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UNIVERSIDADE ESTADUAL DE CAMPINAS
Instituto de Física Gleb Wataghin
Dissertação de Mestrado

The representations of *HOM(2)* and *SIM(2)* in the context of Very Special Relativity

(As representações de *HOM(2)* e *SIM(2)* no contexto da Very Special Relativity)

Gustavo Salinas de Souza

Orientador: Dharam Vir Ahluwalia

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Dissertação apresentada ao Instituto de Física Gleb Wataghin da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Mestre em Física.

Orientador: Dharam Vir Ahluwalia
Coorientador: Pedro Cunha de Holanda

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Abstract

The present work is devoted to a systematic study of the representations of the groups *HOM(2)* and *SIM(2)*, which are subgroups of the Lorentz group. Theories with symmetries given by these subgroups are known to preserve the constancy of the speed of light, this fact being referred as Very Special Relativity. It is shown that there are finite-dimensional reducible representations of *HOM(2)* and *SIM(2)* that are not completely reducible, and thus cannot be obtained entirely from irreducible representations. These are obtained directly from the representations of the Lie algebras *hom(2)* and *sim(2)*, using the knowledge of the universal covering groups of *HOM(2)* and *SIM(2)*, which are also presented in the text.

Resumo

O presente trabalho é dedicado a um estudo sistemático das representações dos grupos *HOM(2)* e *SIM(2)*, que são subgrupos do grupo de Lorentz. É sabido que teorias cujas simetrias são descritas por tais subgrupos preservam a constância da velocidade da luz, esse fato sendo referido como Very Special Relativity. É mostrado que existem representações de *HOM(2)* e *SIM(2)* redutíveis e de dimensão finita, que portanto não podem ser obtidas inteiramente de representações irredutíveis. Estas são obtidas diretamente das representações das álgebras de Lie *hom(2)* e *sim(2)*, usando o conhecimento dos grupos de cobertura universal de *HOM(2)* e *SIM(2)*, que também são apresentados no texto.

Contents

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Chapter 1

Introduction

The theory of Special Relativity describes the inhomogeneous group of Lorentz transformations, i.e, the Poincaré group, as the fundamental symmetry group of physical theories. This comes directly from the causal structure of Minkowski spacetime $\mathcal{M}$, in the sense that any transformation which is not in the Poincaré group will change the causal relations between a subspace of $\mathcal{M}$. In this scenario, a set of *admissible* observers are the ones that can be connected by the transformations in the Poincaré group. For such observers, the principle of relativity is then stated as

The laws of physical systems are the same in every admissible frame of reference.

This implies the less-restrictive affirmation

The speed of light is the same in every admissible frame of reference,

this being what most experiments search in order to confirm the predictions of Special Relativity (e.g., the well-known Michelson-Morley interferometer experiment).

The possibility that the fundamental symmetry group of nature is just a subgroup of the Poincaré group is not excluded by the consideration of causality within $\mathcal{M}$. Furthermore, only invariance under a subgroup denoted as *HOM(2)* is necessary to guarantee the constancy of the speed of light, thus promoting this subgroup to a candidate for describing the symmetries of physical theories. That is the idea of *Very Special Relativity* (*VSR*). In this context, the principle of relativity can still be considered to be valid, if the concept of admissible frames of reference is allowed to be changed. For example, in a *HOM(2)*-invariant context, the admissible observers can be defined as the ones connected by transformations within the *HOM(2)* subgroup. As a result, every admissible observer should construct the same physical laws and the principle of relativity is preserved.

Theories based on VSR break spatial isotropy and, because of this, are generally not considered in the context of Standard Model (SM) physics. In this context, the Lorentz-violating terms need to be regarded as only perturbations, to take into consideration the effects of spatial anisotropy on modern-day particle physics experiments. There is, however, the possibility that VSR is naturally placed outside the Standard Model, where experiments testing spatial isotropy are not available. For example, in it is proposed that VSR provides the correct symmetry group for dark matter. This creates a strong link between SM and non-SM physics in the sense that the fundamental principles of Special Relativity are basically preserved in both, the only difference being the group of fundamental symmetries.

In his well-known work on the unitary irreducible representations of the inhomogeneous Lorentz group, Wigner defended the idea that every physical object (in the form of one-particle states in a Hilbert space) can be described by these representations. In addition, when one considers the quantum theory of fields, also the non-unitary finite-dimensional representations are used to construct the Lagrangian densities of the quantum fields that describe particles. These facts conjure to give a central role to the representations of symmetry groups when constructing physical models from a fundamental perspective. That is the philosophy of the present work. In order to construct VSR-invariant theories from the start, the need is to consider the representations of the VSR subgroups. The intention here is to systematically construct the finite-dimensional representations of the VSR subgroups *HOM(2)* and *SIM(2)*.

The work is divided as follows. In chapter 2, the basic structure and properties of the Lorentz group are discussed and its Lie algebra is obtained. In chapter 3, the necessary concepts of representation theory are considered, with special attention to the concept of *universal covering groups*. Chapter 4 presents the construction of some of the representations of the Lorentz group, using the same method of chapter 5, in which the representations of *HOM(2)* and *SIM(2)* are treated.

Chapter 2

Lorentz group and Very Special Relativity

2.1 Causality in Minkowski spacetime

A physical phenomenon can be pictured as a sequence of events, each of which is characterized by a position in space $\boldsymbol{x} = (x^1, x^2, x^3)$ and an instant of time $t = x^0$. An appropriate setting to study spatiotemporal properties of these events is composed of geometrical objects of the form $x^\mu = (t, \boldsymbol{x}) = (x^0, x^1, x^2, x^3)$, which are called *position four-vectors*. This setting receives the name of *Minkowski spacetime* ($\mathcal{M}$) and is equivalent to a four-dimensional real vector space with inner product defined as

$$x^\mu y_\mu = \eta_{\mu\nu} x^\mu y^\nu = \eta^{\mu\nu} x_\mu y_\nu = -x^0 y^0 + x^1 y^1 + x^2 y^2 + x^3 y^3, \tag{2.1}$$

where $y^\mu = (y^0, y^1, y^2, y^3) \in \mathcal{M}$ and $\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ is called the *Minkowski metric*, as it captures the geometric and causal structure of Minkowski spacetime. This choice of signs in the metric is denoted the $(-, +, +, +)$ signature. It is clear from the definition given in equation 2.1 that the inner product defined in Minkowski spacetime is not *positive definite*, i.e., that $x^\mu x_\mu$ is not in general positive. In fact, a four-vector x^μ is classified as a *time-like* vector if $x^\mu x_\mu < 0$, a *null* (or *light-like*) vector if $x^\mu x_\mu = 0$ or a *space-like* vector if $x^\mu x_\mu > 0$.

The classification of four-vectors with respect to the sign of $x^\mu x_\mu$ has immediate consequences for the physical interpretation of *displacement four-vectors* (the subtraction of position four-vectors). For a displacement $\Delta x^\mu = (\Delta t, \boldsymbol{\Delta x})$ that belongs to the trajectory of a particle moving with constant velocity in Minkowski spacetime, the relation

$$\Delta x^\mu \Delta x_\mu = -(\Delta t)^2 + (\boldsymbol{\Delta x})^2 \begin{cases} < 0, & \text{if the particle moves slower than light} \\ = 0, & \text{if the particle moves at light speed} \end{cases} \tag{2.2}$$

is valid. Thus, the region in Minkowski spacetime that can be causally connected to an event $\bar{x}^\mu = (\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3)$ by signals that move slower than light comprises all four-vectors x^μ that obey

$$\eta_{\mu\nu}(x-\bar{x})^\mu(x-\bar{x})^\nu < 0. \tag{2.3}$$

The boundary of this region, given by

$$\eta_{\mu\nu}(x-\bar{x})^\mu(x-\bar{x})^\nu = 0, \tag{2.4}$$

defines a four-dimensional cone called a *light* (or *null*) *cone* as it is composed of events that can only be reached from the event at $\bar{x}^\mu$ by light-speed signals.

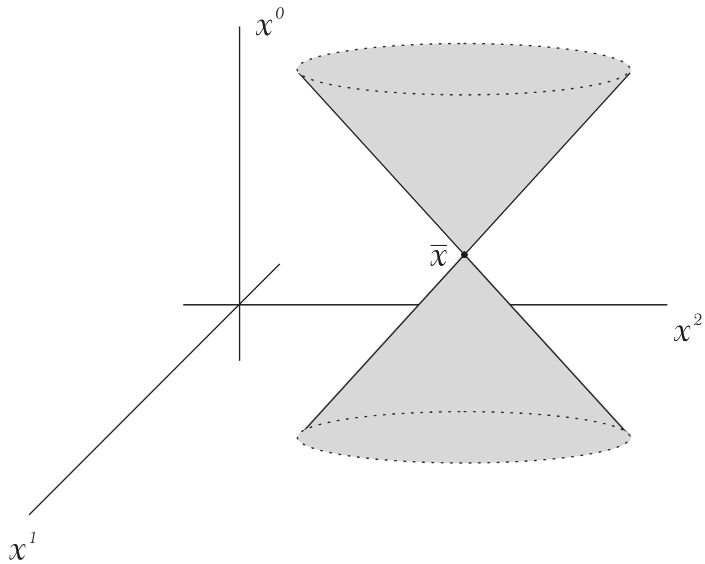

Figure 2.1: Light cone (the third spatial dimension x^3 is omitted)

The same physical system can be described using distinct sets of coordinates $\mathcal{S}(x^0, x^1, x^2, x^3)$, $\mathcal{S}'(x'^0, x'^1, x'^2, x'^3)$,... and these can be taken to represent different *observers* equipped with different *frames of reference*. The connection between different frames of reference is made by a one-to-one map $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ that preserves the causal relations 2.3 and 2.4 between events as well as the time ordering of events that are causally connected.

This kind of transformation is denoted a *causal automorphism* of the four-dimensional Minkowski spacetime. It is evident from equation 2.4 that a causal automorphism also preserves the value of the speed of light.

A *spacetime translation*, defined as

$$x^\mu \to x'^\mu = x^\mu + \alpha^\mu \tag{2.5}$$

($\alpha^\mu \in \mathcal{M}$), is an example of causal automorphism. As displacement four-vectors do not change under translations, the *quadratic form* $\eta_{\mu\nu}(x-\bar{x})^\mu(x-\bar{x})^\nu$ is clearly preserved. In addition, there exist linear transformations,

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{2.6}$$

($\Lambda^\mu{}_\nu$ are real numbers independent of x^ν), that keep the same quadratic form intact, the only condition being

$$\eta_{\sigma\rho}\Lambda^\sigma{}_\mu\Lambda^\rho{}_\nu = \eta_{\mu\nu} \tag{2.7}$$

for the matrix coefficients $\Lambda^\mu{}_\nu$. This defines a *Lorentz transformation.* Also, the relation $\Lambda^0{}_0 > 0$ is needed if the time ordering of events inside a light cone is to be preserved. In fact, two successive events that occur at times x^0 and $\bar{x}^0$ and at the same spatial position as seen by an observer in the frame $\mathcal{S}$ would be seen to have a temporal difference in the frame $\mathcal{S}'$ given by

$$\bar{x}'^0 - x'^0 = \Lambda^0{}_0(\bar{x}^0 - x^0). \tag{2.8}$$

Thus, the sign of the coefficient $\Lambda^0{}_0$ defines either the maintenance (positive values) or the inversion (negative values) of the time ordering of causally connected events.

Other causal automorphisms can maintain the form of relations 2.3 and 2.4 without preserving the quadratic form $\eta_{\mu\nu}(x-\bar{x})^\mu(x-\bar{x})^\nu$. A *dilatation*, having the form

$$x^\mu \to x'^\mu = \kappa x^\mu \tag{2.9}$$

($\kappa \in \mathbb{R}^*_+$), is easily seen to possess this feature, as it just multiplies the displacement four-vectors by a positive factor. A theorem known as *Zeeman's theorem* states that every causal automorphism falls in one of the three cases discussed above, i.e., the only transformations that keep the causal structure of Minkowski spacetime intact are translations, Lorentz transformations (with $\Lambda^0{}_0 > 0$) and dilatations.

2.2 Lorentz group

Lorentz transformations form a *group* under composition that is known as the *Lorentz group* $\mathcal{L}$. The definition 2.7 of a Lorentz transformation can be expressed in matrix form as

$$\Lambda^T \eta \Lambda = \eta, \tag{2.10}$$

where the symbol ^{T} denotes matrix transposition. The composition $\bar{\Lambda}\Lambda$ of two Lorentz transformations Λ and $\bar{\Lambda}$ is also a Lorentz transformation, as can be seen from the relation

$$(\bar{\Lambda}\Lambda)^T \eta (\bar{\Lambda}\Lambda) = \Lambda^T (\bar{\Lambda}^T \eta \bar{\Lambda}) \Lambda = \Lambda^T \eta \Lambda = \eta. \tag{2.11}$$

In addition, taking the determinant of both sides of equation 2.10, the result

$$-\det(\Lambda^T) \det \Lambda = -1 \;\Rightarrow\; (\det \Lambda)^2 = 1 \tag{2.12}$$

is obtained. Consequently, the inverse matrix Λ^{-1} exists and it represents the inverse of the Lorentz transformation Λ. As 2.10 implies

$$\eta = (\Lambda^{-1})^T (\Lambda^T \eta \Lambda) \Lambda^{-1} = (\Lambda^{-1})^T \eta \Lambda^{-1}, \tag{2.13}$$

the inverse of a Lorentz transformation is also a Lorentz transformation.

Closure under composition and inversion is a necessary and sufficient condition to demonstrate the group structure of Lorentz transformations. More specifically, as the Lorentz group has an infinite number of elements that can be mapped one-to-one into finite-dimensional matrices and form a *differentiable manifold* (a geometric object that allows calculus to be applied), it is classified as a *linear Lie group*.

The coefficients of the matrices Λ are real, so that equation 2.12 gives only two alternatives: $\det \Lambda = 1$, defining the *proper* Lorentz transformations, or $\det \Lambda = -1$, for the *improper* Lorentz transformations. In particular, the transformation of *parity*, defined as

$$\mathscr{P}: \; x^0 \to x^0 \;,\; \boldsymbol{x} \to -\boldsymbol{x}, \tag{2.14}$$

is an improper Lorentz transformation. The proper transformations form a *subgroup* of the Lorentz group, as $\det \Lambda = \det \bar{\Lambda} = 1$ implies

$$\det(\bar{\Lambda}\Lambda) = \det \bar{\Lambda} \det \Lambda = 1 \tag{2.15}$$

and

$$\det(\Lambda^{-1}) = (\det \Lambda)^{-1} = 1. \tag{2.16}$$

This subgroup receives the name of *proper Lorentz group* $\mathcal{L}_+$. On the other hand, the improper transformations do not form a subgroup, as their composition is clearly proper.

The restriction 2.7 gives

$$-(\Lambda^0{}_0)^2 + (\Lambda^1{}_0)^2 + (\Lambda^2{}_0)^2 + (\Lambda^3{}_0)^2 = -1 \;\Rightarrow\; |\Lambda^0{}_0| \geq 1. \tag{2.17}$$

For positive values of $\Lambda^0{}_0$, this relation leads to $\Lambda^0{}_0 \geq 1$, thus defining the *orthochronous* Lorentz transformations, which preserve the direction of time. Instead, for negative values of $\Lambda^0{}_0$, the restriction is $\Lambda^0{}_0 \leq -1$, representing the *non-orthochronous* Lorentz transformations. Every non-orthochronous transformation inverts the time ordering of causally connected events, an important example being the transformation of *time-reversal* $\mathscr{T}$, defined as

$$\mathscr{T}: \ x^0 \to -x^0 \ , \ \boldsymbol{x} \to \boldsymbol{x}. \tag{2.18}$$

The orthochronous Lorentz transformations form a subgroup of the Lorentz group called the *orthochronous Lorentz group* $\mathcal{L}^\uparrow$. The consecutive action of two orthochronous transformations Λ and $\bar{\Lambda}$ gives a coefficient

$$(\bar{\Lambda}\Lambda)^0{}_0 = \bar{\Lambda}^0{}_\sigma \Lambda^\sigma{}_0 = \bar{\Lambda}^0{}_0 \Lambda^0{}_0 + \bar{\Lambda}^0{}_i \Lambda^i{}_0 \tag{2.19}$$

that can be shown to be greater than or equal to one, so that the composed transformation is also orthochronous. As the transpose of a Lorentz transformation is also a Lorentz transformation, relation 2.17 implies

$$-(\bar{\Lambda}^0{}_0)^2 + (\bar{\Lambda}^0{}_1)^2 + (\bar{\Lambda}^0{}_2)^2 + (\bar{\Lambda}^0{}_3)^2 = -1 \tag{2.20}$$

for the transformation $\bar{\Lambda}$. As a consequence, the three-vectors $(\Lambda^1{}_0, \Lambda^2{}_0, \Lambda^3{}_0)$ and $(\bar{\Lambda}^0{}_1, \bar{\Lambda}^0{}_2, \bar{\Lambda}^0{}_3)$ have norms $\sqrt{(\Lambda^0{}_0)^2 - 1}$ and $\sqrt{(\bar{\Lambda}^0{}_0)^2 - 1}$, respectively, and the Cauchy-Schwarz inequality gives

$$|\bar{\Lambda}^0{}_i \Lambda^i{}_0| \leq \sqrt{(\bar{\Lambda}^0{}_0)^2 - 1}\sqrt{(\Lambda^0{}_0)^2 - 1}. \tag{2.21}$$

Then, from equation 2.19,

$$(\bar{\Lambda}\Lambda)^0{}_0 \geq \bar{\Lambda}^0{}_0 \Lambda^0{}_0 - \sqrt{(\bar{\Lambda}^0{}_0)^2 - 1}\sqrt{(\Lambda^0{}_0)^2 - 1} = \frac{(\bar{\Lambda}^0{}_0)^2 + (\Lambda^0{}_0)^2 - 1}{\bar{\Lambda}^0{}_0 \Lambda^0{}_0 + \sqrt{(\bar{\Lambda}^0{}_0)^2 - 1}\sqrt{(\Lambda^0{}_0)^2 - 1}}, \tag{2.22}$$

which is easily seen to be positive. So, the only possibility is $(\bar{\Lambda}\Lambda)^0{}_0 \geq 1$. Furthermore, since $\eta^{-1} = \eta$, equation 2.10 gives

$$\Lambda^{-1} = \eta \Lambda^T \eta \ \Rightarrow \ (\Lambda^{-1})^0{}_0 = (-1)\Lambda^0{}_0(-1) = \Lambda^0{}_0, \tag{2.23}$$

which demonstrates that the inverse of an orthochronous Lorentz transformation is also orthochronous.

Not every Lorentz transformation can be continuously obtained from the identity element of the group, represented by the 4×4 identity matrix $\mathbb{1}$. In fact, the transformations that can be connected to the identity form a *subgroup* of the Lorentz group, denoted the *connected Lorentz group*, as inverses and products of these transformations can also be connected to the identity. Since a continuous variation of group elements cannot produce discontinuities neither in the determinant of Λ nor in the value of $\Lambda^0{}_0$, this subgroup do not contain improper and non-orthochronous transformations (the identity is proper and orthochronous). In the next chapter, it will be shown that every proper orthochronous Lorentz transformation is an element of this subgroup, i.e., that the connected Lorentz group is identical to the proper orthochronous Lorentz group $\mathcal{L}_+^\uparrow$.

The transformations of the connected subgroup can be obtained from infinitesimal Lorentz transformations,

$$\lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \tag{2.24}$$

where the real coefficients $\omega^\mu{}_\nu$ are infinitesimal. Inserting equation 2.24 in 2.7 and ignoring terms that are quadratic in ω, the result is

$$\eta_{\sigma\rho}(\delta^\sigma{}_\mu + \omega^\sigma{}_\mu)(\delta^\rho{}_\nu + \omega^\rho{}_\nu) = \eta_{\mu\nu} \;\Rightarrow\; \omega_{\mu\nu} = -\omega_{\nu\mu}, \tag{2.25}$$

with $\omega_{\mu\nu} = \eta_{\sigma\mu}\omega^\sigma{}_\nu$. These coefficients form an antisymmetric matrix that completely determines the infinitesimal Lorentz transformation, so that the number of independent real parameters of this transformation can be obtained from the matrix ω. As this is an antisymmetric 4×4 real matrix, this number is $(4^2 - 4)/2 = 6$ and these parameters can be chosen to be $\theta_1 = \omega_{23}$, $\theta_2 = \omega_{31} = -\omega_{13}$, $\theta_3 = \omega_{12}$ and $\varphi_i = -\omega_{0i}$ for $i = 1, 2, 3$. Consequently, every infinitesimal Lorentz transformation is written in the form

$$\omega = J_1\theta_1 + J_2\theta_2 + J_3\theta_3 + K_1\varphi_1 + K_2\varphi_2 + K_3\varphi_3 = \boldsymbol{J}\cdot\boldsymbol{\theta} + \boldsymbol{K}\cdot\boldsymbol{\varphi}, \tag{2.26}$$

where

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \; J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \; J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$K_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; K_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; K_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{2.27}$$

are called the *generators* of these transformations. These generators form a basis for a six-dimensional real vector space equipped with the operation of matrix commutation (for two square matrices A and B, the commutator is defined as $[A, B] = AB - BA$). This structure is denoted a *real Lie algebra*, and is characterized by the commutation relations between the basis elements 2.27,

$$\begin{aligned} [J_i, J_j] &= -\epsilon_{ijk} J_k, \\ [J_i, K_j] &= -\epsilon_{ijk} K_k, \\ [K_i, K_j] &= \epsilon_{ijk} J_k, \end{aligned} \tag{2.28}$$

with ϵ_{ijk} denoting the *Levi-Civita symbol*, defined as

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } i, j, k \text{ is an even permutation of } 1, 2, 3 \\ -1, & \text{if } i, j, k \text{ is an odd permutation of } 1, 2, 3 \\ 0, & \text{otherwise} \end{cases} . \tag{2.29}$$

Every element of the connected subgroup of the Lorentz group can now be obtained by successive applications of infinitesimal transformations. In particular, a transformation with a finite parameter θ_1 can be constructed as

$$\lim_{N \to +\infty} \left(\mathbb{1} + J_1 \frac{\theta_1}{N} \right)^N = \mathrm{e}^{J_1 \theta_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix} = R_x(\theta_1), \tag{2.30}$$

and represents a rotation by an angle of $0 \leq \theta_1 \leq 2\pi$ around the x-axis. Similarly, the transformations generated by J_2 and J_3 correspond, respectively, to rotations around the y- and the z-axis. Also, the generators K_1, K_2 and K_3 provide the transformations denoted as *Lorentz boosts* along the x, y and z directions, respectively. Specifically, a boost along the x direction has the form

$$B_x(\varphi_1) = \mathrm{e}^{K_1 \varphi_1} = \begin{pmatrix} \cosh\varphi_1 & -\sinh\varphi_1 & 0 & 0 \\ -\sinh\varphi_1 & \cosh\varphi_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.31}$$

and every real value of the parameter φ_1, which receives the name of *rapidity*, represents a different transformation. Thus, the parameter space associated with the Lorentz group is not bounded.

2.3 Translations and conservation of energy-momentum

Spacetime time translations can be joined into the connected Lorentz group to form a larger group known as the *Poincaré group*. For example, if $T_x(a^1)$ denotes the operation of translation along the x-axis, $T_x(a^1)(x^0, x^1, x^2, x^3) = (x^0, x^1 + a^1, x^2, x^3)$, it is clear that

$$T_x(\bar{a}^1)T_x(a^1) = T_x(a^1 + \bar{a}^1), \tag{2.32}$$

and, as $T_x(0) = \mathbb{1}$, this operator can be represented by a matrix of the form

$$T_x(a^1) = \mathrm{e}^{P_1 a^1}, \tag{2.33}$$

where P_1 is the generator of translations along the x-axis. Similarly, an arbitrary spacetime translation can be written as

$$T(a^\mu) = \mathrm{e}^{P_\mu a^\mu}, \tag{2.34}$$

where P_μ ($\mu = 1, 2, 3, 4$) are the generators of spacetime translations, since translations along different spacetime directions commute (i.e., $[P_\mu, P_\nu] = 0$).

The composition of translations and Lorentz transformations has the property

$$\Lambda T(a)x = \Lambda(x + a) = T(\Lambda a)\Lambda x \;\Rightarrow\; T(\Lambda a) = \Lambda T(a)\Lambda^{-1}. \tag{2.35}$$

In particular, if both transformations are infinitesimal and the Lorentz transformation is a rotation around the x-axis, equation 2.35 gives, up to second order in the infinitesimal parameters,

$$\mathbb{1}+P_\mu(\mathbb{1}+\theta_1 J_1)^\mu{}_\nu a^\nu+\frac{1}{2}(P_\mu a^\mu)^2 = \left(\mathbb{1} + \theta_1 J_1 + \frac{1}{2}(\theta_1 J_1)^2\right)\left(\mathbb{1} + P_\mu a^\mu + \frac{1}{2}(P_\mu a^\mu)^2\right)\left(\mathbb{1} - \theta_1 J_1 + \frac{1}{2}(\theta_1 J_1)^2\right). \tag{2.36}$$

The terms of order 1, θ_1, a^μ, θ_1^2 and $(a^\mu)^2$ cancel out and, using 2.27, those of order $\theta_1 a^\mu$ give

$$[J_1, P_\mu] = P_\nu (J_1)^\nu{}_\mu \;\Rightarrow\; \begin{aligned} &[J_1, P_0] = 0, \\ &[J_1, P_1] = 0, \\ &[J_1, P_2] = -P_3, \\ &[J_1, P_3] = P_2. \end{aligned} \tag{2.37}$$

Similarly, 2.35 can be used to find all the commutation relations between the generators of translations P_μ and the generators of Lorentz transformations J_i and K_i. These relations are given by

$$\begin{aligned}
[J_i, P_0] &= 0, \\
[J_i, P_j] &= -\epsilon_{ijk} P_k, \\
[K_i, P_0] &= P_i, \\
[K_i, P_j] &= -\eta_{ij} P_0,
\end{aligned} \tag{2.38}$$

and, together with the commutators in 2.28, they constitute the *real Poincaré algebra.*

Translations of spacetime coordinates are considered as *fundamental symmetries* of any physical theory that assumes spacetime to be homogeneous. This means that the equations of motion of such a theory preserve their dependence on spacetime variables after a translation is performed and are said to be *covariant* under spacetime translations (the theory is said to be *invariant* under translations). In most cases of interest, the equations of motion can be obtained from a functional denoted the *action functional*, defined as

$$\mathcal{S} = \int d^4x \; \mathcal{L}(\phi(x), \partial_\mu \phi) \tag{2.39}$$

($d^4x = dx^0 dx^1 dx^2 dx^3$). The function $\mathcal{L}$ is called the *Lagrangian density* of the system and depends explicitly on a collection of differentiable *fields* represented by $\phi(x)$ and their derivatives, $\partial_\mu \phi$. The so-called *principle of stationary action* states that the classical evolution of a system follows the path of stationary action, $\delta\mathcal{S} = 0$. A translation (or a general symmetry transformation) needs to preserve the result of this extremization process, i.e., the action should be an invariant under such transformation.

The requirement $\delta\mathcal{S} = 0$ gives

$$\begin{aligned}
0 = \delta\mathcal{S} &= \int d^4x \; \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right\} \\
&= \int d^4x \; \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) \right\} \\
&= \int d^4x \; \left\{ \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] \delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) \right\}.
\end{aligned} \tag{2.40}$$

The last term can be disregarded as it gives just a boundary term (taking the the integration over the entire Minkowski spacetime, $\delta\phi$ is assumed to vanish when $x^\mu x_\mu$ goes to infinity). Since 2.40 is true for and arbitrary variation $\delta\phi$, it gives the *Euler-Lagrange* equations,

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \tag{2.41}$$

With the knowledge of the Lagrangian of a physical system, these equations can be used to obtain its classical equations of motion.

Considering a translation of the form

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x), \tag{2.42}$$

where the displacement vector $\epsilon^\mu(x)$ is a function of x^μ, the fields can be expressed in terms of the translated variables x'^μ as

$$\phi(x) \to \phi'(x') = \phi(x' - \epsilon(x)) = \phi(x') - \epsilon^\nu(x)\partial_\nu\phi, \tag{2.43}$$

(the last equality is valid up to first order in ϵ^μ). The Lagrangian density $\mathcal{L}$, then, transforms as

$$\begin{aligned}\mathcal{L} \to \mathcal{L}' &= \mathcal{L} + \frac{\partial\mathcal{L}}{\partial\phi}(-\epsilon^\nu(x)\partial_\nu\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(-\epsilon^\nu(x)\partial_\nu\phi) \\ &= \mathcal{L} - \epsilon^\nu(x)\left(\frac{\partial\mathcal{L}}{\partial\phi}\partial_\nu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu(\partial_\mu\phi)\right) - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\epsilon^\nu\partial_\nu\phi \\ &= \mathcal{L} - \epsilon^\nu(x)\partial_\nu\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\epsilon^\nu\partial_\nu\phi\ ,\end{aligned} \tag{2.44}$$

so that the variation of the action $\mathcal{S}$ is given by

$$\delta\mathcal{S} = \int d^4x \left(-\epsilon^\nu(x)\partial_\nu\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\epsilon^\nu\partial_\nu\phi\right). \tag{2.45}$$

Integrating by parts and disregarding boundary terms,

$$\delta\mathcal{S} = -\int d^4x\ \mathcal{T}^\mu{}_\nu\partial_\mu\epsilon^\nu = -\int d^4x\ \partial_\mu\mathcal{T}^\mu{}_\nu\epsilon^\nu, \tag{2.46}$$

with

$$\mathcal{T}^\mu{}_\nu = \delta^\mu_\nu\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi \tag{2.47}$$

(δ^μ_ν is a Kronecker delta).

Now, taking the parameters ϵ^ν to be constants, the variation in the action should vanish, as it is assumed to be invariant under translations. Since equation 2.46 is valid for an arbitrary displacement ϵ^ν, it gives

$$\partial_\mu\mathcal{T}^\mu{}_\nu = 0. \tag{2.48}$$

Then, each of the $\mathcal{T}^\mu{}_\nu$ (for $\nu = 1, 2, 3, 4$) defines what is called a *conserved current*. Also, the quantities

$$\mathcal{H} = \int d^3x \ \mathcal{T}^0{}_0 \quad \text{and} \quad \mathcal{P}_i = \int d^3x \ \mathcal{T}^0{}_i \tag{2.49}$$

are known as *conserved charges* as their time derivatives vanish. In fact,

$$\frac{d\mathcal{H}}{dt} = \int d^3x \ \partial_0 \mathcal{T}^0{}_0 = -\int d^3x \ \partial_j \mathcal{T}^j{}_0 = 0 \tag{2.50}$$

and

$$\frac{d\mathcal{P}_i}{dt} = \int d^3x \ \partial_0 \mathcal{T}^0{}_i = -\int d^3x \ \partial_j \mathcal{T}^j{}_i = 0, \tag{2.51}$$

as these expressions give just boundary terms. The quantities $\mathcal{H}$ and $\mathcal{P}_i$ represent, respectively, the energy and the three-momentum of the physical system described by the fields $\phi(x)$. For this reason, $\mathcal{T}^\mu{}_\nu$ receives the name of *energy-momentum tensor* and relation 2.48 depicts the conservation of energy and momentum. It is the invariance of the action under spacetime translations that guarantees this conservation.

2.4 The subgroups *HOM(2)* and *SIM(2)*

The theory of Special Relativity is based on the *principle of relativity*, which states that the equations of a physical theory should be covariant under transformations connecting *admissible* frames of reference. In this scenario, these frames are defined as being related to each other by the transformations of the Poincaré group. In this context, the principle of relativity implies that the spacetime of Minkowski is homogeneous and isotropic, since the result of every experiment is the same in translated or rotated frames. In addition, the non-existence of an absolute rest frame is a consequence of covariance under Lorentz boosts.

Since condition 2.4 is preserved under the Poincaré group, it is clear that all observers in admissible frames of reference should agree on the value of the speed of light. In this sense, the definition of admissible frames of reference as being connected by the transformations of the Poincaré group implies the constancy of the speed of light, usually taken as a postulate. However, the converse is not necessarily true. In fact, a universal and isotropic speed of light only implies covariance under a subgroup of the Poincaré group.

It is convenient to express the algebra defined in 2.28 in a different basis, with the generators taken to be $T_1 = K_1 + J_2$, $T_2 = K_2 - J_1$, $\bar{T}_1 = -K_1 + J_2$, $\bar{T}_2 = -K_2 - J_1$, J_3

and K_3. After this change of basis, the commutators in 2.28 become

$$\begin{aligned} [T_1, J_3] &= T_2, & [\bar{T}_1, J_3] &= \bar{T}_2, \\ [T_2, J_3] &= -T_1, & [\bar{T}_2, J_3] &= -\bar{T}_1, \\ [T_1, K_3] &= -T_1, & [\bar{T}_1, K_3] &= \bar{T}_1, \\ [T_2, K_3] &= -T_2, & [\bar{T}_2, K_3] &= \bar{T}_2, \\ [T_1, T_2] &= [\bar{T}_1, \bar{T}_2] = [J_3, K_3] = 0, \\ [T_1, \bar{T}_1] &= [T_2, \bar{T}_2] = -2K_3, \\ [T_1, \bar{T}_2] &= -[T_2, \bar{T}_1] = -2J_3. \end{aligned} \tag{2.52}$$

This algebra is equivalent to the algebra in 2.28, as its elements generate the same group of transformations (the connected Lorentz group).

Under the transformation of parity $\mathscr{P}$, the generators of rotations J_i are preserved but the generators of boosts K_i change sign, as can be seen, for example, from the relations

$$\mathscr{P} R_x(\theta_1) \mathscr{P}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix} = \mathrm{e}^{J_1 \varphi_1} \tag{2.53}$$

and

$$\mathscr{P} B_x(\varphi_1) \mathscr{P}^{-1} = \begin{pmatrix} \cosh\varphi_1 & \sinh\varphi_1 & 0 & 0 \\ \sinh\varphi_1 & \cosh\varphi_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathrm{e}^{(-K_1)\varphi_1}, \tag{2.54}$$

obtained from 2.14, 2.30 and 2.31. In a similar manner, this can also be shown to hold for the transformation of time-reversal $\mathscr{T}$, defined in 2.18. As a consequence, either parity or time-reversal transforms the generator T_1 into $\bar{T}_1$ and T_2 into $\bar{T}_2$.

The subgroup of the connected Lorentz group composed of transformations generated by T_1 and T_2 is denoted by *T(2)*, as it is identical to the group of translations in two dimensions. If a theory is invariant under this subgroup and also under the transformation of parity (or time-reversal), it is also invariant under the transformations generated by $\bar{T}_1$ and $\bar{T}_2$. In addition, the last two rows in 2.52 show that this also implies invariance under the transformations generated by J_3 and K_3. Consequently, a physical theory that is invariant under *T(2)* and parity is invariant under all the transformations in the connected Lorentz group. Furthermore, this is also true for every subgroup that contains *T(2)*, and these are called *Very Special Relativity* (*VSR*) subgroups. The VSR subgroups are *T(2)* itself; *E(2)*, generated by T_1, T_2 and J_3; *HOM(2)*, generated by T_1, T_2 and K_3; *SIM(2)*, generated by T_1, T_2, J_3 and K_3. [1] VSR is equivalent to Special Relativity in the context of parity or time-reversal invariance.

The subgroup *HOM(2)* contains the transformation

$$B_{VSR}(\alpha,\beta,\varphi) = \mathrm{e}^{T_1\alpha}\mathrm{e}^{T_2\beta}\mathrm{e}^{K_3\varphi}, \tag{2.55}$$

which can connect observers moving with arbitrary velocity within the Minkowski spacetime and is sometimes referred as a *VSR boost*. In relation to its own frame of reference, the displacement vector of an observer is given by $\Delta x^\mu = (\Delta t, \mathbf{0})$. Under the transformation defined in 2.55, this vector transforms into

$$B_{VSR}(\alpha,\beta,\varphi)\Delta x^\mu = \gamma\Delta t(1,\boldsymbol{v}), \tag{2.56}$$

where

$$\boldsymbol{v} = \left(-\frac{\mathrm{e}^{-\varphi}}{\gamma}\alpha, -\frac{\mathrm{e}^{-\varphi}}{\gamma}\beta, \frac{\mathrm{e}^{-\varphi}}{\gamma} - 1\right) \tag{2.57}$$

is the velocity of the observer after the transformation and

$$\gamma = \frac{1}{2}\mathrm{e}^{-\varphi}(\alpha^2+\beta^2) + \cosh\varphi = \frac{1}{\sqrt{1-|\boldsymbol{v}|^2}} \geq 1 \tag{2.58}$$

is called the *Lorentz factor* of the transformation B_{VSR}. The expression 2.57 gives an arbitrary velocity, since every triple of components (v_1, v_2, v_3) can be obtained by the choice of parameters

$$\begin{aligned}
\alpha &= -\frac{v_1}{1+v_3},\\
\beta &= -\frac{v_2}{1+v_3},\\
\varphi &= -\log\left[\gamma(1+v_3)\right].
\end{aligned} \tag{2.59}$$

Both the groups *HOM(2)* and *SIM(2)* contain the VSR boost. Since any frames of reference can be connected by a VSR boost followed by a rotation, the assumption of invariance under either one of these subgroups is sufficient to guarantee a universal value

[1] To see that these are indeed subgroups of the connected Lorentz group, it is enough to note that the each corresponding set of generators defines a *subalgebra* of the algebra 2.52.

for the speed of light in every frame of reference. In other words, even that VSR does not imply covariance under all Lorentz transformations, invariance under *HOM(2)* and *SIM(2)* implies the constancy of the speed of light for observers connected by arbitrary Lorentz transformations.

As demonstrated in the last section, invariance under spacetime translations is necessary for energy-momentum conservation. Consequently, it is necessary to include translations in the groups *HOM(2)* and *SIM(2)*, i.e., to consider the *inhomogeneous* groups *IHOM(2)* and *ISIM(2)*, which are subgroups of the Poincaré group. Theories whose symmetry groups are given by these variants of VSR present a preferred direction in Minskowski spacetime, given by the four-vector $n^\mu = (1, 0, 0, 1)$. In fact, from 2.27,

$$(T_1)^\mu_\nu n^\nu = (T_2)^\mu_\nu n^\nu = (J_3)^\mu_\nu n^\nu = 0 \tag{2.60}$$

and

$$(K_3)^\mu_\nu n^\nu = -n^\mu, \tag{2.61}$$

so that n^μ is invariant under the transformations generated by T_1, T_2 and J_3 and the direction of n^μ is maintained under the transformations generated by K_3. As a consequence, the requirement of spatial isotropy excludes these subgroups as fundamental symmetry groups and invariance under the full Poincaré group is implied.

Chapter 3

Group theory and representations

3.1 Basic ideas

As briefly mentioned in last chapter, a group $\mathcal{G}$ is a set of objects in which an operation, called *multiplication*, is defined. In other words, if g and g' are elements of a group $\mathcal{G}$, gg' is also an element of this group and is referred as the *product of* g *with* g'. The multiplication rule needs to be *associative*, i.e.,

$$g(g'g'') = (gg')g'', \tag{3.1}$$

and there needs to exist a unique identity element e, with the property

$$ge = eg = g, \tag{3.2}$$

and an inverse element g^{-1}, which obeys

$$g^{-1}g = gg^{-1} = e, \tag{3.3}$$

for every $g \in \mathcal{G}$. In general, the operation of multiplication is not commutative, in the sense that gg' is not necessarily equal to $g'g$. If all the elements of a group happen to commute under multiplication, this group is called an *abelian* group.

The group structure is usually taken to represent the behavior of mathematical or physical objects under a given set of transformations. An interesting example comes from the symmetry transformations of an equilateral triangle. It is known as the *dihedral group* D_3, and consists of the identity transformation e, two rotations a and b, of 120° and 240°, respectively, around the center of the triangle and three reflections c, d and f with respect to each height of the triangle. The action of these transformations can be represented by matrices, of the form

$$D(a) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}, \; D(b) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \; D(c) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$D(d) = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}, \; D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; D(f) = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \tag{3.4}$$

with composition of transformations being equivalent to matrix multiplication. In particular, the composition of the rotation a with the reflection d gives the reflection f, as can be seen from the relation $D(d)D(a) = D(f)$. The set of matrices in 3.4 is denoted as a (two-dimensional) *representation* of the group D_3 (the dimension of the matrices is referred as the dimension of the representation itself).

For a group $\mathcal{G}$ of general transformations, a representation consists of a map

$$g \to D(g) \tag{3.5}$$

that takes group elements g into finite-dimensional matrices[1] $D(g)$ with the requirement that the multiplication of the group is preserved, i.e., that

$$D(g)D(g') = D(gg'). \tag{3.6}$$

This map is not required to be one-to-one, but when it does, the representation is called *faithful*. Otherwise, if more than one transformation is represented by the same matrix, the representation is *unfaithful*, the simplest example being the trivial representation $D(g) = 1, \forall g \in \mathcal{G}$, which clearly preserves the multiplication of any group $\mathcal{G}$.

The dihedral group D_3 has a finite number of elements, being usually referred as a *finite group*. However, the study of transformations in physical objects more than often asks for a certain kind of group with an infinite number of elements, the so-called *Lie groups*. These groups are characterized by the fact that their elements can be obtained continuously from a set of real *parameters*, so that a differential topology can be associated with these objects. The number of necessary parameters is denoted as the *dimension* of the Lie group. For example, the elements of a n-dimensional Lie group can be written as a function of the parameters n real parameters $g(x_1, x_2, ..., x_n)$. In addition, the structure

[1] In fact, a representation consists of a map into a general linear space, but in this work finite-dimensional ones are emphasized.

of these groups is strongly dependent on the transformations that lie infinitesimally close to the identity, in the sense that they can be obtained from the identity transformation by an infinitesimal change in the parameters. This was already shown in last chapter, in the study of the Lorentz group. The explicit forms of the transformations within this group were determined from the structure of the infinitesimal transformations, exhibited in the Lie algebra 2.28. In particular, if a Lie group presents a faithful representation, it is denoted a *linear* Lie group, since the its transformations are equivalent to linear transformations.

A simpler example of a linear Lie group is the group of rotations on a plane around the origin. This is an abelian group of dimension one with transformations R that can be parametrized by an angle of rotation θ and have the property

$$R(\theta)R(\theta') = R(\theta + \theta'). \tag{3.7}$$

Consequently, these rotations can be represented by

$$D[R(\theta)] = \mathrm{e}^{p\theta}, \tag{3.8}$$

if the condition $\mathrm{e}^{2p\pi} = 1$ is satisfied, i.e., if p is an integer. This condition is necessary since a rotation of 2π produce exactly the same result as the identity transformation[2]. Thus, every integer p gives an additional representation of the group of two-dimensional rotations. It is also possible to construct representations that use matrices of higher dimensions, e.g., the two dimensional representation given by

$$D[R(\theta)] = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{3.9}$$

A matrix of the form 3.9 has determinant equal to unity and inverse identical to its transpose (since both are obtained by making $\theta \to -\theta$). Thus, these matrices are known as two-dimensional *special orthogonal matrices* and form a group under matrix multiplication known as the *special orthogonal group SO(2)*. As these matrices can be one-to-one mapped into rotations on the plane (since the representation 3.9 is faithful), the group of two-dimensional special orthogonal matrices is said to be *isomorphic* to the group of rotations on the plane and both groups are denoted by *SO(2)*.

The group *SO(2)* is a subgroup (a subset of a group that is a group by itself) of the *orthogonal group O(2)* consisting of all rotations and reflections on the plane or,

[2]The elements of a group describe the relation between the conditions before and after a transformation, with no distinction of how this result was obtained. Instead, if one wants to describe the *path* followed by a transformation, it is necessary to define a *curve* in the parameter space of the Lie group.

alternatively, the group of two-dimensional *orthogonal matrices* A, which have real entries and obey the condition

$$A^T = A^{-1}. \tag{3.10}$$

This condition implies

$$\det(AA^{-1}) = (\det A)^2 = 1 \;\Rightarrow\; \det A = \pm 1. \tag{3.11}$$

Thus, every element of *O(2)* is either an element $R(\theta)$ of *SO(2)* (if it has determinant $+1$) or can be written in the form $\mathscr{I}R(\theta)$ (if its determinant is -1), with $\mathscr{I}$ denoting the reflection represented by the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.12}$$

The special orthogonal group consists of the transformations in the orthogonal group that are continuously connected to the identity transformation and, in this sense, *SO(2)* is said to be the *connected subgroup* of *O(2)*. Also, the group *O(2)* is said to be composed of two *connected components*, i.e., continuously connected subsets of elements that can only be mapped into one another by a finite (non-infinitesimal) transformation of the group.

For rotations in three-dimensional space, given by the *special orthogonal group SO(3)*, a similar structure appears. Since results 3.10 and 3.11 also apply in this case, *SO(3)* is the connected part of the *orthogonal group O(3)*, which consists of all orthogonal matrices A of dimension three. A general transformation of this group depends on three real independent parameters, which will be denoted θ_1, θ_2, θ_3, and is either an element $R(\theta_1, \theta_2, \theta_3)$ of *SO(3)* or can be written as a product of the form $\mathscr{P}R(\theta_1, \theta_2, \theta_3)$, with $\mathscr{P}$ representing the transformation of parity in three dimensions,

$$\mathscr{P} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3.13}$$

The real Lie algebra of *SO(3)*, denoted as *so(3)*, can be obtained from the restriction of the generators of rotations in 2.27 to just spatial dimension, given by

$$a_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \; a_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; a_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.14}$$

as

$$[a_i, a_j] = -\epsilon_{ijk} a_k, \tag{3.15}$$

which is of course identical to the first commutator in 2.28.

3.2 Cosets and discrete Lorentz transformations

The results obtained at the end of last section can be expressed more formally in terms of objects known as *cosets*. These objects provide a way of classifying transformations of a group $\mathcal{G}$ using the structure of one of its subgroup $\mathcal{H}$. Specifically, the *left coset* of $\mathcal{H}$ with respect to a fixed element $g \in \mathcal{G}$ is defined to be[3]

$$g\mathcal{H} = \{gh \mid h \in \mathcal{H}\}, \tag{3.17}$$

where h varies over all the elements of the subgroup $\mathcal{H}$. It is easy to see that every element of the group $\mathcal{G}$ can only be in one and only one coset. For example, if $\bar{g}$ was an element of two distinct cosets $g\mathcal{H}$ and $g'\mathcal{H}$,

$$\bar{g} = gh = g'h' \;\Rightarrow\; {g'}^{-1} g = h' h^{-1} \in \mathcal{H}, \tag{3.18}$$

with $h, h' \in \mathcal{H}$. As a consequence, this would imply that the coset $({g'}^{-1} g)\mathcal{H}$ is identical to $\mathcal{H}$, as $\mathcal{H}$ is closed under multiplication, and that $g'[({g'}^{-1} g)\mathcal{H}] = g\mathcal{H} = g'\mathcal{H}$, contradicting the initial assumption that the cosets are different.

Cosets are even more significant when the subgroup $\mathcal{H}$ presents the additional requirement of being an *invariant* subgroup, which is defined by the condition

$$ghg^{-1} \in \mathcal{H}, \tag{3.19}$$

for every $g \in \mathcal{G}$ and every $h \in \mathcal{H}$. In this case, the operation of *coset multiplication* can be defined as

$$(g\mathcal{H})(g'\mathcal{H}) = (gg')\mathcal{H}, \tag{3.20}$$

the condition 3.19 being necessary to guarantee the consistency of the definition 3.20, since an element of $(g\mathcal{H})(g'\mathcal{H})$, of the form

$$ghg'h' = gg'[({g'}^{-1} h g')h'], \tag{3.21}$$

[3]In a similar way, the *right coset* of $\mathcal{H}$ with respect to g is defined as

$$\mathcal{H}g = \{hg \mid h \in \mathcal{H}\}. \tag{3.16}$$

is also an element of $(gg')\mathcal{H}$ if and only if $g'^{-1}hg' \in \mathcal{H}$. The product of cosets provides a group structure for these objects, the group of cosets of $\mathcal{H}$ being called the *factor group* $\mathcal{G}/\mathcal{H}$.

The special subgroup *SO(2)*, discussed in last section, is an invariant subgroup of the orthogonal group *O(2)*. In fact, for arbitrary rotation $R(\theta)$ and orthogonal matrix A, the matrix $AD[R(\theta)]A^{-1}$ is also orthogonal, since (from 3.9 and 3.10)

$$\{AD[R(\theta)]A^{-1}\}^T = (A^{-1})^T\{D[R(\theta)]\}^T A^T = AD[R(\theta)]A^{-1}, \tag{3.22}$$

and its determinant is also +1, as can be seen from

$$\det\{AD[R(\theta)]A^{-1}\} = \det A \det D[R(\theta)] \det A^{-1} = (\det A)^2 = 1, \tag{3.23}$$

where the last equality comes from 3.11. Then, the factor group *O(2)/SO(2)* has only two elements, *SO(2)* and $\mathscr{I}$*SO(2)*, which represent the connected components of the orthogonal group *O(2)*. The possible coset products are given by

$$\begin{aligned} (SO(2))\,(SO(2)) &= SO(2), \\ (SO(2))\,(\mathscr{I}SO(2)) &= \mathscr{I}SO(2), \\ (\mathscr{I}SO(2))\,(SO(2)) &= \mathscr{I}SO(2), \\ (\mathscr{I}SO(2))\,(\mathscr{I}SO(2)) &= SO(2), \end{aligned} \tag{3.24}$$

so that, under the transformation $\mathscr{I}$, the connected components of *O(2)* are taken into each other, as represented in figure 3.1.

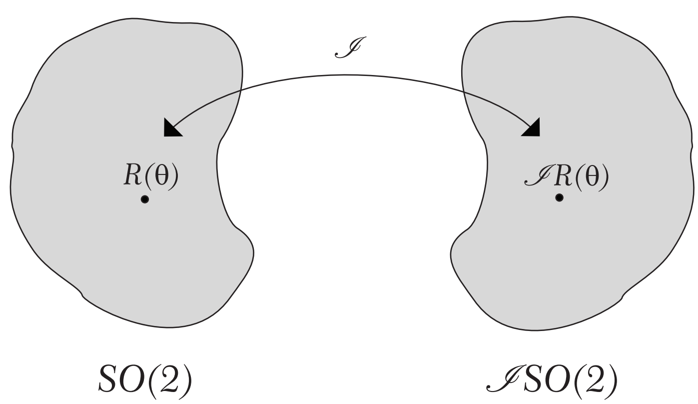

Figure 3.1: Connected components of *O(2)*

For the three-dimensional case of *O(3)*, the factor group *O(3)/SO(3)* posses a structure identical to 3.24, the only difference being the replacement of the two-dimensional reflection $\mathscr{I}$ with the three-dimensional transformation of parity $\mathscr{P}$. As a consequence, the connected components of *O(3)* compose a picture similar to figure 3.1, as depicted in figure 3.2

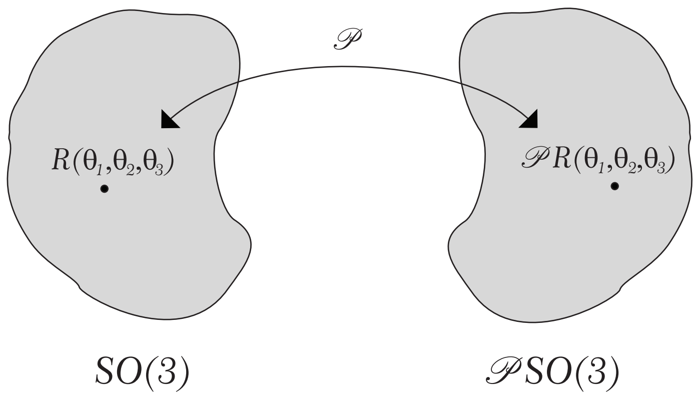

Figure 3.2: Connected components of *O(3)*

The group of three-dimensional rotations can be considered as a subgroup of the Lorentz group, since the Lorentz transformations of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & A & \\ 0 & & & \end{pmatrix} \tag{3.25}$$

form a subgroup isomorphic to *O(3)* (the condition 2.7 implies that A is an orthogonal matrix). Every transformation of this type, can therefore be written either as an element $R(\theta_1, \theta_2, \theta_3) = \mathrm{e}^{\boldsymbol{J}\cdot\boldsymbol{\theta}}$ (according to equations 2.26 and 2.27) or as a product $\mathscr{P}R(\theta_1, \theta_2, \theta_3)$, where $\mathscr{P}$ represents the transformation of parity in Minkowski spacetime, defined in 2.14. This subgroup, of spatial rotations and reflections, preserves the time-like four-vector $(1, 0, 0, 0)$. As a consequence, a general transformation of the Lorentz group can be determined up to a rotation or reflection from its action on this four-vector.

A general Lorentz transformation maps the four-vector $(1, 0, 0, 0)$ into another four-vector $\bar{x}^\mu$, while preserving the quadratic form

$$-(\bar{x}^0)^2 + (\bar{x}^1)^2 + (\bar{x}^2)^2 + (\bar{x}^3)^2 = -1 \;\Rightarrow\; (\bar{x}^0)^2 = 1 + (\bar{x}^1)^2 + (\bar{x}^2)^2 + (\bar{x}^3)^2, \tag{3.26}$$

which defines an hyperboloid of two sheets, sketched in figure 3.3. Every point of this three-dimensional surface determines a Lorentz transformation up to a spatial rotation or reflection. Therefore, each connected component of *O(3)* defines two connected components of the Lorentz group that can be mapped into one another by the transformation of time reversal $\mathscr{T}$, defined in 2.18. From this, it is easy to see that the connected subgroup of the Lorentz group is identical to the proper orthochronous subgroup, as stated in last chapter. The connected components of the Lorentz group $\mathcal{L}$ are, then, the proper orthochronous $\mathcal{L}_+^\uparrow$, the proper non-orthochronous $\mathcal{L}_+^\downarrow$, the improper orthochronous $\mathcal{L}_-^\uparrow$ and the improper non-orthochronous $\mathcal{L}_-^\downarrow$. The transformations in each component can be mapped into each other by the discrete transformations $\mathscr{P}$, $\mathscr{T}$ and $\mathscr{P}\mathscr{T}$ as exhibited in figure 3.4.

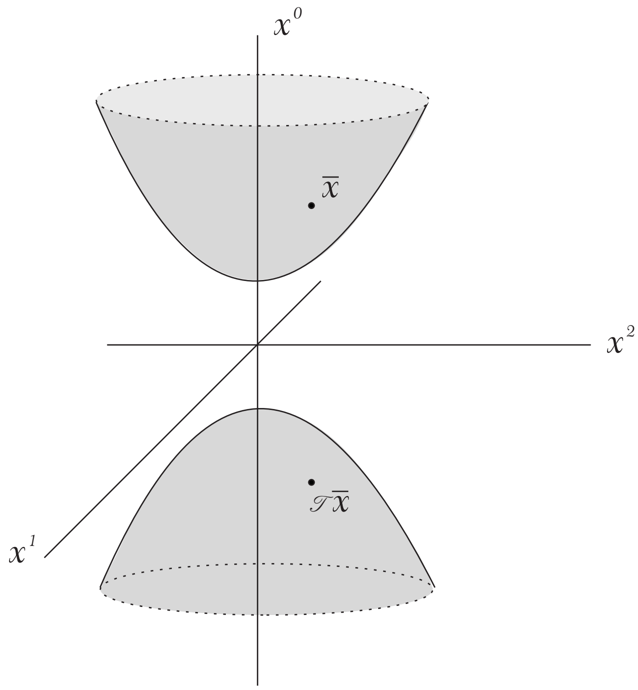

Figure 3.3: Hyperboloid of two sheets

Using the language of cosets, the connected components of the Lorentz group are equivalent to the cosets of the proper orthochronous subgroup $\mathcal{L}_+^\uparrow$, which is of course an invariant subgroup. In this context, the equivalences

$$\begin{aligned} \mathcal{L}_-^\uparrow &= \mathscr{P}\mathcal{L}_+^\uparrow, \\ \mathcal{L}_+^\downarrow &= \mathscr{T}\mathcal{L}_+^\uparrow, \\ \mathcal{L}_-^\downarrow &= \mathscr{P}\mathscr{T}\mathcal{L}_+^\uparrow \end{aligned} \tag{3.27}$$

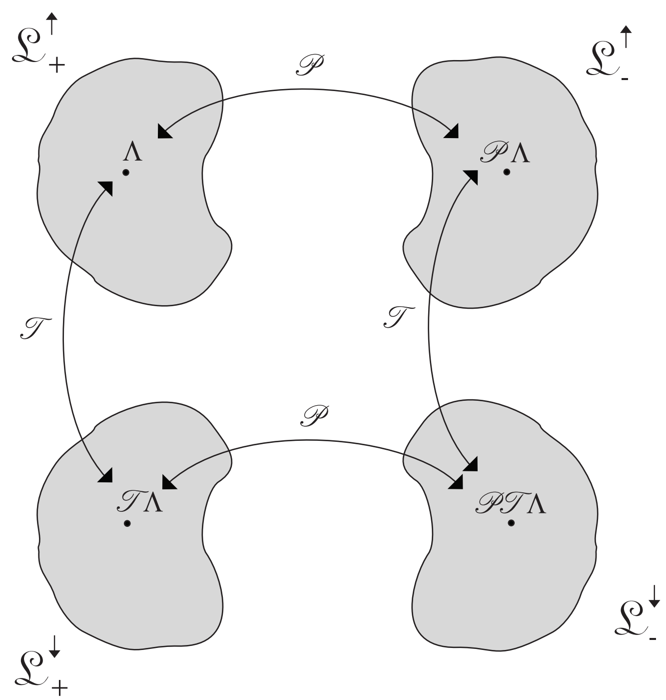

Figure 3.4: Connected components of the Lorentz group

are satisfied.

3.3 Universal covering groups

Every transformation in the connected subgroup of a linear Lie group can be written as a finite product of matrix exponentials of its Lie algebra elements. Hence, it is expected that the representations of any connected linear Lie group can be constructed from the *representations of its algebra* (the set of matrices that represent the commutation relations, as in 2.27). Unfortunately, this is not as direct as it seems. That is because different (non-isomorphic) Lie groups can be associated to the same Lie algebra. As an example, the one-dimensional Lie algebra, composed of only one generator, can be exponentiated to give either the group of rotations on the plane *SO(2)* or the group of positive real numbers under multiplication, denoted by $(\mathbb{R}_{>0}, \times)$. These groups are not isotropic but do present identical Lie algebras. In fact, considering the unique generator to be represented by a real number p, its exponentiation gives 3.8, which is a representation of the multiplicative group of positive real numbers for every p but is only a representation of *SO(2)* when p

is an integer. It is clear then that not all representations of a Lie algebra provide, under matrix exponentiation, representations of an associated linear Lie group.

At this point, it is necessary to introduce the concept of *simply connectedness* for a general connected Lie group. If the dimension of a Lie group is n, the curve given by $x_1 = x_1(t), x_2 = x_2(t), ..., x_n = x_n(t)$ as a function of real parameter $0 \leq t \leq 1$ is said to describe a *continuous path* within the Lie group composed of the elements $g(x_1(t), x_2(t), ..., x_n(t))$. When the group elements on the extreme points of this path coincide, i.e., if $g(x_1(0), x_2(0), ..., x_n(0)) = g(x_1(1), x_2(1), ..., x_n(1)) = g_0$, this path is called a *loop*. A Lie group is then said to be *simply connected* if any loop can be continuously transformed into a point while keeping the extreme points fixed (figure 3.5). For this to be true, there need to exist n functions $f_i(t, s)$, $i = 1, 2, ..., n$, continuous in both variables t and s on the region determined by $0 \leq t, s \leq 1$, and with the properties

$$f_i(t, 0) = x_i(t) \tag{3.28}$$

and

$$g(f_1(t, 1), f_2(t, 1), ..., f_n(t, 1)) = g_0. \tag{3.29}$$

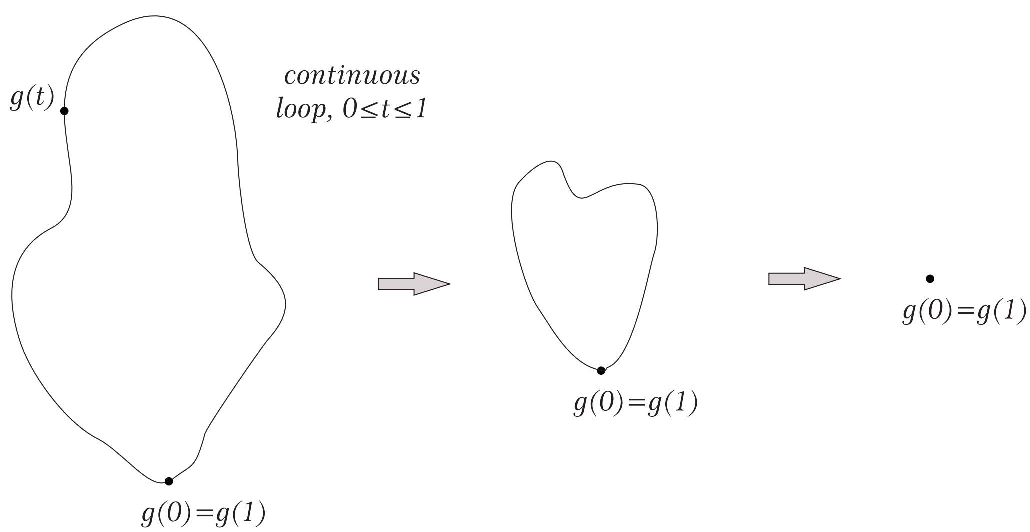

Figure 3.5: A loop being contracted to a point

For the particular case of the group *SO(2)*, a loop is obtained by setting $\theta = 2\pi t$, since $t = 0$ and $t = 1$ both give the identity transformation. Supposing that this loop can be continuously contracted into a point, there should exist a continuous function $f(t,s)$ such that

$$f(t,0) = 2\pi t \tag{3.30}$$

and

$$g(f(t,1)) = g(0) = e \;\Rightarrow\; f(t,1) = 2\pi m, \tag{3.31}$$

where m is an integer and e is the identity transformation of the group *SO(2)*. Continuity implies $f(0,1) = f(0,s) = 0 \Rightarrow m = 0$ and $f(1,1) = f(1,s) = 2\pi \Rightarrow m = 1$, which are contradictory statements. Thus, the loop considered cannot be transformed into a point and the group *SO(2)* is not simply connected. On the other hand, the group $(\mathbb{R}_{>0}, \times)$ is simply connected, even that its algebra is identical to the one of *SO(2)*. A general element of this group can be written, in terms of the real parameter x, as e^x. Consequently, every loop is described by a general continuous function $x(t)$ such that $x(0) = x(1) = x_0$. Every loop of this form is contractible to a point, since the function

$$f(t,s) = (1-s)x(t) + sx_0 \tag{3.32}$$

clearly has the properties 3.28 and 3.29.

For a Lie group, simply connectedness is necessary to guarantee that its representations are furnished by the representations of the associated Lie algebra. It is possible to show that there is only one simply connected group associated to every Lie algebra, this group being called the *universal covering group* of the corresponding algebra. In particular, the universal covering group of the one-dimensional Lie algebra is the multiplicative group $(\mathbb{R}_{>0}, \times)$. This group is also known as the universal cover of *SO(2)*, since it is possible to define a map,

$$\begin{aligned} \phi : (\mathbb{R}_{>0}, \times) &\to SO(2) \\ \mathrm{e}^\theta &\mapsto \phi(\mathrm{e}^\theta) = R(\theta), \end{aligned} \tag{3.33}$$

which clearly preserves products but is not one-to-one. These groups are then said to be *homomorphic* and the map 3.33 is called a *homomorphism*. The *kernel* $\mathrm{Ker}\phi$ of this homomorphism, defined as the subset of elements of $(\mathbb{R}_{>0}, \times)$ that maps into the identity of *SO(2)*, is given by

$$\mathrm{Ker}\phi = \{\mathrm{e}^{2\pi m} \mid m \in \mathbb{Z}\}. \tag{3.34}$$

This is easily seen to define an invariant subgroup, so that the factor group $(\mathbb{R}_{>0}, \times)/\mathrm{Ker}\phi$ is composed of cosets of the form $\mathrm{e}^\theta\mathrm{Ker}\phi$ with $0 \leq \theta < 2\pi$. The group *SO(2)* of rotations on the plane is isomorphic to this factor group, this fact being expressed as

$$SO(2) = (\mathbb{R}_{>0}, \times)/\mathrm{Ker}\phi. \tag{3.35}$$

In general, a universal covering group $\tilde{\mathcal{G}}$ can be associated to an arbitrary linear Lie group $\mathcal{G}$. The group $\tilde{\mathcal{G}}$ needs to be simply connected and have a Lie algebra identical to the one associated with the group $\mathcal{G}$. In addition, it should be possible to find an homomorphic mapping ϕ from $\tilde{\mathcal{G}}$ to $\mathcal{G}$, the kernel (Kerϕ) of which defines an invariant subgroup of $\tilde{\mathcal{G}}$ with the property that $\mathcal{G} = \tilde{\mathcal{G}}/\mathrm{Ker}\phi$. It is also possible to show that every representation of the corresponding Lie algebra exponentiates to a representation of the universal covering group $\tilde{\mathcal{G}}$. As a consequence, if a given representation $\tilde{D}(\tilde{g})$ of $\tilde{\mathcal{G}}$ ($\tilde{g}$ is a general element of this group) associates the same matrix to elements lying in the same coset of $\tilde{\mathcal{G}}/\mathrm{Ker}\phi$, i.e., if

$$\tilde{D}(\mathrm{Ker}\phi) = \mathbb{1}, \tag{3.36}$$

a representation of the group $\mathcal{G}$ is obtained by setting

$$D(\phi(\tilde{g})) = \tilde{D}(\tilde{g}\mathrm{Ker}\phi) \tag{3.37}$$

($\tilde{g}\mathrm{Ker}\phi$ is the coset containing the element $\tilde{g}$). As an example, the one-dimensional representations of *SO(2)* can be obtained from the representations 3.8 of the group $(\mathbb{R}_{>0}, \times)$ if the condition 3.36 is imposed, giving

$$\tilde{D}(\mathrm{Ker}\phi) = \mathrm{e}^{2\pi m p} = 1 \;\Rightarrow\; p \in \mathbb{Z}, \tag{3.38}$$

which exactly reproduces the result obtained in section 3.1. Additionally, the setting

$$D(\phi(\mathrm{e}^{\theta})) = \tilde{D}(\mathrm{e}^{\theta}\mathrm{Ker}\phi) = \mathrm{e}^{p\theta} \tag{3.39}$$

gives the one-dimensional representations of *SO(2)*.

For the case of three-dimensional rotations, the universal cover of *SO(3)* is given by the group *SU(2)* of two-dimensional unitary matrices U with unity determinant, which have complex-valued entries and obey the relation

$$U^{\dagger} = (U^*)^T = U^{-1} \tag{3.40}$$

(the operation symbolized by † receives the name of *Hermitian conjugation*). Every transformation obeying 3.40 can be put into the form

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \tag{3.41}$$

where $\alpha = \alpha_1 + \mathrm{i}\alpha_2$ and $\beta = \beta_1 + \mathrm{i}\beta_2$ are complex numbers that obey the relation

$$|\alpha|^2 + |\beta|^2 = \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1. \tag{3.42}$$

Thus, every transformation U can be associated to a point in the three-dimensional spherical surface determined by 3.42. As every spherical surface (with dimension greater than one) is simply connected, the previous result implies that the group *SU(2)* itself is simply connected. In addition, the Lie algebra of this group is identical to the one of *SO(3)*, depicted in 3.15. If b is a generator of the algebra of *SU(2)*, i.e., if $U = \mathrm{e}^{bx} \in SU(2)$ with x being a real parameter, relation 3.40 implies

$$b^\dagger = -b. \tag{3.43}$$

Then, a basis for this algebra is given by

$$b_1 = \frac{1}{2}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \ , \ b_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \ b_3 = \frac{1}{2}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \tag{3.44}$$

with the commutation relations having the form

$$[b_i, b_j] = -\epsilon_{ijk} b_k, \tag{3.45}$$

which is the same as 3.15.

An homomorphic mapping from *SU(2)* into *SO(3)* can be obtained assuming that the spatial coordinates $x^i (i = 1, 2, 3)$ are transformed under an arbitrary transformation $U \in SU(2)$ into x'^i following the relation

$$\begin{pmatrix} x'^3 & x'^1 - \mathrm{i}x'^2 \\ x'^1 + \mathrm{i}x'^2 & -x'^3 \end{pmatrix} = U \begin{pmatrix} x^3 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & -x^3 \end{pmatrix} U^{-1}. \tag{3.46}$$

Taking the determinant of both sides, this relation implies

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = \left(x'^1\right)^2 + \left(x'^2\right)^2 + \left(x'^3\right)^2. \tag{3.47}$$

Clearly, every transformation $U \in SU(2)$ can be associated with a three-dimensional rotation, $\boldsymbol{x}' = R\boldsymbol{x}$, since both preserve the same characteristic quadratic form. Equation 3.46 can be written as

$$x'^1\sigma_1 + x'^2\sigma_2 + x'^3\sigma_3 = x^1 U\sigma_1 U^{-1} + x^2 U\sigma_3 U^{-1} + x^1 U\sigma_3 U^{-1} \tag{3.48}$$

in terms of the *Pauli matrices* σ_i, given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \ , \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.49}$$

By direct evaluation, the relation

$$\mathrm{Tr}\{\sigma_i \sigma_j\} = \delta_{ij} \tag{3.50}$$

is seen to be valid and can be used, together with equation 3.48, to give

$$x'^i = \frac{1}{2}\,\mathrm{Tr}\{\sigma_i U \sigma_j U^{-1}\} x^j. \tag{3.51}$$

Consequently, the map

$$\begin{aligned} \phi : SU(2) &\to SO(3) \\ U &\mapsto \phi(U) = R, \end{aligned} \tag{3.52}$$

where

$$R^i{}_j = \frac{1}{2}\,\mathrm{Tr}\{\sigma_i U \sigma_j U^{-1}\}, \tag{3.53}$$

is a two-to-one homomorphism with kernel given by $\mathrm{Ker}\phi = \{\mathbb{1}, -\mathbb{1}\}$, in such a way that

$$SO(3) = SU(2)/\{\mathbb{1}, -\mathbb{1}\}. \tag{3.54}$$

Every representation of the Lie algebra of *SO(3)* (or, equivalently, *SU(2)*) provides under matrix exponentiation a representation $\tilde{D}(U)$ of the universal covering group *SU(2)*. If this representation has the property

$$\tilde{D}(\mathbb{1}) = \tilde{D}(-\mathbb{1}) = 1, \tag{3.55}$$

it can be used to construct a representation for the group *SO(3)* as

$$D(R) = \tilde{D}(U) = \tilde{D}(-U). \tag{3.56}$$

Instead, if $\tilde{D}(\mathbb{1}) \neq \tilde{D}(-\mathbb{1})$, it is still possible to obtain representations known as *projective* for the group *SO(3)*. These representations are double-valued, in the sense that it assigns two matrices, $\tilde{D}(U)$ and $\tilde{D}(-U)$, to every transformation $R \in SO(3)$.

3.4 Reducibility of finite-dimensional representations

The matrices representing the transformations of a group can be associated to linear operators acting on a finite-dimensional complex vector space V, which is referred as the *carrier space* of the representation. Supposing that the group $\mathcal{G}$ posses a n-dimensional representation $D(g)$, $g \in \mathcal{G}$, the action of the transformations g can be represented by

$$\Phi(g)\psi^i = [D(g)]^i{}_j\psi^j \tag{3.57}$$

where the vectors ψ^i $(i = 1, 2, ..., n)$ give a basis for the carrier space V and $\Phi(g)$ are linear operators in this space.

The basis for the carrier space of a representation can be altered by performing a *change of basis*

$$\psi^i \to \psi'^i = S^i{}_j\psi^j, \tag{3.58}$$

where the complex-valued matrix S needs to denote an invertible linear transformation, i.e., $\det S \neq 0$. The linear operators $\Phi(g)$ are then transformed into

$$\Phi(g)\psi'^i = S^i{}_j\Phi(g)\psi^j = S^i{}_j[D(g)]^j{}_k(S^{-1})^k{}_l\psi'^l, \tag{3.59}$$

so that the matrices of a representation $D(g)$ are changed into $SD(g)S^{-1}$ (this is called a *similarity transformation*). In this sense, representations of a group that are related by similarity transformations are said to be *equivalent*, since they differ only by a change of basis of the carrier space.

Even though a linear Lie group presents an infinite number of representations, most of the higher-dimensional representations can be constructed from the lower-dimensional ones. Considering, for example, the representation of the group *SO(2)* exhibited in 3.9, a similarity transformation can be applied to give

$$SD[R(\theta)]S^{-1} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \tag{3.60}$$

with

$$S = \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}, \tag{3.61}$$

clearly showing that this two-dimensional representation of *SO(2)* can be decomposed into one-dimensional ones. For this reason, this is referred as a *completely reducible* representation.

For a general group $\mathcal{G}$, a completely reducible representation $D(g)$ acts on its carrier space as

$$D(g)\psi = \begin{pmatrix} D_1(g) & 0 & \cdots & 0 \\ 0 & D_2(g) & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & D_r(g) \end{pmatrix}\psi \tag{3.62}$$

(ψ represents the column matrix $(\psi^1, \psi^2, ..., \psi^n)$), where $D_s(g)$ $(s = 1, 2, ..., r)$ symbolize representations of the group $\mathcal{G}$ that cannot be further decomposed into block-diagonal matrices. Thus, the carrier space of $D(g)$ is a *direct sum* of subspaces that cannot be transformed into each other by elements of the group $\mathcal{G}$, thus being called *invariant subspaces.* If the dimension of $D_s(g)$ is given by n_s, the action of this representation is restricted to the elements ψ^i with $\sum_{s'=1}^{s-1} n_{s'} + 1 \leq i \leq \sum_{s'=1}^{s-1} n_{s'} + n_s$. For the representations $D_s(g)$ there are only two possibilities: either their carrier spaces do not posses invariant subspaces, these being classified as *irreducible* representations, or their carrier spaces present one or more invariant subspaces but cannot be expressed as a direct sum of them. In the latter case, the matrices $D_s(g)$ can be put in the form

$$D_s(g) = \begin{pmatrix} D_s^{(1)}(g) & D_s^{12}(g) & \cdots & D_s^{1n_s}(g) \\ 0 & D_s^{(2)}(g) & & D_s^{2n_s}(g) \\ \vdots & & \ddots & \\ 0 & 0 & & D_s^{(n_s)}(g) \end{pmatrix}, \tag{3.63}$$

with the matrices $D_s^{(p)}(g)$ acting on irreducible representations of the group $\mathcal{G}$. These are referred as *reducible* (but not completely) representations. In fact, every representation that has an invariant subspace is classified as reducible, in such a way that completely reducible representations are of course a subtype of those. It is important to emphasize that the presence of an invariant subspace does not imply complete reducibility for a representation in the most general case.

Chapter 4

Representations of the Lorentz group

In this chapter, a method for obtaining finite-dimensional representations of Lie groups from the respective Lie algebras is presented by the specific treatment of the groups of three-dimensional rotations and of Lorentz transformations.[1] This method will be further used in the next chapter to obtain representations of the VSR subgroups *HOM(2)* and *SIM(2)*.

4.1 Representations of SO(3)

In section 3.3, the universal covering group of *SO(3)* was shown to be the special unitary group *SU(2)*. Then, the representations of the algebra depicted in 3.15 should exponentiate to representations of *SU(2)*. To find representations of this algebra, it is convenient to treat separately cases of fixed dimension. The first case to be considered is the one-dimensional case, in which the generators a_i $(i = 1, 2, 3)$ are given by complex numbers. As every pair of complex numbers commute with each other under multiplication, the only one-dimensional representation of the algebra 3.15 is the trivial one, $a_i = 0$ $(i = 1, 2, 3)$, which of course gives the trivial representation of *SU(2)* under matrix exponentiation.

The case of dimension two is more interesting. As the intention here is to find non-equivalent representations, the freedom of a similarity transformation can always be explored. For example, the generator a_3 can be chosen, assuming that it is diagonizable, to

[1] Both these groups are classified as *semi-simple*, since their Lie algebras do not posses any abelian invariant subalgebra. This implies that every reducible representation is *completely reducible*.

have the form

$$D(a_3) = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}, \tag{4.1}$$

with the coefficients γ_1 and $\gamma_2 \neq \gamma_1$ being the eigenvalues of a_3. The commutator of every matrix X (with coefficients X_{ij}, $i, j = 1, 2$) with a diagonal matrix of the form 4.1 is given by

$$[X, D(a_3)] = \begin{pmatrix} 0 & -X_{12}(\gamma_1 - \gamma_2) \\ X_{21}(\gamma_1 - \gamma_2) & 0 \end{pmatrix} \tag{4.2}$$

so that the generators a_1 and a_2 will be represented by matrices of the form

$$D(a_1) = \begin{pmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{pmatrix}, \; D(a_2) = \begin{pmatrix} 0 & \beta_1 \\ \beta_2 & 0 \end{pmatrix}. \tag{4.3}$$

Then, the form of the algebra 3.15 and relation 4.2 imply

$$\left| \begin{array}{l} \alpha_1 = \beta_1(\gamma_1 - \gamma_2) \\ \alpha_2 = -\beta_2(\gamma_1 - \gamma_2) \\ \beta_1 = -\alpha_1(\gamma_1 - \gamma_2) \\ \beta_2 = \alpha_2(\gamma_1 - \gamma_2) \\ \gamma_1 = -\gamma_2 = \alpha_2\beta_1 - \alpha_1\beta_2 \end{array} \right. . \tag{4.4}$$

As a consequence, $(\gamma_1 - \gamma_2)^2 = -1$ and, without lost of generality, $\gamma_1 - \gamma_2$ can be chosen to be $+\mathrm{i}$, since a similarity transformation can be performed to switch the positions of γ_1 and γ_2 in 4.1. Then, the equations in 5.29 give

$$\left| \begin{array}{l} \alpha_1 = \mathrm{i}\beta_1 \\ \alpha_2 = -\mathrm{i}\beta_2 \\ \gamma_1 = -\gamma_2 = \alpha_2\beta_1 - \alpha_1\beta_2 = \frac{1}{2}\mathrm{i} \end{array} \right. \Rightarrow \left| \begin{array}{l} \alpha_1 = \mathrm{i}\beta_1 \\ \alpha_1\alpha_2 = \beta_1\beta_2 = -\frac{1}{4} \\ \gamma_1 = -\gamma_2 = \frac{1}{2}\mathrm{i} \end{array} \right. \tag{4.5}$$

Another similarity transformation, of the form $D(a_i) \to SD(a_i)S^{-1}$ $(i = 1, 2, 3)$, with

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 2\beta_1 \end{pmatrix}, \tag{4.6}$$

can be performed to set the scale of the coefficients α_1, α_2, β_1 and β_2, so that the result obtained is

$$D(a_1) = \frac{1}{2}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \; , \; D(a_2) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \; , \; D(a_3) = \frac{1}{2}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \tag{4.7}$$

which is identical to the representation shown in 3.44. The conclusion is that every other representation of the algebra *su(2)* is equivalent to the one obtained in 4.7 (at least when $D(a_3)$ is diagonizable). Under matrix exponentiation, this representation gives two-dimensional unitary matrices with determinant one (as can be seen from the discussion in section 3.3). This representation is used in the definition of the matrix group *SU(2)* and, consequently, receives the name of *fundamental representation*.

In the language of section 3.3, the fundamental representation is given by $\tilde{D}(U) = U$, which clearly does not obey relation 3.55. Therefore, it does not give a representation (in the sense defined in last chapter) of *SO(3)*. However, the concept of representation can be expanded to include the so-called *projective representations*, which have the property of preserving group products up to an arbitrary complex phase $\mathrm{e}^{\mathrm{i}\varphi}$, i.e., if $D(g)$ is a projective representation of a group,

$$D(gg') = \mathrm{e}^{\mathrm{i}\varphi(g,g')} D(g)D(g'). \tag{4.8}$$

Consequently, the setting $D(R) = \tilde{D}(U)$, with $R = \phi(U) = \phi(-U)$ and U being arbitrarily chosen to represent every pair $\{U, -U\}$, gives a projective representation for the group of three-dimensional rotations *SO(3)*, since

$$D(R_1 R_2) = \tilde{D}(\pm U_1 U_2) = \pm \tilde{D}(U_1)\tilde{D}(U_2) = \pm D(R_1)D(R_2). \tag{4.9}$$

It may also possible that the matrix representing the generator a_3 is not diagonizable. In such cases, a similarity transformation can be performed to take the matrix $D(a_3)$ into the *Jordan canonical form*

$$\begin{pmatrix} \gamma & 1 \\ 0 & \gamma \end{pmatrix}. \tag{4.10}$$

The commutator of an arbitrary matrix X (with coefficients X_{ij}, $i, j = 1, 2$) with the matrix above is then

$$[X, D(a_3)] = \begin{pmatrix} -X_{21} & X_{11} - X_{22} \\ 0 & X_{21} \end{pmatrix}, \tag{4.11}$$

so that the matrices representing the generators a_1 and a_2 should have the form

$$D(a_1) = \begin{pmatrix} \alpha_1' & \alpha_2' \\ 0 & -\alpha_1' \end{pmatrix} \; , \; D(a_2) = \begin{pmatrix} \beta_1' & \beta_2' \\ 0 & -\beta_1' \end{pmatrix}. \tag{4.12}$$

Inserting these matrices in the commutation relations 3.15, $D(a_i) = 0$ ($i = 1, 2, 3$), i.e., the trivial representation, is the only possible result.

In a similar way, a three-dimensional representation can be constructed for *SO(3)* starting with

$$D(a_3) = \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix}, \tag{4.13}$$

where the eigenvalues γ_1, γ_2 and γ_3 are all different, so that a relation equivalent to 4.2 can be obtained for an arbitrary three-dimensional matrix X (with coefficients X_{ij}, $i, j = 1, 2, 3$) as

$$[X, D(a_3)] = \begin{pmatrix} 0 & -X_{12}(\gamma_1 - \gamma_2) & -X_{13}(\gamma_1 - \gamma_3) \\ X_{21}(\gamma_1 - \gamma_2) & 0 & -X_{23}(\gamma_2 - \gamma_3) \\ X_{31}(\gamma_1 - \gamma_3) & X_{32}(\gamma_2 - \gamma_3) & 0 \end{pmatrix}. \tag{4.14}$$

in such a way that the matrices of the other generators should have the form

$$D(a_1) = \begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ \alpha_4 & 0 & \alpha_3 \\ \alpha_5 & \alpha_6 & 0 \end{pmatrix}, \; D(a_2) = \begin{pmatrix} 0 & \beta_1 & \beta_2 \\ \beta_4 & 0 & \beta_3 \\ \beta_5 & \beta_6 & 0 \end{pmatrix}. \tag{4.15}$$

Inserting these in the commutators 3.15 and assuming that at least one variable in each of the pairs $\{\beta_1, \beta_4\}$ and $\{\beta_3, \beta_6\}$ is not null[2], it follows that

$$\left| \begin{array}{l} (\gamma_1 - \gamma_2)^2 = (\gamma_2 - \gamma_3)^2 = -1 \\ \gamma_1 + \gamma_2 + \gamma_3 = 0 \end{array} \right. , \tag{4.16}$$

where the second equation comes from taking the trace of the commutation relations in 3.15. Without any loss of generality, it is possible to set

$$\gamma_1 = +\mathrm{i} \; , \; \gamma_2 = 0 \; , \; \gamma_3 = -\mathrm{i}. \tag{4.17}$$

In addition, just as in the two-dimensional case, the freedom of similarity transformations can be explored to set the values of some of the other coefficients; in this case, of two

[2]In the cases $\beta_1 = \beta_4 = 0$ and $\beta_3 = \beta_6 = 0$, the representations obtained are completely reducible and can be written in terms of two-dimensional ones.

of them. Therefore, $\beta_1 = \beta_3 = 1/\sqrt{2}$ can be conveniently chosen without excluding any non-equivalent representations and this gives

$$\left| \begin{array}{l} \alpha_1 = \alpha_3 = \mathrm{i}/\sqrt{2} \\ \alpha_2 = \beta_2 = \alpha_5 = \beta_5 = 0 \\ \alpha_4 = \alpha_6 = -\mathrm{i}\beta_4 = -\mathrm{i}\beta_6 = \dfrac{\mathrm{i}}{\sqrt{2}} \end{array} \right. , \tag{4.18}$$

again using the commutators in 3.15. This provides the representation

$$D(a_1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \mathrm{i} & 0 \\ \mathrm{i} & 0 & \mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} , \; D(a_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} , \; D(a_3) = \begin{pmatrix} \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \end{pmatrix} , \tag{4.19}$$

which can be seen to be equivalent to the representation obtained in 3.14 by the application of the similarity transformation $D(a_i) \to SD(a_i)S^{-1}$, with

$$S = \begin{pmatrix} 1 & 0 & -1 \\ \mathrm{i} & 0 & \mathrm{i} \\ 0 & -\sqrt{2} & 0 \end{pmatrix} . \tag{4.20}$$

In fact, every irreducible three-dimensional representation of the algebra *su(2)* is equivalent to 4.19 (the non-diagonizable case also does not give representations here). Under matrix exponentiation, this representations is clearly seen to give the fundamental representation of the group *SO(3)*, which was discussed in section 3.1.

4.2 *SL(2,*$\mathbb{C}$*)*

The procedure explored in last section can be used to obtain finite-dimensional representations of the Lorentz group $\mathcal{L}$. However, it is necessary to obtain first the universal covering group for the group of Lorentz transformations. To achieve this, it is necessary to extend the homomorphism described in 3.52 and 3.53. Considering a position four-vector x^μ to transform as

$$\begin{pmatrix} x'^3 + x'^0 & x'^1 - \mathrm{i}x'^2 \\ x'^1 + \mathrm{i}x'^2 & -x'^3 + x'^0 \end{pmatrix} = M \begin{pmatrix} x^3 + x^0 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & -x^3 + x^0 \end{pmatrix} M^{-1} \tag{4.21}$$

under the transformation defined by the matrix M, the relation

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = -\left(x'^0\right)^2 + \left(x'^1\right)^2 + \left(x'^2\right)^2 + \left(x'^3\right)^2 \tag{4.22}$$

is valid for all matrices M such that $\det M = 1$. Denoting $\sigma_0 = \mathbb{1}$, equation 4.21 can be written as

$$x'^0\sigma_0 + x'^1\sigma_1 + x'^2\sigma_2 + x'^3\sigma_3 = x^0 M\sigma_0 M^{-1} + x^1 M\sigma_1 M^{-1} + x^2 M\sigma_3 M^{-1} + x^1 M\sigma_3 M^{-1} \tag{4.23}$$

and, using $\mathrm{Tr}\{\sigma_\mu\sigma_\nu\} = \delta_{\mu\nu}$ $(\mu, \nu = 1, 2, 3, 4)$,

$$\Lambda^\mu{}_\nu = \frac{1}{2}\,\mathrm{Tr}\{\sigma_\mu M\sigma_\nu M^{-1}\} \tag{4.24}$$

is seen to define a Lorentz transformation. As a consequence, it is possible to construct the homomorphic mapping given by

$$\begin{aligned} \phi : SL(2,\mathbb{C}) &\to \mathcal{L} \\ M &\mapsto \phi(M) = \Lambda, \end{aligned} \tag{4.25}$$

with the coefficients of the matrix Λ being defined as in 4.24 and *SL(2,$\mathbb{C}$)* denoting the group of two-dimensional matrices M with unity determinant. The kernel of the homomorphism defined in 4.24and 4.25 is given by $\mathrm{Ker}\phi = \{\mathbb{1}, -\mathbb{1}\}$, so that 4.25 is seen to be two-to-one and

$$\mathcal{L} = SL(2,\mathbb{C})/\{\mathbb{1}, -\mathbb{1}\}. \tag{4.26}$$

Every matrix $M \in SL(2,\mathbb{C})$ can be written in the form

$$M = UH, \tag{4.27}$$

where $U \in SU(2)$ and H is an hermitian matrix ($H^\dagger = H$) with unity determinant, this being referred as the *polar decomposition* of M. The matrix H can always be taken to have a positive trace. In fact, if the trace of this matrix happens to be negative, one can redefine $U \to -U$ and $H \to -H$ to get $\mathrm{Tr}\, H > 0$. Every two-dimensional hermitian matrix can be written as

$$H = \begin{pmatrix} y^0 + y^3 & y^1 + \mathrm{i}y^2 \\ y^1 - \mathrm{i}y^2 & y^0 + y^3 \end{pmatrix}. \tag{4.28}$$

where y_0, y_1, y_2, y_3 denote real parameters. As $\det H = 1$ and $\mathrm{Tr}\, H > 0$, these parameters obey

$$(y^0)^2 - (y^1)^2 - (y^2)^2 - (y^3)^2 = 1\ ,\ y^0 > 0, \tag{4.29}$$

which clearly define a simply connected space. This implies that the group of two-dimensional hermitian matrices with determinant one and positive trace is simply connected. Furthermore, as every element of *SL(2,ℂ)* can be written in the form 4.27 and *SU(2)* is simply connected, the group *SL(2,ℂ)* is seen to also be simply connected, thus proving that it defines the universal covering group of the Lorentz group $\mathcal{L}$.

4.3 Scalar representation

The first case to be treated for the Lorentz group is that of one-dimensional representations. Considering the algebra 2.28, its only one-dimensional representation is clearly seen to be the trivial

$$J_i = K_i = 0 \ , \ i = 1, 2, 3, \tag{4.30}$$

which exponentiates to

$$D(\Lambda) = 1, \tag{4.31}$$

for every $\Lambda \in \mathcal{L}$. This is usually called the *scalar representation* of the group of Lorentz transformations.

4.4 Spinor representations

The algebra *su(2)* can be thought as a *subalgebra* of the algebra 2.28, in such a way that a two-dimensional representation for the algebra of $\mathcal{L}$ necessarily needs to associate the matrices in 4.7 (or equivalent ones) to the generators of rotations J_i, $(i = 1, 2, 3)$. In this context, it can be immediately set

$$D(J_1) = \frac{1}{2}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \ , \ D(J_2) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \ D(J_3) = \frac{1}{2}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \tag{4.32}$$

and these can be used to find the other matrices in the representation. As the generator K_3 commutes with J_3, the matrix representing it can be diagonalized without changing 4.32. Denoting it by

$$D(K_3) = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \tag{4.33}$$

the second line commutation relations in 2.28 can be used to give

$$D(K_1) = \frac{(\kappa_1 - \kappa_2)}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ D(K_2) = \frac{\mathrm{i}(\kappa_1 - \kappa_2)}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.34}$$

Inserting these into the third line commutators in 2.28, it follows

$$(\kappa_1 - \kappa_2)^2 = 1. \tag{4.35}$$

In addition, noting that every matrix in this representation is traceless[3]. it can be seen that

$$\kappa_1 + \kappa_2 = 0, \tag{4.36}$$

so that

$$\kappa_1 = \frac{1}{2} = -\kappa_2 \quad \text{or} \quad \kappa_1 = -\frac{1}{2} = -\kappa_2. \tag{4.37}$$

Since there is no freedom to exchange the order of κ_1 and κ_2 (as this was already explored to set the form of $D(J_3)$), both cases in 4.37 give non-equivalent representations of the Lie algebra of $\mathcal{L}$. The first one, which will be denoted as $+$, gives

$$D_+(K_1) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;,\; D_+(K_2) = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \;,\; D_+(K_3) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.38}$$

This representation is characterized by the relation

$$D_+(J_i) = +\mathrm{i}D_-(K_i), \tag{4.39}$$

thus justifying the symbol $+$. In addition, the vectors ψ_L of the two-dimensional carrier space associated with this representation are denoted as *left-handed Weyl spinors*. On the other hand, the second case, denoted as $-$, implies

$$D_-(K_1) = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \;,\; D_-(K_2) = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \;,\; D_-(K_3) = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.40}$$

which in turn obeys the relation

$$D_-(J_i) = -\mathrm{i}D_-(K_i), \tag{4.41}$$

with the elements of its carrier space being called *right-handed Weyl spinors* ψ_R. These representations differ only in the form of the matrices representing the generators of boost, with rotations being expressed as $D_+(J_i) = D_-(J_i) = D(J_i)$ for $i = 1, 2, 3$.

All representations equivalent to D_+ obey equation 4.39, as any similarity transformation preserves the form of this equation. This is also true for D_-, if relation 4.41 is

[3] This is true because $\mathrm{Tr}(AB) = \mathrm{Tr}(BA) \Rightarrow \mathrm{Tr}[A, B] = 0$, for every square matrices A and B.

considered. In this sense, the two-dimensional representations of the algebra 2.28 can be divided in two sets, one in which every representation is equivalent to D_+ and the other with all elements equivalent to D_-. It is possible to define discrete transformations that connect these non-equivalent representations, one example being the transformation of complex conjugation $\mathscr{K}$, which clearly interchanges equations 4.39 and 4.41.

In spinor representations, the transformations generated by J_3 are given by

$$\tilde{D}(M) = \mathrm{e}^{D(J_3)\theta_3} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta_3/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\theta_3/2} \end{pmatrix}, \tag{4.42}$$

in such a way that $\theta_3 = 2\pi$ does not give the identity matrix and relation 3.36 is then seen to be violated. That is because the spinor representation is in fact a projective representation of the Lorentz group. This implies that a rotation by an angle of 4π is necessary to map a Weyl spinor back into itself and that under a 2π-rotation it acquires a minus sign. It can also be said that the spinor representation defines a *double-valued* representation of the Lorentz group.

4.5 Three-dimensional representations

For the case of three-dimensional representations, calculations can be performed exactly as in last section. Starting with the diagonal matrix form for the generator K_3,

$$D(K_3) = \begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 \end{pmatrix}, \tag{4.43}$$

and assuming the rotation generators to be represented by the matrices in 4.19, i.e.,

$$D(J_1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \mathrm{i} & 0 \\ \mathrm{i} & 0 & \mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \; D(J_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \; D(J_3) = \begin{pmatrix} \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \end{pmatrix}, \tag{4.44}$$

the commutators in 2.28 give

$$
\begin{aligned}
D(K_1) = -[D(J_2), D(K_3)] &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \kappa_1 - \kappa_2 & 0 \\ \kappa_1 - \kappa_2 & 0 & \kappa_2 - \kappa_3 \\ 0 & \kappa_2 - \kappa_3 & 0 \end{pmatrix}, \\
D(K_2) = [D(J_1), D(K_3)] &= \frac{\mathrm{i}}{\sqrt{2}} \begin{pmatrix} 0 & -(\kappa_1 - \kappa_2) & 0 \\ \kappa_1 - \kappa_2 & 0 & -(\kappa_2 - \kappa_3) \\ 0 & \kappa_2 - \kappa_3 & 0 \end{pmatrix},
\end{aligned} \tag{4.45}
$$

in such a way that the commutator $[D(J_1), D(K_1)] = 0$ implies

$$\kappa_1 + \kappa_3 = 2\kappa_2 \tag{4.46}$$

and, as all generators are traceless,

$$\kappa_1 + \kappa_2 + \kappa_3 = 0. \tag{4.47}$$

Thus, $\kappa_2 = 0$ and $\kappa_3 = -\kappa_1$, and, finally, the commutator $[K_1, K_2] = J_3$ gives

$$\kappa_1^2 = 1 \;\Rightarrow\; \kappa_1 = 1 \text{ or } \kappa_1 = -1. \tag{4.48}$$

Both these values provide non-equivalent representations for the algebra of the Lorentz group obeying

$$D_-(K_i) = +\mathrm{i}D_-(J_i) \quad \text{and} \quad D_+(K_i) = -\mathrm{i}D_+(J_i), \tag{4.49}$$

respectively, for $\kappa_1 = -1$ and $\kappa_1 = +1$.

4.6 Vector representation

From the two-dimensional representations obtained in section 4.4, it is possible to construct a four-dimensional representation given by

$$D_{+-}(\Lambda) = D_+(\Lambda) \otimes D_-(\Lambda), \tag{4.50}$$

where the symbol $\otimes$ denotes the operation known as *direct product* of matrices. Explicitly, these matrices are defined to be

$$D_{+-}(\Lambda) = \begin{pmatrix} D_+(\Lambda)_{11}D_-(\Lambda)_{11} & D_+(\Lambda)_{11}D_-(\Lambda)_{12} & D_+(\Lambda)_{12}D_-(\Lambda)_{11} & D_+(\Lambda)_{12}D_-(\Lambda)_{12} \\ D_+(\Lambda)_{11}D_-(\Lambda)_{21} & D_+(\Lambda)_{11}D_-(\Lambda)_{22} & D_+(\Lambda)_{12}D_-(\Lambda)_{21} & D_+(\Lambda)_{12}D_-(\Lambda)_{22} \\ D_+(\Lambda)_{21}D_-(\Lambda)_{11} & D_+(\Lambda)_{21}D_-(\Lambda)_{12} & D_+(\Lambda)_{22}D_-(\Lambda)_{11} & D_+(\Lambda)_{22}D_-(\Lambda)_{12} \\ D_+(\Lambda)_{21}D_-(\Lambda)_{21} & D_+(\Lambda)_{21}D_-(\Lambda)_{22} & D_+(\Lambda)_{22}D_-(\Lambda)_{21} & D_+(\Lambda)_{22}D_-(\Lambda)_{22} \end{pmatrix}. \tag{4.51}$$

In the previous representation, the generators a_i $(i = 1, .., 6)$ of the algebra are given by

$$D_{+-}(a_i) = \lim_{x_i \to 0} \frac{\partial D_{+-}(\Lambda)}{\partial x_i} = D_{-}(a_i) \otimes \mathbb{1} + \mathbb{1} \otimes D_{+}(a_i) \tag{4.52}$$

(x_i stands for the parameter of the transformation). Thus, using equations 4.19, 4.38 and 4.40, the explicit form of the matrices representing the generators of rotations and boosts is

$$D_{+-}(J_1) = \frac{1}{2}\begin{pmatrix} 0 & \mathrm{i} & \mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 & \mathrm{i} \\ \mathrm{i} & 0 & 0 & \mathrm{i} \\ 0 & \mathrm{i} & \mathrm{i} & 0 \end{pmatrix}, \ D_{+-}(J_2) = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$$
$$D_{+-}(J_3) = \begin{pmatrix} \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \end{pmatrix}, \ D_{+-}(K_1) = \frac{1}{2}\begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \tag{4.53}$$
$$D_{+-}(K_2) = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \ D_{+-}(K_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and these can be transformed into the matrices 2.27 by a similarity transformation $D(a_i) \to SD(a_i)S^{-1}$ with

$$S = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ \mathrm{i} & 0 & 0 & \mathrm{i} \\ 0 & -1 & -1 & 0 \end{pmatrix}. \tag{4.54}$$

This means that the fundamental representation of the Lorentz group, i.e., the vector representation, can be constructed as a direct product of two-dimensional spinor representations.

Chapter 5

HOM(2) and *SIM(2)*

As discussed in chapter 2, the subgroups *HOM(2)* and *SIM(2)* of the Lorentz group $\mathcal{L}$ can be considered as fundamental symmetry groups in the context of Very Special Relativity. This chapter is devoted to the construction of finite-dimensional representations of these subgroups. The method for such construction is identical to the one used in last chapter for the rotation group $SO(3)$ and the Lorentz group $\mathcal{L}$, i.e., the representations of the groups are obtained from the representations of the associated Lie algebra. In this scenario, the universal covering groups of *HOM(2)* and *SIM(2)* are of fundamental importance. They are presented in the following section.

5.1 The universal covering groups of *HOM(2)* and *SIM(2)*

In section 4.2, a homomorphism (4.25) from *SL(2,*$\mathbb{C}$*)* onto $\mathcal{L}$ was utilized to demonstrate that *SL(2,*$\mathbb{C}$*)* is itself the universal covering group of the Lorentz group. This comes with the additional requirement that the group *SL(2,*$\mathbb{C}$*)* be simply connected, a fact that was also shown in section 4.2. As *HOM(2)* and *SIM(2)* are subgroups of $\mathcal{L}$, the subgroups of *SL(2,*$\mathbb{C}$*)* that map onto *HOM(2)* and *SIM(2)* under the homomorphism 4.25 are natural candidates for their universal covers. Since a homomorphism is already established, the only condition left is regarding the simply connectedness of such candidates.

Equation 4.24 can be transformed into a isomorphism between the Lie algebras of *SL(2,*$\mathbb{C}$*)* and $\mathcal{L}$ by taking the derivative of both sides with respect to an arbitrary parameter (of rotations, θ_i, or boosts, φ_i, $i = 1, 2, 3$). In fact, if the elements of the *sl(2,*$\mathbb{C}$*)* algebra are denoted by b_i and the elements of the algebra of the Lorentz groups by a_i

(both with $i = 1, ..., 6$), it easy to see that

$$(a_i)^\mu{}_\nu = \frac{1}{2}\,\mathrm{Tr}\{\sigma_\mu(b_i\sigma_\nu + \sigma_\nu b_i)\}. \tag{5.1}$$

Relation 5.1 maps elements of the two algebras into each other in a one-to-one way, thus defining a isomorphism between them.

For the case of *HOM(2)*, the elements of the *sl(2,*$\mathbb{C}$*)* algebra that map into the generators $T_1 = K_1 + J_2$, $T_2 = K_2 - J_1$ and K_3 (in the form depicted in 2.27) are given by

$$\tau_1 = \frac{-\sigma_1 + \mathrm{i}\sigma_2}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \;,\; \tau_2 = \frac{-\sigma_2 - \mathrm{i}\sigma_1}{2} = \begin{pmatrix} 0 & 0 \\ \mathrm{i} & 0 \end{pmatrix}$$
$$\kappa_3 = -\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.2}$$

and these define the commutations relations

$$[\tau_1, \tau_2] = 0 \quad , \quad [\tau_1, \kappa_3] = -\tau_1 \quad , \quad [\tau_2, \kappa_3] = -\tau_2. \tag{5.3}$$

Furthermore, if matrix exponentiation is performed on the elements depicted in 5.2, the transformations of the form

$$M_{y_1,y_2,y_3} = \begin{pmatrix} \mathrm{e}^{y_3} & 0 \\ y_1 + \mathrm{i}y_2 & \mathrm{e}^{-y_3} \end{pmatrix} \tag{5.4}$$

(y_1, y_2 and y_3 are real parameters) are seen to define the subgroup of *SL(2,*$\mathbb{C}$*)* that maps onto *HOM(2)* under the homomorphism defined in 4.25. This subgroup contains the identity $\mathbb{1}$ but not the element $-\mathbb{1}$ of the kernel of this homomorphism. As a consequence, the transformations depicted in 5.4 can be one-to-one mapped into *HOM(2)* transformations, thus showing that the subgroup of matrices of the form 5.4 is isomorphic to *HOM(2)*.

Every matrix such as 5.4 can be written as

$$M_{y_1,y_2,y_3} = \begin{pmatrix} \mathrm{e}^{y_3} & 0 \\ 0 & \mathrm{e}^{-y_3} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \mathrm{e}^{y_3}(y_1 + \mathrm{i}y_2) & 1 \end{pmatrix}, \tag{5.5}$$

where the first term is an element of a subgroup isomorphic to the multiplicative group of real numbers and the second one to the additive group of complex numbers. Both of these groups are known to be simply connected, thus implying that *HOM(2)* itself is simply connected (as the considered matrix group was shown to be isomorphic to *HOM(2)*). As a consequence, *HOM(2)* is seen to be the universal covering group of the Lie algebra 5.3. This directly implies that every representation of this algebra exponentiates to a (*single-valued*) representation of *HOM(2)*.

The same procedure can be applied to construct the basis of the *sim(2)* algebra given by

$$\tau_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \ \tau_2 = \begin{pmatrix} 0 & 0 \\ \mathrm{i} & 0 \end{pmatrix} \ , \ \kappa_3 = -\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \ \zeta_3 = \frac{\mathrm{i}}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.6}$$

thus defining the extra commutators

$$[\kappa_3, \zeta_3] = 0 \quad , \quad [\tau_1, \zeta_3] = \tau_2 \quad , \quad [\tau_2, \zeta_3] = -\tau_1. \tag{5.7}$$

These exponentiate to transformations of the form

$$M_{z,w} = \begin{pmatrix} \mathrm{e}^z & 0 \\ w & \mathrm{e}^{-z} \end{pmatrix}, \tag{5.8}$$

with $z = y_3 + \mathrm{i}y_4$ and $w = y_1 + \mathrm{i}y_2$ being arbitrary complex numbers (y_1, y_2, y_3, y_4 are real parameters). This subgroup is two-to-one mapped onto *SIM(2)* under the homomorphism 4.25. Additionally, every element of the form 5.8 can be written as

$$M_{z,w} = \begin{pmatrix} \mathrm{e}^z & 0 \\ 0 & \mathrm{e}^{-z} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \mathrm{e}^z w & 1 \end{pmatrix}, \tag{5.9}$$

the first factor defines a subgroup isomorphic to the multiplicative group com complex numbers, which is not simply connected. This can be seen by considering the path defined by $y_4(t) = 2\pi t$, $y_1(t) = y_2(t) = y_3(t) = 0$, as shown in figure 5.1. The shaded region represents the possible values of e^z, and this excludes only the origin, which in the figure is represented by a hole. Clearly, the considered path cannot be continuously contracted into a point, as it goes around the hole at the origin. The conclusion in, then, that the subgroup of matrices of the form 5.8, which will be denoted as $SLT_2(\mathbb{C})$ (group of 2×2 special *lower triangular* matrices with complex entries), is not simply connected.

The group of complex number under ordinary multiplication $(\mathbb{C}, \times)$ has as universal cover the additive group of complex numbers $(\mathbb{C}, +)$, as can be seen from the homomorphism

$$\begin{aligned} \phi' : (\mathbb{C}, +) &\to (\mathbb{C}, \times) \\ z &\mapsto \mathrm{e}^z, \end{aligned} \tag{5.10}$$

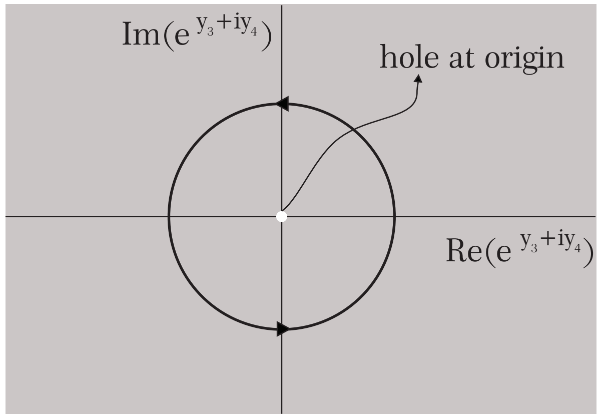

Figure 5.1: A loop that is not contractible into a point

and the fact that $(\mathbb{C},+)$ is indeed simply connected. In this context, the mapping

$$\phi_1 : \tilde{\mathcal{G}} \to SLT_2(\mathbb{C})$$
$$\tilde{M}_{z,w} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -z^*/2 & 1 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-z^*/2} & 0 \\ 0 & 0 & w & \mathrm{e}^{z^*/2} \end{pmatrix} \mapsto M_{z,w} = \begin{pmatrix} \mathrm{e}^{z} & 0 \\ w & \mathrm{e}^{-z} \end{pmatrix}, \tag{5.11}$$

which is clearly an homomorphism, is analogous to the mapping defined in 5.10, as the subgroup of elements $\tilde{M}_{z,0}$ is isomorphic to $(\mathbb{C},+)$ and maps onto the subgroup of $SLT_2(\mathbb{C})$ that is isomorphic to $(\mathbb{C},\times)$. In addition, it is easy to check that the Lie algebras associated with the groups $\tilde{\mathcal{G}}$ and $SLT_2(\mathbb{C})$ are identical. In fact, the definition in 5.11 implies that the Lie algebra of $\tilde{\mathcal{G}}$ is given by

$$\mathsf{T}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \ , \ \mathsf{T}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix}$$
$$\mathsf{K}_3 = -\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ , \ \mathsf{J}_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & -\mathrm{i} \end{pmatrix}. \tag{5.12}$$

Therefore, $\tilde{\mathcal{G}}$ is the universal covering group of *$SLT_2(\mathbb{C})$* and, as a direct consequence, of *SIM(2)*.

The composition of the homomorphism ϕ defined in 4.25 with ϕ_1 depicted in 5.11 gives a homomorphism $\Phi = \phi_1 \circ \phi$ from $\tilde{\mathcal{G}}$ onto *SIM(2)*. This mapping presents a kernel with an infinite number of elements, given by

$$\mathrm{Ker}\Phi = \{\tilde{M}_{\pi \mathrm{i} l,0} \mid l \in \mathbb{Z}\}, \tag{5.13}$$

as can be confirmed by checking that every element $\tilde{M}_{\pi \mathrm{i} l,0}$ is mapped into an element of $\mathrm{Ker}\phi$. Thus, the structure of $\mathrm{Ker}\Phi$ allows multi-valued (even infinite-valued) representations for the group *SIM(2)*.

5.2 From *hom(2)* to *HOM(2)*

5.2.1 One-dimensional case

In last chapter, it was showed that the Lorentz group only admits the trivial representation $D(\Lambda) = 1$ in the case of dimension one. This was because the only one-dimensional solution to the algebra of $\mathcal{L}$ sets all the generators to zero. For *HOM(2)* (and also *SIM(2)*), that is not the case. The algebra represented in the commutators 5.3 admits the one-dimensional solution given by

$$D(T_1) = D(T_2) = 0 \ , \ D(K_3) = k \in \mathbb{C} \text{ (arbitrary)}. \tag{5.14}$$

Under exponentiation, this gives a single-valued representation of *HOM(2)* of the form

$$D(\mathrm{e}^{T_1 \alpha}) = D(\mathrm{e}^{T_2 \beta}) = 1 \ , \ D(\mathrm{e}^{K_3 \varphi}) = \mathrm{e}^{k\varphi}. \tag{5.15}$$

If κ_3 is taken to be purely imaginary, the representations defined in 5.15 is *unitary*, i.e., every matrix in the representation obeys relation 3.40.

5.2.2 Two-dimensional case

The first case to be explored is the one in which the matrix representing the generator K_3 is diagonal, i.e.,

$$D(K_3) = \begin{pmatrix} k & 0 \\ 0 & k' \end{pmatrix}, \tag{5.16}$$

k and k' being the eigenvalues of K_3 in this representation. The commutator of an arbitrary matrix M with coefficients M_{ij} $(i,j = 1,2)$ with $D(K_3)$ is then

$$[M, D(K_3)] = (k - k') \begin{pmatrix} 0 & M_{12} \\ -M_{21} & 0 \end{pmatrix}, \tag{5.17}$$

in such a way that the algebra 5.3 implies that the generators T_1 and T_2 are represented by

$$\begin{gathered} D(T_1) = \begin{pmatrix} 0 & t_1 \\ 0 & 0 \end{pmatrix}, \; D(T_2) = \begin{pmatrix} 0 & t_2 \\ 0 & 0 \end{pmatrix}, \text{ with } k - k' = 1 \\ \text{or} \\ D(T_1) = \begin{pmatrix} 0 & 0 \\ t_1 & 0 \end{pmatrix}, \; D(T_2) = \begin{pmatrix} 0 & 0 \\ t_2 & 0 \end{pmatrix}, \text{ with } k - k' = -1, \end{gathered} \tag{5.18}$$

where t_1 and t_2 are arbitrary complex numbers. However, these can be mapped into each other by a similarity transformation $D \to SDS^{-1}$, with

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{5.19}$$

in such a way that both cases provide equivalent representations. So, it is possible to set

$$D(T_1) = \begin{pmatrix} 0 & t_1 \\ 0 & 0 \end{pmatrix}, \; D(T_2) = \begin{pmatrix} 0 & t_2 \\ 0 & 0 \end{pmatrix}, \; D(K_3) = \begin{pmatrix} k & 0 \\ 0 & k-1 \end{pmatrix}, \tag{5.20}$$

with no non-equivalent representation being lost. Furthermore, if $t_1 \neq 0$ the freedom of similarity transformation can be explored to give

$$D(T_1) \to S' D(T_1) S'^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \ D(T_2) \to S' D(T_2) S'^{-1} = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$$
$$D(K_3) \to S' D(K_3) S'^{-1} = \begin{pmatrix} k & 0 \\ 0 & k-1 \end{pmatrix}, \tag{5.21}$$

where $t = t_2/t_1$ and

$$S' = \begin{pmatrix} 1 & 0 \\ 0 & t_1 \end{pmatrix}. \tag{5.22}$$

Instead, if $t_1 = 0$, the commutations relations in 5.3 imply $t_2 \neq 0$ (otherwise, the representation would be completely reducible) in such a way that every representation is equivalent to

$$D(T_1) = 0 \ , \ D(T_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \ D(K_3) = \begin{pmatrix} k & 0 \\ 0 & k-1 \end{pmatrix}. \tag{5.23}$$

In the case of non-diagonalizable $D(K_3)$, representations similar to the one-dimensional ones of last subsection can be constructed. Setting $D(T_1) = D(T_2) = 0$, the generator K_3 can be represented by an arbitrary matrix and, taking it to be non-diagonalizable, i.e., equivalent to the *Jordan canonical form*

$$D(K_3) = \begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix}, \tag{5.24}$$

the resulting representation is not completely reducible.

As *HOM(2)* is its own universal covering group, every representation of the algebra, of the form 5.21, 5.23 or 5.24, produces a representation of this group under matrix exponentiation. Consequently, it is seen that every two-dimensional representation of *HOM(2)* is equivalent to one of the following representations:

- $D(\mathrm{e}^{T_1\alpha}) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$, $D(\mathrm{e}^{T_2\beta}) = \begin{pmatrix} 1 & t\beta \\ 0 & 1 \end{pmatrix}$ and $D(\mathrm{e}^{K_3\varphi}) = \begin{pmatrix} \mathrm{e}^{k\varphi} & 0 \\ 0 & \mathrm{e}^{(k-1)\varphi} \end{pmatrix}$
- $D(\mathrm{e}^{T_1\alpha}) = \mathbb{1}$, $D(\mathrm{e}^{T_2\beta}) = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$ and $D(\mathrm{e}^{K_3\varphi}) = \begin{pmatrix} \mathrm{e}^{k\varphi} & 0 \\ 0 & \mathrm{e}^{(k-1)\varphi} \end{pmatrix}$

- $D(\mathrm{e}^{T_1\alpha}) = \mathbb{1}$, $D(\mathrm{e}^{T_2\beta}) = \mathbb{1}$ and $D(\mathrm{e}^{K_3\varphi}) = \begin{pmatrix} \mathrm{e}^{k\varphi} & \varphi \\ 0 & \mathrm{e}^{k\varphi} \end{pmatrix}$

5.3 From *sim(2)* to *SIM(2)*

5.3.1 One-dimensional case

The algebra *sim(2)* is composed of the commutators showed in 5.3 and 5.7. Similarly to the case of *hom(2)*, this algebra admits one-dimensional representations that are non-trivial, of the form

$$D(T_1) = D(T_2) = 0 \quad , \quad D(K_3) = k \in \mathbb{C} \text{ and } D(J_3) = j \in \mathbb{C} \text{ (both arbitrary).} \tag{5.25}$$

These matrices generate two-dimensional representations of the universal covering group $\tilde{\mathcal{G}}$, obtained in section 5.1, given by

$$\tilde{D}(\mathrm{e}^{\mathsf{T}_1\alpha'}) = \tilde{D}(\mathrm{e}^{\mathsf{T}_2\beta'}) = \mathbb{1} \quad , \quad \tilde{D}(\mathrm{e}^{\mathsf{K}_3\varphi'}) = \mathrm{e}^{k\varphi'} \quad , \quad \tilde{D}(\mathrm{e}^{\mathsf{J}_3\theta'}) = \mathrm{e}^{j\theta'}, \tag{5.26}$$

where T_1, T_2, K_3 and J_3 are the generators defined in 5.12 and α', β', φ' and θ' are real parameters.

To obtain representations of the VSR subgroup *SIM(2)* from the ones constructed in the last paragraph, it is only necessary to impose condition 3.36, i.e., that all the elements in the kernel 5.13 maps onto the identity. From relations 5.11 and 5.12, it is easy to see that the kernel is composed of the elements with parameters $\theta' = 2\pi l$ ($l \in \mathbb{Z}$), $\alpha' = \beta' = \varphi' = 0$, in such a way that condition 3.36 can be expressed as

$$\mathrm{e}^{2\pi j l} = 1 \ , \ \forall l \in \mathbb{Z} \ \Rightarrow \ j = \mathrm{i}m \ , \ m \in \mathbb{Z}. \tag{5.27}$$

Thus, for integer values of m, the representation depicted in 5.26 gives also a single-valued representation of *SIM(2)*.

It is possible to construct projective representations of *SIM(2)* from $\tilde{\mathcal{G}}$ that are double-valued, triple-valued,..., even infinite-valued. For example, if relation 5.27 is chosen to be valid only for even values of the integer l, in such a way that half-integer values of m are also allowed, the resulting representation is double-valued. On the other hand, taking that relation to be valid for values of l that are multiple of 3, a triple-valued representation is obtained. Also, if no restriction is imposed on the value of j, 5.27 provides an infinite-valued representation for *SIM(2)*.

5.3.2 Two-dimensional case

Once again, it is convenient to work on a basis in which the matrix representing the generator J_3 is in diagonal form,

$$D(J_3) = \begin{pmatrix} j & 0 \\ 0 & j' \end{pmatrix} . \qquad (5.28)$$

If the matrices in 5.28 belong to a representation of *SIM(2)*, relation 5.27 implies $j = \mathrm{i}m$ and $j' = \mathrm{i}m'$. In addition, as *hom(2)* is a subalgebra of *sim(2)*, the matrices representing the other generators can be taken to have one of the three forms obtained in last subsection. The first type, given in 5.21, implies, together with the commutators in 5.7,

$$\left| \begin{array}{l} j' - j = t \\ (j' - j)t = -1 \end{array} \right. \Rightarrow j' - j = t = \pm\mathrm{i}. \qquad (5.29)$$

In addition, it is necessary that

$$j = \mathrm{i}m \ , \ j' = \mathrm{i}m' \quad \text{with} \quad m, m' \in \mathbb{Z} \qquad (5.30)$$

in order to obtain representations of the group *SIM(2)*, of the form

$$\begin{array}{c} D_\pm(\mathrm{e}^{T_1\alpha}) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \ , \ D_\pm(\mathrm{e}^{T_2\beta}) = \begin{pmatrix} 1 & \pm\mathrm{i}\beta \\ 0 & 1 \end{pmatrix} \\ D_\pm(\mathrm{e}^{K_3\varphi}) = \begin{pmatrix} \mathrm{e}^{k\varphi} & 0 \\ 0 & \mathrm{e}^{(k-1)\varphi} \end{pmatrix} \ , \ D_\pm(\mathrm{e}^{J_3\theta}) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}m\theta} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}(m\pm1)\theta} \end{pmatrix} . \end{array} \qquad (5.31)$$

These representations, D_+ and D_-, are nor equivalent and can be obtained from each other by complex conjugation.

For the second type, given in equation 5.23, the commutators in 5.7 imply $\tau_2 = [\tau_1, \zeta_3] = 0$, which gives a contradiction. Thus, no additional representation of the algebra *sim(2)* is obtained in this case. On the other hand, the third type, discussed in the paragraph of relation 5.24, provides representations with

$$D(K_3) = \begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix} \ , \ D(J_3) = \begin{pmatrix} \mathrm{i}m & 0 \\ 0 & \mathrm{i}m \end{pmatrix} , \qquad (5.32)$$

as condition 3.36 is clearly satisfied.

Chapter 6

Conclusion

Special Relativity is frequently taken to be the natural consequence of the causal structure of Minkowski spacetime. However, the further requirement of spatiotemporal isotropy is necessary to promote the Lorentz group to the fundamental symmetry group of nature. If this assumption is not imposed, *HOM(2)* and *SIM(2)* need also to be considered. In this work, it was shown that the representation theory associated with these subgroups brings new elements for the construction of physical models. In fact, most of the representations obtained cannot be extended to the Lorentz group, in the sense that they cannot be expressed as restrictions of representations of the full Lorentz group to *HOM(2)* or *SIM(2)*. This creates the possibility that Very Special Relativity is the natural place for objects that lie outside the Standard Model (in , it was presented as the symmetry group of dark matter, for example).

It is easy to see that the two-dimensional representations of *HOM(2)* and *SIM(2)* obtained at the end of last chapter are reducible. However, these cannot be obtained from the one-dimensional representations (also obtained in last chapter), since the matrices composing them are not equivalent to block diagonal ones (for example, 5.32). In comparison with the case of the Lorentz group, this is a remarkable difference. Every reducible representation of the Lorentz group can be constructed as a direct sum of lower-dimensional irreducible representations (in the language of chapter 3, it can be said that these are completely reducible representations). In this scenario, the irreducible representations of the groups *HOM(2)* and *SIM(2)* (which are all one-dimensional) are not enough to construct models of physical objects, it is also necessary to consider reducible representations.

It is also important to notice that the structure of the universal covering groups of *HOM(2)* and *SIM(2)* strongly determines the possible multi-valued representations. The universal cover of *HOM(2)*, being isomorphic to *HOM(2)* itself, only allows single-valued representations of this group. On the other hand, for the case of *SIM(2)*, multi-valued representations can be of any kind, even infinite-to-one (with a continuous eigenvalue of J_3 in this case). This extends even more the possibilities for the construction of physical theories in a VSR-invariant scenario.

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