)} |F_\nabla|^2(x) \leq C\varepsilon, \tag{3.3.3}$$

for some constant $C > 0$ depending at most on n, G and g near p.

We start defining the following function:

$$\begin{aligned} f : [0,1] &\to [0, \infty[\\ r &\mapsto (1-r)^2 \sup_{x \in B_r(p)} |F_\nabla|(x). \end{aligned}$$

Since $1 = \rho \leq r_p < \mathrm{inj}_g(p)$, it follows that the smooth function $|F_\nabla| : M \to [0, \infty[$ is uniformly continuous[14] on $\overline{B}_1(p)$. From this simple fact, one easily shows that the map $r \mapsto \sup_{x \in B_r(p)} |F_\nabla|(x)$ is continuous on $[0, 1]$. In particular, it follows that f is continuous on $[0, 1]$ and, therefore, attains its maximum at a certain $r_0 \in [0, 1]$. Now let:

- $b := \sup_{x \in B_{r_0}(p)} |F_\nabla|(x)$.
- $x_0 \in \overline{B}_{r_0}(p)$ be such that $b = |F_\nabla|(x_0)$.
- $\sigma := \frac{1}{2}(1 - r_0)$.

At this point one should note that $f(r_0) = 4\sigma^2 b$. Moreover, it is not difficult to see that the following equivalences holds:

$$F_\nabla = 0 \text{ on } B_1(p) \iff f = 0 \iff b = 0 \iff \sigma = 0 \iff r_0 = 1.$$

In particular, if $f = 0$ then we are done: the desired local bound (3.3.3) follows trivially by taking any $C \geq 0$. So suppose $f \neq 0$, which means e.g. that $\sigma > 0$. Then:

$$\begin{aligned}
\sup_{x \in B_\sigma(x_0)} |F_\nabla|(x) &\leq \sup_{x \in B_{\sigma + r_0}(p)} |F_\nabla|(x) \quad (\text{in fact } B_\sigma(x_0) \subseteq B_{\sigma + r_0}(p)) \\
&= \frac{1}{(1 - (\sigma + r_0))^2} f(\sigma + r_0) \\
&\quad (\text{note that } 1 - (\sigma + r_0) = \sigma > 0 \text{ and } \sigma + r_0 \in [0, 1]) \\
&\leq \frac{1}{\sigma^2} f(r_0) \quad (\text{by definition of } r_0) \\
&= 4b.
\end{aligned} \tag{3.3.4}$$

Claim: $f(r_0) \leq 16$ if $\varepsilon_0 = \varepsilon_0(n, p, g, G) > 0$ is sufficiently small.

Suppose, on the contrary, that $f(r_0) > 16$, i.e. suppose $\sigma\sqrt{b} > 2$. Defining $\tilde{g} := bg$, it is clear that $|F_\nabla|_{\tilde{g}} = b^{-1}|F_\nabla|_g$ and $B_{\sigma\sqrt{b}}(x_0; \tilde{g}) = B_\sigma(x_0; g)$. Thus, our assumption together with (3.3.4) gives

$$\begin{aligned}
\sup_{x \in B_2(x_0; \tilde{g})} |F_\nabla|_{\tilde{g}}(x) &\leq \sup_{x \in B_{\sigma\sqrt{b}}(x_0; \tilde{g})} |F_\nabla|_{\tilde{g}}(x) \\
&= \sup_{x \in B_\sigma(x_0)} b^{-1}|F_\nabla|_g(x) \\
&\leq 4.
\end{aligned} \tag{3.3.5}$$

Now, by hypothesis, ∇ is a Yang-Mills connection with respect to g, i.e. $\mathrm{d}_\nabla *_g F_\nabla = 0$. Noting that $*_{\tilde{g}} = b^{\frac{n}{2}-2} *_g$ on 2−forms, and that b is a constant, it follows that ∇ is a Yang-Mills connection with respect to $\tilde{g}$ too. Moreover, note that we can take $r_p(\tilde{g}) = r_p(g)\sqrt{b}$

[14] Indeed, every closed geodesic ball centered at p with radius less then the injectivity radius of (M, g) at p is compact, and every continuous real-valued function on a compact metric space is uniformly continuous.

and that, by previous facts, $B_2(x;\tilde{g}) \subseteq B_{(\sigma+r_0)\sqrt{b}}(p;\tilde{g}) \subseteq B_{r_p(g)\sqrt{b}}(p;\tilde{g})$. In particular, Lemma 3.3.2 yields:

$$\Delta^-_{\tilde{g}}|F_\nabla|^2_{\tilde{g}} \geq -c|F_\nabla|^2_{\tilde{g}} - c'|F_\nabla|^3_{\tilde{g}} \quad \text{on } B_2(x_0;\tilde{g}),$$

for constants $c, c' > 0$ such that c depends only on n and the supremum bound of R^g near p, and c' depends only on n and G. Applying (3.3.5) in the above estimate, we get

$$\Delta^-_{\tilde{g}}|F_\nabla|^2_{\tilde{g}} \geq -(c+4c')|F_\nabla|^2_{\tilde{g}} \quad \text{on } B_2(x_0;\tilde{g}).$$

Now one can apply Lemma 3.3.3 to obtain

$$1 = |F_\nabla|^2_{\tilde{g}}(x_0) \leq \tilde{c}\int_{B_1(x_0;\tilde{g})}|F_\nabla|^2_{\tilde{g}}\mathrm{d}V_{\tilde{g}}, \tag{3.3.6}$$

where the first equality comes from the definition of x_0, and $\tilde{c} > 0$ is a constant depending only on n, G and g near p.

On the other hand, the assumption $\sigma\sqrt{b} > 2$ (along with $\sigma \leq 1$, $b > 0$, and $1 = \rho \leq r_p$) implies $0 < \dfrac{1}{\sqrt{b}} < \dfrac{1}{2} < r_p$. Thus we can apply the monotonicity formula (Theorem 3.2.1) to get:

$$\begin{aligned}
\int_{B_1(x_0;\tilde{g})}|F_\nabla|^2_{\tilde{g}}\mathrm{d}V_{\tilde{g}} &= \left(\frac{1}{\sqrt{b}}\right)^{4-n}\int_{B_{\frac{1}{\sqrt{b}}}(x_0;g)}|F_\nabla|^2_g\mathrm{d}V_g \quad (\tilde{g} = bg)\\
&\leq \left(\frac{1}{\sqrt{b}}\right)^{4-n} e^{\frac{a}{b}}\int_{B_{\frac{1}{\sqrt{b}}}(x_0;g)}|F_\nabla|^2_g\mathrm{d}V_g \quad (e^{\frac{a}{b}} \geq 1)\\
&\leq \left(\frac{1}{2}\right)^{4-n} e^{\frac{a}{4}}\int_{B_{\frac{1}{2}}(x_0;g)}|F_\nabla|^2_g\mathrm{d}V_g \quad \text{(by monotonicity)}\\
&\leq 2^{n-4}e^{\frac{a}{4}}\varepsilon_0. \quad \text{(by the hypothesis (3.3.2))}.
\end{aligned}$$

Combining with (3.3.6), this implies

$$1 \leq 2^{n-4}\tilde{c}e^{\frac{a}{4}}\varepsilon_0,$$

which is impossible for sufficiently small $\varepsilon_0 > 0$ depending only on n, G and g near p (a,$\tilde{c}$). This proves the claim.

Therefore,

$$f(r) \leq 16, \quad \forall r \in [0,1].$$

In particular, taking $r = 1/2$,

$$\sup_{x\in B_{\frac{1}{2}}(p)}|F_\nabla|(x) \leq 64.$$

Using Lemma 3.3.1 again, this implies

$$\Delta^-_g|F_\nabla|^2 \geq -(c+64c')|F_\nabla|^2 \quad \text{on } B_{\frac{1}{2}}(p),$$

where $c, c' > 0$ are constants such that c depends only on n and the curvature of g near p, and c' depends only on n and G. Then, a final application of Lemma 3.3.3 leads us to the required conclusion (3.3.3).

The last assertion of Theorem 3.3.1 follows from the above proof by noting that, in the compact case, we can take the constants of Lemma 3.3.1 and the constant a of the monotonicity formula (Theorem 3.2.1 (ii)) to be uniform on M. The theorem is proved.

Remark 3.3.4. Tian [Tia00, Theorem 2.2.1, p. 213] claims in general, i.e. without assuming M to be compact, that the constants ε_0 and C of Theorem 3.3.1 depend only on n and M. At the time of writing, I'm not able to give a proof of this claim. $\Diamond$

3.4 Convergence away from the blow-up locus

As we have seen in the last section, the $\varepsilon-$regularity theorem (Theorem 3.3.1) provides *a priori* local $L^\infty-$bounds on the curvature of a Yang-Mills connection provided its normalized L^2-energy is sufficiently small. Given a sequence $\{\nabla_i\}$ of Yang-Mills connections with uniformly L^2-bounded curvatures, this previous result allows us to define a set in which the convergence of the sequence necessarily has to fail. (cf. [Nak88], [Tia00, Lemma 3.1.3] and [Wal15]).

Definition 3.4.1. The **blow-up locus** (or **energy concentration set**[15]) of $\{\nabla_i\}$ is the set

$$S = S\left(\{\nabla_i\}\right) := \left\{ x \in M : \liminf_{i\to\infty} e^{ar^2} r^{4-n} \int_{B_r(x)} |F_{\nabla_i}|^2 \mathrm{d}V_g \geq \varepsilon_0, \ \forall 0 < r \leq r_x \right\}, \quad (3.4.1)$$

where r_x is the usual (previously fixed) constant[16], a is the constant given by the monotonicity formula (Theorem 3.2.1) and ε_0 is the constant given by the $\varepsilon-$regularity theorem (Theorem 3.3.1).

Remark 3.4.2. We caution the reader that the notation and terminology used here differ from those used in the main reference [Tia00]. Indeed, Tian denotes the set $S\left(\{\nabla_i\}\right)$ by $S_b\left(\{\nabla_i\}\right)$ and reserves the name 'blow-up locus' for a certain subset of $S_b\left(\{\nabla_i\}\right)$ which he denotes by S_b. The latter, in turn, is what we will define to be the 'bubbling locus' Γ of $\{\nabla_i\}$ (see Definition 4.1.3 and Remark 4.1.5). In fact, we follow the terminology and notations of the recent work of Walpuski [Wal15], which uses the same sort of 'blow-up analysis', based on Lin's paper [Lin99], that is explored in Tian's paper [Tia00], albeit in the context of Fueter sections. The reader will find out in Chapter 4 the main reason for the terminology 'bubbling locus' (cf. Theorem 4.3.6 and Proposition 4.4.2). $\Diamond$

[15] This terminology is used in the context of harmonic map theory, see e.g. [Lin99, p. 787].

[16] See the discussion preceding Theorem 3.2.1.

Remark 3.4.3. When M is compact, by the previously fixed conventions (i.e. $r_x = \delta_0(M,g)$ and $c_x = c_0(M,g)$ for every $x \in M$) we have

$$S = S(\{\nabla_i\}) = \bigcap_{0<r\leq\delta_0} \left\{ x \in M : \liminf_{i\to\infty} e^{ar^2} r^{4-n} \int_{B_r(x)} |F_{\nabla_i}|^2 \mathrm{d}V_g \geq \varepsilon_0 \right\},$$

where $a \geq O(1)c_0$ is uniform on M. ♢

☞ **Convention:** *Henceforth, we denote by $\mathscr{H}^{n-4}$ the $(n-4)-$dimensional Hausdorff measure*[17] *of the connected Riemannian $n-$manifold (M,g).*

Theorem 3.4.4 (Uhlenbeck-Nakajima)**.** *Suppose $\{\nabla_i\} \subseteq \mathfrak{U}(E)$ is a sequence of Yang-Mills connections with uniformly L^2-bounded curvature, say $\mathcal{YM}(\nabla_i) \leq \Lambda$. Then:*

(1) The blow-up locus S of $\{\nabla_i\}$ (cf. Definition 3.4.1) is a closed subset of M. Furthermore, if M is compact then $\mathscr{H}^{n-4}(S) \leq C(M,g,\Lambda) < \infty$.

(2) There exist a subsequence of $\{\nabla_i\}$, still denoted by $\{\nabla_i\}$, a sequence of gauge transformations $g_i \in \mathcal{G}(E|_{M\setminus S})$, and a smooth Yang-Mills connection ∇ on the restriction of E over $M \setminus S$, such that $g_i^*\nabla_i$ converges to ∇ in $C^\infty_{loc}-$topology outside S.

Proof. (1): We divide the proof into two parts:

(i) S is closed.

(ii) If M is compact then $\mathscr{H}^{n-4}(S) \leq C(M,g,\Lambda) < \infty$.

To prove (i), pick $x_0 \in M \setminus S$; then, there exist $0 < r_0 \leq r_{x_0}$ and a sequence $i_j \to \infty$ such that

$$\sup_j r_0^{4-n} \int_{B_{r_0}(x_0)} |F_{\nabla_{i_j}}|^2 \mathrm{d}V_g < \varepsilon_0.$$

Applying the curvature estimate (3.3.1) we get

$$\sup_j \sup_{x\in B_{\frac{r_0}{4}}(x_0)} |F_{\nabla_{i_j}}(x)|^2 < C_0\varepsilon_0 r_0^{-4}.$$

Thus, if we let $K > 0$ be such that $\mu_g(B_r(x_0)) \leq r^n K$ for all $r \leq r_0/8$, we have

$$\begin{aligned} \sup_j \sup_{x\in B_{\frac{r_0}{8}}(x_0)} e^{ar^2} r^{4-n} \int_{B_r(x)} |F_{\nabla_{i_j}}|^2 \mathrm{d}V_g &\leq e^{ar^2} r^4 K C_0 \varepsilon_0 r_0^{-4} \\ &= \text{const.} e^{ar^2} r^4. \end{aligned}$$

In particular, there exists some $0 < r \leq r_0/8$ small enough so that

$$\sup_j \sup_{x\in B_{\frac{r_0}{8}}(x_0)} e^{ar^2} r^{4-n} \int_{B_r(x)} |F_{\nabla_{i_j}}|^2 \mathrm{d}V_g < \frac{\varepsilon_0}{2},$$

[17]See Definition A.2.2 and Example A.2.3 of Appendix A

whence we conclude that $B_{\frac{r_0}{8}}(x_0) \subseteq M \setminus S$.

Now we prove (ii). Note that, since M is compact, it follows by (i) that S is compact. Let $0 < \delta < \min\{1, \delta_0\}$ be arbitrary. Then we can find a finite covering $\{B_{2\delta}(x_\alpha)\}$ of S such that $x_\alpha \in S$ for each α, and (†) $B_\delta(x_\alpha) \cap B_\delta(x_\beta) = \emptyset$ for each $\alpha \neq \beta$ (see Lemma A.3.1). Thus

$$\begin{aligned}
\sum_\alpha \delta^{n-4} &\leq \frac{e^{a\delta^2}}{\varepsilon_0} \sum_\alpha \liminf_{i\to\infty} \int_{B_\delta(x_\alpha)} |F_{\nabla_i}|^2 \mathrm{d}V_g \quad (x_\alpha \in S \text{ for each } \alpha)\\
&\leq \frac{e^{a\delta^2}}{\varepsilon_0} \liminf_{i\to\infty} \sum_\alpha \int_{B_\delta(x_\alpha)} |F_{\nabla_i}|^2 \mathrm{d}V_g\\
&\leq \frac{e^{a\delta^2}}{\varepsilon_0} \liminf_{i\to\infty} \int_{\bigcup_\alpha B_\delta(x_\alpha)} |F_{\nabla_i}|^2 \mathrm{d}V_g \quad (\dagger)\\
&\leq \frac{e^{a}}{\varepsilon_0} \liminf_{i\to\infty} \int_M |F_{\nabla_i}|^2 \mathrm{d}V_g\\
&\leq \frac{e^{a}\Lambda}{\varepsilon_0}. \quad (\mathcal{YM}(\nabla_i) \leq \Lambda \text{ for all } i)
\end{aligned}$$

Since $\{B_{2\delta}(x_\alpha)\}$ covers S, we get

$$\mathscr{H}^{n-4}_{2\delta}(S) \leq \sum_\alpha 2^{n-4}\delta^{n-4} \leq \frac{2^{n-4}e^a\Lambda}{\varepsilon_0} =: C,$$

where $C = C(M, g, \Lambda) > 0$ is *independent* of δ (the uniformity of the constants ε_0 and a is due to the compactness assumption on M). Thus, it follows that

$$\mathscr{H}^{n-4}(S) = \lim_{\delta\downarrow 0} \mathscr{H}^{n-4}_{2\delta}(S) \leq C.$$

Remark 3.4.5. In [Tia00, Lemma 3.1.3 (ii), p. 220], Tian claims that $\mathscr{H}^{n-4}(S) < \infty$ with no further hypothesis on M. His proof can be found in [Tia00, proof of Proposition 3.1.2, pp. 219-220], and follows the same line of argument of the above proof. Unfortunately, Tian does not argues about the uniformity of the involved constants (a and ε_0) in this general context. ♢

(2): By the proof of (1)-(i) above, for each $x \in M \setminus S$ there exist a neighborhood U_x of x in $M \setminus S$ and a subsequence $\{i_j^{(x)}\} \subseteq \{i\}$ such that $|F_{\nabla_{i_j^{(x)}}}|$ is uniformly bounded on U_x. Thus, invoking Theorem 3.1.10, we can find a single subsequence $\{i_j\} \subseteq \{i\}$, gauge transformations g_{i_j} of E over $M \setminus S$ and a smooth Yang-Mills connection ∇ on $E|_{M\setminus S}$ such that $g_{i_j}^*\nabla_{i_j}$ converges to ∇ in C^∞_{loc}−topology outside S.

■

Remark 3.4.6 (Uhlenbeck compactness of the moduli space of flat connections)**.** An easy application of the above theorem shows that the moduli space of flat connections $\mathcal{M}_{\mathrm{flat}}(E) := \{\nabla \in \mathfrak{U}(E) : F_\nabla = 0\}/\mathcal{G}(E)$ is compact in the natural topology of C^∞_{loc}−convergence modulo gauge transformations. Indeed, if $\{\nabla_i\}$ is a sequence of flat connections on the $G-$bundle E, then $\{\nabla_i\}$ trivially satisfies the hypothesis of Theorem 3.4.4 and, furthermore, $S(\{\nabla_i\}) = \emptyset$ (cf. Definition 3.4.1). Thus, after passing to a subsequence, we can find a Yang-Mills connection $\nabla \in \mathfrak{A}(E)$ and gauge transformations $g_i \in \mathcal{G}(E)$ such that $g_i^*\nabla_i$ converges to ∇ in $C^\infty_{\text{loc}}-$topology on M. Clearly, the limit connection ∇ is necessarily flat, thereby proving the claim. ◊

Remark 3.4.7. A special case is when $n = 4$. Suppose M is a *compact* $4-$manifold. Since $\mathscr{H}^0$ is simply the counting measure, Theorem 3.4.4 (i) implies that the blow-up set of any sequence of Yang-Mills connections with uniformly bounded L^2-energy on a $G-$bundle over M^4 is necessarily finite. ◊

In the next chapter, we will examine the causes of this non-compactness phenomenon along the blow-up set. As for the next section, we focus attention in the special type of singular limit connections arising in the above theorem.

3.5 Admissible Yang-Mills connections

Roughly speaking, Theorem 3.4.4 indicates that in order to compactify *moduli spaces* of Yang-Mills connections[18] we need to consider certain *singular* Yang-Mills connections whose singularities are supported on sets of Hausdorff codimension at least four. Following [Tia00, p. 215] (also see [TY02, p. 11]), we introduce the appropriate definition of such singular connections and study some of its basic properties.

Definition 3.5.1 (Admissible connections)**.** A pair (∇, S), where $S \subseteq M$ is a closed subset and ∇ is a smooth connection on $E|_{M\setminus S}$, is called an **admissible connection** on E with *singular set* S when the following holds:

(i) $\mathscr{H}^{n-4}(S \cap K) < \infty$, for each compact subset $K \subset M$.

(ii) The curvature F_∇ on $M \setminus S$ is L^2-integrable:

$$\int_{M\setminus S} |F_\nabla|^2 \mathrm{d}V_g < \infty.$$

An **admissible Yang-Mills connection** is an admissible connection (∇, S) such that ∇ is Yang-Mills outside S, i.e. such that $\mathrm{d}^*_\nabla F_\nabla = 0$ on $M \setminus S$. Analogously, in case M is endowed with a closed $(n-4)-$form $\Xi \in \Omega^{n-4}(M)$, an **admissible $\Xi-$ASD instanton** is an admissible connection (∇, S) such that ∇ is a $\Xi-$ASD instanton on $M \setminus S$.

Remark 3.5.2. It is worth noting that a similar concept of singular connection appears much earlier in the seminal work [BS94] in the form of singular Hermitian metrics on holomorphic vector bundles over compact Kähler manifolds. ◊

[18] i.e. quotients of the form $\mathcal{M} := \{\nabla \in \mathfrak{A}(E) : \mathrm{d}^*_\nabla F_\nabla = 0\} / \mathcal{G}(E)$.

Remark 3.5.3. Let μ_g be the canonical Radon measure on (M, g) associated to $\mathrm{d}V_g$. Recall that μ_g differs from $\mathscr{H}^n$ by a constant factor α_n (cf. Example A.2.3). We now make two simple observations about conditions (i) and (ii) of Definition 3.5.1.

First, we note that (i) implies $\mu_g(S) = \mathscr{H}^n(S) = 0$. Indeed, by Proposition A.2.5, (i) implies $\mathscr{H}^n(S \cap K) = 0$ for each compact subset $K \subseteq M$. Since S is closed, the claim follows by inner regularity of μ_g (cf. Theorem A.1.8). In particular, it follows that F_∇ is defined μ_g-a.e. on M.

The second observation is that (ii) implies

$$\lim_{r\downarrow 0} \int_{N_r(S)} |F_\nabla|^2 \mathrm{d}V_g = 0, \tag{3.5.1}$$

where $N_r(S) := \{x \in M : d_g(x, S) \leq r\}$ denotes the closed $r-$neighborhood of S with respect to the Riemannian distance function d_g of (M, g), for each $r > 0$. To see this, first observe that $\mu_g(S) = 0$ implies, by the outer regularity of μ_g, that $\mu_g(N_r(S)) < \infty$ for each $r > 0$ sufficiently small. Thus, using Theorem A.1.3 (3.b), we have

$$0 = \mu_g(S) = \mu_g(\bigcap_{r>0} N_r(S)) = \lim_{r\downarrow 0} \mu_g(N_r(S)). \tag{3.5.2}$$

Now, the L^2-integrability condition (ii) means that $|F_\nabla|^2$ is in $L^1(\mu_g)$; thus, given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $E \subseteq M$ satisfies $\mu_g(E) < \delta$ then

$$\int_E |F_\nabla|^2 \mathrm{d}V_g < \varepsilon.$$

Using (3.5.2), for each $r > 0$ sufficiently small we do have $\mu_g(N_r(S)) < \delta$ and, therefore,

$$\int_{N_r(S)} |F_\nabla|^2 \mathrm{d}V_g < \varepsilon.$$

The claim follows by the arbitrariness of $\varepsilon > 0$. $\Diamond$

Remark 3.5.4. Let (∇, S) be an admissible connection on E. Since $|F_\nabla|$ is a μ_g-measurable function which is μ_g-almost everywhere defined, it follows that

$$\mu_\nabla := |F_\nabla|^2 \mu_g \tag{3.5.3}$$

defines a measure on M. Furthermore, the L^2-integrability of F_∇ ensures that μ_∇ is a *Radon* measure on M (cf. Lemma A.1.6).

Given the codimension 4 nature of the singular set S of ∇, a natural way to detect a point $x \in S$ is by looking at the $(n-4)-$upper density $\Theta^*(\mu_\nabla, x)$ of μ_∇ at x (cf. Definition A.3.3). Indeed, note that if ∇ is smooth near x then $|F_\nabla|$ is bounded near x. Thus, for $0 < r \ll 1$, recalling that $\mu_g(B_r(x)) \leq \text{const.} r^n$, we have:

$$\Theta^{*n-4}(\mu_\nabla, x) = \limsup_{r\downarrow 0} r^{4-n} \int_{B_r(x)} |F_\nabla|^2 \mathrm{d}V_g \leq \text{const.} \limsup_{r\downarrow 0} r^4 = 0.$$

This shows that if $\Theta^{*n-4}(\mu_\nabla, x) > 0$ then ∇ is not smooth in a neighborhood of x.

To finish this remark, we note that if ∇ is a *smooth* Yang-Mills connection with finite L^2-energy, then the monotonicity formula (Theorem 3.2.1) implies that, for each $x \in M$,

$$0 < \sigma < \rho \leq r_x \quad \Rightarrow \quad e^{a\sigma^2}\sigma^{4-n}\mu_\nabla(B_\sigma(x)) \leq e^{a\rho^2}\rho^{4-n}\mu_\nabla(B_\rho(x)). \quad \diamondsuit$$

By definition, an admissible connection (∇, S) on E defines a smooth connection precisely when $S = \emptyset$. Now the interesting thing is that the singular set S may not be preserved by gauge transformations, i.e. even if $S \neq \emptyset$ there may be a gauge transformation $g \in \mathcal{G}(E|_{M\setminus S})$ such that $g^*\nabla$ is the restriction of a smooth connection over all of M. In this case we say that the singularity set of ∇ is **removable**. In general, if $S' \subsetneq S$ is a closed subset, we say that (∇, S) **extends** to a smooth connection on $M \setminus S'$, modulo gauge transformations, when there exists a smooth gauge transformation $g \in \mathcal{G}(E|_{M\setminus S})$ such that $g^*\nabla$ is the restriction of a smooth connection on $E|_{M\setminus S'}$.

Extending the usual gauge equivalence notion between smooth connections (cf. Section 1.1), we will say that two admissible connections (∇_1, S_1) and (∇_2, S_2) are **gauge equivalent** if there is a gauge transformation g of E over $M\setminus(S_1\cup S_2)$ such that $\nabla_2 = g^*\nabla_1$ outside $S_1 \cup S_2$.

The high codimension of the singular set, together with the L^2-integrability condition on the curvature, allows us to prove that every admissible Yang-Mills connection is a *weak* Yang-Mills connection (cf. [Yan03, p. 358]):

Proposition 3.5.5. *Every admissible Yang-Mills connection* (∇, S) *satisfies the weak Yang-Mills equation:*

$$\int_M \langle \mathrm{d}_\nabla \xi, F_\nabla \rangle \mathrm{d}V_g = 0, \quad \forall \xi \in \Gamma_0(T^*M \otimes \mathfrak{g}_E).$$

Proof. Let K be a compact subset of M containing the support of ξ in its interior, and let $0 < \delta \ll 1$ be a small parameter. Then, since $S \cap K$ is a compact set with $\mathscr{H}^{n-4}(S \cap K) < \infty$, we can find *finitely* many open geodesic balls $B_{r_i}(x_i)$ such that $r_i < \delta$, $x_i \in S \cap K \subseteq \bigcup_i B_{r_i}(x_i)$ and (†) $\sum_i r_i^{n-3} \leq \text{const.}\delta^{1/2}$. For each i, choose a bump function $\phi_i \in C^\infty(M)$ for $M \setminus B_{2r_i}(x_i)$ supported in $M \setminus \overline{B}_{r_i}(x_i)$, with $|d\phi_i(x)| \leq \text{const.}r_i^{-1}\chi_{B_{2r_i}(x_i)}(x)$, for each $x \in M$. Then, $\phi = \prod_i \phi_i \in C^\infty(M)$ is a bump function for $M \setminus N_{2\delta}(S \cap K)$ with $\text{supp}(\phi) \subseteq M \setminus \bigcup_i \overline{B}_{r_i}(x_i)$. Furthermore, we have the pointwise bound

$$|\mathrm{d}\phi(x)| \leq \text{const.} \sum_i r_i^{-1}\chi_{B_{2r_i}(x_i)}(x). \tag{3.5.4}$$

By dominated convergence, note that

$$\int_M \langle \mathrm{d}_\nabla \xi, F_\nabla \rangle = \lim_{\delta \downarrow 0} \int_M \phi_\delta \langle \mathrm{d}_\nabla \xi, F_\nabla \rangle, \tag{3.5.5}$$

since $\phi(x)=\phi_\delta(x)\to 1$ as $\delta\downarrow 0$ for μ_g-a.e. $x\in M$, and $|\phi\langle \mathrm{d}_\nabla\xi, F_\nabla\rangle|$ is dominated by const.$\chi_K|F_\nabla|$ which is in $L^1(\mu_g)$, by the L^2-integrability of F_∇ and Hölder's inequality.

Now, since (∇, S) is an admissible Yang-Mills connection, the following holds on $M\setminus S$:

$$\begin{aligned}
\mathrm{d}\left(\phi\mathrm{tr}(\xi\wedge *F_\nabla)\right) &= \mathrm{d}\phi\wedge\mathrm{tr}(\xi\wedge *F_\nabla)+\phi\mathrm{d}\mathrm{tr}(\xi\wedge *F_\nabla)\\
&= \mathrm{d}\phi\wedge\mathrm{tr}(\xi\wedge *F_\nabla)+\phi\left(\mathrm{tr}(\mathrm{d}_\nabla\xi\wedge *F_\nabla)+\mathrm{tr}(\xi\wedge \mathrm{d}_\nabla * F_\nabla)\right)\\
&= \mathrm{d}\phi\wedge\mathrm{tr}(\xi\wedge *F_\nabla)+\phi\mathrm{tr}(\mathrm{d}_\nabla\xi\wedge *F_\nabla). \quad \text{(Yang-Mills on } M\setminus S)
\end{aligned}$$

Noting that $\mathrm{supp}(\phi)\cap\mathrm{supp}(\xi)\subseteq \mathring{K}\setminus\bigcup_i \overline{B}_{r_i}(x_i)\subseteq M\setminus S$, we get:

$$\begin{aligned}
\int_M \phi\langle \mathrm{d}_\nabla\xi, F_\nabla\rangle &= -\int_M \phi\mathrm{tr}(\mathrm{d}_\nabla\xi\wedge *F_\nabla)\\
&= -\int_{\mathring{K}\setminus\bigcup_i \overline{B}_{r_i}(x_i)} \mathrm{d}\left(\phi\mathrm{tr}(\xi\wedge *F_\nabla)\right)+\int_{\mathring{K}\setminus\bigcup_i \overline{B}_{r_i}(x_i)}\mathrm{d}\phi\wedge\mathrm{tr}(\xi\wedge *F_\nabla)\\
&= \int_{\mathring{K}\setminus\bigcup_i \overline{B}_{r_i}(x_i)}\mathrm{d}\phi\wedge\mathrm{tr}(\xi\wedge *F_\nabla) \quad \text{(by Stokes' theorem)}\\
&= \int_M \mathrm{d}\phi\wedge\mathrm{tr}(\xi\wedge *F_\nabla). \qquad (3.5.6)
\end{aligned}$$

We can estimate the last integral as follows:

$$\begin{aligned}
\left|\int_M \mathrm{d}\phi\wedge\mathrm{tr}(\xi\wedge *F_\nabla)\right| &\le \text{const.}\int_M |\mathrm{d}\phi||F_\nabla|\mathrm{d}V_g \quad (\xi \text{ has compact support})\\
&= \text{const.}\int_{N_{2\delta}(S\cap K)}|\mathrm{d}\phi||F_\nabla|\mathrm{d}V_g \quad (\text{since } \phi\equiv 1 \text{ on } M\setminus N_{2\delta}(S\cap K))\\
&\le \text{const.}\left(\int_{N_{2\delta}(S\cap K)}|\mathrm{d}\phi|^2\mathrm{d}V_g\right)^{1/2}\left(\int_{N_{2\delta}(S\cap K)}|F_\nabla|^2\mathrm{d}V_g\right)^{1/2} \quad \text{(Hölder)}\\
&\le \text{const.}\left(\sum_i\int_{B_{2r_i}(x_i)} r_i^{-2}\mathrm{d}V_g\right)^{1/2}\left(\int_{N_{2\delta}(S\cap K)}|F_\nabla|^2\mathrm{d}V_g\right)^{1/2} \quad \text{(by (3.5.4))}\\
&\le \text{const.}\left(\sum_i r_i^{n-2}\right)^{1/2}\left(\int_{N_{2\delta}(S\cap K)}|F_\nabla|^2\mathrm{d}V_g\right)^{1/2}\\
&\le \text{const.}\delta^{1/2}\left(\int_{N_{2\delta}(S\cap K)}|F_\nabla|^2\mathrm{d}V_g\right)^{1/2}, \quad \text{(by (†))} \qquad (3.5.7)
\end{aligned}$$

where on the last but one inequality we upper-estimated each $\mu_g(B_{2r_i}(x_i))$ by a constant, depending only on g, times r_i^n.

By the L^2-integrability of F_∇, the RHS of (3.5.7) goes to zero as $\delta\downarrow 0$ (see Remark 3.5.3). Thus, using (3.5.6), we conclude that

$$\lim_{\delta\downarrow 0}\int_M \phi_\delta\langle \mathrm{d}_\nabla\xi, F_\nabla\rangle = 0.$$

Combining with (3.5.5), this completes the proof. ■

Generalized first two terms of the Chern character. (cf. [Tia00, Proposition 2.3.1 and Corollary 2.3.2]) Suppose E is a $\mathrm{U}(r)-$bundle over M. Motivated by basic Chern-Weil theory (cf. Section 1.3), we now define (in the sense of currents – see Section A.6) the first two terms of the 'Chern character' of admissible connections, by using their curvature forms.

Definition 3.5.6. If ∇ is an admissible connection on E, we define:

(i) $\mathrm{ch}_1(\nabla) \in \mathscr{D}_{n-2}(M)$ by

$$\mathrm{ch}_1(\nabla)(\varphi) := \frac{i}{2\pi}\int_M \varphi \wedge \mathrm{tr}(F_\nabla), \quad \forall \varphi \in \mathscr{D}^{n-2}(M).$$

(ii) $\mathrm{ch}_2(\nabla) \in \mathscr{D}_{n-4}(M)$ by

$$\mathrm{ch}_2(\nabla)(\varphi) := \frac{-1}{4\pi^2}\int_M \varphi \wedge \mathrm{tr}(F_\nabla \wedge F_\nabla), \quad \forall \varphi \in \mathscr{D}^{n-4}(M).$$

(iii) $c_2(\nabla) \in \mathscr{D}_{n-4}(M)$ by

$$c_2(\nabla)(\varphi) := \frac{1}{8\pi^2}\int_M \varphi \wedge \left(\mathrm{tr}(F_\nabla \wedge F_\nabla) - \mathrm{tr}(F_\nabla) \wedge \mathrm{tr}(F_\nabla)\right), \quad \forall \varphi \in \mathscr{D}^{n-4}(M).$$

Remark 3.5.7. The L^2-integrability of F_∇ ensures the above integrals are well-defined (see Remark 3.5.3). ♢

The following contains [Tia00, Proposition 2.3.1 & Corollary 2.3.2, pp. 216-218].

Theorem 3.5.8 (Tian)**.** *Let* (∇, S) *be an admissible connection on* E*. Then:*

(a) $\mathrm{ch}_1(\nabla)$ *is closed. In particular, the natural* $(n-4)-$*current on* M *defined by* $\mathrm{tr}(F_\nabla)\wedge \mathrm{tr}(F_\nabla)$ *is closed.*

If, furthermore, (∇, S) *is an admissible* Yang-Mills *connection, then:*

(b) $\mathrm{ch}_2(\nabla)$ *is closed. In particular,* $c_2(\nabla)$ *is closed.*

The rest of this paragraph is devoted to the proof of Theorem 3.5.8.

(a) Recalling Definition A.6.6, we want to show that $[\partial \mathrm{ch}_1(\nabla)](\varphi) = \mathrm{ch}_1(\nabla)(\mathrm{d}\varphi) = 0$, for every test form $\varphi \in \mathcal{D}^{n-3}(M)$. Let $\varphi \in \mathscr{D}^{n-3}(M)$ and let K be a compact subset of M containing the support of φ in its interior. Using the fact that (∇, S) is an admissible connection, choose a cut-off function $\phi = \phi_\delta$ on M exactly in the same way as we did in the proof of Theorem 3.5.5 (same notation). Note that the following holds on $M \setminus S$:

$$\begin{aligned}
\mathrm{d}\left(\phi\varphi \wedge \mathrm{tr}(F_\nabla)\right) &= \mathrm{d}\left(\phi \mathrm{tr}(\varphi \wedge F_\nabla)\right) \\
&= \mathrm{d}\phi \wedge \mathrm{tr}(\varphi \wedge F_\nabla) + (-1)^{n-3}\phi\left(\mathrm{tr}(\varphi \wedge \mathrm{d}_\nabla F_\nabla) + \mathrm{tr}(\mathrm{d}\varphi \wedge F_\nabla)\right) \\
&= \mathrm{d}\phi \wedge \mathrm{tr}(\varphi \wedge F_\nabla) + \phi \mathrm{d}\varphi \wedge \mathrm{tr}(F_\nabla) \quad \text{(by Bianchi identity).}
\end{aligned}$$

Thus, by Stokes' theorem,

$$\int_M \phi \operatorname{tr}(F_\nabla) \wedge \mathrm{d}\varphi = -\int_M \mathrm{d}\phi \wedge \operatorname{tr}(F_\nabla \wedge \varphi).$$

We can now estimate the RHS of the last equation in the same way as we did in the proof of Theorem 3.5.5 to conclude

$$\int_M \mathrm{d}\varphi \wedge \operatorname{tr}(F_\nabla) = 0.$$

We omit the details.

For the last assertion of (a), just note that $\mathrm{d}\varphi \wedge \operatorname{tr}(F_\nabla) = \mathrm{d}(\varphi \wedge \operatorname{tr}(F_\nabla))$ and $\varphi \wedge \operatorname{tr}(F_\nabla) \in \mathscr{D}^{n-3}(M)$ whenever $\varphi \in \mathscr{D}^{n-5}(M)$. This completes the proof of (a).

☞ **Warning:** *To the extent of what we have established so far, the following proof is only rigorous if M is compact*[19].

(b) This case is more subtle. The method used in the proof of (a) fails here because we would need more than just L^2-integrability of F_∇ to use Hölder's inequality in the estimates. Thus, in this case, we follow Tian's proof [Tia00, pp. 216-218], which uses the $\varepsilon-$regularity theorem 3.3.1 together with Uhlenbeck's local Coulomb gauge theorem [Uhl82b, Theorem 2.7] to overcome this problem. In the following proof, C will denote a positive uniform constant.

If $n = 4$ there is nothing to prove. So, suppose $n \geq 5$. Let $\varphi \in \mathscr{D}^{n-5}(M)$ and let $K \subseteq M$ be a compact subset containing the support of φ in its interior. By means of a standard partition of unity argument, we may suppose M is an open ball in $\mathbb{R}^n$ and E is a trivial bundle over M. Choosing a global trivialization for E, we write $\nabla = \mathrm{d} + A$, where $A \in \Omega^1(M, \mathfrak{g})$.

Fix any $0 < \varepsilon \leq \varepsilon_0$ sufficiently small, where ε_0 is the constant given by Theorem 3.3.1. For $0 < r \ll 1$, we define

$$E_r := \{x \in K : r^{4-n} e^{ar^2} \int_{B_r(x)} |F_\nabla|^2 \mathrm{d}V_g \geq \varepsilon\},$$

where a is the constant giving by the monotonicity formula (Theorem 3.2.1).

Lemma 3.5.9. *The following holds:*

(i) $0 < r \leq r' \ll 1$ *implies* $E_r \subseteq E_{r'}$;

(ii) E_r *is compact;*

(iii) $\bigcap_{r>0} E_r \cap (M \setminus S) = \emptyset$.

[19] There are issues regarding the non-uniformity of the constants involved in the monotonicity formula (Theorem 3.2.1) and the $\varepsilon-$regularity theorem (Theorem 3.3.1)

Proof. Part (i) follows directly from the monotonicity formula (Theorem 3.2.1). For (ii) just note that if (x_n) is a sequence of points in E_r which converges to $x \in K$, then $\chi_{B_r(x_n)}$ converges to $\chi_{B_r(x)}$ pointwise as $n \to \infty$, so that by the L^2-integrability of F_∇ we may apply dominated convergence to get

$$r^{4-n}e^{ar^2}\int_{B_r(x)}|F_\nabla|^2 \mathrm{d}V_g = \lim_{n\to\infty} r^{4-n}e^{ar^2}\int_{B_r(x_n)}|F_\nabla|^2 \mathrm{d}V_g \geq \varepsilon,$$

where the last inequality follows since each x_n lies in E_r. Thus E_r is a closed subset of the compact set K, showing E_r is compact.

To prove (iii), let $x \in M \setminus S$. Since ∇ is smooth on the open set $M \setminus S$, we have $\Theta^{*n-4}(\mu_\nabla, x) = 0$ (cf. Remark 3.5.4). Thus, $x \notin E_r$ for each $r > 0$ sufficiently small. ■

Let $\phi \in C^\infty(\mathbb{R})$ be a cut-off function satisfying the following: (1) $\phi(t) = 0$ for $t \leq 1$, (2) $\phi(t) = 1$ for $t \geq 2$, and (3) $0 \leq \phi'(t) \leq 2$ for all $t \in \mathbb{R}$. In particular, note that ϕ is a nondecreasing function. Now define, for each $0 < r \ll 1$, the continuous function $\phi_r \in C^0(M)$ given by

$$\phi_r(x) := \phi\left(\frac{d_g(x, (S\cap K)\cup E_r)}{3r}\right).$$

Using properties (1) and (2) of ϕ, for each $0 < r \ll 1$ we have[20]

$$\phi_r\Big|_{N_{3r}((S\cap K)\cup E_r)} \equiv 0 \quad \text{and} \quad \phi_r\Big|_{M\setminus N_{6r}((S\cap K)\cup E_r)} \equiv 1.$$

Moreover, by the monotonicity of ϕ and Lemma 3.5.9 (i), we have

$$\phi_r(x) \uparrow 1 \quad \text{as } r \downarrow 0, \quad \forall x \in M.$$

In particular, by dominated convergence,

$$\int_M \mathrm{d}\varphi \wedge \mathrm{tr}(F_\nabla \wedge F_\nabla) = \lim_{r\downarrow 0}\int_M \phi_r \mathrm{d}\varphi \wedge \mathrm{tr}(F_\nabla \wedge F_\nabla). \tag{3.5.8}$$

Now fix any $0 < r \ll 1$. By Lemma 3.5.9 (ii), the set $E_r \cup (S\cap K)$ is compact, so that it admits a finite cover $\{B_{2r}(x_\alpha)\}_{1\leq\alpha\leq N_r}$ such that $x_\alpha \in E_r \cup (S\cap K)$ and $B_r(x_\alpha)\cap B_r(x_\beta) = \emptyset$ for each $\alpha \neq \beta$ (cf. Lemma A.3.1). Notice that, for each fixed index α_0 of the cover, the number of x_α ($\alpha = 1, \ldots, N_r$) such that $B_{8r}(x_\alpha) \cap B_{8r}(x_{\alpha_0}) \neq \emptyset$ is bounded by a constant independent of r (†), in fact, depending only on n and M.

(In what follows, for notational simplicity, we will ignore points outside K, since $\mathrm{supp}(\varphi) \subseteq \mathring{K}$ and we want to estimate the integral (3.5.8)) For each $x \notin \bigcup_\alpha B_{2r}(x_\alpha)$, we have

$$r^{4-n}\int_{B_r(x)}|F_\nabla|^2 \mathrm{d}V_g < \varepsilon.$$

[20]For each $A \subseteq (M, g)$ and $\delta > 0$, we let $N_\delta(A) := \{x \in M : d_g(x, A) \leq \delta\}$ be the closed $\delta-$neighborhood of A in M with respect to the Riemannian distance function d_g (M is connected).

Thus, by the local curvature estimate (Theorem 3.3.1),

$$|F_\nabla|(x) \leq \frac{C}{r^2}\left(r^{4-n}\int_{B_r(x)}|F_\nabla|^2 \mathrm{d}V_g\right)^{1/2} < \frac{C\sqrt{\varepsilon}}{r^2}. \tag{3.5.9}$$

Since $0 < \varepsilon \leq \varepsilon_0$ is sufficiently small (independent of r), we can apply Uhlenbeck's local Coulomb gauge theorem [Uhl82b, Theorem 2.7] to find, for each $x \in M \backslash N_{3r}\left((S \cap K) \cup E_r\right)$, a gauge transformation g_x over $B_r(x)$ such that

$$|g_x^* A|(y) \leq \frac{C}{r}\left(r^{4-n}\int_{B_r(x)}|F_\nabla|^2 \mathrm{d}V_g\right)^{1/2}, \quad \forall y \in B_r(x).$$

Now, by standard methods (see e.g. [DK90, p. 162]), we can glue these g_x appropriately so that, for each α, we obtain a gauge transformation g_α over $B_{8r}(x_\alpha) \backslash N_{3r}\left((S \cap K) \cup E_r\right)$ such that

$$|g_\alpha^* A|(x) \leq \frac{C}{r}\left(r^{4-n}\int_{B_r(x)}|F_\nabla|^2 \mathrm{d}V_g\right)^{1/2}, \tag{3.5.10}$$

for each $x \in B_{8r}(x_\alpha) \setminus N_{3r}\left((S \cap K) \cup E_r\right)$. From (3.5.10), using the basic relation (1.1.7), we get

$$|\mathrm{d}g_\alpha \cdot g_\beta^{-1}| \leq \frac{2C\sqrt{\varepsilon}}{r} \quad \text{on } \left(B_{8r}(x_\alpha) \cap B_{8r}(x_\beta)\right) \setminus N_{3r}\left((S \cap K) \cup E_r\right).$$

Hence, by modifying g_α slightly on the overlaps, we may assume that $g_\alpha \cdot g_\beta^{-1}$ is constant on each connected component of $\left(B_{8r}(x_\alpha) \cap B_{8r}(x_\beta)\right) \setminus N_{3r}\left((S \cap K) \cup E_r\right)$ for any $\alpha \neq \beta$.

Recalling (1.1.8), (1.1.18) and the definition of the standard Chern-Simons 3−form (see e.g. [BM94, pp. 284-285]), for each $\alpha \leq N_r$ we have

$$\begin{aligned} \mathrm{tr}(F_\nabla \wedge F_\nabla)(x) &= \mathrm{tr}(F_{g_\alpha^*\nabla} \wedge F_{g_\alpha^*\nabla})(x) \\ &= \mathrm{dtr}\left(g_\alpha^* A \wedge F_{g_\alpha^*\nabla} + \frac{1}{3} g_\alpha^* A \wedge g_\alpha^* A \wedge g_\alpha^* A\right)(x), \end{aligned}$$

for all $x \in B_{8r}(x_\alpha) \setminus N_{3r}\left((S \cap K) \cup E_r\right)$.

For any $\alpha \neq \beta$, since $g_\alpha \cdot g_\beta^{-1}$ is piecewise constant, on the overlap $\left(B_{8r}(x_\alpha) \cap B_{8r}(x_\beta)\right) \setminus N_{3r}\left((S \cap K) \cup E_r\right)$ we have

$$\begin{aligned} &\mathrm{tr}\left(g_\alpha^* A \wedge F_{g_\alpha^*\nabla} + \frac{1}{3} g_\alpha^* A \wedge g_\alpha^* A \wedge g_\alpha^* A\right) \\ = \ &\mathrm{tr}\left(g_\beta^* A \wedge F_{g_\beta^*\nabla} + \frac{1}{3} g_\beta^* A \wedge g_\beta^* A \wedge g_\beta^* A\right). \end{aligned}$$

Therefore, we get a globally defined Chern-Simons form Ψ on $M \setminus N_{3r}\left((S \cap K) \cup E_r\right)$, such that

$$\mathrm{d}\Psi = \mathrm{tr}(F_\nabla \wedge F_\nabla)$$

and

$$\Psi(x) = \mathrm{tr}\left(g_\alpha^* A \wedge F_{g_\alpha^*\nabla} + \frac{1}{3} g_\alpha^* A \wedge g_\alpha^* A \wedge g_\alpha^* A\right)(x),$$

whenever $x \in B_{8r}(x_\alpha) \setminus N_{3r}((S \cap K) \cup E_r)$. Using the estimates (3.5.9) and (3.5.10), for each α and $x \in B_{6r}(x_\alpha) \setminus N_{3r}((S \cap K) \cup E_r)$ we have

$$|\Psi(x)| \leq Cr^{-3}\left(r^{4-n}\int_{B_r(x)}|F_\nabla|^2 \mathrm{d}V_g\right)^{\frac{3}{2}} \leq Cr^{1-n}\int_{B_{8r}(x_\alpha)}|F_\nabla|^2 \mathrm{d}V_g.$$

Note that in the second inequality we applied (3.5.9) and used that $B_r(x) \subseteq B_{8r}(x_\alpha)$.

From the above analysis, we can estimate:

$$\begin{aligned}\left|\int_M \phi_r \mathrm{d}\varphi \wedge \mathrm{tr}(F_\nabla \wedge F_\nabla)\right| &\leq \int_{3r \leq d_g(x,(S\cap K)\cup E_r)\leq 6r} \frac{1}{3r}|\Psi||\mathrm{d}\varphi|\mathrm{d}V_g \\ &\leq C\sup_M|\mathrm{d}\varphi|\sum_{\alpha=1}^{N_r}\int_{B_{8r}(x_\alpha)}|F_\nabla|^2\mathrm{d}V_g.\end{aligned}$$

Recalling (†) and combining with (3.5.8), it follows that

$$\left|\int_M \mathrm{d}\varphi \wedge \mathrm{tr}(F_\nabla \wedge F_\nabla)\right| \leq C\sup_M|\mathrm{d}\varphi| \lim_{r\downarrow 0}\int_{N_{8r}((S\cap K)\cup E_r)}|F_\nabla|^2\mathrm{d}V_g.$$

Finally, noting that the parts (iii) and (i) of Lemma 3.5.9 imply that $\bigcap_{r>0} N_{8r}(S \cup E_r) \subseteq S$ and $N_{8r}(S \cup E_r) \subseteq N_{8r}(S \cup E_{r'})$ for $0 < r \leq r' \ll 1$, it follows that the last integral converges to zero as $r \downarrow 0$. Therefore

$$\int_M \mathrm{d}\varphi \wedge \mathrm{tr}(F_\nabla \wedge F_\nabla) = 0.$$

This completes the proof that $\mathrm{ch}_2(\nabla)$ is closed.

Using the second part of (a), and the fact that the linear combination of closed currents is a closed current, the last assertion of (b) follows. Theorem 3.5.8 is proved.

Remark 3.5.10. Tian [Tia00, proof of Corollary 2.3.2, p. 218] provides a different proof of Theorem 3.5.8 (a) assuming (∇, S) is an admissible Yang-Mills connection. In this case, the 2−form $\mathrm{tr}(F_\nabla)$ is L^2−integrable and *harmonic* on $M \setminus S$. Tian claims that this implies, by standard elliptic theory, that $\mathrm{tr}(F_\nabla)$ extends to a smooth form on the whole M, thereby proving the result. Unfortunately, I was not able to come up with a detailed proof of Tian's claim. Since this fact will be important for us in the next chapter, let me at least *sketch* a tentative explanation for the compact case. What follows may not be entirely rigorous but should contain a grain of truth concerning the problem.

There is a strong singularity removal result of Harvey-Polking [HP70, Theorem 4.1 (b)] that implies the following. Suppose $P(x, D)$ is a partial differential operator of order 2 on an open subset U of $\mathbb{R}^n$ and let Σ be a closed subset of U with $\mathscr{H}^{n-2}(\Sigma) = 0$. Then, for every $f \in L^1_{\mathrm{loc}}(U)$ satisfying $Pf = 0$ weakly on $U \setminus \Sigma$, one has that $Pf = 0$ holds weakly on the whole U (see Appendix B for basic definitions).

We want to apply the obvious analogue version of this result for the Laplacian Δ_g acting on 2−forms of M. Since $\mathscr{H}^{n-4}\lfloor S$ is locally finite, we do have $\mathscr{H}^{n-2}(S) = 0$. Moreover, since $\mathrm{tr}(F_\nabla)$ is L^2−integrable, the natural functional $\varphi \mapsto \langle \varphi, \mathrm{tr}(F_\nabla)\rangle_{L^2}$ defines a distribution on $\mathscr{D}^2(M)$ and, by the hypothesis, $\Delta_g \mathrm{tr}(F_\nabla) = 0$ holds strongly on $M \setminus S$. If M is compact and the result of Harvey-Polking applies to our setting, then it would follow that $\Delta_g \mathrm{tr}(F_\nabla) = 0$ holds weakly on M. Thus, we can invoke the elliptic regularity result [Joy07, Theorem 1.4.1, p. 13] to conclude $\mathrm{tr}(F_\nabla)$ is in fact a smooth harmonic form on the whole M. ♢

Weak convergence of admissible Yang-Mills connections. Having the compactness-type result of Theorem 3.4.4 in mind, we now define a sensible notion of convergence for admissible Yang-Mills connections (cf. [Tia00, p. 219]).

Definition 3.5.11. A sequence $\{(\nabla_i, S_i)\}$ of admissible Yang-Mills connections is said to **converge weakly** to an admissible Yang-Mills connection (∇, S) modulo gauge transformations if the following holds.

(i) $\int_M |F_{\nabla_i}|^2 \mathrm{d}V_g \leq \Lambda$, for some constant $\Lambda > 0$ uniform on i.

(ii) There exist (smooth) gauge transformations g_i of the G−bundle E over $M \setminus S$ such that, for any compact subset $K \subseteq M \setminus S$, the connections $g_i^*\nabla_i$ extends smoothly across K for i sufficiently large and converges to ∇ in C^∞−topology on K as $i \to \infty$.

The definition clearly implies that for any smooth $\mathfrak{g}_E$−valued 1−form α with compact support on M we have

$$\lim_{i\to\infty} \int_M \langle F_{g_i^*\nabla_i}, \mathrm{d}\alpha\rangle \mathrm{d}V_g = \int_M \langle F_\nabla, \mathrm{d}\alpha\rangle \mathrm{d}V_g,$$

which in some sense justifies the adjective ‘weak’. In particular, we have the following basic result.

Lemma 3.5.12. *Weak limits of admissible Yang-Mills connections are unique modulo gauge transformations.*

Chapter 4

Structure of blow-up loci

☞ *In this chapter we will make use of some basic results in geometric measure theory. For the sake of completeness and textual linearity, we collect these results together with the relevant definitions in Appendix A.*

In this chapter we study the structure of the blow-up set S of a weakly convergent sequence of Yang-Mills connections $\nabla_i \rightharpoonup \nabla$. More specifically, we study the causes of the formation of S, its rectifiability and some of its geometry. We follow closely chapters 3 and 4 of Tian's paper [Tia00].

In Section 4.1, we start showing that, after passing to a subsequence if necessary, the Radon measures $\mu_i := |F_{\nabla_i}|^2 \mathrm{d}V_g$ have a weak* limit $\mu = |F_\nabla|^2 \mathrm{d}V_g + \nu$, where $\nu = \theta \mathscr{H}^{n-4} \lfloor S$ for some nonnegative $\mathscr{H}^{n-4}-$integrable function $\theta : S \to [0, \infty[$. We then proceed to show that S decomposes into two closed pieces:

$$S = \Gamma \cup \operatorname{sing}(\nabla),$$

where $\Gamma := \operatorname{spt}(\nu)$ and $\operatorname{sing}(\nabla)$ is the support of the $(n-4)-$density of $|F_\nabla|^2 \mathrm{d}V_g$. Further, $\operatorname{sing}(\nabla)$ is shown to be an $\mathscr{H}^{n-4}-$negligible set. Next, in Section 4.2, we show a first regularity result for the blow-up locus: Γ is a countably $\mathscr{H}^{n-4}-$*rectifiable* set, i.e. at $\mathscr{H}^{n-4}-$a.e. $x \in \Gamma$ the approximate $(n-4)-$dimensional tangent space $T_x\Gamma$ exists, and ν can be written as $\nu = \Theta(\mu, \cdot)\mathscr{H}^{n-4} \lfloor \Gamma$, where $\Theta(\mu, \cdot)$ is the $(n-4)-$density function of μ, an upper semi-continuous function $S \to [\varepsilon_0, \infty[$, where $\varepsilon_0 > 0$ is given by Theorem 3.3.1.

Section 4.3 is the core of this chapter. We analyze the behavior of ∇_i for $i \gg 1$ near a smooth point $x \in \Gamma$, i.e. a point $x \notin \operatorname{sing}(\nabla)$ at which $T_x\Gamma$ is well-defined. The main result is that, at $\mathscr{H}^{n-4}-$a.e. smooth point[1] $x \in \Gamma$, there is a blowing up of the sequence $\{\nabla_i\}$ around the point x whose limit $B(x)$ is a non-flat Yang-Mills connection

[1]In fact, the result holds for all smooth points $x \in \Gamma$ such that $\Theta(\mu, \cdot)|_\Gamma$ is $\mathscr{H}^{n-4}-$approximately continuous at x.

on T_xM which is, modulo gauge transformations, the pull-back of a connection $I(x)$ on $T_x\Gamma^\perp$ satisfying the energy inequality

$$\mathcal{YM}(I(x)) \leq \Theta(\mu, x).$$

At this stage, we know that at each point x of the blow-up locus S the sequence $\{\nabla_i\}$ loses energy via bubbling and/or develops a singularity.

In Section 4.4 we turn to the case in which $\{\nabla_i\}$ is a sequence of $\Xi-$anti-self-dual instantons, for some closed $(n-4)-$form Ξ on the base. In this case, we are able to show that, at $\mathscr{H}^{n-4}-$a.e. $x \in \Gamma$, the tangent space $T_x\Gamma$ is calibrated by Ξ, and the bubbling connection $I(x)$ is a non-flat ASD instanton. Moreover, when $G \subseteq U(r)$, we prove that the natural $(n-4)-$current $c_2(\Gamma, \Theta)$ defined by the triple $(\Gamma, \Xi, \frac{1}{8\pi^2}\Theta)$ is a *closed integral current* and the following conservation of the instanton charge density holds:

$$c_2(\nabla_i) \rightharpoonup c_2(\nabla) + c_2(\Gamma, \Theta).$$

These results, due to G. Tian [Tia00], show a striking relationship between gauge theory and calibrated geometry: if Ξ is a calibration then the blow-up locus of a sequence of $\Xi-$anti-self-dual instantons with uniformly bounded L^2-energy defines a $\Xi-$calibrated cycle, i.e. a generalized (possibly very singular) $\Xi-$calibrated submanifold. ($\Xi-$ASD instantons $\rightsquigarrow$ $\Xi-$calibrated submanifolds.) This is the climax of this work.

Finally, in Section 4.5, following [Tia00, §4.5 and §5.1], we introduce *stationary* admissible Yang-Mills connections by showing that the blow-up locus of a general sequence of Yang-Mills connections with uniformly bounded L^2-energy defines a minimal cycle if, and only if, the weak limit connection is stationary. (Yang-Mills connections $\rightsquigarrow$ minimal submanifolds.)

☞ **Convention:** *Throughout this chapter, unless otherwise stated,* (M, g) *denotes a connected,* compact*, oriented, Riemannian* $n-$*manifold, with* $n \geq 4$*, and* E *denotes a* $G-$*bundle over* M*, where* G *is a compact Lie group.*

4.1 Decomposition of blow-up loci

Throughout this section we consider a sequence of Yang-Mills connections $\{\nabla_i\}$ on the $G-$bundle E with uniformly bounded L^2-energy, say $\mathcal{YM}(\nabla_i) \leq \Lambda$, for some uniform constant $\Lambda > 0$. By Uhlenbeck-Nakajima Theorem 3.4.4, after passing to a subsequence if necessary, we may assume ∇_i converges weakly to an admissible Yang-Mills connection (∇, S), where $S = S(\{\nabla_i\})$ denotes the blow-up locus of $\{\nabla_i\}$ (cf. Definition 3.4.1). We want to investigate the causes of the formation of the set S.

A key idea to study S is to look at the Radon measures $\{\mu_i := |F_{\nabla_i}|^2 \mathrm{d}V_g\}$ and $\mu_\nabla := |F_\nabla|^2 \mathrm{d}V_g$ (cf. Remark 3.5.4) associated to the sequence $\{\nabla_i\}$ and its weak-limit ∇. In this language, recalling Remark 3.4.3, S is given by

$$S = \bigcap_{0<r\leq\delta_0} \left\{ x \in M : \liminf_{i\to\infty} e^{ar^2} r^{4-n} \mu_i(B_r(x)) \geq \varepsilon_0 \right\}.$$

A basic observation is that, by the uniform L^2-boundedness hypothesis, $\mathcal{YM}(\nabla_i) \leq \Lambda$, after passing to a subsequence, we may assume that μ_i converges weakly* to a Radon measure μ on M (cf. Theorem A.4.4). Then, since $\nabla_i \rightharpoonup (\nabla, S)$, using Fatou's lemma we get

$$\int_M f \mathrm{d}\mu_\nabla = \int_M f|F_\nabla|^2 \mathrm{d}V_g \leq \liminf_{i\to\infty} \int_M f|F_{\nabla_i}|^2 \mathrm{d}V_g = \int_M f \mathrm{d}\mu, \quad \forall f \in C_c(M).$$

Therefore, applying Riesz's representation theorem (see Remark A.4.2), there exists a unique (nonnegative) Radon measure ν on M such that[2]

$$\mu = \mu_\nabla + \nu.$$

ν is called the **defect measure** associated to the weakly convergent sequence $\{\nabla_i\}$.

In what follows we will see that the weak* limit measure μ, and its components μ_∇ and ν, play a fundamental role in the study of S. The next two lemmas summarizes some crucial facts about these objects (compare with [Tia00, (proof of) Lemma 3.1.4, pp. 221-223]).

Lemma 4.1.1.

(i) *μ inherits the monotonicity property of each μ_i: for all $x \in M$,*

$$0 < \sigma < \rho \leq \delta_0 \quad \Rightarrow \quad e^{a\sigma^2}\sigma^{4-n}\mu(B_\sigma(x)) \leq e^{a\rho^2}\rho^{4-n}\mu(B_\rho(x)),$$

where $a \geq 0$ and $\delta_0 > 0$ are, as usual, the constants given in Theorem 3.2.1. In particular, the $(n-4)-$density of μ at x,

$$\Theta(\mu, x) := \Theta^{n-4}(\mu, x) = \lim_{r\downarrow 0} r^{4-n}\mu(B_r(x)),$$

exists and is bounded by $e^{a\delta_0^2}\delta_0^{n-4}\Lambda$ for every $x \in M$. Moreover, $\Theta(\mu, \cdot)$ defines an upper semi-continuous function on M.

(ii) *Given $x \in M$, the following are equivalent:*

[2] More explicitly, ν is the unique Radon measure on M such that

$$\int_M f \mathrm{d}\nu = \int_M f \mathrm{d}\mu - \int_M f \mathrm{d}\mu_\nabla, \quad \forall f \in C_c(M).$$

(ii.1) $x \in S$.

(ii.2) $\Theta(\mu, x) \geq \varepsilon_0$.

(ii.3) $\Theta(\mu, x) > 0$.

Proof. (i): Let $x \in M$ and $0 < \sigma < \rho \leq \delta_0$. Then, for each $i \in \mathbb{N}$, we know from Price's monotonicity (Theorem 3.2.1) that

$$e^{a\sigma^2}\sigma^{4-n}\mu_i(B_\sigma(x)) \leq e^{a\rho^2}\rho^{4-n}\mu_i(B_\rho(x)). \tag{4.1.1}$$

Also, since $\mu_i \rightharpoonup \mu$, we have (cf. Theorem A.4.5 (i)):

$$\mu(B_\sigma(x)) \leq \liminf_{i\to\infty} \mu_i(B_\sigma(x)). \tag{4.1.2}$$

Now let $\mathscr{R}_{x,\delta_0}(\mu) \subseteq \,]0, \delta_0]$ be defined as in Theorem A.4.5 (iv). If $\rho \notin \mathscr{R}_{x,\delta_0}(\mu)$, then (4.1.1) and (4.1.2) immediately imply

$$e^{a\sigma^2}\sigma^{4-n}\mu(B_\sigma(x)) \leq e^{a\rho^2}\rho^{4-n}\mu(B_\rho(x)).$$

The general case follows by an approximation argument. Indeed, since $\mathscr{R}_{x,\delta_0}(\mu)$ is countable, if $\rho \in \mathscr{R}_{x,\delta_0}(\mu)$ then we can find $\{\rho_j\} \subseteq \,]\sigma, \rho]$, with $\rho_j \uparrow \rho$, such that $\rho_j \notin \mathscr{R}_{x,\delta_0}(\mu)$ for all $j \in \mathbb{N}$. Therefore, on the one hand, by the monotone convergence theorem,

$$\mu(B_\rho(x)) = \lim_{j\to\infty} \mu(B_{\rho_j}(x)). \tag{4.1.3}$$

On the other hand, since $\rho_j \notin \mathscr{R}_{x,\delta_0}(\mu)$ for all $j \in \mathbb{N}$,

$$e^{a\sigma^2}\sigma^{4-n}\mu(B_\sigma(x)) \leq e^{a\rho_j^2}\rho_j^{4-n}\mu(B_{\rho_j}(x)), \quad \forall j \in \mathbb{N}.$$

Now make $j \to \infty$ in the above inequality and use (4.1.3). This completes the proof of the monotonicity property.

From the above it is immediate that

$$0 \leq \Theta_*(\mu, x) = \Theta^*(\mu, x) \leq e^{a\delta_0^2}\delta_0^{4-n}\Lambda, \quad \forall x \in M.$$

To see that $\Theta(\mu, \cdot)$ is upper semi-continuous, suppose $\{x_m\}$ is a sequence of points in M with $x_m \to x \in M$ as $m \to \infty$. Let $\varepsilon > 0$ and $0 < r \leq \delta_0$. Then, by the monotonicity, for $m \gg 1$ we have

$$\Theta(\mu, x_m) \leq e^{ar^2}r^{4-n}\mu(B_r(x_m)) \leq e^{ar^2}r^{4-n}\mu(B_{r+\varepsilon}(x)).$$

Hence, $\limsup_{m\to\infty} \Theta(\mu, x_m) \leq e^{ar^2}r^{4-n}\mu(B_r(x))$. Taking the limit as $r \downarrow 0$, we arrive at the desired conclusion. This completes the proof of (i).

(ii): We prove the equivalences (ii.1) $\iff$ (ii.2) and (ii.2) $\iff$ (ii.3).

(ii.1) $\Rightarrow$ (ii.2): We show that

$$e^{ar^2}r^{4-n}\mu(B_r(x)) \geq \varepsilon_0, \quad \forall r \in \,]0,\delta_0]. \tag{4.1.4}$$

If $r \notin \mathscr{R}_{x,\delta_0}(\mu)$ then (cf. Theorem A.4.5 (iv))

$$e^{ar^2}r^{4-n}\mu(B_r(x)) = \lim_{i\to\infty} e^{ar^2}r^{4-n}\mu_i(B_r(x)),$$

so that equation (4.1.4) holds due to $x \in S$. If $r \in \mathscr{R}_{x,\delta_0}(\mu)$, we can proceed by an approximation argument just as in the proof of (i): since $\mathscr{R}_{x,\delta_0}(\mu)$ is countable, we can find $\{r_j\} \subseteq \,]0,r[$, with $r_j \uparrow r$, such that $r_j \notin \mathscr{R}_{x,\delta_0}(\mu)$ for all $j \in \mathbb{N}$. Then, on the one hand, by the monotone convergence theorem,

$$\mu(B_r(x)) = \lim_{j\to\infty} \mu(B_{r_j}(x)).$$

On the other hand, by the choice of the sequence $\{r_j\}$,

$$e^{ar_j^2}r_j^{4-n}\mu(B_{r_j}(x)) \geq \varepsilon_0, \quad \forall j \in \mathbb{N}.$$

Letting $j \to \infty$ this proves (4.1.4) as we wanted.

(ii.2) $\Rightarrow$ (ii.1): This implication follows trivially from (i) and the fact that $\mu(B_r(x)) \leq \liminf_{i\to\infty} \mu_i(B_r(x))$.

(ii.2) $\Rightarrow$ (ii.3): Trivial.

(ii.3) $\Rightarrow$ (ii.2): Suppose, by contradiction, that $\Theta(\mu, x) < \varepsilon_0$. Using (i), this means that

$$e^{ar^2}r^{4-n}\mu(B_r(x)) < \varepsilon_0, \quad \forall r \in \,]0,\delta_0].$$

Now pick $r_0 \notin \mathscr{R}_{x,\delta_0}(\mu)$; then

$$\lim_{i\to\infty} e^{ar_0^2}r_0^{4-n}\mu_i(B_{r_0}(x)) < \varepsilon_0.$$

Thus, there exists $i_0 \in \mathbb{N}$ such that

$$i \geq i_0 \quad \Rightarrow \quad e^{ar_0^2}r_0^{4-n}\mu_i(B_{r_0}(x)) < \varepsilon_0.$$

By the $\varepsilon-$regularity theorem it follows that

$$i \geq i_0 \quad \Rightarrow \quad \sup_{y\in B_{\frac{r_0}{4}}(x)} |F_{\nabla_i}|^2(y) \leq C\varepsilon_0 r_0^{-4}.$$

Therefore, for every $r \in \,]0, r_0/4[$ and $i \geq i_0$ we have

$$e^{ar^2}r^{4-n}\mu_i(B_r(x)) \leq \text{const.} e^{ar^2}r^4.$$

Now using the approximation trick for $\mathscr{R}_{x,\frac{r_0}{8}}(\mu)$, we get

$$e^{ar^2}r^{4-n}\mu(B_r(x)) \leq \text{const.} e^{ar^2}r^4, \quad \forall r \in \,]0, \frac{r_0}{8}].$$

Thus, letting $r \downarrow 0$, we conclude $\Theta(\mu, x) = 0$. Contradiction. This concludes the proof of (ii).

■

Lemma 4.1.2.

(i) $\mu_\nabla(S) = 0$ *and* $\nu(M \setminus S) = 0$ *(thus* μ_∇ *and* ν *are mutually singular measures). In particular,* $\operatorname{spt}(\nu) \subseteq S$.

(ii) $\Theta^*(\mu_\nabla, x) := \Theta^{*n-4}(\mu_\nabla, x) = 0$ *for* $\mathscr{H}^{n-4}-$*a.e.* $x \in S$.

(iii) $\mathscr{H}^{n-4}\lfloor S \ll \nu \ll \mathscr{H}^{n-4}\lfloor S$, *so that*

$$\nu = \theta \mathscr{H}^{n-4}\lfloor S \quad \text{and} \quad \mathscr{H}^{n-4}\lfloor S = \tilde{\theta}\nu,$$

for $\mathscr{H}^{n-4}-$*integrable functions* $\theta, \tilde{\theta} : S \to \mathbb{R}$ *such that, for* $\mathscr{H}^{n-4}-$*a.e.* $x \in S$,

$$\theta(x) = \lim_{r\downarrow 0} \frac{\nu(B_r(x))}{\mathscr{H}^{n-4}(S \cap B_r(x))} \quad \text{and} \quad \tilde{\theta}(x) = \lim_{r\downarrow 0} \frac{\mathscr{H}^{n-4}(S \cap B_r(x))}{\nu(B_r(x))}.$$

In particular, for $\mathscr{H}^{n-4}-$*a.e.* $x \in S$ *the density* $\Theta(S, x) := \Theta^{n-4}(S, x)$ *exists and*

$$\theta(x)\Theta(S, x) = \Theta(\nu, x) = \Theta(\mu, x), \quad \textit{for } \mathscr{H}^{n-4}-\textit{a.e. } x \in S. \tag{4.1.5}$$

Proof. (i): The first assertion is clear from the fact that $\mu_g(S) = 0$ (cf. Theorem 3.4.4 and Remark 3.5.3). We prove that $\nu(M \setminus S) = 0$. Since $M \setminus S$ is an open set, by Theorem A.4.1 it suffices to prove that for every $f \in C_c(X)$ such that $\operatorname{supp}(f) \subseteq M \setminus S$ and $\|f\|_\infty \leq 1$, we have

$$\lim_{i\to\infty} \int_M f|F_{\nabla_i}|^2 \mathrm{d}V_g = \int_M f|F_\nabla|^2 \mathrm{d}V_g.$$

Denote by K the (compact) support of f in $M \setminus S$, and consider, for each $i \in \mathbb{N}$, the functions

$$h_i := f|F_{\nabla_i}|^2 \quad \text{and} \quad g_i := \chi_K |F_{\nabla_i}|^2.$$

It's clear that $|h_i| \leq g_i$ for each $i \in \mathbb{N}$.

By the weak convergence $\nabla_i \rightharpoonup \nabla$, there exists a sequence $\{g_i\} \subseteq \mathcal{G}(E|_{M\setminus S})$ such that $g_i^* \nabla_i \to \nabla$ in C^∞_{loc} on $M \setminus S$. Thus, using the invariance $|F_{\nabla_i}| = |F_{g_i^*\nabla_i}|$, (and from the fact that $\mu_g(S) = 0$,) when $i \to \infty$ we have

$$h_i \to h := f|F_\nabla|^2 \quad \mu_g-\text{a.e. on } M,$$

and

$$g_i \to g := \chi_K |F_\nabla|^2 \quad \text{uniformly on } M.$$

Moreover, from the uniform bound $\mathcal{YM}(\nabla_i) \leq \Lambda$, we automatically have $h_i, g_i, h, g \in L^1(\mu_g)$.

Finally, note that the uniform convergence $g_i \to g$ implies

$$\lim_{i\to\infty}\int_M g_i \mathrm{d}V_g = \int_M g \mathrm{d}V_g,$$

since the g_i are supported in a compact set and μ_g is Radon. Therefore, from a well-known version of the dominated convergence theorem (see [Fol13, p. 59, exercise 20.]), it follows that

$$\lim_{i\to\infty}\int_M h_i \mathrm{d}V_g = \int_M h \mathrm{d}V_g,$$

as we wanted.

The assertion that $\mathrm{spt}(\nu) \subseteq S$ follows from the fact that $M \setminus S$ is an *open* subset with $\nu(M \setminus S) = 0$ (cf. Definition A.1.2).

(ii): Recall from Theorem 3.4.4 that S is a closed subset of M. Thus, $M \setminus S$ is trivially μ_∇−measurable (note that μ_∇ is a Radon measure and, in particular, a Borel measure). Now, since $\mathcal{YM}(\nabla_i) \leq \Lambda$ and $\nabla_i \rightharpoonup \nabla$, it follows from Fatou's lemma that $\mu_\nabla(M \setminus S) = \|F_\nabla\|^2_{L^2(M\setminus S)} \leq \Lambda < \infty$. Finally, noting that (i) implies $\mu_\nabla = \mu_\nabla \lfloor (M \setminus S)$, the desired result follows from Theorem A.3.8 applied to μ_∇ and $M \setminus S$.

(iii): On the one hand, using (ii) it follows that

$$\Theta(\nu, x) = \Theta(\mu, x), \quad \text{for } \mathscr{H}^{n-4}\text{−a.e. } x \in S.$$

On the other hand, from parts (i) and (ii) of Lemma 4.1.1, we know that

$$0 < \varepsilon_0 \leq \Theta(\mu, x) \leq e^{a\delta_0^2}\delta_0^{4-n}\Lambda, \quad \forall x \in M.$$

Therefore

$$0 < \varepsilon_0 \leq \Theta(\nu, x) \leq e^{a\delta_0^2}\delta_0^{4-n}\Lambda, \quad \text{for } \mathscr{H}^{n-4}\text{−a.e. } x \in S. \tag{4.1.6}$$

Now, since $\mathrm{spt}(\nu) \subseteq S$ (by (i)), it follows from Theorem A.3.7 that $\mathscr{H}^{n-4}\lfloor S \ll \nu \ll \mathscr{H}^{n-4}\lfloor S$. Thus, the existence of the functions $\tilde{\theta}, \theta : S \to \mathbb{R}$ follows from Theorem A.4.9. In particular, for $\mathscr{H}^{n-4}$−a.e. $x \in S$ we have

$$\Theta(S, x) = \lim_{r\downarrow 0} r^{4-n}\mathscr{H}^{n-4}(S \cap B_r(x)) = \lim_{r\downarrow 0}\left(r^{4-n}\nu(B_r(x))\right)\left(\frac{\mathscr{H}^{n-4}(S \cap B_r(x))}{\nu(B_r(x))}\right) = \Theta(\nu, x)\tilde{\theta}(x).$$

Noting that $\tilde{\theta}(x) = (\theta(x))^{-1}$ for $\mathscr{H}^{n-4}$−a.e. $x \in S$, the equation (4.1.5) follows. The Lemma is proved. ■

With the above results in mind, we now introduce some terminology.

Definition 4.1.3. Let $\{\nabla_i\}$, ∇, $\{\mu_i\}$ and $\mu = \mu_\nabla + \nu$ be as above. Then:

- $\Gamma := \mathrm{spt}(\nu)$ is called the **bubbling locus** of $\{\nabla_i\}$, and $\Theta(\mu, \cdot)$ is called the **multiplicity** of Γ.

- $\mathrm{sing}(\nabla) := \left\{x \in M : \Theta^{*n-4}(\mu_\nabla, x) > 0\right\}$ is called the **singular set** of ∇.

Proposition 4.1.4 (Decomposition of the blow-up locus)**.** *The blow-up locus S decomposes as*

$$S = \Gamma \cup \mathrm{sing}(\nabla),$$

and $\mathscr{H}^{n-4}(\mathrm{sing}(\nabla)) = 0$.

Proof. ($\supseteq$): By Lemma 4.1.2 (i), we have $\Gamma \subseteq S$. So, it suffices to prove that $\mathrm{sing}(\nabla) \subseteq S$. Now, by Remark 3.5.4, if ∇ is smooth in a neighborhood of x, then $x \notin \mathrm{sing}(\nabla)$. Since S is closed and ∇ is smooth on $M \setminus S$, the desired inclusion follows.

($\subseteq$) Let $x \in S$. Then, by Lemma 4.1.1 (ii), we know that $\Theta(\mu, x) \geq \varepsilon_0 > 0$. Since $\mu = \mu_\nabla + \nu$, we have:

- if $x \notin \Gamma$, then $\Theta(\nu, x) = 0$ (cf. Remark A.3.6) and, therefore, $\Theta^*(\mu_\nabla, x) = \Theta(\mu, x) \geq \varepsilon_0 > 0$; thus $x \in \mathrm{sing}(\nabla)$.

- if $x \notin \mathrm{sing}(\nabla)$, i.e. if $\Theta^*(\mu_\nabla, x) = 0$, then $\Theta(\nu, x) = \Theta(\mu, x) \geq \varepsilon_0 > 0$, so that $x \in \Gamma$ (again by Remark A.3.6).

Finally, the last assertion of the theorem follows immediately from Lemma 4.1.2 (ii), since $\mathrm{sing}(\nabla) \subseteq S$. ■

Observe that, by the gauge invariance of $|F_\nabla|$, the singular set $\mathrm{sing}(\nabla)$ is invariant under gauge transformations, so that it consists of *non-removable* singularities of ∇. On the other hand, since we can write the energy concentration set as $S = \{x \in M : \Theta(\mu, x) \geq \varepsilon_0\}$ (cf. Lemma 4.1.1 (ii)), one should interpret $\Theta(\nu, x)$ as the energy density lost by the sequence $\{\nabla_i\}$ around $x \in S$. Thus, the above result shows that the noncompactness along S has two sources: one involving loss of energy and one involving the formation of non-removable singularities.

Remark 4.1.5. As a corollary of the above proposition, we now establish the relation between the notation employed by Tian [Tia00] and the notation introduced in the present work, as claimed in Remark 3.4.2. Define[3]

$$S_b := \overline{\{x \in S : \Theta(\mu_\nabla, x) = 0\}}.$$

We want to show that

$$\Gamma = S_b.$$

[3] Since $x \in S$ implies $\Theta(\mu, x) > 0$ (Lemma 4.1.1 (ii)), notice the redundancy on Tian's original definition of S_b [Tia00, (3.1.11), p. 223].

Since S is closed and $\operatorname{sing}(\nabla) \subseteq S$, it immediately follows from the definitions of S_b and $\operatorname{sing}(\nabla)$ that

$$S = S_b \cup \operatorname{sing}(\nabla).$$

Moreover, using the characterization of Lemma 4.1.1 (ii) for S and that $\mu = \mu_\nabla + \nu$, we have:

$$S_b = \overline{\{x \in M : 0 < \Theta(\mu, x) = \Theta(\nu, x)\}}.$$

Recalling Remark A.3.6 and the fact that $\Gamma = \operatorname{spt}(\nu)$ is closed, it follows that $S_b \subseteq \Gamma$. Now, by Proposition 4.1.4, we know that $S = \Gamma \cup \operatorname{sing}(\nabla)$. So it remains to show that $\operatorname{sing}(\nabla) \setminus S_b \subseteq \operatorname{sing}(\nabla) \setminus \Gamma$. Let $x \in \operatorname{sing}(\nabla) \setminus S_b$. Then, there exists an open subset U of M such that $x \in U \cap S \subseteq \operatorname{sing}(\nabla)$. Thus

$$\left[\mathscr{H}^{n-4} \lfloor S\right](U) \leq \mathscr{H}^{n-4}(\operatorname{sing}(\nabla)) = 0,$$

where in the last step we used Proposition 4.1.4. Since $\nu \ll \mathscr{H}^{n-4}\lfloor S$ (Lemma 4.1.2 (iii)), it follows that $\nu(U) = 0$. This means that $x \notin \operatorname{spt}(\nu) = \Gamma$, as we wanted. ♢

Remark 4.1.6. When $n = 4$, Proposition 4.1.4 implies that $S = \Gamma$, so that the causes of the noncompactness along S only involves energy loss in this case. This is in accordance with the classical removable singularity theorem of Uhlenbeck [Uhl82b]. ♢

4.2 Rectifiability of bubbling loci

As a first step towards understanding the noncompactness phenomenon involving energy loss, in this short section we show an important regularity result about the set Γ at which this phenomenon occurs.

Theorem 4.2.1 (Rectifiability of the bubbling locus)**.** *The bubbling locus* Γ *is countably* $\mathscr{H}^{n-4}$-rectifiable *(cf. Definition A.5.6) and*

$$\nu = \Theta(\mu, \cdot)\mathscr{H}^{n-4}\lfloor\Gamma.$$

Proof. By Lemma 4.1.1 and Lemma 4.1.2 (ii), we know that (see the proof of Lemma 4.1.2 (iii))

$$0 < \varepsilon_0 \leq \Theta^{n-4}(\nu, x) \leq e^{a\delta_0^2}\delta_0^{4-n}\Lambda < \infty, \quad \text{for } \mathscr{H}^{n-4} - \text{a.e. } x \in S.$$

Since $\Gamma = \operatorname{spt}(\nu)$ and $\nu \ll \mathscr{H}^{n-4}\lfloor S$ (cf. Lemma 4.1.2 (iii)), we get

$$0 < \varepsilon_0 \leq \Theta^{n-4}(\nu, x) \leq e^{a\delta_0^2}\delta_0^{4-n}\Lambda < \infty, \quad \text{for } \nu - \text{a.e. } x \in \Gamma. \tag{4.2.1}$$

By the Nash embedding theorem [Nas56], we can suppose that, for some $N \in \mathbb{N}$ big enough, (M, g) is an embedded Riemannian submanifold of $(\mathbb{R}^N, g_0)$, where g_0 is the standard flat metric. Now extend ν to the whole $\mathbb{R}^N$ in the trivial manner: set $\nu(A) = 0$ for every $A \subseteq \mathbb{R}^N \setminus M$. It is easy to verify ν is still a Radon measure satisfying (4.2.1), so that it satisfies the hypothesis of Theorem A.5.17. Hence, $\nu = \Theta(\nu, \cdot)\mathscr{H}^{n-4}\lfloor\Gamma$ and Γ is a countably $\mathscr{H}^{n-4}-$rectifiable set in $\mathbb{R}^N$. Since (M, g) is *isometrically* embedded in $\mathbb{R}^N$, and $\Gamma \subseteq M$, it follows that Γ is countably $\mathscr{H}^{n-4}-$rectifiable in (M, g) (see Remark A.5.7). From here we can just forget the chosen embedding. Finally, since $\Theta^{n-4}(\mu, x) = \Theta^{n-4}(\nu, x)$ for $\mathscr{H}^{n-4}-$a.e. $x \in \Gamma$, we conclude $\nu = \Theta(\mu, \cdot)\mathscr{H}^{n-4}\lfloor\Gamma$. ∎

Remark 4.2.2. Theorem 4.2.1 corresponds to Proposition 3.3.3 of Tian's paper [Tia00]. Instead of invoking the deep result of D. Preiss, Tian gives an independent proof of the above result, devoting the entire Section 3.3 of his paper for that. There is only one (technical) lemma which is developed in his proof that will be useful for us later [Tia00, Lemma 3.3.2]. This will be stated and proved separately as Lemma 4.3.4. ◊

Remark 4.2.3. This theorem is trivial for $n = 4$: a set is countably $\mathscr{H}^0-$rectifiable if, and only if, it is at most countable, and according to Remark 3.4.7 this is indeed the case. In truth, as expected, the analysis of this chapter has content only when $n > 4$. ◊

Since $\mathscr{H}^{n-4}(S \setminus \Gamma) = 0$ (cf. Proposition 4.1.4), it follows that the blow-up locus itself is countably $\mathscr{H}^{n-4}-$rectifiable, and for $\mathscr{H}^{n-4}-$a.e. $x \in S$ the energy density lost by the sequence around the point x is measured by $\Theta^{n-4}(\mu, x)$.

Using Theorem A.5.16, we get the following consequence of Theorem 4.2.1:

Corollary 4.2.4. *At $\mathscr{H}^{n-4}$-a.e. $x \in \Gamma$, the bubbling locus has a well-defined tangent space $T_x\Gamma \subseteq T_xM$ and ν has a unique tangent measure, i.e. the limit*

$$T_x\nu := \lim_{\lambda \to 0} \lambda^{4-n}(\exp_x \circ \tau_\lambda)^*\nu$$

exists and

$$T_x\nu = \Theta(\mu, x)\mathscr{H}^{n-4}\lfloor T_x\Gamma.$$

Here τ_λ denotes the scaling map on T_xM taking v to λv.

Definition 4.2.5. We will say that $x \in \Gamma$ is a **smooth point** when the following holds:

(i) The tangent space $T_x\Gamma \subseteq T_xM$ is well-defined.

(ii) $x \notin \mathrm{sing}(\nabla)$.

By Theorem 4.2.1 and Lemma 4.1.2 (ii), it follows that the set of points in Γ which are not smooth in the above sense is $\mathscr{H}^{n-4}-$negligible.

4.3 Bubbling analysis

In this section we analyze the structure of ∇_i near smooth points of Γ when $i \gg 1$. We deduce the existence of non-trivial connections bubbling off transversely to Γ.

We start introducing some notations. Recall that we suppose M to be a *compact* manifold. In particular, the Riemannian manifold (M, g) is *complete*: for each $x \in M$, the exponential map $\exp_x$ is well-defined on the whole tangent space T_xM.

Now fix an arbitrary point $x \in \Gamma$. Firstly, simplifying the notation of Definition A.5.10 in Appendix A, if σ is a Radon measure on M then for each $\lambda \in \mathbb{R}_+$ we define the scaled measure $\sigma_{x,\lambda}$ on T_xM by

$$\sigma_{x,\lambda} := \lambda^{4-n}(\exp_x \circ \tau_\lambda)^*\sigma,$$

where $\tau_\lambda : v \mapsto \lambda v$. For each $\lambda \in \mathbb{R}_+$, we define the following scaled objects on T_xM:

$$\begin{aligned}
\tilde{g}_{x,\lambda} &:= (\exp_x \circ \tau_\lambda)^* g, \\
g_{x,\lambda} &:= \lambda^{-2}(\exp_x \circ \tau_\lambda)^* g, \\
\nabla_{i,x,\lambda} &:= (\exp_x \circ \tau_\lambda)^* \nabla_i, \quad \forall i \in \mathbb{N}.
\end{aligned}$$

Below we list a few elementary (but useful) observations:

- $g_{x,\lambda}$ converges to the flat metric $g_{x,0} = g|_{T_xM}$ in $C^\infty_{\text{loc}}-$topology on T_xM as $\lambda \downarrow 0$.
- For $0 < \delta < \text{inj}_g(x)$, $\tilde{g}_{x,\lambda}$ and $g_{x,\lambda}$ are Riemannian metrics on the ball $B_{\lambda^{-1}\delta}(0; g_{x,0}) \subseteq T_xM$, and $\exp_x \circ \tau_\lambda$ maps $(B_{\lambda^{-1}\delta}(0; \tilde{g}_{x,0}), \tilde{g}_{x,\lambda})$ *isometrically* onto $(B_\delta(x; g), g)$. In particular, since $g_{x,\lambda} := \lambda^{-2}\tilde{g}_{x,\lambda}$, we have
$$B_r(0; g_{x,\lambda}) = B_r(0; g_{x,0}) \quad \forall \lambda r < \text{inj}_g(x).$$
- $\nabla_{i,x,\lambda}$ is a Yang-Mills connection w.r.t. $g_{x,\lambda}$ on the pullback bundle $(\exp_x \circ \tau_\lambda)^* E$ over $B_{\lambda^{-1}\delta}(0; g_{x,0})$; in fact:
$$\begin{aligned}
\mathrm{d}_{\nabla_{i,x,\lambda}} *_{g_{x,\lambda}} F_{\nabla_{i,x,\lambda}} &= c(\lambda)\mathrm{d}_{\nabla_{i,x,\lambda}} *_{\tilde{g}_{x,\lambda}} F_{\nabla_{i,x,\lambda}} \\
&= c(\lambda)(\exp_x \circ \tau_\lambda)^*(\mathrm{d}_\nabla *_g F_\nabla) = 0.
\end{aligned}$$
- $\mathrm{d}V_{x,\lambda} := \mathrm{d}V_{g_{x,\lambda}} = \lambda^{-n}(\exp_x \circ \tau_\lambda)^*\mathrm{d}V_g$ on $B_{\lambda^{-1}\delta}(0; g_{x,0})$.
- $\displaystyle\int_{B_r(0;g_{x,\lambda})} |F_{\nabla_{i,x,\lambda}}|^2 \mathrm{d}V_{x,\lambda} = \lambda^{4-n} \int_{B_{\lambda r}} |F_{\nabla_i}|^2_g \mathrm{d}V_g$, for all $\lambda r < \text{inj}_g(x)$.
- $|F_{\nabla_{i,x,\lambda}}|^2_{g_{x,\lambda}} \mathrm{d}V_{x,\lambda} \rightharpoonup \mu_{x,\lambda}$ on $B_{\lambda^{-1}\delta}(0; g_{x,\lambda})$, whenever $0 < \delta < \text{inj}_g(x)$. In the sequel we omit the metric subscript $g_{x,\lambda}$ on the norm of $F_{\nabla_{i,x,\lambda}}$ for clarity.

- $\lambda^{4-n}(\exp_x \circ \tau_\lambda)^*\mu_i = |F_{\nabla_{i,x,\lambda}}|^2 \mathrm{d}V_{x,\lambda}$.
- For each $0 < \delta < \mathrm{inj}_g(x)$ and $z \in B_{\lambda^{-1}\delta}(0; g_{x,0})$, we can take the monotonicity constant $a(n, z, g_{x,\lambda}) = \lambda^2 a(n, \exp_x(\lambda z), g) = \lambda^2 a$.[4]

With the above definitions, we have the following easy consequence of Corollary 4.2.4.

Lemma 4.3.1. *If $x \in \Gamma$ is a smooth point then we can find a null-sequence*[5] $\{\lambda_i\} \subseteq]0, 1[$ *such that*

$$T_x\nu = \lim_{i\to\infty} |F_{\nabla_{i,x,\lambda_i}}|^2 dV_{x,\lambda_i}. \tag{4.3.1}$$

Proof. By Corollary 4.2.4, ν has a unique tangent measure $T_x\nu := \lim_{\lambda\downarrow 0} \lambda^{4-n}(\exp_x \circ \tau_\lambda)^*\nu$. Since $x \notin \mathrm{sing}(\nabla)$, we have

$$\lim_{\lambda\downarrow 0} \lambda^{4-n}(\exp_x \circ \tau_\lambda)^*\nu = \lim_{\lambda\downarrow 0} \lambda^{4-n}(\exp_x \circ \tau_\lambda)^*\mu.$$

Thus, since $\mu_i \rightharpoonup \mu$,

$$T_x\nu = \lim_{\lambda\downarrow 0} \lim_{i\to\infty} \lambda^{4-n}(\exp_x \circ \tau_\lambda)^*\mu_i = \lim_{i\to\infty} \lambda^{4-n}(\exp_x \circ \tau_{\lambda_i})^*\mu_i,$$

for some null-sequence $\{\lambda_i\}$. This shows the claim since

$$\lambda^{4-n}(\exp_x \circ \tau_{\lambda_i})^*\mu_i = |F_{\nabla_{i,x,\lambda_i}}|^2 \mathrm{d}V_{x,\lambda_i}.$$

∎

Now we prove two preliminary lemmas. The following is a consequence of the upper semi-continuity of $\Theta(\mu, \cdot)$ and the finiteness of $\mathscr{H}^{n-4}(S)$ (cf. [Tia00, Lemma 3.3.2, p. 225]).

Lemma 4.3.2. *The density function $\Theta(\mu, \cdot)|_\Gamma$ is $\mathscr{H}^{n-4}-$approximately continuous at $\mathscr{H}^{n-4}-$a.e. $x \in \Gamma$, i.e. at $\mathscr{H}^{n-4}-$a.e. $x \in \Gamma$, for any $\varepsilon > 0$ we have*

$$\lim_{r\downarrow 0} r^{4-n}\mathscr{H}^{n-4}\left(\{y \in B_r(x) \cap \Gamma : |\Theta(\mu, y) - \Theta(\mu, x)| > \varepsilon\}\right) = 0. \tag{4.3.2}$$

Proof. Let $\varepsilon > 0$. For each $k \in \mathbb{N}$, define

$$E_k := \left\{x \in \Gamma : \frac{(k-1)\varepsilon}{2} \leq \Theta(\mu, x) < \frac{k\varepsilon}{2}\right\}.$$

Since $0 \leq \Theta(\mu, x) \leq e^{a\delta_0^2}\delta_0^{4-n}\Lambda < \infty$ for all $x \in M$ (cf. Lemma 4.1.1 (i)), we can write $\Gamma = \bigcup_k E_k$. Then, by $\sigma-$additivity of $\mathscr{H}^{n-4}$, it suffices to check (4.3.2) for $\mathscr{H}^{n-4}-$a.e. $x \in E_k$, for each $k \in \mathbb{N}$.

Let $k \in \mathbb{N}$. For every $x, y \in E_k$, it is clear that $|\Theta(\mu, y) - \Theta(\mu, x)| < \varepsilon/2$, so that

$$\{y \in B_r(x) \cap \Gamma : |\Theta(\mu, y) - \Theta(\mu, x)| > \varepsilon\} \subseteq B_r(x) \cap (\Gamma \setminus E_k), \quad \forall x \in E_k. \tag{4.3.3}$$

Moreover, since $\Theta(\mu, \cdot)$ is upper semi-continuous[6] (cf. Lemma 4.1.1 (i)) and $\Gamma = \mathrm{spt}(\nu)$

[4] In fact, this is the key reason we will make use of Theorem 3.2.1 instead of Price's original monotonicity formula [Pri83, Theorem 1'].

[5] A sequence $\{\alpha_i\} \subseteq \mathbb{R}_+$ of positive real numbers is said to be a *null-sequence* if $\alpha_i \downarrow 0$ as $i \to \infty$.

[6] This means $\Theta(\mu, \cdot)^{-1}\left(]-\infty, \alpha[\right)$ is open for every $\alpha \in \mathbb{R}$.

is a closed set, it follows that each E_k is a Borel set[7] (so does $\Gamma \setminus E_k$). Recalling that $\mathscr{H}^{n-4}(\Gamma) \leq \mathscr{H}^{n-4}(S) < \infty$ (cf. Theorem 3.4.4 (i)), it follows from Theorem A.3.8 that

$$\lim_{r\downarrow 0} r^{4-n}\mathscr{H}^{n-4}\left(B_r(x)\cap(\Gamma\setminus E_k)\right) = 0, \quad \text{for } \mathscr{H}^{n-4}\text{−a.e. } x \in E_k. \tag{4.3.4}$$

Combining (4.3.3) and (4.3.4) we get the desired conclusion. The result follows. ∎

The next result, corresponding to [Tia00, Lemma 3.2.3, p. 225], is essentially due to Lin [Lin99, Lemma 2.4, p. 804].

Lemma 4.3.3. *Let $x \in \Gamma$ be such that $x \notin \operatorname{sing}(\nabla)$ and $\Theta(\mu,\cdot)|_\Gamma$ is $\mathscr{H}^{n-4}$−approximately continuous at x. Then, if $0 < r \leq \delta_0$ is sufficiently small (depending only on x and n), we may find $n-4$ points $x_1,\ldots,x_{n-4} \in B_r(x)\cap\Gamma$ satisfying the following two conditions:*

(i) $\Theta(\mu,x_j) \geq \Theta(\mu,x) - \varepsilon(r)$ for $j = 1,\ldots,n-4$, where $\varepsilon(r)\downarrow 0$ as $r \downarrow 0$;

(ii) There exists $s = s(n) \in \,]0,\frac{1}{2}[$, depending only on n, such that $d(x_1,x) \geq sr$ and $d(x_k,\exp_x(V_{k-1})) \geq sr$ for $2 \leq k \leq n-4$, where

$$V_l := \operatorname{span}_{\mathbb{R}}\left\{(\exp_x|_{B_r(0;g_{x,0})})^{-1}(x_1),\ldots,(\exp_x|_{B_r(0;g_{x,0})})^{-1}(x_l)\right\}, \quad l = 1,\ldots,n-4.$$

Proof. We start noting that, since $\Theta(\mu,\cdot)|_\Gamma$ is $\mathscr{H}^{n-4}$−approximately continuous at x, for all sufficiently small $0 < r \ll 1$ (depending only on x and n) we can find $\varepsilon(r) > 0$ such that

$$r^{4-n}\mathscr{H}^{n-4}\left(\{y \in B_r(x)\cap\Gamma : |\Theta(\mu,y)-\Theta(\mu,x)| \geq \varepsilon(r)\}\right) \leq \frac{s(n)}{2} < \frac{1}{2},$$

where $s(n) > 0$ is a positive number to be determined later, and $\varepsilon(r)\downarrow 0$ as $r \downarrow 0$.

Now we argue by contradiction. Suppose the lemma is false. Then, in particular, there would be a null-sequence $\{r_k\} \subseteq \,]0,1[$, of sufficiently small positive numbers $0 < r_k \ll 1$, such that for each $k \in \mathbb{N}$ one cannot find $n-4$ points inside the set

$$\{y \in B_{r_k}(x)\cap\Gamma : |\Theta(\mu,y)-\Theta(\mu,x)| < \varepsilon(r_k)\} \tag{4.3.5}$$

satisfying condition (ii) of the Lemma. Therefore, for each $k \in \mathbb{N}$, the set (4.3.5) is contained in the sr−neighborhood of $\exp_x(L_k)$ for some $(n-5)$−dimensional subspace $L_k \leq T_xM$. Thus, for each $k \in \mathbb{N}$, given $y \in B_{r_k}(x)\cap\Gamma$ one has either $|\Theta(\mu,y)-\Theta(\mu,x)| \geq \varepsilon(r)$ or $y \in N_{sr_k}(\exp_x|_{B_{r_k}(0;g_{x,0})}(L_k))$.

Now we wish to estimate $\mu(B_{r_k}(x)\cap\Gamma)$ for $k \in \mathbb{N}$ sufficiently large (therefore r_k small enough). By the upper semi-continuity of $\Theta(\mu,\cdot)$, for r_k small enough we have

$$\Theta(\mu,y) \leq 2\Theta(\mu,x), \quad \forall y \in B_{r_k}(x).$$

[7]Note that we can write E_k as $\Gamma\cap\Theta(\mu,\cdot)^{-1}(]-\infty,\frac{k\varepsilon}{2}[)\setminus\Theta(\mu,\cdot)^{-1}(]-\infty,\frac{(k-1)\varepsilon}{2}[)$.

In particular, since $\mu\lfloor\Gamma = \nu = \Theta(\mu,\cdot)\mathscr{H}^{n-4}\lfloor\Gamma$, we get

$$\begin{aligned} &\mu \quad (\{y \in B_{r_k}(x)\cap\Gamma : |\Theta(\mu,y)-\Theta(\mu,x)| \geq \varepsilon(r_k)\}) \\ \leq \;& 2\Theta(\mu,x)\mathscr{H}^{n-4}\left(\{y \in B_{r_k}(x)\cap\Gamma : |\Theta(\mu,y)-\Theta(\mu,x)| \geq \varepsilon(r_k)\}\right) \\ \leq \;& \Theta(\mu,x)s(n)r_k^{n-4}. \end{aligned} \tag{4.3.6}$$

Next, since L_k is an $(n-5)-$dimensional subspace of T_xM, we may cover $N_{sr_k}(\exp_x|_{B_{r_k}(0;g_{x},0)}(L_k))$ by $s^{5-n}C(n)$ geodesic balls of radius sr_k, where $C(n)$ is a uniform constant independent of $s(n)$. Let $\{B_j^k\}_{j=1}^{N_k}$ be such a cover with $N_k \leq s^{5-n}C(n)$ and $B_j^k = B_{sr_k}(y_j^k)$, with $y_j^k \in B_{r_k}(x)$ $(j = 1,\ldots,N_k)$. Then

$$\mu\left(N_{sr_k}(\exp_x|_{B_{r_k}(0;g_x,0)}(L_k))\right) \leq \sum_{j=1}^{N_k}\mu(B_j^k).$$

To estimate this last term we note that, by the monotonicity of μ, there exists $0 < \delta_x \leq \delta_0$ such that

$$0 < r \leq \delta_x \quad \Rightarrow \quad r^{4-n}\mu(B_r(x)) \leq \frac{3}{2}\Theta(\mu,x). \tag{4.3.7}$$

If $r_k \ll \delta_x$ then

$$\begin{aligned} \mu(B_j^k) \;&=\; e^{-a(sr_k)^2}(sr_k)^{n-4}e^{a(sr_k)^2}(sr_k)^{4-n}\mu(B_{sr_k}(y_j^k)) \\ &\leq\; (sr_k)^{n-4}e^{a(\delta_x-r_k)^2}(\delta_x-r_k)^{4-n}\mu(B_{\delta_x-r_k}(y_j^k)) \quad \text{(by monotonicity and } e^{-a(sr_k)^2}\leq 1) \\ &\leq\; (sr_k)^{n-4}e^{a(\delta_x-r_k)^2}(\delta_x-r_k)^{4-n}e^{-a\delta_x^2}\delta_x^{n-4}e^{a\delta_x^2}\delta_x^{4-n}\mu(B_{\delta_x}(x)) \quad \text{(note that } y_j^k \in B_{r_k}(x)) \\ &\leq\; (sr_k)^{n-4}e^{a(\delta_x-r_k)^2}(\delta_x-r_k)^{4-n}\delta_x^{n-4}\frac{3}{2}\theta(\mu,x) \quad \text{(using (4.3.7) and } e^{-a\delta_x^2}\leq 1) \\ &\leq\; 2(sr_k)^{n-4}\Theta(\mu,x). \quad (r_k \ll \delta_x) \end{aligned}$$

Therefore,

$$\mu\left(N_{sr_k}(\exp_x|_{B_{r_k}(0;g_x,0)}(L_k))\right) \leq s^{5-n}C(n)2(sr_k)^{n-4}\Theta(\mu,x) = 2s(n)C(n)r_k^{n-4}\Theta(\mu,x). \tag{4.3.8}$$

Combining (4.3.6) and (4.3.8), we conclude that for all k sufficiently large

$$\mu(B_{r_k}(x)\cap\Gamma) \leq s(n)(2C(n)+1)r_k^{n-4}\Theta(\mu,x) < \frac{1}{2}r_k^{n-4}\Theta(\mu,x),$$

provided we choose $s(n) < 2^{-1}(2C(n)+1)^{-1}$; also note that $0 \neq \Theta(\mu,x) \geq \varepsilon_0 > 0$, since $x \in \Gamma$. However:

$$\lim_{k\to\infty} r_k^{4-n}\mu(B_{r_k}(x)\cap\Gamma) = \Theta(\nu,x) = \Theta(\mu,x), \tag{4.3.9}$$

since $r_k \to 0$ as $k \to \infty$ and $x \notin \operatorname{sing}(\nabla)$. Contradiction. The lemma is proved. ■

For the next two lemmas, we fix the following notations. Let $x \in \Gamma$ be a smooth point (cf. Definition 4.2.5) such that $\Theta(\mu,\cdot)|_\Gamma$ is $\mathscr{H}^{n-4}$approximately continuous at x (cf.

Lemma 4.3.2), and let $\{\lambda_i\} \subseteq]0,1[$ be a null-sequence such as in Lemma 4.3.1. Moreover, we put $V := T_x\Gamma$ and write $T_xM = V \times V^\perp$. Thus, each $z \in T_xM$ is identified with a pair (z', z''), where $z' \in V$ and $z'' \in V^\perp$. Also, we choose orthonormal coordinates $(z^1, \ldots, z^n)$ on $(T_xM, g_{x,0})$ such that $(z^1, \ldots, z^{n-4})$ are coordinates for V (and, therefore, $(z^{n-3}, \ldots, z^n)$ are coordinates for $V^\perp$).

Lemma 4.3.4 ([Tia00, Lemma 3.3.2, p. 231]). *In the above situation, for any $\alpha \leq n-4$ we have*

$$\lim_{i\to\infty} \int_{B_4(0;g_{x,\lambda_i})} \left| \frac{\partial}{\partial z^\alpha} \lrcorner F_{\nabla_{i,x,\lambda_i}} \right|^2 \mathrm{d}V_{g_{x,\lambda_i}} = 0.$$

Proof. We start noting that (4.3.1) implies, for all $\delta > 0$,

$$\lim_{i\to\infty} \int_{B_4(0;g_{x,\lambda_i})\setminus T_\delta(V)} \left| F_{\nabla_{i,x,\lambda_i}} \right|^2 \mathrm{d}V_{x,\lambda_i} = 0, \tag{4.3.10}$$

where $T_\delta(V) := \{y \in T_xM : d_{g_{x,0}}(y, V) \leq \delta\}$ denotes the $\delta-$tubular neighborhood of V in $(T_xM, g_{x,0})$.

For each i sufficiently large, by Lemma 4.3.3, we may find $n-4$ points $x_1^i, \ldots, x_{n-4}^i \in B_{\lambda_i}(x) \cap \Gamma$ such that for $j = 1, \ldots, n-4$ we have

(1) $\Theta(\mu, x_j^i) \geq \Theta(\mu, x) - \varepsilon(\lambda_i)$, where $\varepsilon(r) \downarrow 0$ as $r \downarrow 0$;

(2) $d(x_j^i, \exp_x(V_{j-1}^i)) \geq s\lambda_i$, where $V_0^i := \{0\}$ and for $l = 1, \ldots, n-4$

$$V_l^i := \mathrm{span}_{\mathbb{R}} \left\{ \tilde{\xi}_1^i := (\exp_x|_{B_{\lambda_i}(0;g_{x,0})})^{-1}(x_1^i), \ldots, \tilde{\xi}_l^i := (\exp_x|_{B_{\lambda_i}(0;g_{x,0})})^{-1}(x_l^i) \right\}.$$

It is clear that, for all $i \gg 1$ and each $j \in \{1, \ldots, n-4\}$, we have $\xi_j^i := \lambda_i^{-1}\tilde{\xi}_j^i \in B_1(0; g_{x,0})$. Thus, after passing to a subsequence, $\lambda_i^{-1}\xi_j^i$ converges to some $\xi_j \in \overline{B}_1(0; g_{x,0})$ with respect to $g_{x,0}$, for each $j \in \{1, \ldots, n-4\}$. It follows from (2) that (i) $V = \mathrm{span}_{\mathbb{R}}\{\xi_1, \ldots, \xi_{n-4}\}$, (ii) $d_{g_{x,0}}(\xi_j, 0) \geq s$ for each j, and (iii) $d_{g_{x,0}}(\xi_j, \xi_k) \geq s$ for each $j \neq k$.

For $r > 0$, using the definition of ξ_j^i, the monotonicity of μ and (1), we deduce that for each $i \gg 1$

$$\begin{aligned} e^{a\lambda_i^2 r^2} r^{4-n} \mu_{x,\lambda_i}(B_r(\xi_j^i; g_{x,\lambda_i})) &= e^{a(\lambda_i r)^2} (\lambda_i r)^{4-n} \mu(B_{\lambda_i r}(x_j^i)) \\ &\geq \Theta(\mu, x_j^i) \geq \Theta(\mu, x) - \varepsilon(\lambda_i). \end{aligned}$$

Since, for each j, we also have $x_j^i \to x$ and $e^{a(\lambda_i r)^2}(\lambda_i r)^{4-n}\mu(B_{\lambda_i r}(x_j^i)) \downarrow \Theta(\mu, x)$ as $i \to \infty$, by increasing $\varepsilon(\lambda_i)$ if necessary, we may assume that for $i \gg 1$

$$\left| e^{a\lambda_i^2 r^2} r^{4-n} \mu_{x,\lambda_i}(B_r(\xi_j^i; g_{x,\lambda_i})) - \Theta(\mu, x) \right| \leq \varepsilon(\lambda_i).$$

Now since $|F_{\nabla_{i,x,\lambda}}|^2 \mathrm{d}V_g \rightharpoonup \mu_{x,\lambda}$, for $i \gg 1$ we have

$$\left| e^{a\lambda_i^2 r^2} r^{4-n} \int_{B_r(\xi_j^i; g_{x,\lambda_i})} |F_{\nabla_{i,x,\lambda_i}}|^2 \mathrm{d}V_{x,\lambda_i} - \Theta(\mu, x) \right| \leq 2\varepsilon(\lambda_i), \quad \forall r > 0.$$

Thus, using the monotonicity formula (Theorem 3.2.1) for ∇_{i,x,λ_i} and g_{x,λ_k} we get, for all $r > s$,

$$\int_{B_r(\xi_j^i;g_{x,\lambda_i})\setminus B_s(\xi_j^i;g_{x,\lambda_i})} e^{a\lambda_i^2(\rho_j^i)^2}(\rho_j^i)^{4-n}\left|\frac{\partial}{\partial\rho_j^i}\lrcorner F_{\nabla_{i,x,\lambda_i}}\right|^2 \mathrm{d}V_{x,\lambda_i} \leq \varepsilon(\lambda_i), \tag{4.3.11}$$

where ρ_j^i denotes the distance from ξ_j^i with respect to g_{x,λ_i}. Using (i), the result follows from (4.3.11) and (4.3.10). ∎

Lemma 4.3.5 ([Tia00, Lemma 4.1.2, p. 235])**.** *There are points $z_i' \in V \cap B_{\frac{1}{2}}(0;g_{x,0})$ with $\lim_{i\to\infty} z_i' = 0$ such that*

$$\lim_{i\to\infty}\left(\sup_{0<r\leq\frac{1}{2}} r^{4-n}\int_{V\cap B_r(z_i';g_{x,0})\times V^\perp\cap B_{\frac{1}{2}}(0;g_{x,0})}\left(\sum_{\alpha=1}^{n-4}\left|\frac{\partial}{\partial z_\alpha}\lrcorner F_{\nabla_{i,x,\lambda_i}}\right|^2\right)\mathrm{d}V_{x,\lambda_i}\right) = 0. \tag{4.3.12}$$

Proof. By contradiction, suppose that the lemma is false. Then we can find $\delta > 0$ and $s \in \,]0,\frac{1}{2}[$ such that for each i sufficiently large and $z' \in V \cap B_s(0;g_{x,0})$, there exists $r = r(i,z') \in \,]0,\frac{1}{2}]$ such that

$$r^{4-n}\int_{V\cap B_r(z';g_{x,0})\times V^\perp\cap B_{\frac{1}{2}}(0;g_{x,0})}\left(\sum_{\alpha=1}^{n-4}\left|\frac{\partial}{\partial z_\alpha}\lrcorner F_{\nabla_{i,x,\lambda_i}}\right|^2\right)\mathrm{d}V_{x,\lambda_i} \geq \delta. \tag{4.3.13}$$

Since $V \cap B_r(z';g_{x,0}) \times V^\perp \cap B_{\frac{1}{2}}(0;g_{x,0}) \subseteq B_2(0;g_{x,0})$, it follows from Lemma 4.3.4 that $r(i,z') \to 0$ as $i \to \infty$ for each fixed z'. Now, for each $i \gg 1$ we can find a finite cover $\mathcal{C}_i = \{V \cap B_{2r(i,z'_{i,\alpha})}(z'_{i,\alpha};g_{x,0}) : \alpha = 1,\ldots,m_i\}$ of $V \cap B_{\frac{1}{2}}(0;g_{x,0})$ where (i) $z'_{i,\alpha} \in V \cap B_s(0;g_{x,0})$ for all α, and (ii) $V \cap B_{r(i,z'_{i,\alpha})}(z'_{i,\alpha};g_{x,0}) \cap B_{r(i,z'_{i,\beta})}(z'_{i,\beta};g_{x,0}) = \emptyset$ for all $\alpha \neq \beta$. Thus

$$\begin{aligned}
0 < \delta\left(\frac{s}{2}\right)^{n-4} &\leq \delta 2^{4-n}\sum_{\alpha=1}^{m_i}(2r(i,z'_{i,\alpha}))^{n-4} \quad (\text{since } \mathcal{C}_i \text{ is a cover of } V\cap B_{\frac{1}{2}}(0;g_{x,0}))\\
&= \delta\sum_{\alpha=1}^{m_i} r(i,z'_{i,\alpha})^{n-4}\\
&\overset{(4.3.13)}{\leq} \sum_{\alpha=1}^{m_i}\int_{V\cap B_{r(i,z'_{i,\alpha})}(z'_{i,\alpha};g_{x,0})\times V^\perp\cap B_{\frac{1}{2}}(0;g_{x,0})}\left(\sum_{\beta=1}^{n-4}\left|\frac{\partial}{\partial z_\beta}\lrcorner F_{\nabla_{i,x,\lambda_i}}\right|^2\right)\mathrm{d}V_{x,\lambda_i}\\
&\leq \int_{B_2(0;g_{x,0})}\left(\sum_{\beta=1}^{n-4}\left|\frac{\partial}{\partial z_\beta}\lrcorner F_{\nabla_{i,x,\lambda_i}}\right|^2\right)\mathrm{d}V_{x,\lambda_i},
\end{aligned} \tag{4.3.14}$$

where for the last inequality we use properties (i) and (ii) of the cover $\mathcal{C}_i$. However, by Lemma 4.3.4, the last term of (4.3.14) goes to zero as $i \to \infty$, arriving to a contradiction. ∎

Now we can state and prove the main theorem of this section (cf. [Tia00, Proposition 4.1.1, p. 235]).

Theorem 4.3.6 (Tian)**.** *Let $x \in \Gamma$ be a smooth point of the bubbling locus such that $\Theta(\mu, \cdot)|_\Gamma$ is $\mathscr{H}^{n-4}$approximately continuous at x. Then there exist linear automorphisms $\sigma_i : T_xM \to T_xM$such that a subsequence of $\sigma_i^* \exp_x^* \nabla_i$ converges to a Yang-Mills connection $B(x)$ on $(T_xM, g_{x,0})$ which is, modulo gauge transformations, the pull-back of a non-flat connection $I(x)$ on $T_x\Gamma^\perp$ by the natural orthogonal projection $\pi : T_x\Gamma \to T_x\Gamma^\perp$. Moreover,*

$$\mathcal{YM}(I(x)) \leq \Theta(\mu, x).$$

We call the connection $B(x)$ a **bubbling connection** *at $x \in \Gamma$.*

Proof. If $V \coloneqq T_x\Gamma$, recall that we can find a null-sequence $\{\lambda_i\} \subseteq]0, \frac{1}{2}[$ such that

$$|F_{\nabla_{i,x,\lambda_i}}|^2 \mathrm{d}V_{x,\lambda_i} \rightharpoonup \Theta(\mu, x)\mathscr{H}^{n-4} \lfloor V.$$

Moreover, since $\nabla_i \rightharpoonup (\nabla, S)$, it follows that (modulo gauge transformations) ∇_{i,x,λ_i} converges uniformly to zero on compact subsets in $T_xM \setminus V$.

Write $T_xM = V \oplus V^\perp$, with respect to the flat metric $g_{x,0}$, and let $z^1, \dots, z^n$ be orthonormal coordinates on $(T_xM, g_{x,0})$ such that $z^1, \dots, z^{n-4}$ are coordinates for V. Then, by Lemma 4.3.4,

$$\lim_{i\to\infty} \left(\sum_{\alpha=1}^{n-4} \int_{B_2(0;g_{x,\lambda_i})} \left| \frac{\partial}{\partial z^\alpha} \lrcorner F_{\nabla_{i,x,\lambda_i}} \right|^2 \mathrm{d}V_{g_{x,\lambda_i}} \right) = 0. \tag{4.3.15}$$

Moreover, by Lemma 4.3.5, there are points $z_i' \in V \cap B_{\frac{1}{2}}(0; g_{x,0})$ with $\lim_{i\to\infty} z_i' = 0$ satisfying (4.3.12). Observe that, if $\delta > 0$ is a fixed number, then for all i large

$$\max_{z'' \in V^\perp \cap B_{\frac{1}{2}}(0;g_{x,0})} \delta^{4-n} \int_{B_\delta(z_i'+z'';g_{x,0})} |F_{\nabla_{i,x,\lambda_i}}|^2 \mathrm{d}V_{x,\lambda_i} \geq \varepsilon_0,$$

while for fixed $i \in \mathbb{N}$ and $z'' \in V^\perp \cap B_{\frac{1}{2}}(0; g_{x,0})$ we have

$$\lim_{\delta \downarrow 0} \delta^{4-n} \int_{B_\delta(z_i'+z'';g_{x,0})} |F_{\nabla_{i,x,\lambda_i}}|^2 \mathrm{d}V_{x,\lambda_i} = 0.$$

Hence, we can find a null-sequence $\{\delta_i\} \subseteq]0, \frac{1}{2}[$ and a sequence of points $\{z_i''\} \subseteq V^\perp \cap B_{\frac{1}{4}}(0; g_{x,0})$, with $z_i'' \to 0$ as $i \to \infty$, such that for each $i \in \mathbb{N}$, the maximum

$$\max_{z'' \in V^\perp \cap B_{\delta_i}(0;g_{x,0})} \delta_i^{4-n} \int_{B_{\delta_i}(z_i'+z'';g_{x,0})} |F_{\nabla_{i,x,\lambda_i}}|^2 \mathrm{d}V_{x,\lambda_i} = \frac{\varepsilon_0}{4}$$

is achieved at z_i'':

$$\delta_i^{4-n} \int_{B_{\delta_i}(z_i'+z_i'';g_{x,0})} |F_{\nabla_{i,x,\lambda_i}}|^2 \mathrm{d}V_{x,\lambda_i} = \max_{z'' \in V^\perp \cap B_{\delta_i}(0;g_{x,0})} \delta_i^{4-n} \int_{B_{\delta_i}(z_i'+z'';g_{x,0})} |F_{\nabla_{i,x,\lambda_i}}|^2 \mathrm{d}V_{x,\lambda_i} = \frac{\varepsilon_0}{4}. \tag{4.3.16}$$

Define a sequence of connections $\{B_i(x)\}$ given by

$$B_i(x) \coloneqq (\exp_x \circ \sigma_i)^* \nabla_i,$$

where $\sigma_i : T_xM \to T_xM$ is the linear automorphism given by

$$\sigma_i(u) = \tau_{\lambda_i}(z_i' + z_i'' + \delta_i u) = \lambda_i(z_i' + z_i'' + \delta_i u), \quad \forall u \in T_xM.$$

Note that each $B_i(x)$ is a Yang-Mills connection with respect to the scaled metric $g_{x,\lambda_i\delta_i} = \delta_i^{-2}\tau_{\delta_i}^* g_{x,\lambda_i}$ on $B_{4R_i}(0; g_{x,0})$, where $R_i := (4\delta_i)^{-1}$. Moreover, the based Riemannian manifolds $(B_{4R_i}(0; g_{x,0}), g_{x,\lambda_i\delta_i}, z_i' + z_i'')$ converges to $(T_xM, g_{x,0}, 0)$ as $i \to \infty$.

From (4.3.12) and (4.3.16) we get, respectively,

$$\lim_{i\to\infty}\left(\sum_{\alpha=1}^{n-4}\int_{B_{R_i}(0;g_{x,0})}\left|\frac{\partial}{\partial z^\alpha} \lrcorner F_{B_i(x)}\right|^2 \mathrm{d}V_{g_{x,\lambda_i\delta_i}}\right) = 0 \tag{4.3.17}$$

and

$$\int_{B_1(0;g_{x,0})}|F_{B_i(x)}|^2\mathrm{d}V_{x,\lambda_i\delta_i} = \max_{z''\in V^\perp\cap B_{R_i^{-1}}(0;g_{x,0})}\int_{B_1(z'';g_{x,0})}|F_{B_i(x)}|^2\mathrm{d}V_{x,\lambda_i\delta_i} = \frac{\varepsilon_0}{4}. \tag{4.3.18}$$

It follows from the monotonicity formula that

$$\sup_i \int_{B_R(0;g_{x,0})}|F_{B_i(x)}|^2\mathrm{d}V_{x,\lambda_i\delta_i} \le C(\Lambda)R^{n-4},$$

for $0 < R < R_i$, where $C(\Lambda) > 0$ is a constant depending only on Λ. Thus, by taking a subsequence if necessary, $B_i(x)$ converges weakly to an admissible Yang-Mills connection $B(x)$ on $(T_xM, g_{x,0})$ and, by (4.3.18), $B(x)$ is smooth on the band $[V \cap B_1(0; g_{x,0})] \times V^\perp$. Moreover, (4.3.17) implies that

$$v \lrcorner F_{B(x)} = 0 \quad \forall v \in V, \text{ whenever } B(x) \text{ is well-defined.} \tag{4.3.19}$$

On $[V \cap B_1(0; g_{x,0})] \times V^\perp$ we may write

$$B(x) = \sum_{\alpha=1}^{n} B_\alpha dz^\alpha, \quad \text{where } B_\alpha \in \mathfrak{g}.$$

We now show that $B(x)$ extends to a smooth connection on all of T_xM by showing that it is, modulo gauge transformations, the pull-back of some non-flat connection $I(x)$ on $V^\perp$. We eliminate B_α for $\alpha \le n-4$ inductively. First, by taking a gauge transformation, we may assume that $B_1 = 0$. Then, using (4.3.19), we deduce that all B_α are independent of z^1. Again, taking a gauge transformation if necessary, we can get rid of B_2, and so on. Eventually, by finitely many gauge transformations, we arrive at a connection, still denoted by $B(x)$, which is the pull-back of some connection $I(x)$ on $V^\perp$.

Finally, since (4.3.18) holds, both $B(x)$ and $I(x)$ are necessarily non-flat connections and, furthermore, it follows that $\mathcal{YM}(I(x)) \le \Theta(\mu, x)$. We are done. ■

Taking into account the results that have been proven so far, this completes the proof of Theorem A stated in the introduction of this work.

4.4 Blow-up loci of instantons and calibrated geometry

Throughout this section we assume further that M is endowed with a closed $(n-4)-$form Ξ and that $\{\nabla_i\}$ is a sequence of $\Xi-$ASD instantons on the $G-$bundle E. In this setting, we show important extensions of the results derived so far for general Yang-Mills connections.

We begin with the following simple

Lemma 4.4.1. *In the above setting, the weak limit connection ∇ is in fact an admissible $\Xi-$ASD instanton.*

Proof. By assumption, there exists a sequence of gauge-transformations $\{g_i\} \subseteq \mathcal{G}(E|_{M\setminus S})$, $S = S(\{\nabla_i\})$, such that $g_i^*\nabla_i$ converges to ∇ in $C^\infty_{\mathrm{loc}}-$topology outside S. In particular, recalling Remark 3.5.10, it follows that $\mathrm{tr}(F_{\nabla_i})$ converges to $\mathrm{tr}(F_\nabla)$ in $C^\infty-$topology on M, and $\mathrm{tr}(F_\nabla)$ is harmonic. Since $*_\Xi = *(\Xi\wedge\cdot)$ is clearly a continuous operator with respect to the $C^\infty-$topology, and each ∇_i is $\Xi-$ASD, the result follows from the equivariance of $*_\Xi$. ∎

Next, we note that there are more information on the bubbling connections prescribed in Proposition 4.3.6 (cf. [Tia00, Theorem 4.2.1]).

Proposition 4.4.2 (Bubbling $\Xi-$ASD connections)**.** *Let $B(x) = \pi^*I(x)$ be a bubbling connection at a smooth point $x \in \Gamma$ such that $\Theta(\mu,\cdot)|_\Gamma$ is $\mathscr{H}^{n-4}$approximately continuous at x (cf. Proposition 4.3.6). Then $B(x)$ is a (non-flat) Ξ_x-ASD instanton on $(T_xM, g_{x,0})$ with* $\mathrm{tr}(F_{B(x)}) = 0$.

Proof. Recall that $B_i(x) := \sigma_i^*\exp_x^*\nabla_i$, where $\sigma_i : T_xM \to T_xM$ is of the form $v \mapsto \lambda_i(z_i', z_i'') + \lambda_i\delta_i v$, with $(z_i', z_i'') \to 0$ and $\lambda_i, \delta_i \to 0$ as $i\to\infty$. Since $\mathrm{tr}(F_{\nabla_i}) \to \mathrm{tr}(F_\nabla)$ uniformly as $i\to\infty$, it follows that $\mathrm{tr}(F_{B_i(x)}) \to 0$ uniformly as $i\to\infty$.

On the other hand, note that $B_i(x)$ is $(\lambda_i\delta_i)^{4-n}\sigma_i^*\exp_x^*\Xi-$ASD with respect to the metric $g_{x,\lambda_i\delta_i}$. Moreover, since $\lambda_i(z_i', z_i'')$ converges to zero, we have that $\sigma_i^*\exp_x^*\Xi \to \Xi_x$ as $i\to\infty$.

In conclusion, recalling that $g_{x,\lambda_i\delta_i} \to g_{x,0}$ as $i\to\infty$, it follows that the limit connection $B(x)$ is Ξ_x-ASD with respect to the flat metric $g_{x,0}$, and $\mathrm{tr}(F_{B(x)}) = 0$. ∎

The combination of this last result with Proposition 2.3.6 immediately yields:

Corollary 4.4.3. *At each smooth point $x\in\Gamma$ such that $\Theta(\mu,\cdot)|_\Gamma$ is $\mathscr{H}^{n-4}$approximately continuous at x, there is a choice of orientation on $(T_x\Gamma, g|_{T_x\Gamma})$ with respect to which it* *is calibrated by* Ξ_x*. Furthermore, if* $B(x) = \pi^* I(x)$ *is a bubbling connection then* $I(x)$ *is a non-trivial ASD instanton on* $(T_x\Gamma^{\perp}, g|_{T_x\Gamma^{\perp}})$ *with respect to the induced orientation* $*\Xi_x|_{T_x\Gamma^{\perp}}$*.*

Finally, we conclude the proof of Theorem B (stated in the introduction), by proving the following (cf. [Tia00, Theorem 4.3.2]):

Theorem 4.4.4 (Tian)**.** *Suppose* $G \subseteq \mathrm{U}(r)$ *and let* $c_2(\Gamma, \Theta) \in \mathscr{D}_{n-4}(M)$ *be defined by*

$$c_2(\Gamma,\Theta)(\varphi) \coloneqq \frac{1}{8\pi^2}\int_\Gamma \langle \varphi, \Xi|_\Gamma\rangle \Theta\, d\left(\mathscr{H}^{n-4}\lfloor\Gamma\right), \quad \forall\varphi\in\mathscr{D}^{n-4}(M).$$

Then $c_2(\Gamma,\Theta)$ *is a closed integral current (cf. Definition A.6.16) and the following conservation of the instanton charge density holds:*

$$c_2(\nabla_i) \rightharpoonup c_2(\nabla) + c_2(\Gamma,\Theta). \tag{4.4.1}$$

Remark 4.4.5. Recall from Section 3.5 that $c_2(\nabla_{(i)}) \in \mathscr{D}_{n-4}(M)$ is defined by

$$c_2(\nabla_{(i)})(\varphi) = \frac{1}{8\pi^2}\int_M \varphi\wedge\left(\mathrm{tr}(F_{\nabla_{(i)}}\wedge F_{\nabla_{(i)}}) - \mathrm{tr}(F_{\nabla_{(i)}})\wedge\mathrm{tr}(F_{\nabla_{(i)}})\right), \quad \forall\varphi\in\mathscr{D}^{n-4}(M).$$

We note that

$$\mathrm{tr}(F_{\nabla_{(i)}}\wedge F_{\nabla_{(i)}}) = \mathrm{tr}(F^0_{\nabla_{(i)}}\wedge F^0_{\nabla_{(i)}}) + \frac{1}{r}\mathrm{tr}(F_{\nabla_{(i)}})\wedge\mathrm{tr}(F_{\nabla_{(i)}}).$$

Since $\mathrm{tr}(F_{\nabla_i})$ converges to $\mathrm{tr}(F_\nabla)$ in $C^\infty-$topology, for every $\varphi\in\mathscr{D}^{n-4}(M)$ we get:

$$8\pi^2\left[c_2(\nabla_i) - c_2(\nabla)\right](\varphi) = \lim_{i\to\infty}\int_M \varphi\wedge\left(\mathrm{tr}(F^0_{\nabla_i}\wedge F^0_{\nabla_i}) - \mathrm{tr}(F^0_\nabla\wedge F^0_\nabla)\right). \tag{4.4.2}$$

In particular, equation (4.4.1) is equivalent to

$$\frac{1}{8\pi^2}\lim_{i\to\infty}\int_M\varphi\wedge\mathrm{tr}(F^0_{\nabla_i}\wedge F^0_{\nabla_i}) = \frac{1}{8\pi^2}\int_M\varphi\wedge\mathrm{tr}(F^0_\nabla\wedge F^0_\nabla) + c_2(\Gamma,\Theta)(\varphi), \tag{4.4.3}$$

for every $\varphi\in\mathscr{D}^{n-4}(M)$. Since M is compact, we can apply equation (4.4.3) in $8\pi^2\Xi\in\mathcal{D}^{n-4}(M)$ to deduce the following L^2-**energy conservation**:

$$\lim_{i\to\infty}\int_M|F_{\nabla_i}|^2\mathrm{d}V_g = \int_M|F_\nabla|^2\mathrm{d}V_g + \int_\Gamma\Theta\mathrm{d}\left(\mathscr{H}^{n-4}\lfloor\Gamma\right),$$

i.e. $\int_\Gamma\Theta\mathrm{d}\mathscr{H}^{n-4}\lfloor\Gamma$ is precisely the L^2-energy lost by the weakly convergent sequence $\{\nabla_i\}$ as $i\to\infty$. ♢

Proof. We start showing that $\Theta(\mu,x)\in 8\pi^2\mathbb{Z}$ at $\mathscr{H}^{n-4}-$a.e. $x\in\Gamma$. Indeed, let $x\in\Gamma$ be a smooth point such that $\Theta(\mu,\cdot)|_\Gamma$ is $\mathscr{H}^{n-4}-$approximately continuous at x. We will prove that $\Theta(\mu,x)\in 8\pi^2\mathbb{Z}$.

Recall that ν has a unique $(n-4)-$tangent measure at x given by $T_x\nu = \Theta(\mu,x)\mathscr{H}^{n-4}\lfloor T_x\Gamma$. Furthermore, by Lemma 4.3.1, there exists a null-sequence $\{\lambda_i\} \subseteq]0,1[$ such that (4.3.1) holds. In particular, it follows that

$$\Theta(\mu,x) = T_x\nu(B_1(0;g_{x,0})) = \lim_{i\to\infty}\int_{B_1(0;g_{x,0})}|F_{\nabla_{i,x,\lambda_i}}|^2 \mathrm{d}V_{x,\lambda_i}. \tag{4.4.4}$$

Moreover, since $\nabla_i \rightharpoonup \nabla$, by taking gauge transformations, we may assume that ∇_{i,x,λ_i} converges to zero uniformly on compact sets outside $V := T_x\Gamma$. Then, for each $z' \in V \cap B_1(0;g_{x,0})$, the restriction of ∇_{i,x,λ_i} over $\{z'\}\times V^\perp \cap B_{\sqrt{1-|z'|^2}}(0;g_{x,0})$ converges to zero uniformly away from $(z',0)$. In particular, by standard transgression arguments, this implies that

$$\lim_{i\to\infty}\frac{1}{8\pi^2}\int_{\{z'\}\times V^\perp\cap B_{\sqrt{1-|z'|^2}}(0;g_{x,0})}\mathrm{tr}(F_{\nabla_{i,x,\lambda_i}}\wedge F_{\nabla_{i,x,\lambda_i}}) \in \mathbb{Z}, \tag{4.4.5}$$

and such integer is a topological number which does not depend on z' $(*)$.

Denote by $F^{V^\perp}_{\nabla_{i,x,\lambda_i}}$ the curvature of the restricted connection $\nabla|_{\{z'\}\times V^\perp}$. Since ∇_{i,x,λ_i} is $\tau_\lambda^*\exp_x^*\Xi-$ASD with respect to g_{x,λ_i}, and $g_{x,\lambda_i}\to g_{x,0}$ in $C^\infty_{\mathrm{loc}}-$topology as $i\to\infty$, we obtain:

$$\begin{aligned}\frac{1}{8\pi^2}|F_{\nabla_{i,x,\lambda_i}}|^2\mathrm{d}V_{x,\lambda_i} &= \frac{1}{8\pi^2}\mathrm{tr}(F_{\nabla_{i,x,\lambda_i}}\wedge F_{\nabla_{i,x,\lambda_i}})\wedge(\tau_{\lambda_i}^*\exp_x^*\Xi) \\ -\frac{1}{8\pi^2}&\left\{-\mathrm{tr}(F^{V^\perp}_{\nabla_{i,x,\lambda_i}}\wedge F^{V^\perp}_{\nabla_{i,x,\lambda_i}}) + |F_{\nabla_{i,x,\lambda_i}}|\left(O(1)\sum_{\alpha=1}^{n-4}\left|\frac{\partial}{\partial z^\alpha}\lrcorner F_{\nabla_{i,x,\lambda_i}}\right| + o(1)|F_{\nabla_{i,x,\lambda_i}}|\right)\right\}\mathrm{d}V_{x,\lambda_i},\end{aligned} \tag{4.4.6}$$

where $o(1)$ denotes a quantity which converges to 0 as $i\to\infty$. Plugging (4.4.6) into (4.4.4), using Lemma 4.3.4 and $(*)$ we conclude that:

$$\begin{aligned}\frac{1}{8\pi^2}\Theta(\mu,x) = &\lim_{i\to\infty}\int_{V\cap B_1(0;g_{x,0})}\mathrm{d}\left(\mathscr{H}^{n-4}\lfloor V\right) \\ &\cdot\left(\lim_{i\to\infty}\frac{1}{8\pi^2}\int_{\{z'\}\times V^\perp\cap B_{\sqrt{1-|z'|^2}}(0;g_{x,0})}\mathrm{tr}(F_{\nabla_{i,x,\lambda_i}}\wedge F_{\nabla_{i,x,\lambda_i}})\right).\end{aligned}$$

Therefore, by (4.4.5), we conclude that $\Theta(\mu,x)\in 8\pi^2\mathbb{Z}$ as we wanted.

Now we prove that $c_2(\Gamma,\Theta)$ is a closed current. By Theorem 3.5.8 (together with the facts that weak*-limits of closed currents are closed, and that linear combination of closed currents are closed), it suffices to show that (4.4.1) holds. Equivalently, it suffices to show that (4.4.3) holds. Actually, the $C^\infty-$convergence $\mathrm{tr}(F_{\nabla_i})\to\mathrm{tr}(F_\nabla)$ allow us to suppose, without loss of generality, that G is semi-simple.

For each $i\in\mathbb{N}$, define the current $T_i \in \mathscr{D}_{n-4}(M)$ given by

$$T_i := c_2(\nabla_i) - c_2(\nabla).$$

Being a linear combination of closed currents, T_i is closed for each i. Moreover, the mass $\mathbf{M}(T_i)$ of T_i is uniformly bounded: indeed, for any $\varphi\in\mathscr{D}^{n-4}(M)$ with $\|\varphi\|_{C^0}\leq 1$ we have:

$$|T_i(\varphi)| \leq \frac{\mathrm{const.}}{8\pi^2}\left(\|F_{\nabla_i}\|^2_{L^2}+\|F_\nabla\|^2_{L^2}\right) \leq \frac{\mathrm{const.}\Lambda}{4\pi^2} =: C(n,\Lambda).$$

Therefore, by Lemma A.6.12, taking a subsequence if necessary, we can suppose $T_i \rightharpoonup T$. It follows that $\partial T = 0$ and $\mathbf{M}(T) \leq \liminf \mathbf{M}(T_i) \leq C(n, \Lambda)$.

At this point, Tian applies Theorem A.6.17 to conclude T is rectifiable (see [Tia00, Theorem 4.2.3, pp. 239-241]). However, we still need to verify that

$$\Theta^{*n-4}(\|T\|, x) > 0 \quad \text{for } \|T\|\text{−a.e. } x \in M. \tag{4.4.7}$$

In order to do this, first note that the convergence $\nabla_i \rightharpoonup \nabla$ implies $\mathrm{spt}(T) \subseteq S$. In particular, we have that $\mathrm{spt}(\|T\|) \subseteq S$. We claim further that $\|T\| \ll \mathscr{H}^{n-4}\lfloor S$. Indeed, let $0 < r \ll 1$ and let $x \in S$. Then, whenever $\varphi \in \mathscr{D}^{n-4}(M)$ is such that $\|\varphi\|_{C^0} \leq 1$ and $\mathrm{supp}(\varphi) \subseteq B_r(x)$ we have

$$|T_i(\varphi)| \leq \text{const.} \left(\int_{B_r(x)} |F_{\nabla_i}|^2 \mathrm{d}V_g + \int_{B_r(x)} |F_{\nabla}|^2 \mathrm{d}V_g \right).$$

Since $\Theta(\mu, \cdot)$ is upper semi-continuous, the function $\Theta(\mu, \cdot)|_{S \cap \overline{B}_r(x)}$ attains its maximum, so recalling that $\mu_i \rightharpoonup \mu = \mu_\nabla + \Theta(\mu, \cdot)\mathscr{H}^{n-4}\lfloor S$, it follows that

$$|T(\varphi)| \leq \text{const.} \left(\mu_\nabla(B_r(x)) + \mathscr{H}^{n-4}(\Gamma \cap B_r(x)) \right),$$

for all $\varphi \in \mathscr{D}^{n-4}(M)$ such that $\|\varphi\|_{C^0} \leq 1$ and $\mathrm{supp}(\varphi) \subseteq B_r(x)$. In particular,

$$\|T\|(B_r(x)) \leq \text{const.} \left(\mu_\nabla(B_r(x)) + \mathscr{H}^{n-4}(\Gamma \cap B_r(x)) \right).$$

Now, since $\mathscr{H}^{n-4}(S) < \infty$, by Theorem A.3.9 we know that $\Theta^{*n-4}(S, x) \leq 1$ for $\mathscr{H}^{n-4}$−a.e. $x \in S$. Moreover, by Lemma 4.1.2 (ii), we have that $\Theta^{*n-4}(\mu_\nabla, x) = 0$ for $\mathscr{H}^{n-4}$−a.e. $x \in S$. Thus, it follows that

$$\Theta^{*n-4}(\|T\|, x) \leq \text{const.} \quad \text{for } \mathscr{H}^{n-4}\text{−a.e. } x \in S.$$

Therefore, the claim follows by Theorem A.3.7.

Now let $f \in \mathscr{D}(M)$ and take $\varphi = f\Xi$; then:

$$\begin{aligned}
T(f\Xi) &= \lim_{i\to\infty} T_i(f\Xi) \\
&= \frac{1}{8\pi^2} \lim_{i\to\infty} \int_M f \left(\mathrm{tr}(F_{\nabla_i} \wedge F_{\nabla_i}) - \mathrm{tr}(F_\nabla \wedge F_\nabla)\right) \wedge \Xi \\
&= \frac{1}{8\pi^2} \lim_{i\to\infty} \int_M f \left(-\mathrm{tr}(F_{\nabla_i} \wedge *F_{\nabla_i}) + \mathrm{tr}(F_\nabla \wedge *F_\nabla)\right) \quad \text{(by the } \Xi\text{−ASD condition)} \\
&= \frac{1}{8\pi^2} \lim_{i\to\infty} \int_M f \left(|F_{\nabla_i}|^2 - |F_\nabla|^2\right) \\
&= \frac{1}{8\pi^2} \left(\int_M \mathrm{d}\mu - \int_M f|F_\nabla|^2 \mathrm{d}V_g \right) \\
&= \frac{1}{8\pi^2} \int_\Gamma f\Theta(\mu, \cdot)\mathrm{d}\left(\mathscr{H}^{n-4}\lfloor\Gamma\right). \quad \text{(since } \mu_i \rightharpoonup \mu = \mu_\nabla + \Theta(\mu, \cdot)\mathscr{H}^{n-4}\lfloor\Gamma) \quad (4.4.8)
\end{aligned}$$

Thus, if $x \in \Gamma$ and $r > 0$, choosing a bump function $\tilde{f}$ for $\overline{B}_{\frac{r}{2}}(x)$ supported in $B_r(x)$, and letting[8] $f := \|\Xi\|_{C^0}^{-1}\tilde{f}$, we get

$$\|T\|(B_r(x)) \geq T(f\Xi) \geq \frac{1}{8\pi^2}\int_{\overline{B}_{\frac{r}{2}}(x)\cap\Gamma}\Theta(\mu,\cdot)\mathrm{d}\left(\mathscr{H}^{n-4}\lfloor\Gamma\right) \geq \frac{1}{8\pi^2}\varepsilon_0\mathscr{H}^{n-4}(\Gamma\cap\overline{B}_{\frac{r}{2}}(x)).$$

Since $\mathscr{H}^{n-4}(\Gamma) < \infty$, by Theorem A.3.9 we know that $\Theta^{*n-4}(\Gamma,x) \geq 2^{4-n} > 0$ for $\mathscr{H}^{n-4}-$a.e. $x \in \Gamma$. Thus, it follows that $\Theta^{*n-4}(\|T\|,x) > 0$ for $\mathscr{H}^{n-4}-$a.e. $x \in \Gamma$. Hence, since $\mathscr{H}^{n-4}(S\setminus\Gamma) = 0$ and $\|T\| \ll \mathscr{H}^{n-4}\lfloor S$, the hypothesis 4.4.7 of Theorem A.6.17 is indeed verified. Therefore, we can find a triple (Γ',Θ',ξ) such that

$$T(\varphi) = \frac{1}{8\pi^2}\int_{\Gamma'}\langle\varphi,\xi\rangle\Theta'\mathrm{d}\left(\mathscr{H}^{n-4}\lfloor\Gamma'\right), \quad \forall\varphi\in\mathscr{D}^{n-4}(M)$$

where

- $\Gamma' \subseteq M$ is $\mathscr{H}^{n-4}-$measurable and countably $\mathscr{H}^{n-4}-$rectifiable;
- $\Theta' : \Gamma \to [0,\infty[$ is locally $\mathscr{H}^{n-4}-$integrable;
- $\xi : \Gamma \to \Lambda^k TM$ is $\mathscr{H}^{n-4}-$measurable and such that $\xi(x)$ orients the approximate $(n-4)-$tangent space $T_x\Gamma'$ for $\mathscr{H}^{n-4}-$a.e. $x \in \Gamma'$.

In particular, for every $f \in \mathscr{D}(M)$ we have

$$T(f\Xi) = \frac{1}{8\pi^2}\int_{\Gamma'} f\langle\Xi,\xi\rangle\Theta'\mathrm{d}\left(\mathscr{H}^{n-4}\lfloor\Gamma'\right).$$

Comparing with (4.4.8), we conclude that $\Gamma = \Gamma'$ and $\langle\Xi,\xi\rangle\Theta' = \Theta(\mu,\cdot)$. Finally, since $\Xi|_\Gamma$ is one of the volume forms of Γ we get $\Theta' = \Theta$, so that $T = c_2(\Gamma,\Theta)$. This proves (4.4.1) and, using the first part of the proof, concludes that $c_2(\Gamma,\Theta)$ is a closed integral current. ∎

For convenience, below we state a corollary of Theorem B which summarizes some of its key results (see [Tia00, Theorem 4.3.2, p. 242]).

Theorem 4.4.6 (Tian)**.** *Let (M,g) be a compact, connected and oriented Riemannian $n-$manifold, $n \geq 4$, endowed with a calibration $\Xi \in \Omega^{n-4}(M)$. Let E be a $G-$bundle over M, where $G \subseteq U(r)$ is a compact Lie group, and let $\{\nabla_i\} \subseteq \mathfrak{U}(E)$ be a sequence of $\Xi-$anti-self-dual instantons with uniformly bounded L^2-energy. Then by taking a subsequence if necessary, ∇_i converges weakly to an admissible $\Xi-$anti-self-dual instanton ∇ with the bubbling locus (Γ,Θ), such that $c_2(\Gamma,\Theta)$ is a $\Xi-$calibrated cycle (cf. Definition 2.2.15), and satisfies*

$$c_2(\nabla_i) \rightharpoonup c_2(\nabla) + c_2(\Gamma,\Theta).$$

[8] Here, of course, it suffices to consider the case $\Xi \neq 0$.

In particular, the bubbling locus is mass-minimizing (cf. Proposition 2.2.17) and its components are $\Xi-$submanifolds except for singular sets of Hausdorff codimension at least 2 (cf. Theorem 2.2.18). Aside from the clear analogies between $\Xi-$ASD instantons and $\Xi-$calibrated submanifolds that we have seen in Chapter 2, this Tian's result shows a concrete and remarkable relation between gauge theory and calibrated geometry.

In what follows we state important corollaries of Theorem 4.4.6 in the special cases where (M, g) is a Riemannian manifold with a special holonomy group $\mathrm{Hol}(g) = \mathrm{U}(m)$, G_2 or $\mathrm{Spin}(7)(\supseteq \mathrm{SU}(4))$. In what follows, suppose that $G \subseteq \mathrm{U}(r)$ is compact Lie group.

Theorem 4.4.7. *Let (Z, ω) be a compact Kähler $m-$fold, and let $\{\nabla_i\}$ be a sequence of Hermitian-Yang-Mills connections with uniformly bounded L^2-energy on a $G-$bundle E over Z. Then, by taking a subsequence if necessary, ∇_i converges weakly to an admissible Hermitian-Yang-Mills connection ∇ with bubbling locus (Γ, Θ) such that $\Gamma = \cup_\alpha \Gamma_\alpha$ and $\Theta|_{\Gamma_\alpha} = 8\pi^2 m_\alpha$, where each Γ_α is a complex subvariety in Z and m_α is a positive integer. Moreover, for any smooth real test form $\varphi \in \mathscr{D}^{2m-4}(M)$,*

$$\lim_{i\to\infty} c_2(\nabla_i)(\varphi) = c_2(\nabla)(\varphi) + \sum_\alpha m_\alpha \int_{S_\alpha} \varphi.$$

Proof. By Lemma 2.3.12 we can apply Theorem 4.4.6 to the sequence $\{\nabla_i\}$. Then, using Corollary 2.2.20 and a result of Harvey-Shiffman [HS74, Theorem 2.1] the theorem follows. ∎

For G_2-manifolds the following result is immediate:

Theorem 4.4.8. *Let (Y^7, ϕ) be a compact G_2-manifold and let $\{\nabla_i\}$ be a sequence of G_2-instantons with uniformly bounded L^2-energy on a $G-$bundle E over Y. Then, by taking a subsequence if necessary, ∇_i converges weakly to an admissible G_2-instanton ∇ such that the associated bubbling locus (Γ, Θ) defines an* associative *cycle $c_2(\Gamma, \Theta)$ and*

$$c_2(\nabla_i) \rightharpoonup c_2(\nabla) + c_2(\Gamma, \Theta).$$

For Calabi-Yau 4−folds, using Lemma 2.3.32 we get the following:

Theorem 4.4.9. *Let (Z^8, ω, Υ) be a compact Calabi-Yau $4-$fold, and let $\{\nabla_i\}$ be a sequence of complex ASD instantons with uniformly bounded L^2-energy on a $G-$bundle E over Z. Then, by taking a subsequence if necessary, ∇_i converges weakly to an admissible complex anti-self-dual instanton ∇ such that the associated bubbling locus (Γ, Θ) defines a* Cayley *cycle $c_2(\Gamma, \Theta)$ and*

$$c_2(\nabla_i) \rightharpoonup c_2(\nabla) + c_2(\Gamma, \Theta).$$

In fact, we have more generally:

Theorem 4.4.10. *Let* (X^8, Φ) *be a compact* Spin(7)*−manifold and let* $\{\nabla_i\}$ *be a sequence of* Spin(7)*−instantons with uniformly bounded* L^2*−energy on a* G*−bundle* E *over* X*. Then, by taking a subsequence if necessary,* ∇_i *converges weakly to an admissible* Spin(7)*−instanton* ∇ *such that the associated bubbling locus* (Γ, Θ) *defines a* Cayley *cycle* $c_2(\Gamma, \Theta)$ *and*

$$c_2(\nabla_i) \rightharpoonup c_2(\nabla) + c_2(\Gamma, \Theta).$$

The interesting problems in this direction are to see Tian's result in practice and to reverse the process: given a possible limit $\Xi-$calibrated submanifold $\Gamma \hookrightarrow M$ and an admissible $\Xi-$ASD instanton ∇, one asks when does Γ appear as a bubbling locus of a sequence of $\Xi-$ASD instantons with uniformly bounded L^2-energy weakly converging to ∇, and to what extent can we understand the limit connections.

In his thesis [Wal13a] (also see [Wal12, Wal14]), Walpuski gave sufficient conditions for an (unobstructed) associative/Cayley submanifold in a G_2/Spin(7)−manifold to appear as the bubbling locus of a sequence of G_2/Spin(7)−instantons, related to the existence of a Fueter section of a bundle of ASD instanton moduli spaces over said submanifold.

Hence associative/Cayley submanifolds and connections on them arise as building blocks for constructing G_2/Spin(7)−instantons by gluing methods. This has been successfully implemented on both Joyce's construction and Kovalev's twisted connected sums (cf. [Wal13b] and [SEW15]). One can also attempt to construct invariants of G_2-manifolds by "counting" G_2-instantons and associative submanifolds [DT98, DS09], but this is still currently speculative [Wal13a, Chapter 6].

4.5 General blow-up loci and stationary connections

In this final section, we give a brief summary, without proofs, of the main results in [Tia00, §4.5 and §5.1]. We note that the key analytical tools necessary to reproduce Tian's proofs for the results we cite in this section have already been developed in the previous sections.

The following shows the existence of geometrical constraints on the support of bubbling loci of general sequences of smooth Yang-Mills connections.

Theorem 4.5.1 ([Tia00, Theorem 4.5.1, p. 247])**.** *Let* $\{\nabla_i\}$ *be a sequence of smooth Yang-Mills connections converging weakly to an admissible Yang-Mills connection* ∇ *with bubbling locus* (Γ, Θ)*. For any vector field* X *with compact support in* M*,*

$$-\int_\Gamma \mathrm{div}_\Gamma X.\Theta_\mu d\mathscr{H}^{n-4} = \int_M \left(|F_\nabla|^2 \mathrm{div}\ X - 4 \sum_{i,j=1}^{n} \langle F_\nabla(D_{e_i}X, e_j), F_\nabla(e_i, e_j)\rangle \right) \mathrm{d}V_g, \tag{4.5.1}$$

where $\mathrm{div}_\Gamma X$ *denotes the divergence of* X *along* Γ*, i.e. if* $T_x\Gamma$ *exists and* $\{v_i\}$ *is any orthonormal basis of* $T_x\Gamma$*,*

$$(\mathrm{div}_\Gamma X)\,(x) = \sum_{j=1}^{n-4} (D_{v_i}X, v_i)(x).$$

Note that the RHS of (4.5.1) is precisely the RHS of (3.2.3), the first variational formula of $\mathcal{YM}$ along X when ∇ is a *smooth* connection. This motivates the following defintion.

Definition 4.5.2 (Stationary connections)**.** An admissible Yang-Mills connection ∇ on E is called **stationary** if for every vector field X on M with compact support we have

$$\int_M \left(|F_\nabla|^2 \mathrm{div}\, X - 4\sum_{i,j=1}^{n} \langle F_\nabla(D_{e_i}X, e_j), F_\nabla(e_i, e_j)\rangle \right) \mathrm{d}V_g = 0,$$

where the integrand is written in an orthonormal local frame $\{e_i\}$ of TM.

Example 4.5.3. Every smooth Yang-Mills connection is stationary by (3.2.3). More generally, if $n > 4$ then every admissible Yang-Mills connection with *discrete* singular set is stationary.

Remark 4.5.4. It is worth noting that a highly nontrivial removable singularity theorem proved by Tao-Tian [TT04] imply that a stationary admissible Yang-Mills connection (∇, S) on E extends, modulo gauge transformations, to a smooth Yang-Mills connection on a $G-$bundle $\tilde{E}$ over $M \setminus \mathrm{sing}(\nabla)$ which is isomorphic to E over $M \setminus S$. Since the stationary property is automatic for admissible Yang-Mills connections in dimension $n = 4$, such result generalizes Uhlenbeck's removable singularity theorem [Uhl82b]. $\Diamond$

Of course, a direct consequence of Theorem 4.5.1 and Definition 4.5.2 is:

Corollary 4.5.5. *Let* $\{\nabla_i\}$ *be a sequence of smooth Yang-Mills connections converging weakly to an admissible Yang-Mills connection* ∇ *with bubbling locus* (Γ, Θ)*. Then* ∇ *is stationary if, and only if,* Γ *has no boundary on* M *and its generalized mean curvature vanishes (i.e.* Γ *defines a minimal cycle).*

The next proposition shows that the above corollary is indeed a generalization of Theorem 4.4.6 albeit with a slightly weaker conclusion.

Proposition 4.5.6 ([Tia00, Proposition 5.1.2, p. 251])**.** *If* (M, g) *is endowed with a closed* $(n-4)-$*form* Ξ*, then any* $\Xi-$*ASD admissible connection is stationary.*

Corollary 4.5.7. *If* $\{\nabla_i\}$ *is a sequence of* $\Xi-$*ASD connections converging weakly to an admissible* $\Xi-$*ASD connection* ∇ *with bubbling locus* (Γ, Θ)*, then* Γ *defines a minimal cycle.*

As we observed earlier in Remark 3.2.2 of Chapter 3, Price's monotonicity formula is also available for stationary connections.

Theorem 4.5.8. *Let $p \in M$, and let r_p and $c(p)$ be as in §3.2. Then there exists a nonnegative constant $a = a(n,p,g) \geq O(1)c(p)$ such that the following holds. Let ∇ be a stationary admissible connection on E. Then for all $0 < \sigma < \rho \leq r_p$ we have:*

$$\begin{aligned} & e^{a\rho^2}\rho^{4-n}\int_{B_\rho(p)} |F_\nabla|^2 \,\mathrm{d}V_g - e^{a\sigma^2}\sigma^{4-n}\int_{B_\sigma(p)} |F_\nabla|^2 \,\mathrm{d}V_g \\ \geq \ & 4\int_{B_\rho(p)\setminus B_\sigma(p)} e^{ar^2} r^{4-n} \left| \frac{\partial}{\partial r} \lrcorner F_\nabla \right|^2 \mathrm{d}V_g, \end{aligned}$$

where r denotes the radial distance function on $B_{r_p}(p)$. Furthermore:

(i) If $(M,g) = (\mathbb{R}^n, g_0)$, where g_0 denotes the standard flat metric, then we can take $a = 0$ and the above inequality holds for every $\rho \in]0,\infty[$.

(ii) If M is compact, we can choose uniform *constants $a \geq 0$ and $\delta_0 > 0$ so that the above holds for every $0 < \sigma < \rho \leq \delta_0$.*

We finish citing a generalization of Theorem 3.4.4 for stationary admissible Yang-Mills connections cf. [TY02, Proposition 3.3, p. 12].

Theorem 4.5.9. *Let $\{(\nabla_i, S_i)\}$ be a sequence of stationary admissible Yang-Mills connections on E with $\mathcal{YM}(\nabla_i) \leq \Lambda$. Suppose further that $S_{cls} := \overline{\limsup_{i\to\infty} S_i}$ satisfies $\mathscr{H}^{n-4}(S_{cls}) = 0$. Then, there exist a closed subset $S \subseteq M$ of locally finite $(n-4)-$dimensional Hausdorff measure and an admissible Yang-Mills connection $(\nabla, S \cup S_{cls})$ such that, after passing to a subsequence, (∇_i, S_i) converges weakly to $(\nabla, S \cup S_{cls})$ modulo gauge transformations.*

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Appendix A

Geometric measure theory

In this appendix we collect some basic facts from Geometric Measure Theory (GMT) to support the material of Chapter 4. Let me stress that I have no intention to make a complete systematic exposition here. This is just a quick guide to organize and summarize the main definitions and results, and to fix our notations and conventions. As a consequence, we will omit almost all proofs and refer the reader to standard texts. Good references for the material in this appendix are Simon's notes [Sim83], Federer's classic [Fed69], and the more recent books [MF96] and [DL08].

Notation. Throughout X will denote a metric space with distance function d. For any subset $A \subseteq X$, we denote by $\overline{A}$, $\mathring{A}$ and ∂A, respectively, the topological closure, interior and boundary of A. For each $x \in X$ and $r \in \mathbb{R}_+$, we write $B_r(x)$, $\overline{B}_r(x)$ and $\partial B_r(x)$ to denote, respectively, the *open* ball with center x and radius r, its closure and its boundary. If B is an open (resp. closed) ball in X of center x and radius r, then for each positive real number $\lambda > 0$ we write λB for the open (resp. closed) ball in X of center x and radius λr. The distance between two subsets $A, B \subseteq X$ is denoted by $d(A, B)$, and the diameter of A is denoted by $\operatorname{diam}(A)$. Finally, we shall use the extended real number system $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$ with the obvious ordering and arithmetical operations partially extended, e.g., as in [Fol13, p. 11].

A.1 Basic concepts

Definition A.1.1 (Measures, $\sigma-$additivity and measurable sets)**.** A **measure** (or *outer* measure) on X is a set function $\mu : 2^X \to [0, \infty]$ satisfying the following conditions:

(i) $\mu(\emptyset) = 0$;

(ii) (Monotonicity) $A \subseteq B,\ A, B \subseteq X \Rightarrow \mu(A) \leq \mu(B)$;

(iii) (Subadditivity) For each countable collection $\{A_i\}_{i\in\mathbb{N}} \subseteq X$,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

Given a family $\mathcal{F}$ of subsets of X, we say that μ is $\sigma-$**additive on** $\mathcal{F}$ whenever

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i),$$

for each countable collection of *disjoint* sets $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{F}$.

Finally, we say that a subset $A \subseteq X$ is $\mu-$**measurable** if

$$\mu(E) = \mu(E \setminus A) + \mu(E \cap A), \quad \text{for each } E \subseteq X.$$

We denote by $\mathcal{M}_\mu$ the collection of all $\mu-$measurable sets.

Definition A.1.2. Let μ be a measure on X. We define the **support** of μ to be the following closed subset of X:

$$\operatorname{spt}(\mu) \coloneqq X \setminus \bigcup\{U \subseteq X : U \text{ is open and } \mu(U) = 0\}.$$

Given a measure μ on X, a sentence of the form

"(...) holds for $\mu-$almost every point $x \in X$"

or, briefly,

"(...) holds for $\mu-$**a.e.** $x \in X$"

means that the subset, say A, of X for which (...) doesn't hold is a $\mu-$**negligible set**, i.e. $\mu(A) = 0$.

Recall that a $\sigma-$**algebra** Σ on a set Y is a collection of subsets of Y, containing the empty set $\emptyset$ and Y itself, that is closed under the set operations of taking complements and countable unions. When Y is a topological space, the smallest $\sigma-$algebra $\mathcal{B}(Y)$ containing the topology of Y is called the **Borel** $\sigma-$algebra and its elements are the **Borel sets**.

In the next result we collect some well-known basic facts about general measures (see e.g. [Fol13, §1]).

Theorem A.1.3. *If μ is a measure on X, then $\mathcal{M}_\mu$ is a $\sigma-$algebra on X. Moreover, we have the following properties:*

(1) If $\mu(A) = 0$, $A \subseteq X$, then $A \in \mathcal{M}_\mu$ (i.e., every $\mu-$negligible set is $\mu-$measurable).

(2) μ is $\sigma-$additive on $\mathcal{M}_\mu$.

(3) If $\{A_i\} \subseteq \mathcal{M}_\mu$*, then*

(3.a) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i)$ *provided* $A_1 \subseteq A_2 \subseteq \ldots$*.*

(3.b) $\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i)$ *provided* $A_1 \supseteq A_2 \supseteq \ldots$ and $\mu(A_1) < \infty$*.*

In particular, given a measure μ on X we can always find a $\sigma-$algebra $\mathcal{M}_\mu$ restricted to which μ is $\sigma-$additive. Reciprocally, given a $\sigma-$algebra Σ and a $\sigma-$additive measure $\mu : \Sigma \to [0, \infty]$, we can extend μ to the whole power set of X as follows: for each $A \subseteq X$, define

$$\mu(A) := \inf\left\{\sum_i \mu(S_i) : \{S_i\}_{i\geq 1} \subseteq \Sigma \text{ with } A \subseteq \bigcup_i S_i\right\}.$$

It is straightforward to check this indeed defines a measure on X whose restriction to Σ is the originally given measure.

Definition A.1.4 ($\mu-$measurable functions)**.** Let Y be a topological space and μ a measure on X. A function $f : X \to Y$ is said to be $\mu-$**measurable** when $f^{-1}(U)$ is a $\mu-$measurable set in X for every open subset $U \subseteq Y$.

Given a measure μ on X, we do integration theory with respect to μ by restricting ourselves to the natural measure space $(X, \mathcal{M}_\mu, \mu|_{\mathcal{M}_\mu})$ determined by μ. It is beyond the scope of this appendix to reproduce all the definitions and main theorems of the usual Lebesgue integration theory on measure spaces. The interested reader is referred to [Fol13].

In the next definition we introduce some important properties that a given measure may satisfy.

Definition A.1.5. Let μ be a measure on X; we say that μ is:

- **locally finite** if $\mu(K) < \infty$ for each compact subset $K \subseteq X$;
- **metric** if $\mu(A \cup B) = \mu(A) + \mu(B)$, for each $A, B \subseteq X$ such that $d(A, B) > 0$;
- **Borel** if all Borel sets are $\mu-$measurable, i.e. $\mathcal{B}(X) \subseteq \mathcal{M}_\mu$;
- **Borel regular** if it is a Borel measure and if for every $A \subseteq X$ there is a Borel set $B \subseteq X$ such that $A \subseteq B$ and $\mu(A) = \mu(B)$;
- **Radon** if it is a locally finite and Borel regular measure.

Let μ be a measure on X. If $f : X \to \mathbb{R}$ is a nonnegative ($f \geq 0$) $\mu-$measurable function, then we can form a new measure $f\mu$ on X such that

$$[f\mu](A) := \int_A f \mathrm{d}\mu, \quad \forall A \in \mathcal{M}_\mu.$$

In particular, when $A \subseteq X$ is a $\mu-$measurable subset, we denote by $\mu\lfloor A$ the measure $\chi_A\mu$, i.e.

$$[\mu\lfloor A](E) := \mu(A \cap E), \quad \forall E \subseteq X.$$

As one may check directly, all $\mu-$measurable sets are $(\mu\lfloor A)-$measurable sets. Also, if μ is a Borel regular measure, then so is $\mu\lfloor A$. Moreover, it is not difficult to show the following:

Lemma A.1.6. *If μ is a Radon measure and $f \in L^1(\mu)$ is a nonnegative function, then $f\mu$ is a Radon measure.*

A key tool to check Borel sets are $\mu-$measurable is the following [Sim83, p. 3, Theorem 1.2]:

Theorem A.1.7 (Carathéodory's criterion)**.** *Let μ be a measure on the metric space X. Then,*

$$\mu \textit{ is a Borel measure} \iff \mu \textit{ is a metric measure.}$$

Finally, we state a very useful approximation result for Borel regular measures [Sim83, Theorem 1.3 and Remark 1.4]:

Theorem A.1.8 (Inner and outer approximation)**.** *Suppose μ is a Borel regular measure on X and $X = \cup_{j\geq 1} V_j$, where $\mu(V_j) < \infty$ and V_j is open for each $j \in \mathbb{N}$. Then:*

(i) $\mu(A) = \inf\{\mu(U) : U \supseteq A,\ U \textit{ open}\}$, *for any* $A \subseteq X$.

(ii) $\mu(A) = \sup\{\mu(C) : C \subseteq A,\ C \textit{ closed}\}$, *for any* $A \in \mathcal{M}_\mu$.

In particular, if X is a second countable and locally compact metric space[1] *(e.g. when X is a manifold) and μ is a Radon measure on X then (i) holds and (ii) can be improved to*

(ii') $\mu(A) = \sup\{\mu(K) : K \subseteq A,\ K \textit{ compact}\}$, *for any* $A \in \mathcal{M}_\mu$.

A.2 Hausdorff measure and dimension

We start describing a standard general process for constructing metric measures on metric spaces, called **Carathéodory's construction**. The input for this method is the following. Suppose we are given a pair $(\mathcal{F}, \rho)$, where $\mathcal{F}$ is a collection of subsets of X, $\rho : \mathcal{F} \to [0, \infty]$ and

(i) for each $\delta > 0$, there exists a countable cover $\{E_i\}_{i\in\mathbb{N}} \subseteq \mathcal{F}$ of X such that $\mathrm{diam}(E_i) \leq \delta$.

[1] In particular, X admits an open covering $\{V_j\}$ such that $\overline{V}_j$ is compact and contained in V_{j+1}, for each $j \in \mathbb{N}$; see e.g. [War13, proof of Lemma 1.9, p. 9].

(ii) for each $\delta > 0$, there exists an element $E \in \mathcal{F}$ such that $\rho(E) \leq \delta$ and $\operatorname{diam}(E) \leq \delta$.

For example, if $\mathcal{F}$ contains all non-empty open balls of X and X is separable, then (i) is easily seen to be verified. If, moreover, one has $\rho(\cdot) = C\operatorname{diam}(\cdot)$, for some uniform constant $C \leq 1$, then (ii) is also checked trivially.

For each $\delta > 0$, define

$$\mathcal{F}_\delta := \{E \in \mathcal{F} : \operatorname{diam}(E) \leq \delta\},$$

and construct preliminary measures ν_δ on X putting, for each $A \subseteq X$,

$$\nu_\delta(A) := \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : A \subseteq \bigcup_{i=1}^{\infty} E_i \text{ and } \{E_i\}_{i \in \mathbb{N}} \subseteq \mathcal{F}_\delta \right\}.$$

We note that $0 < \delta \leq \delta'$ implies $\mathcal{F}_\delta \subseteq \mathcal{F}_{\delta'}$, so that $\nu_\delta \geq \nu_{\delta'}$. Therefore, it is well-defined (possibly ∞)

$$\nu(A) := \lim_{\delta \searrow 0} \nu_\delta(A) = \sup_{\delta > 0} \nu_\delta(A), \quad \text{for each } A \subseteq X.$$

It is straightforward to check that ν is a measure on X. Moreover, we claim that ν is in fact a metric measure (therefore, by Theorem (A.1.7), ν is Borel): indeed, if $A, B \subseteq X$ are such that $d(A, B) > \delta > 0$, then

$$\nu_\delta(A \cup B) \geq \nu_\delta(A) + \nu_\delta(B),$$

because whenever $\mathcal{C} = \{E_i\}$ is a covering of $A \cup B$ with $\operatorname{diam}(E_i) < \delta$, the collections

$$\mathcal{C} \cap \{E : E \cap A \neq \emptyset\} \text{ and } \mathcal{C} \cap \{E : E \cap B \neq \emptyset\}$$

are clearly disjoint, covering A and B respectively. Thus the claim follows from the definition of ν.

Example A.2.1 (Lebesgue measure)**.** Let $X = \mathbb{R}^n$ and d be the usual Euclidean distance:

$$d(x, y) = \left(\sum_{i=1}^{n} (y_i - x_i)^2 \right)^{1/2}, \quad \text{for each } x, y \in \mathbb{R}^n.$$

Define

$$\mathcal{F} := \left\{ \prod_{i=1}^{n} [a_i, b_i] : a_i, b_i \in \mathbb{R}, a_i < b_i, i = 1, \ldots, n \right\},$$

i.e. $\mathcal{F}$ is the collection of all (non-degenerated) closed $n-$cubes on $\mathbb{R}^n$, and take $\rho : \mathcal{F} \to [0, \infty]$ defined by

$$\rho(\prod_{i=1}^{n} [a_i, b_i]) := \prod_{i=1}^{n} (b_i - a_i).$$

Then, the resulting measure of Carathéodory's construction applied to $(\mathcal{F}, \rho)$ is the $n-$**dimensional Lebesgue measure** $\mathscr{L}^n$ on $\mathbb{R}^n$.

A well-known characterization for $\mathscr{L}^n$ is the following: $\mathscr{L}^n$ is the unique Borel regular, translation-invariant measure on $\mathbb{R}^n$, normalized so that the measure of the unit cube $[0, 1]^n$ is 1 (see [Fed69] and [Sim83, p. 8]).

We now proceed to define the Hausdorff $s-$dimensional measure $\mathscr{H}^s$ on an arbitrary *separable* metric space (X, d).

Definition A.2.2 (Hausdorff measure)**.** Let $s \in \mathbb{R}_{\geq 0}$. The $s-$**dimensional Hausdorff measure** $\mathscr{H}^s$ of a separable metric space (X, d) is the measure on X generated by Carathéodory's construction when $\mathcal{F}$ is taken to be the collection of all non-empty subsets of X and ρ is given by

$$\rho(A) := 2^{-s}\mathrm{diam}(A)^s, \quad \text{for each } \emptyset \neq A \subseteq X.$$

More explicitly, for each $A \subseteq X$,

$$\mathscr{H}^s(A) := \lim_{\delta \downarrow 0} \mathscr{H}^s_\delta(A),$$

where

$$\mathscr{H}^s_\delta(A) := 2^{-s} \inf \left\{ \sum_{i=1}^{\infty} (\mathrm{diam}(E_i))^s : A \subseteq \bigcup_{i=1}^{\infty} E_i, \text{ and } \mathrm{diam}(E_i) \leq \delta \right\}, \quad \forall \delta > 0.$$

Example A.2.3 ($n-$dimensional Hausdorff measure of a connected Riemannian $n-$manifold)**.** Let (M, g) be a connected Riemannian $n-$manifold. Then, on the one hand, viewing M as a metric space with the natural Riemannian distance function d_g induced by g (see e.g. [Aub82, §2.1]), we get associated $s-$dimensional Hausdorff measures $\mathscr{H}^s$ on M for each nonnegative real number s; in particular, we get $\mathscr{H}^n$.

On the other hand, supposing further that M is oriented, we get a Riemannian volume $n-$form $\mathrm{d}V_g$ on (M, g), which in turn induces a canonical Radon measure μ_g on M resulting from the application of Riesz's representation theorem (see Remark A.4.2) on the integration functional

$$\begin{aligned} I_g : C_c(M; \mathbb{R}) &\rightarrow \mathbb{R} \\ f &\mapsto \int_M f \mathrm{d}V_g. \end{aligned}$$

Now, for all $s \in \mathbb{R}_{\geq 0}$, we define

$$\alpha_s := \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2} + 1\right)},$$

where $\Gamma(z) := \int_0^\infty x^{z-1} e^{-x} dx$ (defined for each $z \in \mathbb{C}$ with positive real part) is the so-called *Euler gamma function.* Note that when $s = k \in \mathbb{N}_0$ is a nonnegative integer, α_k is precisely the Lebesgue measure $\mathscr{L}^k(B_1(0))$ of the unit ball in $\mathbb{R}^k$.

In this setting, we can state the following relation between $\mathscr{H}^n$ and μ_g:

Proposition A.2.4. *On a connected, oriented, Riemannian $n-$manifold (M, g), the $n-$dimensional Hausdorff measure $\mathscr{H}^n$ multiplied by the constant factor α_n equals the Riemannian volume measure μ_g.*

The reader interested in a proof of this fact may consult [BBI01, p. 196, Theorem 5.5.5] (see also [Sim83, p. 10, Theorem 2.6] for the $M = \mathbb{R}^n$ case).

The following proposition is immediate from the above definitions:

Proposition A.2.5. *Let $A \subseteq X$ and $s \in \mathbb{R}_{\geq 0}$. Then:*

(i) $\mathscr{H}^s(A) < \infty \Rightarrow \mathscr{H}^{s'}(A) = 0$, $\forall s' > s$.

(ii) $\mathscr{H}^s(A) > 0 \Rightarrow \mathscr{H}^{s''}(A) = \infty$, $\forall s'' < s$.

We can then make the following definition:

Definition A.2.6 (Hausdorff dimension)**.** The **Hausdorff dimension** $\dim_{\mathscr{H}}(A)$ of a subset $A \subseteq X$ is the extended real number given by

$$\dim_{\mathscr{H}}(A) := \inf\{s \in \mathbb{R}_{\geq 0} : \mathscr{H}^s(A) = 0\} = \sup\{s \in \mathbb{R}_{\geq 0} : \mathscr{H}^s(A) = \infty\}.$$

In other words, if $A \subseteq X$ then the Hausdorff dimension $\dim_{\mathscr{H}}(A)$ of A is the unique extended real number in $[0, \infty]$ such that

$$s < \dim_{\mathscr{H}}(A) \quad \Rightarrow \quad \mathscr{H}^s(A) = \infty,$$
$$s > \dim_{\mathscr{H}}(A) \quad \Rightarrow \quad \mathscr{H}^s(A) = 0.$$

A priori, in case $s = \dim_{\mathscr{H}}(A)$, all the three possibilities $\mathscr{H}^s(A) = 0$, $0 < \mathscr{H}^s(A) < \infty$ and $\mathscr{H}^s(A) = \infty$ are admissible. On the other hand, if we can find s such that $0 < \mathscr{H}^s(A) < \infty$, then certainly $\dim_{\mathscr{H}}(A) = s$. Also, if $s \in \mathbb{R}_{\geq 0}$ is such that $\mathscr{H}^s(A) < \infty$ then $\dim_{\mathscr{H}}(A) \leq s$.

Some immediate properties the Hausdorff dimension satisfies are the following:

- (Monotonicity) If $A \subseteq B \subseteq X$, then $\dim_{\mathscr{H}}(A) \leq \dim_{\mathscr{H}}(B)$;
- (Stability w.r.t. countable unions) If $\{A_i\}$ is a countable collection of subsets $A_i \subseteq X$, then
$$\dim_{\mathscr{H}}(\bigcup_i A_i) = \sup_i \dim_{\mathscr{H}}(A_i).$$

In particular, if $S \subseteq X$, is such that $S = \bigcup_{i \geq 1} A_i$ with $\mathscr{H}^s(A_i) < \infty$ (for each $i \geq 1$), then $\dim_{\mathscr{H}}(S) \leq s$.

A.3 Densities and covering theorems

We start this section summarizing the covering theorems that are particularly useful for this work and then introduce the notion(s) of (lower and upper) density(ies) of measures. We finish with results relating appropriate information about the upper density of a measure and relations between such measure and the Hausdorff measure, as well as estimates on the upper density of the Hausdorff measure on appropriate sets.

Covering theorems. The first lemma we will prove is a simple result in metric space topology which is fairly used in chapters 3 and 4.

Lemma A.3.1. *Let K be a compact subspace of a metric space (X, d). Given $r > 0$, we can find a finite set of points $\{x_1, \dots, x_m\} \subseteq K$ such that the following holds:*

(i) $K \subseteq \bigcup_{i=1}^{m} B_{2r}(x_i)$, *and*

(ii) $B_r(x_i) \cap B_r(x_j) = \emptyset$, *for each* $i \neq j$, $i, j \in \{1, \dots, m\}$.

Proof. We describe an explicit algorithm to construct the $\{x_i\}$. In the first step, fix some $x_1 \in K$. In the second step, consider

$$C_2 \coloneqq K \setminus B_{2r}(x_1).$$

If $C_2 = \emptyset$, stop the algorithm; the set $\{x_1\}$ will do the job. If $C_2 \neq \emptyset$, then choose $x_2 \in C_2$ and go to the next step. In general, when we arrive at the $j-$th step, $j \geq 2$, the first $j-1$ points $x_1, \dots, x_{j-1} \in K$ are already constructed, so we consider

$$C_j \coloneqq K \setminus \bigcup_{i=1}^{j-1} B_{2r}(x_i).$$

If $C_j = \emptyset$, stop the algorithm; the set $\{x_1, \dots, x_{j-1}\}$ is clearly the set of points in K we are looking for. If $C_j \neq \emptyset$, choose $x_j \in C_j$ and go to the next step.

We claim this process ends in a finite number of steps, i.e. we always arrive at the case $C_j = \emptyset$, for some $j \in \mathbb{N}$ large enough. Otherwise, the algorithm just described would give rise to a sequence $\{x_i\}_{n=1}^{\infty}$ in K which does not admit a convergent subsequence: if $n, m \in \mathbb{N}$ are such that $n < m$, then $d(x_n, x_m) > 2r$ because $x_m \notin \bigcup_{i=1}^{m-1} B_{2r}(x_i) \supseteq B_{2r}(x_n)$ by construction. This contradicts the compacity of K. ■

Another important covering theorem is the following (cf. [Sim83, Theorem 3.3]).

Theorem A.3.2 (5$r-$covering lemma)**.** *Suppose (X, d) is a* separable *metric space. If $\mathscr{B}$ is an arbitrary family of (closed or open) balls in X satisfying*

$$\sup_{B \in \mathscr{B}} \operatorname{diam}(B) < \infty,$$

then there exists a countable *and (pairwise)* disjoint *subcollection $\mathscr{B}' \subseteq \mathscr{B}$ such that*

$$\bigcup_{B \in \mathscr{B}} B \subseteq \bigcup_{B \in \mathscr{B}'} 5B,$$

where $5B$ denotes a ball with the same center as B and five times the radius of B.

Densities. We now introduce the notions of upper and lower $k-$dimensional density of a measure at a point. The reference for this part is [Sim83, §3].

Definition A.3.3 (Upper and lower densities)**.** Let $s \in \mathbb{R}_{\geq 0}$ and let μ be a measure on X. We define the **upper** (resp. **lower**) $s-$**dimensional density** of μ at $x \in X$ by

$$\Theta^{*s}(\mu, x) \coloneqq \limsup_{r \downarrow 0} r^{-s} \mu(B_r(x)).$$

$$\Big(\text{resp. } \Theta_*{}^s(\mu, x) \coloneqq \liminf_{r \downarrow 0} r^{-s} \mu(B_r(x)).\Big)$$

Whenever $\Theta^{*s}(\mu, x) = \Theta_*{}^s(\mu, x)$, we denote the common value by

$$\Theta(\mu, x) \coloneqq \lim_{r \downarrow 0} r^{-s} \mu(B_r(x))$$

and simply speak of the $s-$**density of** μ **at** x.

For an arbitrary subset $A \subseteq X$, we define the **upper** (resp. **lower**) $s-$**dimensional density** of A at x by

$$\Theta^{*s}(A, x) \coloneqq \Theta^{*s}(\mathscr{H}^s \lfloor A, x).$$

$$(\text{resp. } \Theta_*{}^s(A, x) \coloneqq \Theta_*{}^s(\mathscr{H}^s \lfloor A, x).)$$

When the upper and lower $s-$dimensional densities of A at x are equal we write the common value by $\Theta^s(A, x)$.

Remark A.3.4. Some authors (including L. Simon) define the Hausdorff measure multiplying the one in Definition (A.2.2) by the constant factor α_s of Example A.2.3. In this case, it is convenient to modify the above definition multiplying the densities by α_s^{-1}. ♢

Remark A.3.5. We note that when μ is a Borel measure then $\Theta^{*s}(\mu, \cdot)$ and $\Theta_*{}^s(\mu, \cdot)$ are $\mu-$measurable functions. In fact, for each fixed $r > 0$, the function on X defined by

$$x \mapsto \mu(B_r(x))$$

is upper semi-continuous whenever μ is a Borel measure. Indeed: fix $x \in X$ and $r > 0$; we want to show

$$\mu(B_r(x)) \geq \limsup_{y \to x} \mu(B_r(y)).$$

If $\mu(B_r(x)) = \infty$ the assertion is clearly true, so suppose $\mu(B_r(x)) < \infty$. Let (x_n) be a sequence in X such that $x_n \to x$. Then, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \quad \Rightarrow \quad B_r(x_n) \subseteq B_{r+\varepsilon}(x).$$

Thus, on one hand we have

$$\limsup_{n \to \infty} \mu(B_r(x_n)) \leq \mu(B_{r+\varepsilon}(x)).$$

On the other hand, for $\varepsilon_0 > 0$ small enough, Theorem A.1.3 (3.b) implies

$$\mu(B_r(x)) = \mu\left(\bigcap_{0<\varepsilon<\varepsilon_0} B_{r+\varepsilon}(x)\right) = \lim_{\varepsilon\downarrow 0} \mu(B_{r+\varepsilon}(x)).$$

Therefore

$$\limsup_{n\to\infty} \mu(B_r(x_n)) \leq \mu(B_r(x)).$$

The claim follows. $\Diamond$

Remark A.3.6. If $x \notin \operatorname{spt}(\mu)$ then $\Theta^s(\mu, x) = 0$ for every $0 \leq s < \infty$. Indeed, when $x \notin \operatorname{spt}(\mu)$ there exists an open subset $U \subseteq X$ such that $x \in U$ and $\mu(U) = 0$. Thus, for all sufficiently small $r > 0$ we have $\mu(B_r(x)) = 0$. In particular, $\Theta^s(\mu, x) = 0$ for every $0 \leq s < \infty$. $\Diamond$

The next result tells us that appropriate information about the upper $s-$dimensional density function of a given Borel-regular measure gives estimates of this measure with respect to the $s-$dimensional Hausdorff measure.

Theorem A.3.7. *Let μ be a Borel regular measure on X, and let $s, t \in \mathbb{R}_{\geq 0}$.*

(i) If $A_1 \subseteq A_2 \subseteq X$ and $\Theta^{*s}(\mu\lfloor A_2, x) \geq t$ for all $x \in A_1$, then

$$t\mathscr{H}^s(A_1) \leq \mu(A_2).$$

(ii) If $A \subseteq X$ and $\Theta^{*s}(\mu\lfloor A, x) \leq t$ for all $x \in A$, then

$$\mu(A) \leq 2^s t\mathscr{H}^s(A).$$

In particular, (i) and (ii) imply

$$t_1\mathscr{H}^s(A) \leq \mu(A) \leq 2^s t_2\mathscr{H}^s(A),$$

whenever $A \subseteq X$ is such that $0 \leq t_1 \leq \Theta^{*s}(\mu\lfloor A, x) \leq t_2$ for all $x \in A$.

The proof of the above result uses Theorem A.3.2 for (i) and is elementary for (ii); see [Sim83, Theorem 3.2]. As a corollary of Theorem A.3.7 (i), one can prove the following useful result [Sim83, Theorem 3.5]:

Theorem A.3.8. *If μ is Borel-regular and $A \subseteq X$ is $\mu-$measurable with $\mu(A) < \infty$ then*

$$\Theta^{*s}(\mu\lfloor A, x) = 0 \quad \text{for } \mathscr{H}^s - a.e.\ x \in X \setminus A.$$

Restricting attention to Hausdorff measures, there are some useful estimates for the density on sets of finite measure [Sim83, Theorem 3.6].

Theorem A.3.9. *Let $s \in \mathbb{R}_{\geq 0}$. Then the following assertions holds.*

(i) If $\mathscr{H}^s(A) < \infty$ then $\Theta^{*s}(A,x) \leq 1$ for $\mathscr{H}^s-$a.e. $x \in A$.

(ii) If $\mathscr{H}^s_\delta(A) < \infty$ for each $\delta > 0$, then $\Theta_*{}^s(A,x) \geq 2^{-s}$ for $\mathscr{H}^s-$a.e. $x \in A$.

In particular[2], *if $\mathscr{H}^s(A) < \infty$ then*

$$2^{-s} \leq \Theta^{*s}(A,x) \leq 1 \quad \textit{for } \mathscr{H}^s - \textit{a.e. } x \in A.$$

A.4 Radon measures

In this subsection we assume X is a *locally compact* and *separable* metric space.

Let H denote a finite dimensional real *Hilbert space* with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Denote by $C_c(X;H)$ the space of continuous functions $X \to H$ with *compact* support in X. We endow $C_c(X;H)$ with the topology of uniform convergence on compact sets: if $\{f_n\}_{n\in\mathbb{N}} \subset C_c(X;H)$, then $f_n \to f \in C_c(X;H)$ if, and only if,

1. there exists a compact subset $K \subseteq X$ such that $\operatorname{supp}(f_n) \subseteq K$, for each $n \in \mathbb{N}$; and
2. $\sup\ \{\|f_n(x) - f(x)\|: x \in K\} \to 0$ as $n \to \infty$.

Given a Radon measure μ on X and a $\mu-$measurable function $v : X \to H$ with $\|v(x)\|= 1$ for μ-a.e. $x \in X$, then

$$L : f \mapsto \int_X \langle f, v\rangle d\mu$$

defines a continuous linear functional on $C_c(X;H)$: indeed, let $K \subseteq X$ be a compact set and suppose $f \in C_c(X;H)$ is such that $\operatorname{supp}(f) \subseteq K$. Since f is continuous and μ is Radon, we have $\|f\|_\infty= \sup_{x\in K}\|f(x)\|< \infty$ and $C_K := \mu(K) < \infty$. Moreover, by the hypothesis on v and the Cauchy-Schwarz inequality,

$$|\langle f(x), v(x)\rangle| \leq \|f(x)\|\|v(x)\|= \|f(x)\|, \quad \text{for } \mu\text{-a.e. } x \in X.$$

Therefore,

$$|L(f)| \leq \int_K |\langle f(x), v(x)\rangle| d\mu(x) \leq C_K\|f\|_\infty.$$

Conversely, we have [Sim83, Theorem 4.1, p.18]:

Theorem A.4.1. *(Riesz) Let $L : C_c(X;H) \to \mathbb{R}$ be a linear functional such that*

$$\sup\{L(f) : f \in C_c(X;H),\ \|f\|_\infty\leq 1,\ \operatorname{supp}(f) \subseteq K\} < \infty, \quad \forall K \subseteq X \textit{ compact.} \quad (*).$$

[2]Note that $\mathscr{H}^s \geq \mathscr{H}^s_\delta \geq \mathscr{H}^s_\infty$.

Then there is a Radon measure μ on X and a $\mu-$measurable function $v : X \to H$ with $\|v(x)\|= 1$ for μ-a.e. $x \in X$ such that

$$L(f) = \int_X \langle f, v\rangle \mathrm{d}\mu, \quad \forall f \in C_c(X; H).$$

Moreover, the Radon measure μ is unique; in fact, in the above conditions we have

$$\mu(V) = \sup\{L(f) : f \in C_c(X; H),\ \|f\|_\infty \leq 1 \textit{ and } \operatorname{supp}(f) \subseteq V\},$$

for every open subset $V \subseteq X$; μ is called the **total variation measure** *associated with the functional L.*

Remark A.4.2. When $H = \mathbb{R}$, if we replace the hypothesis $(*)$ in Theorem A.4.1 by the condition that $Lf \geq 0$ whenever $f \geq 0$ (in case L is called a *nonnegative* functional), then we can in fact find $v : X \to \mathbb{R}$ such that $v \equiv 1$ $\mu-$a.e. and, therefore, conclude that

$$L(f) = \int_X f \, d\mu, \quad \forall f \in C_c(X; \mathbb{R}).$$

Such version of the Riesz representation theorem can be found, for example, in Folland's book [Fol13, Theorem 7.2, p.212] (see also [Rud86, Theorem 2.14, p.40]). In particular, we can identify the set of Radon measures on X with the set of nonnegative linear functionals on $C_c(X) \coloneqq C_c(X; \mathbb{R})$. ◊

It is then natural to endow the space of Radon measures on X with the weak* topology of the topological dual of $C_c(X)$:

Definition A.4.3 (Weak* convergence)**.** Given a sequence of Radon measures $\{\mu_i\}$ we say that μ_i **converges weakly*** to a Radon measure μ, and we write $\mu_i \rightharpoonup \mu$, when

$$\lim_{i\to\infty} \int_X f \, d\mu_i = \int_X f \, d\mu, \quad \forall f \in C_c(X).$$

Having Remark A.4.2 in mind, the following theorem is a fairly easy application of the general Banach-Alaoglu theorem.

Theorem A.4.4 (Weak* Compactness of Radon Measures)**.** *If $\{\mu_i\}$ is a sequence of Radon measures on X satisfying*

$$\sup\{\mu_i(U) : i \geq 1\} < \infty, \quad \forall U \Subset X,$$

then $\{\mu_i\}$ admits a weakly convergent subsequence.*

The following basic result is of fundamental importance and will be used repeatedly in Chapter 4 [DL08, Proposition 2.7, p.8].

Theorem A.4.5. *Let $\{\mu_i\}$ be a sequence of Radon measures on X such that $\mu_i \rightharpoonup \mu$.*

(i) If $U \subseteq X$ is an open subset then

$$\mu(U) \leq \liminf_{i\to\infty} \mu_i(U).$$

(ii) If $K \subseteq X$ is a compact subset then

$$\mu(K) \geq \limsup_{i\to\infty} \mu_i(K).$$

In particular,

(iii) If $U \Subset X$ is a precompact open subset with $\mu(\partial U) = 0$, then

$$\mu(U) = \lim_{i\to\infty} \mu_i(U).$$

(iv) Given $x \in X$ and $\delta > 0$, then

$$\mathscr{R}_{x,\delta}(\mu) \coloneqq \{r \in \,]0,\delta] : \mu(\partial B_r(x)) > 0\}$$

is at most countable and

$$\mu(B_r(x)) = \lim_{i\to\infty} \mu_i(B_r(x)), \quad \forall r \in \,]0,\delta] \setminus \mathscr{R}_{x,\delta}(\mu).$$

We end this section with a theorem which requires the following definitions.

Definition A.4.6. Let $\mathscr{B}$ be a collection of balls in X. We define the **set of centres** of $\mathscr{B}$ to be

$$C_{\mathscr{B}} \coloneqq \{x \in X : B_r(x) \in \mathscr{B} \text{ for some } r > 0\}.$$

A subset $A \subseteq X$ is said to be covered **finely** by $\mathscr{B}$ if for every $x \in A$ and every $\varepsilon > 0$ there exists a ball $B \in \mathscr{B}$ such that $x \in B$ and $\operatorname{diam}(B) < \varepsilon$.

Definition A.4.7. Let μ be a Radon measure on X. We say that X has the **symmetric Vitali property** relative to μ if for every collection of balls $\mathscr{B}$ which covers its set of centres $C_{\mathscr{B}}$ finely and with $\mu(C_{\mathscr{B}}) < \infty$, there is a countable pairwise disjoint subcollection $\mathscr{B}' \subseteq \mathscr{B}$ covering $\mu-$almost all of $C_{\mathscr{B}}$.

Example A.4.8. If X is locally compact, Hausdorff and second countable (e.g. if X is a manifold) then X has the symmetric Vitali property relative to every Radon measure on X.

The following is a useful result about differentiation of measures due to Besicovitch [Sim83, p. 24, Theorem 4.7].

Theorem A.4.9 (Besicovitch differentiation of measures)**.** *Suppose μ_1 and μ_2 are Radon measures on X, where X has the symmetric Vitali property with respect to μ_1. Then*

$$\frac{d\mu_2}{d\mu_1}(x) := \lim_{r \downarrow 0} \frac{\mu_2(B_r(x))}{\mu_1(B_r(x))}$$

exists (possibly ∞) μ_1-almost everywhere and defines a μ_1-measurable function on X. Furthermore, the Radon-Nikodyn decomposition of μ_2 with respect to μ_1 is given by

$$\mu_2 = \frac{d\mu_2}{d\mu_1}\mu_1 + \mu_2 \lfloor Z, \tag{A.4.1}$$

where Z is a Borel set of μ_1-measure zero. Moreover, in case X also has the symmetric Vitali property with respect to μ_2 then $\dfrac{d\mu_2}{d\mu_1}$ also exists μ_2-almost everywhere and we may take $Z = \left\{x : \dfrac{d\mu_2}{d\mu_1}(x) = \infty\right\}$ in (A.4.1).

A.5 Rectifiable sets and measures

Perhaps the most relevant class of functions in the context of geometric measure theory is the class of Lipschitz functions.

Definition A.5.1 (Lipschitz maps)**.** A map $f : (X, d) \to (X', d')$ between metric spaces is called $\lambda-$**Lipschitz**, for some $\lambda \in [0, \infty[$, when

$$d'(f(x), f(y)) \leq \lambda d(x, y), \quad \forall x, y \in X.$$

Whenever

$$\mathrm{Lip}(f) := \inf\{\lambda \in [0, \infty[: f \text{ is } \lambda - \text{Lipschitz}\} < \infty,$$

f is called a **Lipschitz** funtion.

Lemma A.5.2. *Let X and X' be metric spaces and $E \subseteq X$ an arbitrary subset. If $f : E \to X'$ is a Lipschitz map, then*

$$\mathscr{H}^s(f(E)) \leq \mathrm{Lip}(f)^s \mathscr{H}^s(E).$$

In particular, a Lipschitz map takes $\mathscr{H}^s-$negligible sets to $\mathscr{H}^s-$negligible sets.

Next, we give a simple extension result.

Lemma A.5.3. *Let $A \subseteq X$ and $n \in \mathbb{N}$. Then, every $\lambda-$Lipschitz map admits a $\sqrt{n}\lambda-$Lipschitz extension $\overline{f} : X \to \mathbb{R}^n$.*

Sketch of proof. For $n = 1$, we simply define

$$\overline{f}(x) := \inf\{f(a) + \lambda f(x) : a \in A\}, \quad \forall x \in X.$$

It is straightforward to verify $\overline{f}$ is well-defined and satisfy the desired properties.

For $n \geq 2$ one writes $f = (f_1, \dots, f_n)$ and extends each $f_i : A \to \mathbb{R}$ separately as above. ■

For Lipschitz maps between Euclidean spaces we have the following important result [Sim83, Theorem 5.2, p.30].

Theorem A.5.4 (Rademacher)**.** *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz map, then f is differentiable for $\mathscr{L}^n-$a.e. $x \in \mathbb{R}^n$.*

Using the above theorem (and other results), one can prove [Sim83, Theorem 5.3, p.32]:

Theorem A.5.5. *If $U \subseteq \mathbb{R}^n$ is open and if $f : U \to \mathbb{R}$ is differentiable $\mathscr{L}^n-$a.e. in U, then for each $\varepsilon > 0$ there is a closed set $A \subseteq U$ and a C^1-function $g : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\mathscr{L}^n(U \setminus A) < \varepsilon,\ f|_A = g|_A \text{ and } (\operatorname{grad} f)|_A = (\operatorname{grad} g)|_A.$$

We now introduce a concept of great importance in geometric measure theory, which can be seen as a measure-theoretic notion of smoothness.

Definition A.5.6 (Rectifiable sets and measures)**.** Let $k \in \mathbb{N}_0$. A subset $\Gamma \subseteq X$ is called **countably $\mathscr{H}^k-$rectifiable** if there exists a sequence of Lipschitz maps $f_i : A_i \subseteq \mathbb{R}^k \to X$ such that

$$\mathscr{H}^k\left(\Gamma \setminus \bigcup_i f_i(A_i)\right) = 0.$$

A Radon measure ν on X is called **$\mathscr{H}^k-$rectifiable** if $\nu = \theta \mathscr{H}^k \lfloor \Gamma$ for some countably $\mathscr{H}^k-$rectifiable set Γ and some Borel function $\theta : \Gamma \to [0, \infty[$.

By Lemma A.5.2, it follows that if $\Gamma \subseteq X$ is a countably $\mathscr{H}^k-$rectifiable set then $\mathscr{H}^k \lfloor \Gamma$ is $\sigma-$finite and, therefore, $\dim_{\mathscr{H}} \Gamma \leq k$. Note also that any Borel subset of a countably $\mathscr{H}^k-$rectifiable set is countably $\mathscr{H}^k-$rectifiable. Moreover, a countable union of countably $\mathscr{H}^k-$rectifiable sets is again a countably $\mathscr{H}^k-$rectifiable set.

Remark A.5.7. By definition, the property of being countably $\mathscr{H}^k-$rectifiable is *intrinsic*, i.e. if (X, d) is isometrically embedded in another metric space (X', d'), then $\Gamma \subseteq X$ is countably $\mathscr{H}^k-$rectifiable in X if, and only if, Γ is countably $\mathscr{H}^k-$rectifiable in X'. ◊

Here is a slightly different characterization of rectifiable sets that uses as A_i compact sets and that shows that the collection $\{f_i(A_i)\}$ can be disjoint [Lan07, Proposition 9.2, p.20].

Theorem A.5.8. *Suppose X is a locally complete metric space and $\Gamma \subseteq X$ an $\mathscr{H}^k-$measurable and countably $\mathscr{H}^k-$rectifiable set. Then there exists a countable family of bi-Lipschitz*[3] *maps $f_i : K_i \to f(K_i) \subseteq \Gamma$, with $K_i \subseteq \mathbb{R}^k$ compact, such that the images $f_i(K_i)$ are pairwise disjoint and*

$$\mathscr{H}^k\left(\Gamma \setminus \bigcup_i f_i(K_i)\right) = 0.$$

For subsets of Euclidean spaces, using Theorem A.5.5, one has the following characterization of countably $\mathscr{H}^k-$rectifiable sets [Sim83, Lemma 11.1, p.59].

Theorem A.5.9. *Let $E \subseteq \mathbb{R}^n$, where $n \geq k$. Then, E is countably $\mathscr{H}^k-$rectifiable if, and only if, there exists a sequence of $k-$dimensional C^1-submanifolds N_i of $\mathbb{R}^n$ such that*

$$\mathscr{H}^k\left(E \setminus \bigcup_i N_i\right) = 0.$$

More generally, in arbitrary complete Riemannian manifolds, one has important characterizations of rectifiability for both sets and measures in terms of *approximate tangent spaces*. In what follows, we will give a brief account of this topic. For a detailed discussion of the concept of rectifiablity and its characterizations in Euclidean spaces the reader is encouraged to see DeLellis' lecture notes [DL08] (also see Simon's notes [Sim83, p.60-66]). Here we will adapt the relevant definitions and results to the context of Riemannian manifolds.

In what follows, let (M, g) be a connected, *complete*, Riemannian $n-$manifold. For each $s \in \mathbb{R}_{\geq 0}$, we let $\mathscr{H}^s$ denote the $s-$dimensional Hausdorff measure on M associated to the induced Riemannian distance function d_g.

Definition A.5.10 ($s-$tangent measures)**.** Let ν be a Radon measure on M, and let $s \in \mathbb{R}_{\geq 0}$. Given $x \in M$ and $\lambda \in \mathbb{R}_+$, we write τ_λ for the linear scaling map on T_xM taking v to λv, and define the scaled and translated measure $\nu_{x,\lambda} := (\exp_x \circ \tau_\lambda)^* \nu$ on T_xM by

$$\nu_{x,\lambda}(E) = \nu(\exp_x(\lambda E)), \quad \forall E \subseteq T_xM.$$

We say that a Radon measure η on T_xM is a $s-$**tangent measure of** ν **at** x when there exists a null-sequence $\{\lambda_i\} \subseteq \mathbb{R}_+$ such that

$$\lambda_i^{-s} \nu_{x,\lambda_i} \rightharpoonup \eta.$$

We let $\mathrm{Tan}_s(\nu, x)$ denote the set of all $s-$tangent measures of ν at x.

[3] i.e. f_i is Lipschitz, injective and such that $f_i^{-1}|_{f_i(K_i)}$ is Lipschitz.

Remark A.5.11. Note that if $(M,g) = (\mathbb{R}^n, g_0)$, where g_0 is the standard Euclidean metric, then $\nu_{x,\lambda}(E) = \nu(x+\lambda E)$. ♢

Theorem A.5.12 (Marstrand)**.** *Let ν be a Radon measure on M, let $s \in \mathbb{R}_{\geq 0}$ and let $\Gamma \subseteq M$ be a Borel set with $\nu(\Gamma) > 0$. Suppose*

$$0 < \Theta^s_*(\nu, x) = \Theta^{*s}(\nu, x) < \infty \quad \textit{for } \nu\textit{−a.e. } x \in \Gamma.$$

Then $s = k \in \mathbb{N}_0$. Moreover, for ν−a.e. $x \in \Gamma$, there exists a k−dimensional subspace $V_x \leq T_xM$ such that $\Theta^k(\nu,x)\mathscr{H}^k\lfloor V_x \in \mathrm{Tan}_k(\nu, x)$.

From now on k will denote a nonnegative integer.

Definition A.5.13 (Approximate tangent spaces)**.** Let $\Gamma \subseteq M$ be an $\mathscr{H}^k$−measurable set, and let $\Theta : \Gamma \to]0,\infty[$ be a locally $\mathscr{H}^k$−integrable function. A k−dimensional subspace $V_x \leq T_xM$ is called the **approximate k−tangent space** for Γ at x with multiplicity $\Theta(x)$ if $\mathrm{Tan}_k(\Theta\mathscr{H}^k\lfloor\Gamma, x) = \{\Theta(x)\mathscr{H}^k\lfloor V_x\}$, i.e. if

$$\lambda^{-k}(\Theta\mathscr{H}^k\lfloor\Gamma)_{x,\lambda} \rightharpoonup \Theta(x)\mathscr{H}^k\lfloor V_x \quad \text{as } \lambda \downarrow 0.$$

Let ν be a Radon measure on M. A k−dimensional subspace $V_x \leq T_xM$ is called the **approximate k−tangent space** for ν at x with multiplicity $\Theta(x) \in]0,\infty[$ if $\mathrm{Tan}_k(\nu,x) = \{\Theta(x)\mathscr{H}^k\lfloor V_x\}$, i.e. if

$$\lambda^{-k}\nu_{x,\lambda} \rightharpoonup \Theta(x)\mathscr{H}^k\lfloor V_x \quad \text{as } \lambda \downarrow 0.$$

Remark A.5.14. Let ν be a Radon measure on M and let $x \in M$. We claim that if $\eta := \Theta(x)\mathscr{H}^k\lfloor V_x \in \mathrm{Tan}_k(\nu,x)$ for some $\Theta(x) \in]0,\infty[$ and some k−dimensional subspace $V_x \leq T_xM$, then $\Theta^k(\nu,x)$ exists and equals $\Theta(x)$.

Pick $r \in \mathbb{R}_+$ such that $\eta(\partial B_r(0)) = 0$. Then

$$\begin{aligned}\Theta(x)r^k = \eta(B_r(0)) &= \lim_{\lambda\downarrow 0}\lambda^{-k}\nu_{x,\lambda}(B_r(0))\\ &= \lim_{\lambda\downarrow 0}\lambda^{-k}\nu(B_{\lambda r}(x)).\end{aligned}$$

Therefore,

$$\Theta(x) = \lim_{\lambda\downarrow 0}(\lambda r)^{-k}\nu(B_{\lambda r}(x)) = \lim_{\delta\downarrow 0}\delta^{-k}\nu(B_\delta(x)),$$

from which our claim follows, since $\Theta(x) \in]0,\infty[$. ♢

We are now in position to state a number of rectifiability criteria.

Theorem A.5.15. *Let $\Gamma \subseteq M$ be an $\mathscr{H}^k$−measurable set. Then, Γ is countably $\mathscr{H}^k$−rectifiable if, and only if, there exists an $\mathscr{H}^k$−integrable function $\Theta : \Gamma \to]0,\infty[$ such that $\nu := \Theta\mathscr{H}^k\lfloor\Gamma$ has approximate k−tangent space V_x for $\mathscr{H}^k$−a.e. $x \in \Gamma$.*

Theorem A.5.16. *Let ν be a Radon measure on M. Then, ν is $\mathscr{H}^k-$rectifiable if, and only if, for $\nu-$a.e. $x \in M$, there exist a positive constant $\Theta(x) \in]0,\infty[$ and a $k-$dimensional subspace $V_x \leq T_xM$ such that V_x is the approximate $k-$tangent space for ν at x with multiplicity $\Theta(x)$.*

The last result we cite is highly non-trivial and was proved in [Pre87] by David Preiss.

Theorem A.5.17 (Preiss)**.** *Let ν be a locally finite Borel measure on $\mathbb{R}^n$. Suppose that, for $k \in \mathbb{N}$, $k \leq n$,*

$$0 < \Theta^k_*(\nu, x) = \Theta^{k^*}(\nu, x) < \infty, \quad \textit{for } \nu\textit{−a.e. } x \in \operatorname{spt}(\nu).$$

Then ν is $\mathscr{H}^k-$rectifiable, i.e. $\nu \ll \mathscr{H}^k$ and $\operatorname{spt}(\nu)$ is countably $\mathscr{H}^k-$rectifiable.

A.6 Currents

In this section we introduce the basics about de Rham's theory of currents. Our goal is to establish the rectifiability Theorem A.6.17. We develop the theory in the framework of open subsets of $\mathbb{R}^n$ and at the end we explain how to pass from this context to arbitrary smooth manifolds.

The spaces $\Omega^k(U)$ and $\mathscr{D}^k(U)$. Let U be an open subset of $\mathbb{R}^n$. For each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ we associate the differential operator

$$D^\alpha := \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \circ \ldots \circ \left(\frac{\partial}{\partial x^n}\right)^{\alpha_n},$$

whose *order* is

$$|\alpha| := \alpha_1 + \ldots + \alpha_n.$$

If $|\alpha| = 0$, then $D^\alpha = \mathbb{1}$.

As usual, we denote by $\Omega^k(U)$ the real vector space of *smooth* $k-$forms on U. We topologize $\Omega^k(U)$ with the $C^\infty_{\mathbf{loc}}-$**topology** which makes $\Omega^k(U)$ into a Fréchet space[4]. This is done by choosing an exhaustion of U by compact sets $\{K_i\}_{i\in\mathbb{N}}$ and defining, for each $i \in \mathbb{N}$, the seminorm $p_i : \Omega^k(U) \to \mathbb{R}_{\geq 0}$ given by

$$p_i(\varphi) := \sup\left\{|(D^\alpha \varphi_{j_1,\ldots,j_k})(x)| : x \in K_i,\ |\alpha| \leq i,\ 1 \leq j_1 < \ldots < j_k \leq n\right\},$$

for all

$$\varphi = \sum_{j_1<\ldots<j_k} \varphi_{j_1,\ldots,j_k} \mathrm{d}x^{j_1\cdots j_k} \in \Omega^k(U).$$

[4] i.e. a locally convex topological vector space whose topology is induced by a translation-invariant metric which makes the space complete.

Then $\mathscr{P} := \{p_i\}$, being a countable separating family of seminorms on $\Omega^k(U)$, defines a metrizable locally convex topology on $\Omega^k(U)$ admitting a translation-invariant compatible metric (see [Rud91, Theorem 1.37 and Remark (c) of Section 1.38]). A local base for 0 is given by the sets

$$V_i := \left\{\varphi \in \Omega^k(U) : p_i(\varphi) < \frac{1}{i}\right\}, \quad i \in \mathbb{N}.$$

One may readily check that every Cauchy sequence on $\Omega^k(U)$ has a limit in $\Omega^k(U)$, so that $\Omega^k(U)$ is in fact a Fréchet space.

For each compact $K \subseteq U$,

$$\mathscr{D}^k_K(U) := \Omega^k(U) \cap \{\varphi : \mathrm{supp}(\varphi) \subseteq K\}$$

is a closed subspace of $\Omega^k(U)$, and therefore is also a Fréchet space. The union of the spaces $\mathscr{D}^k_K(U)$, as K ranges over all compact subsets of U, is denoted by

$$\mathscr{D}^k(U) := \Omega^k(U) \cap \{\varphi : \ \varphi \text{ has compact support in } U\}.$$

This is clearly a vector space under the usual operations. We endow $\mathscr{D}^k(U)$ with the largest topology making the inclusion maps $\mathscr{D}^k_K(U) \hookrightarrow \mathscr{D}^k(U)$ continuous (cf. [Fed78, §6]); this is called the C^∞**−topology** on $\mathscr{D}^k(U)$. It can be shown that this topology makes $\mathscr{D}^k(U)$ into a locally convex topological vector space. Moreover:

Proposition A.6.1. *Given* $\{\varphi^i\}_{i\in\mathbb{N}} \subseteq \mathscr{D}^k(U)$*, where*[5]

$$\varphi^i = \sum_J \varphi^i_J \mathrm{d}x^J, \quad \text{for each } i \in \mathbb{N},$$

then $\varphi^i \to 0$ *in* $\mathscr{D}^k(U)$ *if, and only if, the following holds:*

(i) there exists a compact subset $K \subseteq U$ *with supp*$(\varphi^i) \subseteq K$*, for all* $i \in \mathbb{N}$*.*

(ii) $\sup_{x\in K} |(D^\alpha \varphi^i_J)(x)| \to 0$ *as* $i \to \infty$*, for all* J *and* $\alpha \in \mathbb{N}^n_0$*.*

Proposition A.6.2. *Let* $T : \mathscr{D}^k(U) \to Y$ *be a linear map into a locally convex space* Y*. Then the following are equivalent:*

(a) T *is continuous.*

(b) If $\varphi^i \to 0$ *in* $\mathscr{D}^k(U)$ *then* $T\varphi^i \to 0$ *in* Y*.*

Remark A.6.3. The approach given above for the spaces $\Omega^k(U)$ and $\mathscr{D}^k(U)$ is an adaptation of Rudin's approach [Rud91, §1.46 and §6.2-6.8] for the corresponding spaces of functions. In this spirit, the reader interested in a proof of the above results may want to compare Proposition A.6.1 with [Rud91, Theorem 6.5 (f), pp. 154-155], and Proposition A.6.2 with [Rud91, Theorem 6.6 (a),(c), p. 155]. ◊

[5] Here the sum runs over all $J = \{1 \leq j_1 < \ldots < j_k \leq n\}$.

An element $\varphi \in \mathscr{D}^k(U)$ is also known as a *test form*.

Definition A.6.4 (Current)**.** A $k-$**current** T on U is an element of the topological dual $\mathscr{D}_k(U) := \left(\mathscr{D}^k(U)\right)'$, i.e. a continuous linear functional of $\mathscr{D}^k(U)$. When $k = 0$ we use the notations $\mathscr{D}(M) := \mathscr{D}^0(U)$ and $\mathscr{D}'(M) := \mathscr{D}_0(U)$.

$0-$currents are also known as *distributions*.

Definition A.6.5 (Weak* convergence)**.** A sequence of $k-$currents $\{T_i\} \subseteq \mathscr{D}_k(U)$ **converges weakly*** to $T \in \mathscr{D}_k(U)$, and we write $T_i \rightharpoonup T$, if

$$\lim_{i\to\infty} T_i(\varphi) = T(\varphi), \quad \forall \varphi \in \mathscr{D}^k(U).$$

Let $T \in \mathscr{D}_k(M)$. The **support** $\operatorname{spt}(T)$ of T is the intersection of all closed subsets $F \subseteq M$ satisfying:

$$\operatorname{spt}(\varphi) \cap F = \emptyset,\ \varphi \in \mathscr{D}^k(M) \quad \Longrightarrow \quad T(\varphi) = 0.$$

Note that every compactly supported $k-$current extends to a continuous linear functional on $\Omega^k(U)$.

Definition A.6.6 (Boundary)**.** Let $k \in \mathbb{N}$. If $T \in \mathscr{D}_k(U)$, the **boundary** of T is the current $\partial T \in \mathscr{D}_{k-1}(U)$ given by

$$\partial T(\varphi) := T(\mathrm{d}\varphi), \quad \forall \varphi \in \mathscr{D}^{k-1}(U).$$

We define the boundary of a $0-$current to be the zero function. A current $T \in \mathscr{D}_k(U)$ is said to be **closed** if $\partial T = 0$.

Remark A.6.7. We list some elementary observations concerning Definition A.6.6.

- $\partial \circ \partial = 0$, as a direct consequence of $\mathrm{d} \circ \mathrm{d} = 0$. In particular, there is an associated complex:
$$\ldots \to \mathscr{D}^{k+1}(U) \xrightarrow{\partial} \mathscr{D}^k(U) \xrightarrow{\partial} \mathscr{D}^{k-1}(U) \to \ldots$$
- $\operatorname{supp}(\partial T) \subseteq \operatorname{supp}(T)$ (since $\operatorname{supp}(\mathrm{d}\eta) \subseteq \operatorname{supp}(\eta)$);
- $T_i \rightharpoonup T \Longrightarrow \partial T_i \rightharpoonup \partial T$: indeed, given $\varphi \in \mathscr{D}^{k-1}(U)$ we have
$$\partial T_i(\varphi) = T_i(\mathrm{d}\varphi) \to T(\mathrm{d}\varphi) = \partial T(\varphi),$$
whenever $T_i \rightharpoonup T$. ♢

The concept of a $k-$current on U is the measure-geometric generalization of the concept of an oriented $k-$submanifold on U with locally finite $k-$dimensional Hausdorff measure. This is the motivation for the definition of boundary for currents we have given above, as the following example illustrates:

Example A.6.8. For $k \geq 1$, let $N^k \subseteq U$ be an oriented $k-$submanifold with boundary ∂N in U and orientation ξ. Suppose $\mathscr{H}^k \lfloor N$ is locally finite (or, equivalently, a Radon measure). Then, N naturally induces a $k-$current $[[N]]$ on U given by

$$[[N]](\varphi) := \int_N \langle \varphi, \xi(x) \rangle \alpha_k \mathrm{d}\mathscr{H}^k = \int_N \varphi, \quad \forall \varphi \in \mathscr{D}^k(U).$$

Analogously, the boundary of N with the induced orientation induces a $(k-1)-$current $[[\partial N]]$ on U. Now, for each $\phi \in \mathscr{D}^{k-1}(U)$, by Stokes' theorem we have

$$[[\partial N]](\phi) = \int_{\partial N} \phi = \int_N \mathrm{d}\phi = [[N]](\mathrm{d}\phi) = \partial[[N]](\phi).$$

This shows that the boundary of the current determined by N, as per Definition A.6.6, equals the current determined by the boundary ∂N of N.

Definition A.6.9 (Total variation measure and mass of a current)**.** Let $T \in \mathscr{D}_k(U)$. For an open subset $W \subseteq U$ and any $A \subseteq U$, we define

$$\|T\|(W) := \sup\{T(\varphi) : \mathrm{spt}(\varphi) \subseteq W,\ \|\varphi\|_{C^0} \leq 1\},$$

$$\|T\|(A) := \inf\{\|T\|(W) : A \subseteq W,\ W \text{ open}\}.$$

The resulting Borel regular (outer) measure $\|T\|$ is called the **total variational measure** of T. The (extended) number

$$\mathbf{M}(T) := \|T\|(U) = \sup\{T(\varphi) : \|\varphi\|_{C^0} \leq 1, \varphi \in \mathscr{D}^k(U)\} \in [0, \infty].$$

is called the **mass** of T.

Note that when $T = [[N]]$ is induced by a $k-$submanifold $N \subseteq U$ as in the above example, the total variation measure of T is simply $\|T\| = \mathscr{H}^k \lfloor N$, which shows that the mass generalizes the area of a submanifold.

Definition A.6.10 (Finite mass currents)**.** Let $k \in \mathbb{N}_0$. We define the space $\mathbf{M}_k(U)$ of **finite mass** $k-$currents on U by

$$\mathbf{M}_k(U) := \{T \in \mathscr{D}_k(U) : \|T\| \text{ is a finite measure}\}.$$

$\mathbf{M}_k(U)$ has a natural structure of normed space induced by the **mass norm** $\mathbf{M}(T) := \|T\|$.

More generally, we define the space $\mathbf{M}_{k,\mathrm{loc}}(U)$ of **locally finite mass** $k-$currents on U by

$$\mathbf{M}_{k,\mathrm{loc}}(U) := \{T \in \mathscr{D}_k(U) : \|T\| \text{ is a Radon measure}\}.$$

$\mathbf{M}_{k,\mathrm{loc}}(U)$ has a natural topology induced by the family of semi-norms $\{\mathbf{M}_W\}_{W \Subset U}$, where $\mathbf{M}_W(T) := \|T\|(W)$.

A current $T \in \mathscr{D}_k(U)$ is said to be **representable by integration** when there exist a Radon measure μ_T over U and a μ_T-measurable function $\xi : U \to \Lambda^k\mathbb{R}^n$, with $|\xi|= 1$ μ_T-a.e., such that

$$T(\varphi) = \int_U \langle \varphi, \xi \rangle \mathrm{d}\mu_T.$$

In such case, one may prove that $\mu_T = \|T\|$. In particular, when T is representable by integration then $T \in \mathbf{M}_{k,\mathrm{loc}}(U)$. The converse follows from the Riesz representation theorem (Theorem A.4.1). Thus:

Theorem A.6.11 (Integral representation)**.** *Let $T \in \mathscr{D}_k(U)$. Then, T is representable by integration if, and only if, $T \in \mathbf{M}_{k,\mathrm{loc}}(U)$.*

Moreover, we have the following standard weak* compactness result, which follows from the standard Banach-Alaoglu theorem (cf. [Sim83, Lemma 26.14, p. 135]).

Lemma A.6.12. *Let $\{T_i\} \subseteq \mathbf{M}_{k,\mathrm{loc}}(U)$ be such that*

$$\sup_{i\geq 1}\|T_i\|(W) < \infty, \qquad \textit{for each } W \Subset U.$$

Then, after passing to a subsequence, there exists $T \in \mathbf{M}_{k,\mathrm{loc}}(U)$ such that $T_i \rightharpoonup T$.

Next we introduce various important spaces of currents.

Definition A.6.13 (Normal currents)**.** Let $k \in \mathbb{N}_0$. We define the space $\mathbf{N}_k(U)$ of **normal** $k-$currents on U by

$$\mathbf{N}_k(U) := \{T \in \mathscr{D}_k(U) : \|T\|+\|\partial T\| \text{ is a finite measure}\}.$$

$\mathbf{N}_k(U)$ has a natural structure of normed space induced by $\mathbf{N}(T) := \mathbf{M}(T) + \mathbf{M}(\partial T)$.

More generally, we define the space $\mathbf{N}_{k,\mathrm{loc}}(U)$ of **locally normal** $k-$currents on U by

$$\mathbf{N}_{k,\mathrm{loc}}(U) := \{T \in \mathscr{D}_k(U) : \|T\|+\|\partial T\| \text{ is a Radon measure}\}.$$

$\mathbf{N}_{k,\mathrm{loc}}(U)$ has a natural topology induced by the family of semi-norms $\{\mathbf{N}_W\}_{W\Subset U}$, where $\mathbf{N}_W(T) := \mathbf{M}_W(T) + \mathbf{M}_W(\partial T)$.

Definition A.6.14 (Integer rectifiable currents)**.** A $k-$current $T \in \mathscr{D}_k(U)$ is called **locally integer rectifiable** if there is a triple (Γ, ξ, Θ) such that:

(i) $\Gamma \subseteq U$ is $\mathscr{H}^k-$measurable and countably $\mathscr{H}^k-$rectifiable;

(ii) $\Theta : \Gamma \to [0, \infty[$ is locally $\mathscr{H}^k-$integrable and such that $\Theta(x) \in \mathbb{Z}$ for $\mathscr{H}^k-$a.e. $x \in \Gamma$;

(iii) $\xi : \Gamma \to \Lambda^k\mathbb{R}^n$ is $\mathscr{H}^k-$measurable and such that $\xi(x)$ *orients* the approximate $k-$tangent space $T_x\Gamma$ for $\mathscr{H}^k-$a.e. $x \in \Gamma$, that is, for $\mathscr{H}^k-$a.e. $x \in \Gamma$, $\xi(x) \in \Lambda^k\mathbb{R}^n$ is simple, unitary and represents the approximate $k-$tangent space $T_x\Gamma$;

(iv) the current T is given by

$$T(\varphi) \coloneqq \int_\Gamma \langle \varphi, \xi \rangle \Theta \mathrm{d}\mathscr{H}^k, \quad \forall \varphi \in \mathscr{D}^k(U).$$

We call Θ the **multiplicity** of T and ξ the **orientation** of T; we write $T = (\Gamma, \xi, \Theta)$.

The set of *locally integer rectifiable* $k-$currents on U is denoted by $\mathbf{R}_{k,\mathrm{loc}}(U)$. The set of **integer rectifiable** $k-$currents on U is defined by

$$\mathbf{R}_k(U) \coloneqq \mathbf{R}_{k,\mathrm{loc}}(U) \cap \mathbf{M}_k(U).$$

Remark A.6.15. In the literature, the space $\mathbf{R}_{k,\mathrm{loc}}(U)$ is sometimes simply called the space of *locally rectifiable* $k-$currents on U (note the missing of 'integer'). ♢

In general, if $T \in \mathbf{R}_{k,\mathrm{loc}}$ it need not be true that $\partial T \in \mathbf{R}_{k-1,\mathrm{loc}}$.

Definition A.6.16 (Integral currents)**.** The space of **locally integral** $k-$currents on U is defined by

$$\mathbf{I}_{k,\mathrm{loc}}(U) \coloneqq \{T \in \mathbf{R}_{k,\mathrm{loc}} : \partial T \in \mathbf{R}_{k-1,\mathrm{loc}}\}, \quad \text{if } k \geq 1,$$

and we set

$$\mathbf{I}_{0,\mathrm{loc}}(U) \coloneqq \mathbf{R}_{0,\mathrm{loc}}.$$

The space $\mathbf{I}_k(U)$ of **integral** $k-$currents on U is defined by

$$\mathbf{I}_k(U) \coloneqq \mathbf{I}_{k,\mathrm{loc}}(U) \cap \mathbf{N}_k(U).$$

The following theorem gives an important criterion for a $k-$current to be rectifiable [Sim83, Theorem 32.1, pp. 183-187].

Theorem A.6.17 (Rectifiability Theorem)**.** *If $T \in \mathscr{D}_k(U)$ is such that*

(i) $T \in \mathbf{N}_{k,\mathrm{loc}}(U)$ (i.e. $\|T\|(W) + \|\partial T\|(W) < \infty$, $\forall W \Subset U$), and

(ii) $\Theta^{*k}(\|T\|, x) > 0$ for $\|T\|-$a.e. $x \in U$,

then T is rectifiable, *i.e. T is defined by a triple (Γ, ξ, Θ), in the sense that*

$$T(\varphi) = \int_\Gamma \langle \varphi, \xi \rangle \Theta \mathrm{d}\mathscr{H}^k \lfloor \Gamma, \quad \forall \varphi \in \mathscr{D}^k(U),$$

where

1. $\Gamma \subseteq U$ is $\mathscr{H}^k-$measurable and countably $\mathscr{H}^k-$rectifiable;

2. *$\Theta : \Gamma \to [0, \infty[$ is locally $\mathscr{H}^k-$integrable;*

3. *$\xi : \Gamma \to \Lambda^k \mathbb{R}^n$ is $\mathscr{H}^k-$measurable and such that $\xi(x)$ orients the approximate $k-$tangent space $T_x\Gamma$ for $\mathscr{H}^k-$a.e. $x \in \Gamma$.*

Definition A.6.18 (Cycles, boundaries, etc.)**.** For $k \geq 1$, define the abelian groups

$$\mathscr{Z}_k(U) \coloneqq \{T \in \mathbf{I}_k(U) : \partial T = 0\},$$
$$\mathscr{B}_k(U) \coloneqq \{\partial S : S \in \mathbf{I}_{k+1}(U)\} \subseteq \mathscr{Z}_k(U).$$

An element of $\mathscr{Z}_k(U)$ is called a **cycle**; an element of $\mathscr{B}_k(U)$ is called a **boundary**. Two cycles $T, T' \in \mathbf{I}_k(U)$ are called **homologous** if $T - T'$ is a boundary.

Currents on manifolds. We now explain how the definition of currents on open subsets of Euclidean spaces can be transported to general manifolds. The key observation is the following. Let $F : U \to V$ be a coordinate change between two coordinate systems $(U, x^1, \dots, x^n)$ and $(V, y^1, \dots, y^n)$, where $U, V \subseteq \mathbb{R}^n$ are open subsets. Thus, F is a diffeomorphism such that $x^i = y^i \circ F$, for each $i = 1, \dots, n$.

Claim. Let $F : U \to V$ be a diffeomorphism; then the natural induced map

$$F^* : \mathscr{D}^k(V) \to \mathscr{D}^k(U)$$

is an isomorphism of topological vector spaces.

Since F is a diffeomorphism, it is clear that F^* is a linear isomorphism. Moreover, since $(F^*)^{-1} = (F^{-1})^*$, in order to prove F^* is a homeomorphism it suffices to show that F^* is continuous (then the same argument will apply to the inverse map switching the roles of F and F^{-1}). By Proposition A.6.2, showing the continuity of F^*, in turn, boils down to proving the following: if $\phi^i \to 0$ in $\mathscr{D}^k(V)$ then $\varphi^i \coloneqq F^*(\phi^i) \to 0$ in $\mathscr{D}^k(U)$.

Now suppose that $\phi^i \to 0$ in $\mathscr{D}^k(V)$. In particular, by Proposition A.6.1 (i), there exists a compact subset $\tilde{K} \subseteq V$ such that $\mathrm{supp}(\phi^i) \subseteq \tilde{K}$ for each i. It follows that the compact subset $K \coloneqq F^{-1}(\tilde{K}) \subseteq U$ is such that $\mathrm{supp}(\varphi^i) \subseteq K$, for each i. We write

$$\phi^i = \sum_J \phi^i_J \mathrm{d}y^J,$$

so that

$$\varphi^i = \sum_J \left(\phi^i_J \circ F\right) \mathrm{d}x^J \eqqcolon \sum_J \varphi^i_J \mathrm{d}x^J.$$

Since

$$\frac{\partial}{\partial y^i} = F_* \frac{\partial}{\partial x^i},$$

it follows from Proposition A.6.1 (ii) that

$$\sup_{x \in K} |(D^\alpha \varphi^i_J)(x)| = \sup_{y \in \tilde{K}} |(\tilde{D}^\alpha \phi^i_J)(y)| \to 0 \quad \text{as } i \to \infty,$$

for all J and $\alpha \in \mathbb{N}_0^n$, where

$$\begin{aligned} D^\alpha &:= \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \circ \ldots \circ \left(\frac{\partial}{\partial x^n}\right)^{\alpha_n}, \quad \text{and} \\ \tilde{D}^\alpha &:= \left(\frac{\partial}{\partial y^1}\right)^{\alpha_1} \circ \ldots \circ \left(\frac{\partial}{\partial y^n}\right)^{\alpha_n}. \end{aligned}$$

Therefore, by Proposition A.6.1, $\varphi^i := F^*(\phi^i) \to 0$ in $\mathscr{D}^k(U)$. This proves the continuity of F^*, completing the proof of the claim.

Now let M be a smooth $n-$manifold. Then, by making use of local charts on M, and the above observation, we get a well-defined $C^\infty-$topology on the space $\mathscr{D}^k(M)$ of **smooth compactly supported** $k-$**forms** on M. Thus we can define:

Definition A.6.19. A $k-$**current** T on M is an element of the topological dual $\mathscr{D}_k(M) := \big(\mathscr{D}^k(M)\big)'$, i.e. a continuous linear functional $T : \mathscr{D}^k(M) \to \mathbb{R}$.

$\mathscr{D}_k(M)$ endowed with the weak* topology is called the **space of** $k-$**currents on** M.

The previous definitions and results of this section are then obviously adapted to this context.

Appendix B

Background analysis

In this brief appendix we collect some basic terminology and facts from analysis of PDEs which are specially used in Sections 1.1, 1.4 and 3.1. The main references for this appendix are [Weh04] and [Nic].

B.1 Partial differential operators

In this section, we provide some definitions concerning (linear) partial differential operators (PDOs) on manifolds. We follow the algebraic point of view of Nicolaescu's lecture notes [Nic, Chapter 10].

Let $E_i \to M$ be a $\mathbb{K}-$vector bundle over a smooth manifold M, $i = 1, 2$. We start letting $\mathbf{Op}(E_1, E_2)$ be the natural $\mathbb{K}-$vector space whose underlying set is given by

$$\mathbf{Op}(E_1, E_2) \coloneqq \{P : \Gamma(E_1) \to \Gamma(E_2) : P \text{ is } \mathbb{K} - \text{linear}\}.$$

In what follows, we will regard PDOs from sections of E_1 to sections of E_2 as elements of $\mathbf{Op}(E_1, E_2)$ that interact in a specific way with the $C^\infty(M, \mathbb{K})-$module structure of $\Gamma(E_1)$ and $\Gamma(E_2)$.

Let $\mathrm{Hom}(E_1, E_2)$ denote the space of vector bundle homomorphisms from E_1 to E_2, i.e. the space of all $P \in \mathbf{Op}(E_1, E_2)$ such that P is $C^\infty(M, \mathbb{K})-$linear. Then we can write

$$\mathrm{Hom}(E_1, E_2) = \{P \in \mathbf{Op}(E_1, E_2) : \mathrm{ad}(f)P = 0, \forall f \in C^\infty(M, \mathbb{K})\} =: \ker \mathrm{ad},$$

where

$$\begin{aligned} \mathrm{ad}(f) : \mathbf{Op}(E_1, E_2) &\to \mathbf{Op}(E_1, E_2) \\ P &\mapsto [P, f] \coloneqq P \circ f - f \circ P. \end{aligned}$$

Here we are regarding f as the natural $C^\infty(M, \mathbb{K})-$module multiplication operator it induces on $\Gamma(E_1)$ and $\Gamma(E_2)$ where appropriate.

Definition B.1.1 (PDOs)**.** Let $E_1, E_2 \to M$ be $\mathbb{K}-$vector bundles. For each $m \in \mathbb{N}_0$, we let

$$\mathbf{PDO}^{(m)}(E_1, E_2) := \{P \in \mathbf{Op}(E_1, E_2) : \ \mathrm{ad}(f_0)\mathrm{ad}(f_1)\cdots\mathrm{ad}(f_m)P = 0, \ \forall f_i \in C^\infty(M, \mathbb{K})\}$$

and we set

$$\mathbf{PDO}(E_1, E_2) := \bigcup_{m \geq 0} \mathbf{PDO}^{(m)}(E_1, E_2).$$

An element $P \in \mathbf{PDO}(E_1, E_2)$ is called a **partial differential operator** from E_1 to E_2.

Definition B.1.2 (Formal adjoint)**.** Suppose (M, g) is an oriented Riemannian manifold and let $E_i \to M$ be a $\mathbb{K}-$vector bundle over M endowed with a metric $\langle\cdot,\cdot\rangle_i$, $i = 1, 2$. Given $P \in \mathbf{PDO}(E_1, E_2)$, we say that $Q \in \mathbf{PDO}(E_2, E_1)$ is a **formal adjoint** of P whenever

$$\int_M \langle Pu, v\rangle_2 \mathrm{d}V_g = \int_M \langle u, Qv\rangle_1 \mathrm{d}V_g,$$

for each $u \in \Gamma(E_1)$ and $v \in \Gamma(E_2)$ one of which has compact support[1] in M.

Proposition B.1.3 (Existence and uniqueness of formal adjoints)**.** *Suppose (M, g) is an oriented Riemannian manifold and let $E_i \to M$ be a $\mathbb{K}-$vector bundle over M endowed with a metric $\langle\cdot,\cdot\rangle_i$, $i = 1, 2$. Then for any $P \in \mathbf{PDO}(E_1, E_2)$ there exists a unique formal adjoint $P^* \in \mathbf{PDO}(E_2, E_1)$ of P.*

B.2 Sobolev spaces

We now introduce Sobolev spaces of sections of *vector* bundles, and state the corresponding so-called Sobolev embedding theorems. This is the minimal background material to deal with Sobolev spaces of connections on $G-$bundles (see §1.1 of Chapter 1). For a definition of Sobolev spaces of sections of general fiber bundles[2], as well as other details, we refer the reader to [Weh04, Appendix B].

Let M be an oriented $n-$manifold endowed with a Riemannian metric g, and let $\pi : F \to M$ be a $\mathbb{K}-$vector bundle over M endowed with a metric $h = \langle\cdot,\cdot\rangle$ and associated pointwise norm $|\cdot|$. Henceforth, we use the notations introduced in Chapter 1.

Definition B.2.1 (L^p-sections)**.** Let $1 \leq p \leq \infty$. We define the **Lebesgue space** $L^p(M, F)$ of L^p-sections of $F \to M$ to be the natural $\mathbb{K}-$vector space whose underlying set consists of all (equivalence classes, modulo the relation of equality μ_g-almost everywhere, of) Borel measurable maps $u : M \to F$ such that the following holds.

[1] When $\partial M \neq \emptyset$, one assumes that the compact support lies in the *interior* of the manifold M.

[2] This would cover, for instance, the case of Sobolev spaces of gauge transformations of $G-$bundles.

(i) $(\pi \circ u)(x) = x$, for μ_g-almost all $x \in M$.

(ii) The function $|u| : M \to \mathbb{R}$ defines an element in $L^p(\mu_g)$.

The L^p-**norm** $\|\cdot\|_p$ on $L^p(M, F)$ is given by

$$\|u\|_p := \begin{cases} \left(\int_M |u|^p \mathrm{d}V_g\right)^{1/p}, & \text{if } 1 \le p < \infty, \\ \operatorname{ess\,sup}_M |s|, & \text{if } p = \infty. \end{cases}$$

More generally, we define the space $L^p_{\mathrm{loc}}(M, F)$ of *locally* L^p-integrable sections of $F \to M$ by

$$L^p_{\mathrm{loc}}(M, F) := \{u : fu \in L^p(M, F) \text{ for all } f \in C_c^\infty(M)\}.$$

Given an exhaustion $\{\Omega_i\}$ of M by precompact open subsets $\Omega_i \Subset M$, the space $L^p_{\mathrm{loc}}(M, F)$ is endowed with the natural Fréchet topology induced by the family of seminorms $\{\|\cdot\|_{p,\Omega_i}\}_{i\in\mathbb{N}}$ given by

$$\|u\|_{p,\Omega_i} := \int_{\Omega_i} |u|^p \mathrm{d}V_g, \quad \forall u \in L^p_{\mathrm{loc}}(M, F).$$

Remark B.2.2. Suppose $1 \le p, q \le \infty$ are *Hölder conjugate*, i.e. $1/p + 1/q = 1$, and let $u \in L^p(M, F)$ and $v \in L^q(M, F)$. Then, using the Cauchy-Schwarz inequality together with Hölder's inequality for functions, one gets

$$\int_M |\langle u, v\rangle| \mathrm{d}V_g \le \|u\|_p \|v\|_q.$$

More generally, let $F_i \to M$ $(i = 1, \dots, l)$ be vector bundles with metrics and consider $F_1^* \otimes \ldots \otimes F_l^*$ endowed with the induced tensor product metric. If $\Omega \in L^{p_0}(F_1^* \otimes \ldots \otimes F_l^*)$ for some $1 \le p_0 \le \infty$, then for every $1 \le p_1, \dots, p_l \le \infty$ such that

$$1 - \frac{1}{p_0} = \frac{1}{p_1} + \ldots + \frac{1}{p_l},$$

and for every $u_i \in L^{p_i}(M, F_i)$, $i = 1, \dots, l$, one can prove that

$$\int_M |\Omega(u_1, \dots, u_l)| \, \mathrm{d}V_g \le \|\Omega\|_{p_0} \|u_1\|_{p_1} \cdots \|u_l\|_{p_l}.$$

$\diamondsuit$

Lemma B.2.3. *The Lebesgue space* $(L^p(M, F), \|\cdot\|_p)$ *is a Banach space which is reflexive for* $1 < p < \infty$.

Now fix a smooth connection ∇ on F compatible with h. In what follows, we still denote by ∇ the tensor product connections (1.1.19) induced by ∇ and the Levi-Civita connection D^g of (M, g).

Definition B.2.4. Let $u \in L^1_{\text{loc}}(M,F)$ and let $v \in L^1_{\text{loc}}(M, \bigotimes^j T^*M \otimes F)$. We say that $\nabla^j u = v$ **weakly** if

$$\int_M \langle u, (\nabla^j)^*\phi \rangle \mathrm{d}V_g = \int_M \langle v, \phi \rangle \mathrm{d}V_g, \quad \forall \phi \in \Gamma_0(\bigotimes^j T^*M \otimes F),$$

where $(\nabla^j)^*$ denotes the formal adjoint of $\nabla^j \in \mathbf{PDO}^{(j)}(F, \bigotimes^j T^*M \otimes F)$.

Definition B.2.5 ($W^{k,p}-$sections)**.** Let $1 \leq p \leq \infty$ and let $k \in \mathbb{N}_0$. We define the **Sobolev space** $W^{k,p}(M,F)$ of $W^{k,p}-$sections of $F \to M$ to be the natural $\mathbb{K}-$vector space whose underlying set consists of all $u \in L^p(M,F)$ such that, for each $1 \leq j \leq k$, there exists $v_j \in L^p(M, \bigotimes^j T^*M \otimes F)$ satisfying $\nabla^j u = v_j$ weakly. The **Sobolev** $W^{k,p}-$**norm** $\|\cdot\|_{p,k}$ on $W^{k,p}(M,F)$ is given by

$$\|u\|_{k,p} := \sum_{j=0}^{k} \|\nabla^j u\|_p.$$

Note that, by definition, $W^{0,p}(M,F) = L^p(M,F)$. Moreover:

Theorem B.2.6. *$W^{k,p}(M,F)$ is a Banach space which is reflexive for $1 < p < \infty$.*

By the Banach-Alaoglu theorem of functional analysis, one gets:

Corollary B.2.7. *If $1 < p < \infty$, then every bounded sequence in $W^{k,p}(M,F)$ has a weakly convergent subsequence.*

Let $\Gamma_0(F)$ denote the space of *compactly supported* sections of $F \to M$. The next result implies that we could have defined $W^{k,p}(M,F)$ as the norm completion of $(\Gamma_0(F), \|\cdot\|_{k,p})$.

Proposition B.2.8. *If $1 \leq p < \infty$, then $\Gamma_0(F)$ is dense on $W^{k,p}(M,F)$.*

Contrary to what our notation suggests so far, the spaces $W^{k,p}(M,F)$ may heavily depend on the choices of a metric g on M, a metric h on E and, in case $k \geq 1$, the choice of a compatible connection ∇ on F. In fact, when M is non-compact, this dependence has to be seriously taken into account. On the other hand, it turns out that for *compact* base manifolds M these spaces are independent of these choices and, although their norms always depend on the various choices of g, h and (possibly) ∇ (all of which will be clear in the context), a change of choices always gives equivalent norms. Indeed, we have the following (cf. [Nic, p.251, Theorem 10.2.36]):

Theorem B.2.9. *Let $F \to M$ be a vector bundle over a* compact*, oriented, $n-$manifold M. Suppose that g_i is a Riemannian metric on M, h_i is a metric on F and that ∇_i is a smooth connection on F compatible with h_i, where $i = 1, 2$. Then we have the set equality*

$$W^{k,p}(M,F; g_1, h_1, \nabla_1) = W^{k,p}(M,F; g_2, h_2, \nabla_2)$$

and the identity map between these Banach spaces is a bounded linear map.

We finish this appendix stating the so-called Sobolev embeddings (cf. [Weh04, Theorem B.2, p. 182]). In what follows, we suppose M to be a *compact*, oriented, $n-$manifold. Moreover, for each $j \in \mathbb{N}_0$, we let $C^j(M, F)$ be the space of C^j-sections of $F \to M$, i.e. the space of all maps $u : M \to F$ of class C^j such that $\pi \circ u = \mathbb{1}_M$. We endow $C^j(M, F)$ with the uniform C^j-topology induced by the $W^{j,\infty}-$norm.

Theorem B.2.10 (Sobolev embeddings)**.** *Let $0 \leq j < k$ be integers and let $1 \leq p, q < \infty$ be real numbers.*

(i) If $k - \frac{n}{p} \geq j - \frac{n}{q}$ then the natural inclusion

$$W^{k,p}(M, F) \hookrightarrow W^{j,q}(M, F)$$

is a bounded linear map. Moreover, if strictly inequality holds this inclusion map is compact.

(ii) If $k - \frac{n}{p} > j$ then there is a compact bounded inclusion map

$$W^{k,p}(M, F) \hookrightarrow C^j(M, F).$$