Universidade Estadual de Campinas
Instituto de Computação

Italos Estilon da Silva de Souza

Stability Analysis of Hedonic Games

Análise da Estabilidade de Jogos Hedônicos

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Italos Estilon da Silva de Souza

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Análise da Estabilidade de Jogos Hedônicos

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Resumo

Jogos hedônicos são jogos de formação de coalizão nos quais os agentes apenas se importam ou são influenciados pelos agentes na mesma coalizão que eles estão. Os agentes podem formar qualquer coalizão que eles queiram e cada agente tem um perfil de preferência, uma ordem fraca sobre o conjunto de coalizões que o contém indicando sua preferência. Um jogo hedônico é definido por um conjunto de agentes e seus perfis de preferência. Classicamente, o resultado de tais jogos é uma partição do conjunto de agentes.

Nesta dissertação, nós revisamos alguns resultados da literatura a respeito da existência de resultados Nash estáveis, do preço da anarquia e estabilidade, da existência de partições no núcleo e da complexidade de computar um resultado que está no núcleo.

Estudamos o modelo de jogos hedônicos que permite a formação de coalizões com sobreposição. Esta extensão permite a representação de vários cenários como interações sociais, grupos de trabalhos e formação de redes. Nós apresentamos um modelo para jogos fracionários com sobreposição de coalizões e mostramos que o núcleo não é vazio para jogos representados por circuitos, caminhos e grafos bipartidos com emparelhamento perfeito. Nós também apresentamos um modelo para jogos hedônicos aditivamente separáveis com sobreposição de coalizões. Mais ainda, mostramos que, para jogos hedônicos aditivamente separáveis simétricos com sobreposição de coalizões, o bem-estar social de qualquer estrutura de coalizão é no máximo o bem-estar social ótimo da versão do jogo sem sobreposição de coalizões.

Abstract

Hedonic games are coalition formation games where the agents only care or are influenced by agents in the same coalition as they are. Agents may form any coalition they want, and every agent has a preference profile, a weak ordering on the set of coalitions that contains it. A hedonic game is defined by a set of agents and their profile preferences.

Classically, the outcome of such games is a partition of the agent set. We review some literature results regarding the existence of Nash stable outcomes, the price of anarchy and stability, the existence of core stable partitions, and the complexity to compute a Core stable outcome.

We extend the hedonic games model by allowing the formation of overlapping coalitions. This extension permits the representation of many scenarios by hedonic games, such as social interactions, working groups, and network formation. We give a model for fractional hedonic games with overlapping coalitions and we show that the core is not empty for games represented by cycles, paths, and bipartite graphs with perfect matching. We also give a model for additively separable hedonic games with overlapping coalitions. Moreover, we show that for symmetric additively separable hedonic games with overlapping coalitions, the social welfare of any coalition structure is at most the optimal social welfare of the game version without overlapping coalitions.

List of Figures

List of Tables

Contents

Chapter 1

Introduction

Morgenstern and Von Neumann founded the field of *game theory* by publishing the book *Theory of Games and Economic Behavior* [22]. According to Tardos and Vazirani [30], "game theory aims to model situations in which participants interact or affect each other's outcomes". We call these participants *agents* or *players*. Agents are considered to be self-interested, and they want to maximize their own gain. Sometimes they can cooperate if this helps them to achieve a better result. The actions that an agent can do are called *strategies*. John Nash was one of the first mathematicians to win the Nobel prize for his work with game theory. He defined a concept of stability that is achieved when every agent cannot improve her gain, if she, acting alone, changes her strategy. He also gave sufficient conditions for a game to achieve such stability.

In many scenarios, it is common to have people joining together to achieve shared and non-shared goals. For example, imagine four people that are going from a place to another. Instead of each one using her own car and paying for the costs alone, they can use only one car and share the costs. A union is another example of people acting together. This way they have more power to negotiate with the employer since the union is more powerful than any individual worker. As a team, they may achieve goals that otherwise would be difficult to accomplish. We call a group formed to achieve some objective a *coalition*. This concept is usual in politics, where politicians with similar ideology (or goals) group themselves in parties so they can have more power and influence. Another example is social life, where people arrange themselves into friendship groups considering similarities and other criteria that lead them to appreciate each other. Morgenstern and von Neumann analyzed coalition formation games in their work *Theory of Games and Economic Behaviour* [22].

Since different coalitions may arise, agents may prefer some over others. The preference of an agent over the coalitions that she can form with other agents can influence the way these coalitions are formed, as well as the satisfaction of the agents involved in each of these coalitions.

An agent has a *hedonic preference* if she only cares about the agents in her coalition. She does not care or is not influenced by the composition of the other coalitions. The word *hedonic* comes from Greek *hēdonikós* and stands for pleasurable [1]. Hedonic formation games were first considered by Dreze and Grenberg [16]. Since then, many works were published analyzing the stability and optimality of this class of games. Different types of hedonic games are obtained considering different possibilities of agents preferences. The simplicity and representativity of hedonic games facilitate their application in many scenarios such as games in wireless networks,

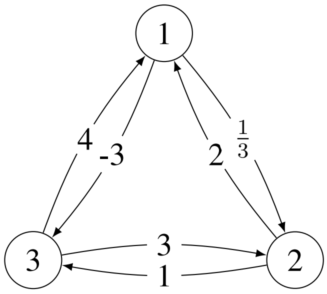


Figure 1.1: Example of a graphical hedonic game.

social networks and team formation [27, 13, 25].

An *outcome* of a hedonic game is a set of coalitions, which is called a *coalition structure*. A coalition structure for classical hedonic games is a partition of the agent set. In this work, we analyze coalition structures where overlapping coalitions are allowed, that is the outcome is not a partition of the agent set. An agent is called *deviant* if she changes her current coalition for another. A coalition structure is considered *stable* if one or more agents do not have an incentive to deviate from their current coalitions. The meaning of incentive to deviate depends on the concept of stability. Since agents may cooperate to deviate as a group, stability concepts that capture this behavior are interesting to be considered. One of the most important concepts of stability for coalition formation games is the core [10]. A coalition structure is in the *core* if there is no group of agents that strictly prefer a new coalition that could be formed by their deviation. Unfortunately, in many classes of hedonic games, the core can be empty. In the literature, there exist many other concepts of stability for hedonic games. See [12, Chapter 15] for a survey.

Graphical hedonic games are those where agents preferences can be represented as a graph. In Figure 1.1 we have an example of a graphical hedonic game where the vertices represent the agents and an arc weight represents how much an agent values another. In this work, we analyze two classes of graphical hedonic games: fractional hedonic games (FHG) and additive separable hedonic games (ASHG). Figure 1.1 can represent both an FHG and an ASHG.

Fractional hedonic games can be represented as a direct graph where an arc between two agents represents the hedonic relation among them. The weight of each arc, if there is one, accounts for how much an agent appreciate the other. The total satisfaction of an agent participating in a coalition is the sum of appreciation the agent has for all others in the same coalition divided by the coalition size, that is, the average appreciation of its neighbors. A fractional hedonic game is *simple* if the edge weights are either 0 or 1, and *symmetric* if the hedonic relation among the agents is symmetrical. For simple symmetric fractional hedonic games induced by graphs of maximum degree at most 2, forests, k-partite complete graphs and graphs with girth at least 5, Aziz et al. [7] proved that the core is not empty. They also showed that for general fractional hedonic games, the core can be empty. Brandl et al. [11] gave an instance of a simple symmetric fractional hedonic game with an empty core. Brandl et al. [11] also showed that deciding if a symmetric fractional hedonic game has a non-empty core is NP-hard. Results regarding other stability concepts such as Nash Stability can be found in the literature [11, 8, 20, 9].

Additively separable hedonic games is another widely studied class of such games. This class is similar to the fractional hedonic, except that the satisfaction of an agent is the sum of appreciation the agent has for the others in the same coalition. The representation of ASHGs is similar to FHG. Aziz, Brandt, and Seedig [4] gave an example of symmetric ASHG with an empty core and they proved that even with symmetric preferences, checking the emptiness of the core is NP-hard. Other results regarding the existence of stable outcomes and the complexity to compute such results can be found in the literature [10, 29, 28, 4, 15].

As Aziz and Savani [12, Chapter 15] said, in many realistic scenarios agents may be part of more than one coalition. For example, an agent may be part of different teams to complete a set of tasks. So, given that the agent has a finite amount of time available, she could divide her time between different coalitions. Chalkiadakis et al. [14] give an example of a game of overlapping team formation where agents have resources they have to invest together with other agents in tasks to complete them, sharing the gain. Other interesting scenarios are social networks, where it is usual for a person to be part of many friendship groups. In order to understand how these groups are formed and what structures emerge, it is necessary to model situations where agents can participate in multiple coalitions and develop new theoretical results for this model. Thus, the objective of this work is to extend hedonic games by allowing agents to form overlapping coalitions and to analyze properties of the games with such an extension.

The rest of the text is organized as follows. In Chapter 2, basic concepts of game theory, social choice theory, coalition formation games, and overlapping coalition formation games are defined. These concepts are relevant to understanding the rest of the text. In Chapter 3 we present results regarding the stability of fractional hedonic games. In Chapter 4 fractional hedonic games with overlapping coalitions are defined. We also give a definition for the core of such games and we present some game classes for which the core is non-empty. Chapter 5 presents results regarding the stability of additionally separable hedonic games and defines additionally separable hedonic games with overlapping coalitions. Finally, Chapter 6 concludes the text by presenting possible lines of research that can be followed for the study of hedonic games.

Chapter 2

Preliminaries

In this chapter, we present basic definitions of game theory, computational social choice, and coalition formation games.

2.1 Game Theory

We give an example of a situation that can be modeled as a game and we use this example to explain the definitions related to game theory, such as agents (players), strategies, utility, and stability.

Imagine that countries have agreed that pollution is causing damage to the lives of their citizens, provoking climate change, and producing economic impacts. Each country can take actions to reduce pollution, investing in new energy sources, regularizing production sectors or in some other way. However, a country may consider that the costs of reducing pollution cannot be borne by its population, or that it will hurt trade and may choose not to reduce pollution. As the world is very integrated, if a country continues to pollute, it ends up influencing the other countries, so that if a country is not polluting, it still has to pay the cost of pollution caused by polluting countries. Let us say that the cost of one country to reduce the pollution is 3 and the cost it suffers per each other polluting country is 1. Let n be the number of countries and k be the number of polluting countries, thus the cost for a non-polluting country is $3 + k$, and the cost for a polluting country is k.

Observe that if a country wants to reduce its cost, it must choose to pollute. But, if every country chooses to pollute, then the cost for every country is n. And if all of them choose not to pollute, they all would have cost 3. Later on, we will analyze the situation where no country chooses to pollute.

Definition 2.1. Let $\mathcal{N} = \{1, 2, \ldots, n\}$ be a set of agents (players). Each agent i has a *set of possibles strategies* S_i. We denote the *strategy vector* by $s = (s_1, s_2, \ldots, s_n)$, where $s_i \in S_i$ is the strategy chosen by agent i. We use $S = \times_{i \in \mathcal{N}} S_i$ to denote the *set of possible strategy vectors* that can be picked by the agents. Each agent has a *utility function* $u_i : S \to \mathbb{R}$ that maps a strategy vector to a real value. We denote by $u = \{u_1, \ldots, u_n\}$ the *set of utility functions*. A game $\mathcal{G}$ is a triple $(\mathcal{N}, S, u)$. We assume that agents want to maximize their utility.

In our example, the agents are the countries and the set of possible strategies for each country is pollute and not pollute. As the objective of an agent is to maximize her utility, in our example,

Girl \ Boy	A	B
A	6 5	1 1
B	2 2	5 6

Table 2.1: Matrix representation of the battle of the sexes.

we model the utility function as follows, where for a given strategy vector s, k_s is the number of countries that choose to pollute:

$$u_i(s) = \begin{cases} -(k+3), & \text{if } i \text{ chooses to not pollute} \\ -k_s, & \text{otherwise.} \end{cases}$$

Sometimes when the utility is negative, we call it *cost*, and we denote by $c_i = -(u_i)$ the cost of agent i. When we are dealing with cost functions, the objective of an agent is to minimize the cost. For a given strategy vector s and an agent i, we denote the strategies picked by all agents, except i, by s_{-i}, that is, $s_{-i} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$. For convenience, sometimes we denote $u_i(s)$ as $u_i(s_i, s_{-i})$.

For an agent i, a strategy s_i is said to be *dominant* if, given that the other agents are playing s_{-i}, any other strategy in S_i cannot give a better utility for i than s_i does.

Definition 2.2. For an agent i, a strategy $s_i \in S_i$ is a *dominant strategy* if, for every $s' \in S$, it holds that

$$u_i(s_i, s'_{-i}) \geq u_i(s'_i, s'_{-i}).$$

A strategy vector s is a *dominant strategy solution* if, for each player i, s_i is a dominant strategy.

Not all games have a single dominant strategy solution, and requiring a game to have one is a stringent demand [30]. Our pollution game has a single dominant strategy solution, which is every player pollutes. Observe that a dominant strategy solution does not guarantee optimal utility for the players, that is the best utility that a player can get. This is the case in our game. The best utility for each player is achieved when they all choose not to pollute.

Now we give an example of a game without a dominant strategy solution. Consider two friends (a boy and a girl) who want to go to the movies and have to decide which movie they are going to watch. The boy prefers movie A and the girl prefers movie B. Both prefer to watch the same movie than to see different ones. Table 2.1 shows the utility for the boy and for the girl for every movie choice they make. In the literature, this game is commonly called the battle of the sexes. Note that the best strategy for an agent of this game depends on the strategy that the other agent chooses, so this game has no dominant strategy solution.

Given that games rarely have a dominant strategy solution, we need another solution concept less stringent. Nash [23] introduced a concept of stability that captures the idea that an agent cannot improve her utility by herself by changing her strategy. If a strategy vector has this property, then it is said to be a *Nash equilibrium*.

Definition 2.3. A strategy vector s is a *Nash equilibrium* if, for each player i and each alternate strategy s'_i, we have that

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}).$$

In our pollution game, if every country chooses to pollute, then the utility of every agent i is $-n$. If an agent j chooses not-pollute, she would have utility $-3-(n-1)$, and therefore, she would not improve her utility. Hence, when every player chooses to pollute, we have a Nash equilibrium (the only one this game has). Observe that a dominant strategy solution is always a Nash equilibrium.

A game may have none or more that one Nash equilibrium and an agent may have different utility values for each of them. Because of this, we do not know which equilibrium will emerge. In the battle of the sexes, when the boy and the girl choose to see the same movie, this is a Nash equilibrium. We then have two different stable results.

The Nash equilibrium defined above is called *pure strategy equilibrium*, where an agent deterministically chooses her strategy. But, we can consider the case where the agent chooses her strategy at random. For this, each agent i has a probability distribution over her strategy set S_i.

Definition 2.4. For an agent i, a *mixed strategy* σ_i is a probability distribution over S_i. A *mixed strategy vector* is denoted by $\sigma = (\sigma_i)_{i \in \mathcal{N}}$. For convenience, σ_{-i} denotes the strategy vector of every agent except i.

Given that a mixed strategy is a probability distribution, we have that the *expected utility* of an agent is given as

$$\mathbb{E}[u_i(\sigma)] = \sum_{s \in S} u_i(s)\mathbb{P}_\sigma(s),$$

where $\mathbb{P}_\sigma(s) = \prod_j \sigma_j(s_j)$. For convenience, we also denote $\mathbb{E}[u_i(\sigma)]$ as $\mathbb{E}[u_i(\sigma_i, \sigma_{-i})]$. Observe that a pure strategy is a mixed strategy with probability 1 for some strategy. We assume that the agents are risk neutral, acting only to maximize their expected utility.

A mixed strategy σ_i is a *best response* for σ_{-i} if, for every alternate mixed strategy σ', we have that

$$\mathbb{E}[u_i(\sigma_i, \sigma_{-i})] \geq \mathbb{E}[u_i(\sigma'_i, \sigma_{-i})].$$

Definition 2.5. A mixed strategy vector σ is a *mixed Nash equilibrium* if for every i, σ_i is a best response for σ_{-i}.

Nash [23] showed that for a finite game with a finite set of strategies, a mixed Nash equilibrium always exists. This is one of the most import results in game theory. The proof is based on the Brouwer's theorem which states that for a unit ball $B \subset \mathbb{R}^n$, for any continuous function $f : B \to B$, there is a point x_0 for which $f(x_0) = x_0$. The point x_0 is called a *fixed point*. The proof consists in constructing a function f from the space of n-tuples that contains σ to itself where the fixed points of f are the mixed Nash equilibria.

2.2 Computational Social Choice

The field of social choice theory is interested in studying mechanisms of aggregation of individual preferences in a collective choice, which includes developing and analyzing voting rules [12]. An example of a social choice mechanism is the Brazilian presidential election. Each Brazilian citizen has a preference over the candidates who are running for the election. Then the citizen informs his preference by voting on one of the candidates (observe that a voter can vote for a candidate which is not whom she prefers the most). The mechanism then chooses the candidate who received the greatest number of valid votes. That is, the Brazilian election aggregate the preferences of the citizens by choosing a candidate to be president.

Let $\mathcal{C} = \{1, \ldots, n\}$ be a set of *individuals* (*agents*), and let A be a finite set of *alternatives*. The set of all *weak orders* $\succsim$ on A, binary relations that are complete and transitive, is denoted by $\mathcal{R}(A)$, and the set of all *linear orders* $\succ$ on A, binary relations that are total orders, is denoted by $\mathcal{L}(A)$. A *preference profile* $\succ_i$ of an individual i is a preference linear order of i on A. Observe that ties are allowed in weak orders, but they are not allowed in linear orders. A *social welfare function* receives a sequence of preference profiles $P = (\succ_1, \ldots, \succ_n)$, one for each individual, and maps it to a single preference order, that is, a social welfare function aggregates the preferences of individuals in a single order that represents the preference of society.

Definition 2.6. A *social welfare function (SWF)* is a function of the form $f : \mathcal{L}(A)^n \to \mathcal{R}(A)$. We refer to the outcome $f(\succ_1, \ldots, \succ_n)$ of a SWF f as *social preference order*.

Ties are allowed in social preference orders, but the preference profiles in the input are expected to be strict. In a SWF f representing the Brazilian presidential election, f aggregates voter preferences in a social preference order in which the first choice is the candidate who received the most votes.

Arrow [2] determined that a reasonable SWF must have at least two properties: to be weakly Paretian and independent of irrelevant alternatives. A SWF f is *weakly Paretian* if all individuals have the same preference between two alternatives $a, b \in A$, then, f ranks a and b accordantly to this preference.

Definition 2.7. A social welfare function f is *weakly Paretian* if for all profile preference $P = (\succ_1, \ldots, \succ_n)$ and for all $i \in \mathcal{N}$, $a \succ_i b$, then $a \succ b$ where $\succsim = f(\succ_1, \ldots, \succ_n)$.

A SWF f is *independent of irrelevant alternatives* if, for any two alternatives $a, b \in A$, the rank of a with respect to b given by f takes into consideration just how individuals ranked a in respect to b, without considering any other alternative c.

Definition 2.8. Let $P = (\succsim_1, \ldots, \succsim_n)$ and $P' = (\succsim'_1, \ldots, \succsim'_n)$ be sequences of preference profiles. A social welfare function f is *independent of irrelevant alternatives* if for every P and P' and for all $a, b \in A$, we have that if $a \succ_i b \iff a \succ'_i b$ for all i, then $a \succsim b \iff a \succsim' b$ where $\succsim = f(P)$ and $\succsim' = f(P')$.

Imagine the following situation of social choice. An entity is conducting an election to choose the greatest villain of all time. The alternatives are Voldemort (v), Sauron (s), Palpatine (p), and Big Jim (b). Each individual should inform a strict preference order on the alternatives. The SWF f ranks alternatives by the number of times it appeared at the top of some individual's order of preference. We show that f is not independent of irrelevant alternatives. Preference profiles are given as follows:

$$\succ_1 = p \succ b \succ v \succ s,$$
$$\succ_2 = p \succ b \succ s \succ v,$$
$$\succ_3 = v \succ b \succ s \succ p,$$
$$\succ_4 = v \succ b \succ p \succ s.$$

By the definition of f, the social preference order $\succsim = f(\succ_1, \ldots, \succ_n)$ ranks the alternatives in the following way: $v \succsim p \succ b \succsim s$. If agent 2 changes her preference profile to

$$\succsim_2' = b \succ p \succ s \succ v,$$

the social preference order would be $v \succ b \succsim p \succ s$. However, the preference between p and v did not change for all agents. Hence, f is not independent of irrelevant alternatives. Note that the Brazilian election for president is similar to f.

The above example illustrates that it is not easy for a social welfare function to be weakly Paretian and independent of irrelevant alternatives. A dictatorship has these properties. In a *dictatorship*, one of the preference profiles of the individuals is chosen by f to represent the social preference order.

Definition 2.9. A social welfare function f is a *dictatorship* if there is an individual $i \in \mathcal{N}$ such that if $a \succ_i b$ then $a \succ b$ under $\succsim = f(\succ_1, \ldots, \succ_n)$, for any $a, b \in A$.

In fact, Arrow [2] has shown that if the size of the set of alternatives is at least 3, any social welfare function f that is weakly Paretian and independent of irrelevant alternatives is a *dictatorship*.

Theorem 2.10 (Arrow [2])**.** *For a set of alternatives A with $|A| \geq 3$, every SWF that is weakly Paretian and independent of irrelevant alternatives must be a dictatorship.*

Arrow's theorem is considered to be the birth of the modern theory of social choice, with the following works focusing mainly on axiomatic and normative aspects of voting rules [12]. However, these studies neglected the computational complexity to calculate the result of these voting rules. The practical acceptability of a voting rule or a fair allocation mechanism should take into account not only its fairness but also the amount of time to compute a result [12]. Computer science can then make a relevant contribution to the field of social choice.

David Gale and Lloyd Shapley started the stable matching theory [18], which studies how agents from different sets can be matched taking into account the agents preferences. An example of a situation studied by the matching theory is the problem of assigning students to universities, first defined by Gale and Shapley [18].

Let $S = \{s_1, \ldots s_n\}$ be a set of students and $U = \{u_1, \ldots, u_m\}$ be a set of universities. Each university u_i has a capacity c_i, which is the number of students that it can accept. Let $E \subseteq S \times U$

be the set of *acceptable* pairs. Each student $s_i \in S$ has an *acceptable universities set* given by

$$A(s_i) = \{u_j \in U : (s_i, u_j) \in E\},$$

and s_i also has a linear preference order on $A(s_i)$ denoted by $\succ_{s_i}$. Each university $u_j \in U$ has an *acceptable students set* given by

$$A(u_j) = \{s_i \in S : (s_i, u_j) \in E\},$$

and u_j also has a strict preference order on $A(u_j)$ denoted by $\succ_{u_j}$. An *assignment* M is a subset of E. For a pair $(s_i, u_j) \in M$, we say that s_i is *assigned* to u_j and that u_j is *assigned* to s_i. The set of assignees for each $a_k \in S \cup U$ is denoted by $M(a_k)$. A student s_i is *unassigned* if $M(s_i) = \emptyset$. A university u_j is *undersubscribed* if $|M(u_j)| < c_j$ and it is *full* if $|M(u_j)| = c_j$. An assignment M is a *matching* if for ever $s_i \in S$ and every $u_j \in U$, it holds that $|M(s_i)| \leq 1$ and $|M(u_j)| \leq c_j$.

Definition 2.11 (Gale and Shapley [18]). An assignment M of students to universities is *unstable* if there exist a pair $(s_i, u_j) \in E \setminus M$ such that s_i is unassigned or prefers u_j to a member of $M(s_i)$, and u_j is undersubscribed or prefers s_i to at least one member of $M(u_j)$.

Gale and Shapley [18] showed that a stable matching always exists and proposed an efficient (polynomial) algorithm to compute such an assignment. The algorithm proposed by them is used in many scenarios such as matching doctors and hospitals, students and universities, and housing allocation.

The housing allocation problem consists of a set of applicants who have strict preference orders in relation to a set of houses, and we have to allocate houses to applicants. This problem differs from stable matching problem because houses do not have preference orders on the applicants. Formally, let $C = \{c_1, \ldots, c_n\}$ be a set of applicants and $H = \{h_1, \ldots, h_m\}$ be a set of houses. Let $E \subseteq C \times H$ be the set of *acceptable* pairs. Each applicant $c_i \in C$ has an *acceptable houses set* given by $A(c_i) = \{h_j \in H : (c_i, h_j) \in E\}$, and c_i also has a strict preference order on $A(c_i)$ denoted by $\succ_{c_i}$. Each house $h_j \in H$ has an *acceptable applicants set* given by $A(h_j) = \{c_i \in C : (c_i, h_j) \in E\}$. An *assignment* M is a subset of E. For an pair $(c_i, h_j) \in M$, we say that c_i is *assigned* to h_j and that h_j is *assigned* to c_i. The set of assignees for each $p_k \in C \cup H$ is denoted by $M(p_k)$. An assignment M is a *matching* if for every $c_i \in C$ and $h_j \in H$, $|M(c_i)| \leq 1$ and $|M(h_j)| \leq 1$. For convenience, we denote the house assigned to an applicant c_i by $M(c_i)$, and the applicant assigned to a house h_j by $M(h_j)$ if M is a matching.

The *house marketing problem* is a specific case of the housing allocation problem where the number of applicants equals the number of houses. Let M and M' be matchings for a house marketing problem. We say that M' *blocks* M with respect to a subset $S \subseteq C$ of applicants if:

- it holds that $\{M(c_i) : c_i \in S\} = \{M'(c_i) : c_i \in S\}$, that is, the members of S can only exchange houses between them;
- for every $c_i \in S$, $M'(c_i) \succsim_{c_i} M(c_i)$;
- there exists at least one $c_k \in S$, for which $M'(c_k) \succ_{c_k} M(c_k)$.

The members of S only can exchange houses between themselves, every member of S must be assigned to a house that she prefers at most as she prefers her old one, and there must exist a member of S that is assigned to a house that she prefers more than she prefers her old one.

We refer to S as a coalition, and we say that S is a *blocking coalition* if M' blocks M with respect to S.

Definition 2.12. A matching M is in the *core* if M admits no blocking coalition.

Shapley and Scarf [26] showed that the core of house-marketing problems is not empty by constructing a matching that is in the core using David Gale's algorithm called *Top Trading Cycle*. Given an initial matching, the algorithm consists of constructing a directed graph as follows: the vertices are the applicants, and there is an arc from i to j if the most preferred house of i is assigned to j. Since there is at least one cycle (note that a vertex can be the tail and the head of the same arc), the algorithm executes the exchanges within each cycle and removes these agents. In the sequence, it repeats the process over the directed graph with the remaining applicants. The algorithm stops when there are no more applicants.

2.3 Coalition Formation Games

In the book *Theory of Games and Economic Behavior* [22], von Neumann and Morgenstern analyzed a class of games that models situations where individuals join themselves in coalitions through the framework of n-person games with transferable utility (TU games). A *TU game* has a set of n agents and a characteristic function that maps each possible coalition to the value that their members can achieve by acting together. This value can be transferred between the coalition's members without loss.

Definition 2.13. A *TU game* is a tuple $(\mathcal{N}, v)$ where $\mathcal{N} = \{1, 2, \ldots, n\}$ is a set of agents and $v : 2^{\mathcal{N}} \to \mathbb{R}$ is a characteristic function, mapping every coalition $C \subseteq \mathcal{N}$ to a value $v(C)$ representing the total utility that C can achieve if its members cooperate. By convention $v(\emptyset) = 0$. Coalition $C = \mathcal{N}$ is called the *grand coalition*.

Early work with TU games supposed that the function v was superadditive, that is, the sum of the value of two coalitions is at most the value of the union of this two coalitions. With such property, it is expected that the grand coalition $\mathcal{N}$ emerge as the outcome. Therefore, studies focused on how to distribute the gain of the grand coalition [19]. However, many scenarios are not superadditive. For instance, political parties. As a political party gets larger, the more difficult it is to find an agreement. Hence, subgroups tend to emerge.

The assumption that utility can be freely transferred depends on a commodity whose value is proportional to its quantity and does not depend on any other factor [19]. Moreover, each agent may have a different utility for the commodity distributed by the coalition. In this way, it might be better to represent a coalition by a vector that determines the utility of each agent for this coalition, rather than simply giving it a unique value. We have then the non-transferable utility games.

Definition 2.14 (Hajduková [19])**.** A *non-transferable utility game (NTU game)* is a tuple $(\mathcal{N}, V)$ where $\mathcal{N} = \{1, 2, \ldots, n\}$ is a set of agents and V is a utility map that assigns to every coalition $S \subseteq \mathcal{N}$ a subset $V(S)$ of $\mathbb{R}^S$ such that V satisfies the following conditions for all $S \neq \emptyset$:

1. $V(S)$ is a nonempty, closed, and convex subset of $\mathbb{R}^S$,
2. $V(S)$ is comprehensive, that is, if $x \in V(S)$ and $y \leq x$, then $y \in V(S)$,
3. $V(S) \cap \mathbb{R}^S_+$ is bounded.

Condition 2 states that if a coalition S can achieve a payoff vector x, it can achieve a smaller payoff vector y, and condition 3 limits the payoff that a coalition can achieve. Observe that a non-transferable utility game can represent a transferable utility game by defining $V(S) = \{x \in \mathbb{R}^S : \sum_{i \in S} x_i \leq v(S)\}$ for all $S \subseteq \mathcal{N}$, with $S \neq \emptyset$. Hence, NTU games are a generalization of TU games.

It is not always possible to determine the value of a coalition for an agent. For example, when people are dividing themselves into work groups, it is difficult to determine a value for each group (coalition) or utility vectors for each person. In that case, the benefit to the person is to be part of the coalition. However, a person may prefer one group to another, which will determine which group she will try to be a part of. In a *coalition formation game*, each agent has a preference order over all possible coalitions she can join. This behavior is captured by the concept of *profile preference of an agent*, which is a weak ordering of coalitions containing such an agent. Observe that the number of possible coalitions that can be formed is exponential in the number of agents.

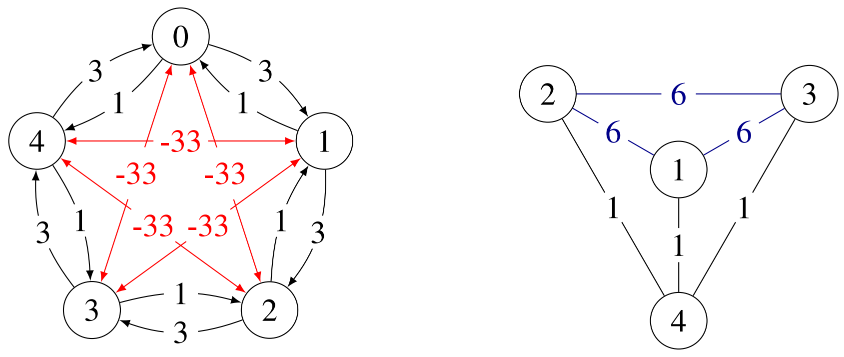
Definition 2.15. Let $\mathcal{N} = \{1, 2, \ldots, n\}$ be the set of agents and $\mathcal{C}_i \subseteq 2^{\mathcal{N}}$ be the set of all possible coalitions that contain i for $i \in \mathcal{N}$. The set of all possible coalitions is denoted by $\mathcal{C} = 2^{\mathcal{N}}$. A *profile preference* $\succsim_i$ is a weak ordering over $\mathcal{C}_i$. For any $C, D \in \mathcal{C}_i$, $C \succ_i D$ denotes that agent i *strictly prefers* coalition C over coalition D, $C \sim_i D$ denotes that i is *indifferent* between coalitions C and D, and $C \succsim_i D$ denotes that i *weakly prefers* coalition C over D, that is $C \succ_i D$ or $C \sim_i D$. For an agent i, we say that a coalition $C \in \mathcal{C}_i$ is *unacceptable* if $\{i\} \succ_i C$, that is, i prefers being alone than being part of C.

A *coalition structure* is a partition of $\mathcal{N}$ and represents a possible way the agents could organize themselves, and it is said to be *stable* if the agents have no incentive to deviate from their current coalitions [12]. Later, we present some stability concepts studied in the literature.

Definition 2.16. For a set $\mathcal{N}$ of agents, a *coalition structure* π is a partition of $\mathcal{N}$ where $\pi(i)$ denotes the coalition of agent $i \in \mathcal{N}$.

When we assume that the preference order is over the coalitions instead of over the coalition structures (an agent only care about her coalition and it is not affected by how others coalitions are organized), the game has hedonic profile preferences. The hedonic aspect was introduced by Dreze and Greenberg [16]. If a coalition formation game has hedonic preferences, it is said to be a *hedonic game*.

Definition 2.17. Let $\mathcal{N}$ be a finite set of agents and let $\succsim = \{\succsim_1, \succsim_2, \ldots, \succsim_n\}$ be a set of profile preferences on coalitions where $\succsim_i$ is the profile preference of agent $i \in \mathcal{N}$. A *hedonic game* is a tuple $(\mathcal{N}, \succsim)$ and an *outcome* of a hedonic game is a coalition structure π.



(a) Example of a fractional hedonic game.

(b) Example of a symmetric fractional hedonic game.

Figure 2.1: Examples of fractional hedonic games.

One must note that the representation of a hedonic game may be very large because of the size of their profile preferences. Some alternative representations have been proposed in the literature, see [12] for a survey of some of those representations.

2.3.1 Fractional Hedonic Games

Aziz et al. [7] proposed a class of hedonic games where the utility of an agent for a coalition she is part of is the average of the value she gives for the agents in the coalition. An agent values herself with 0. A fractional hedonic game can be represented by a directed graph as we can see in Figure 2.1a. A vertex represents an agent and an arc weight represents how much the agent in the arc's tail values the agent in the arc's head. In the game in Figure 2.1a, every agent values the agent that comes after (clockwise direction) more than she values the agent that comes before, and she pretty much dislikes everyone else.

Now, we define formally this class of games. Let $\mathcal{N}$ be a set of agents, a *coalition* C is a subset of $\mathcal{N}$, and a *coalition structure* π is a partition of $\mathcal{N}$. For any agent $i \in \mathcal{N}$, we use $\pi(i)$ to denote the coalition of agent i in π.

Definition 2.18. For each $i \in \mathcal{N}$, let $u_i : \mathcal{N} \to \mathbb{R}$ be a *valuation function* that denotes how much agent i values each other agent $j \in \mathcal{N}$. We define that $u_i(i) = 0$. Given a coalition C, the utility of agent i is defined as $u_i(C) = \frac{\sum_{j \in C} u_i(j)}{|C|}$. We use $u_i(\pi)$ as a shorthand for $u_i(\pi(i))$. A *Fractional Hedonic Game* is defined as a pair $\mathcal{G} = (\mathcal{N}, u)$. The *outcome* of $\mathcal{G}$ is a coalition structure π.

In a fractional hedonic game $\mathcal{G}$, an agent $i \in \mathcal{N}$ weakly prefers coalition C to D if and only if $u_i(C) \geq u_i(D)$, and i is indifferent between C and D if and only if $u_i(C) = u_i(D)$. A fractional hedonic game is said to be *symmetric* if for all $i, j \in \mathcal{N}$, $u_i(j) = u_j(i)$ and is *simple* if for all $i, j \in \mathcal{N}$, $u_i(j) \in \{0, 1\}$.

Observe that a simple symmetric fractional hedonic game can be represented by a simple graph. The vertices represent the agents and there is an edge between agent i and agent j if and only if $u_i(j) = 1$. We denote by $\mathcal{G}(G)$ the game represented by a graph G. For an agent $i \in \mathcal{N}$, we denote by $N(i)$ the set of neighbors of i. For convenience, we can restrict the neighborhood of i to a subset S of $\mathcal{N}$, then $N_S(i)$ denotes the neighborhood of i in S.

Fractional hedonic games are a useful tool to model scenarios where large coalitions tend to become unstable, that is, large coalitions tend to divide in small ones. For instance, in a large political party, it might be difficult to achieve a consensus. Thus this political party might split. That is, the cost to add someone in the coalition might not be justified by the value this person brings to the coalition. See the game in Figure 2.1b. In a coalition with agents 1, 2, and 3, each one of them has utility $\frac{12}{3}$. If agent 4 would join this coalition the utility of 1, 2, and 3 would decrease, in spite of they valuing agent 4 positively.

Solution Concepts

In the literature, there exist many concepts of stability for hedonic games. In this Section, we review some of the most common solution concepts. We define solution concepts to fractional hedonic games, but the definition for any class of hedonic games is similar.

The strongest concept of stability is the *perfect* coalition structure where every agent does not want to leave her current coalition because there exists no other possible coalition that she strictly prefers. If a coalition structure has this property, then it is also stable for any other solution concept presented in this section [3]. Although a perfect structure would be a very good outcome of a hedonic game, it is uncommon for a game to have this property.

Definition 2.19 (Aziz, Brandt, and Harrenstein [5])**.** A coalition structure π is *perfect* if for each agent $i \in \mathcal{N}$, there exists no $C \in \mathcal{C}_i$ such that $u_i(C) > u_i(\pi)$.

The game in Figure 2.1b is an example of a game that admits no perfect coalition structure. The coalition that agent 1 prefers the most is $\{1, 2, 3\}$, and the coalition that agent 4 prefers the most is $\{1, 2, 3, 4\}$. Since both agents cannot be in their most preferred coalition at the same time, this game has no perfect coalition structure.

Since a perfect structure is not guaranteed to exist in a hedonic game, we should analyze the existence of other stability concepts. One of them, individual rationality, guarantees that there exists a coalition structure where each agent prefers the coalition she is now as much as staying alone.

Definition 2.20 (Elkind and Wooldridge [17])**.** A coalition structure π is *individually rational* if for each agent $i \in \mathcal{N}$, we have that $u_i(\pi) \geq u_i(\{i\}) = 0$.

A very important and widely studied concept of stability is the Nash stability, where every agent in the coalition structure has no incentive to deviate individually [10]. An agent may deviate from her coalition to another that is more pleasurable, forming a new coalition structure. The creation of a new coalition structure may incentive some other agents to deviate, changing, this way, the organization of coalitions again. In a Nash stable coalition structure, this does not happen.

Definition 2.21. A coalition structure π is *Nash stable* if for each agent $i \in \mathcal{N}$ and for all $C \in \pi \cup \{\emptyset\}$, we have that $u_i(\pi) \geq u_i(C \cup \{i\})$.

This concept does not consider group deviations where more than one agent collaborates to deviate together. For the game in Figure 2.1b, the grand coalition is Nash stable because any agent of this game has positive utility for the grand coalition and utility 0 for being alone.

Observe that Nash stability is similar to Nash equilibrium (Section 2.1) in the sense that, in both solution concepts, an agent cannot improve her utility by acting alone.

The next concept also captures the idea of agents acting alone. In this case, the agent may have an incentive to deviate, but if she does, then she will harm the agents in the new coalition. The idea of this concept is that the agent which wants to deviate must be accepted by the members of the coalition she intends to join. A coalition structure π is *individually stable* [10] if no agent can move to another coalition $C \in \pi \cup \{\emptyset\}$, without making at least one of the agents in C worse off. Note that if this deviation could be performed without permission, the agents in the new coalition could have an incentive to also deviate.

Definition 2.22. A coalition structure π is *individually stable* if for each agent $i \in \mathcal{N}$, and for any coalition $C \in \pi \cup \{\emptyset\}$ such that $u_i(C \cup \{i\}) > u_i(\pi)$, there is at least one agent $j \in C$ such that $u_j(C) > u_j(C \cup \{i\})$.

In the game in Figure 2.1b, the coalition structure $\{\{4\}, \{1, 2, 3\}\}$ is not Nash stable because agent 4 has an incentive to join $\{1, 2, 3\}$. Nonetheless, it is individually stable because agents 1, 2, and 3 will not allow a deviation that will make them worse off. And the three of them are in the most preferred coalition, therefore, they do not want to leave it. In fact, individually stability is a necessary condition for Nash stability but it is not a sufficient one.

Agents also may deviate as a group instead of individually, thus, stability concepts that capture this behavior are interesting to be considered. One of the most important concepts of stability for coalition formation games is the *core* [10]. A coalition structure is in the *core* if there is no group of agents that strictly prefer a new coalition that could be formed by their deviation.

Definition 2.23. A coalition $C \subseteq \mathcal{N}$ *blocks* a coalition structure π if for every $i \in C$, it holds that $u_i(C) > u_i(\pi)$. A coalition structure π is in the *core* if there is no coalition that blocks π. A coalition structure π that is in the core is said to be *core stable*.

For instance, in the game in Figure 2.1b, the coalition structure $\{\{1, 2, 3, 4\}\}$ is blocked by coalition $\{1, 2, 3\}$ because these agents prefer $\{1, 2, 3\}$ instead of grand coalition. So they have an incentive to deviate by forming this coalition and, thus, the coalition structure $\{\{1, 2, 3, 4\}\}$ is not in the core.

The Pareto optimality is another stability concept for group deviation. A coalition structure is Pareto optimal [16] if there exists no coalition structure where every agent values at least as much as her current coalition structure, and there is one agent that values strictly more than her current coalition structure.

Definition 2.24. A coalition structure π is *Pareto optimal* if there is no coalition structure π' such that for each agent $i \in \mathcal{N}$,

$$u_i(\pi'(i)) \geq u_i(\pi(i))$$

and at least for one agent $j \in \mathcal{N}$,

$$u_j(\pi'(j)) > u_j(\pi(j)).$$

In Figure 2.1b, the coalition structure $\{\{1, 2, 3\}, \{4\}\}$ is Pareto optimal because, in any other coalition structure, 1, 2, 3 will be worse given that $\{1, 2, 3\}$ is the coalition they prefer the

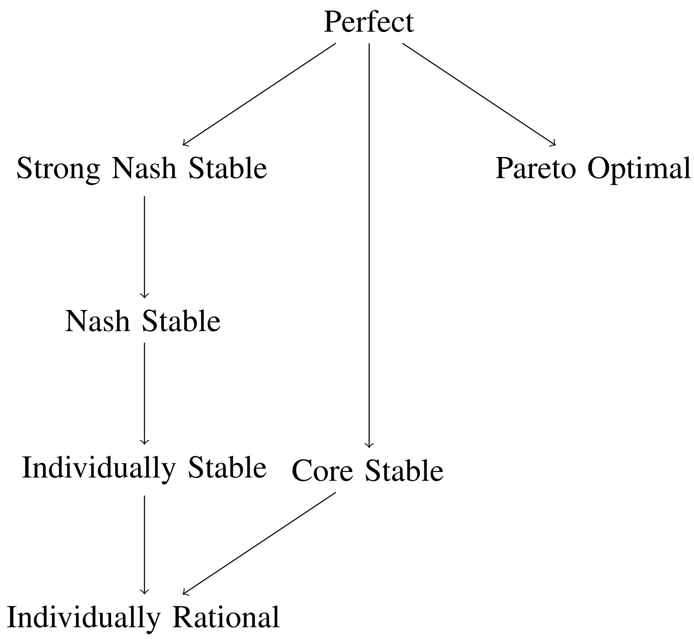


Figure 2.2: The relation among the solution concepts. An arc indicates that a solution concept implies the other. Source: Aziz and Brandl [3].

most and any coalition agent 4 wants to form has to be with at least one of them, but all of them become worse off with agent 4.

A coalition structure π' is reachable from another coalition π by the deviation of a group of agents H if any two agents in $\mathcal{N} \setminus H$ are in the same coalition in π' only if they were in the same coalition in π.

Definition 2.25 (Karakaya [21])**.** For a coalition structure π, another coalition structure $\pi' \neq \pi$ is *reachable* from π by the deviation of $H \subseteq \mathcal{N}$, if for all $i, j \in \mathcal{N} \setminus H$, with $i \neq j$, $\pi(i) = \pi(j)$ if and only if $\pi'(i) = \pi'(j)$.

The Nash stability concept has been extended to capture group deviations. If there exists no structure π' reachable from π such that every agent in the deviation group strictly prefers her new coalition in π', then π is strong Nash stable [21].

Definition 2.26. A non-empty subset of agents $H \subseteq \mathcal{N}$ *strongly Nash blocks* π if there exists π' reachable from π by the deviation of H such that for each $i \in H$, $u_i(\pi') > u_i(\pi)$. A structure π is *strong Nash stable* if it is not strongly Nash blocked by any $H \subseteq \mathcal{N}$.

Aziz and Brandl [3] analyzed the relation between these solution concepts, as shown in Figure 2.2, and they analyzed the existence of some of them for some classes of hedonic games. We refer to Bogomolnaia and Jackson [10] for more results regarding stability in hedonic games.

2.3.2 Additively Separable Hedonic Games

In an additively separable hedonic game [10], the preference of agents are denoted by a value she has for every other agent. The utility of a coalition is the sum of the preferences an agent has for the other agents in the same coalition.

Definition 2.27. An *Additively Separable Hedonic Game* is a pair $G = (\mathcal{N}, u)$ such that for each $i \in \mathcal{N}$, $u_i : \mathcal{N} \to \mathbb{R}$ is a valuation function that denotes how much agent i values all other $j \in \mathcal{N}$. We define that $u_i(i) = 0$. Given a coalition C, the utility of agent i is given as $u_i(C) = \sum_{j \in C} u_i(j)$. Let u be a set of utility functions u_i for each $i \in \mathcal{N}$. An outcome of a game G is a coalition structure π.

For all coalition S, T that contains i, it holds that $S \succsim_i T \iff u_i(S) \geq u_i(T)$. The social welfare of a coalition C is given as $SW(C) = \sum_{j \in C} u_j(C)$. The social welfare of a coalition structure π is given by

$$\mathcal{SW}(\pi) = \sum_{C \in \pi} SW(C).$$

An additively separable hedonic game is *symmetric* if $u_i(j) = u_j(i)$ for every $i, j \in \mathcal{N}$. The solution concepts defined in Section 2.3.1 also apply to additively separable hedonic games.

2.4 Overlapping Coalition Formation Games

Although the models presented in Section 2.3 can model many situations, they fail to model scenarios where an agent can be in more than one coalition. For instance, a country can be part of different trade agreements. We can see each agreement as a coalition. Another scenario is friendship groups, since a person can have a group of friends from work and a group from where she lives. In order to provide a framework for analyzing scenarios where coalitions can overlap, Chalkiadakis et al. [14] generalize the non-transferable utility games framework.

In their model, each agent has a limited amount of resource that she can split between the coalitions she is part of. Thus, a *coalition* is a vector in which each dimension represents the contribution of an agent. There is a *characteristic function* that maps to each coalition a value that this coalition can achieve through the cooperation of its members.

Definition 2.28. Let $\mathcal{N} = \{1, 2, \ldots, n\}$ be a set of agents. A *(partial) coalition* is given by a vector $r = (r_1, \ldots, r_n)$ where r_i is the fraction of resource that agent i contributed to this coalition. If $r_i = 0$, then agent i is not a member of r. The support of a coalition r is denoted by $\text{supp}(r)$ and is defined as $\{i \in \mathcal{N} : r_i > 0\}$. A *characteristic function* $v : [0,1]^n \to \mathbb{R}$ assigns to each coalition $r \in [0,1]^n$ a real value. An *overlapping coalition formation game (OCF) G* is a tuple $(\mathcal{N}, v)$.

Observe that a classic coalition $C \subseteq \mathcal{N}$ can be represented by a partial coalition r where $r_i = 1$ if $i \in C$ and $r_i = 0$ otherwise.

In a non-overlapping coalition setting, a coalition structure of a game is a partition of $\mathcal{N}$. For an OCF game, a *coalition structure* is a list of vectors (the coalitions).

Definition 2.29. For a set of agents $T \subseteq \mathcal{N}$, a *coalition structure* on T is a list of vectors

$$\pi_T = (r^1, \ldots, r^k), \text{ for some } k \in \mathbb{N}^*,$$

that satisfies the following conditions:

- $r^i \in [0,1]^n$,

- $\mathrm{supp}(r^i) \subseteq T$ for $i = 1, 2, \ldots, k$,
- $\sum_{i=1}^{k} r_j^i \leq 1$ for all $j \in T$.

We denote by $|\pi_T| = k$ the size of a coalition structure on T. The set of all possible coalition structures on T is denoted by $\boldsymbol{\pi}_T$.

Observe that an agent is not required to allocate all of her resources. It can be the case that $\sum_{i=1}^{k} r_j^i < 1$. Also observe that there can be infinitely many different coalitions and, therefore, infinitely many coalition structures. Thus, it is impossible to find the coalition structure that maximizes the social welfare (the sum of the utilities of all agents) by enumerating all coalition structures because a social welfare maximizing coalition structure might not even exist. Then, we extend the definition of v to coalition structures as $v(\pi) = \sum_{r \in \pi} v(r)$. Furthermore, for any subset $S \subseteq \mathcal{N}$, we define $v^*(S) = \sup_{\pi \in \boldsymbol{\pi}_S} v(\pi)$, that is, an upper bound on the value that a subset S can achieve by forming a coalition structure. We say that v is *bounded* if $v^*(\mathcal{N}) < \infty$.

Associated with each coalition structure there is a list of vectors that determine how the coalition gains will be divided among its members. This list is called *imputation*.

Definition 2.30. Given a coalition structure π on $\mathcal{N}$, with $|\pi| = k$, an *imputation* for π is a k-tuple $x = (x^1, x^2, \ldots, x^k)$, where $x^i \in \mathbb{R}^n$ for $i = 1, \ldots, k$ such that

- for every $r^i \in \pi$, $\sum_{j=1}^{n} x_j^i = v(r^i)$ and if $r_j^i = 0$ then $x_j^i = 0$,
- (*individual rationality*) the utility of an agent j is at least as much as she can gain by herself, that is, $\sum_{i=1}^{k} x_j^i \geq v^*(\{j\})$.

The set of all imputations for π is denoted by $I(\pi)$.

Observe that an agent only have positive utility for a coalition that she is part of. This prohibits the transfer of utility to outside the coalition.

An outcome of an OCF game is given by a coalition structure and an imputation.

Definition 2.31. A *feasible agreement* (outcome) for a set of agents $S \subseteq \mathcal{N}$ is a tuple (π, x) such that $\pi \in \boldsymbol{\pi}_S$, $|\pi| = k$ for some $k \in \mathbb{N}$, and $x = (x^1, x^2, \ldots, x^k) \in I(\pi)$. The set of all feasible agreements for a set S is denoted by $\mathcal{F}(S)$. The *utility* of an agent j under a feasible agreement (π, x) is given by $u_j(\pi, x) = \sum_{i=1}^{k} x_j^i$.

We are interested in some feasible agreements, such as those that have the property of being stable. Chalkiadakis et al. [14] proposed three definitions of stability. Here, we present one of them, which is based on the core of NTU games. An outcome is core stable if there is no subset of agents that can form a coalition between them for which their utility is strictly greater than the utility for the previous outcome.

Definition 2.32. Given an OCF-game $G = (\mathcal{N}, v)$, and a subset of agents $S \subseteq \mathcal{N}$, let (π, x) and (π', y) be two outcomes of G, such that for any coalition $r^i \in \pi'$, either $\mathrm{supp}(r^i) \subseteq S$ or $\mathrm{supp}(r^i) \subseteq \mathcal{N} \setminus S$. We say that (π', y) is a *profitable deviation* of a subset S from (π, x), if for all $j \in S$, $u_j(\pi', y) > u_j(\pi, x)$. A feasible agreement (π, x) is in the *core* if no $S \subseteq \mathcal{N}$ has a profitable deviation.

Observe that in a profitable deviation, the deviant agents can only form coalitions between themselves. This constraint simulates the behavior of deviant agents in the core of NTU games. For those games, deviant agents are all in the same possible blocking coalition.

Chapter 3

Fractional Hedonic Games

In this chapter, we present literature results about core and Nash stability for fractional hedonic games. We also give a proof that the core of simple symmetric fractional hedonic games represented by pseudoforest is non-empty.

3.1 Core stability

In this section, we present some results regarding the existence of core stable coalition structures for fractional hedonic games. We begin by showing that the core can be empty for fractional hedonic games in general. Then, we show some classes for which the core is non-empty.

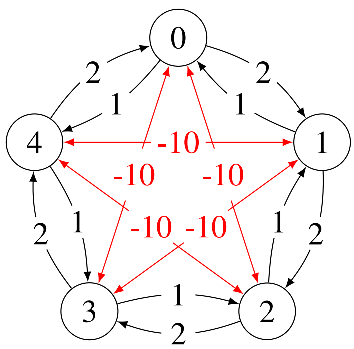


Figure 3.1: Example of a fractional hedonic game with empty core.

Theorem 3.1 (Aziz et al. [7]). *For fractional hedonic games, the core can be empty.*

Proof. We show that the game in Figure 3.1 has an empty core. Since core stability implies individual rationality, we only have to show that no individually rational coalition structure is in the core. As every coalition with more than two agents is unacceptable for at least two of them, no coalition with size at least three is individually rational. If there is no coalition S with $|S| \geq 3$, then there exists at least one coalition with size 1. Let i be an agent that is alone, and let j be the agent such that $u_j(i) = 2$. Note that j has an incentive to deviate to join i and i clearly has an incentive to form a coalition with j, so such coalition structures are not stable. Thus, every coalition structure is blocked by some coalition, then the core is empty. □

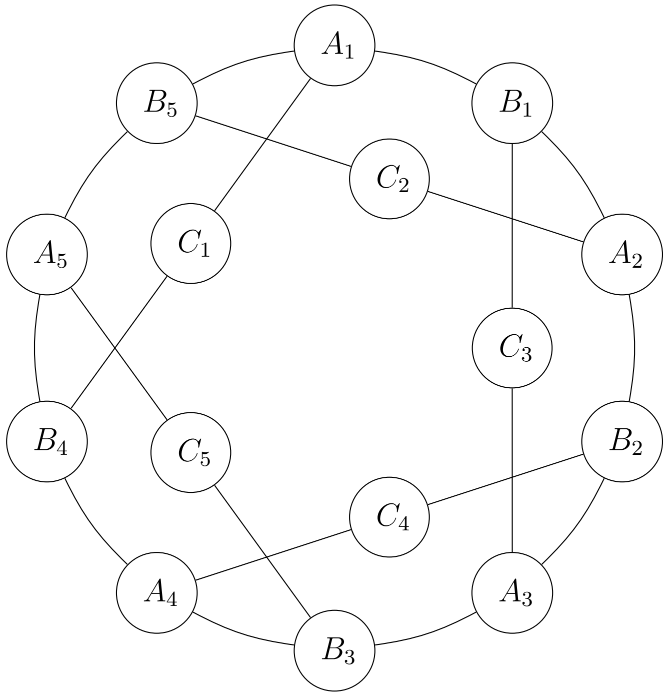


Figure 3.2: Example of a simple symmetric fractional hedonic game with empty core. For all $l \in \{1, \ldots, 5\}$, A_l and C_l are isomorphic to K_3 and B_l is isomorphic to K_2.

Even restricting the valuations to be simple and symmetric does not guarantee the core to be non-empty. Brandl et al. [11] showed that the game represented by the graph in Figure 3.2 has an empty core. The proof consists of an extensive analysis of cases.

Theorem 3.2 (Brandl et al. [11])**.** *For a simple symmetric fractional hedonic game the core can be empty.*

Now we show classes of simple symmetric fractional hedonic games for which the core is non-empty. We begin with games represented by cycles and paths.

Theorem 3.3 (Aziz et al. [7])**.** *For a simple symmetric fractional hedonic game represented by a graph G with $\Delta(G) \leq 2$, the core is non-empty.*

Proof. We give an algorithm to construct a core stable coalition structure. Observe that every connected component of G is either a path or a cycle. Let V_1 be the set of vertices that are in some connected component isomorphic to K_3. Form a coalition for each connected component isomorphic to K_3. Thus, the utility of an agent in V_1 is $\frac{2}{3}$. Let $G' = G - V_1$. Let M be a maximum matching of G'. Let V_2 be the vertices of G' covered by M and V_3 the remaining vertices of G'. Form a coalition for each edge of M. Leave the vertices in V_3 alone. Hence, the utility of an agent in V_2 is $\frac{1}{2}$ and the utility of an agent in V_3 is 0. Now we show that this coalition structure is in the core.

The agents in V_1 have the best utility possible, as they have no neighbor outside their coalition, therefore, they have no incentive to deviate. A blocking coalition formed only by vertices in V_3 does not exist because there is no edge between vertices in V_3 as, otherwise, the matching M would not be maximum. A blocking coalition formed by vertices only in V_2 is also not

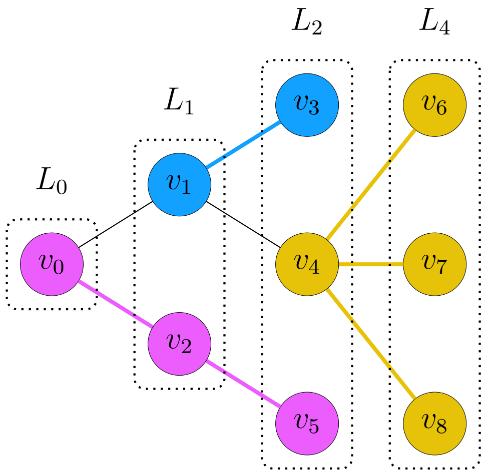


Figure 3.3: An output example of the algorithm to compute a core stable coalition structure for games represented by forests. Each color represents a coalition.

possible, because every coalition with size greater than 2 has at least one agent with utility less than or equal to $\frac{1}{2}$. If a coalition has size at least 4, the best utility an agent can have is $\frac{2}{4} = \frac{1}{2}$. Hence, any possible blocking coalition must be formed by one vertex from V_3 (every agent in V_2 has at most one neighbor in V_3) and two vertices from V_2. But, for one of the vertices of V_2, the utility is $\frac{1}{3}$ in this coalition, because one of them is not adjacent to the vertex from V_3. Hence, this coalition is not blocking. □

Another class of games for which the core is non-empty is the class containing the games represented by forests.

Theorem 3.4 (Aziz et al. [7])**.** *For a simple symmetric fractional hedonic game represented by a forest, the core is not empty.*

Proof. We first give an algorithm to construct a core stable coalition structure π. Let $G = (V, E)$ be the forest that represents the game. We can suppose that G is a tree, otherwise, we can apply the same argument for each connected component. At first, choose some vertex $v_0 \in V(G)$. Let L_k be the set of vertices at distance k from v_0. Let L_l be the last layer. For each vertex $v \in L_{l-1}$ which has a child in L_l let C_v be the set $\{u : \{v, u\} \in E(G),\ u \in L_l\} \cup \{v\}$. For each C_v, form a coalition with the vertices in it and remove it from G. Repeat this process until no more layers are left. If v_0 is left alone, put it in the smallest coalition that has one of its neighbors. Figure 3.3 illustrates how it is a coalition structure formed by this algorithm.

We show that π is in the core. First, we show that no vertex from a coalition containing only vertices from the lowermost two layers L_{l-1} and L_l can be in a blocking coalition. Suppose that there exists a blocking coalition S with a vertex u that is from L_l. Observe that S must contain vertex u and its parent v from layer L_{l-1} and $|S| < \pi(u)|$. Thus, S is not blocking for v, as $u_v(S) < u_v(\pi)$. If vertex u is from layer L_{l-1}, a possible blocking coalition S has all u's children from layer L_l and its parent, but S would not be blocking for the children c of u, $u_c(S) < u_c(\pi)$.

We remove the vertices from coalitions formed only by vertices from L_l and $L_{l-1}$and repeat the argument inductively.

Now, we show that there is no blocking coalition that contains v_0. Let us consider separately the two cases: $u_{v_0}(\pi) < \frac{1}{2}$ and $u_{v_0}(\pi) \geq \frac{1}{2}$. If $u_{v_0}(\pi) < \frac{1}{2}$, then the algorithm reached v_0 when all its children were already in some coalition and v_0 was placed in the smallest one. The only way that v_0 can increase its utility is if it is in a smaller coalition than its current one with at least one of its children, or to be with more than one of its children. But by the way π was built, a child of v_0 would only increase its utility by being in this coalition if its own children are too, but this coalition would not be blocking for them. If $u_{v_0}(\pi) \geq \frac{1}{2}$, a possible blocking coalition S that contains v_0 must contain more children of v_0 than $\pi(v_0)$ contains. Thus, S is not blocking for a child of v_0 that is in S and $\pi(v_0)$. Hence, $S \cup \pi(v_0) = \{v_0\}$. But, for every child v_i of v_0 in S, $u_{v_i}(\pi) \geq \frac{1}{2}$, then for $u_{v_i}(S) \geq \frac{1}{2}$, S must contain children of v_i, but S would not be blocking for the children of v_i. □

Aziz et al. [7] showed that games represented by bipartite graphs that admit a perfect matching have a non-empty core by proving that the perfect matching induces a core stable coalition structure.

Theorem 3.5 (Aziz et al. [7]). *For a simple symmetric fractional hedonic game represented by a bipartite graph that admits a perfect matching, the core is not empty.*

Proof. Let G be a bipartite graph that admits a perfect matching. Let $\{\mathcal{N}', \mathcal{N}''\}$ be the parts of G. For every coalition $S \subseteq \mathcal{N}$, either $\frac{|S \cap \mathcal{N}'|}{|S|} \leq \frac{1}{2}$ or $\frac{|S \cap \mathcal{N}''|}{|S|} \leq \frac{1}{2}$. Hence, S has one agent i with $u_i(S) \leq \frac{1}{2}$. Let M be a perfect matching of G. Let each edge of M induce a coalition. This way, M induces a coalition structure, where every agent has utility $\frac{1}{2}$. Given that in any coalition $S \subseteq \mathcal{N}$, there is an agent with a utility of at most $\frac{1}{2}$, then there is no blocking coalition for the coalition structure induced by M. □

We now introduce the concept of packing in graphs. Let $\mathcal{F}$ be a set of graphs. A *$\mathcal{F}$-packing* of a graph G is a subgraph H of G such that each component of H is isomorphic to some element of $\mathcal{F}$. We can see each component of H as a coalition, this way, if H is a spanning subgraph of G, then H can induce a coalition structure. A *star* S_k is a bipartite complete graph $K_{1,k-1}$. A *center* of a star S_k is a vertex with degree equal to $k-1$ and vertices with degree different of $k-1$ are called *leaves*. We say that a star S_k, with $k > 2$, has one center c and $k-1$ leaves $l_1, l_2, \ldots, l_{k-1}$. We assume that S_2 has two centers and no leaves. A *star-packing* is a $\mathcal{F}$-packing such that $\mathcal{F} = \{S_1, S_2, S_3, \ldots\}$, where S_i is a star.

The *girth* of a graph G is the length of a shortest cycle of G. Aziz et al. [7] showed that games represented by graphs with large girth (at least 5) have a non-empty core by giving an algorithm to construct a star-packing that induces a core stable partition.

Theorem 3.6 (Aziz et al. [7]). *For a fractional hedonic game represented by a graph with girth at least* 5*, the core is non-empty.*

Proof. Let $G = (V, E)$ be the graph that represents the game. We denote each star-packing by π and we associate with each π an objective vector $\vec{x}(\pi) = (x_1, x_2, \ldots, x_{|V|})$ such that $x_i \leq x_j$ if and only if $1 \leq i \leq j \leq |V|$, and there is a bijection $f : V \to \{1, \ldots, |V|\}$ with $u_v(\pi) = x_{f(v)}$. This way, in $\vec{x}(\pi)$, the vertices' utility for π are in non-decreasing order. The idea is to compute a star-packing that maximizes its objective vector with respect to the lexicographical order. In

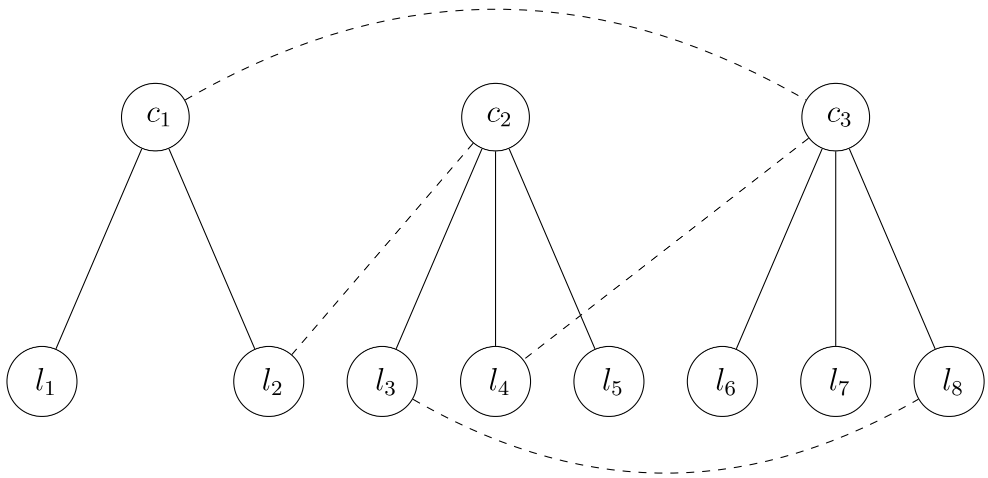


Figure 3.4: Non-optimal star-packing of a graph with girth 5. The packing is indicated by the solid edges.

star-packings with optimal objective vectors, we expect the stars to be balanced in size, without needlessly leaving a vertex alone.

Given that the number of star-packings for a graph G is finite, we have a star-packing π with an optimal objective vector. Now we show that such packing induces a core stable coalition structure.

First, we show that for two leaves l, l' of stars in π, we have that l and l' are not neighbors in G. For the sake of contradiction, suppose that this is not the case. Thus, l and l' have no neighbors in common, otherwise G would have a triangle. Therefore, l and l' are leaves of different stars. We have that l and l' are from stars isomorphic to S_k and $S_{k'}$, respectively, with $k > 2$ and $k' > 2$, because S_2 has no leaves. Hence, $u_l(\pi) < \frac{1}{2}$ and $u_{l'}(\pi) < \frac{1}{2}$, but if l and l' form a star by themselves, they both would have utility $\frac{1}{2}$. Observe that the centers of $\pi(l)$ and $\pi(l')$ would still be centers after l and l' leave their stars, and the leaves of $\pi(l)$ and $\pi(l')$ would improve their utility after l and l' leave their star. This means that this new star-packing has an objective vector greater than the optimal. Hence, we know that two leaves l, l' of π are not neighbors in G. For an illustration, see Figure 3.4. Observe that if l_3 and l_8 became centers, we have a packing better than the one shown on the figure.

For a contradiction, suppose that π is not in the core. Thus, there exists a blocking coalition S for which $u_i(S) > u_i(\pi)$ for all $i \in S$.

If $i \in \mathcal{N}$ is such that $u_i(\pi) = 0$, then i must have no neighbors in G. Otherwise, we could put i in any coalition that has one of its neighbors, which would increase its utility and we would have an objective vector greater than the optimal. Therefore, S only contains vertices that are not isolated in G.

We can divide the proof in three cases: S contains no center; S contains only one center; and S contains more than one center.

If S only contains leaves, then the utility of every vertex in S is 0, since leaves are not neighbors in G. Therefore S is not blocking.

If S contains only one center of π, we show that $\vec{x}(\pi)$ is not optimal. Let S consists of one center c and m leaves $l_1, \ldots, l_m$. Since there are no edges between leaves, we have that S induces a star. Let l be a leaf of S and let c' be the center of $\pi(l)$. We construct a star-packing π' where we move l to the coalition that contains c. We claim that $\vec{x}(\pi')$ is lexicographically larger than $\vec{x}(\pi)$. Observe that it suffices to show that $u_l(\pi') > u_l(\pi)$ and $u_k(\pi') \geq u_l(\pi')$, for all k with $u_k(\pi') < u_k(\pi)$.

To show that $u_l(\pi') > u_l(\pi)$, note that $u_c(\pi) < u_c(S)$ since c is a center in π and in S, and S is blocking. Then, we have that

$$u_c(\pi) = \frac{|\pi(c)| - 1}{|\pi(c)|} < \frac{|S| - 1}{|S|} = u_c(S).$$

Moreover,

$$u_l(\pi) = \frac{1}{|\pi(l)|} < \frac{1}{|S|} = u_l(S).$$

Thus, $|\pi(c)| < |S| < |\pi(l)|$. It follows that

$$|\pi'(l)| = |\pi(c) \cup \{l\}| \leq |S| < |\pi(l)|.$$

Hence, $u_l(\pi') > u_l(\pi)$.

Let k be such that $u_k(\pi') < u_k(\pi)$, we will show that $u_k(\pi') \geq u_l(\pi')$. Either k is c', the center of $\pi(l)$, or k is a leaf of $\pi(c)$. As c' is a center in $\pi(l)$ and l is a leaf, it means that there is other leaf different of l in $\pi(l)$, therefore, c' is still a center in π'. Hence, $u_{c'}(\pi') \geq \frac{1}{2} > u_l(\pi')$. Now assume that k is a leaf in $\pi(c)$, then in $\pi'(c)$ both k and l are leaves of the same star, therefore $u_k(\pi') = u_l(\pi')$.

If S contains more than one center in π, let us say c and c', then $u_c(\pi) \geq \frac{1}{2}$ and $u_{c'}(\pi) \geq \frac{1}{2}$. Hence, either $|S| = 2k + 2$ or $|S| = 2k + 3$ for some $k \geq 1$, since if $|S| \leq 3$, then $u_c(S) \leq \frac{1}{2}$ or $u_{c'}(S) \leq \frac{1}{2}$ because girth is at least 5. As both $u_c(S) > \frac{1}{2}$ and $u_{c'}(S) > \frac{1}{2}$, then

$$|\{i \in S : (c, i) \in E\}| \geq k + 2$$

and

$$|\{i : (c', i) \in E\}| \geq k + 2.$$

If $N_S(c) \cap N_S(c') = \emptyset$, then $|S| \leq 2k + 4$, which is a contradiction to the fact that $|S| \leq 2k + 3$. Therefore, if $N_S(c) \cap N_S(c') = x \geq 1$, then $|S| \geq 2k + 6 - x$. For $|S| \leq 2k + 3$, $x \geq 3$. Then, c and c' have at least three neighbors in common. But then G has a cycle with size at most 4, which is a contradiction to the fact that G has girth at least 5. Then, π is core stable. □

A *pseudoforest* is a graph where every connected component has at most one cycle. A *pseudotree* is a connected component of a pseudoforest. Below, we show that games represented by pseudoforests has a non-empty core by giving an algorithm to construct a core stable coalition structure.

Theorem 3.7. *For a simple symmetric fractional hedonic game represented by a pseudoforest, the core is not empty.*

Proof. We show how to construct a coalition structure π that is in the core. Let G be the graph that represents the game. We can suppose that G is a pseudotree, otherwise, we can apply the same argument for each pseudotree.

The case where G has no cycle is covered by Theorem 3.4, therefore we can suppose that G has a cycle. We can suppose that G has girth at most 4 because Theorem 3.6 covers the case where G has girth at least 5. Let A be the cycle that is a subgraph of G. For every vertex i of A, let T_i be a maximal tree that contains i and is a subgraph of $G \setminus \{j \,|\, j \in A, j \neq i\}$. Perform a breadth first search on T_i beginning at i. Let L_k be the set of vertices at distance k from i. Let L_l be the last layer. For each vertex $v \in L_{l-1}$ which has a child in L_l let C_v be the set $\{u \,|\, \{v, u\} \in E(G),\, u \in L_l\} \cup \{v\}$. For each C_v, form a coalition with the vertices in it and remove it from G. Repeat this process until no more layers are left.

Now, we consider two cases: (i) if A is a triangle and every vertex of A is either alone or in a coalition of size at most 2, form a coalition with the vertices of A and leave their previous partners alone; (ii) otherwise, let A' be the subgraph of A composed by the vertices of A that are alone in some coalition. Take a maximum matching of A', let us say M. Create a coalition for each edge of M. The vertices of A' not covered by M are left alone.

Now we prove that π is in the core. We can see every vertex i of A as a root of tree T_i. Then we can apply an argument similar to the one gave in the proof of Theorem 3.4 to show that there is no blocking coalition formed only by vertices of T_i. In fact, since there are no edges between vertices in T_i and vertices outside of T_i (except for i), any possible blocking coalition cannot contain vertices of T_i, except maybe for i. Now we show that the vertices in A are not in a blocking coalition.

If A is a triangle and $A \in \pi$, then by the construction of π, we know that for $i \in A$, we have two possibilities: either every neighbor of i has a utility at least $\frac{1}{2}$ for π or there are at most one neighbor of i with utility 0 for π. This happens because the coalition A is formed only if every vertex of A was either left alone or was put in a coalition of size at most 2. Let S be some subset of $\mathcal{N}$. We show that if $A \subseteq S$, then S is not a blocking coalition for π. Since vertices of A do not share neighbors outside of A, for at least one $j \in A$, $u_j(S) < \frac{2}{3} = u_j(A)$. Thus, S is not blocking. Now, we analyze the case where $A \not\subseteq S$. If $|S \cap A| = 1$, let $i \in A \cap S$. By the proof of Theorem 3.4, since S only contains vertices from tree T_i, we know that S is not blocking. If $|S \cap A| = 2$, let $i, j \in A$. As i and j have no neighbor in common in S, at least one of them has utility of at most $\frac{1}{2}$ for S. Then, S is not blocking.

If A is a triangle but not a coalition in π, then by the construction of π in the second case, there is a vertex $j \in A$, such that $u_j(\pi) \geq \frac{2}{3}$. A possible blocking coalition $S \subseteq \mathcal{N}$ such that $A \subseteq S$ is not a blocking coalition since S is not blocking for the neighbors of j that are not in A, and if $N_S(j) = A \setminus \{j\}$, then $u_j(S) \leq \frac{2}{3} \leq u_j(\pi)$. As j is equivalent to v_0 in the proof of Theorem 3.4, we have that j is not in a blocking coalition with its neighbors outside A. If there is a vertex $k \in A$ with $u_k(\pi) = 0$, then by the proof of Theorem 3.4, we have that k cannot be in a blocking coalition with its neighbors outside of A. For two distinct vertices $s, t \in A \setminus \{j\}$ we have that $u_s(\pi) \geq \frac{1}{2}$ or $u_t(\pi) \geq \frac{1}{2}$. Thus s and t cannot be in a blocking coalition together because one of them (maybe both) has utility at least $\frac{1}{2}$ for π.

If A is a square, there is at least one agent $i \in A$ such that $u_i(\pi) \geq \frac{1}{2}$, because of the way π was built. Therefore, A is not a blocking coalition. If a vertex $j \in A$ has utility 0, then, by the way π was built, all of its neighbors have utility at least $\frac{1}{2}$. Hence, a subset of A is not blocking. Since we can see the vertices of A as roots of trees, we know by the proof of Theorem 3.4 that they cannot be in a blocking coalition with a vertex from outside of A. Thus, π is not blocked by any coalition. □

Next, we show that computing a core stable partition for a symmetric fractional hedonic game is NP-hard. For this, we present a class of ASHGs called *aversion to enemies*. An ASHG is *aversion to enemies* if for all $i, j \in \mathcal{N}$ with $i \neq j$, $u_i(j) \in \{1, -n\}$. We show that a class of FHGs is equivalent to aversion to enemies games, thus, any computational result regarding the core stability of aversion to enemies games is also valid for FHGs.

Theorem 3.8 (Aziz et al. [7]). *Computing a core stable coalition structure is NP-hard for a symmetric fractional hedonic game, and deciding if a coalition structure is core stable is coNP-complete.*

Proof. Let $\mathcal{N}$ be a set of agents. Let $G = (V, E)$ be a graph such that $V = \mathcal{N}$. Let v be a valuation function such that for all $i, j \in \mathcal{N}$ with $i \neq j$, $v_i(j) = 1$ if and only if $\{i, j\} \in E$, and $v_i(j) = -n$ otherwise. Let be $\mathcal{G} = (\mathcal{N}, v)$ be an ASHG and $\mathcal{G}' = (\mathcal{N}, v)$ be an FHG. For some agent i, let $S \in \mathcal{N}_i$ be a coalition. If S is unacceptable in $\mathcal{G}$ for i, then S contains an agent j for which $v_i(j) = -n$. Hence S is also unacceptable for i in $\mathcal{G}'$. Observe that the converse is also true. If S is acceptable for i in $\mathcal{G}$, then i values non-negatively every agent in S. Let $T \in \mathcal{N}_i$ be another acceptable coalition for i in $\mathcal{G}$. Now we show that $S \succsim_i T$ in $\mathcal{G}$ if and only if $S \succsim_i T$ in $\mathcal{G}'$. Observe that if $S \succsim_i T$ in $\mathcal{G}$, then $|S| \geq |T|$. Therefore, $\frac{|S|-1}{|S|} \geq \frac{|T|-1}{|T|}$, which implies that $S \succsim_i T$ in $\mathcal{G}'$. The converse is clearly true. Thus, we have that a coalition structure π is core stable in $\mathcal{G}$ if and only if is core stable in $\mathcal{G}'$.

Dimitrov et al. [15] showed that the core of $\mathcal{G}$ is non-empty, but computing a core stable coalition structure for $\mathcal{G}$ is NP-hard. And Sung and Dimitrov [28] showed that deciding if a coalition structure is core stable for $\mathcal{G}$ is coNP-complete. □

According to Brandl et al. [11], deciding whether a symmetric fractional hedonic game has a non-empty core is NP-hard and it is NP-complete to decide if there is a Nash stable coalition structure. They also proved that deciding if a fractional hedonic game has an individually stable structure is NP-complete. Table 3.1 summarizes results regarding the complexity of checking the stability of fractional hedonic games.

Operation	**Complexity**
Computing a core stable coalition structure (SFHG)	NP-hard [7]
Checking if a coalition structure is in the core (SFHG)	coNP-complete [7]
Deciding emptiness of the core (SFHG)	NP-hard [11]
Deciding the existence of a Nash stable coalition structure (SFHG)	NP-complete [11]
Deciding the existence of a individually stable coalition structure (FHG)	NP-complete [11]

Table 3.1: The complexity of checking the stability of fractional hedonic games (FHG) and symmetric fractional hedonic games (SFHG).

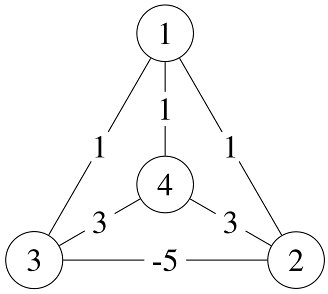


Figure 3.5: Example of a fractional hedonic game with no Nash stable coalition structure.

3.2 Nash stability

For fractional hedonic games with positive valuations only, the grand coalition is a Nash stable outcome. In this case, it's impossible for an agent alone to have utility greater than the one she has in the grand coalition, since the worst utility she can have for the grand coalition is 0, which is her utility for being alone. The following theorem follows from this fact.

Theorem 3.9 (Bilò et al. [8]). *For any graph G with positive weights only, the fractional hedonic game $\mathcal{G}(G)$ has a Nash stable coalition structure.*

The next theorem shows that if a game has negative valuations, then it may not have a Nash stable coalition structure.

Theorem 3.10 (Brandl et al. [11]). *There exists a symmetric fractional hedonic game with negative valuations such that it has no Nash stable coalition structure.*

Proof. We show that the game in Figure 3.5 has no Nash stable coalition structure. Observe that any coalition structure where agents 2 and 3 are in the same coalition is not stable because such coalition is unacceptable for both of them. Also, note that every coalition structure where agent 1 is alone is not stable because she prefers any coalition to be alone.

Without loss of generality, due to symmetry, we only have to consider the following possible coalition structures:

$$\begin{aligned}
\pi_1 &= \{\{1,2\},\{3,4\}\},\\
\pi_2 &= \{\{1,4\},\{2\},\{3\}\},\\
\pi_3 &= \{\{1,2,4\},\{3\}\}, \text{ and}\\
\pi_4 &= \{\{1,2\},\{3\},\{4\}\}.
\end{aligned}$$

Partition π_1 is not stable because agent 1 can improve her utility by joining 3 and 4. Partition π_2 is not stable because any of 2 or 3 can improve her utility by joining 1 and 4. Partition π_3 is not stable because agent 4 can improve her utility by joining agent 3. Partition π_4 is not stable because agent 3 can improve her utility by joining agent 4. □

3.2.1 Price of Stability and Anarchy

Social welfare is a measure of how well a society is. In our case, the social welfare is the sum of the utility of each agent. The price of stability and the price of anarchy are widely studied measures of loss efficiency caused by Nash stable outcomes in social welfare. The price of stability measures how good Nash stable outcomes can be with respect to the optimal social welfare, and the price of anarchy measures how bad Nash stable outcomes can be with respect to the optimal social welfare. In the first one, the social welfare from an optimal coalition structure is divided by the social welfare of some best Nash stable outcome, and in the second, the social welfare from an optimal coalition structure is divided by the social welfare of some worst Nash stable outcome.

Definition 3.11. The *social welfare* $\mathcal{SW}$ of a coalition structure π of a fractional hedonic game $(\mathcal{N}, u)$ is

$$\mathcal{SW}(\pi) = \sum_{i \in \mathcal{N}} u_i(\pi).$$

A coalition structure π is *optimal* if there exits no π' such that $\mathcal{SW}(\pi') > \mathcal{SW}(\pi)$.

Definition 3.12. Let π^* be an optimal coalition structure for an FHG $(\mathcal{N}, u)$. Let NS be the set of Nash stable coalition structures of a fractional hedonic game $(\mathcal{N}, u)$. The *price of anarchy PoA* is defined as

$$PoA(\mathcal{N}, u) = \max_{\pi \in NS} \frac{\mathcal{SW}(\pi^*)}{\mathcal{SW}(\pi)}.$$

And the *price of stability PoS* is defined as

$$PoS(\mathcal{N}, u) = \min_{\pi \in NS} \frac{\mathcal{SW}(\pi^*)}{\mathcal{SW}(\pi)}.$$

Bilò et al. [8] gave an upper bound of $n-1$ on the price of anarchy for fractional hedonic games represented by a weighted graph with no negatives weights.

Theorem 3.13 (Bilò et al. [8])**.** *For any weighted graph with non-negative edge weights G, it holds that $PoA(\mathcal{G}(G)) \leq n-1$.*

Proof. For an agent i, let $W_i = \max_{j \in \mathcal{N}} u_i(j)$. For any Nash stable coalition structure π, we have that $u_i(\pi) \geq \frac{W_i}{n}$, otherwise, agent i would join the coalition where the agent that defines W_i is. Hence, $\mathcal{SW}(\pi) \geq \frac{1}{n} \sum_{i \in \mathcal{N}} W_i$. We also have that for an optimal coalition structure π^*, $u_i(\pi^*) \leq \frac{n-1}{n} W_i$ for all $i \in \mathcal{N}$. Thus, $\mathcal{SW}(\pi^*) \leq \frac{n-1}{n} \sum_{i \in \mathcal{N}} W_i$. Therefore,

$$PoA(\mathcal{G}(G)) \leq \frac{\mathcal{SW}(\pi^*)}{\mathcal{SW}(\pi)} = \frac{\frac{n-1}{n} \sum_{i \in \mathcal{N}} W_i}{\frac{1}{n} \sum_{i \in \mathcal{N}} W_i} = n-1.$$

□

Bilò et al. [8] provided a lower bound of $\Omega(n)$ for the price of anarchy of fractional hedonic games by showing that for any $n \geq 2$ there exists a simple path P_n such that $PoA(P_n) = \Omega(n)$. They also proved that for any unweighted tree the price of stability is 1, and they provided an algorithm to compute an optimal Nash stable structure in polynomial time for this type of graph. Bilò et al. [8] also gave a lower bound of $\Omega(n)$ on the price of stability for symmetrical fractional hedonic games by showing that for any $n \geq 2$, there exists a game with n agents for which, the price of stability is $\Omega(n)$.

Theorem 3.14 (Bilò et al. [8])**.** *For any integer $n \geq 2$, there exists a FHG $G = (\mathcal{N}, v)$ with $|\mathcal{N}| = n$ for which, $PoS(\mathcal{G}(G)) = \Omega(n)$.*

Proof. Let $\mathcal{N}$ be an agent set with $|\mathcal{N}| = n$. Let v be a symmetric valuation function such that for all $i, j \in \{1, \ldots, n-2\}$, $v_i(n) = v_n(i) = 1$, $v_i(j) = 0$, $v_i(n-1) = v_{n-1}(i) = 0$, and $v_n(n-1) = v_{n-1}(n) = W$, for some W sufficiently greater than n. Let $G = (\mathcal{N}, v)$ be a FHG and π be a coalition structure where all agents are in the same coalition. Observe that π is the only Nash stable coalition structure for G. In every coalition structure where $i = 1, \ldots, n-1$ is not in the same coalition as n, i can improve her utility by joining the coalition that contains n. Hence $\mathcal{SW}(\pi) = \frac{2(W+n-2)}{n}$. A optimal coalition structure π^* is where n and $n-1$ are alone in one coalition. Thus, $\mathcal{SW}(\pi^*) = W$. Therefore, we have that

$$PoS(G) = \frac{\mathcal{SW}(\pi^*)}{\mathcal{SW}(\pi)} = \frac{nW}{2(W+n-2)} = \Omega(n).$$

□

Bilò et al. [9] provided upper and lower bounds on the price of stability for the games represented by bipartite graphs. They proved that for any bipartite graph G, $PoS(\mathcal{G}(G)) > 1.003$ and that $PoS(\mathcal{G}(G)) \leq 6(3 - 2\sqrt{2}) \approx 1.0294$. They also showed an upper bound of 4 on the price of stability for triangle-free graphs.

Kaklamanis et al. [20] proved that the price of stability of symmetric simple hedonic games is at least $1 + \frac{\sqrt{6}}{2} - \epsilon$ for all $\epsilon > 0$. They also proved that the price of stability for games represented by a simple symmetric graph with girth at least 5 is 1.

Olsen [24] introduced the following variation of a hedonic game where the calculation of the utility does not take into account the agent herself:

$$u_i(\pi) = \begin{cases} \frac{\sum_{j \in \pi(i)} u_i(j)}{|\pi(i)|-1}, & |\pi(i)| > 1 \\ 0, & |\pi(i)| \leq 1 \end{cases}$$

Kaklamanis et al. [20] proved that for any fractional hedonic game, the price of stability is 1 with the above utility function.

Chapter 4

Fractional Hedonic Games with Overlapping Coalitions

Now, we present a model that generalizes fractional hedonic game by allowing overlapping coalitions. We model those games as an agent having a resource that she can divide as she wants among the coalitions. Time is an example of a resource that an agent can invest in a coalition. This way, a coalition is determined by the amount of resource every agent contributes to that coalition. We say that an agent is part of a coalition if she contributes with strictly more than 0. Note that if we constrain the contribution to be either 0 or 1, we have the classical model.

Definition 4.1. Let $\mathcal{N} = \{1, 2, \ldots, n\}$ be a set of agents of size $|\mathcal{N}| = n$. A *coalition* (or *partial coalition*) is a vector $r \in [0, 1]^n$, where r_i denotes the participation of agent i in r. The support of a coalition is denoted by $\mathrm{supp}(r) = \{j \in \mathcal{N} : r_j \neq 0\}$. A coalition structure π on a set $S \subseteq \mathcal{N}$, denoted by π_S, is a list of k vectors $\pi = (r^1, \ldots, r^k)$ such that for all $r^k \in \pi$, $\mathrm{supp}(r^k) \subseteq S$ and for each $i \in S$, $\sum_{j=1}^{k} r_i^j = 1$. We denote $\pi_{\mathcal{N}}$ simply by π.

The main difference of our model to the model without overlapping coalition is how the utility of an agent is calculated. The utility of an agent considers the amount of her participation in a coalition as well as the participation of the other agents in this coalition.

Definition 4.2. As before, let $u_i : \mathcal{N} \to \mathbb{R}$ be a valuation function that denotes how much agent i values each other agent $j \in \mathcal{N}$ and let $u_i(i) = 0$. Given a coalition r, the utility of agent i is given as $u_i(r) = r_i \frac{\sum_{j \in \mathcal{N}} u_i(j) \cdot r_j}{|\mathrm{supp}(r)|}$. The utility function of agent i for a coalition structure $\pi = (r^1, \ldots, r^k)$ is given as $u_i(\pi) = \sum_{j=1}^{k} u_i(r^j)$. A *Fractional Hedonic Game with Overlapping Coalition* (FHGO) is defined as a pair $\mathcal{G} = (\mathcal{N}, u)$. The *outcome* of a game $\mathcal{G}$ is a coalition structure π.

We introduce the idea of the potential of a coalition for an agent as the utility this coalition would have to her if she were integrally in this coalition. This concept is relevant to simplify some calculations.

Definition 4.3. Let r be a coalition. For an agent $i \in \mathrm{supp}(r)$, we say that the potential of r is the utility that i would get from r, if i were totally in r. More formally, the potential of r to i is defined as $\alpha_i(r) = \frac{\sum_{j \in \mathcal{N}} r_j u_i(j)}{|\mathrm{supp}(r)|}$.

Our definition of the core is based on the definition of the conservative core of Chalkiadakis et al. [14]. Our core is similar to the classical one in the sense that the deviants, agents that are in a blocking set, can only form coalitions among themselves. That is, there is no coalition formed by deviants and non-deviants.

Definition 4.4. For a FHGO, a coalition structure $\pi = (r^1, \ldots, r^k)$ is blocked by a subset $S \subseteq \mathcal{N}$ if there is a coalition structure π' such that for each coalition $r^i \in \pi'$, either $\mathrm{supp}(r^i) \subseteq S$ or $\mathrm{supp}(r^i) \subseteq \mathcal{N} \setminus S$, and for all $j \in S$, $u_j(\pi') > u_j(\pi)$. If a coalition structure π has no blocking set $S \subseteq \mathcal{N}$, then π is said to be in the *core*.

Let π be a coalition structure that is blocked by some set S. We denote by π'_S the coalition structure on S, such that for each $i \in S$, $u_i(\pi'_S) > u_i(\pi)$. We show that if a set S blocks a coalition structure π, then for each agent in S, there exists a coalition in π'_S for which the potential of this coalition is greater than the utility of π.

Lemma 4.5. *For a FHGO represented by a simple graph and a coalition structure π, a set $S \subseteq \mathcal{N}$ blocks π with overlapping coalitions if and only if for every $i \in S$ there is a coalition $r \in \pi'_S$ for which holds that $i \in \mathrm{supp}(r)$ and $\alpha_i(r) > u_i(\pi)$.*

Proof. In a fractional hedonic game with overlapping coalitions, the utility of an agent $i \in \mathcal{N}$ can be seen as the convex combination of the potential of each coalition this agent is part of. Thus, for $u_i(\pi'_S) > u_i(\pi)$ there must exist a coalition $r \in \pi'_S$ such that $\alpha_i(r) > u_i(\pi)$. □

In the following sections, we present results regarding the existence of core stable coalition structures.

4.1 Cycles and Paths

We analyze fractional hedonic games with overlapping coalitions represented by graphs such that every connected component is either a cycle or a path. We show that the algorithm given by Aziz et al. [7] produces core stable coalition structures. The idea of the algorithm is to decompose the graph into disjoint cliques.

Theorem 4.6. *For a FHGO represented by a simple graph G with maximum degree at most 2, the core is non-empty.*

Proof. We will use the algorithm provided by Aziz et al. [7] to construct a coalition structure π and show that it is in the core. First, for each component C isomorphic to K_3 form a coalition r with $r_i = 1$ for all $i \in C$. Let V_1 be a subset of $V(G)$ such that $v \in V_1$ if v is in a K_3. Let $G' = G - V_1$. Let M be a maximum matching of G'. For each edge $\{u, v\} \in E(M)$, form a coalition with u and v. The vertices not covered by M are left alone. Let V_2 be a subset of $V(G)$ such that $v \in V_2$ if v is covered by M. Note that, if an agent is in a K_3, she has utility $\frac{2}{3}$ and if an agent is covered by a maximum matching of G', she has utility $\frac{1}{2}$.

Now we prove that there is no subset $S \subseteq \mathcal{N}$ that blocks π. In Theorem 3.3, we proved that an integral coalition formed by all the agents in S does not block π. Hence, the agents in S must form at least two partial coalitions. An agent of a K_3 component has the best possible utility because she is in a coalition with all the other agents she values with 1. Therefore, agents from a K_3 component are not in S.

We only have to worry about the vertices from G'. Now we show that a vertex from a path or a cycle is not in S. Let i be a vertex from G'. If i is covered by M then i has utility $\frac{1}{2}$. Observe that a vertex from V_2 that has degree 1 cannot be in a blocking set. The only way i can have a utility greater than $\frac{1}{2}$ is when she is with her two neighbors in some coalition because every coalition with only two agents has utility (and potential) at most $\frac{1}{2}$. By Lemma 4.5, we know that i needs some coalition with potential greater than $\frac{1}{2}$. Let us call this coalition r. We know that $|\operatorname{supp}(r)| \geq 3$ since the potential of i for r must be greater than $\frac{1}{2}$. Let $x = \sum_{j \in \mathcal{N}} u_i(j) \cdot r_j$. We need that $\frac{x}{3} > \frac{1}{2}$. But there are only two agents that i values with 1, and both of them have to contribute to r with at least $\frac{1}{2}$. Observe that one of the neighbors of i has utility $\frac{1}{2}$ for π, let us say t. As $\alpha_t(r) \leq \frac{1}{3}$, t also needs to be in some other coalition, let us say r'. Note that $\alpha_t(r') \leq \frac{2}{3}$. But, since $r_t \geq \frac{1}{2}$, $u_t(r') \leq \frac{1}{2} \cdot \frac{2}{3}$. Hence, the utility of t for a coalition structure π_S on S is at most $\frac{1}{2}\left(\frac{1}{3} + \frac{2}{3}\right)$. Therefore, S is not blocking for t.

If i is not covered by M, then she has utility 0. She needs only another agent to improve her utility. Let j be such an agent. Since a maximum matching of a path or a cycle does not cover at most one vertex, j has utility $\frac{1}{2}$ for π and S is not blocking by the argument given above. □

4.2 Forests

In this section, we show that for fractional hedonic games with overlapping coalitions represented by forests the core is non-empty. For this, we show that the algorithm used in the proof of Theorem 3.4 produces a core stable coalition structure.

Theorem 4.7. *For a FHGO represented by a forest, the core is non-empty.*

Proof. We use the same algorithm used in the proof of Theorem 3.4 to construct a coalition structure π. Observe that the support of every coalition of π induces a star. Now we show that π is core stable. The proof follows by induction in the number of layers.

We begin by the base cases, where there are two and three layers. If there are two layers, for vertex v_0 to be in a blocking set S, it needs more neighbors in S than there is in the support of its coalition in π. But, all the neighbors of v_0 are already in the same coalition in π. If there are three layers, we have two cases: either $u_{v_0}(\pi) < \frac{1}{2}$ or $u_{v_0}(\pi) \geq \frac{1}{2}$. If, $u_{v_0}(\pi) < \frac{1}{2}$, every neighbor of v_0 has utility greater than $\frac{1}{2}$ for π, then, for a neighbor v_i of v_0 to be in a blocking set S, v_i needs for all its neighbors and v_0 to be in S, but since v_0 is only neighbor of v_i, then S is not blocking for the neighbors of v_i. If $u_{v_0}(\pi) \geq \frac{1}{2}$, then v_0 has neighbors that are centers and neighbors that are leaves in the stars induced by the support of their coalitions. Let $r \in \pi$ be the coalition of π such that $v_0 \in \operatorname{supp}(r)$. For v_0 to be in a blocking set S, it needs that $|S \cap N_S(v_0)| \geq |\operatorname{supp}(r)|$. Hence, at least a neighbor of v_0 that is a center has to be in S, let us say v_k. Moreover, for S to block, it needs that the neighbors which are in the same coalition as v_k in π to be in S. However, S is not blocking for these neighbors of v_k since they value each vertex in S with 0, except v_k.

Now, we show that no vertex from a coalition formed only by vertices from L_l and L_{l-1} can be in a blocking set S. Let v be a vertex from L_{l-1}. For vertex v to be in S, it needs that its parent and all of its children to be in S and to be in some coalition (partial or not) with it. This is because v needs some coalition with potential greater than the utility it has for π, thus, it needs to be in some coalition with all of its children and its parent. But the children of v will not improve their utility because the only neighbor they have is v, therefore S would not be blocking. Let u be a vertex from L_l, let t be the parent of u. Vertex u would be in S only if its parent t is in S and some child of t is not. But, S is only blocking for t if its parent and all of its children are in S. Hence, S is not blocking for u. Remove all vertices from coalitions formed only by vertices from L_l and L_{l-1} and repeat the argument inductively. □

4.3 Bipartite graphs with perfect matching

In this section, we prove that for fractional hedonic games with overlapping coalitions represented by bipartite graphs with a perfect matching, the core is non-empty by showing that a coalition structure in which the coalitions induce edges of a perfect matching is core stable.

Lemma 4.5 provided an important condition for the existence of a blocking set. The main proof of this section is built upon showing that this condition can not be satisfied. The proof idea is to show that in every coalition, the number of agents for whom the coalition potential is greater than a half is less than the number of agents that contribute with more than a half for this coalition. So we show that this coalition structure fails to satisfy the condition of Lemma 4.5.

Now we prove that, if there exists some blocking set S, then for each coalition r in a blocking coalition structure on S, the set of agents for whom the potential of r is greater than one half is an independent set. More formally, for every coalition $r \in \pi'_S$, let $H_r = \{i : i \in \mathrm{supp}(r), \alpha_i(r) > u_i(\pi)\}$. We denote the neighborhood of some vertex i on a subset $T \subseteq \mathcal{N}$ by $N_T(i)$.

Lemma 4.8. *Let π be a coalition structure for a FHGO represented by a bipartite graph G that admits a perfect matching, such that the coalitions of π induce a perfect matching of G. If π is not in the core, it holds that H_r induces an independent set for each $r \in \pi'_S$, such that S is a blocking set for π.*

Proof. Suppose that there exists some coalition $r \in \pi'_S$ for which H_r does not induce an independent set. Let $i, j \in H_r$ such that i and j are adjacent in G, thus $N_{\mathrm{supp}(r)}(i) \cap N_{\mathrm{supp}(r)}(j) = \emptyset$ because $\mathrm{supp}(r)$ induces a subgraph of G and G is bipartite. We know that both i and j must be adjacent to strictly more than a half of the agents in $\mathrm{supp}(r)$, because $\alpha_i(r) > \frac{1}{2}$ and $\alpha_j(r) > \frac{1}{2}$. Thus, i and j must have a neighbor in common, which is a contradiction with the fact that G is bipartite. Hence, H_r must induce an independent set. □

Now we show that the number of agents that contribute to a coalition r with more than $\frac{1}{2}$ is strictly greater than the number of agents for whom r has potential greater than $\frac{1}{2}$.

Lemma 4.9. *Let π be a coalition structure for a FHGO represented by a bipartite graph G that admits a perfect matching, such that the coalitions of π induce a perfect matching of G. If π is blocked by some set S, then for every coalition $r \in \pi'_S$ and for every agent $i \in H_r$, let $U_r(i) = |\{j : j \in N_{\mathrm{supp}(r)}(i), r_j > \frac{1}{2}\}|$. It holds that $|U_r(i)| > |H_r|$.*

Proof. Let $r \in \pi'_S$ be a coalition. For contradiction, suppose that there exists some agent $i \in H_r$ for which $|U_r(i)| \leq |H_r|$. We know by Lemma 4.8 that H_r induces an independent set. Now we show that $\alpha_i(r) \leq \frac{1}{2}$. If $|U_r(i)| \leq |H_r|$, even if each agent in $U_r(i)$ contributes with 1 to r, there is at least the same number of vertices $j \in H_r$ with $u_i(j) = 0$, since $U_r(i) \cap H_r = \emptyset$, hence $\alpha_i(r) \leq \frac{1}{2}$. This is a contradiction because, by definition, $\alpha_i(r) > \frac{1}{2}$. Hence, for every $i \in H_r$, $|U_r(i)| > |H_r|$. □

Below, we prove that a coalition structure that induces a perfect matching on the graph representing the game is in the core by showing that any subset $S \subseteq \mathcal{N}$ fails to form a coalition structure that satisfies the condition stated in Lemma 4.5.

Theorem 4.10. *For a FHGO $\mathcal{G} = (\mathcal{N}, u)$ represented by a bipartite graph G that admits a perfect matching, the core is non-empty.*

Proof. Let M be a perfect matching of G. Let π be a coalition structure where each coalition of π corresponds to an edge of M. Note that, for every $i \in \mathcal{N}$, $u_i(\pi) = \frac{1}{2}$. Suppose for the sake of contradiction that π is not in the core. Then, there must be some blocking set $S \subseteq \mathcal{N}$ for π. Let π_S be a coalition structure such that for every $i \in S$, $u_i(\pi_S) > u_i(\pi)$. Let D be a directed graph where there is one vertex for each agent of S and a vertex for each coalition of π_S. An arc of D goes from the vertex that represents an agent i to the vertex that represents a coalition r only if $r_i > \frac{1}{2}$, and it goes from the vertex that represents r to the vertex that represents i only if $\alpha_i(r) > \frac{1}{2}$. Observe that the out-degree of a vertex that represents an agent is at most 1. Note that, by Lemma 4.5, each vertex that represents an agent must have in-degree at least 1. By Lemma 4.9, we know that the in-degree of a vertex that represents a coalition is strictly greater than the out-degree. Since in a directed graph the sum of the in-degrees must equal the sum of the out-degrees, there must exist some vertex that represents an agent with in-degree equals to 0. Then, there must exists some agent $i \in S$ for which there exists no $r \in \pi_S$ such that $\alpha_i(r) > \frac{1}{2}$. Therefore, there is no blocking set S for π. □

Chapter 5

Additively Separable Hedonic Games

In this chapter, we present some results regarding the stability of additively separable hedonic games and our model for additively separable hedonic games with overlapping coalitions. We begin by showing that a ASHG with symmetric preferences has a Nash stable outcome, and therefore it also has an individually stable outcome.

Theorem 5.1 (Bogomolnaia and Jackson [10]). *An additively separable hedonic game with symmetric preferences has an individually stable coalition structure, as well as a Nash stable structure.*

Proof. As shown in Figure 2.2, the Nash Stability implies individual stability, so we have only to show that there is a Nash stable coalition structure.

Let π^* be a coalition structure that maximizes the social welfare. We will show that π^* is Nash stable. Suppose, by contradiction, that there exists an agent i that can deviate forming the coalition structure π such that $\sum_{j\in\pi(i)} u_i(j) = u_i(\pi) > u_i(\pi^*) = \sum_{j\in\pi^*(i)} u_i(j)$.

Note that, except for the coalition from where i left and the coalition to which i joined, any other coalition is equal in π and π^*, so the social welfare of those coalitions are equal in the two coalition structures. Then

$$\mathcal{SW}(\pi) - \mathcal{SW}(\pi^*) = u_i(\pi) - u_i(\pi^*) + \sum_{k\in\pi(i)} u_k(i) - \sum_{j\in\pi^*(i)} u_j(i) = 2\cdot u_i(\pi) - 2\cdot u_i(\pi^*) > 0,$$

which is a contradiction since $\mathcal{SW}(\pi^*)$ is maximum. Note that the last equivalence is true because the game is symmetric. □

The price of stability and anarchy can be defined for additively separable games in a similar way as they were defined for fractional hedonic games in Section 3.2.

Corollary 5.2. *The price of stability for an additively separable hedonic game with symmetric preferences is 1.*

Proof. It follows from the proof of Theorem 5.1. □

Sung and Dimitrov [29] proved that deciding whether an additively separable hedonic game has a Nash stable coalition structure or has a coalition structure in the core is NP-complete in the strong sense. Deciding if a coalition structure is in the core is coNP-complete according to Sung and Dimitrov [28]. Aziz et al. [4] showed that even with symmetric preferences, checking the emptiness of the core and the strict core is NP-hard in the strong sense and verifying whether the grand coalition is Pareto optimal is coNP-complete. The problem of checking if a structure is Pareto optimal is coNP-complete in the strong sense, even if the preferences are symmetric and strict, according to Aziz, Brandt, and Seedig [6]. They also used the concept of *serial dictatorship* to show that a Pareto optimal coalition structure, under strict preferences, can be computed in polynomial time. Table 5.1 summarizes the results regarding the complexity of checking the stability of additively separable hedonic games.

Operation	**Complexity**
Checking if a coalition structure is in the core	coNP-complete [28]
Deciding emptiness of the core	NP-complete [29]
Deciding the existence of a Nash stable coalition structure	NP-complete [29]
Checking if a coalition structure is Pareto optimal	coNP-complete [4]
Computing a Pareto optimal coalition structure with strict preference	P [6]

Table 5.1: The complexity of checking stability for additively separable hedonic games.

5.1 Additively Separable Hedonic Games with Overlapping Coalitions

In this section, we present a model that generalizes additively separable hedonic games by allowing overlapping coalitions. The model is similar to the model from fractional hedonic games. The definitions of coalition and coalition structure are identical to the definitions given in Chapter 4. We can interpret the participation of an agent in a partial coalition as the probability the agent is in the non-partial version of this coalition. Thus, each agent has a probability distribution over all possible non-partial coalitions. An agent's utility for a coalition structure can be seen as the expected utility if the coalitions are formed at random using the probability distributions given by the agents.

Definition 5.3. An *Additively Separable Hedonic Game with Overlapping Coalition* (ASHGO) is a pair $G = (\mathcal{N}, u)$ such that for each $i \in \mathcal{N}$, $u_i : \mathcal{N} \to \mathbb{R}$ is a valuation function that denotes how much agent i values every other agent $j \in \mathcal{N}$. We say that $u_i(i) = 0$. Given a coalition r, the utility of agent i is given as

$$u_i(r) = r_i \sum_{j \in \mathcal{N}} u_i(j) \cdot r_j.$$

The outcome of a game G is a coalition structure π. The utility function can be overloaded to define utility of a coalition structure $\pi = (r^1, \ldots, r^k)$ as

$$u_i(\pi) = \sum_{j=1}^{k} u_i(r^j).$$

The social welfare of a coalition r is given as

$$SW(r) = \sum_{j \in \mathcal{N}} u_j(r),$$

and the social welfare of a coalition structure $\pi = (r^1, \ldots, r^k)$ is given as

$$\mathcal{SW}(\pi) = \sum_{j=1}^{k} SW(r^j).$$

For an agent i and a coalition r, we say that r_i is the *participation* of agent i in r. If $r_i = 1$, we say that r_i is an *integral participation*.

We give a definition of potential of coalition for an agent in a similar way we did for FHGOs. This concept is relevant to simplify some calculations.

Definition 5.4. Let r be a coalition. For an agent $i \in \text{supp}(r)$, we say that the potential of r is the utility that i would get from r, if i were totally in r. More formally, the potential of r to i is defined as $\alpha_i(r) = \sum_{j \in \mathcal{N}} r_j u_i(j)$.

The following theorem shows that for an additively separable hedonic game with overlapping coalitions, the expected social welfare is limited by the optimal social welfare of the classical version of the game. Which means that the expected social welfare is equal to the optimal social welfare.

Theorem 5.5. *The social welfare of a symmetric ASHGO is less than or equal to the optimal social welfare of this game without overlapping coalitions.*

Proof. Let π be some coalition structure of some ASHGO. The proof follows by induction in the number of non-integral participation in coalitions of π. If all participations are integral, then every agent is part of only one coalition. Trivially, the social welfare of this outcome is less than or equal to the optimal social welfare of the version of this game without overlapping coalitions.

Let i be an agent and π a coalition structure such that for at least two partial coalitions r^k and r^l we have $r^k_i > 0$ and $r^l_i > 0$. Without loss of generality, suppose that $\alpha_i(r^k) \geq \alpha_i(r^l)$.

Let π' be a coalition structure constructed in the following way: $r^{k'}$ is a partial coalition such that $r^{k'}_i = r^k_i + r^l_i$, and for all $j \neq i$, $r^{k'}_j = r^k_j$; and $r^{l'}$ is a partial coalition such that $r^{l'}_i = 0$ and for all $j \neq i$, $r^{l'}_j = r^l_j$; any other partial coalition from π is also in π'.

We show that $\mathcal{SW}(\pi') \geq \mathcal{SW}(\pi)$. Suppose by contradiction that $\mathcal{SW}(\pi') < \mathcal{SW}(\pi)$. Hence, $\mathcal{SW}(\pi) - \mathcal{SW}(\pi') > 0$.

We have that

$$\mathcal{SW}(\pi) - \mathcal{SW}(\pi') = u_i(r^k) + u_i(r^l) + r^k_i \left(\sum_{j \in \text{supp}(r^k)} r^k_j u_j(i) \right) + r^l_i \left(\sum_{s \in \text{supp}(r^l)} r^l_s u_s(i) \right)$$
$$- u_i(r^{k'}) - (r^k_i + r^l_i) \left(\sum_{j \in \text{supp}(r^k)} r^k_j u_j(i) \right).$$

By the symmetry of preferences, we have that

$$
\begin{aligned}
\mathcal{SW}(\pi) - \mathcal{SW}(\pi') &= 2u_i(r^k) + 2u_i(r^l) - u_i(r^{k'}) - r_i^k \sum_{j \in \mathrm{supp}(r^k)} r_j^k u_j(i) \\
&\quad - r_i^l \sum_{j \in \mathrm{supp}(r^k)} r_j^k u_j(i) \\
&= 2u_i(r^k) + 2u_i(r^l) - 2r_i^k \sum_{j \in \mathrm{supp}(r^k)} r_j^k u_j(i) - 2r_i^l \sum_{j \in \mathrm{supp}(r^k)} r_j^k u_j(i) \\
&= 2u_i(r^l) - 2r_i^l \sum_{j \in \mathrm{supp}(r^k)} r_j^k u_j(i) \\
&= 2r_i^l \sum_{s \in \mathrm{supp}(r^l)} r_s^l u_s(i) - 2r_i^l \sum_{j \in \mathrm{supp}(r^k)} r_j^k u_j(i) \\
&= 2r_i^l \alpha_i(r^l) - 2r_i^l \alpha_i(r^k).
\end{aligned}
$$

Given that $\alpha_i(r^k) \geq \alpha_i(r^l)$, we have

$$\mathcal{SW}(\pi) - \mathcal{SW}(\pi') \leq 0.$$

This is a contradiction. Then, we have that $\mathcal{SW}(\pi) \leq \mathcal{SW}(\pi')$. As π' has less non-integral participations than π, then by induction hypothesis, the social welfare of π is less than or equal to the optimal social welfare of this game without overlapping coalitions.

□

Chapter 6

Conclusion

In many relevant scenarios that can be modeled with hedonic games, it is common that an agent is part of more than one coalition at the same time, therefore allowing overlapping coalitions in hedonic games is a straightforward generalization of the model of such games. We initiated the study of hedonic games with overlapping coalitions, approaching these two appealing classes of games. This work focused on one definition of core, but others may be considered as well as non-cooperative stability concepts. We showed that for fractional hedonic games with overlapping coalitions represented by cycles and paths, forests, and bipartite graphs with perfect matching, the core is non-empty. We also showed that for fractional hedonic games represented by pseudoforests the core is non-empty. For symmetric additively separable hedonic games with overlapping coalitions, we showed that the social welfare of any coalition structure is at most the optimal social welfare of the game version without overlapping coalitions.

Next, we highlight some open problems. It is still unknown if for symmetric fractional hedonic games there always exists an individually stable outcome. It would be interesting to identify which are the classes of fractional hedonic games with negative valuations for which Nash stable results exist. For simple FGHs represented by bipartite graphs that do not admit perfect matching and with girth 4, it is still unknown whether the core is non-empty. There are many classes of hedonic games with overlapping coalitions that can be analyzed to verify the emptiness of the core, in particular, those for which the core is non-empty in the non-overlapping version. We highlight in particular the class of FHGO represented by a graph with girth at least 5. Following in this direction, it is interesting to verify whether, for a FHGO, the core is non-empty if the core of the non-overlapping version is also non-empty.

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