UNIVERSIDADE ESTADUAL DE CAMPINAS
FACULDADE DE ENGENHARIA ELÉTRICA E DE COMPUTAÇÃO

FRANCISCO RAIMUNDO ALBUQUERQUE PARENTE

STATISTICAL APPROXIMATIONS TO SUMS OF CORRELATED RAYLEIGH AND EXPONENTIAL RANDOM VARIABLES WITH APPLICATION TO DIVERSITY-COMBINING SCHEMES

APROXIMAÇÕES ESTATÍSTICAS PARA SOMAS DE VARIÁVEIS ALEATÓRIAS CORRELACIONADAS DOS TIPOS RAYLEIGH E EXPONENCIAL COM APLICAÇÃO A ESQUEMAS DE COMBINAÇÃO DE DIVERSIDADE

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ADVISOR/ORIENTADOR: PROF. DR. JOSÉ CÂNDIDO SILVEIRA SANTOS FILHO

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ABSTRACT

Sums of random variables are widely applied to wireless communications systems. Examples include linear equalization, signal detection, interference phenomena, and diversity-combining schemes. However, the exact formulation for the statistical functions of these sums, such as the probability density function and the cumulative distribution function, requires in general a complicated mathematical treatment, which has motivated the search for simple approximate solutions. Although there are several approximate proposals available in the literature, many of which obtained through the traditional moment-matching technique, they do not offer a good fit under the regime of high signal-to-noise ratio. It is well-known that this regime is a paramount region for the performance analysis of communications systems in terms of important metrics such as bit error rate and outage probability. More recently, in order to circumvent this limitation, a new promising technique known as asymptotic matching was proposed, capable of providing approximations for statistics of the sum of random variables with an excellent fit under the regime of high signal-to-noise ratio. Even so, this technique was initially proposed for the sum of mutually independent variables only, and thus it has not been applicable to sums of correlated variables. In this work, a novel asymptotic analysis is proposed, from which it is possible to generalize the application of asymptotic matching to the correlated case. The proposed analysis is illustrated for sums of Rayleigh and sums of exponential variables with arbitrary correlation and arbitrary fading parameters. Furthermore, closed-form asymptotic expressions are derived in order to obtain new simple and precise approximations under the regime of high signal-to-noise ratio. As application examples, practical diversity-combining schemes are addressed, namely, equal-gain combining and maximal-ratio combining. Finally, numerical results show the excellent performance of the proposed approximations in comparison to the approximations obtained via moment matching.

Keywords: Asymptotic analysis, correlation, diversity combining, fading channels, sums of random variables.

RESUMO

Somas de variáveis aleatórias são amplamente aplicadas em sistemas de comunicação sem fio. Exemplos incluem equalização linear, detecção de sinais, fenômenos de interferência e esquemas de combinação de diversidade. No entanto, a formulação exata para as funções estatísticas dessas somas, como a função densidade de probabilidade e a função distribuição acumulada, requer em geral um tratamento matemático complicado, o que tem motivado a busca por soluções aproximadas mais simples. Apesar de haver várias propostas de aproximação disponíveis na literatura, muitas das quais obtidas usando-se a tradicional técnica de casamento de momentos, elas não oferecem um bom ajuste em regime de alta relação sinal-ruído. Sabe-se, porém, que essa é uma região primordial para a análise de desempenho de sistemas de comunicação em termos de métricas importantes como taxa de erro de bit e probabilidade de interrupção. Mais recentemente, com o intuito de contornar essa limitação, foi proposta uma nova técnica promissora conhecida como casamento de assíntotas, capaz de fornecer aproximações para estatísticas de somas de variáveis aleatórias positivas com um ótimo ajuste em regime de alta relação sinal-ruído. Ainda assim, essa técnica foi inicialmente implementada apenas para o caso de somas de variáveis independentes, não sendo até então aplicável para somas de variáveis correlacionadas. Neste trabalho, uma nova análise assintótica é proposta, a partir da qual é possível generalizar o uso do casamento de assíntotas para o caso correlacionado. A análise proposta é ilustrada para somas de variáveis Rayleigh e somas de variáveis exponenciais com correlação e parâmetros de desvanecimento arbitrários. Além disso, deduzem-se expressões assintóticas em forma fechada com o intuito de obter novas aproximações simples e precisas em regime de alta relação sinal-ruído. Como exemplos de aplicação, esquemas práticos de combinação de diversidade são abordados, quais sejam, combinação por ganho igual e combinação por razão máxima. Por fim, resultados numéricos mostram o excelente desempenho das aproximações propostas em comparação com as aproximações obtidas via casamento de momentos.

Palavras-chave: Análise assintótica, canais de desvanecimento, combinação de diversidade, correlação, somas de variáveis aleatórias.

LIST OF FIGURES

LIST OF ABBREVIATIONS AND ACRONYMS

BER	Bit-Error Rate
CF	Characteristic Function
cdf	Cumulative Distribution Function
EGC	Equal-Gain Combining
5G	Fifth Generation
MRC	Maximal-Ratio Combining
OP	Outage Probability
pdf	Probability Density Function
RV	Random Variable
SNR	Signal-to-Noise Ratio

LIST OF SYMBOLS

$\tilde{X}$	random variable that approximates a random variable X
$\sim$	asymptotically equal to (around zero)
Ω	average power
Ω_i	average power of the ith component (or branch)
$J_0(\cdot)$	Bessel function of the first kind and zeroth order
$\Phi_X(\cdot)$	characteristic function of a random variable X
$*$	convolution operator
ρ	correlation coefficient
$F_X(\cdot)$	cumulative distribution function of a random variable X
dB	decibel
$\triangleq$	definition operator
$\det(\cdot)$	determinant operator
$d_{i,j}$	distance between ith and jth antennas
$\equiv$	equivalence operator
$\exp(\cdot)$	exponential function
$\mathscr{F}\{\cdot\}$	Fourier transform
g_i	gain at the ith MRC diversity branch
$\Gamma(\cdot)$	gamma function
j	imaginary unit number (equals $\sqrt{-1}$)
$\mathscr{F}^{-1}\{\cdot\}$	inverse Fourier transform
W_i	ith exponential random variable
R_i	ith Rayleigh random variable
$\mathbb{E}[X^k]$	kth moment of a random variable X
$\boldsymbol{K}^{-1}$	inverse of a matrix $\boldsymbol{K}$
N	mean noise power
$f_{\boldsymbol{X}}(\cdot)$	multivariate probability density function of a vector random variable $\boldsymbol{X}$
M	amount of random variables in the sum
R_{EGC}	output of the equal-gain combiner
R_{MRC}	output of the maximal-ratio combiner
$\tilde{\mu}$	parameter of the gamma and α-μ approximate distributions
$\tilde{\Omega}$	parameter of the gamma, Nakagami-m, Weibull, and α-μ approximate distributions
σ_i	parameter of the ith Rayleigh RV
$\tilde{m}$	parameter of the Nakagami-m approximate distribution
m	parameter of the Nakagami-m distribution
$\tilde{\alpha}$	parameter of the Weibull and α-μ approximate distributions

α	parameter of the α-μ distribution
μ	parameter of the α-μ distribution
$f_X(\cdot)$	probability density function of a random variable X
$\mathbb{N}$	set of natural numbers
Γ_i	signal-to-noise ratio at the ith MRC diversity branch
Γ	signal-to-noise ratio at the MRC output
$\mathscr{N}$	total noise power
$(\cdot)^T$	transpose operator
$\mathbb{V}[\cdot]$	variance operator
λ	carrier wavelength

LIST OF PUBLICATIONS

- F. R. A. Parente e J. C. S. Santos Filho, "Aproximações estatísticas para somas de variáveis Rayleigh correlacionadas e aplicação," *Anais do XXXVI Simpósio Brasileiro de Telecomunicações e Processamento de Sinais (SBrT'18)*, Campina Grande, Brasil, Set. 2018, pp. 563–567.

- F. R. A. Parente and J. C. S. Santos Filho, "Asymptotically exact framework to approximate sums of positive correlated random variables and application to diversity-combining receivers," *IEEE Wireless Communications Letters*, in press.

CONTENTS

Chapter 1

INTRODUCTION

Nowadays people live surrounded by electronic devices that keep them continuously interconnected. It is well-known that wireless technologies have disrupted mobile communications, bringing altogether the world to a new technological stage. In fact, due recent developments and research, mobile devices are increasingly more robust and efficient, whose performance is comparable to that of fixed stations.

Nevertheless, the wireless environment is chaotic by nature. The channel itself may drastically distorts the propagation signal, which undergoes path loss and several other phenomena, such as scattering, reflection, and diffraction [1, 2]. Due to such phenomena, the signal reaches the receiver with a large number of scattered, reflected, and diffracted waves, coming from diverse paths, with random amplitudes and phases, generating what is called multipath propagation. The combination of these factors stochastically alters both amplitude and phase of the received signal, an effect known as fading [2, 3].

One way to overcome the limitations imposed by fading in wireless systems consists of using diversity-combining schemes [3, 4]. Basically, these schemes provide the receiver with multiple replicas (branches) of the transmitted signal, which are then combined to obtain a resulting signal of better quality.

There are several types of diversity-combining techniques, such as equal-gain combining (EGC) and maximal-ratio combining (MRC). These two schemes are additive, and therefore their performance analysis in terms of bit-error rate (BER) and outage probability (OP) requires knowledge of sum statistics, namely, the probability density function (pdf) or, equivalently, the cumulative distribution function (cdf). However, the exact computation of these statistics is rather cumbersome, since it involves a multifold integration over the multivariate pdf of the summands [5]. As the number of random variables (RVs) in the sum increases, the exact formulation may prove unfeasible, which has motivated the search for simple approximate solutions.

Sums of RVs can be applied not only to diversity combining but also to many other communications schemes, such as signal detection and linear equalization. Due to its

importance, several works have proposed approximations to sums considering a variety of fading scenarios. On this concern, the next section presents a brief review of literature with the main research achievements to date and the motivation for this work.

1.1 Literature Review and Motivation

Since the first approximations for sums of RVs were proposed by Nakagami [6], researchers have tried to find good approximate solutions for a variety of fading scenarios. For instance, some works proposed approximations to the sum of non-identical independent Nakagami-m RVs by using either the Nakagami-m distribution itself [7] or the generalized α-μ distribution [8]. Some accurate approximations have also been obtained for sums of many other distributions, such as the sum of Ricean [9] and α-μ [10] RVs.

Nonetheless, many approximations proposed in the literature have been obtained under the constraint that the summands are mutually independent and by using the traditional moment-matching technique [7–10]. Such technique has been designed to provide a good fit in the distribution body, but it loses track of the distribution tail. This region corresponds to the regime of high signal-to-noise ratio (SNR), which is a compelling scenario to compare different communications systems in terms of BER and OP.

In order to overcome this drawback of moment-based approximations, it has been recently proposed a new approach known as asymptotic matching [11]. In this technique, the asymptote of the approximate distribution is matched to the asymptote of the exact sum distribution, guaranteeing an outstanding fit at the high-SNR regime. Even though asymptotic matching offers better approximations at the distribution tail, its use is very recent and has been limited to the independent case only.

More recently, several works have addressed correlated fading scenarios, which are a more realistic assumption to model emerging communication techniques over massive multiple-input multiple-output systems [12, 13]. In such systems, some undesirable correlation between the input-output links may arise due to insufficiently spaced antennas [14]. For instance, considering some diversity-combining schemes over particular fading distributions, asymptotic expressions to approximate performance metrics in the high-SNR regime were derived in [15–21]. Specifically, it was observed in [16–21] that the asymptotic system performance over the correlated channels addressed therein is a scaled version of the asymptotic system performance over independent channels, with the scale factor depending on the correlation matrix. Interestingly, though, it has been overlooked so far that a broad class of positive correlated RVs behaves asymptotically as an equivalent set of mutually independent RVs, which is an insightful and fundamental result explored herein. Another very recent work [22] aimed to approximate the body of the distribution of sums of correlated Weibull RVs by using expressions in terms of the Meijer G-function.

However, as highlighted in [23], this approach may notably depart from the exact distribution tail, and even lead to computationally erroneous results near the origin. This region corresponds to the important regime of high SNR, as one can move toward the distribution tail either by reducing the value of the instantaneous SNR or by increasing the value of the average SNR [16].

From the above reasons, it is important to obtain new accurate approximations for the challenging correlated scenario, specially in the high-SNR region, of most practical interest. In this way, capitalizing on a new asymptotic result for sums of correlated RVs, we propose a unified, general approach to design approximations that render an excellent fit at the high-SNR regime (i.e., at the cdf tail). Particularly, we investigate sums of correlated Rayleigh RVs and sums of correlated exponential RVs with arbitrary fading parameters in both cases. Various candidate approximate distributions are proposed and discussed. As application examples, we analyze the performance of EGC and MRC operating over correlated Rayleigh fading channels. These and other contributions of this work are outlined next.

1.2 Contributions

In this work, the following contributions are provided:

(i) Capitalizing on a new fundamental result elaborated herein, the asymptotic-matching scheme is extended to the correlated scenario, allowing for accurate statistical approximations to general sums near the origin, or, equivalently, at high SNR.

(ii) Asymptotically optimal approximations are proposed to sums of Rayleigh RVs and sums of exponential RVs with arbitrary correlation and arbitrary fading parameters. These approximations keep a good track of the body of the exact sum distribution while ensuring an outstanding fit at the distribution tail, i.e., at high SNR.

(iii) For comparison purposes, the performance of some candidate approximate distributions are evaluated, namely, Nakagami-m, gamma, Weibull, and α-μ distributions.

(iv) New simple, closed-form, asymptotic expressions are derived and applied to EGC and MRC schemes operating over correlated Rayleigh fading.

1.3 Structure

The remainder of this work is organized as follows.

- **Chapter 2:** This chapter introduces the problem formulation for sums of RVs. Considering both independent and correlated fading scenarios, we revisit the exact solution to the problem as well as some approximate approaches available in the literature.

- **Chapter 3:** A newly fundamental insight on sums of arbitrarily correlated RVs is introduced in this chapter. Capitalizing on this novel result, asymptotic matching is performed in order to provide optimal approximations around the origin to sums of correlated Rayleigh and exponential RVs. Various candidate approximate distributions are presented. Finally, the analysis is applied to analyze output statistics of two different diversity-combining schemes, namely, EGC and MRC.

- **Chapter 4:** This chapter illustrates the excellent performance of the proposed approximations for many scenarios. The exact sum statistics are approximated by the Nakagami-m, gamma, Weibull, and α-μ distributions. Numerical results show that the new approximations outclass conventional moment-based approximations, especially at high SNR.

- **Chapter 5:** The main conclusions are summarized in this chapter. It is also presented some final considerations as well as some topics for future work.

Chapter 2

SUMS OF RANDOM VARIABLES

There are several applications of sums of RVs in wireless communications, such as signal detection, linear equalization, and diversity-combining schemes. In these scenarios, the evaluation of system performance in terms of BER and OP requires knowledge of the sum pdf or the sum cdf, whose exact formulation may prove unfeasible. This chapter introduces the exact general solution to find the statistics of sums of RVs, addressing both independent and correlated cases. Afterwards, two methods that provide approximate solutions to circumvent the intricacy of the exact approach are discussed, namely, the traditional moment-matching and the new asymptotic-matching techniques.

2.1 Problem Formulation

Let S be the sum of M arbitrarily correlated RVs S_i, $i \in \{1, \ldots, M\}$, i.e.,

$$S = \sum_{i=1}^{M} S_i. \tag{1}$$

The problem consists of finding the sum pdf and the sum cdf of S, denoted as $f_S(\cdot)$ and $F_S(\cdot)$, respectively. As the cdf can be determined from the pdf in a straightforward manner, the analysis herein is developed based on the pdf alone.

2.2 Exact Solutions

The general formulation to obtain the exact sum pdf $f_S(\cdot)$ of S requires knowledge of the multivariate pdf $f_{\boldsymbol{S}}(\cdot) \triangleq f_{S_1,\ldots,S_M}(\cdot, \ldots, \cdot)$ of $\boldsymbol{S} \triangleq [S_1 \cdots S_M]^T$. In this section, the analysis to obtain the exact sum statistics of arbitrarily correlated RVs is described. Previously, though, the independent scenario is revisited.

2.2.1 Independent Case

For the particular case of mutually independent RVs, the exact sum pdf $f_S(\cdot)$ of S is given by either the convolution of the marginal pdfs $f_{S_i}(\cdot)$ or the inverse Fourier transform of the product of the individual characteristic functions (CFs) $\Phi_{S_i}(\cdot)$ of S_i [24].

The first approach is the multidimensional convolution of the marginal pdfs $f_{S_i}(\cdot)$ of the summands, i.e.,

$$f_S(s) = f_{S_1}(s_1) * f_{S_2}(s_2) * \cdots * f_{S_M}(s_M). \tag{2}$$

Let the characteristic function $\Phi_{S_i}(\cdot)$ of S_i be defined as [24]

$$\Phi_{S_i}(\omega) \triangleq \int_{-\infty}^{\infty} f_{S_i}(s_i) \exp(j\omega s_i) ds_i, \tag{3}$$

where $j \triangleq \sqrt{-1}$ is the imaginary unit. Note from (3) that the characteristic function $\Phi_{S_i}(\cdot)$ of S_i can be viewed as the Fourier transform of the pdf $f_{S_i}(\cdot)$ of S_i (with a reversal in the sign of the exponent), i.e.,

$$\Phi_{S_i}(\omega) = \mathscr{F}\{f_{S_i}(s_i)\}. \tag{4}$$

In this way, taking the Fourier transform of (2), it yields

$$\Phi_S(\omega) = \prod_{i=1}^{M} \Phi_{S_i}(\omega). \tag{5}$$

Furthermore, from the Fourier transform inversion formula, the pdf $f_{S_i}(\cdot)$ of S_i is given by [24]

$$f_{S_i}(s_i) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{S_i}(\omega) \exp(-j\omega s_i) d\omega, \tag{6}$$

and it follows that

$$f_{S_i}(s_i) = \mathscr{F}^{-1}\{\Phi_{S_i}(\omega)\}. \tag{7}$$

Note from (4) and (7) that the pdf $f_{S_i}(\cdot)$ and the CF $\Phi_{S_i}(\cdot)$ of S_i form a unique Fourier transform pair. This approach provides another way to obtain the statistics of the sum S. For instance, assuming knowledge of the CF $\Phi_{S_i}(\cdot)$ of each RV S_i, the exact sum pdf $f_S(\cdot)$ of S can be attained by taking the inverse Fourier transform of (5), i.e.,

$$f_S(s) = \mathscr{F}^{-1}\{\Phi_S(\omega)\}. \tag{8}$$

Therefore, when the RVs are mutually independent, (2) and (8) provide two ways to obtain the exact pdf $f_S(\cdot)$ of the sum S. However, when the RVs are mutually correlated,

these approaches cannot be applied. In this case, a formulation known as Brennan's integral should be used instead, which is described next.

2.2.2 Correlated Case

Considering the scenario when the summands are positive RVs, it was shown in [5] by using a geometric approach that the pdf $f_S(\cdot)$ and the cdf $F_S(\cdot)$ of the sum S can be formulated as

$$f_S(s) = \int_0^s \int_0^{s-s_M} \cdots \int_0^{s-\sum_{i=3}^M s_i} f_{\boldsymbol{S}}\left(s - \sum_{i=2}^M s_i, s_2, \ldots, s_M\right) ds_2 \cdots ds_{M-1} ds_M \tag{9a}$$

$$F_S(s) = \int_0^s \int_0^{s-s_M} \cdots \int_0^{s-\sum_{i=3}^M s_i} \int_0^{s-\sum_{i=2}^M s_i} f_{\boldsymbol{S}}\left(s_1, s_2, \ldots, s_M\right) ds_1 ds_2 \cdots ds_{M-1} ds_M. \tag{9b}$$

The integral in (9) is known as Brennan's integral, which is a general formulation to obtain the exact pdf $f_S(\cdot)$ and cdf $F_S(\cdot)$ of the sum of either independent or correlated RVs. Note that $f_S(\cdot)$ and $F_S(\cdot)$ are expressed as a multidimensional integral of the multivariate pdf $f_{\boldsymbol{S}}(\cdot)$. Therefore, even though Brennan's formulation is general and exact, it provides closed-form solutions only for particular cases. Furthermore, its implementation in computing softwares may prove unfeasible when the number M of summands increases (e.g., $M > 5$).

In order to circumvent this limitation, many approximate approaches have been proposed in the literature. In Section 2.3, two methods to provide approximate solutions for both independent and correlated scenarios are presented.

2.3 Approximation Techniques

In this section, two approaches used to approximate the exact sum distribution of either independent or correlated summands are covered. Initially, the classical moment-matching technique is presented. Then, a more recent approach known as asymptotic matching is discussed. In both cases, we assume that a certain candidate distribution $f_{\tilde{S}}(\cdot)$ has been selected to approximate the exact sum. So the only remaining task is to adjust the approximate distribution parameters in order to render a good fit.

2.3.1 Moment Matching

A well-known approach used to approximate the statistics of sums of RVs is called moment matching [7–10]. In this method, some moments of the exact sum S are matched to the corresponding moments of the candidate approximate RV $\tilde{S}$, i.e.,

$$\mathbb{E}[\tilde{S}^k] = \mathbb{E}[S^k], \tag{10}$$

where $\mathbb{E}[\tilde{S}^k]$ is the kth moment of the approximate distribution, and $\mathbb{E}[S^k]$ is the kth moment of the exact sum distribution, $k \in \mathbb{N}$. Particularly, should the RVs in the sum be independent, $\mathbb{E}[S^k]$ can be obtained from the individual moments of the summands as [7, eq. (6)]

$$\mathbb{E}[S^k] = \sum_{k_1=0}^{k} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{M-1}=0}^{k_{M-2}} \binom{k}{k_1}\binom{k_1}{k_2}\cdots\binom{k_{M-2}}{k_{M-1}} \mathbb{E}[S_1^{k-k_1}]\mathbb{E}[S_2^{k_1-k_2}]\cdots\mathbb{E}[S_M^{k_{M-1}}]. \tag{11}$$

However, should the RVs in the sum be mutually correlated and the CF of S be known, $\mathbb{E}[S^k]$ can be obtained from the moment theorem as [24]

$$\mathbb{E}[S^k] = \frac{1}{j^k}\frac{d^k}{d\omega^k}\Phi_S(\omega)\bigg|_{\omega=0}. \tag{12}$$

Moment-based approximations guarantee a good fit mainly in the distribution body. On the other hand, it loses track of the distribution tail at high-SNR regime, which is a compelling scenario to compare different communication systems. To overcome such limitation, a new method called asymptotic matching has been proposed, which is presented next.

2.3.2 Asymptotic Matching

Assuming a scenario where the summands are mutually independent, an approach known as asymptotic matching has been recently proposed [11]. In this method, the parameters of the approximate distribution are adjusted so that its asymptote equals the asymptote of the exact sum distribution.

Let the Maclaurin series expansion of the marginal pdf $f_{S_i}(\cdot)$ of S_i be given by

$$f_{S_i}(s_i) = \sum_{n=0}^{\infty} a_{i,n} s_i^{b_{i,n}} \sim a_{i,0} s_i^{b_{i,0}}, \tag{13}$$

and the Maclaurin series expansion of the sum pdf $f_S(\cdot)$ of S be expressed by

$$f_S(s) = \sum_{n=0}^{\infty} a_n s^{b_n} \sim a_0 s^{b_0}, \tag{14}$$

where the symbol "$\sim$" denotes "asymptotically equal to (around zero)". Since the sum pdf is expressed as the multidimensional convolution of the marginal pdfs in the independent case, the asymptote (around the origin) $a_0 s^{b_0}$ of $f_S(\cdot)$ in (14) is the multidimensional convolution of the M corresponding asymptotes $a_{i,0} {s_i}^{b_{i,0}}$ of each marginal pdf in (13). More specifically, it is shown in [11] (using a similar procedure as in the proof sketch in [16, Proposition 4]) that a_0 and b_0 are given by

$$a_0 = \frac{\prod_{i=1}^{M} a_{i,0} \Gamma\left(b_{i,0} + 1\right)}{\Gamma\left(M + \sum_{i=1}^{M} b_{i,0}\right)} \tag{15a}$$

$$b_0 = (M - 1) + \sum_{i=1}^{M} b_{i,0}, \tag{15b}$$

where $\Gamma(\cdot)$ denotes the gamma function. Note that the sum's asymptotic parameters (a_0 and b_0) are given exclusively in terms of the number of summands (M) and their marginal asymptotic parameters ($a_{i,0}$ and $b_{i,0}$, $i \in \{1, \ldots, M\}$). Moreover, in a log-scale plot, note from (14) that a_0 and b_0 determine the linear and angular coefficients of the asymptote of the sum pdf, respectively.

In order to perform asymptotic matching, the parameters of the approximate pdf are adjusted so that its asymptote, say $f_{\tilde{S}}(\cdot) \sim \tilde{a}_0 s^{\tilde{b}_0}$, equals the asymptote of the exact sum pdf in (14). This is achieved by forcing

$$\tilde{a}_0 = a_0 \tag{16a}$$

$$\tilde{b}_0 = b_0. \tag{16b}$$

Assuming the distribution parameters of each summand are known, we can then adjust the parameters of the approximate distribution by solving the system of equations in (16). This matching guarantees that both the exact and approximate distributions are asymptotically the same, providing an excellent fit around the origin, i.e., at high SNR.

Since the asymptotic-matching approach has been proposed under the independent constraint, its use is in principle not applicable to the correlated case. However, due to a novel insight on sums of correlated RVs introduced in Chapter 3, this technique can be exploited in the correlated scenario as well.

Chapter 3

PROPOSED APPROXIMATIONS

Several works in the literature have attempted to accurately approximate sums of RVs. Considering the case of arbitrarily correlated summands, this task has proven even more challenging. Some recent works (cf. [15–21]) derived asymptotic expressions to approximate performance metrics of diversity-combining schemes operating over correlated fading scenarios. Although capable of describing the system performance for particular fading scenarios, these results are neither general nor distribution-oriented. In this chapter a new general framework is introduced in order to design approximate distributions that well fit the whole body of the exact sum distribution while being asymptotically exact near the origin. The analysis is then applied to the sum of correlated Rayleigh and exponential RVs. For illustrative purposes, new simple, closed-form, asymptotic expressions are derived and applied to the performance analysis of two classical diversity-combining techniques, namely, EGC and MRC.

3.1 Preliminaries

Let us assume that the asymptote of the multivariate pdf of $\boldsymbol{S}$ can be expressed in the form

$$f_{\boldsymbol{S}}(\boldsymbol{s}) \sim a \prod_{i=1}^{M} s_i^{b_i}, \tag{17}$$

where a and b_i are constants.[1] This implies that the asymptote of the sum pdf can be expressed as a multidimensional convolution, i.e.,

$$f_S(s) \sim a \left(s_1^{b_1} * s_2^{b_2} * \cdots * s_M^{b_M} \right). \tag{18}$$

[1] This is a mild condition that holds true for many popular fading models, such as the Rayleigh, Rice, Nakagami-m, Hoyt, Weibull, and α-μ distributions.

We can restate (17) in a more convenient form, i.e.,

$$f_{\boldsymbol{S}}(\boldsymbol{s}) \sim \prod_{i=1}^{M} \hat{a}_{i,0} s_i^{\hat{b}_{i,0}}, \tag{19}$$

where

$$\hat{a}_{i,0} \triangleq a^{\frac{1}{M}} \tag{20a}$$

$$\hat{b}_{i,0} \triangleq b_i. \tag{20b}$$

Note the implications raised by (19). We can view the ith term $\hat{a}_{i,0} s_i^{\hat{b}_{i,0}}$ in the product as the asymptote of an equivalent marginal pdf. And the product of these M terms is asymptotically equal to the multivariate pdf of $\boldsymbol{S}$. In other words, (19) implies that, around the origin, the correlated RVs S_i in the sum behaves as an equivalent set of mutually independent RVs. Accordingly, the asymptote $a_0 s^{b_0}$ of the sum pdf in (14) is given by the convolution of the M equivalent marginal asymptotes $\hat{a}_{i,0} s_i^{\hat{b}_{i,0}}$ in (19). This is a novel and general asymptotic result for sums of positive correlated RVs, with many further implications. For instance, we can apply (15) (obtained for the independent case) to the correlated scenario. To this end, we just replace $a_{i,0}$ and $b_{i,0}$ in (15) by $\hat{a}_{i,0}$ and $\hat{b}_{i,0}$ in (20), respectively, so as to determine a_0 and b_0 for the correlated case.

Once the asymptote $a_0 s^{b_0}$ of the sum pdf is determined, we can match it with the asymptote $\tilde{a}_0 s^{\tilde{b}_0}$ of the approximate pdf, i.e., we can force (16). This guarantees an asymptotically optimal fit in the high-SNR regime. Furthermore, as a candidate approximate distribution may have more than two parameters to be adjusted, and as the asymptotic matching provides only two equations, it may be necessary to use asymptotic matching along with other existing methods. Since the moment matching can provide a good approximation in the body of the pdf, we propose its use in order to complement the asymptotic matching. In this way, when the approximate distribution has $l > 2$ parameters, (10) can provide the remaining $l - 2$ equations to complete the system of equations and find the distribution parameters accordingly.

Our proposed analysis can be used for designing statistical approximations to sums of a broad class of positive correlated RVs. As a case study, next we investigate two kinds of correlated sums, namely, sums of Rayleigh RVs and sums of exponential RVs.

3.2 Sums of Correlated Rayleigh Random Variables

In this section, we initially discuss the sums of correlated Rayleigh RVs in order to apply our analysis. Thereafter, some candidate distributions are provided to approximate

the exact sum. Even though our framework is suitable for a variety of candidate distributions, we illustrate the development by using the generalized, versatile α-μ distribution and two of its particular cases, namely, Nakagami-m ($\alpha = 2$, $\mu = m$) and Weibull ($\mu = 1$) distributions [25]. The analysis can then be used to evaluate the performance of an EGC scheme operating over correlated Rayleigh fading, as discussed at the end of the session.

3.2.1 Exact Sum Statistics

Let R ($\equiv S$) be the sum of M arbitrarily correlated Rayleigh RVs R_i ($\equiv S_i$), $i \in \{1, \ldots, M\}$, i.e.,

$$R = \sum_{i=1}^{M} R_i. \tag{21}$$

The marginal pdf of each RV R_i is given by

$$f_{R_i}(r_i) = \frac{r_i}{\sigma_i^2} \exp\left(-\frac{r_i^2}{2\sigma_i^2}\right), \; r_i \geq 0, \tag{22}$$

where $\sigma_i > 0$ is a scale parameter, and $\Omega_i \triangleq \mathbb{E}[R_i^2] = 2\sigma_i^2$ is the average power. In order to specify the multivariate Rayleigh pdf $f_{\boldsymbol{R}}(\cdot) \triangleq f_{R_1,\ldots,R_M}(\cdot, \ldots, \cdot)$ of $\boldsymbol{R} \triangleq [R_1 \cdots R_M]^T$, it is appropriate to decompose each RV in terms of its in-phase and quadrature components, i.e.,

$$R_i = \sqrt{X_i^2 + Y_i^2}, \tag{23}$$

where X_i and Y_i are independent and identically distributed Gaussian RVs for each i, with zero mean and variance $\mathbb{V}[X_i] = \mathbb{V}[Y_i] = \sigma_i^2$ [26]. Note that in general (X_i,X_j), (Y_i,Y_j), and (X_i,Y_j) are pairs of correlated RVs, $i \neq j$. We can arrange the components X_i and Y_i into the vector form

$$\boldsymbol{X} \triangleq [X_1 \cdots X_M]^T \text{ and } \boldsymbol{Y} \triangleq [Y_1 \cdots Y_M]^T, \tag{24}$$

so that their marginal and joint statistics can be specified by the covariance matrix of $\boldsymbol{X}$, the covariance matrix of $\boldsymbol{Y}$, and the cross-covariance matrix between $\boldsymbol{X}$ and $\boldsymbol{Y}$ — $\boldsymbol{K_{XX}} \triangleq \mathbb{E}[\boldsymbol{X}\boldsymbol{X}^T]$, $\boldsymbol{K_{YY}} \triangleq \mathbb{E}[\boldsymbol{Y}\boldsymbol{Y}^T]$, and $\boldsymbol{K_{XY}} \triangleq \mathbb{E}[\boldsymbol{X}\boldsymbol{Y}^T]$, respectively. These three matrices can then be rearranged into a unique (symmetric and non-singular) matrix defined as

$$\boldsymbol{K} \triangleq \begin{bmatrix} \boldsymbol{K_{XX}} & \boldsymbol{K_{XY}} \\ \boldsymbol{K_{XY}^T} & \boldsymbol{K_{YY}} \end{bmatrix}. \tag{25}$$

Hence, the multivariate Rayleigh pdf can be expressed as a function of the matrix $\boldsymbol{K}$ only [26]

$$f_{\boldsymbol{R}}(\boldsymbol{r}) = \frac{\prod_{i=1}^{M} r_i}{(2\pi)^M [\det(\boldsymbol{K})]^{\frac{1}{2}}} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \exp\left[-\frac{1}{2} g\left(\boldsymbol{r}, \boldsymbol{\phi}\right)\right] d\phi_1 \cdots d\phi_M, \tag{26}$$

where $\boldsymbol{r} \triangleq [r_1 \cdots r_M]^T \in [0, \infty)^M$, $\boldsymbol{\phi} \triangleq [\phi_1 \cdots \phi_M]^T \in [-\pi, \pi)^M$, and $g\left(\boldsymbol{r}, \boldsymbol{\phi}\right)$ is given by [26]

$$\begin{aligned} g\left(\boldsymbol{r}, \boldsymbol{\phi}\right) = & \sum_{i=1}^{M} (A_{ii} \cos^2\phi_i + C_{ii} \sin^2\phi_i + 2B_{ii} \cos\phi_i \sin\phi_i) r_i^2 \\ & + \sum_{\substack{i,j=1 \\ i \neq j}}^{M} (A_{ij} \cos\phi_i \cos\phi_j + C_{ij} \sin\phi_i \sin\phi_j + 2B_{ij} \cos\phi_i \sin\phi_j) r_i r_j, \end{aligned} \tag{27}$$

with A_{ij}, B_{ij}, and C_{ij} being obtained from

$$\boldsymbol{A} \triangleq \left(\boldsymbol{K_{XX}} - \boldsymbol{K_{XY}} \boldsymbol{K_{YY}^{-1}} \boldsymbol{K_{XY}^T}\right)^{-1} \tag{28}$$

$$\boldsymbol{B} \triangleq -\left(\boldsymbol{K_{XX}} - \boldsymbol{K_{XY}} \boldsymbol{K_{YY}^{-1}} \boldsymbol{K_{XY}^T}\right)^{-1} \boldsymbol{K_{XY}} \boldsymbol{K_{YY}^{-1}} \tag{29}$$

$$\boldsymbol{C} \triangleq \left(\boldsymbol{K_{YY}} - \boldsymbol{K_{XY}^T} \boldsymbol{K_{XX}^{-1}} \boldsymbol{K_{XY}}\right)^{-1}. \tag{30}$$

Using the Maclaurin series expansion of the integrand in (26) and then taking its first term, the asymptote of the multivariate Rayleigh pdf is obtained as

$$f_{\boldsymbol{R}}(\boldsymbol{r}) \sim \frac{\prod_{i=1}^{M} r_i}{[\det(\boldsymbol{K})]^{\frac{1}{2}}}. \tag{31}$$

Therefore, from (31), we have for the Rayleigh case that

$$a = \frac{1}{[\det(\boldsymbol{K})]^{\frac{1}{2}}} \tag{32a}$$

$$b_i = 1. \tag{32b}$$

Replacing $a_{i,0}$ and $b_{i,0}$ by $\hat{a}_{i,0}$ and $\hat{b}_{i,0}$ in (15), respectively, and using the results from (20) and (32), we obtain

$$a_0 = \frac{1}{[\det(\boldsymbol{K})]^{\frac{1}{2}} \Gamma(2M)} \tag{33a}$$

$$b_0 = 2M - 1. \tag{33b}$$

In order to apply moment matching, we can use the first moment $\mathbb{E}[R]$ of the sum, which is sufficient for the scope of this work and is easily obtained from (21) and (22) as

$$\mathbb{E}[R] = \sqrt{\pi/2} \sum_{i=1}^{M} \sigma_i. \tag{34}$$

Finally, using (33) and (34), we can obtain approximations to the exact sum distribution by performing the matching techniques accordingly. Once performed, the matching techniques provide the parameters of the approximate distribution in terms of those of the exact sum distribution. Hence, one can adjust the approximate pdf/cdf by properly setting its parameters. This is illustrated next for three different approximations.

3.2.2 Weibull Approximation

In the first proposed approximation, the sum R of correlated Rayleigh RVs is approximated by a Weibull RV $\tilde{R}$, whose pdf is given by [27, eq. (4-43)]

$$f_{\tilde{R}}(r) = \frac{\tilde{\alpha} r^{\tilde{\alpha}-1}}{\tilde{\Omega}} \exp\left(-\frac{r^{\tilde{\alpha}}}{\tilde{\Omega}}\right), \tag{35}$$

where $\tilde{\alpha} > 0$ is the shape (fading) parameter, and $\tilde{\Omega} = \mathbb{E}[\tilde{R}^{\tilde{\alpha}}]$ is the scale parameter of the distribution. Our objective is to find the values of the parameters $\tilde{\alpha}$ and $\tilde{\Omega}$ of the Weibull pdf such that $f_{\tilde{R}}(\cdot)$ renders a good approximation to the exact sum pdf $f_R(\cdot)$.

In order to guarantee a good adjustment at the high-SNR regime, one can perform asymptotic matching. To this end, taking the Maclaurin series expansion of the exponential function in (35), the coefficients $\tilde{a}_0$ and $\tilde{b}_0$ can be obtained as

$$\tilde{a}_0 = \frac{\tilde{\alpha}}{\tilde{\Omega}} \tag{36a}$$

$$\tilde{b}_0 = \tilde{\alpha} - 1. \tag{36b}$$

Finally, substituting (33) and (36) into (16), and solving the system of equations for the parameters $\tilde{\alpha}$ and $\tilde{\Omega}$, we obtain

$$\tilde{\alpha} = 2M \tag{37a}$$

$$\tilde{\Omega} = 2M[\det(\boldsymbol{K})]^{\frac{1}{2}}\Gamma(2M). \tag{37b}$$

3.2.3 Nakagami-m Approximation

In the second proposed approximation, the sum R of correlated Rayleigh RVs is approximated by a Nakagami-m RV $\tilde{R}$, whose pdf is expressed by [6, eq. (3)]

$$f_{\tilde{R}}(r) = \frac{2\tilde{m}^{\tilde{m}} r^{2\tilde{m}-1}}{\Gamma(\tilde{m})\tilde{\Omega}^{\tilde{m}}} \exp\left(-\frac{\tilde{m} r^2}{\tilde{\Omega}}\right), \tag{38}$$

where $\tilde{\Omega} = \mathbb{E}[\tilde{R}^2]$ and $\tilde{m} \triangleq \tilde{\Omega}^2/\mathbb{V}[\tilde{R}^2]$ are the parameters of the distribution. Similarly as for the previous Weibull approximation, our objective is to find the values of the parameters $\tilde{m}$ and $\tilde{\Omega}$ of the Nakagami-m pdf such that $f_{\tilde{R}}(\cdot)$ renders a good approximation to the exact sum pdf $f_R(\cdot)$.

Hence, taking the Maclaurin series expansion of the exponential function in (38), the coefficients $\tilde{a}_0$ and $\tilde{b}_0$ can be obtained as

$$\tilde{a}_0 = \frac{2\tilde{m}^{\tilde{m}}}{\Gamma(\tilde{m})\tilde{\Omega}^{\tilde{m}}} \tag{39a}$$

$$\tilde{b}_0 = 2\tilde{m} - 1. \tag{39b}$$

Substituting (33) and (39) into (16), and solving the system of equations for the parameters $\tilde{m}$ and $\tilde{\Omega}$, we obtain

$$\tilde{m} = M \tag{40a}$$

$$\tilde{\Omega} = M \left\{ \frac{2[\det(\boldsymbol{K})]^{\frac{1}{2}}\Gamma(2M)}{\Gamma(M)} \right\}^{\frac{1}{M}}. \tag{40b}$$

3.2.4 α-μ Approximation

Since both Weibull and Nakagami-m pdfs have only two parameters, the asymptotic matching itself is sufficient to solve the system of equations. Nevertheless, in order to obtain more degrees of freedom, it is important to investigate distributions containing more than two parameters. For illustrative purposes, we depict this case by using the generalized α-μ distribution, whose pdf is [25]

$$f_{\tilde{R}}(r) = \frac{\tilde{\alpha}\tilde{\mu}^{\tilde{\mu}} r^{\tilde{\alpha}\tilde{\mu}-1}}{\Gamma(\tilde{\mu})\tilde{\Omega}^{\tilde{\mu}}} \exp\left(-\frac{\tilde{\mu} r^{\tilde{\alpha}}}{\tilde{\Omega}}\right), \tag{41}$$

where $\tilde{\alpha} > 0$, $\tilde{\Omega} = \mathbb{E}[\tilde{R}^{\tilde{\alpha}}]$, and $\tilde{\mu} \triangleq \tilde{\Omega}^2/\mathbb{V}[\tilde{R}^{\tilde{\alpha}}]$ are the parameters of the distribution.

The analysis here is similar to that of the two previous approximations, except that now we have one more parameter. In this way, one more equation is needed, which can

be provided by the moment-matching technique. Therefore, from (10) and (16), it is necessary to obtain $\tilde{a}_0$, $\tilde{b}_0$, and $\mathbb{E}[\tilde{R}^k]$ of the approximate distribution in order to perform the matching techniques accordingly.

Initially, by taking the Maclaurin series expansion of the exponential function in (41), the coefficients $\tilde{a}_0$ and $\tilde{b}_0$ are easily obtained as

$$\tilde{a}_0 = \frac{\tilde{\alpha}\tilde{\mu}^{\tilde{\mu}}}{\Gamma(\tilde{\mu})\tilde{\Omega}^{\tilde{\mu}}} \tag{42a}$$

$$\tilde{b}_0 = \tilde{\alpha}\tilde{\mu} - 1, \tag{42b}$$

and its kth moment is given by [25]

$$\mathbb{E}[\tilde{R}^k] = \frac{\tilde{\Omega}^{\frac{k}{\tilde{\alpha}}}\Gamma\left(\frac{k}{\tilde{\alpha}} + \tilde{\mu}\right)}{\tilde{\mu}^{\frac{k}{\tilde{\alpha}}}\Gamma(\tilde{\mu})}. \tag{43}$$

As the first moment is sufficient for the scope of this work, we can set $k = 1$ in (43), which gives

$$\mathbb{E}[\tilde{R}] = \frac{\tilde{\Omega}^{\frac{1}{\tilde{\alpha}}}\Gamma\left(\frac{1}{\tilde{\alpha}} + \tilde{\mu}\right)}{\tilde{\mu}^{\frac{1}{\tilde{\alpha}}}\Gamma(\tilde{\mu})}. \tag{44}$$

These results can then be combined into a set of three transcendental equations by substituting (33), (34), (42), and (44) into (10) and (16). Even though there is no closed-form solution for this set of equations, one can in principle solve it numerically by using a computing software such as Mathematica or MATLAB, obtaining the parameters of the approximate pdf in terms of those of the exact sum pdf.

3.2.5 Application to Equal-Gain Combining (EGC)

The proposed analysis can be directly applied to the study of diversity-combining schemes. As an application example, we investigate the EGC technique operating over correlated Rayleigh fading channels.

Considering an EGC scheme with M arbitrarily correlated Rayleigh fading branches R_i, one can express its output envelope R_{EGC} by [1]

$$R_{EGC} = \frac{1}{\sqrt{M}}\sum_{i=1}^{M} R_i = \frac{R}{\sqrt{M}}, \tag{45}$$

where $\sqrt{M}$ is a normalization factor that accounts for the increased output noise. The EGC output in (45) is simply the sum in (21) normalized by $\sqrt{M}$. For simplicity, we drop the normalization, as this is just a scale factor that can be handled through a trivial

transformation of variables. Therefore, the analysis based on the sum R in (21) is directly applicable to the EGC output R_{EGC} in (45).

By applying the proposed analysis to EGC schemes operating over correlated Rayleigh fading, one can obtain approximate pdfs and cdfs to the EGC output. In the high-SNR regime, these approximations are asymptotically optimal and can then be used to evaluate the EGC performance in terms of BER and OP.

3.3 Sums of Correlated Exponential Random variables

Similarly as for the Rayleigh case above, this section introduces sums of correlated exponential RVs in order to apply our analysis. For illustrative purposes, the exact sum distribution is approximated by the α-μ distribution and two of its particular cases, namely, gamma ($\alpha = 1$) and Weibull distributions [25]. As detailed at the end of the section, the analysis can be readily applied to evaluate the performance of an MRC scheme operating over correlated Rayleigh fading.

3.3.1 Exact Sum Statistics

Let W ($\equiv S$) be the sum of M arbitrarily correlated exponential RVs W_i ($\equiv S_i$), $i \in \{1, \ldots, M\}$, i.e.,

$$W = \sum_{i=1}^{M} W_i, \tag{46}$$

where the exponential RV is defined as $W_i \triangleq R_i^2$. The multivariate exponential pdf $f_{\boldsymbol{W}}(\cdot) \triangleq f_{W_1,\ldots,W_M}(\cdot,\ldots,\cdot)$ of $\boldsymbol{W} \triangleq [W_1 \cdots W_M]^T$ is expressed by [26]

$$f_{\boldsymbol{W}}(\boldsymbol{\gamma}) = \frac{1}{(4\pi)^M [\det(\boldsymbol{K})]^{\frac{1}{2}}} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \exp\left[-\frac{1}{2} h\left(\boldsymbol{\gamma}, \boldsymbol{\phi}\right)\right] d\phi_1 \cdots d\phi_M, \tag{47}$$

where $\boldsymbol{\gamma} \triangleq [\gamma_1 \cdots \gamma_M]^T \in [0, \infty)^M$, and $h\left(\boldsymbol{\gamma}, \boldsymbol{\phi}\right)$ is given by [26]

$$\begin{aligned} h\left(\boldsymbol{\gamma}, \boldsymbol{\phi}\right) = & \sum_{i=1}^{M} (A_{ii} \cos^2\phi_i + C_{ii} \sin^2\phi_i + 2B_{ii} \cos\phi_i \sin\phi_i)\gamma_i \\ & + \sum_{\substack{i,j=1 \\ i \neq j}}^{M} (A_{ij} \cos\phi_i \cos\phi_j + C_{ij} \sin\phi_i \sin\phi_j + 2B_{ij} \cos\phi_i \sin\phi_j)(\gamma_i \gamma_j)^{\frac{1}{2}}, \end{aligned} \tag{48}$$

with A_{ij}, B_{ij}, and C_{ij} defined, as before, from (28)–(30).

Taking the Maclaurin series expansion of the integrand in (47), it is straightforward to show that the asymptote of the multivariate exponential pdf is expressed by

$$f_{\boldsymbol{W}}(\boldsymbol{\gamma}) \sim \frac{1}{2^M[\det(\boldsymbol{K})]^{\frac{1}{2}}}. \tag{49}$$

Hence, from (49), we have for the exponential case that

$$a = \frac{1}{2^M[\det(\boldsymbol{K})]^{\frac{1}{2}}} \tag{50a}$$

$$b_i = 0. \tag{50b}$$

Replacing $a_{i,0}$ and $b_{i,0}$ by $\hat{a}_{i,0}$ and $\hat{b}_{i,0}$ in (15), respectively, and using the results from (20) and (50), we obtain

$$a_0 = \frac{1}{2^M[\det(\boldsymbol{K})]^{\frac{1}{2}}\Gamma(M)} \tag{51a}$$

$$b_0 = M - 1. \tag{51b}$$

Considering the average Rayleigh power defined in Section 3.2, i.e., $\Omega_i \triangleq \mathbb{E}[R_i^2] = \mathbb{E}[W_i]$, the first moment $\mathbb{E}[W]$ of the sum is easily obtained as

$$\mathbb{E}[W] = \sum_{i=1}^{M} \Omega_i. \tag{52}$$

Using (51) and (52), one can attain approximations to the exact sum distribution by performing the matching techniques in a similar manner as in Section 3.2. In fact, the development detailed therein is general and readily applicable to sums of exponential RVs as well, as illustrated next.

3.3.2 Weibull Approximation

In order to approximate the sum W of correlated exponential RVs by a Weibull RV $\tilde{W}$, one can follow the procedure described in Subsection 3.2.2. In this way, taking the Maclaurin series expansion of the Weibull pdf in (35), the coefficients $\tilde{a}_0$ and $\tilde{b}_0$ required for asymptotic matching are given by (36).

Therefore, substituting (36) and (51) into (16), and solving the system of equations

for the parameters $\tilde{\alpha}$ and $\tilde{\Omega}$, we obtain

$$\tilde{\alpha} = M \tag{53a}$$

$$\tilde{\Omega} = 2^M M[\det(\boldsymbol{K})]^{\frac{1}{2}}\Gamma(M). \tag{53b}$$

3.3.3 Gamma Approximation

As the exponential RV is a squared Rayleigh RV, it is interesting to investigate the approximation of the sum W (of correlated exponential RVs) not by the Nakagami-m RV as in Subsection 3.2.3, but by its squared version. The squared Nakagami-m RV is a gamma RV $\tilde{W}$ [25], whose pdf $f_{\tilde{W}}(\cdot)$ is [27, eq. (4-34)]

$$f_{\tilde{W}}(\gamma) = \frac{\tilde{\mu}^{\tilde{\mu}}\gamma^{\tilde{\mu}-1}}{\Gamma(\tilde{\mu})\tilde{\Omega}^{\tilde{\mu}}} \exp\left(-\frac{\tilde{\mu}\gamma}{\tilde{\Omega}}\right), \tag{54}$$

where $\tilde{\Omega} = \mathbb{E}[\tilde{W}]$ and $\tilde{\mu} \triangleq \tilde{\Omega}^2/\mathbb{V}[\tilde{W}]$ are the parameters of the distribution.

Using the Maclaurin series expansion of the exponential function in (54), the coefficients $\tilde{a}_0$ and $\tilde{b}_0$ can be obtained as

$$\tilde{a}_0 = \frac{\tilde{\mu}^{\tilde{\mu}}}{\Gamma(\tilde{\mu})\tilde{\Omega}^{\tilde{\mu}}} \tag{55a}$$

$$\tilde{b}_0 = \tilde{\mu} - 1. \tag{55b}$$

Finally, substituting (51) and (55) into (16), and solving the system of equations for the parameters $\tilde{\Omega}$ and $\tilde{\mu}$, we obtain

$$\tilde{\Omega} = 2M[\det(\boldsymbol{K})]^{\frac{1}{2M}} \tag{56a}$$

$$\tilde{\mu} = M. \tag{56b}$$

3.3.4 α-μ Approximation

In order to provide more degrees of freedom during the adjustment process, one can choose an approximate distribution with more than two parameters. For illustrative purposes, we depict this case in the same way as in Subsection 3.2.4, where the generalized α-μ distribution was used. The analysis here is similar to the previous one.

The α-μ pdf has three parameters, namely, $\tilde{\alpha}$, $\tilde{\mu}$, and $\tilde{\Omega}$. Hence, three equations are necessary to solve the system of equations and find the distribution parameters properly. As for the asymptotic-matching step, the coefficients $\tilde{a}_0$ and $\tilde{b}_0$ given by (42) are used. In order to provide the third equation, moment matching can be performed by using the

first moment of the α-μ distribution, which is given by (44).

Therefore, one can combine these results into a set of three transcendental equations by substituting (42), (44), (51), and (52) into (10) and (16). Although there is no closed-form solution for this system of equations, it can be solved numerically by using a computing software, obtaining the parameters of the approximate pdf in terms of those of the exact sum pdf.

3.3.5 Application to Maximal-Ratio Combining (MRC)

The analysis developed on sums of correlated exponential RVs is of great importance in wireless communications. For instance, it can be applied to study the performance analysis of the MRC technique operating over correlated Rayleigh fading channels, as follows.

Let us assume an MRC scheme consisting of M arbitrarily correlated Rayleigh fading branches R_i. Its output R_{MRC} can be expressed by [1]

$$R_{MRC} = \sum_{i=1}^{M} g_i R_i, \tag{57}$$

where g_i is the gain at the ith branch. An important performance measure is the SNR Γ_i, defined for each branch i as

$$\Gamma_i \triangleq \frac{\text{local mean signal power}}{\text{mean noise power}}, \tag{58}$$

where the local mean signal power is given by $R_i^2/2$ [1]. Assuming the presence of Gaussian noise with mean power $N_i = N$ in each branch, then

$$\Gamma_i = \frac{R_i^2}{2N}. \tag{59}$$

The total noise power $\mathscr{N}$ at the MRC output is given by

$$\mathscr{N} = N \sum_{i=1}^{M} g_i^2. \tag{60}$$

Hence, the resulting SNR Γ is

$$\Gamma = \frac{R_{MRC}^2}{2\mathscr{N}} = \frac{1}{2} \frac{\left(\sum_{i=1}^{M} g_i R_i\right)^2}{N \sum_{i=1}^{M} g_i^2}. \tag{61}$$

Furthermore, it is shown in [1] that the SNR Γ is maximized if each gain g_i is equal to the ratio of the signal voltage to noise power of the respective branch, i.e.,

$$g_i = \frac{R_i}{N}. \tag{62}$$

Therefore, substituting (62) into (61), it follows that

$$\Gamma = \frac{1}{2}\frac{\left(\sum_{i=1}^{M} R_i^2/N\right)^2}{N\sum_{i=1}^{M}(R_i/N)^2} = \sum_{i=1}^{M}\frac{R_i^2}{2N} = \sum_{i=1}^{M}\Gamma_i. \tag{63}$$

The result in (63) shows that the output SNR Γ of the MRC is the sum of the SNRs Γ_i in each branch.

Comparing (63) to (46), note that $\Gamma_i \equiv W_i$ and $\Gamma \equiv W$. Consequently, the analysis developed based on the sum W in (46) is also applicable to the MRC output SNR Γ in (63). To this end, by applying the proposed analysis to MRC schemes operating over correlated Rayleigh fading, one can obtain approximate pdfs and cdfs to the MRC output SNR. In the high-SNR regime, these approximations are asymptotically optimal and can then be used to evaluate the MRC performance in terms of BER and OP.

Chapter 4

NUMERICAL RESULTS

This chapter presents several numerical results in order to evaluate the performance of the statistical approximations proposed in this work. We compare our approximations with those obtained by using the traditional moment-based approach (cf. [7–10]). The exact solution shown in the plots has been computed by numerically integrating Brennan's formula, reproduced in (9).

We present curves of pdf and cdf for sums of correlated Rayleigh and sums of correlated exponential RVs. The pdfs are shown in terms of envelope level for Rayleigh sums and in terms of power level for exponential sums. The cdfs are plotted in terms of average power (per branch) for both sums, since this is a common practice in the literature and shows the high-SNR regime. As for the combining techniques discussed in Chapter 3, the distribution of the sum envelope level corresponds to the distribution of the EGC output, and the distribution of the sum power level corresponds to the distribution of the MRC output. For illustrative purposes, we let

$$\mathbb{E}[X_i Y_j] = 0, \qquad \forall i, j, \tag{64a}$$

$$\mathbb{E}[X_i X_j] = \mathbb{E}[Y_i Y_j] = \rho_{i,j}, \forall i \neq j. \tag{64b}$$

Note that $\rho_{i,j}$ is the correlation coefficient between the ith and jth components [24], which can be modeled by

$$\rho_{i,j} = J_0\left(\frac{2\pi d_{i,j}}{\lambda}\right), \tag{65}$$

where $J_0(\cdot)$ is the Bessel function of the first kind and zeroth order, $d_{i,j}$ is the distance between ith and jth antennas, and λ is the wavelength of the carrier signal. For simplicity, we fix $\rho_{i,j} = \rho$, $\forall i, j$, where $\rho \in \{0.1, 0.5, 0.9\}$. All curves are in a log-scale plot and were obtained by using the software Mathematica (version 11.1.1.0).

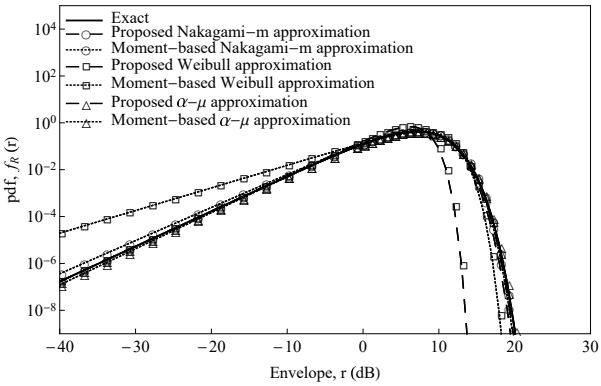
4.1 Sums of Rayleigh Random Variables

Initially, we consider the sum of correlated Rayleigh RVs. Figures 1, 2, and 3 show the exact and approximate pdfs and cdfs of the sum of two, three, and four RVs, respectively. For the pdf curves, we fix the average power per input branch at $\Omega_i = 2\sigma_i^2 = 2$, $\forall i$, and vary the sum envelope level r. For the cdf curves, we fix the sum envelope level at $r = 1$ and vary the average power per input branch $\Omega_i = \Omega$, $\forall i$.

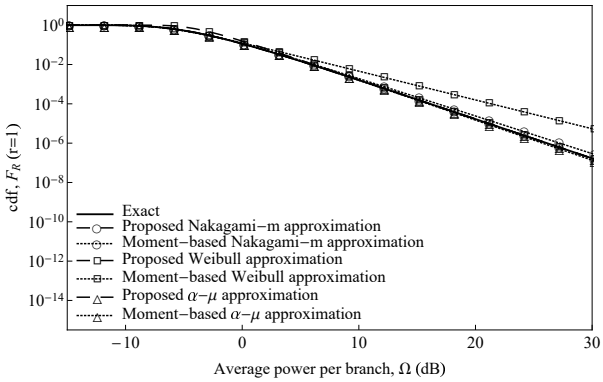
We provide three candidate distributions to approximate the exact sum, namely, the Nakagami-m, Weibull, and α-μ distributions. These approximations are obtained by using two different approaches: our proposed analysis and moment matching alone. On the one hand, as for our proposed approach, only asymptotic matching is required for obtaining the Nakagami-m and Weibull approximations, whereas the proposed α-μ approximation uses asymptotic matching along with the first moment of the sum (to perform the moment-matching step). On the other hand, in order to obtain the moment-based approximations, we have chosen for simplicity (i) the first and second moments for the Nakagami-m and Weibull approximations, and (ii) the first, second, and third moments for the α-μ approximation.

From the pdf plots, note that our approximations are asymptotically optimal, matching the exact curve near the origin — at pdf left tail. In this region, our proposed Nakagami-m, Weibull, and α-μ approximations outclass the moment-based Nakagami-m, Weibull, and α-μ counterparts, respectively. Particularly, the proposed Nakagami-m and Weibull approximations near the origin outperform the proposed α-μ approximation as M and ρ increase. This shows that, in the region of most practical interest, the two approximations that use asymptotic matching alone provide better results than the proposed α-μ approximation, where moment matching was introduced. In fact, there is a trade-off between asymptotic-matching and moment-matching: the former provides a better fit in the left pdf tail — as expected by design —, and the latter offers a better fit in the right pdf tail. However, the left pdf tail is of most practical interest when comparing different communications systems in terms of BER and OP, since it corresponds to the quintessential high-SNR regime.

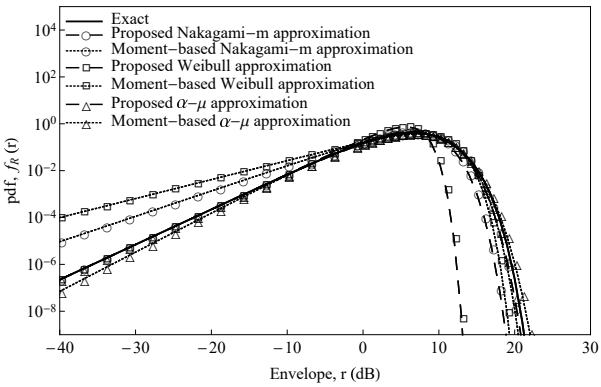
Similar results can be noticed from the cdf plots. In this case, the cdf right tail corresponds to the high-SNR regime, where our proposed approximations are asymptotically optimal. When applied to the EGC scheme, these cdf curves reveal that our proposed approximations keep track of the diversity (slope) and coding (offset) gains of the exact combining output, clearly outperforming the moment-based approximations.



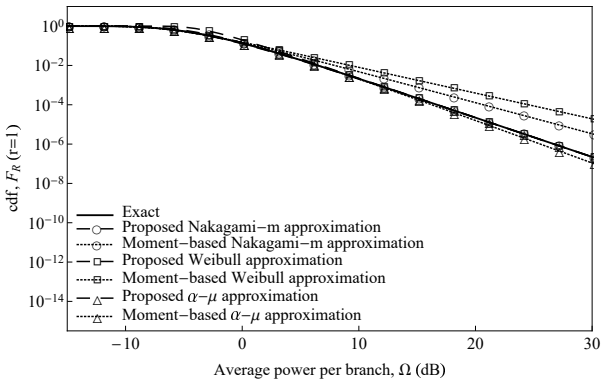
(a) Pdf of the sum with $\rho = 0.1$.



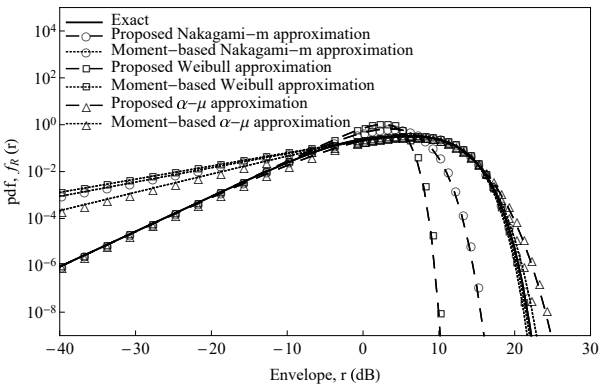
(b) Cdf of the sum with $\rho = 0.1$.



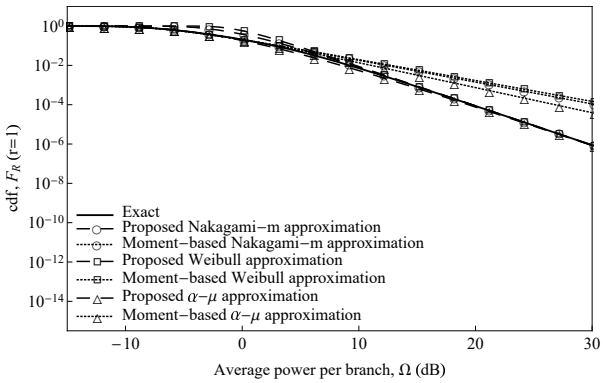
(c) Pdf of the sum with $\rho = 0.5$.



(d) Cdf of the sum with $\rho = 0.5$.

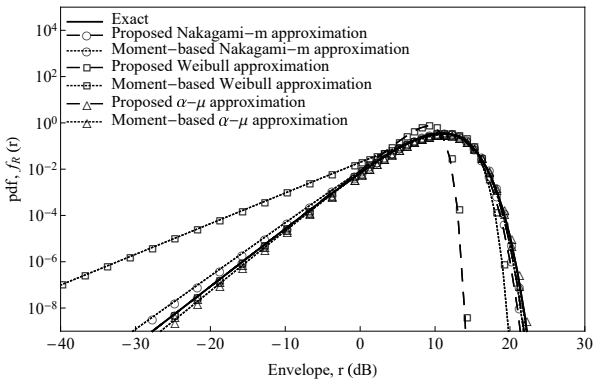


(e) Pdf of the sum with $\rho = 0.9$.

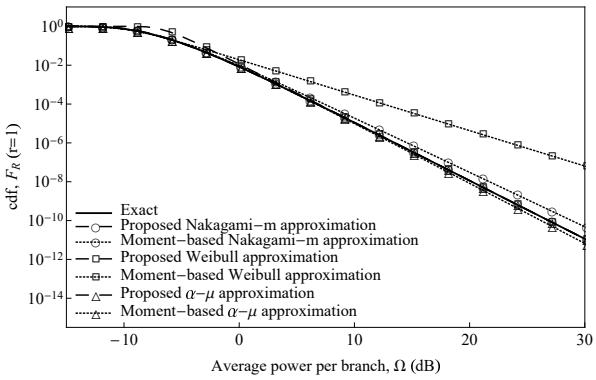


(f) Cdf of the sum with $\rho = 0.9$.

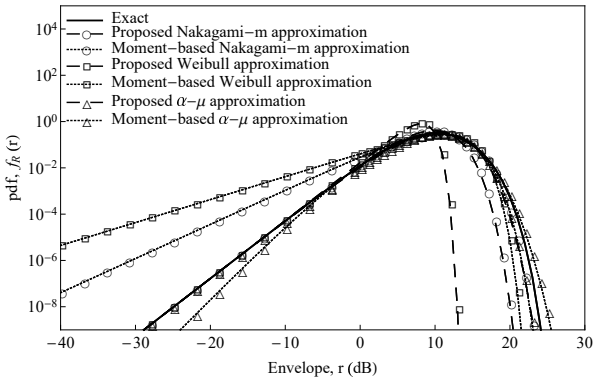
Figure 1 - Sum statistics of two correlated Rayleigh RVs.



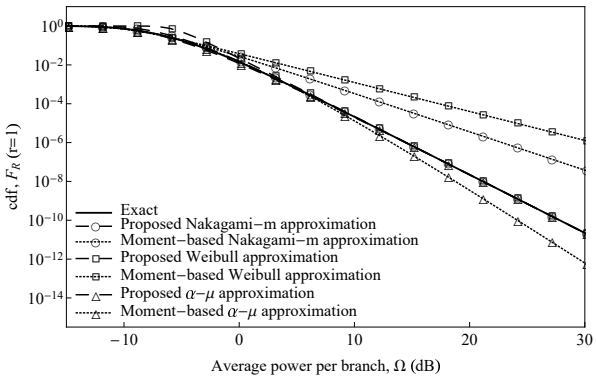
(a) Pdf of the sum with $\rho = 0.1$.



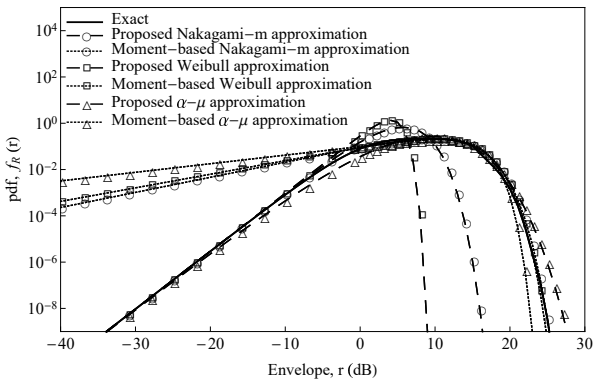
(b) Cdf of the sum with $\rho = 0.1$.



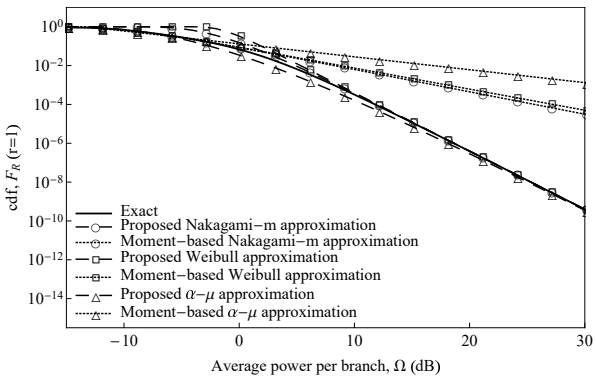
(c) Pdf of the sum with $\rho = 0.5$.



(d) Cdf of the sum with $\rho = 0.5$.

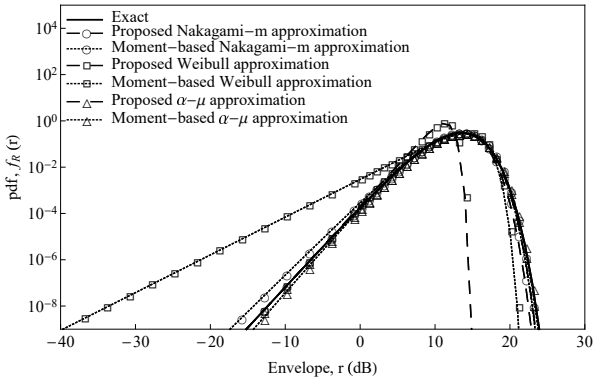


(e) Pdf of the sum with $\rho = 0.9$.

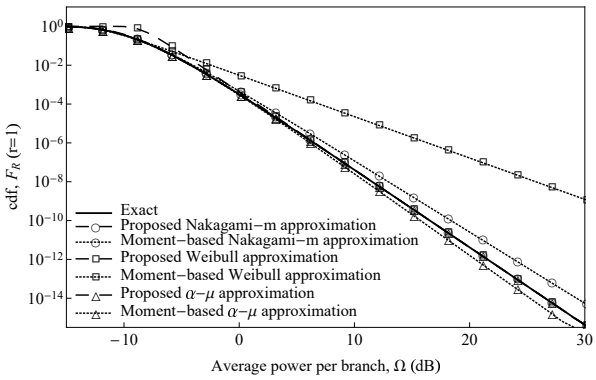


(f) Cdf of the sum with $\rho = 0.9$.

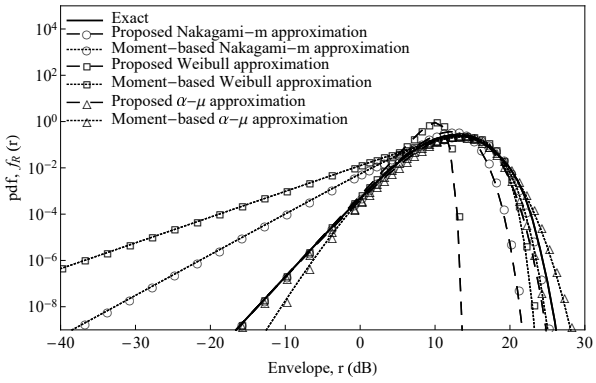
Figure 2 - Sum statistics of three correlated Rayleigh RVs.



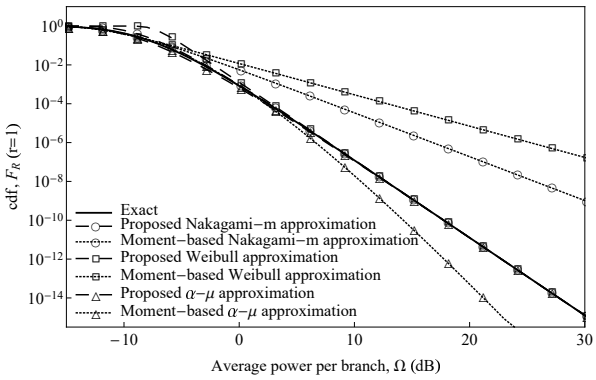
(a) Pdf of the sum with $\rho = 0.1$.



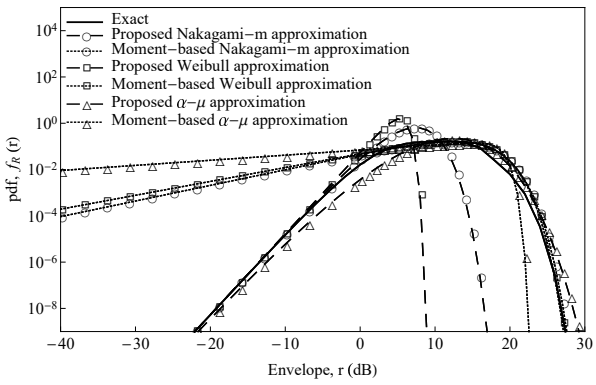
(b) Cdf of the sum with $\rho = 0.1$.



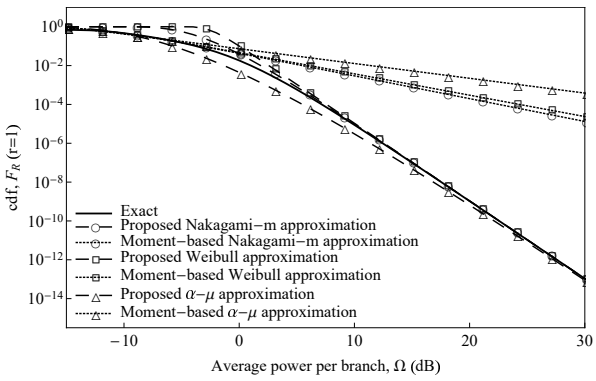
(c) Pdf of the sum with $\rho = 0.5$.



(d) Cdf of the sum with $\rho = 0.5$.



(e) Pdf of the sum with $\rho = 0.9$.



(f) Cdf of the sum with $\rho = 0.9$.

Figure 3 - Sum statistics of four correlated Rayleigh RVs.

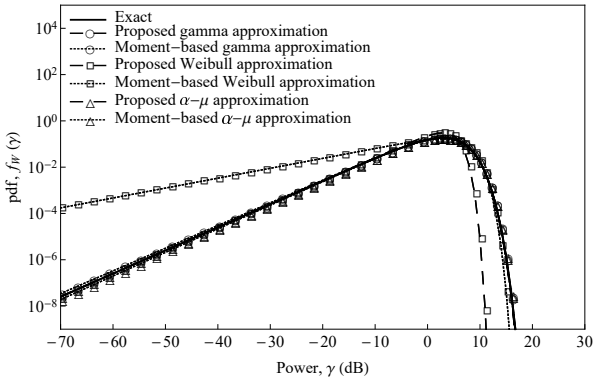
4.2 Sums of Exponential Random Variables

We now evaluate the performance of the approximations to sums of correlated exponential RVs. Figures 4, 5, and 6 show the exact and approximate pdfs and cdfs of the sum of two, three, and four RVs, respectively. For the pdf curves, we fix the average power per input branch at $\Omega_i = 2\sigma_i^2 = 2$, $\forall i$, and vary the sum power level γ. For the cdf curves, we fix the sum power level at $\gamma = 1$ and vary the average power per input branch $\Omega_i = \Omega$, $\forall i$.

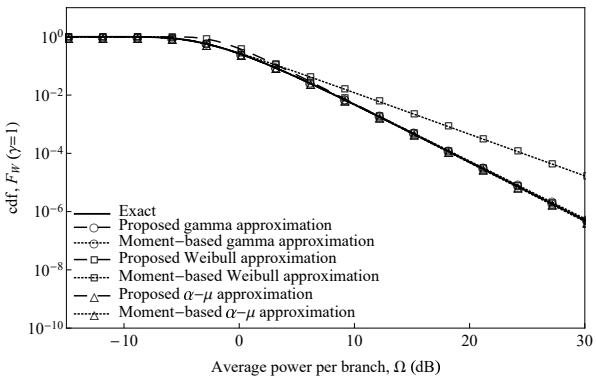
We provide three candidate distributions to approximate the exact sum, namely, the gamma, Weibull, and α-μ distributions. Similarly as in the previous section, these approximations are obtained by using two different approaches: our proposed analysis and moment matching alone. As for our proposed approach, only asymptotic matching is required for obtaining the gamma and Weibull approximations, whereas the proposed α-μ approximation uses asymptotic matching along with the first moment of the sum. Again, in order to obtain the moment-based approximations, we have chosen for simplicity (i) the first and second moments for the gamma and Weibull approximations, and (ii) the first, second, and third moments for the α-μ approximation.

Note from the pdf plots that our approximations are asymptotically optimal, keeping track of the exact curve near the origin, as expected. In this region, our proposed gamma, Weibull, and α-μ approximations outclass once again the moment-based gamma, Weibull, and α-μ counterparts, respectively. As discussed in the previous section, the proposed gamma and Weibull approximations near the origin outperform the proposed α-μ approximation as M and ρ increase.

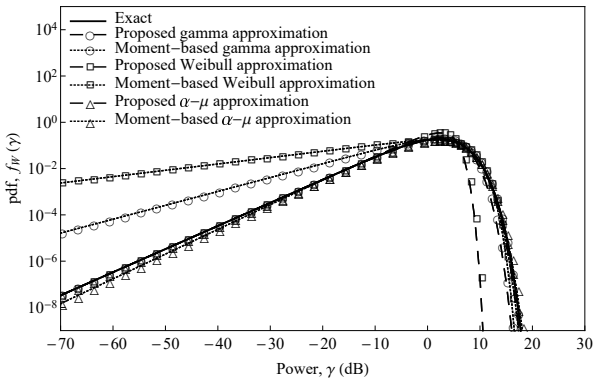
These results can also be noticed from the cdf plots, which reveal that our proposed approximations keep track of the diversity and coding gains of the exact MRC output. As expected, our proposed approximations provide an asymptotically optimal performance and clearly outperform the moment-based approximations.



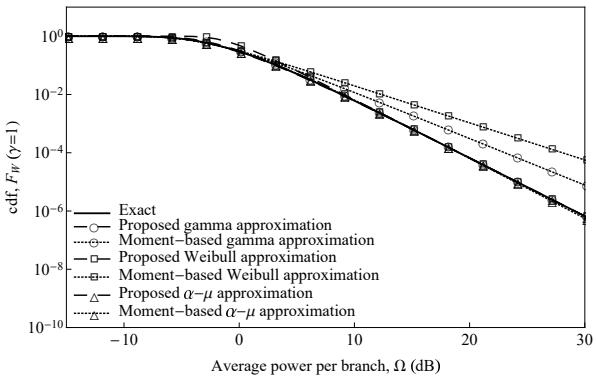
(a) Pdf of the sum with $\rho = 0.1$.



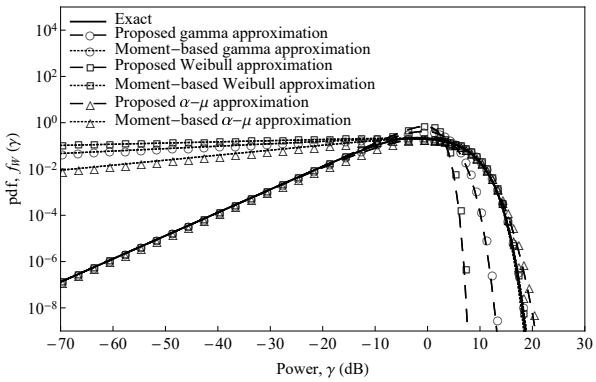
(b) Cdf of the sum with $\rho = 0.1$.



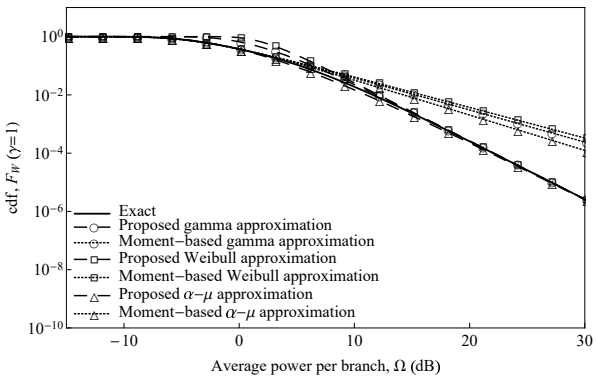
(c) Pdf of the sum with $\rho = 0.5$.



(d) Cdf of the sum with $\rho = 0.5$.

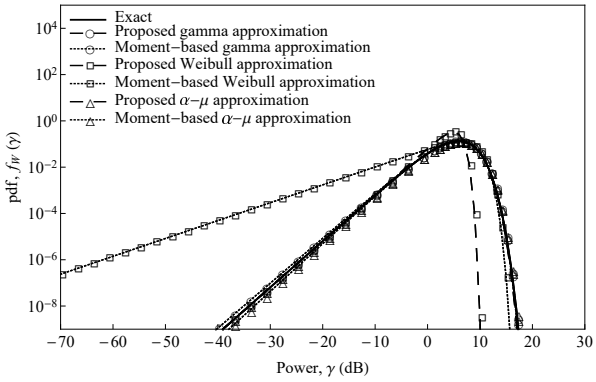


(e) Pdf of the sum with $\rho = 0.9$.

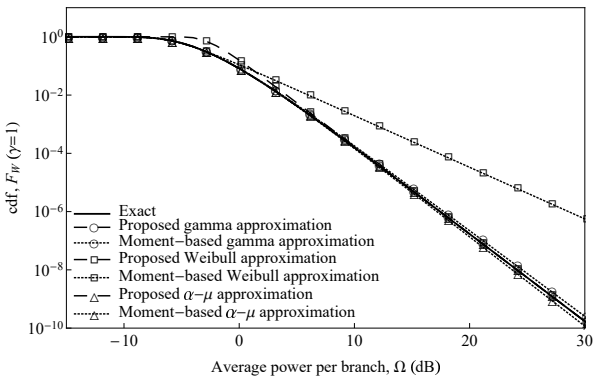


(f) Cdf of the sum with $\rho = 0.9$.

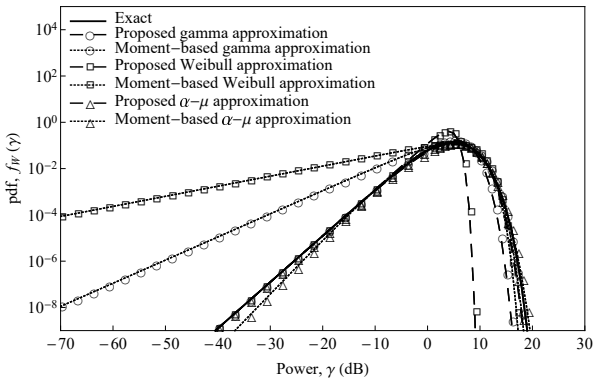
Figure 4 - Sum statistics of two correlated exponential RVs.



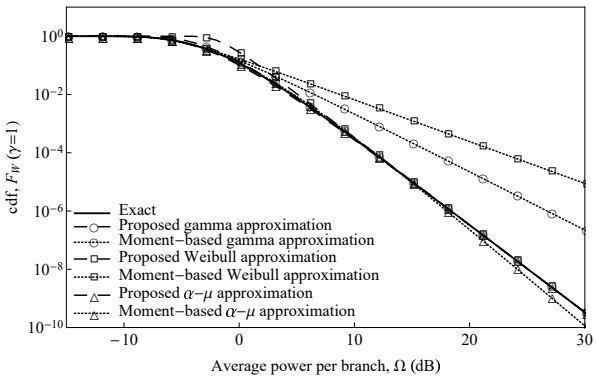
(a) Pdf of the sum with $\rho = 0.1$.



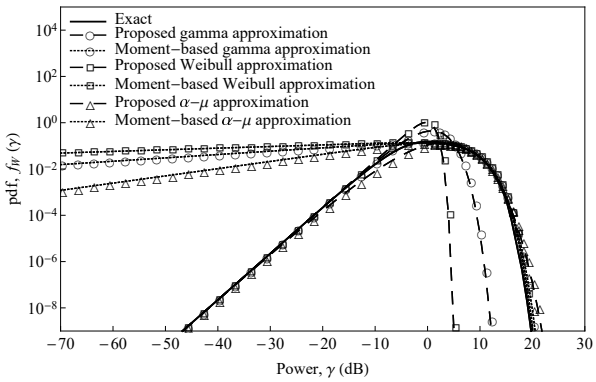
(b) Cdf of the sum with $\rho = 0.1$.



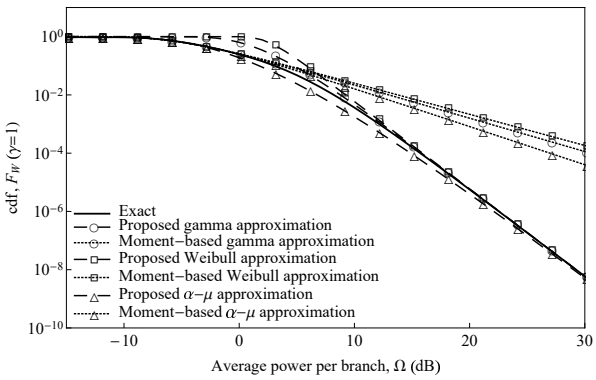
(c) Pdf of the sum with $\rho = 0.5$.



(d) Cdf of the sum with $\rho = 0.5$.

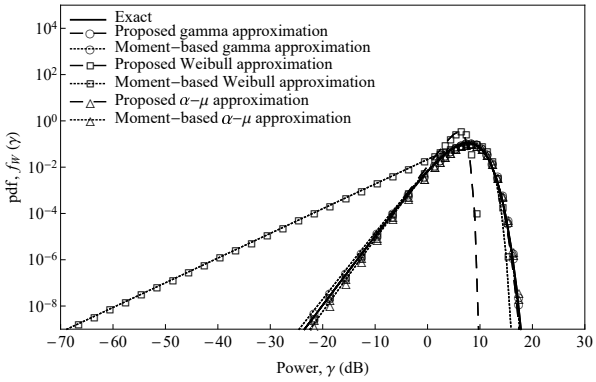


(e) Pdf of the sum with $\rho = 0.9$.

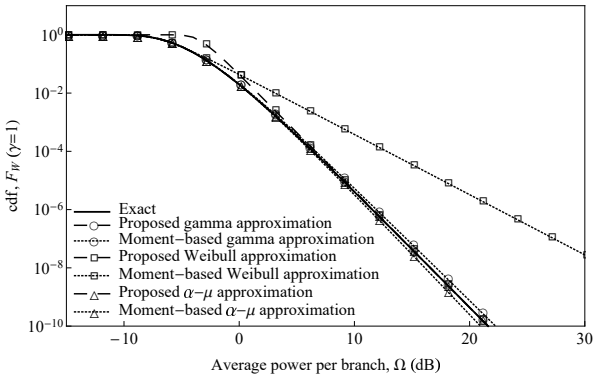


(f) Cdf of the sum with $\rho = 0.9$.

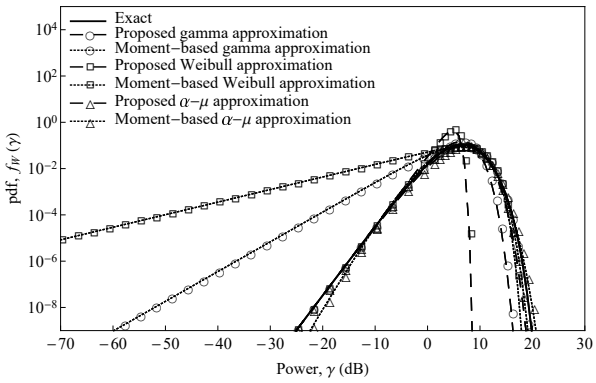
Figure 5 - Sum statistics of three correlated exponential RVs.



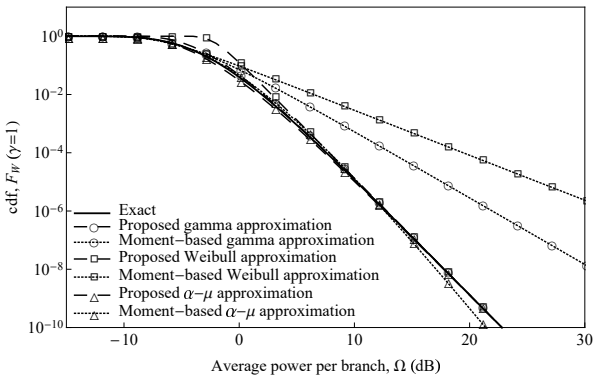
(a) Pdf of the sum with $\rho = 0.1$.



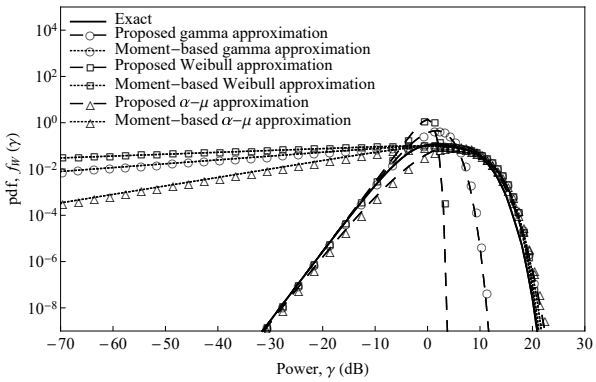
(b) Cdf of the sum with $\rho = 0.1$.



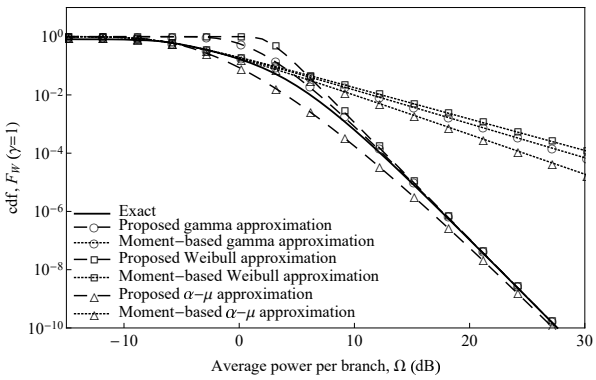
(c) Pdf of the sum with $\rho = 0.5$.



(d) Cdf of the sum with $\rho = 0.5$.



(e) Pdf of the sum with $\rho = 0.9$.



(f) Cdf of the sum with $\rho = 0.9$.

Figure 6 - Sum statistics of four correlated exponential RVs.

Chapter 5

CONCLUSIONS

This work has addressed statistical sums of arbitrarily correlated RVs. Initially, the general formulation to obtain the exact sum statistics was discussed for both independent and correlated scenarios. As this formulation is inherently intricate, some methods that provide accurate approximations were revisited, namely, the traditional moment-matching and the new asymptotic-matching approaches. The moment-matching technique yields a good fit in the distribution body, but it loses track of the distribution tail. As this region corresponds to the important regime of high SNR, it has been highly desirable to circumvent such limitation experienced by moment-based approximations. In this way, a new approach known as asymptotic matching has been recently proposed, which is capable of providing an outstanding fit near the origin. However, this technique has not been applicable to the most general and challenging correlated scenario. This work aimed to fill this gap.

5.1 Final Considerations

It has been apparently overlooked so far that some positive correlated RVs behave asymptotically near the origin as an equivalent set of independent RVs. This is a key insight elaborated on and much explored herein to propose asymptotically optimal approximations to sums of Rayleigh and sums of exponential RVs with arbitrary correlation. In this way, one can exploit the asymptotic matching, which can be used along with traditional techniques so as to well approximate the body of the exact sum distribution while guaranteeing an excellent fit in the distribution tail and, consequently, at high-SNR regime. As application examples, new simple, closed-form, asymptotic expressions were derived for the output statistics of EGC and MRC schemes operating over correlated Rayleigh fading, proving highly accurate and greatly outperforming the standard moment-matching approach.

Various candidate distributions to approximate the exact sum were presented, namely, Nakagami-m, gamma, Weibull, and α-μ distributions. In the proposed Nakagami-m, gamma, and Weibull approximations, closed-form expressions for the parameters of the approximate distribution were obtained, differently from the proposed α-μ approximation, whose parameters had to be obtained by solving a system of transcendental equations. It has also been observed from the numerical results that, as M and ρ increase, the proposed α-μ approximation earlier departs from the distribution tail, as compared to the other proposed approximations, which tend to keep better track. This observation reveals that, even though the α-μ distribution provides one more degree of freedom during the matching process, its inherent complexity may prove undesirable, being outperformed by the simpler Nakagami-m, gamma, and Weibull approximations.

5.2 Future Work

We should emphasize that the analysis developed herein is rather general. It can be readily used to design asymptotically exact pdfs and cdfs for a variety of fading scenarios and candidate approximate distributions. Moreover, it can be applied to many different transmission systems, such as those envisioned by the fifth generation of cellular mobile communications (5G) [28, 29].

As a non-exhaustive list of research topics to extend the results discussed herein, we provide the following:

1. An asymptotic characterization of other multivariate fading models, both traditional (Hoyt, Nakagami-m, Rice, and Weibull) and generalized (α-μ, η-μ, and κ-μ).

2. A general asymptotic characterization for the pdf and cdf of wireless link arrays that compose advanced-generation communication systems (like 5G) — sums (simple, hybrid, and hierarchical), products, maxima, minima, harmonic averages, and mixed combinations, among others —, applicable to generalized fading models and correlated scenarios. The asymptotic characterization allows to overcome the inherent difficulties of the exact statistical treatment, while guaranteeing an excellent approximation for operational regimes of practical interest, from medium to high SNR.

3. Corresponding (asymptotic and approximate) expressions for the BER and OP of each link array over different fading scenarios.

4. Optimal resource allocation strategies for each of those link arrays, in order to minimize the BER or the OP in the high-SNR regime.

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