Universidade Estadual de Campinas
Instituto de Computação

Francisco Jhonatas Melo da Silva

Game-Theoretic Analysis of Transportation Problems

Análise de Problemas de Transporte sob a Perspectiva da Teoria de Jogos

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Dissertação apresentada ao Instituto de Computação da Universidade Estadual de Campinas como parte dos requisitos para a obtenção do título de Mestre em Ciência da Computação.

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Resumo

Problemas relacionados com meios de transporte são comumente encontrados na área de Otimização Combinatória, como, por exemplo, o Problema do Caixeiro Viajante e o Problema de Roteamento de Veículos. Nesta dissertação, consideramos um problema de transporte sob a perspectiva da teoria de jogos algorítmica onde todos os jogadores querem ser transportados a um destino em comum o mais rápido possível, e para isso eles devem escolher um dentre os ônibus disponíveis.

Revisamos alguns resultados quanto à existência e à ineficiência de equilíbrios puros de Nash em relação a duas funções sociais. Então, apresentamos limitantes para o Preço de Anarquia para uma nova função social, chamada de função utilitária.

Consideramos também o jogo na forma extensiva, o qual chamamos de jogo de transporte sequencial e apresentamos limitantes para o Preço da Anarquia Sequencial considerando três funções sociais, para instâncias métricas e não-métricas.

Abstract

Problems related to transportation, such as the Traveling Salesman Problem and the Vehicle Routing Problem, commonly appear in the Combinatorial Optimization area. In this master's thesis, we present a game-theoretic analysis of a transportation game where all players want to be transported to a common destination as quickly as possible and, in order to achieve this goal, they have to choose one of the available buses.

We review some results concerned with the existence and inefficiency of Pure Nash Equilibria in relation with two social functions. Then, we give bounds on the Price of Anarchy to a new social function, called the utilitarian function.

Furthermore, we consider the game in its extensive form, which we call sequential transportation game, and then we provide bounds for the Sequential Price of Anarchy considering three social functions in both metric and non-metric instances.

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Chapter 1

Introduction

John von Neumann and Oskar Morgenstern published in 1944 the book *Theory of Games and Economic Behavior* [1], which is considered the pioneer book in Game Theory. Since then, this field has been developed by scientists ranging from economy to biology and more recently computer science. The first Nobel Prize in Economic Sciences given to game theorists was awarded, in 1994, to John Harsanyi, John Nash, and Reinhard Selten "for their pioneering analysis of equilibria in the theory of non-cooperative games" [2]. Moreover, in the past years, the number of Nobel prizes in the area has increased, with the last one being given to Jean Tirole in 2014 "for his analysis of market power and regulation" [2].

The study of how rational agents behave when dealing with situations of conflict and cooperation is the main objective of Game Theory. Those agents, usually called players, want to maximize their gains, and in order to do that, they may act selfishly. A game is a mathematical model where those players choose one strategy from a set of strategies aiming to maximize their own payoff or minimize their cost. For example, we can imagine a group of people who want to go to a common destination like an airport. Here, each person (player) can choose to travel in a bus (strategy), from a set of available buses (set of strategies), trying to get to the final destination as fast as possible (with the time taken representing the cost). An outcome of this game can be seen as an attribution of the players into the buses.

It is not difficult to see that, as the number of players and the set of strategies grow, a game can have an exponential number of possible outcomes. Therefore, questions that commonly emerge are: is it possible to, given a game, calculate an outcome in which every player is "happy" with their choices? By "happy" we mean that a player cannot get a better result by changing her strategy choice, given that the other players remain with their strategies unchanged. If we could calculate such outcome, can it be done efficiently?

In this aspect, Computer Science brings tools to help the analysis of issues that arise in Game Theory. For instance, the Theory of Computation can help to prove if a problem can be solved in polynomial time, and for problems that cannot be computed in polynomial time, it may provide ways of finding good solutions efficiently. Thus, analyzing problems of Game Theory from the point of view of Computer Science and vice versa characterizes the field called Algorithmic Game Theory.

In this work, we aim to investigate the environment where players are competing against each other for the use of shared resources, commonly called as *Resource Allocation Games*. In this setting, generally the resources either are limited or are associated with a cost, so that resource sharing is necessary or desirable. More specifically, we study a family of resource allocation games called *transportation games*.

The class of Transportation Games was recently introduced by Fotakis *et al.* [3], and they model situations motivated by ridesharing systems like *Uber, Dial-a-ride*, or *Blablacar*. Those systems are also important because of their direct impact on the environment in general as they can induce less pollutant gas emission and reduce traffic congestion [4]. Problems which are related to transportation commonly appear in the Combinatorial Optimization area, such as the *Traveling Salesman Problem* [5] and the *Vehicle Routing Problem* [6], which are both known to be NP-Hard problems, because of their practical applications and theoretical challenges.

In this master's thesis, we focus on the existence of Pure Nash equilibria and also on the properties of these equilibria, if they exist. Moreover, for the analysis of the inefficiency of equilibrium, we use the concepts of the Price of Anarchy and the Price of Stability, which compare the worst value of an equilibrium (resp. the best value of an equilibrium) with the value of an optimal outcome. We also analyze these concepts applied to *sequential transportation games*, which are games where all players irrevocably choose their strategies one by one in sequence and, when making their decisions, they only know the decisions made by their predecessors.

The organization of this thesis is as follows. In Section 1.1 we list our main contributions. Next, in Chapter 2, we give the main concepts of Game Theory, and also we present the formal model of the transportation games and its sequential version. We categorize the results into two groups: the existence of equilibria (Chapter 3) and the inefficiency of equilibria (Chapter 4 and Chapter 5). Finally, in Chapter 6 we summarize our results and give future research directions.

1.1 Main Contributions

First, we extend the results of Fotakis *et al.* [3] by defining a new social function for the game, called utilitarian function (in Section 2.2). Furthermore, we analyze the inefficiency of equilibria associated with this function by giving bounds on the Price of Anarchy (PoA) (see Section 4.4).

Second, we introduce a capacitated version of the transportation games (Section 3.1), and we show that, even for metric instances, the existence of (Pure) Nash Equilibrium is not guaranteed.

Finally, we also present the sequential version of the transportation games (Section 2.3). For this new version, we show in Chapter 5 bounds for the *Sequential Price of Anarchy* (SPoA) for all social functions considered in this master's thesis. In short, we first show that the value of the SPoA is unbounded for non-metric instances, and then we proceed to show the value of the SPoA for metric instances, which is tight for two of the social functions considered. In Table 1.1, we summarize the lower (LB) and upper (UB) bounds for the inefficiency of equilibria for the metric instances of transportation games, where the columns of this table represent all measures we use to analyze the inefficiency of equilibria: Price of Anarchy (PoA), Price of Stability (PoS), Sequential Price of Anarchy (SPoA), and Myopic Sequential Price of Stability (MSPoS).

	PoA		PoS		SPoA		MSPoS	
Function	LB	UB	LB	UB	LB	UB	LB	UB
D	n	n	n	n	$\boldsymbol{n}$	$\boldsymbol{n}$	?	?
	Prop. 4.7	Cor. 4.5	Prop. 4.7	Prop. 4.7	Thm. 5.2	Thm. 5.2		
E	$2\lceil \frac{n}{m} \rceil - 1$	$2\lceil \frac{n}{m} \rceil - 1$	$O(n/m)$	$\Omega(n/m)$	$\boldsymbol{2n-1}$	$\boldsymbol{2n-1}$	?	$\mathbf{2}$
	Prop. 4.11	Prop. 4.11	Cor. 4.10	Cor. 4.10	Thm. 5.3	Thm. 5.3		Thm. 5.5
U	?	$\boldsymbol{2n-1}$	?	?	?	$\boldsymbol{2n-1}$	?	?
		Cor. 4.14				Cor. 5.6		

Table 1.1: Summary of the bounds for the inefficiency of equilibria. The values in boldface are results of this thesis and the symbol ? assigns open questions.

Chapter 2

Preliminaries

In this section, we present the general concepts from Game Theory, including the basic definitions as well as the class of problems of our interest. Most of these definitions are somehow related to the definitions given by Nisan et al. [7]. Throughout this section, as we exhibit the concepts, we will give some aspects of the bibliographic history.

2.1 Basic Definitions from Game Theory

Before presenting the formal definitions from Game Theory that are relevant in this thesis, let us see an example of a classical game.

Example 2.1.1 [Prisoners' Dilemma] Suppose that there are two prisoners A and B detained by the police because of a supposed crime they have committed. As the police does not have enough evidence to incriminate them, the prisoners will be questioned in separated rooms and will be given the option of either confessing or remaining in silence. If both prisoners remain in silence, they will be charged for 2 years because of public disturbance and minor offenses. If both of them confess, then they will stay in prison for 4 years each. Now, if one of them confess while the other one remains in silence, then the one who confessed will be charged with 1 year and the other will be charged with 5 years.

In this example, we can think that the prisoners A and B are the players, and that they have a set of two strategies: confess or remain in silence. Their costs are computed according to strategies picked by them. For example, for the case where both of them remain in silence, they will have a cost of 2. We get that this game has four possible outcomes, and we can summarize them in Table 2.1 where the content of each cell represents the costs of players A and B, respectively. We tend to think that the outcome where both players remain in silence is the most beneficial for them, but we will see later that this outcome is not stable in the sense that each one of them will have an incentive to switch strategies and get a reduced cost.

Now we present the formal definition of these kinds of games.

Definition 2.1 A *game* $\mathcal{G}$ is defined by a tuple $(N, \mathcal{S}, c)$, where N is the finite set of players, and $\mathcal{S} = \times_{i\in N}\mathcal{S}_i$ is the finite set of strategy profiles, with $\mathcal{S}_i$ being the strategy

S_A	S_B *confess*	*silent*
confess	$4, 4$	$1, 5$
silent	$5, 1$	$2, 2$

Table 2.1: Prisoners' dilemma game in matrix form. The rows are associated with the set of strategies S_A of player A while the columns are associated with the set of strategies of player B.

set of player i. We call $\sigma \in \mathcal{S}$ as a strategy profile of game $\mathcal{G}$, which is also said to be the result or the outcome of the game. Finally, $c = (c_i)_{i \in N}$ is a vector with $c_i : \mathcal{S} \to \mathbb{R}$ being a cost function which maps into a real value all strategies chosen by the players.

We say that a game $\mathcal{G}$ is *simultaneous* if players chose their strategies simultaneously, without knowing the other players‘ strategies choices. Back to Example 2.1.1, the Prisoners‘ Dilemma is a classical example of a simultaneous game.

We assume that players are both rational and selfish. By this, we mean that players will choose a strategy that maximizes their gains or minimizes their costs according to the game regardless of the gains/costs of the other players. Let us suppose we are dealing with a game where players want to minimize their costs. Given a strategy profile σ, we use σ_{-i} to denote the vector representing the strategies played by others players excluding player i. Also, we say that the set $\mathcal{S}_{-i}$ is formed by all strategy sets of the players excluding player i, i.e. the set of all σ_{-i}. If a player $i \in N$ chooses a strategy $\sigma_i \in \mathcal{S}_i$, then we use $c_i(\sigma)$, with profile $\sigma = (\sigma_i, \sigma_{-i})$, to represent the cost incurred to i when choosing strategy σ_i. Using these notations, we present the next definition.

Definition 2.2 We say that a strategy $\sigma_i \in \mathcal{S}_i$ is a *best response* with respect to $\sigma_{-i} \in \mathcal{S}_{-i}$, if we have

$$c_i(\sigma_i, \sigma_{-i}) \leq c_i(\sigma_i', \sigma_{-i}), \forall \sigma_i' \in \mathcal{S}_i.$$

If player i knows for sure that the other players are choosing σ_{-i}, then the best she can do is to play a best response σ_i. For instance, in Example 2.1.1, in the profile (*silent*, *silent*) player A has a best response to the strategy chosen by B, which is to choose the strategy confess and by doing so, she gets a cost of 1 instead of 2. If σ_i is always a best response, then it is called a dominant strategy.

Definition 2.3 We say that a strategy $\sigma_i \in \mathcal{S}_i$ is a *dominant strategy* if for each $\sigma'_{-i} \in \mathcal{S}_{-i}$ and $\sigma_i' \in \mathcal{S}_i$, we have

$$c_i(\sigma_i, \sigma'_{-i}) \leq c_i(\sigma_i', \sigma'_{-i}).$$

Back to Example 2.1.1, the strategy confess is a best response against both strategies confess and silent, so it is a dominant strategy. However, not all games possess dominant strategies. Thus, we need a less constrained concept to help us analyze games in general. One important notion is the Nash Equilibrium, which states that stable solutions are those where all players are playing best responses against the strategies chosen by the others.

Definition 2.4 We say that a strategy profile $\sigma \in \mathcal{S}$ is a *(Nash) Equilibrium*, if for every player $i \in N$ and each strategy $\sigma'_i \in \mathcal{S}_i$, we have

$$c_i(\sigma_i, \sigma_{-i}) \leq c_i(\sigma'_i, \sigma_{-i}).$$

Putting this into words, in an equilibrium, a player i does not want to change or deviate unilaterally from her strategy σ_i to σ'_i since it would not benefit her, supposing all other players are still playing their strategies in σ_{-i}. Hence, all players in an equilibrium are satisfied with their choices. Again, observe that the profile (*confess*, *confess*) in Example 2.1.1 is an equilibrium.

On the one hand, if within a game an equilibrium is reached by the players choosing their strategy deterministically, then this equilibrium is called *Pure Nash Equilibrium* (PNE), which is the way we assumed in Definition 2.4. On the other hand, if this outcome is achieved by players' choices in a randomized way over their set of strategies, then this equilibrium is called *Mixed Nash Equilibrium* (MNE). This is an important concept because of the following theorem proved by Nash [8], one of the most relevant results of Game Theory.

Theorem 2.5 [Nash [8]] Every game with finite sets of players and strategies has a Mixed Nash Equilibrium.

It is worth noticing that both assumptions in the previous theorem are important because games with an infinite set of players or games with a finite set of players having access to an infinite set of strategies, may not have an MNE.

While in an equilibrium we are only concerned about individual deviations, there are some refinements of equilibrium which deal with group deviations. One of them is the concept of *Strong (Nash) Equilibrium* (SE), introduced by Aumann [9], where given an outcome of a game, no set C of players can jointly deviate such that all players in C improve their cost. Before presenting the formal definition, let us define σ_C as a strategy profile of the players in a set C of players and σ_{-C} as the strategy profile of the players not in C. Then, we say that $c_i(\sigma_C, \sigma_{-C})$ is the cost of player $i \in C$ under profile σ.

Definition 2.6 We say that a strategy profile $\sigma \in \mathcal{S}$ is a *Strong Nash Equilibrium*, if there is no non-empty set C of players, and each strategy $\sigma'_i \in \mathcal{S}_i$ for $i \in C$ forming a profile σ'_C, such that

$$c_i(\sigma'_C, \sigma_{-C}) < c_i(\sigma), \forall i \in C.$$

Another refinement is the *Super Strong (Nash) Equilibrium* (SSE), in which instead of all players in C having an improvement as in SE, there cannot exist a joint deviation of C in such way that at least one player of C improves her cost while all others do not have an increase in their costs.

Definition 2.7 We say that a strategy profile $\sigma \in \mathcal{S}$ is a *Super Strong Nash Equilibrium*, if there is no non-empty set C of players and $j \in C$, and each strategy $\sigma'_i \in \mathcal{S}_i$ for $i \in C$ forming a profile σ'_C, such that

$$c_i(\sigma'_C, \sigma_{-C}) \leq c_i(\sigma), \forall i \in C,$$

and

$$c_j(\sigma'_C, \sigma_{-C}) < c_j(\sigma).$$

We also have the notion of *better-response dynamics* in the following sense: until the current strategy profile σ is not a stable state (PNE or SE), pick an arbitrary player i and one of her beneficial deviation σ'_i, which is any strategy that decreases her actual cost, and move to the outcome $\sigma = (\sigma'_i, \sigma_{-i})$. If σ'_i is chosen in the way that minimizes i's cost, given σ_{-i}, then we call this dynamics as *best-response dynamics*. Moreover, if this dynamics halts, then we have arrived at a PNE. Also, it cycles in any game without a PNE, and it can even cycle in games that possess a PNE depending on the initial profile.

Another important concept is the *social function* $f : \mathcal{S} \to \mathbb{R}$, which is responsible for evaluating strategy profiles, since most of the games have an enormous number of possible outcomes. Therefore, it is important to have such a definition as it gives us a way of ranking or comparing the quality of different strategy profiles of a game for the society. Then, for example, when dealing with cost minimization games, the *optimal social cost* is the cost of a strategy profile $\sigma \in \mathcal{S}$ such that $f(\sigma)$ is minimum.

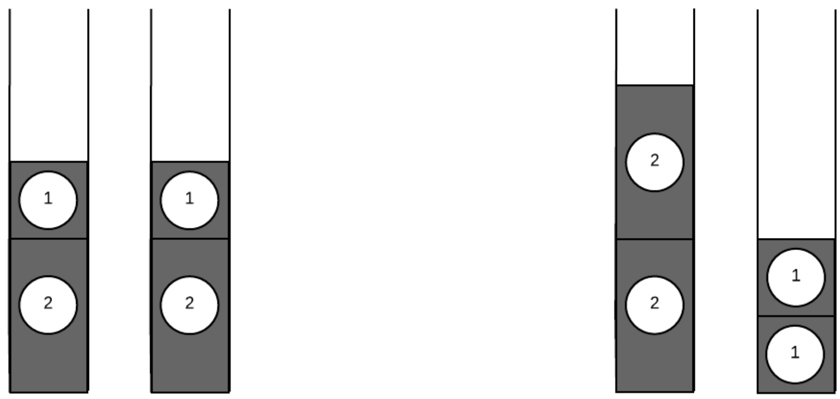
Next, we introduce the problem called *Selfish Load Balancing Games* [10], and we give an example adapted from Nisan et al. [7] of this game to show the application of some of the definitions seen until now.

A simultaneous *Load Balancing Game* $\mathcal{J}$ is defined by a tuple (N, M, w), where $N = \{1, \ldots, n\}$ is the set of tasks, $M = \{1, \ldots, m\}$ is the set of machines, and the vector $w = (w_{ij})_{i \in N, j \in M}$ represents the values of processing time of task i in machine j. Let us say each player is responsible for one of the tasks, and their goal is to have their tasks processed in a machine with the lowest load. Therefore, the set of strategies $\mathcal{S}_i$ is M. Here, we have the strategy profiles meaning an attribution of tasks into machines, $A : N \to M$, and $A(i)$ shows the machine where task i will be processed.

Let A_j be the set of tasks allocated in machine j. Then, the load of a machine j, $l_j(A)$, is calculated as $l_j(A) = \sum_{i \in A_j} w_{ij}$. The cost c_i associated with each player i is the load of the machine $A(i)$. Finally, we have the social cost under attribution A defined by the maximum load over all machines, also called *makespan*, denoted by $c(A) = \max\{l_j(A) : j \in M\}$.

Example 2.1.2 Consider an instance of the load balancing game with two identical machines and four tasks, with two of them having a processing time of 1 and the other two having processing time 2. Figure 2.1 shows the only two attributions of this instance in equilibrium.

In Figure 2.1 (a), it is shown an optimal attribution A with $c(A) = 3$. Notice that A is an equilibrium since any task cannot improve her cost by changing to another machine (e.g., if one of the tasks with processing time 1 changes to another machine, it will have



(a) Attribution A.　　(b) Attribution A'.

Figure 2.1: Two attributions A and A' in equilibrium for the instance in Example 2.1.2.

a cost worst than its current value – 4 instead of 3). Another equilibrium of this instance is showed in Figure 2.1 (b) which, under attribution A', has a makespan of $c(A') = 4$. Note that both assignments are SE and SSE since there is no coalition of players that can benefit with they jointly deviating from their current strategies.

From Example 2.1.2, we can see that different equilibria can have different social values. Because of it, tools have been proposed for evaluating inefficiency of equilibria. Koutsoupias and Papadimitriou [11] introduced the term *Price of Anarchy*, which will be formally defined next, as being the largest ratio among all instances of a game between the worst equilibrium and the optimal social outcome of it.

Definition 2.8 Given a function f representing the social function of a game $\mathcal{G}$ and let $\mathrm{PNE}(\mathcal{G})$ be the set of all PNE of $\mathcal{G}$. The *Price of Anarchy (PoA)* is defined as

$$\mathrm{PoA}(f, \mathcal{G}) = \frac{\max_{\sigma \in \mathrm{PNE}(\mathcal{G})} f(\sigma)}{\min_{\sigma^* \in \mathcal{S}} f(\sigma^*)}.$$

Informally speaking, the PoA provides us the information of how much of social cost is impacted in the outcome, given the selfishness of the players. For example, when the PoA has its value far away from 1, it means that the players' selfish behavior is provoking a significant disturbance in the social outcome.

Another relevant measure of the inefficiency of equilibria is the *Price of Stability (PoS)*. It was proposed by Anshelevich et al. [12] and, unlike PoA, it evaluates the ratio between the best equilibrium and the optimal social outcome of a game. As a consequence, we have that PoA $\geq$ PoS ≥ 1. We will use only $\mathrm{PoA}(f)$ and $\mathrm{PoS}(f)$ when the game $\mathcal{G}$ is clear from the context.

Definition 2.9 Given a function f representing the social function of a game $\mathcal{G}$ and let $\mathrm{PNE}(\mathcal{G})$ be the set of all PNE of $\mathcal{G}$. The *Price of Stability (PoS)* is defined as

$$\mathrm{PoS}(f, \mathcal{G}) = \frac{\min_{\sigma \in \mathrm{PNE}(\mathcal{G})} f(\sigma)}{\min_{\sigma^* \in \mathcal{S}} f(\sigma^*)}.$$

The PoS is important because there are games in which cooperation may be an option for the players, and this behavior can lead to an optimal equilibrium. Back to Example 2.1.2, observe that the instance presents PoS $= 1$ and PoA $= \frac{4}{3}$.

Even though every finite game has an MNE, there exist finite games without any PNE. Therefore, the existence of PNE is an interesting issue when analyzing games. Indeed, an important tool for this purpose is the exact potential function.

Definition 2.10 An *exact potential function* is a function $\Phi : \mathcal{S} \to \mathbb{R}$, such that, for every $\sigma \in \mathcal{S}$ and every player $i \in N$,

$$\Phi(\sigma_i, \sigma_{-i}) - \Phi(\sigma'_i, \sigma_{-i}) = c_i(\sigma_i, \sigma_{-i}) - c_i(\sigma'_i, \sigma_{-i}), \forall \sigma'_i \in S_i. \tag{2.1}$$

In other words, when player i changes her strategy from σ_i to σ'_i, then the change in her cost is exactly equal to the change occurred in the potential function. Games possessing an exact potential function associated with, called *potential games*, have two main properties: they always have a PNE and converge to a PNE through better-response dynamics, as shown in next theorem.

Theorem 2.11 [Tardos and Wexler [13]] Let $\mathcal{G}$ be a finite potential game. Then, the better-response dynamics always converge to an equilibrium.

Proof. Let us consider a strategy profile σ. If σ is not in equilibrium, then there exists a player i which is not satisfied and desires to change her strategy σ_i to another strategy σ'_i. By Definition 2.10, a potential game has an exact potential function Φ that satisfies Equation 2.1. Because player i decreased her cost, we have that $\Phi(\sigma_i, \sigma_{-i}) > \Phi(\sigma'_i, \sigma_{-i})$ and hence the deviation done by player i has made the potential of the new strategy profile (σ'_i, σ_{-i}) be strictly smaller than the previous one. In each iteration, an improving move is played and therefore a strategy profile is not evaluated more than once. As a result, since the set of strategies of the game is finite, this sequence of better response dynamics will reach an equilibrium eventually. ◻

Moreover, the method of providing potential functions for games has been used to show the existence of PNE in the literature, such as Congestion Games [14], Global Connection Games [12], Cost-Sharing Scheduling Games [15], and so forth. Another use of potential functions is that they can be used to give bounds on the PoS as shown in next theorem.

Theorem 2.12 [Tardos and Wexler [13]] Let $\mathcal{G}$ be a finite potential game and f be a social function. If for any profile σ we have that

$$\frac{f(\sigma)}{A} \leq \Phi(\sigma) \leq B \cdot f(\sigma) \tag{2.2}$$

for some constants $A, B > 0$, then the PoS of this game is at most AB.

Proof. Let σ be a strategy profile that minimizes the potential function Φ of this game. As a corollary of Theorem 2.11, we have that σ is an equilibrium, and then $\Phi(\sigma) \leq \Phi(\sigma^*)$ where σ^* is an optimal social outcome of this game. By assumption, we have that $\frac{f(\sigma)}{A} \leq \Phi(\sigma)$. Now, following our assumption, the second inequality give us that $\Phi(\sigma^*) \leq B \cdot f(\sigma^*)$. Combining those inequalities we get that $f(\sigma) \leq ABf(\sigma^*)$. Hence, we have that PoS $\leq AB$. □

Until now, we were considering simultaneous games, since all players announce their strategies simultaneously. We also say that these games are in *normal-form*. In these settings, we do not have any sense of the sequence of the actions performed by the players. As an alternative concept, we have games represented in *extensive form*, which can also be called as *tree* form, where the notion of time and sequence is explicit. This notion will be important for the understanding of the model of *sequential transportation games* in Section 2.3.

Informally, a game in extensive form is represented like a rooted tree, where each internal node represents a choice of one of the players, an edge represents one possible action made by a player, and the leaves represent the possible outcomes of the game with the cost/gain of each player in that possible outcome. We next present the formal definition and other concepts associated with it, which can be found in Shoham and Leyton-Brown [16].

Definition 2.13 A (finite) game in *extensive form* is a tuple $\mathcal{G} = (N, A, H, Z, \chi, \rho, \tau, c)$ where N is a set of n players; A is a set of actions; H is a set of nonterminal choice nodes; Z is a set of terminal nodes (disjoint from H); $\chi : H \to 2^A$ is the action function, which assigns to each choice node a set of possible actions; $\rho : H \to N$ is a function that assigns to each choice node a player $i \in N$ who chooses an action at that node; $\tau : H \times A \to H \cup Z$ is a function that maps a choice node h and an action $a \in \chi(h)$ to a new choice node or terminal node such that for all $h_1, h_2 \in H$ and $a_1, a_2 \in A$, if $\tau(h_1, a_1) = \tau(h_2, a_2)$, then $h_1 = h_2$ and $a_1 = a_2$; and $c = (c_1, \ldots, c_n)$ where $c_i : Z \to \mathbb{R}$ is a cost function for player i on the terminal nodes of Z.

It is possible to see that Definition 2.13 induces a tree $T_{\mathcal{G}} = (H \cup Z, \tau)$ according to function τ since it does not create cycles. In that sense, we can see the sequenciality of the actions taken by the players and a node's *history* can be seen as the path from the root to this node. Also, in Definition 2.13 we could have an array of utility functions u instead of having an array of cost functions c. Utility functions, on the other hand, mean the payoff or gain which a player earns in maximization games, which are games where the players want to maximize their utilities.

Example 2.1.3 Let us consider the following *ultimatum game*. In this game, we have two players a and b who wish to divide a set of 3 indivisible items. Player a proposes a split, which can be one of the following: she gives 1 or 2 of the items to b, and then b decides whether or not to accept it. If b accepts it, then both players receive the number of items proposed in the deal. Otherwise, both players receive 0 (zero) items, as they entered into a disagreement. Assuming both players value the items equally and additively, the extensive form of this game can be seen in Figure 2.2.

In games in extensive form, the set of pure strategies of a player is a full characterization of which actions should deterministically be chosen by her at all nodes that belong to her. The formal definition is given next.

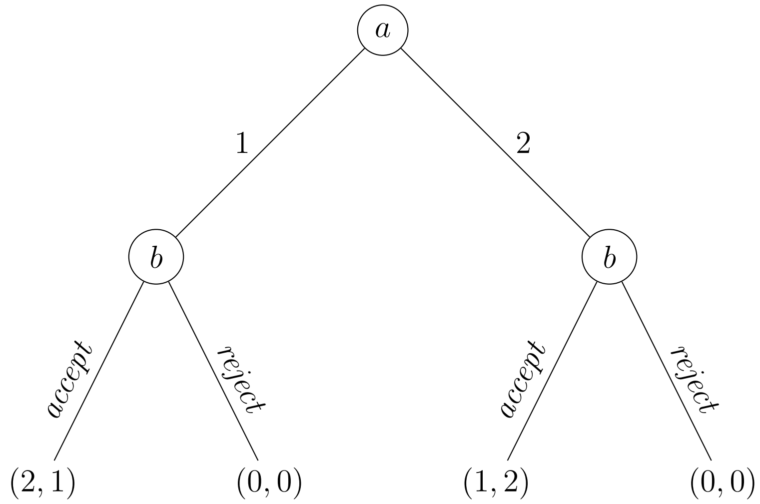


Figure 2.2: The ultimatum game in its extensive form. Here, we have $N = \{a, b\}$, and $A = \{1, 2, accept, reject\}$. Let $1, 2, \ldots, 7$ be the nodes from top to bottom, and from left to right. Then, the nonterminal nodes are $H = \{1, 2, 3\}$, and the terminal nodes are $Z = \{4, 5, 6, 7\}$. The actions for each nonterminal node are $\chi(1) = \{1, 2\}$, $\chi(2) = \{accept, reject\}$, and $\chi(3) = \{accept, reject\}$. Also, we get that $\rho(1) = a$, $\rho(2) = b$, and $\rho(3) = b$. Now, the edges of tree are formed by $\tau(1, 1) = 2$, $\tau(1, 2) = 3$, $\tau(2, accept) = 4$, $\tau(2, reject) = 5$, $\tau(3, accept) = 6$, $\tau(3, reject) = 7$. Finally, the terminal nodes are represented by the utilities of the players.

Definition 2.14 Let $\mathcal{G} = (N, A, H, Z, \chi, \rho, \tau, c)$ be a game in extensive form. Then, the *set of pure strategies* $\mathcal{S}_i$ of player i is formed by the Cartesian product $\times_{h \in H, \rho(h)=i} \chi(h)$.

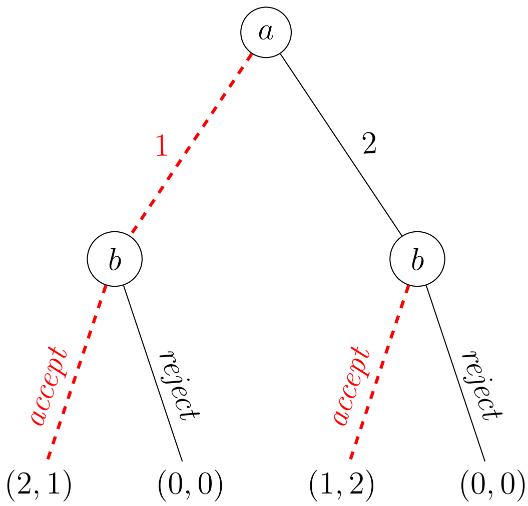
Observe that Definition 2.14 requires that a decision should be made at each choice node even if it is not possible to reach that particular node given all other choice nodes. In Example 2.1.3, player a has two pure strategies ($S_a = \{1, 2\}$) and player b has four pure strategies ($S_b = \{(accept, accept), (accept, reject), (reject, accept), (reject, reject)\}$).

For games in extensive form, the concepts of best response (Def. 2.2) and Nash equilibrium (Def. 2.4) given for the normal form are the same. Moreover, every game in extensive form can be converted into an equivalent game in normal form by tabulating the set of pure strategies of the players and recording their costs/payoffs, which is their costs/utilities according with each possible profile. Table 2.2 shows the normal form, in matrix form, of the ultimatum game described in Example 2.1.3, where the rows and columns are associated with S_a and S_b, respectively, and the content of each cell represents the payoff of players a and b, respectively. Observe that the profiles $(1, (accept, accept))$, $(1, (accept, reject))$, and $(2, (reject, accept))$ are PNE. Let us then analyze what happens with those PNE in this game in its extensive form, and for that we draw two of them in Figure 2.3.

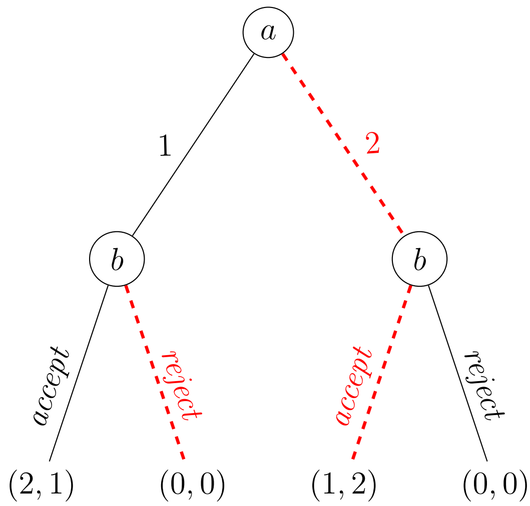
Beginning with the profile $(1, (accept, accept))$, if player a chooses to give 1 item for player b, then b would prefer to *accept* and be better off than nothing. Hence, we notice that the strategy $(accept, accept)$ played by b is a best response against strategy played by a and vice versa, and so it is indeed an equilibrium. Taking into consideration now the profile depicted in Figure 2.3 (b), in this scenario player b is threatening player a by

	S_b			
S_a	$(accept, accept)$	$(accept, reject)$	$(reject, accept)$	$(reject, reject)$
1	$2, 1$	$2, 1$	$0, 0$	$0, 0$
2	$1, 2$	$0, 0$	$1, 2$	$0, 0$

Table 2.2: Ultimatum game in normal form.



(a) Profile $(1, (accept, accept))$.



(b) Profile $(2, (reject, accept))$.

Figure 2.3: Representation of both equilibria of the game from Example 2.1.3. Dashed red edges indicate the action chosen at each node.

choosing to reject the offer of receiving only 1 item, and thus making player a chooses to give her 2 items. However, player a may think that this threat is not credible: would player b really reject the offer of receiving 1 item, and by doing so reducing her own utility? That is why not all PNE "make sense" in the context of games in extensive form.

In order to see why there are PNE that are not an equilibrium in games in extensive form, we first present the definition of subgame.

Definition 2.15 Let $\mathcal{G}$ be a game in extensive form. Then, the *subgame* of $\mathcal{G}$ rooted at node h is the restriction of $\mathcal{G}$ to the descendants of h. Also, we have that the set of subgames of $\mathcal{G}$ is formed by all subgames of $\mathcal{G}$ rooted at some node in $\mathcal{G}$.

Now we are able to present the notion of subgame-perfect equilibrium, which is a refinement of the Nash equilibrium in games in extensive form.

Definition 2.16 Let $\mathcal{G}$ be a game in extensive form. The *subgame-perfect equilibria (SPE)* of $\mathcal{G}$ are all strategy profiles σ such that for any subgame $\mathcal{G}'$ of $\mathcal{G}$, the restriction of σ to $\mathcal{G}'$ is a Nash equilibrium of $\mathcal{G}'$.

By Definitions 2.15 and 2.16, we get that every SPE are Nash equilibria since $\mathcal{G}$ is its own subgame. One can compute an SPE using backward induction, well known

```
BackwardInduction(node h)
1  if h ∈ Z
2      return u(h)
3  best_cost = +∞
4  for each a ∈ χ(h)
5      cost_at_child = BackwardInduction(τ(h, a))
6      if cost_at_child[ρ(h)] < best_cost[ρ(h)]
7          best_cost = cost_at_child
8  return best_cost
```

Figure 2.4: [16] Algorithm for finding the value of an SPE in a game in extensive form.

as “Zermelo’s algorithm”, which basically starts solving recursively the equilibria in the “bottom-most” subgame trees, and assume that indeed that is the action that will be played by the player responsible for the root of each one of those subgames. Using this procedure we are guaranteed to find an SPE in linear time on the size of the game in extensive form.

In Figure 2.4, we see the algorithm which can be used to compute the value of an SPE. The variable *best_cost* is an array that stores costs for each player; similarly, *cost_at_child* is an array denoting the costs of each player at the child node, and so $cost\_at\_child[\rho(h)]$ denotes the cost of player $\rho(h)$ who is responsible to take action in node h. Notice that BackwardInduction does not return a strategy profile, but we can use labels at each update in line 6 for keeping track of the actions chosen at each node, and thus getting an SPE. Therefore, every game in extensive form has at least one SPE and this SPE is also a PNE since there is only one player ($\rho(h)$) into consideration at each turn, and all players know what has been played by their predecessors, a deterministic move is optimal. All of these implications are done by a seminal result due to Zermelo [17].

Theorem 2.17 [Zermelo [17]] Every game in extensive form has a Pure Nash Equilibrium.

In Example 2.1.3 we have that the strategy profile $(1, (accept, reject))$ is not a SPE since, for example, player b did not choose a strategy that maximizes her utility when player a chooses to give her 2 items. In fact, the only SPE of this game is the profile strategy $(1, (accept, accept))$.

2.2 Transportation Games

The following transportation game, which is the base of our research, was recently introduced by Fotakis et al. [3] and its model is given next. We say that a graph $G = (V, E)$ is metric if there is a distance function $d : E \to \mathbb{R}_+$ that assigns metric values for the edges, i.e., for every $x, y, w \in V$, $d(x, y) + d(y, w) \geq d(x, w)$.

A simultaneous *transportation game* Γ is a tuple (N, M, G), where $N = \{1, \ldots, n\}$ is the set of n players; $M = \{1, \ldots, m\}$ is the set of $m \geq 2$ buses; $G = (V, E)$ is a complete undirected graph with a source node s and a destination node t, where $V = N \cup \{s, t\}$; and $d : E \to \mathbb{R}_+$ is a distance function. Each player is placed in a vertex of G, and they have as a goal to be transported from their location to t with the lowest cost.

A strategy profile in Γ is an assignment $\sigma : N \to M$ in which a player i chooses one bus j that will pick her up, and we call $\mathcal{S}$ the set of all those profiles. Considering a strategy profile $\sigma \in \mathcal{S}$, player' i cost under this profile, $c_i(\sigma)$, is defined as the distance traveled by σ_i, the bus chosen by player i in profile σ, between the location of i and destination t.

In order to determine the routes for the buses, we suppose that each bus $j \in M$ has an algorithm $\mathcal{A}_j$, which, given $V' \subseteq V$, calculates its route which starts on node s, goes through vertices of V', and finishes its route on node t. We consider that, as in Fotakis et al. [3], each algorithm $\mathcal{A}_j$, for $j \in M$, is based on a permutation $\pi_j : N \to N$, which is given as input, and that a bus j will only visit players that have chosen it to travel with, so j will eventually do some shortcuts in its permutation whenever it is possible.

Example 2.2.1 Consider the metric instance depicted in Figure 2.5, where the cost of the edges not shown are the value of the minimun path between any pair of nodes. In this instance we have $N = \{1, \ldots, 5\}$ and $M = \{1, 2\}$ as the set of available buses. Let π_j, for $j \in M$, be the identity permutation, i.e. $\pi_j = (1, 2, 3, 4, 5)$, where we can interpret it as bus j following the path $s \to 1 \to 2 \to 3 \to 4 \to 5 \to t$. Notice that player 4 will always choose a different bus than the one chosen by player 5 because if she travels together with player 5, then her cost would value 7, but if she chooses a different bus than player 5, then her cost would be 3. Let us analyze profile $\sigma = (1, 1, 1, 2, 1)$. Observe that under σ, buses 1 and 2 are going to perform shortcuts in their routes. For example, bus 2 will follow the path $s \to 4 \to t$. Here, we have the following costs: $c_1(\sigma) = 14$, $c_2(\sigma) = 8$, $c_3(\sigma) = 5$, $c_4(\sigma) = 3$, and $c_5(\sigma) = 3$. Under σ, just player 1 is willing to deviate and does so. Now, with this changing, we have the profile $\sigma' = (2, 1, 1, 2, 1)$, and the improved cost of player 1 is $c_1(\sigma') = 4$. Since under profile σ' no one wants to do an unilateral deviation, this is an equilibrium.

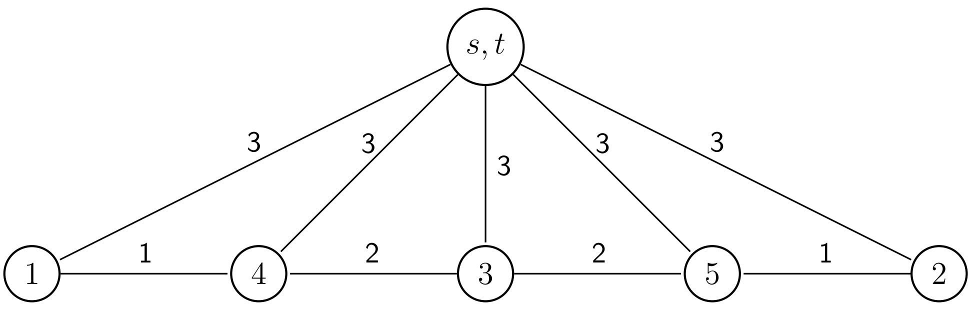


Figure 2.5: Metric instance with five players and their distances.

We also use three different social functions, where the first two of them were defined and analyzed by Fotakis et al. [3], and both functions neglect the distance between s and the first client. The first one is described as *Vehicle Kilometers Travelled*, which reflects the environmental impact of the game's outcome. Given a strategy profile σ

and for $j \in M$, let $(j_1, \ldots, j_{n_j})$ be the ordering that players are picked up by bus j, where $n_j = |\{i : \sigma_i = j\}|$ is the number of players choosing to travel in bus j. We define

$$D(\sigma) = \sum_{j=1}^{m} \sum_{i=1}^{n_j} d(j_i, j_{i+1}), \tag{2.3}$$

where $j_{n_j+1} = t$ for all buses. Indeed, function $D(\sigma)$ represents the total distance travelled by the buses when taking to destination t at least one player, except the distance from s to the first traveller. The *Egalitarian* cost $E(\sigma)$ is a classical social function which, in our context, will show the player who will spend more time traveling. It is defined as

$$E(\sigma) = \max_{i \in N} c_i(\sigma). \tag{2.4}$$

This function also represents the maximum distance travelled by a single bus.

Utilitarian social functions, denoted here by $U(\sigma)$, are also very common in game-theoretic analysis, and examples of them can be found in several works [18, 19, 20]. This function represents, in the context of transportation games, the sum of the times that players will get to t, and it can be defined as

$$U(\sigma) = \sum_{i \in N} c_i(\sigma). \tag{2.5}$$

The main results from Fotakis et al. [3] are divided into two cases: (1) existence and computation of an equilibrium and (2) its quality measured by the PoA and the PoS for functions D and E. We review those results in Chapter 3 and Chapter 4, respectively.

2.3 Sequential Transportation Games

Motivated by the framework given by Leme et al. [21], we analyze the transportation game in its sequential version, i.e., in its extensive form. We consider that the decisions will be made from players 1 to n. Furthermore, this version of the game can be seen as an m-ary tree, where in each level i $(1 \leq i \leq n)$, player i is responsible for taking decisions of all nodes in her level, in other words, she has to choose one of the m buses knowing only about the decisions of her $i-1$ predecessors. On the leaves (terminal nodes) are the information about the cost of each player according to that outcome.

Leme et al. [21] also introduced the notion of Sequential Price of Anarchy (SPoA), which corresponds to the largest ratio between a Subgame Perfect Equilibrium (SPE) and the optimal social outcome of it related to some social function.

Definition 2.18 Given a function f representing the social function of a game $\mathcal{G}$. Let $\mathrm{SPE}(\mathcal{G})$ be the set of all SPE of $\mathcal{G}$. The *Sequential Price of Anarchy (SPoA)* is defined as

$$\mathrm{SPoA}(f, \mathcal{G}) = \frac{\max_{\sigma \in \mathrm{SPE}(\mathcal{G})} f(\sigma)}{\min_{\sigma^* \in S} f(\sigma^*)}.$$

We will use again only $\mathrm{SPoA}(f)$ when the game $\mathcal{G}$ is clear from the context. We show next an example of an instance of a sequential transportation game.

Example 2.3.1 Consider Figures 2.6 and 2.7. There are $n = 4$ players and $m = 2$ buses with $\pi_1 = \pi_2 = (1, 2, 4, 3)$. In Figure 2.7 we have the game depicted in its extensive form where it is also indicated the SPE $\sigma = (1, 1, (2, 1), (2, 1, 2, 1))$. Also, the leaves represent the cost associated with each of players 1, 2, 3, and 4, respectively. Under Egalitarian social function, we can see that, for this instance, SPoA $\geq 5/3$ as an optimal solution values 3, which is the outcome when players 1 and 2 are together in one bus and on the other bus are players 3 and 4. For instance, an optimal profile would be $\sigma^* = (1, 1, (2, 2), (2, 2, 2, 2))$.

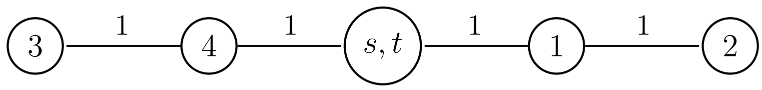


Figure 2.6: Instance with four players on a line.

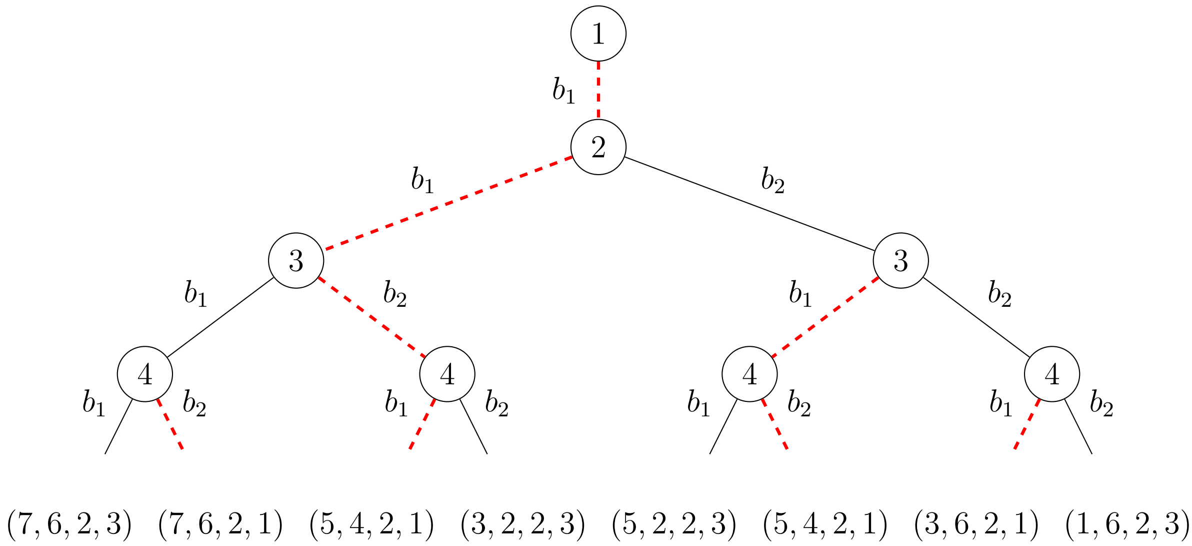


Figure 2.7: Game in extensive form. Because the permutations are equal, the tree is symmetric so we draw here just one half of it, where player 1 chooses the first bus. Dashed red edges indicate the action chosen at each node for SPE $\sigma = (1, 1, (2, 1), (2, 1, 2, 1))$.

Our results about the values of the SPoA regarding all three social functions D, E, and U, for non-metric and metric instances, are presented in Chapter 4.

Chapter 3

Existence of Pure Nash Equilibria

In this chapter, most of the discussion is concerned with the results from Fotakis et al. [3] about the existence of equilibria for transportation games. Basically, the authors deal with two situations when trying to prove the existence of equilibria: games where the bus' permutations are identical or not, and distances in the graph being metric or not.

In Section 3.1, we introduce the capacitated version of the transportation games as well as we show our results about the existence of PNE in this new version of the game.

As we can see in the next theorem, independently of the distances being metric or not, if all buses have the same permutation, then the best-response dynamics converge to a Strong Nash Equilibrium. Whether this dynamics converges in polynomial time or not is an open question.

Theorem 3.1 [Fotakis et al. [3]] If all buses have the same permutation π, then the best-response dynamics converges to an SE. Moreover, an SE can be built in $O(nm)$ time.

Proof. First, let us suppose, without loss of generality, that $\pi_j = (n, n{-}1, \ldots, 1)$ for $j \in M$. Now, for a profile σ, we create a vector $\vec{V}(\sigma)$ of dimension n, and we define the value of every coordinate i of $\vec{V}(\sigma)$ as $c_i(\sigma)$. If in a profile σ there is a group of players C who would benefit by deviating to a new profile σ', then we can show that $\vec{V}(\sigma')$ is lexicographically smaller than $\vec{V}(\sigma)$. Let k represent the player of smaller index in C, so we have that $c_i(\sigma) = c_i(\sigma')$ for $i < k$, since the cost of those players are not influenced by players who are picked up before them, and $c_k(\sigma) > c_k(\sigma')$, as player k has decreased her cost. Thus, $\vec{V}(\sigma')$ is lexicographically smaller than $\vec{V}(\sigma)$, and this dynamics will eventually converge to an SE.

In order to build an SE, we start with an empty profile σ and begin to assign the players one by one following the order in π. For each player i, we assign her to the bus which will give her the lowest cost (breaking ties assigning players to the bus of smaller index). If a player i chooses a bus j, then her cost $c_i(\sigma)$ will be $d(i, t)$ if she is alone on j or $d(i, i') + c_{i'}(\sigma)$, where i' is the player with largest index among those travelling in bus j. Then, player i does not interfere in the cost of those already assigned players and her choice will be the one giving her the lowest cost. Therefore, σ is lexicographically minimal and so we conclude it is an SE, which is built in $O(nm)$ time. ∎

We point out that the previous result cannot be extended to show the existence of an SSE because the authors show a metric instance which does not contain any SSE.

Proposition 3.2 [Fotakis et al. [3]] There exists a metric instance of the transportation game with $m = 2$ buses having identical permutations which does not admit any Super Strong Equilibrium.

Proof. Consider the instance depicted in Figure 3.1, and that there are $m = 2$ buses, with $\pi_1 = \pi_2 = (5, 4, 3, 2, 1)$. Both players 1 and 2 have dominant strategies by traveling in different buses, so without loss of generality let $\sigma_1 = 1$ and $\sigma_2 = 2$. If a player has the same cost in the two buses, then bus 1 is chosen in order to break ties. It is possible to see that among players 3, 4, and 5, there is always a non-empty group of players C where at least one player in C decreases her utility while all others players do not have their costs increased. For example, let us consider the profile $\sigma = (1, 2, 1, 2, 2)$ where we get that $c_3(\sigma) = 4$, $c_4(\sigma) = 5$, and $c_5(\sigma) = 7$. If players 3, 4, and 5 change their strategies, then we get $\sigma' = (1, 2, 2, 1, 1)$ where we have that $c_3(\sigma') = 4$, $c_4(\sigma') = 3$, and $c_5(\sigma') = 5$. ∎

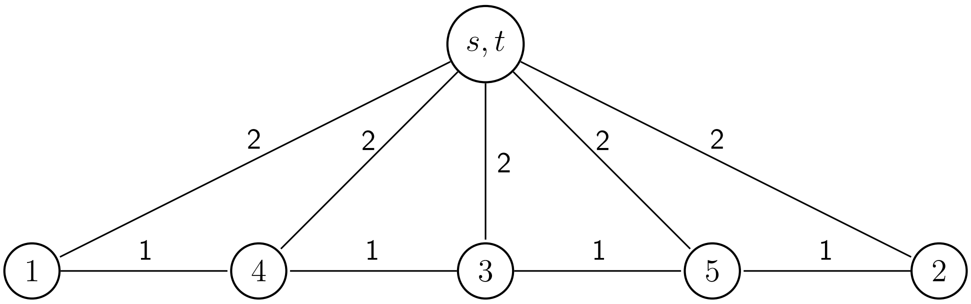


Figure 3.1: The instance used in the proof of Proposition 3.2.

In the case of non-metric instances with buses not having identical permutations, there is also an instance in which does not have PNE.

Proposition 3.3 [Fotakis et al. [3]] There exists a non-metric instance of the transportation game with $m = 2$ buses and $n = 3$ players which does not admit any PNE.

Proof. Let us consider players 1, 2, and 3, and buses 1 and 2 with permutations $\pi_1 = (3, 2, 1)$ and $\pi_2 = (1, 2, 3)$. Figure 3.2 shows the distances between players and nodes s and t. Here, if player 2 is alone on a bus or she is the last one to be picked up by a bus, then she always prefers to deviate. Then, we get four remaining possible profiles: $\sigma^1 = (1, 1, 1^*)$, $\sigma^2 = (1^*, 1, 2)$, $\sigma^3 = (2^*, 2, 2)$, and $\sigma^4 = (1, 2^*, 2)$, where a star alongside a player's choice indicates that she benefits by deviating. Therefore, we have that this instance does not admit any PNE. ∎

Because of the previous negative result about the existence of PNE in non-metric instances, from now on until the end of this section, we will only deal with metric instances. Therefore, we begin by showing that if a metric instance of the transportation game has only $m = 2$ buses, then we have that better-response dynamics converges to a PNE.

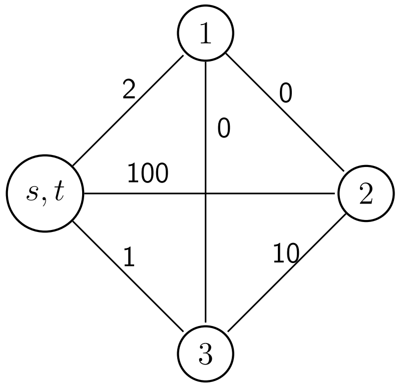


Figure 3.2: The instance used in the proof of Proposition 3.3.

Theorem 3.4 [Fotakis et al. [3]] For the transportation game with $m = 2$ buses and metric distances, better-response dynamics converges to a PNE.

Proof. By contradiction, suppose that the better-response dynamics cycles for some initial profile σ, and so the dynamics does not converge to a PNE. Let N_0 be the set of players who never change their strategies during the dynamics, and observe that $N_0 \neq \emptyset$ since for the last player i in a permutation π_j, for $j \in \{1, 2\}$, it is a dominant strategy to choose j because $d(i, t)$ is the minimum cost she would get by the triangle inequality. Let $N_1 = N \setminus N_0$ and let $p_j \in N_1$, for $j \in \{1, 2\}$, be the player coming last in permutation of bus j. Also, let d_j, for $j \in \{1, 2\}$, be the player from N_0 who is just after p_j in the permutation of bus j. Then, we have that the cost of player p_j, if she chooses bus j, is $d(p_j, d_j) + c_{d_j}$, where c_{d_j} is invariant in the cycle. Observe also that $p_1 \neq p_2$ because if it was not the case, then p_1 would be in N_0, as her cost does not depend on the actions taken by the players in $N_1 \setminus \{p_1\}$, and in this case she would have a dominant strategy that is to choose a bus j that minimizes $d(p_j, d_j) + c_{d_j}$.

Since we have a cycle in the dynamics and players do unilateral deviations, there is a state in which both players p_1 and p_2 are on the same bus. Without loss of generality, suppose a state in which p_2 changes her strategy and goes to bus 1 where p_1 is and that in π_1 player p_2 comes before p_1 (we can suppose this because $p_1 \neq p_2$, and one of them should be before the other in a permutation, so we can always rename them). Then, we know that

$$d(p_2, d_2) + c_{d_2} > d(p_2, p_1) + d(p_1, d_1) + c_{d_1}. \tag{3.1}$$

After a while, p_1 profitably moves to bus 2 and we get that

$$d(p_1, d_1) + c_{d_1} > d(p_1, d_2) + c_{d_2}. \tag{3.2}$$

By combining Inequalities 3.1 and 3.2 we get that $d(p_2, d_2) > d(p_2, p_1) + d(p_1, d_2)$, which violates the triangle inequality. □

As a result of the previous theorem, the authors left as open questions (1) to provide a potential function for the case of metric transportation games with two buses and also (2) to show if this dynamics converges in polynomial time.

Next, we show that we can compute one PNE, for metric transportation games with $m = 2$ buses, in polynomial time. The idea behind the proof is that after we let a player change her strategy (if profitable), she will not regret it and, as we do this once for each player, it takes n steps to reach a PNE.

Corollary 3.5 [Fotakis et al. [3]] For $m = 2$ buses and metric distances, the transportation game has a PNE that can be computed in $O(n)$.

Proof. Let 1 and 2 be the two buses. We start by assigning all players in bus 1. Then, we consider one by one in the reverse order that they appear in the permutation of bus 2. There are two choices for a player: she can stay on bus 1 or she can move to bus 2. For both cases, we can show that she will not regret it afterwards. In the first case, if she decides to stay on bus 1, later on her cost could only remain the same or decrease. In the second case, using arguments similar to those in Theorem 3.4, we can see why she would not regret her decision if she decides to move to bus 2. Therefore, this procedure computes a PNE in n steps. $\square$

We now exhibit why we cannot extend the results from Theorem 3.4 to all metric instances with any number m of buses by showing an instance with $m = 3$ buses that does not contain any PNE. An instance is said to be Euclidean if we consider that all nodes of the graph are in a 2-dimensional plane and that between all pair of nodes $x, y \in V$, with their coordinates being (p_1, q_1) and (p_2, q_2) respectively, their distance is $d(x, y) = \sqrt{(p_1 - p_2)^2 + (q_1 - q_2)^2}$. Note that all Euclidean instances are metric.

Proposition 3.6 [Fotakis et al. [3]] There exists an Euclidean instance with $m = 3$ buses which does not admit any PNE.

Proof. (sketch)The authors present an Euclidean instance of the game with $n = 8$ players and by an exhaustive examination, we can see that none of all the possible profiles is a PNE. $\square$

As a result of what was shown in Proposition 3.6, Fotakis et al. [3] conjectured that any instance with at most 7 players and $m = 3$ buses admits a PNE. On the other hand, they show that in this case, the dynamics may not converge to a PNE.

Proposition 3.7 [Fotakis et al. [3]] There exists a metric instance with $m = 3$ buses and $n = 7$ players for which the better-response dynamics may cycle.

Proof. The metric space and the distribution of the seven players are depicted in Figure 3.3. Let 1, 2, and 3 be the three buses with their permutation as being $\pi_1 = (6, 7, 2, 1, 4, 3, 5)$, $\pi_2 = (5, 7, 4, 1, 3, 2, 6)$, and $\pi_3 = (5, 6, 1, 2, 4, 3, 7)$, respectively. Here, players 5, 6, and 7 have dominant strategies (they choose buses 1, 2, and 3, respectively) because of their positions on the permutations and the triangle inequality. In Table 3.1, we see the 10 profiles that composes the cycle during the dynamics. Because of their dominant strategies, we do not show players 5, 6, and 7 in Table 3.1. A star alongside a player indicates that she wants to change to another bus. $\square$

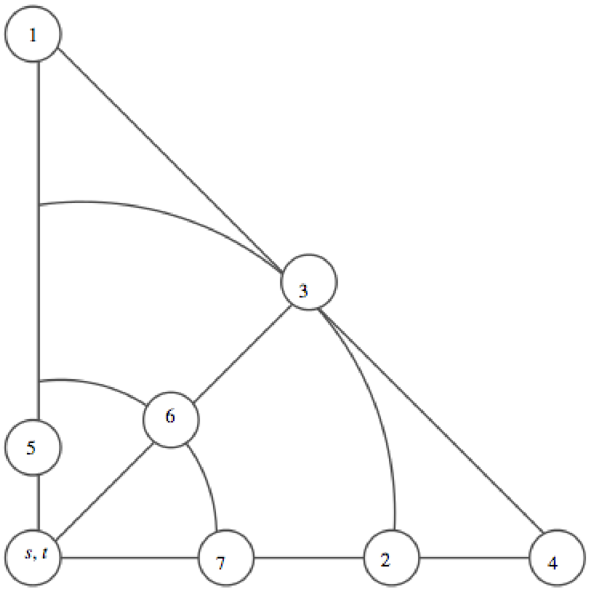


Figure 3.3: Metric instance of Proposition 3.7 [3].

	bus 1	bus 2	bus 3
σ^1	2	1^* 3	4
σ^2	2^* 1	3	4
σ^3	1	3^* 2	4
σ^4	1	2	4^* 3
σ^5	1 4	2	3^*
σ^6	1 4 3	2^*	
σ^7	1^* 4 3		2
σ^8	4^* 3	1	2
σ^9	3^*	1	2 4
σ^{10}		1 3	2^* 4

Table 3.1: Cycles of 10 profiles during the dynamics.

3.1 Capacitated Transportation Games

Based in settings arising from real-life scenarios, capacitated constraints have been analyzed in Game Theory [22, 23, 24, 25]. In this section, we introduce the model of capacitated transportation games, which is almost the same as described in Section 2.2, but now all buses $j \in M$ have a capacity k_j associated with it. This means that a bus j can carry at most k_j players. Here, if more than k_j players choose bus j to travel with, bus j will pick up only the first k_j players in its permutation π_j, and then the remaining players will have cost equal to ∞, meaning that they will not be picked up by bus j or any other bus.

For feasibility reasons, we will only consider games with $\sum_{j \in M} k_j \geq n$. Because of it, we also can work only with profiles σ that do not break the capacity constraint since a player that has cost ∞ can always choose a bus with free space to travel with.

Surprisingly enough, in contrast with the positive results of Fotakis et al. [3] showed in Theorems 3.1 and 3.4, we show that in the capacitated version even with $m = 2$ buses having the same permutation π, there exists a metric instance with no PNE.

Proposition 3.8 There exists a metric instance of the capacitated transportation game with $m = 2$ buses having identical permutations which does not admit any PNE.

Proof. Consider the metric instance showed in Figure 3.4 with $n = 4$ players, $m = 2$ buses, and π being equal to the identity permutation, i.e. $\pi_1 = \pi_2 = (1, 2, 3, 4)$. Also, consider that all remaining edges not shown here have a distance cost of 2, and let $k_1 = k_2 = 2$.

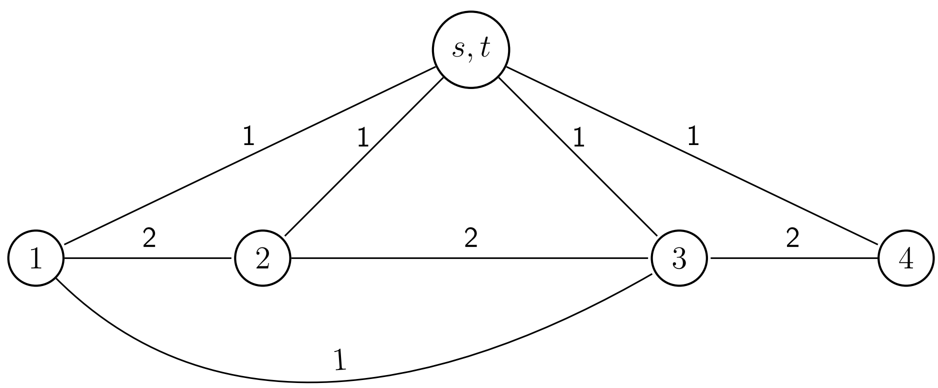


Figure 3.4: Instance of the capacitated transportation game with no PNE.

This instance has 16 possible profiles, but only 6 of them are feasible because of the capacity constraint. Hence, it suffices for us to show that none of them is a PNE. In the following, consider that the star alongside a player's choice indicates that that player wants do change to the other bus.

- $\sigma^1 = (1^*, 1, 2, 2)$: Player 1 wants to change since in bus 2 she will get cost of 2 instead of 3;
- $\sigma^2 = (2, 1^*, 2, 1)$: Player 2 wants to change since in bus 2 she will get cost of 1 instead of 3;

- $\sigma^3 = (2^*, 2, 1, 1)$: Symmetric to the case of σ^1;
- $\sigma^4 = (1, 2^*, 1, 2)$: Symmetric to the case of σ^2;
- $\sigma^5 = (1, 2^*, 2, 1)$: Player 2 wants to change since in bus 1 she will get cost of 1 instead of 3;
- $\sigma^6 = (2, 1^*, 1, 2)$: Symmetric to the case of σ^5.

$\square$

Chapter 4

Inefficiency of Equilibria in the Simultaneous Transportation Games

In this chapter, we analyze the inefficiency of equilibria by first reviewing the main results from Fotakis et al. [3] for the simultaneous version of the transportation game. In Section 4.1 we show that, for non-metric instances, the PoS is unbounded for social functions D, E, and U. In Sections 4.2 and 4.3 we see the results about the values of the bounds of PoA and PoS for both social functions D and E, respectively. Then, in Section 4.4 we show an upper bound on the PoA for the function U, which is our contribution since this is a new social function for the game.

4.1 Non-metric Instances

We start by showing that for both functions D and E the PoS is unbounded when dealing with non-metric instances, even in the case where all permutations are equal. First, observe that if we are dealing with instances where the number of players is $n \leq 2$ and the number of buses is $m \geq 2$, then it is possible to verify that the PoS = PoA = 1 even for non-metric instances.

Proposition 4.1 [Fotakis et al. [3]] For every $n \geq 3$, the PoS is unbounded for D and E if the distance is not metric, even if all permutations are identical.

Proof. Let us analyze the case where $n > m \geq 2$. We give an instance where $n = m + 1$. Consider the graph shown in Figure 4.1 where the distance between player $m + 1$ and the other players in $N \setminus \{m + 1\}$ values $1/\varepsilon$ and the remaining edges have their distances equal to ε. Let $\pi_j = (m + 1, m, \dots, 2, 1)$, for every $j \in [1, m]$. In these instances, a PNE σ does not contain two players i, i', with $1 \leq i < i' \leq m$, on the same bus since each one of them can be alone in a bus with cost ε. Now, no matter which bus player $m + 1$ chooses, her cost will be $1/\varepsilon + \varepsilon$, and therefore $f(\sigma) \geq 1/\varepsilon$ for $f \in \{D, E\}$. However, in an optimal solution σ^*, player $m + 1$ is alone in a bus, two players are in another bus, and the remaining $m - 2$ buses are picking up just one player. Therefore, $E(\sigma^*) = 2\varepsilon$ and $D(\sigma^*) = (m - 1)\varepsilon + 2\varepsilon = (m + 1)\varepsilon$. We get that, for both functions D and E, when $\varepsilon \to 0$ the PoS tends to ∞.

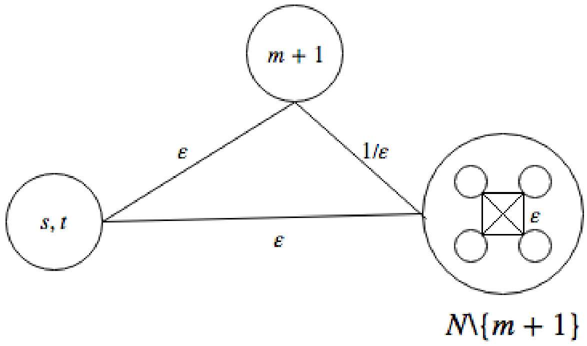


Figure 4.1: Graph G with the following distances, where $\varepsilon \in (0, 1)$: $d(m + 1, i) = 1/\varepsilon$ for $i \in [1, m]$, and every other distance values ε.

Let us analyze the case where $3 \leq n \leq m$, and again let $\varepsilon \in (0, 1)$. We consider a graph G where all node distances value ε, with exception of the distances involving nodes 2 and 3, which can be seen in Figure 4.2. Also, let $\pi_j = (m, \ldots, 2, 1)$, for every $j \in [1, m]$. In these instances, in a PNE σ, player 2 is on the same bus selected by player 1 since otherwise her cost would be 3ε. Because of this, player 3 will have a cost of at least $1/\varepsilon$ as she cannot be picked up right before player 1. As a consequence, we have that $f(\sigma) \geq 1/\varepsilon$ for $f \in \{D, E\}$. However, in an optimal solution σ^*, player 2 is in a bus which is different than the one chosen by player 1, so in σ^* all edges traversed by a bus have distance value of ε or 3ε. We get that, for both functions D and E, when $\varepsilon \to 0$, the PoS tends to ∞. □

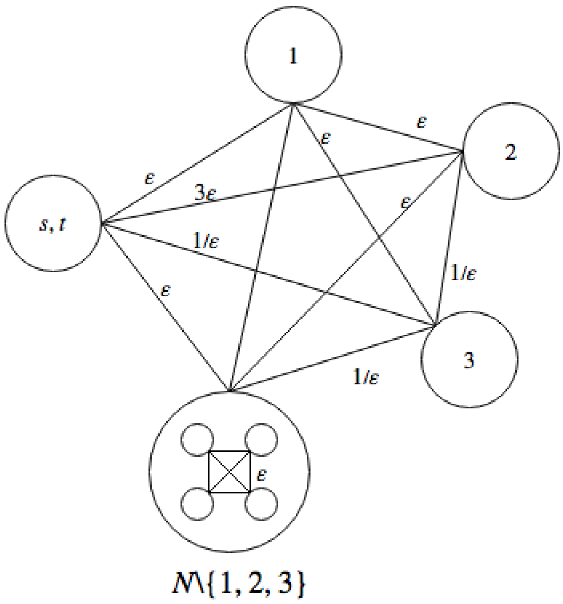


Figure 4.2: Graph G with the following distances, where $\varepsilon \in (0, 1)$: $d(2, t) = 3\varepsilon$, $d(2, i) = \varepsilon$ for $i \in N \setminus \{3\}$, $d(3, i) = 1/\varepsilon$ for $i \in \{N \setminus \{1\}\} \cup \{s, t\}$, and every other distance values ε.

It is possible to use the instances in the proof of Proposition 4.1 to show that the PoS is also unbounded for the utilitarian function U.

Corollary 4.2 For every $n \geq 3$, the PoS(U) is unbounded if the distance is not metric, even if all permutations are identical.

Because of Proposition 4.1 and Corollary 4.2, we will investigate, from now on, the inefficiency of equilibria of metric instances, until otherwise stated.

4.2 Function D with Metric Instances

Given a strategy profile σ and for $j \in M$, let $(j_1, \dots, j_{n_j})$ be the ordering that players are picked up by bus j, where $n_j = |\{i : \sigma_i = j\}|$ represents the number of players choosing to travel in bus j. Recall that $D(\sigma)$ is defined as

$$D(\sigma) = \sum_{j=1}^{m} \sum_{i=1}^{n_j} d(j_i, j_{i+1}),$$

where $j_{n_j+1} = t$ for all buses.

We start by giving an upper bound on the distance between every pair of nodes of any metric graph instances.

Lemma 4.3 [Fotakis et al. [3]] Let σ^* be an optimal profile. If d is metric, then $d(x, y) \leq D(\sigma^*)$ holds for all nodes $x, y \in N \cup \{t\}$.

Proof. We have two cases to consider, depending on how players x and y are being transported to t in the optimal solution σ^*. In the first case, assume that both players are traveling together in the same bus b and, without loss of generality, b picks up player y before player x. In this case, $D(\sigma^*)$ is at least the total route length done by b, given by the sum of edges on it, which is at least $d(x, y)$. The remaining case is the one where players are on different buses. Let us say player x is on bus b' and player y is on bus b''. Then, again, $D(\sigma^*)$ is at least the sum of the routes done by b' and b'', which is at least $d(x, t) + d(y, t) \geq d(x, y)$ by the triangle inequality. □

Now, with Lemma 4.3 we can give an upper bound on the value of function D for any profile σ of the metric game.

Proposition 4.4 [Fotakis et al. [3]] Let σ^* be an optimal profile. If d is metric, then $D(\sigma) \leq nD(\sigma^*)$ holds for every profile σ.

Proof. In equation $D(\sigma) = \sum_{j=1}^{m} \sum_{i=1}^{n_j} d(j_i, j_{i+1})$, we have that n_j, $1 \leq j \leq m$, form a partition of n (the number of players), so we get that function D is computed by n terms. By Lemma 4.3, each of them is at most $D(\sigma^*)$. □

By putting together the results from Lemma 4.3 and Proposition 4.4, we obtain that the PoA(D) $\leq n$.

Corollary 4.5 [Fotakis et al. [3]] The PoA(D) of the transportation game on n players with metric distances and $m \geq 2$ buses is upper bounded by n.

We next present a family of instances for the case where $2 \leq m \leq n$, which will help us to give upper bounds on the PoS (and consequently a lower bound on the PoA) of both functions D and E.

Example 4.2.1 [Fotakis et al. [3]] We construct the following metric instance considering that k is a positive integer. The set of players is $N = L \cup R$ where $L = \{l_{i,j} : 1 \leq i \leq k, 1 \leq j \leq m\}$ and $R = \{r_{i,j} : 1 \leq i \leq k, 1 \leq j \leq m\}$. Therefore, we are dealing with $n = 2km$ players. Next, we decompose L and R into k levels, where $L_i = \{l_{i,j} : 1 \leq j \leq m\}$ and $R_i = \{r_{i,j} : 1 \leq j \leq m\}$. Consider the graph G depicted in Figure 4.3.

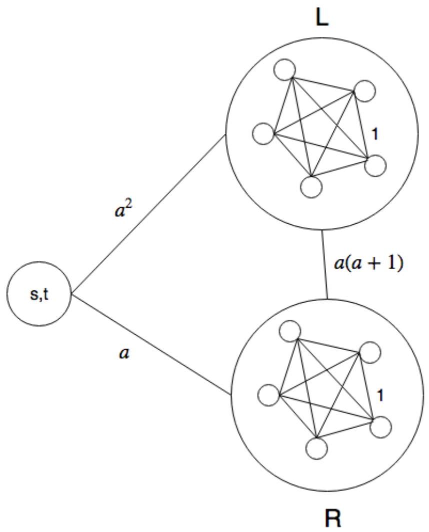


Figure 4.3: Graph G with the following distances, where a is some positive integer: $d(u,v) = 1$ if $u,v \in L$ or $u,v \in R$; $d(v,t) = a$ if $v \in R$; $d(v,t) = a^2$ if $v \in L$; $d(u,v) = a(a+1)$ if $u \in L$ and $v \in R$.

We abuse notation in order to show how the buses' permutations are formed: we will construct permutations π composed of an intercalation of sets L_i and R_i, $1 \leq i \leq k$, but this is to be seen as a sequence of the players in nondecreasing order of their indexes within each set. For $j \in M$, let $\pi_j = (R_k, L_k, R_{k-1}, L_{k-1}, \ldots, R_1, L_1)$, where players in R_i (resp. L_i) appear in π_j as $(r_{i,m}, r_{i,m-1}, \ldots, r_{i,1})$ (resp. $(l_{i,m}, l_{i,m-1}, \ldots, l_{i,1})$).

Lemma 4.6 [Fotakis et al. [3]] In the instances presented in Example 4.2.1, a profile σ is a PNE if and only if each bus contains exactly one player of each level of L and one player of each level of R.

Proof. By contradiction, suppose there is a profile σ that is a PNE in which there is at least one bus that does not follow the claim. Let $q \geq 1$ be the first level where the condition does not hold. We have then two possibilities to consider:

1. W.l.o.g., suppose that b is the bus that does not pick up any player from L_q. Then there exists another bus b' that is picking up at least two players, say $l_{q,j}$ and $l_{q,j'}$ with $j' > j$, from L_q consecutively. We have by construction that the cost of player $l_{q,j'}$ is $c_{l_{q,j'}}(\sigma) = d(l_{q,j'}, l_{q,j}) + c_{l_{q,j}}(\sigma) = 1 + c_{l_{q,j}}(\sigma)$, but if she moves to bus b, her cost would be at most $c_{l_{q,j}}(\sigma)$, which contradicts the fact of σ being a PNE.

2. Analogously, we can analyze what happens if the claim does not hold to R_q in the same way.

□

Finally, by Proposition 4.7, we see that the bound in Corollary 4.5 is asymptotically tight.

Proposition 4.7 [Fotakis et al. [3]] For any $n \geq 2$, there are metric instances of the transportation game with n players and $m \geq 2$ buses where the PoS(D) is asymptotically n, even if all buses have the same permutation.

Proof. First, if $2 \leq n \leq m$, we analyze the following metric instance played on a graph G where $d(i,t) = 1$, for all $i \in N$ and $d(i,i') = \varepsilon$, for all $i, i' \in N$, with $\varepsilon \in (0,1)$. Let π_j, for $j \in M$, be the identity permutation, i.e., $\pi_j = (1, \ldots, n)$. Observe that an optimal solution σ^* is the one where all players choose to travel in the same bus, and therefore $D(\sigma^*) = 1 + (n-1)\varepsilon$. Since there are as many buses as the number of players and the instance is metric, in all PNE σ each bus is being used by one single player. Then, $D(\sigma) = n$ and, as a result, we have that PoS(D) $= \lim_{\varepsilon \to 0} \frac{n}{1+(n-1)\varepsilon} = n$.

For the remaining case where $2 \leq m \leq n$, we use the instance showed in Example 4.2.1. From Lemma 4.6, we know the structure of every PNE σ of those instances, so we get that in each bus there are exactly $2k$ players and then $D(\sigma) = m((2k-1)(a^2+a)+a^2) = 2kma^2 + 2kma - am = na^2 + a(n-m)$. However, in an optimal profile σ^*, there are only two buses being used: one for all players of L and one for all players of R. We get that there are km players in both buses, and then $D(\sigma^*) = (km-1)+a^2+(km-1)+a = a^2+a+n-2$. Therefore, we have that PoS(D) $= \lim_{a \to \infty} \frac{D(\sigma)}{D(\sigma^*)} = n$. □

4.3 Function E with Metric Instances

We recall that the social function E is defined as $E(\sigma) = \max_{i \in N} c_i(\sigma)$. There are basically two ideas behind the main result of this section. First, Lemma 4.8 gives us an upper bound for the distance of every edge of the graph's instance by relating it with the value $E(\sigma^*)$ of an optimal solution σ^*. Second, Lemma 4.9 provides us the maximum cost that a player will have in any PNE.

Lemma 4.8 [Fotakis et al. [3]] Let σ^* be an optimal profile. If d is metric, then $d(x,y) \leq 2E(\sigma^*)$ holds for every pair of nodes $x, y \in N$, and $d(x,t) \leq E(\sigma^*)$ holds for every node $x \in N$.

Proof. In the first part, we have two cases to consider, depending on how players x and y are being transported to t in an optimal solution σ^*. In the first case, assume that both players are traveling together in the same bus b and, without loss of generality, b picks up player x before player y. In this case, the route which b follows has cost of at least $d(x,y)$ plus the cost of player y, $c_y(\sigma^*)$. Therefore, whether or not b is the bus which characterizes σ^*, i.e., the bus containing the player with maximum cost, we get that $d(x,y) \leq d(x,y) + c_y(\sigma^*) \leq E(\sigma^*) \leq 2E(\sigma^*)$. Now, if players x and y are on different buses in σ^*, then we know that the cost of both players, $c_x(\sigma^*)$ and $c_y(\sigma^*)$, are upper bounded by $E(\sigma^*)$. Therefore, we have that $d(x,y) \leq d(x,t) + d(y,t) \leq c_x(\sigma^*) + c_y(\sigma^*) \leq 2E(\sigma^*)$.

In the remaining part, we know that, by Equation 2.4, the cost of any player x is upper bounded by $E(\sigma^*)$. Hence, as d is metric, $d(x,t) \leq c_x(\sigma^*) \leq E(\sigma^*)$. □

Now we are able to show an upper bound on the cost of every player according to social cost function E.

Lemma 4.9 [Fotakis et al. [3]] Let σ^* be an optimal profile. In any Pure Nash Equilibrium, the cost of a player is at most $(2\lceil\frac{n}{m}\rceil - 1)E(\sigma^*)$.

Proof. Suppose that, by contradiction, there is a PNE σ in which there is a player i choosing to travel in a bus b and that $c_i(b,\sigma_{-i}) > (2\lceil\frac{n}{m}\rceil - 1)E(\sigma^*)$. Let k be the number of players bus b will pick up between i and the final destination t. Then, by Lemma 4.8, we get that $c_i(b,\sigma_{-i}) \leq (2k-1)E(\sigma^*)$ which means that $k > \lceil\frac{n}{m}\rceil$. However, if bus b is going to pick up more than $\lceil\frac{n}{m}\rceil$ players, than it must exist another bus b' taking less than $\lceil\frac{n}{m}\rceil$ players. Hence, if player i changes her strategy and goes to bus b', her cost would be $c_i(b',\sigma_{-i}) < (2\lceil\frac{n}{m}\rceil - 1)E(\sigma^*)$, which contradicts the stability of σ. □

Next, using Lemma 4.9 and Lemma 4.6, we can give the value of the PoS for social function E.

Corollary 4.10 [Fotakis et al. [3]] The PoS(E) of the transportation game is $\Theta(\frac{n}{m})$.

Proof. The upper bound comes from Lemma 4.9. For the lower bound, we use the instance given in Example 4.2.1. Recall, from Lemma 4.6, that any PNE σ in those instances has a unique structure, so we get that $E(\sigma) = (2k-1)(a^2+a) + a^2$. Recall that in this instance, in an optimal profile σ^* there are only two buses being used: one for all players of L and one for all players of R. We get that there is km players in both buses, which gives us that $E(\sigma^*) = (km-1) + a^2$. Hence, we have that PoS(E) $= \lim_{a\to\infty} \frac{E(\sigma)}{E(\sigma^*)} = 2k = \frac{n}{m}$. □

We conclude this section by showing the value of the Price of Anarchy with relation to social function E.

Proposition 4.11 [Fotakis et al. [3]] For the transportation game, PoA(E) $= 1$ if $n \leq m$, and PoA(E) $= 2\lceil\frac{n}{m}\rceil - 1$ if $n > m$.

Proof. If $n \leq m$, then every player chooses a different bus, and by Lemma 4.8, we have that PoA(E) $= 1$. Now, for the case $n > m$, we get the upper bound from Lemma 4.9. Next, we construct the following metric instance in order to give the matching lower bound, where $m = 2p$, $n = 2pq$, q is an even number, and p is any positive integer.

- We decompose N into m sets of size q: $N_j = \{1^j, 2^j, \ldots, q^j\}, j = 1 \ldots 2p$;
- Consider the graph G depicted in Figure 4.4.

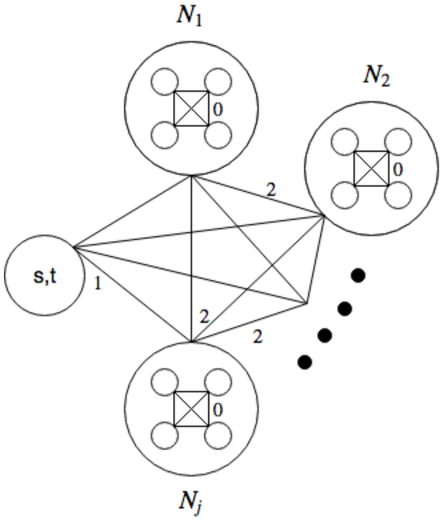


Figure 4.4: Graph G with the following distances: $d(u,t) = 1$, for $u \in N$; $d(u,v) = 2$ if $u \in N_j$ and $v \in N_{j'}$ with $j \neq j'$; $d(u,v) = 0$ if $u, v \in N_j$.

- We define $\pi^{even} = (2^1, 2^2, \ldots, 2^{2p}, 4^1, \ldots)$ as the sequence of players of even indexes, and we also define $\pi_j^{even} = (2^j, 4^j, \ldots, q^j)$. The same is done with players of odd indexes, which defines π^{odd}. We again abuse notation to define the buses' permutations. Let $\pi_j^{'} = N \setminus \{\pi_{j+1}^{odd}, \pi_j^{odd}\}$ be any permutation of the players in N without the players in π_{j+1}^{odd} and π_j^{odd}, and in the same way let $\pi_j^{''} = N \setminus \{\pi_{j-1}^{even}, \pi_j^{even}\}$. Then,
  - if $j \in [1, 2p]$ is odd: $\pi_j = (\pi_j^{'}, (q-1)^{j+1}, (q-1)^j, \ldots, 1^{j+1}, 1^j)$.
  - Otherwise: $\pi_j = (\pi_j^{''}, q^{j-1}, q^j, \ldots, 2^{j-1}, 2^j)$.

Observe that a profile σ where each bus $j \in [1, 2p]$ is selected by the last q players in its permutation π_j is a PNE because in σ every player has a cost of at most $2(q-1)+1 = 2q-1$. Thus, if a player decides to change her strategy, then her cost would be $2((q+1)-1)+1 = 2q+1$. Hence, σ is stable.

Therefore, we know that in those instances there is a PNE σ with $E(\sigma) = 2q-1$. Now, in an optimal solution σ^*, a bus j is being used only by the players of N_j, and therefore $E(\sigma^*) = 1$ since the cost of every player in σ^* is 1. Then, the PoA(E) is at least $2q - 1 = 2\frac{n}{m} - 1$. □

4.4 Function U with Metric Instances

Now, let us consider the utilitarian function $U(\sigma)$ which represents the sum of the times that players will get to t. It is computed by the following expression:

$$U(\sigma) = \sum_{i \in N} c_i(\sigma) = \sum_{j \in M} \sum_{i \in \sigma_j} c_i(\sigma).$$

Then, $U(\sigma)/n$ can be seen as the average time a player will take to reach her final destination.

We will analyze its inefficiency by giving bounds on the PoA(U). We start by showing a lower bound on the value of an optimal profile, and then we prove that the PoA is at most $2n-1$, where n is the number of players.

Proposition 4.12 Let σ^* be an optimal profile. If d is metric, then $\sum_{i\in N} d(i,t) \leq U(\sigma^*)$.

Proof. Because of the triangle inequality, for a player i we have that $d(i,t) \leq c_i(\sigma^*)$. Then, the result follows. □

Theorem 4.13 For metric transportation games and for any profile σ, we have that $U(\sigma) \leq (2n-1)U(\sigma^*)$, where σ^* is an optimal profile.

Proof. For each bus $j \in M$, let $(j_1, \ldots, j_{n_j})$ be the sequence of players bus j is going to pick up, where $n_j = |\{i : \sigma_i = j\}|$ represents the number of players choosing to travel in bus j. Then,

$$
\begin{aligned}
U(\sigma) &= \sum_{j=1}^{m} \sum_{i\in\sigma_j} c_i(\sigma) \\
&= \sum_{j=1}^{m} \sum_{i=1}^{n_j} i \cdot d(j_i, j_{i+1}) && (4.1) \\
&\leq \sum_{j=1}^{m} \sum_{i=1}^{n_j} i(d(j_i,t) + d(t, j_{i+1})) && (4.2) \\
&= \sum_{j=1}^{m} \sum_{i=1}^{n_j} (2i-1) d(j_i, t) \\
&\leq (2n-1) \sum_{j=1}^{m} \sum_{i=1}^{n_j} d(j_i,t) \\
&\leq (2n-1)U(\sigma^*). && (4.3)
\end{aligned}
$$

In (4.1) we consider that $j_{n_j+1} = t$ for all $j \in M$, and Inequality 4.2 follows from the triangle inequality. Finally, we get the Inequality 4.3 from the fact that the summations are composed by n terms of $d(j_i,t)$, and then we use Proposition 4.12 to get the final result. □

From Theorem 4.13, we get as a corollary an upper bound on the PoA for transportation games when being analyzed by the social function U.

Corollary 4.14 For metric transportation games with n players, the PoA(U) is at most $2n-1$.

Chapter 5

Inefficiency of Equilibria in the Sequential Transportation Games

All the results listed in this chapter are our main contributions for this work. Here, we analyze the Sequential Price of Anarchy for the sequential transportation games by considering the social functions D, E, and U. We begin by showing that the SPoA is unbounded for all those social functions when dealing with non-metric instances, and then we discuss our results about the value of the SPoA for metric cases of the sequential transportation games.

5.1 Non-metric Instances

As one could imagine, following the results of Fotakis et al. [3] stating that the PoA is unbounded for non-metric instances of the transportation game in its simultaneous version, we show that the SPoA is also unbounded for social functions D, E, and U when dealing with non-metric instances, even if all permutations are equal.

Proposition 5.1 The SPoA is unbounded for D, E, and U if the distance is not metric, even if all permutations are equal.

Proof. Consider the instance showed in Figure 5.1 with $n = m = 2$ and π being equal to the identity permutation, i.e. $\pi_1 = \pi_2 = (1, 2)$. In Figure 5.2 we can see the game in its extensive form where the leaves represents the cost of players 1 and 2 respectively. Observe that player 2 will have cost 1 no matter which bus she chooses to travel with, so one possible SPE is the one where player 2 always chooses the same bus as player 1 and, therefore, player 1 will have cost $X + 1$. Now we analyze the SPoA for functions D, E, and U.

- Functions D and U: The optimal solution is the one where the players are traveling in different buses, which costs 2. In the SPE described before, the social cost is $X + 2$. Thus, $\text{SPoA}(D) = \text{SPoA}(U) = \lim_{X\to\infty} \frac{X+2}{2} = \infty$.
- Function E: The optimal solution is also the one where the players are traveling in different buses, which costs 1. In the SPE described before, the social cost is $X + 1$. Thus, $\text{SPoA}(E) = \lim_{X\to\infty} \frac{X+1}{1} = \infty$.

□

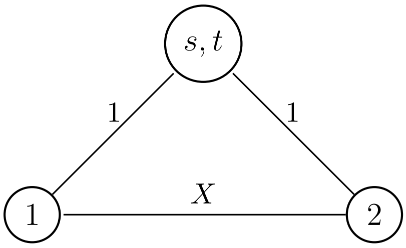


Figure 5.1: Non-metric instance with $n = 2$ players, where X is any positive integer.

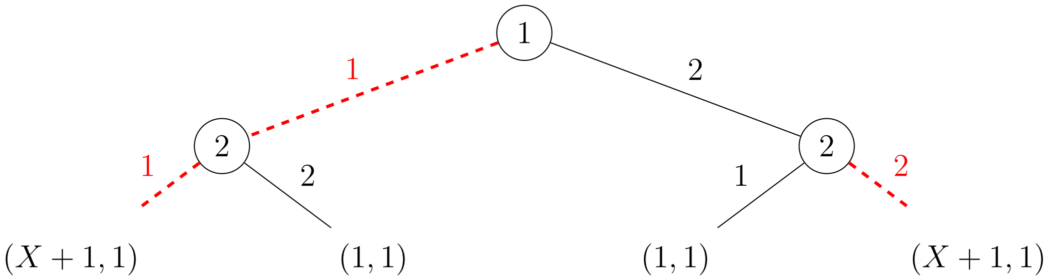


Figure 5.2: Representation of the SPE $(1,(1,2))$. Dashed red edges indicate the action chosen at each node.

5.2 Function D with Metric Instances

Next, we show the value of the SPoA for metric instances in relation with social function D, and it turns out that this value is n, which is equal to the value of the PoA(D) for its simultaneous version [3].

Theorem 5.2 For metric transportation games with n players, SPoA$(D) = n$.

Proof. The upper bound comes from Proposition 4.4, where we have that $D(\sigma) \leq nD(\sigma^*)$ for any profile σ and optimal solution σ^*. For the lower bound, consider an instance (N, M, G) where $|N| = n$, $M = \{1, \ldots, n\}$, and the graph is the one showed in Figure 5.3. Let π_j, for $j \in M$, be any permutation. It is possible to see that an optimal solution σ^* is the one where all players are traveling together in a single bus, and thus we get $D(\sigma^*) = 1$.

Now, it suffices to show that those instances possess an SPE with value n, i.e., a solution where each bus is being used by one single player. Observe that a player i will always have a cost of 1 no matter what the other players do. Consequently, consider the profile σ where player i chooses bus i at each choice node she is responsible for. So, in σ the cost of each player is 1 and there is not another bus in which she can get a better cost. Therefore, σ is an SPE with $D(\sigma) = n$. □

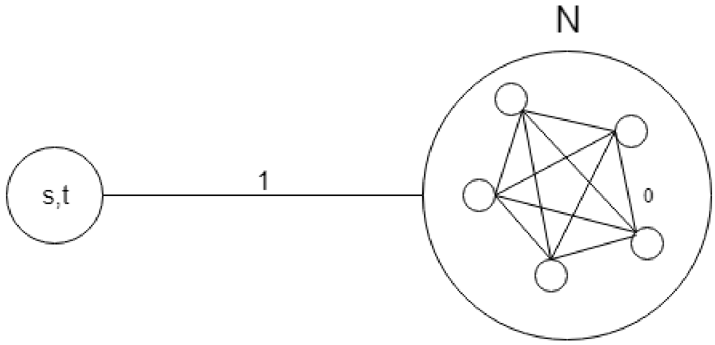


Figure 5.3: Graph G where $d(s,u) = d(u,t) = 1$ for all $u \in N$ and $d(u,v) = 0$ for all $u, v \in N$.

5.3 Function E with Metric Instances

In contrast with the results on the value of PoA related to function E presented by Fotakis et al. [3] ($\text{PoA}(E) = 2\lceil \frac{n}{m} \rceil - 1$ for $n > m$ and $\text{PoA}(E) = 1$ if $n \leq m$), next theorem shows that the value of the SPoA is worse than in its simultaneous version for function E even when $n = m$.

Theorem 5.3 For metric transportation games with n players, $\text{SPoA}(E) = 2n - 1$.

Proof. Let σ^* be an optimal solution. The maximum value a solution can achieve is when all players choose to travel on a single bus, and we argument that it is an upper bound on the $\text{SPoA}(E)$ of $(2n-1)E(\sigma^*)$. This value comes from Lemma 4.8, which states that $d(u,v) \leq 2E(\sigma^*)$ for all pairs $u, v \in N$ and $d(u,t) \leq E(\sigma^*)$ for all $u \in N$. Then, since all players are on a single bus, we have that the path which will be used by it has only one edge directly connected to t and the remaining $(n-1)$ edges are used to pick up all players. Hence, this path values at most $(n-1)2E(\sigma^*) + E(\sigma^*) = (2n-1)E(\sigma^*)$.

Now, for the lower bound, we provide a family of instances containing one SPE with value that matches the given upper bound. These instances are given by (N, M, G) where $|N| = |M| = n$, and the graph depicted in Figure 5.4. Let π_j, for $j \in M$, be the identity permutation, i.e., $\pi_j = (1, \ldots, n)$. Observe that an optimal solution σ^* is the one where each bus is being used by a single player, and hence we have that $E(\sigma^*) = 1$.

We will show, by backward induction on the player's index, that there exists at least one SPE σ where all players choose the same bus, and therefore we get that $\text{SPoA}(E) \geq (2n-1)$. For the last player n, since she is the last player to be picked up in all buses, her cost $c_n(\sigma)$ will always be 1 despite the bus she chooses to travel in. Then, she can choose bus σ_{n-1} (bus chosen by player $n-1$) at each choice node she is responsible for and still gets $c_n(\sigma) = 1$.

Now, suppose the claim is valid for all players $k+1, \ldots, n$, and consider player k. Here, observe that the actions taken by her predecessors do not influence her cost because all of them are being picked up before her according to permutations π. She has m options of buses which will all give her a cost $c_k(\sigma)$ of $(2n - 2k + 1)$. By the induction hypothesis, players $k+1, \ldots, n$ will choose the same bus as player k. Therefore, as player k cannot avoid traveling with all of them, she can choose to travel on bus σ_{k-1}, since her cost will not be influenced by the decisions of her predecessors.

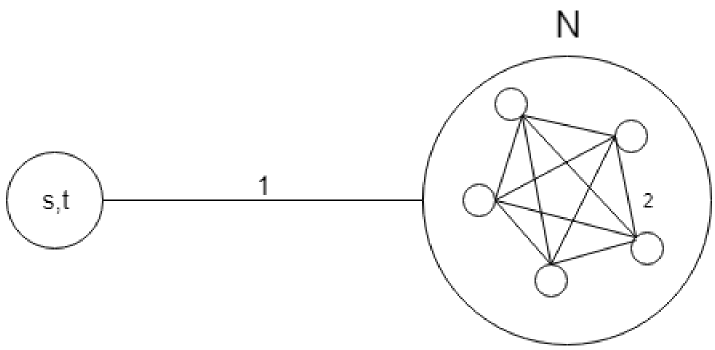


Figure 5.4: Graph G where $d(s,u) = d(u,t) = 1$ for all $u \in N$ and $d(u,v) = 2$ for all $u, v \in N$.

This will lead to a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_1)$ where all players are choosing bus σ_1 at each choice node of the extensive form this game. Thus, the value of the SPE σ is at most $(2n-1)E(\sigma^*)$.

To have a better insight, take a look at the decision tree for an instance with 3 players, depicted in Figure 5.5. □

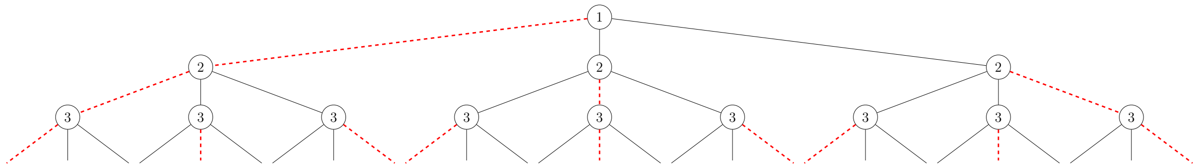


Figure 5.5: Decision tree for the game with three players. Consider that each of the three edges of each node is labeled as follows, from left to right: bus 1, 2, and 3. Then, the dashed red edges indicate the action chosen at each node.

For this particular social function E, we also study the best-case order in which players can arrive, which was introduced by Mamageishvili and Mihalák [26]. In this scenario, this version of the game resembles it in its extensive form, but now it is assumed that there exists an authority who can control the function ρ from Definition 2.13, that is, who takes action at each node. Therefore this authority can control the order in which the players will make their actions in the decision tree, but not the actions taken by the players. Also, here the final outcome of the game in general is not an SPE. Mamageishvili and Mihalák [26] also introduced the notion of **Myopic Sequential Price of Stability** (MSPoS), which is defined by them as the best-case ratio of the costs of a strategy profile σ that can be obtained by ordering the players as in a permutation π and letting player $\pi(i)$ choose the best-response $\sigma_{\pi(i)}$ (bus which is currently taking her to t more quickly) in the game induced by the first i players $\pi(1), \ldots, \pi(i)$ and of an optimal profile. For the formal definition presented next, let MSP($\mathcal{G}$) be the set of all such strategy profiles of a game $\mathcal{G}$.

Definition 5.4 Given a function f representing the social function of a game $\mathcal{G}$. The *Myopic Sequential Price of Stability* (MSPoS) is defined as

$$\mathrm{MSPoS}(f, \mathcal{G}) = \frac{\min_{\sigma \in \mathrm{MSP}(\mathcal{G})} f(\sigma)}{\min_{\sigma^* \in \mathcal{S}} f(\sigma^*)}.$$

We next present our result about the MSPoS according to social function E, in which the ideas behind the proof comes from one of the results from Mamageishvili and Mihalák [26].

Theorem 5.5 The MSPoS(E) for metric transportation games is at most 2.

Proof. Consider an instance of the transportation game (N, M, G) where $|N| = n$ and $|M| = m$. We have that the structure of any feasible solution σ is a tree T formed by m disjunct paths on the vertices of G that start at node s and finish at node t, so T is rooted in t with this root having degree at most m. Each one of those m paths is associated with the route done by one of the buses in σ, and the length of those routes is the total distance traveled by that specific bus according to its permutation. Thus, T has at most m leaves. Let T^* be the tree formed by an optimal solution σ^* and $dist_{T^*}(u, v)$ be the distance between nodes (players) u and v using only edges of T^*. Also, let $cost(T^*)$ be the sum of edges' distances of T^*. Then, we have $E(\sigma^*) \leq cost(T^*)$, since the optimum value is characterized by the total distance between t and one of its m leaves.

Without loss of generality, consider that the players are numbered as in a post-order traversal of T^*, and let the players enter the game in this order. To prove the claimed bound, we will show that every player in the subgame perfect equilibrium chooses a strategy which guarantees her a cost of at most $2cost(T^*)$. For example, player i has the option of choosing the same bus σ_{i-1}, which will result in a solution with edges of T^*. In that situation, her cost would be equal to the distance between i and $i-1$ plus the cost of player $i-1$. Any other alternative should cost at most that value, so we have the following chain of inequalities:

$$c_i \leq dist_{T^*}(i, i-1) + c_{i-1}, i = 2, \ldots, n.$$

For the first player, since she is the last player to be picked up, we have that $c_1 \leq dist_{T^*}(1, t)$. By summing up all those inequalities, we get that

$$c_n \leq \sum_{i=1}^{n} dist_{T^*}(i, i-1),$$

with $i-1 = 0$ representing t. In this summation, there are edges that are not contained in T^*, but their cost can be bounded by the cost of edges that are. For example, if edge $(i, i-1)$ is not in T^*, then it can be bounded by edges (i, t) and $(t, i-1)$ because of the triangle inequality. Therefore, this summation values at most $2cost(T^*)$, and we get the claimed bound. $\square$

5.4 Function U with Metric Instances

Observe that Theorem 4.13 gives us also an upper bound for the SPoA(U). It is left as an open question whether this bound is tight or not.

Corollary 5.6 For metric transportation games with n players, SPoA(U) is at most $2n-1$.

Chapter 6

Conclusion

In this thesis, we have analyzed a transportation game proposed by Fotakis et al. [3]. In Chapter 3 we reviewed their results about the existence of Nash Equilibrium for both metric and non-metric instances. Also, in Chapter 4 (until Section 4.3) we revisited their results concerning the inefficiency of equilibrium considering both social functions D (Vehicle Kilometers Travelled) and E (Egalitarian), for metric and non-metric instances.

We then have extended their results by first analyzing the inefficiency of equilibria for a new social function, the utilitarian function U, by giving an upper bound on the Price of Anarchy for it. We also presented a capacitated version of this game and showed that it is not guaranteed that metric instances with only $n = 2$ players have a PNE.

Another contribution is the study of this game in its extensive form, defined here as sequential transportation games. As a result of it, we were able to give bounds for all three social functions considered in this thesis. Furthermore, for social function E, we showed an upper bound for the myopic SPoS, which is a concept of stability that is interested in the best-case order in which players can arrive in a sequential game.

As a future direction, we also believe, as Fotakis et al. [3] do, that analyzing the game with different ways of computing the routes of the buses would be of great interest. Also, for the Price of Anarchy and the Sequential Price of Anarchy related with function U, closing the gap or proving that it is tight is another direction for the research.

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