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Group gradings on triangular matrices and graded identities of Universal algebras

Graduações por grupo nas álgebras de matrizes triangulares e identidades graduadas de álgebras Universais

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"C'est avec la logique que nous prouvons
et avec l'intuition que nous trouvons."
(Henri Poincaré)

Resumo

Nesta tese, classificamos as graduações por um grupo nas álgebras triangulares superiores, vistas como álgebras de Lie e de Jordan, sobre um corpo arbitrário e um grupo arbitrário. A partir deste resultado, e assumindo condições mais fortes, obtivemos a classificação das graduações por um grupo na álgebra das matrizes triangulares em blocos, vista como uma álgebra de Lie e de Jordan.

Nós calculamos o comportamento assintótico da sequência de codimensões graduadas de cada graduação na álgebra associativa de matrizes triangulares superiores. Obtemos um resultado parcial para a sequência de codimensões graduadas, para as graduações elementares no caso de Lie. Para as demais graduações nos casos de Lie e Jordan, fomos capazes de calcular o seu expoente graduado.

Finalmente, investigamos o problema de determinar uma álgebra simples a partir de suas identidades polinomiais. Nós provamos que Ω-álgebras de dimensão finita graduadas, que são graduadas-primas, sobre um corpo algebricamente fechado, são unicamente determinadas por suas identidades polinomiais graduadas.

Abstract

In this thesis, we classify group gradings on the algebra of upper triangular matrices, viewed as Lie and Jordan algebras, over an arbitrary field and arbitrary grading group. Using this result, and assuming stronger conditions, we were able to obtain the classification of group gradings on the algebra of block-triangular matrices, viewed as Lie and Jordan algebras.

We compute the asymptotic behavior of the graded codimension sequence for any grading on the associative algebra of upper-triangular matrices. For the Lie case, we obtain a partial result for the asymptotic behavior of graded codimensions, and we compute the graded exponent of all gradings on the upper triangular matrices, as Lie and Jordan algebras.

Finally, we investigate the problem of determining a simple algebra by its polynomial identities. We prove that finite-dimensional graded Ω-algebras, which are graded-prime, over an algebraically closed field are uniquely determined by their graded polynomial identities.

Contents

Introduction

Gradings are a natural structure which arises in several contexts in Mathematics. Gradings by a group (or a semigroup) is a natural generalization of the notion of the degree in polynomial algebras. Basically, a grading is an attribution of some number (or an element of a semigroup) to some elements of the algebra, so that the multiplication of the algebra is compatible with the operation of the semigroup. As an example, the product of monomials is compatible with the sum of integers: $x^n \cdot x^m = x^{n+m}$.

An interesting example appears whenever we study finite-dimensional Lie algebras. If $\mathfrak{g}$ is a Lie algebra, then one obtains a natural decomposition by its root system, $\mathfrak{g} = \sum_{\lambda\in\Phi} \mathfrak{g}_\lambda$. Given λ_1, $\lambda_2 \in \Phi$, one sets $\mathfrak{g}_{\lambda_1+\lambda_2} = 0$, if $\lambda_1 + \lambda_2 \notin \Phi$. It is known that $[g_{\lambda_1}, g_{\lambda_2}] \subset g_{\lambda_1+\lambda_2}$. In this way one obtains a natural grading on the Lie algebra $\mathfrak{g}$. This structure of grading plays an extremely important role in the classification of finite-dimensional semisimple Lie algebras over an algebraically closed field.

Physicists are interested in the so-called superstructures, which are $\mathbb{Z}_2$-graded algebras where the product is appropriately "twisted". It is also interesting to adapt the applicability of gradings by other abelian groups other than $\mathbb{Z}_2$. Thus graded algebras are interesting for various branches of research.

It is natural to ask what are all possible gradings on a given algebra. E. Zelmanov asked the following question: classify all possible semigroup gradings on matrix algebras.

In this direction, among the influential works, yet not the first publications on the subject, we cite Y. Bahturin, S. Sehgal and M. Zaicev [15], and Y. Bahturin, I. Shestakov and M. Zaicev [16]. The former provided the classification of group gradings on matrix algebras over an algebraically closed field, under some additional mild conditions. The latter paper proposed a technique to dealing with group gradings on simple Lie algebras, using the result of the former one. These two works gave an insight to starting an extensive theory, giving complete results in the classification of group gradings on matrix algebras over algebraically closed fields, on the simple Lie algebras, and on the simple Jordan algebras. A comprehensive reference for the theory is the recent monograph [35].

However, several important algebras are not simple. The block-triangular matrices represent a special kind of an algebra, which is not simple in general. It plays a prominent role in several branches of Mathematics. As a particular kind of block-triangular matrices, we cite the upper triangular matrices. Group gradings on the associative algebra of upper triangular matrices were classified by A. Valenti and M. Zaicev [63], for arbitrary base field and arbitrary group grading; O. Di Vincenzo, P. Koshlukov, and A. Valenti gave the classification of these gradings up to isomorphism [29]. Also, under some restrictions, A. Valenti and M. Zaicev described the group gradings on the upper block-triangular matrices [64]. The same algebra had its gradings classified up to isomorphism by A. Borges, C. Fidelis and D. Diniz [25].

In this thesis we provide a classification of the group gradings on the algebra of upper triangular matrices, viewed as a Lie algebra and as a Jordan algebra, over an arbitrary field and with an arbitrary group grading. Imposing some conditions, we also obtain a complete result for the upper block-triangular matrices, viewed as Lie and Jordan algebras.

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Algebras with polynomial identities (PI algebras for short) will play a crucial role in this thesis. The theory of PI algebras was initiated, albeit implicitly, in the twenties and thirties, with works by Dehn [28] and Wagner [65]. Later on Albert, Jacobson, Kaplansky and Levitzki started the research on PI algebras in its proper sense. We direct the interested reader to the introductory parts of the monographs [33] and [38] for further details on the development of PI theory until recent years. Given a polynomial in the non-commutative variables $f = f(x_1, \ldots, x_m)$ (usually we will consider non-associative and non-commutative variables), we say that f is a polynomial identity of a given algebra A if $f(a_1, \ldots, a_m) = 0$ for all a_1, …, $a_m \in A$. In this case we say that A is a PI-algebra. An interesting question (but also very hard to solve, and unsolved for most of the algebras) is computing the set $\mathrm{Id}(A)$ of all polynomial identities of a given algebra A. Let us only state that little is known about the set of polynomial identities satisfied by the simple associative algebras. If we consider algebras over an infinite field one can show it is sufficient to consider matrix algebras. But the polynomial identities satisfied by matrix algebras are known only when the size of the matrices is up to two (and for matrices of order two, one adds the restriction that the field must be of characteristic different from 2).

The PI-algebras are a natural generalization of the commutative algebras. Also PI-algebras represent a class of algebras large enough to include important examples (say all finite dimensional algebras), and sufficiently good to be workable. Some deep structure theorems which are true for commutative or finite-dimensional algebras are also true for the PI-algebras, see for example the first chapter of [38] for a glimpse on the structure theory of PI algebras. These are just a few motivations to justify the study of PI-algebras.

One of the most important problems in PI theory is known as the Specht problem: does every algebra have a finite set of polynomial identities, generating its ideal of polynomial identities? In the context of associative algebras in characteristic zero, Kemer gave a positive answer to the question (see [46], or [4]). In his theory, Kemer proved that for any algebra A, there exists a finite-dimensional $\mathbb{Z}_2$-graded algebra B such that $\mathrm{Id}(A) = \mathrm{Id}(G(B))$ where $G(B)$ stands for the so-called Grassmann envelope of B. This result was fundamental in establishing the Specht property for any associative algebra over a field of characteristic zero, thus reinforcing the importance of graded algebras. It is known that even in the associative setting, the Specht property does not hold as long as the field is of positive characteristic. It does hold for ample classes of Lie and Jordan algebras (including all finite dimensional Lie algebras).

As a natural generalization, one can study polynomial identities with additional structure, for example, graded polynomial identities, polynomial identities with derivations, and so on. The graded analog of Kemer's Theorem was obtained independently by E. Aljadeff and A. Kanel-Belov [5], and I. Sviridova [61].

Another natural question is whether the set of polynomial identities can determine uniquely an algebra. This question is false if asked in its generality but it becomes interesting if we restrict to simple (or prime) algebras over an algebraically closed field. A lot of research has been done in this direction but it turns that Razmyslov gave the strongest answer in his monograph [58, Chapter 1, Paragraph 5]. The same question was investigated in the context of graded algebras in several recent papers. However, we prove that the solution in the graded context can be derived from Razmyslov's Theorem. Our argument is simple but it seems that it was unnoticed and overlooked.

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The codimension sequence is an important numerical invariant of a PI algebra. Let P_m denote the vector space of multilinear polynomials in the variables $x_1, \ldots, x_m$, for each $m \in \mathbb{N}$. Then P_m becomes a left S_m-module under the usual permutation action of the symmetric group S_m. It is clearly isomorphic to the regular S_m-module. An easy but extremely useful remark is that in characteristic 0, the polynomial identities of an algebra A are determined by its multilinear identities, that is by the intersections $\mathrm{Id}(A)\cap P_m$. Since $\mathrm{Id}(A)$ is invariant under permutations of the variables we obtain that $\mathrm{Id}(A)\cap P_m$ is a submodule. Thus in order to study the identities of A one can employ the well developed theory of S_m-representations. A big problem arises though: if A is an associative PI algebra then $\mathrm{Id}(A)\cap P_m$ tends to grow very fast with m. That is why one studies the quotient S_m-module $P_m(A)=P_m/(\mathrm{Id}(A)\cap P_m)$ instead. Given a PI-algebra A, we define $c_m(A)=\dim P_m(A)$. The sequence $c_m(A)$ is called the codimension sequence of A. A celebrated theorem due to A. Regev states that the codimension sequence of an associative PI-algebra is exponentially bounded [59]. In other words if A is PI then there exists a constant C such that $c_m(A)\le C^m$ for each m. (In fact if d is the degree of a PI satisfied by A then one can take $C=(d-1)^2$.) We draw the reader's attention to the fact that $\dim P_m = m!$ which grows faster than C^m. As an application of this exponential bound for the codimension sequence, Regev obtained also that the tensor product of two PI-algebras is again a PI-algebra. Thus, from the exponential bound, $0\le c_m(A)\le C^m$ for some constant C, and for every $m\in\mathbb{N}$. Hence one has $0\le \sqrt[m]{c_m(A)}\le C$ for every $m\in\mathbb{N}$. Knowing that in several instances $\sqrt[m]{c_m(A)}$ is a convergent sequence, with an integer limit, Amitsur asked whether it is always true that $\lim \sqrt[m]{c_m(A)}$ exists and is an integer, for every associative PI algebra in characteristic 0.

A positive answer to Amitsur's question was given by A. Giambruno and M. Zaicev, see, for example [38], for the complete account of their important result. The limit $\lim \sqrt[m]{c_m(A)}$ is called the (PI) exponent of A, denoted by $\exp(A)$. This result gave rise to several new branches of research, for example, minimal varieties with respect to their codimensions, algebras with polynomial growth or almost polynomial growth, and so on. It is known that block-triangular matrices are an essential algebra in the study of minimal varieties. In fact this is another evidence of the importance of this algebra. The existence of the graded exponent was settled in the papers by E. Aljadeff, A. Giambruno and D. La Mattina [1], and E. Aljadeff and A. Giambruno [2]. The Amitsur's conjecture was extended and proved in other contexts as well. However, it is known that in the non-associative case, the codimension sequence may grow faster than exponentially, hence the exponent may not even exist. Moreover there exist examples of Lie algebras whose codimensions grow "erratically", that is $3<\liminf \sqrt[m]{c_m(A)}\neq \limsup \sqrt[m]{c_m(A)}<4$.

A related question is the investigation of the asymptotic behaviour of the codimension sequence. Given two functions f, $g\colon \mathbb{N} \to \mathbb{N}$, we denote $f \sim g$ if $\lim_{n\to\infty} f(n)/g(n) = 1$. Regev conjectured that for every PI algebra A there exist constants c and a half integer g such that $c_m(A) \sim cm^g d^m$ where $d = \exp(A)$. This is indeed the case for unitary PI-algebras, as proved by Berele and Regev for finitely generated algebras satisfying some Capelli identity in [21], and by Berele in [20] in the general case of unitary PI-algebras. In [40], the authors proved a weaker version of the previous result for any PI-algebra.

We investigate such problems for the upper triangular matrices, endowed with any G-grading, and viewed as an associative, Lie or Jordan algebra. In the associative case, we are able to compute the asymptotic behaviour of the graded codimension sequence, for any G-grading. As a consequence, we obtain that the graded exponent always coincides with n, the fixed size of the upper triangular matrix algebra.

For the Lie case, we provide a lower and an upper bound for the asymptotic behaviour of the graded codimension sequence for any elementary grading. In particular, we prove that the graded exponent for these gradings is always $n-1$. For the other gradings in the Lie case (which are not elementary), we compute the graded exponent; it coincides with $n-1$. Finally in the Jordan case we prove that the graded exponent for any grading is always equal to n.

This thesis is divided as follows. Chapter 1 contains preliminary results that are needed in our exposition. While it is neither complete nor exhaustive we do recommend, for the parts concerning gradings, the recent monograph [35]. For a thorough treatment of PI theory the monographs [33] and [38] are recommended.

Chapter 2 contains the investigation of a combinatorial machinery concerning commutators in the free associative algebra. The results from Chapter 2 are used throughout the thesis. These have been submitted for publication in [68].

Chapter 3 deals with the group gradings on upper triangular matrices, viewed as Lie and Jordan algebras. We classify the gradings in both cases. In contrast with the associative case here it turns out there appear non-elementary ones. We call these gradings *Type II* ones as they are not restriction of associative gradings. The results of this chapter were published in [71, 72, 73].

In chapter 4 we investigate group gradings on the algebra of block-triangular matrices. The first part is published in [74]. The last part of the work has been submitted for publication [69].

In chapter 5, we study the asymptotic behaviour of the graded codimension sequence of the gradings on upper triangular matrices. The results of this chapter were published in [70].

Finally, the last chapter investigates the problem of determining an algebra by its polynomial identities. The results have been accepted for publication [67].

Chapter 1
Preliminaries

"As coisas mais belas não são feitas da noite para o dia."
J. Honda

In this chapter we define some notions, and state some important results concerning graded algebras. We let G be an arbitrary group with multiplicative notation and neutral element 1, and A is an arbitrary (not necessarily associative) algebra.

1. Graded algebras

Definition 1.1.1. We say that A is G-graded if there exist vector subspaces $\{A_g\}_{g\in G}$ where some of the A_g can be zero, such that

$$A = \bigoplus_{g\in G} A_g,$$

and $A_g A_h \subset A_{gh}$, for all g, $h \in G$. The subspaces A_g are called *homogeneous*, and a non-zero element $x \in A_g$ is called homogeneous of degree g. We denote $\deg_G x = g$, or $\deg x = g$, whenever no ambiguity can arise.

A vector subspace $S \subset A$ is called graded if $S = \bigoplus_{g\in G} A_g \cap S$. Equivalently, S is graded if and only if given any $s \in S$, if $s = s_1 + \ldots + s_m$ is the decomposition of s into homogeneous components, then $s_1, \ldots, s_m \in S$.

If $I \subset A$ is an ideal and a graded subspace, then we call I a *graded ideal.* In this case the quotient A/I inherits from A a natural structure of G-graded algebra. We say that A is *graded-simple* if it does not admit non-trivial graded ideals.

Let $B = \bigoplus_{g\in G} B_g$ be any G-graded algebra, and let $f\colon A \to B$ be any map. We say that f is a homomorphism of G-graded algebras (or a homomorphism, for short) if f is a homomorphism of algebras and $f(A_g) \subset B_g$ for every $g \in G$. In the special case where f is an isomorphism of algebras, then we call f an isomorphism of G-graded algebras (or just an isomorphism, for short), and we say that A and B are G-graded isomorphic (or isomorphic, for short). We use similar names when f is an automorphism.

The isomorphism theorem holds in the graded context. If $f\colon A \to B$ is a homomorphism of G-graded algebras, then it is easy to verify that $\operatorname{Ker} f$ is a graded ideal of A, $f(A)$ is a graded subalgebra of B, and $A/\operatorname{Ker} f$ and $f(A)$ are G-graded isomorphic.

Now assume that B is an ungraded algebra and let $f\colon A \to B$ be an isomorphism of ordinary algebras. Then f induces a G-grading on $B = \bigoplus_{g\in G} B_g$, if we put $B_g = f(A_g)$. Moreover, endowed with this grading, A and B are isomorphic, and f is a graded isomorphism between them.

Sometimes it is convenient to name a grading. Let $\Gamma\colon A = \oplus_{g\in G} A_g$ and $\Gamma'\colon \oplus_{g\in G} A'_g$ be two G-gradings on A. We say that Γ' is a *refinement* of Γ if for each $g' \in G$, there exists $g \in G$ such that $A'_{g'} \subset A_g$. In this case, we say that Γ is a *coarsening* of Γ'.

Let H be any group and let $\alpha\colon G \to H$ be a homomorphism of groups. Then α induces a H-grading, say $A = \bigoplus_{h\in H} A'_h$, on the G-graded algebra A if we define

$$A'_h = \sum_{g\in\alpha^{-1}(h)} A_g.$$

The H-grading is called the coarsening of Γ induced by the homomorphism α.

Finally, we state two simple results. The next proposition is easy to prove, and it will be essential in various steps of our main proofs.

Proposition 1.1.2. *Let A be a not necessarily associative algebra and S, $I \subset A$ graded subspaces. Then the left annihilator $\operatorname{Ann}^l_S(I) = \{s \in S \mid sb = 0$, for every $b \in I\}$ is a graded subspace.* □

Similar results hold for right and two-sided annihilators.

It is a standard and well known fact that in every graded associative unital algebra, the unit element is homogeneous of degree 1. In analogy, we have the following result.

Proposition 1.1.3. *Let A be an associative graded algebra and assume that A admits a left unit. Then A admits a homogeneous left unit.*

Proof. Let e be the left unit and write $e = x_1 + \ldots + x_m$ as a sum of homogeneous components. For every homogeneous $y \in A$, one has $y = ey = x_1 y + \cdots + x_m y$. Since the right-hand side is a combination of homogeneous elements, one must have $y = x_i y$ for some i. Necessarily $\deg x_i = 1$. In particular, $y = x_i y$, for each homogeneous $y \in A$, hence x_i is a homogeneous left unit of A. □

Example 1. Consider the algebra

$$A = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix},$$

with the $\mathbb{Z}_2$-grading given by $\deg e_{11}+e_{12} = \bar{0}$ and $\deg e_{12} = \bar{1}$. Note that $e_{11} = (e_{11}+e_{12})-e_{12}$ is a left unit of A, and $e_{11}+e_{12}$ is a homogeneous left unit of A.

2. Duality between gradings and actions

Let K be an algebraically closed field of characteristic zero, and let G be a finitely generated abelian group. Let $\hat{G} = \{\varphi\colon G \to K^\times \mid \varphi \text{ is a group homomorphism}\}$ be the group of characters of G. Since G is finitely generated, $\hat{G}$ is an algebraic group. Fix a finite-dimensional algebra A. Thus the automorphism group $\mathrm{Aut}(A)$ is an algebraic group as well. In this setting G-gradings on A are equivalent to actions of $\hat{G}$ on A.

Fix a G-grading on $A = \bigoplus_{g\in G} A_g$. For each $\chi \in \hat{G}$, define the map $\psi_\chi\colon A \to A$ linearly by $\psi_\chi(a_g) = \chi(g)a_g$, for each $a_g \in A_g$. Then ψ_χ is readily seem to be a graded automorphism of A. Thus one obtains a homomorphism of algebraic groups $\hat{G} \to \mathrm{Aut}(A)$.

Conversely, assume a homomorphism of algebraic groups $\hat{G} \to \mathrm{Aut}(A)$ is given. Since G is abelian, and the base field is algebraically closed of characteristic zero, the action of $\hat{G}$ is reductive. So A decomposes as a direct sum of irreducible $\hat{G}$-invariant subspaces. These subspaces are indexed by $\mathfrak{X}(\hat{G}) \simeq G$ where

$$\mathfrak{X}(G) = \{\varphi\colon G \to K^\times \mid \varphi \text{ is a homomorphism of algebraic groups}\}.$$

Thus, we obtain a G-grading on A.

3. Graded polynomial identities

Let G be any group, and define $X^G = \{x_i^{(g)} \mid i \in \mathbb{N}, g \in G\}$. The (absolutely) free G-graded algebra is the (absolutely) free algebra freely generated over K by the set of variables X^G, and it will be denoted $K\langle X^G\rangle$.

Let A be a G-graded algebra. For any map $\psi\colon X^G \to A$, such that $\psi(x_i^{(g)}) \in A_g$, there exists unique homomorphism of G-graded algebras $\bar{\psi}\colon K\langle X^G\rangle \to A$ extending ψ. Hence we can make graded evaluations. We say that $f(x_1^{(g_1)}, \ldots, x_m^{(g_m)}) \in K\langle X^G\rangle$ is a graded polynomial identity for A if $f(a_1, \ldots, a_m) = 0$, for all $a_1 \in A_{g_1}$, …, $a_m \in A_{g_m}$.

Let $P_1^G = \mathrm{Span}\{x_1^{(g)} \mid g \in G\}$, and for $m \in \mathbb{N}$, $m > 1$, define P_m^G inductively by

$$P_m^G = \mathrm{Span}\{p_{m_1}(x_{\sigma(1)}^{(g_1)}, \dots, x_{\sigma(m_1)}^{(g_{m_1})})p_{m_2}(x_{\sigma(m_1+1)}^{(g_{m_1+1})}, \dots, x_{\sigma(m)}^{(g_m)})\}.$$

Here m_1, $m_2 \in \mathbb{N}$, $m_1 + m_2 = m$, and $p_{m_1} \in P_{m_1}^G$, $p_{m_2} \in P_{m_2}^G$, $\sigma \in \mathcal{S}_m$.

The elements of P_m^G are called *graded multilinear polynomials*.

Chapter 2
Combinatorial Properties of Lie commutators

"Coisas não sérias não precisam ser sérias..."
E. Hitomi

1. Introduction.

For $m \in \mathbb{N}$, let $I_m = \{1, 2, \ldots, m\}$ and let $\mathcal{S}_m$ denote the set of all bijections of I_m. We assume that the elements of $\mathcal{S}_m$ act on the left-hand side on I_m, that is if σ, $\tau \in \mathcal{S}_m$ then $\sigma \circ \tau$ stands for applying τ first and then σ.

We are interested in studying the subset $\mathscr{T}_m$ of $\mathcal{S}_m$ where given an associative algebra A and given elements x_1, x_2, $\ldots$, $x_m \in A$,

$$[x_1, x_2, \ldots, x_m] = \sum_{\sigma \in \mathscr{T}_m} \pm x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(m)}$$

where the long commutator is left normed, that is, we define

$$\begin{aligned}
[x_1, x_2] &= x_1 x_2 - x_2 x_1, \\
[x_1, x_2, \ldots, x_m] &= [[x_1, x_2, \ldots, x_{m-1}], x_m], \quad \text{for } m > 2.
\end{aligned}$$

The set $\mathscr{T}_m$, its properties, and relations with Lie algebras have been extensively investigated, see for example survey [23].

We will state some equivalent definitions of $\mathscr{T}_m$, and then we will derive properties concerning actions by permutations on sequences restricted to $\mathscr{T}_m$. The main result of this chapter, Theorem 2.3.2 is essential in describing isomorphism classes of group gradings on the upper triangular matrices, as Lie algebras. Also, the equivalent descriptions of the set $\mathscr{T}_m$ will be helpful in computing the asymptotics of codimension growth of gradings on the same algebras.

2. Some equivalences.

Following [23], let

$$\mathscr{T}_m = \{\sigma \in \mathcal{S}_m \mid \sigma(1) > \cdots > \sigma(t) = 1, \sigma(t+1) < \cdots < \sigma(m)\}.$$

There are several manners (all of them equivalent) to define the above set.

Lemma 2.2.1 ([23])**.** *Let $\sigma \in \mathcal{S}_m$. The following conditions are equivalent:*

(1) *$\sigma \in \mathscr{T}_m$;*

(2) *there exists r such that: $\sigma(j) > \sigma(j+1)$ if and only if $1 \le j \le r$;*

(3) *there exists $j_1 > j_2 > \cdots > j_r > 1$ such that*

$$\sigma = (j_r \dots 1) \cdots (j_1 \dots 1).$$

Moreover, $j_i = \sigma(i)$ for $i = 1, 2, \dots, r$.

Also given an associative algebra A and $x_1, x_2, \dots, x_m \in A$,

$$[x_1, x_2, \dots, x_m] = \sum_{\sigma \in \mathscr{T}_m} (-1)^{\sigma^{-1}(1)-1} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(m)}.$$

□

The last assertion of the previous lemma says that the set $\mathscr{T}_m$ is indeed the set we want to study.

If we write the elements of $\mathcal{S}_m$ using the two row notation, like

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & m \\ \sigma(1) & \sigma(2) & \dots & \sigma(m) \end{pmatrix},$$

then we can easily recognize if σ is an element of $\mathscr{T}_m$ or not, by definition. Also, it is easy to see that for every $r = 1, 2, \dots, m$, the numbers $1, 2, \dots, r$ in the second row appear together, in “only one block".

We draw the reader's attention that in general $\mathscr{T}_m$ is not even a subsemigroup of $\mathcal{S}_m$.

Example 2. Consider the permutations

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad \sigma_2 \circ \sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

Then, using the remark above, $\sigma_1, \sigma_2 \in \mathscr{T}_3$ but $\sigma_2 \circ \sigma_1 \notin \mathscr{T}_3$. Hence $\mathscr{T}_m$ is not in general a subgroup of $\mathcal{S}_m$.

It is not hard to derive the following equivalence:

Lemma 2.2.2. *Let $\sigma \in \mathcal{S}_m$. Then $\sigma \in \mathscr{T}_m$ if and only if σ satisfies the following condition: there exists an integer t, with $1 \le t \le m$, such that*

(i) $\sigma(t) = 1$;

(ii) *for every positive integers* k_1, $k_2 \geq 0$ *such that* $k_1 + k_2 < m$ *and*

$$\{\sigma(t-k_1), \sigma(t-k_1+1), \ldots, \sigma(t+k_2)\} = \{1, 2, \cdots, k_1+k_2+1\},$$

it holds that either

- $t - k_1 - 1 \geq 1$ *and* $\sigma(t-k_1-1) = k_1+k_2+2$ *or*
- $t + k_2 + 1 \leq m$ *and* $\sigma(t+k_2+1) = k_1+k_2+2$.

□

We denote $\mathscr{T}_m^{(t)} = \{\sigma \in \mathscr{T}_m \mid \sigma(t) = 1\}$.

The previous lemma is useful for applying inductive arguments concerning the elements of $\mathscr{T}_m$.

The following lemma is convenient for our applications:

Lemma 2.2.3. *Let* r_1, r_2, $\ldots$, r_m *be strictly upper triangular matrix units such that their associative product* $r_1 r_2 \cdots r_m \neq 0$. *Then*

(i) $r_{\sigma^{-1}(1)} r_{\sigma^{-1}(2)} \cdots r_{\sigma^{-1}(m)} \neq 0$ *if and only if* $\sigma = 1$;

(ii) $[r_{\sigma^{-1}(1)}, r_{\sigma^{-1}(2)}, \ldots, r_{\sigma^{-1}(m)}] \neq 0$ *if and only if* $\sigma \in \mathscr{T}_m$.

Proof. Let r_1, r_2, $\ldots$, r_m be strictly upper triangular matrices such that their associative product $r_1 r_2 \cdots r_m \neq 0$. Then $r_{\sigma(1)} r_{\sigma(2)} \cdots r_{\sigma(m)} \neq 0$ if and only if $\sigma = 1$, since every $r_l = e_{i_l j_l}$ and $i_l < j_l$. Also, note that $r_l r_k \neq 0$ if and only if $j_l = i_k$.

Now consider the Lie commutator. Let $\sigma \in \mathcal{S}_m$ be such that

(1) $$[r_{\sigma^{-1}(1)}, r_{\sigma^{-1}(2)}, \ldots, r_{\sigma^{-1}(m)}] \neq 0$$

Assume $t = \sigma^{-1}(1)$. Since $[r_{\sigma^{-1}(1)}, r_{\sigma^{-1}(2)}] \neq 0$, it follows that either $\sigma^{-1}(2) = t+1$ or $\sigma^{-1}(2) = t-1$. Iterating this and using induction we obtain that $\sigma \in \mathscr{T}_m$, by Lemma 2.2.2. The same idea can be used to prove the converse, that is (1) holds for each $\sigma \in \mathscr{T}_m$. □

We remark that if $m_1 \leq m_2$ we can consider $\mathcal{S}_{m_1}$ as a subgroup of $\mathcal{S}_{m_2}$ in the usual manner, that is the elements of $\mathcal{S}_{m_1}$ fix all symbols $t > m_1$. Using the same identification, we can consider $\mathscr{T}_{m_1}$ as a subset of $\mathscr{T}_{m_2}$. By the definition we obtain an interesting consequence.

Corollary 2.2.4. *Let* $\sigma \in \mathscr{T}_m$ *and* $m_1 = \sigma(1)$. *Then* $\sigma \in \mathscr{T}_{m_1}$ *(that is,* $\sigma(t) = t$ *for* $t > m_1$*).*

Proof. A direct consequence of Lemma 2.2.1, item 3. □

Now we consider the following special subset of permutations in $\mathscr{T}_m$:

Definition 2.2.5. For every $i = 1, 2, \ldots, m$, the i-reverse permutation is given by

$$\tau_i = \begin{pmatrix} 1 & 2 & \ldots & i-1 & i & i+1 & i+2 & \ldots & m \\ i & i-1 & \ldots & 2 & 1 & i+1 & i+2 & \ldots & m \end{pmatrix}.$$

We observe that, for every $i = 1, 2, \ldots, m$ one has $\tau_i \in \mathscr{T}_m^{(i)}$ and $\tau_i^2 = 1$. Moreover, $\tau_{i-1} \circ \tau_i = (i \quad i-1 \quad \ldots \quad 1)$. So, by Lemma 2.2.1.(3), we obtain:

Corollary 2.2.6. $\mathscr{T}_m = \{\tau_2^{i_2} \circ \cdots \circ \tau_m^{i_m} \mid i_2, \ldots, i_m \in \{0, 1\}\}$. □

Note that it is easy to obtain the decomposition of elements of $\mathscr{T}_m$ into product of τ_i. Corollary 2.2.6 also tells us when the product of two elements of $\mathscr{T}_m$ still belongs to $\mathscr{T}_m$.

3. Action on sequences.

Let X be any set. Then we have a left action of $\mathcal{S}_m$ on the elements $s = (s_1, s_2, \ldots, s_m) \in X^m$ permuting the order of the elements:

$$\sigma(s_1, s_2, \ldots, s_m) = (s_{\sigma^{-1}(1)}, s_{\sigma^{-1}(2)}, \ldots, s_{\sigma^{-1}(m)}), \quad \sigma \in \mathcal{S}_m.$$

We are interested in the following notion.

Definition 2.3.1. Given two sequences s, $s' \in X^m$, we say that s and s' are mirrored if $\mathscr{T}_m s = \mathscr{T}_m s'$.

Equivalently, s and s' are mirrored if and only if for all τ', $\sigma \in \mathscr{T}_m$, we can find τ, $\sigma' \in \mathscr{T}_m$ such that $\sigma s = \sigma' s'$ and $\tau' s' = \tau s$.

Notation. Given $s \in X^m$, we denote by $\operatorname{rev} s := \tau_m s$, the reverse sequence of s.

We give two examples of mirrored sequences.

Example 3. The trivial example: for any $s \in X^m$, s and s are mirrored.

Example 4. If s, $s' \in X^m$ are mirrored then s and $\operatorname{rev} s'$ are also mirrored.

Proof. Let $\sigma \in \mathscr{T}_m$. Then there exists $\sigma' \in \mathscr{T}_m$ such that $\sigma s = \sigma' s'$, since s and s' are mirrored. Also $\sigma'' := \sigma' \tau_m \in \mathscr{T}_m$, by Corollary 2.2.6, and

$$\sigma'' \operatorname{rev} s' = \sigma' \tau_m \tau_m s' = \sigma' s' = \sigma s.$$

Conversely, given $\tau' \in \mathscr{T}_m$, we have $\tau' \tau_m \in \mathscr{T}_m$ and there exists $\tau \in \mathscr{T}_m$ such that $\tau s = \tau' \tau_m s' = \tau' \operatorname{rev} s'$. Hence s and $\operatorname{rev} s'$ are mirrored. □

We will prove that these two examples give the only ways to produce mirrored sequences. A precise statement of our main result is as follows.

Theorem 2.3.2. *Let X be any set, $m \in \mathbb{N}$, and let s, $s' \in X^m$. Then $\mathscr{T}_m s = \mathscr{T}_m s'$ if and only if $s = s'$ or $s = \operatorname{rev} s'$.*

An equivalent statement is the following:

Corollary 2.3.3. *Let X be any set and s, $s' \in X^m$. Then $s = s'$ or $s = \operatorname{rev} s'$ if and only if for every σ, $\tau' \in \mathscr{T}_m$, we can find σ', $\tau \in \mathscr{T}_m$ such that $\sigma s = \sigma' s'$ and $\tau s = \tau' s'$.* □

We have already proved one of the implications of Theorem 2.3.2 in the two previous examples. The next section is dedicated exclusively to prove the converse.

4. Proof of Theorem 2.3.2.

We fix some notations.

Notation. We denote $I_m = \{1, 2, \ldots, m\}$. Note that each $s \in X^m$ can be viewed as a function $s\colon I_m \to X$, and for each $\sigma \in \mathcal{S}_m$, $\sigma s = s \circ \sigma^{-1}$ (equality of functions). Given integers $0 < m_2 \leq m$, we denote by $I_{-m_2}^{(m)} = \{m, m-1, \ldots, m-m_2+1\}$, and $I_0^{(m)} = \emptyset$ (the last m_2 elements of I_m).

First we deduce several useful properties.

Definition 2.4.1. Let s, $s'\colon I_m \to X$, and set $A = s(1)$. A coincidence of (s, s') is a pair (m_1, m_2) where $m_1 > 0$ and $m_2 \geq 0$ are integers satisfying:

(1) $s(i) = s'(i) = A$, for all $i = 1, 2, \ldots, m_1$, and $i = m, m-1, \ldots, m-m_2+1$,

(2) $s(m_1+1) \neq A$, $s'(m_1+1) \neq A$, $s(m-m_2) \neq A$ and $s'(m-m_2) \neq A$.

In this context, we denote $m' = m_1 + m_2$, $I_m^{(-)} = I_{m_1} \cup I_{-m_2}^{(m)}$ and $I'_m = I_m \setminus I_m^{(-)}$.

Lemma 2.4.2. *Let m_1, $m_2 \geq 0$ with $m_1 + m_2 \leq m$. Then $\sigma \in \mathscr{T}_m$ satisfies:*

- $\sigma(m-i+1) = m-i+1$, *for* $i = 1, 2, \ldots, m_2$,
- $\sigma(i) = m - m_2 - i + 1$, *for* $i = 1, 2, \ldots, m_1$,

if and only if $\sigma \in \mathscr{T}_{m-m_1-m_2} \circ \tau_{m-m_2}$.

Proof. Let $\sigma \in \mathscr{T}_{m-m_1-m_2} \circ \tau_{m-m_2}$, then clearly $\sigma(m-i+1) = m-i+1$, for $i = 1, 2, \ldots, m_2$. Also, writing $\sigma = \sigma' \circ \tau_{m-m_2}$, we have $\sigma(1) = \sigma'(m-m_2) = m - m_2$, since $\sigma' \in \mathscr{T}_{m-m_1-m_2}$, and using the same idea, we see that σ satisfies the second condition.

We can prove the converse using the same idea. □

Example 5. It is easy to represent such a permutation $\sigma \in \mathscr{T}_{m-m_1-m_2} \circ \tau_{m-m_2}$: the last m_2 entries of σ are:

$$\sigma = \begin{pmatrix} \ldots & m-m_2+1 & \ldots & m-1 & m \\ \ldots & m-m_2+1 & \ldots & m-1 & m \end{pmatrix}$$

and the first m_1 entries are:

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & m_1 & \ldots \\ m_0 & m_0 - 1 & \ldots & m_0 - m_1 +' & \ldots \end{pmatrix},$$

where $m_0 = m - m_2$.

The following example serves as a motivation for considering the above permutations and the notion of coincidence:

Example 6. Let (m_1, m_2) be a coincidence for (s, s'), and let $A = s(1)$. Then s and s' are written as:

$$\begin{aligned} s &= (\underbrace{A, \ldots, A}_{m_1 \text{ times}}, B_1, \ldots, B_l, \underbrace{A, \ldots, A}_{m_2 \text{ times}}) \\ s' &= (A, \ldots, A, B'_1, \ldots, B'_l, A, \ldots, A), \end{aligned}$$

where $B_1 \neq A$, $B'_1 \neq A$, $B_l \neq A$ and $B'_l \neq A$. A permutation $\sigma \in \mathscr{T}_{m-m'} \circ \tau_{m-m_2}$ will act like:

$$\sigma s = (C_1, \ldots, C_l, \underbrace{A, \ldots, A}_{m' \text{ times}}),$$

where $C_l \neq A$.

Let $s\colon I_m \to X$. Then there is a coincidence (m_1, m_2) of (s, s), and we can consider the restriction $s_0 = s\mid_{I'_m}$. Denote $\varphi\colon n \in I_{m-m'} \mapsto n + m_1 \in I'_m$, and let

$$\mathscr{T}'_m = \left\{\varphi \circ \sigma \circ \varphi^{-1} \mid \sigma \in \mathscr{T}_{m-m'}\right\}.$$

For every $\sigma' \in \mathscr{T}'_m$, we can define an element $\sigma \in \mathscr{T}_{m-m'} \circ \tau_{m-m_2}$ by

$$\sigma(i) = \begin{cases} \varphi^{-1} \circ \sigma(i)', & \text{if } i \in I'_m, \\ m - i + 1 - m_2, & \text{if } i \in I_{m_1}, \\ i, & \text{if } i \in I^{(m)}_{-m_2}. \end{cases}$$

Hence we can see $\mathscr{T}'_m$ as a subset of $\mathscr{T}_{m-m'} \circ \tau_{m-m_2}$. In particular, both sets have the same cardinality, so we have equality of sets. It is also easy to see what an action of $\sigma \in \mathscr{T}_{m-m'} \circ \tau_{m-m_2}$ on s_0 will be.

Example 7. Given $s : I_m \to X$, write

$$s = (\underbrace{A, \ldots, A}_{m_1 \text{ times}}, B_1, \ldots, B_l, \underbrace{A, \ldots, A}_{m_2 \text{ times}}),$$

with $B_1 \neq A$, and $B_l \neq A$. An element $\sigma \in \mathscr{T}_{m-m'} \circ \tau_{m-m_2}$ acts like

$$\sigma s = (C_1, \ldots, C_l, A, \ldots, A),$$

with $C_l \neq A$. Using this notation, we will have a coincidence (m_1, m_2) of (s, s), more precisely $s_0 = s\mid_{I'_m} = (B_1, \ldots, B_l)$ and $\sigma s_0 = (C_1, \ldots, C_l)$.

Lemma 2.4.3. *Using the above notation, if $\sigma \in \mathscr{T}_m$ satisfies*

$$\{\sigma^{-1}(m), \sigma^{-1}(m-1), \ldots, \sigma^{-1}(m-m'+1)\} = \{1, 2, \ldots, m_1, m, m-1, \ldots, m-m_2+1\}$$

then there exists $\tau \in \mathscr{T}_{m-m'} \circ \tau_{m-m_2}$ such that $\sigma s = \tau s$.

Proof. Note that, in this case, $\varphi \circ \sigma(I'_m) \subset I'_m$, so we can see σ as an element of $\mathscr{T}'_m$, and therefore we can construct one such $\tau \in \mathscr{T}_{m-m'} \circ \tau_{m-m_2}$. □

Lemma 2.4.4. *Let s, $s'\colon I_m \to X$ be mirrored and assume that there is a coincidence (m_1, m_2) of (s, s'). Then $s_0 = s\mid_{I'_m}$ and $s'_0 = s'\mid_{I'_m}$ are mirrored.*

Proof. For every $\sigma \in \mathscr{T}_{m-m'} \circ \tau_{m-m_2}$, we can find $\tau \in \mathscr{T}_m$ such that $\sigma s = \tau s'$. We consider $\sigma \in \mathscr{T}'_m$, hence it is sufficient to prove that τ is an element of $\mathscr{T}'_m$. Using the previous lemma, assume that there is i such that $i \in \{m, m-1, \dots, m-m'+1\}$ and $\tau^{-1}(i) = m_1 + 1$ (or $\tau^{-1}(i) = m - m_2$). By the definition of coincidence, we know that $s'(m_1+1) \neq A$ and $s'(m - m_2) \neq A$, where $A = s(1)$. Also, by the choice of σ, we know that $\sigma^{-1}(i) \in \{1, 2, \dots, m_1, m, m-1, \dots, m-m_2+1\}$. Therefore $\sigma s(i) = A$ (see example 6). Thus $\tau s'(i) \neq A = \sigma s(i)$, a contradiction. This proves that we can find $\tau' \in \mathscr{T}'_m$ with $\tau' r' = \tau r$, and in particular, $\sigma s_0 = \tau' s'_0$. Since a coincidence (s, s') is also a coincidence of (s', s), we can repeat the argument and prove that s_0 and s'_0 are mirrored. □

Definition 2.4.5. Let $s\colon I_m \to X$ and $w\colon I_d \to X$ with $d \leq m$. Let

$$\mathscr{O}(s, w) = \{(\sigma, i) \in \mathscr{T}_m \times I_m \mid \sigma s(i+j) = w(1+j), \quad j = 0, 1, 2, \dots, d-1\},$$

and denote $n\colon (\sigma, i) \in \mathscr{T}_m \times I_m \mapsto i \in I_m$, and finally define

$$o_w(s) = \begin{cases} \min\{n(x) \mid x \in \mathscr{O}(s,w)\}, \text{ if } \mathscr{O}(s,w) \neq \emptyset, \\ \infty, \text{ otherwise.} \end{cases}$$

Lemma 2.4.6. *Let s, $s'\colon I_m \to X$. If there exists $w\colon I_d \to X$ such that $o_w(s) \neq o_w(s')$, then s and s' cannot be mirrored.*

Proof. Assume $i = o_w(s) < o_w(s')$. In this case, there will exist $\sigma \in \mathscr{T}_m$ such that σs has i entries, followed by the entries of w. However it is impossible to get such $\sigma' \in \mathscr{T}_m$ satisfying the same property, since in this case we would obtain $o_w(s') \leq i$, and in particular, $\sigma' s' \neq \sigma s$, for all $\sigma' \in \mathscr{T}_m$. □

Example 8. Let $s = (A, A, B, C, D)$ and $s' = (A, B, C, D, A)$. Then if $w = (A, A)$, we have $o_w(s) = 1$ and $o_w(s') = 4$, hence s and s' can not be mirrored. This can also be seen directly: let $\sigma = id \in \mathscr{T}_m$, then there is no $\sigma' \in \mathscr{T}_m$ such that $\sigma' s' = \sigma s = s$.

Notation.

(i) Given $w_1\colon I_{d_1} \to X$ and $w_2\colon I_{d_2} \to X$, we denote by $w = (w_1, w_2)$ the sequence $w : I_{d_1+d_2} \to X$ defined by

$$w(i) = \begin{cases} w_1(i), \text{ if } 1 \leq i \leq d_1, \\ w_2(i - d_1), \text{ if } i > d_1. \end{cases}$$

(ii) Analogously we define $(w_1, w_2, \dots, w_p)$.

(iii) Given $A \in X$ and $d \in \mathbb{N}$ a positive integer, we denote by $A_d\colon I_d \to X$ the constant sequence $A_d(1) = \cdots = A_d(d) = A$.

We focus now on the special case where $X = \{A, B\}$ has exactly two symbols.

Definition 2.4.7. Let $s : I_m \to X = \{A, B\}$. Let $n_1 \geq 0$ be the largest integer such that

$$s(1) = s(2) = \cdots = s(n_1) = A,$$

and, for this n_1, let $n_2 > 0$ be the largest integer such that

$$s(n_1 + 1) = s(n_1 + 2) = \cdots = s(n_1 + n_2) = B.$$

Continuing this process, we obtain the sequence $\Sigma(s) = (n_1, n_2, \dots, n_{2t-1}, n_{2t})$ where we can have $n_1 = 0$ and we can have $n_{2t} = 0$. We call it the spectrum sequence of s.

Definition 2.4.8. Let $s\colon I_m \to X = \{A, B\}$. For $i \in \{1, 2, \dots, 2t\}$ and $j \in \mathbb{N} \cup \{0\}$, let $\Sigma(s)(l) = 0$, for $l \notin \{1, 2, \dots, 2t\}$, and let

$$e_i^{(j)}(s) = \begin{cases} \Sigma(s)(i), \text{ if } j = 0, \\ \Sigma(s)(i + j) + \Sigma(s)(i - j), \text{ if } j > 0. \end{cases}$$

Furthermore we define $m_A^{(1)}(s) = \max\{\Sigma(s)(2i + 1) \mid i = 0, 1, 2, \dots\}$, and $I_A^{(1)}(s) = \{2i + 1 \mid \Sigma(s)(2i + 1) = m_A^{(1)}(s)\}$, and inductively for $i > 1$,

$$m_A^{(i)}(s) = \begin{cases} \max\{e_l^{(i-1)}(s) \mid l \in I_A^{(i-1)}(s)\}, \text{ if } I_A^{(i-1)}(s) \neq \emptyset, \\ 0, \text{ otherwise} \end{cases},$$

$$I_A^{(i)}(s) = \{l \in I_A^{(i-1)}(s) \mid e_l^{(i-1)}(s) = m_A^{(i)}(s) \text{ and } m_A^{(i)}(s) > 0\}.$$

We perform analogous constructions for the symbol B.

Lemma 2.4.9. *Given s, $s'\colon I_m \to X = \{A, B\}$, if there exists i such that $m_A^{(i)}(s) \neq m_A^{(i)}(s')$ then s and s' cannot be mirrored.*

Proof. Assume that $i \in \mathbb{N}$ is such that $m_A^{(j)}(s) = m_A^{(j)}(s')$, for $j < i$, and $m_A^{(i)}(s) > m_A^{(i)}(s')$. Let

$$w = (A_{n_1}, B_{n_2}, \dots, C_{n_i}),$$

where $n_j = m_A^{(j)}(s)$, for all $j = 1, 2, \dots, i$, $C = A$ if i is odd and $C = B$ if i is even. Then $o_w(s) = 1 \neq \infty = o_w(s')$, hence s and s' are not mirrored. $\square$

An intuitive approach for the numbers $m_A^{(i)}(s)$ is the following. The largest sequence of consecutive A's is $m_A^{(1)}(s)$. Along these maximum number $m_A^{(1)}(s)$, we see the number of B's after and before the sequence of A's, and the largest number of B's is denoted by $m_A^{(2)}(s)$. We continue inductively.

Example 9. Let $s = (A_3, B_3, A_3, B_1)$ and $s' = (A_3, B_1, A_3, B_3)$. Then $m_A^{(1)}(s) = m_A^{(1)}(s') = 3$, $m_A^{(2)}(s) = m_A^{(2)}(s') = 4$, $m_A^{(3)}(s) = m_A^{(3)}(s') = 3$, $m_A^{(j)}(s) = m_A^{(j)}(s') = 0, \forall j > 3$. But we have

$$m_B^{(2)}(s) = 6 \neq 3 = m_B^{(2)}(s'),$$

hence s and s' are not mirrored.

Let us consider once again $s\colon I_m \to X = \{A, B\}$, and the respective spectrum sequence $\Sigma(s) = (n_1, n_2, \ldots, n_{2t-1}, n_{2t})$. Note that

$$I_A^{(1)}(s) \supset I_A^{(2)}(s) \supset \cdots,$$

and that there is $n \in \mathbb{N}$ such that $I_A^{(n)}(s) \neq \emptyset$ and $I_A^{(j)}(s) = \emptyset$ for every $j > n$. In this case, if $i \in I_A^{(n)}(s)$ then this entry satisfies the following condition: $e_i^{(n-1)}$ is either the first non-zero entry of $\Sigma(s)$, or the last non-zero entry of $\Sigma(s)$ or else the sum of the first and the last non-zero entries of $\Sigma(s)$. Note that the last possibility happens if and only if A appears "in the middle" if we consider the spectrum sequence $\Sigma(s)$.

Now write $\Sigma(s) = (n_1, n_2, \ldots, n_{2t-1}, n_{2t})$ and assume $n_1 > 0$ (we can do it renaming A and B, if necessary). Also, define (the "last entry")

$$n_l = \begin{cases} n_{2t-1}, & \text{if } n_{2t} = 0, \\ 0, & \text{otherwise.} \end{cases}$$

We use the analogous notation for a given $s'\colon I_m \to X = \{A, B\}$. Assume that s and s' are mirrored. Then $s' = \sigma s$ for some $\sigma \in \mathscr{T}_m$, and $s = \sigma' s'$ for some $\sigma' \in \mathscr{T}_m$. Since $\sigma^{-1}(m)$, $\sigma'^{-1} \in \{1, m\}$, we can assume that (by Example 4, changing s' with $\operatorname{rev} s'$ if necessary) $n'_1 > 0$.

Since s and s' are mirrored, we can look at the permutations in $\mathscr{T}_{m-n_1-n_l} \circ \tau_{m-n_l}$ and conclude that necessarily $n_1 + n_l = n'_1 + n'_l$. Moreover, since $m_C^{(i)}(s) = m_C^{(i)}(s')$ for all i and for all $C \in \{A, B\}$, we obtain necessarily (by the observation above) $n_i = n'_j$, for at least one pair of i, $j \in \{1, l\}$. As a consequence, using these two equations, we obtain the following

Lemma 2.4.10. *If s, $s' \colon I_m \to X = \{A, B\}$ are mirrored then there is a coincidence in (s, s') or in $(s, \operatorname{rev} s')$.* □

Now we use induction in order to prove the following

Proposition 2.4.11. *Let s, $s' \colon I_m \to X = \{A, B\}$ be mirrored. Then $s = s'$ or $s = \operatorname{rev} s'$.*

Proof. We can assume that (m_1, m_2) is a coincidence for (s, s'), by the previous lemma, changing s' to $\operatorname{rev} s'$, if necessary. Moreover we can consider $s_0 = s|_{I'_m}$ and $s'_0 = |_{I'_m}$, which are mirrored, by Lemma 2.4.4. By the induction hypothesis, we obtain $s_0 = s'_0$ or $s_0 = \operatorname{rev} s'_0$. The latter equality implies the following fact: if $\Sigma(s) = (n_1, n_2, \ldots, n_{2t-1}, n_{2t})$ then $\Sigma(s') = (n_1, n_{2t-1}, \ldots, n_2, n_{2t})$. Since $m_C^{(i)}(s) = m_C^{(i)}(s')$ for all i and for all C, we obtain necessarily $s = s'$ or $s = \operatorname{rev} s'$. □

Below we consider when the general case can be reduced to that of 2 symbols.

Definition 2.4.12. Let $s \colon I_m \to X$ where X is any set, let $X_0 \subset X$ and let R be any symbol (it can be an element of X or not). We define the function

$$\pi_{X_0,R}(s) \colon I_m \to X \cup \{R\}$$

by

$$\pi_{X_0,R}(s)(i) = \begin{cases} s(i), & \text{if } s(i) \in X_0, \\ R, & \text{otherwise.} \end{cases}$$

Note that for any $\sigma \in \mathcal{S}_m$, $\pi_{X_0,R}(\sigma s) = \sigma \pi_{X_0,R}(s)$. In particular, if s and s' are mirrored, then so are $\pi_{X_0,R}(s)$ and $\pi_{X_0,R}(s')$, for any choice of $X_0 \subset X$ and R.

Example 10. Let $s = (A, B, C, D, C)$ and $s' = (A, D, C, B, C)$. It is easy to see that $s \neq s'$ and $s \neq \operatorname{rev} s'$, and s and s' are not mirrored. On the other hand for every choice of $x, y \in \{A, B, C, D\}$ we have $\pi(s) = \pi(s')$ or $\pi(s) = \operatorname{rev} \pi(s')$ where $\pi = \pi_{\{x\},y}$.

The previous example shows that we cannot always reduce to the case where X has two elements.

Definition 2.4.13. Let s, $s' \colon I_m \to X$ where X is any set. An element $A \in \operatorname{Im} s$ is called:

(i) direct for the pair (s, s') if for all $i \in s^{-1}(A)$, one has $s'(i) = A$,

(ii) reverse for the pair (s, s') if for all $i \in s^{-1}(A)$, it holds $\operatorname{rev} s'(i) = A$.

Example 11. Let $s,\ s'\colon I_m \to X$.

(1) If each $A \in \operatorname{Im} s$ is direct for (s, s') then $s = s'$.
(2) If each $A \in \operatorname{Im} s$ is reverse for (s, s') then $s = \operatorname{rev} s'$.
(3) The converses of (1) and (2) hold. That is, if $s = s'$ ($s = \operatorname{rev} s'$, respectively), then each $A \in \operatorname{Im} s$ is direct (reverse, respectively) for (s, s').

The motivation for the previous definition is as follows.

Example 12. Let $s,\ s'\colon I_m \to X$. If there exists an $A \in \operatorname{Im} s$ that is neither direct nor reverse for (s, s') then s and s' are not mirrored.

In this case we can make a reduction: letting $\pi = \pi_{\{A\},B}$, then $s_0 = \pi(s)$ and $s'_0 = \pi(s')$, we have $s_0 \neq s'_0$. The last statement holds since there is an i such that $s(i) = A \neq s'(i)$, hence $\pi(s)(i) = A \neq B = \pi(s')(i)$. By a similar argument, $s_0 \neq \operatorname{rev} s'_0$. In particular, s_0 and s'_0 are not mirrored by Proposition 2.4.11, and this implies that s and s' are not mirrored.

Now we focus on the case where the reduction to the case of two elements is not possible.

Definition 2.4.14. Let $s,\ s'\colon I_m \to X$ where X is any set. We say that (s, s') is a special pair if every $A \in Im\, s$ is either direct or reverse for (s, s').

It is easy to construct examples of special pairs. A particular consequence of the previous examples is the following:

Lemma 2.4.15. *If $s,\ s'\colon I_m \to X$ are mirrored then (s, s') is a special pair.* □

Example 13. Let (s, s') be a special pair, and assume that $A = s(1)$ is direct for (s, s'). Then there is a coincidence in (s, s').

Example 14. Let (s, s') be a special pair. For every $i = 1, 2, \ldots, m$, let $t_i = (1 \quad \tau_m(i)) = (i \quad m - i + 1)$ be a transposition. Let $A = s(i)$, if $s(i) \neq s'(i)$ then A is not direct, and necessarily A is reverse hence $s(i) = t_i s'(i) = s'(m - i + 1)$. Let $B = s(m - i + 1)$. There are two possibilities: B is direct, which implies $A = B$ or B is reverse, and so $s'(i) = B$. In particular, (s, s') is a special pair if and only if (s', s) is a special pair.

Example 15. Let (s, s') be a special pair. Then $(\operatorname{rev} s, s')$ and $(s, \operatorname{rev} s')$ are special pairs as well. Moreover, A is direct (reverse, respectively) for (s, s') if and only if A is reverse (direct, respectively) for $(s, \operatorname{rev} s')$. Analogously for $(\operatorname{rev} s, s')$.

Lemma 2.4.16. *Let $s\colon I_m \to X$ be such that (s, s') is a special pair where $s' = \tau_{m-1}s$. Then $s = s'$.*

Proof. (Sketch) Let $A = s(1)$. If A is direct for (s, s') then $s(1) = s'(1) = A$, and $s(m-1) = \tau_{m-1}s(1) = s'(1) = A$. Hence, as A is direct, $s'(m-1) = A$ and we can proceed the argument inductively.

If A is reverse, then $A = s'(m) = s(m)$ hence $s'(1) = A$. So $s(m-1) = \tau_{m-1}s(1) = s'(1) = A$, hence $s'(2) = A$, which implies $s(m-2) = A$. Continuing the process, the lemma is proved. □

Note that a similar argument can be used to prove the following:

Lemma 2.4.17. *Let (s, s') be a special pair, assume (m_1, m_2) a coincidence of (s, s') and let $s_0 = s\,|_{I'_m}, s'_0 = s'\,|_{I'_m}$. If $s_0 = \mathrm{rev}'\, s'_0$ then $s = s'$.* □

We are in a position to prove the main theorem.

Proof of Theorem 2.3.2. Assume s and s' mirrored. Then (s, s') is a special pair, by Lemma 2.4.15, and we can change s' to $\mathrm{rev}\, s'$ if necessary, to guarantee that there is a coincidence in (s, s').

Now $s_0 = s\,|_{I'_m}$ and $s'_0 =|_{I'_m}$ are mirrored, by Lemma 2.4.4, and by the induction hypothesis, $s_0 = s'_0$ or $s_0 = \mathrm{rev}'\, s'_0$. Both cases imply $s = s'$, by Lemma 2.4.17, proving the theorem. □

Chapter 3
Group gradings on upper triangular matrices

"Para ser um grande escritor, seja antes um grande leitor"
P. Koshlukov

1. Introduction

In this chapter, we classify group gradings on the algebra of upper triangular matrices, viewed as a Lie algebra and as a Jordan algebra.

As an associative algebra, the group gradings on UT_n is classified in two papers. In [63], Valenti and Zaicev prove that every grading on UT_n is isomorphic to a so-called elementary grading. It is worth mentioning that elementary gradings play an important role in the gradings of matrix algebras, and on some of its subalgebras. Di Vincenzo, Koshlukov and Valenti classified the elementary gradings on UT_n up to isomorphism, and computed its graded polynomial identities [29]. Moreover, the authors prove that every grading is uniquely determined by its graded polynomial identities. Thus, one obtains a complete classification of group gradings on the associative algebra of upper triangular matrices. This classification result holds for arbitrary field and arbitrary group.

If we consider the Lie bracket $[a, b]$, then we can view UT_n as a Lie algebra, denoted by $UT_n^{(-)}$. Using the Jordan product $a \circ b = ab + ba$, we view UT_n as a Jordan algebra, denoted by UJ_n.

In this chapter, we work considering an arbitrary field of characteristic not 2, and an arbitrary group. We prove that there are two family of gradings on UJ_n, namely the elementary ones and the so-called type II gradings. Moreover, we classify the gradings up to isomorphism, and prove that each grading can be distinguished by its graded polynomial identities. We also obtain that the support of the grading is commutative.

We obtain similar results for the Lie case $UT_n^{(-)}$. But for the Lie case, we shall deal with its center (see the notion of practically same grading below).

This chapter is divided as follows. We determine the isomorphism classes of gradings in sections 2 and 3. We choose to work in the Jordan case, but similar arguments hold for the

Lie case. In Section 4 we describe the gradings on UJ_n. Section 5 is dedicated to introducing the notion of practical isomorphism, and we prove some of its properties. Finally, in Section 6, we prove the result for $UT_n^{(-)}$.

1.1. **Preliminaries and notations.** We denote by e_{ij} the matrix units. We set $e = e_{11} + \ldots + e_{nn}$ the identity matrix. If x is matrix, we denote by $(x)_{(i,j)}$ its (i,j) entry. We will always work with matrices of a fixed size n. For adequate integers i, m, we set

$$e_{i:m} = e_{i,i+m}, \quad e_{-i:m} = e_{n-m-i+1,n-i+1}$$
$$(x)_{(i:m)} = (x)_{(i,i+m)}, \quad (x)_{(-i:m)} = (x)_{(n-m-i+1,n-i+1)}.$$

We define non-associative products to be left normed. That is, we set, by induction,

$$a_1 \circ a_2 \circ \cdots \circ a_m = (a_1 \circ a_2 \circ \cdots \circ a_{m-1}) \circ a_m,$$
$$[a_1, a_2, \ldots, a_m] = [[a_1, a_2, \ldots, a_{m-1}], a_m].$$

We define the associator of three elements as $(a, b, c) = (a \circ b) \circ c - a \circ (b \circ c)$.

1.1.1. *Automorphism group of* $UT_n^{(-)}$. The automorphisms of the Lie algebra of upper triangular matrices were described by Đoković [31]. (The description given in [31] holds for upper triangular matrix algebra over any commutative ring with 1 having no idempotents apart from 0 and 1.) In order to state Đoković's theorem we need some notation.

Let A be associative and let $a \in A$ be invertible then the map $\mathrm{Int}(a)\colon x \mapsto axa^{-1}$ is an (inner) automorphism of A. One denotes the group of all inner automorphisms of A by G_0. Clearly the elements of G_0 are automorphisms of $A^{(-)}$ as well. Now fix $A = UT_n$. It was shown in [31, Lemma 2] that in this case the kernel of the epimorphism $A^\times \to G_0$, $t \mapsto \varphi_t$, is $\{\alpha e \mid \alpha \in K^\times\}$, the multiplicative group of K. Therefore $G_0 \cong UT_n^\times / K^\times$.

Denote further $S = \{a = (a_1, \ldots, a_n) \mid a_i \in UT_n, a_1 + \cdots + a_n + 1 \neq 0\}$. If $a \in S$ one defines a linear transformation on UT_n by $\psi_a(e_{ij}) = e_{ij} + \delta_{ij} a_i e$, here δ_{ij} is the Kronecker symbol. It was shown in [31] that $G_1 = \{\psi_a \mid a \in S\}$ is a group of automorphisms of $UT_n^{(-)}$. Moreover Proposition 3 of [31] gives that G_0 and G_1 commute element-wise and $G_0 \cap G_1 = 1$.

One defines, as in [31], the automorphism ω_0 of $UT_n^{(-)}$ by $\omega_0(e_{ij}) = -e_{n+1-j,n+1-i}$ for all $i \leq j$. (This is the flip along the second diagonal with a change of sign.) According to Propositions 4 and 5 of [31], the element $\omega_0 \in Aut(UT_n^{(-)})$ normalizes both G_0 and G_1.

Theorem 3.1.1 ([31])**.** *The group of automorphisms of the Lie algebra* $UT_n^{(-)}$ *is isomorphic to* $G_0 \times G_1$ *if* $n = 2$*, and is isomorphic to* $(G_0 \times G_1) \rtimes \langle \omega_0 \rangle$ *whenever* $n \geq 3$*. Here* $\rtimes$ *is the semidirect product of the two groups.* □

In fact we will need a weaker version of the theorem of Đoković. What we need is that the map $G_0 \times G_1 \times \{1, \omega_0\} \to Aut(UT_n^{(-)})$ given by $(\varphi, \psi_a, \alpha) \mapsto \varphi \cdot \psi_a \cdot \alpha$ is well defined and onto.

1.1.2. *Automorphism group of* UJ_n. We recall that, according to [18], every automorphism of UJ_n is given either by an automorphism or an anti-automorphism of UT_n (or, equivalently, by an automorphism of UT_n, or by an automorphism followed by the involution $e_{i:m} \mapsto e_{-i:m}$.) Moreover, according to [27], every automorphism of UT_n is inner (as an associative algebra).

2. Elementary gradings

Let G be any group and K any field. We call a G-grading on UJ_n elementary if all matrix units e_{ij} are homogeneous in the grading.

Lemma 3.2.1. *Let* UJ_n *be equipped with an elementary* G*-grading. Then*

(i) $\deg e_{ii} = 1$, $i = 1, \ldots, n$.
(ii) *The sequence* $\eta = (\deg e_{12}, \deg e_{23}, \ldots, \deg e_{n-1,n})$ *defines completely the grading.*
(iii) *The support of the grading is commutative.*

Proof. The statements of the lemma and their proofs are standard facts, we give these proofs for the sake of completeness.

(i) Since $e_{ii} \circ e_{ii} = 2e_{ii}$ we have $(\deg e_{ii})^2 = \deg e_{ii}$ hence $\deg e_{ii} = 1$.

(ii) It follows from $e_{ij} = e_{i,i+1} \circ e_{i+1,i+2} \circ \cdots \circ e_{j-1,j}$.

(iii) Let $t_1 = \deg e_{12}$, $t_2 = \deg e_{23}$, $\ldots$, $t_{n-1} = \deg e_{n-1,n}$. By (ii), it suffices to prove that $t_i t_j = t_j t_i$ for all i, $j \in \{1, 2, \ldots, n-1\}$. But if $i < j$ then

$$e_{i,i+1} \circ (e_{j,j+1} \circ (e_{i+1,i+2} \circ \cdots \circ e_{j-1,j})) = e_{j,j+1} \circ (e_{i,i+1} \circ e_{i+1,i+2} \circ \cdots \circ e_{j-1,j}).$$

Thus $t_i t_j t_{i+1} \cdots t_{j-1} = t_j t_i t_{i+1} \cdots t_{j-1}$ and $t_i t_j = t_j t_i$. □

Since the support of an elementary grading is commutative, from here on in this section, we assume that G is abelian.

Notation. We denote by (UJ_n, η) the elementary grading defined by $\eta \in G^{n-1}$. This grading is defined by putting $\deg e_{i,i+1} = g_i$, for each i, where $\eta = (g_1, g_2, \ldots, g_{n-1})$. We denote by $\operatorname{rev}\eta = (g_{n-1}, g_{n-2}, \ldots, g_1)$.

Lemma 3.2.2. *Let $\eta \in G^{n-1}$. The map $\varphi\colon (\mathrm{UJ}_n, \eta) \to (\mathrm{UJ}_n, \operatorname{rev}\eta)$ given by $e_{ij} \mapsto e_{n-j+1,n-i+1}$ is an isomorphism of G-graded algebras.*

Proof. The proof is a direct and easy verification. □

Using same argument as Lemma 2.2.3, one can prove

Lemma 3.2.3. *Let r_1, …, r_m be strictly upper triangular matrix units such that the associative product $r_1 \cdots r_m \neq 0$, and let $\sigma \in S_m$. Then $r_{\sigma^{-1}(1)} \circ \cdots \circ r_{\sigma^{-1}(m)} \neq 0$ if and only if $\sigma \in \mathscr{T}_m$.* □

In analogy with [29] we define

Definition 3.2.4. Let G be a group and let (UJ_n, η) be an elementary G-grading. Let $\mu = (a_1, \ldots, a_m) \in G^m$ be any sequence.

(1) (See [29]) The sequence μ is associative η-good if there exist strictly upper triangular matrix units r_1, …, $r_m \in UT_n$ such that $r_1 \cdots r_m \neq 0$ and $\deg r_i = a_i$ for every $i = 1$, …, m. Otherwise μ is associative η-bad sequence.
(2) The sequence μ is Jordan η-good if there exist strictly upper triangular matrix units r_1, …, r_m such that $r_1 \circ \cdots \circ r_m \neq 0$ and $\deg r_i = a_i$, for every $i = 1$, …, m. Otherwise μ is Jordan η-bad sequence.

Definition 3.2.5. If $\mu = (a_1, a_2, \cdots, a_m) \in G^m$ we define

$$f_\mu = f_1^{(a_1)} \circ f_2^{(a_2)} \circ \cdots \circ f_m^{(a_m)}$$

where

$$f_h^{(a)} = \begin{cases} (x_{3h-2}^{(1)}, x_{3h-1}^{(1)}, x_{3h}^{(1)}), & \text{if } a = 1, \\ x_h^{(a)}, & \text{if } a \neq 1 \end{cases}.$$

The following lemma is proved exactly in the same way as Proposition 2.2 of [29].

Lemma 3.2.6. *A sequence μ is Jordan η-bad if and only if f_μ is a G-graded identity for (UJ_n, η).* □

The unique non-zero associative product of $n-1$ strictly upper triangular matrix units of UT_n is $e_{12}e_{23}\cdots e_{n-1,n}$ (see [29]), so combining this fact, Lemma 3.2.3, and Lemma 3.2.6, we obtain

Lemma 3.2.7. *A sequence $\mu \in G^{n-1}$ is Jordan η-good for (UJ_n, η) if and only if $\mu \in \mathscr{T}_{n-1}\eta$.* □

Combining Lemma 3.2.7 and Corollary 2.3.3, we obtain

Corollary 3.2.8. *Let η, $\eta' \in G^{n-1}$ with $\eta \neq \eta'$ and $\eta \neq \operatorname{rev}\eta'$. Then $(\mathrm{UJ}_n, \eta) \not\cong (\mathrm{UJ}_n, \eta')$.*

Proof. By Corollary 2.3.3, there exists $\sigma \in \mathscr{T}_m$ such that $\sigma\eta \neq \sigma'\eta'$ for each $\sigma' \in \mathscr{T}_m$, interchanging η and η' if necessary. By Lemma 3.2.7, $\sigma\eta$ is Jordan η-good sequence but Jordan η'-bad sequence, hence $f_{\sigma\eta}$ is not a graded identity for (UJ_n, η), but it is a graded identity for (UJ_n, η'). In particular, $(\mathrm{UJ}_n, \eta) \not\cong (\mathrm{UJ}_n, \eta')$. □

In this way we have a classification of the elementary gradings on UJ_n:

Theorem 3.2.9. *The support of an elementary G-grading on UJ_n is commutative.*

Let G be an abelian group and define the equivalence relation on G^{n-1} as follows. Let μ_1 and $\mu_2 = (a_1, a_2, \ldots, a_{n-1}) \in G^{n-1}$, then $\mu_1 \sim \mu_2$ whenever $\mu_1 = \mu_2$ or $\mu_1 = (a_{n-1}, \cdots, a_2, a_1)$.

Then there is 1–1 correspondence between $G^{n-1}/\sim$ and the class of non-isomorphic elementary G-gradings on UJ_n. □

Remark. Given $\mu = (a_1, \ldots, a_m) \in G^m$, the equivalent Lie polynomial of Definition 3.2.5 is

$$f_\mu = [f_1^{(a_1)}, \ldots, f_m^{(a_m)}],$$

where

$$f_h^{(a)} = \begin{cases} [x_{2h-1}^{(1)}, x_{2h}^{(1)}], & \text{if } a = 1, \\ x_h^{(a)}, & \text{if } a \neq 1 \end{cases}.$$

3. Type II gradings

We fix an arbitrary group G and a field of characteristic not 2.

Notation. If i, $m \in \mathbb{N}$ we denote $X_{i:m}^{+} = e_{i:m} + e_{-i:m}$, and $X_{i:m}^{-} = e_{i:m} - e_{-i:m}$.

Remark. In the above notation, if $n - m$ is odd then $X_{i:m}^{+} = 2e_{i:m} = 2e_{-i:m}$, and $X_{i:m}^{-} = 0$ for $i = (n - m + 1)/2$.

Definition 3.3.1. A G-grading on UJ_n is of type II if all $X^+_{i:m}$, $X^-_{i:m}$ are homogeneous and $\deg X^+_{i:m} \neq \deg X^-_{i:m}$.

Lemma 3.3.2. *One has* $X^+_{i:1} \circ X^+_{i+1:1} \circ \cdots \circ X^+_{i+m-1:1} = \lambda X^+_{i:m}$ *for some* $\lambda = 2^p$, $p \in \mathbb{Z}$.

Proof. Induction on m. When $m = 1$ the statement is trivial, so assume $m > 1$. If $X^+_{i:1} \circ X^+_{i+1:1} \circ \cdots \circ X^+_{i+m-1:1} = \lambda X^+_{i:m} = \lambda(e_{i:m} + e_{-i:m})$ then $(\lambda X^+_{i:m}) \circ X^+_{i+m:1} = \lambda'(e_{i:m+1} + e_{-i:m+1})$. □

Lemma 3.3.3. *Let a G-grading on* UJ_n *be of type II, then*

(i) $\deg X^+_{i:0} = 1$ *for every* i, *and* $\deg X^-_{1:0} = \deg X^-_{2:0} = \cdots = \deg X^-_{\lfloor \frac{n}{2} \rfloor:0} = t$ *is an element of order 2.*

(ii) *Let* $q = \lceil \frac{n-1}{2} \rceil$, *then the sequence* $\eta = (\deg X^+_{1:1}, \deg X^+_{2:1}, \ldots, \deg X^+_{q:1})$ *and the element* $t = \deg X^-_{1:0}$ *completely define the grading.*

(iii) *The support of the grading is commutative.*

Moreover, if the elements $X^\pm_{i:m}$, *for each* i *and for* $m = 0$ *and* $m = 1$, *are homogeneous, with* $\deg X^+_{i:m} \neq \deg X^-_{i:m}$, *then the grading is necessarily of type II.*

Proof. (i) The equalities $X^-_{1:0} \circ X^+_{1:1} = X^-_{1:1}$, $(X^\pm_{i:0})^2 = 2X^+_{i:0}$ and $X^-_{i:0} \circ X^\pm_{i:1} = X^-_{i+1:0} \circ X^\pm_{i:1}$ yield the proof.

(ii) It follows from Lemma 3.3.2.

(iii) According to (ii), the elements $\deg X^-_{1:0}$ and $\deg X^+_{i:1}$, for all i, generate the support of the group. Using Lemma 3.3.2 and the same idea as of Lemma 3.2.1.(iii), we prove the statement. □

Since the support of a type II grading is commutative, we assume from now on in this section G abelian. We denote by (UJ_n, t, η) the type II grading defined by $t \in G$ and the sequence $\eta \in G^q$.

It is well known that, if we have an associative algebra with involution $(A, *)$, then the decomposition of A into symmetric and skew-symmetric elements with respect to $*$ gives rise to a $\mathbb{Z}_2$-graded algebra. If, moreover, A is endowed with an H-grading and $*$ is a graded involution (that is, $\deg a^* = \deg a$, for all homogeneous $a \in A$), then the decomposition cited yields an $H \times \mathbb{Z}_2$-graded Jordan algebra.

The upper triangular matrices possess a natural involution, given by $\psi : e_{i:m} \in UT_n \mapsto e_{-i:m} \in UT_n$. For an elementary grading η on UT_n, ψ will be a graded involution if and only if $\eta = \operatorname{rev}\eta$. It is easy to see that the obtained grading by the involution is a type II grading.

Example 16. Let $G = \mathbb{Z}_4$ and take the type II grading on UJ_4 given by $\deg X^{+}_{i:1} = 1 \in \mathbb{Z}_4$, and $\deg X^{-}_{i:0} = 2 \in \mathbb{Z}_4$, for every i. Since $\mathbb{Z}_4$ is an indecomposable group, it cannot be written in the form $\mathbb{Z}_2 \times H$. Therefore there exist type II gradings that cannot be given by the involution.

Below we classify all type II gradings. Note that the ideal J of all strictly upper triangular matrices is invariant under all automorphisms of UJ_n.

Lemma 3.3.4. *Let η, $\eta' \in G^q$ where $q = \lceil \frac{n-1}{2} \rceil$ and t_1, $t_2 \in G$ are elements of order 2. If $t_1 \neq t_2$ then $(\mathrm{UJ}_n, t_1, \eta) \not\simeq (\mathrm{UJ}_n, t_2, \eta')$.*

Proof. If $\psi\colon (\mathrm{UJ}_n, t_1, \eta) \to (\mathrm{UJ}_n, t_2, \eta')$ is a graded isomorphism then $\bar{\psi}\colon \mathrm{UJ}_n/J \to \mathrm{UJ}_n/J$ will be a graded isomorphism which is impossible when $t_1 \neq t_2$. □

Lemma 3.3.5. *Let $t \in G$ be an element of order 2 and let $\eta = (g_1, \ldots, g_q)$, $\eta' = (g'_1, \ldots, g'_q) \in G^q$ where $q = \lceil \frac{n-1}{2} \rceil$. Assume that one of the following holds:*

- *there is an i, $1 \le i \le \lfloor \frac{n-1}{2} \rfloor$ such that $g_i \not\equiv g'_i \pmod{\langle t \rangle}$, or*
- *n is even and $g_q \neq g'_q$.*

Then $(\mathrm{UJ}_n, t, \eta) \not\simeq (\mathrm{UJ}_n, t, \eta')$.

Proof. Let $\varphi\colon G \to G_0 = G/\langle t \rangle$ be the canonical projection. The induced G_0-grading on UJ_n by φ and by (UJ_n, t, η) coincides with the elementary G_0-grading (UJ_n, η_0) where

$$\eta_0 = \begin{cases} (\varphi(g_1), \varphi(g_2), \ldots, \varphi(g_q), \varphi(g_q), \varphi(g_{q-1}), \ldots, \varphi(g_1)), & \text{if } n \text{ is odd},\\ (\varphi(g_1), \varphi(g_2), \ldots, \varphi(g_{q-1}), \varphi(g_q), \varphi(g_{q-1}), \ldots, \varphi(g_1)), & \text{if } n \text{ is even}. \end{cases}$$

A G-graded isomorphism $\psi\colon (\mathrm{UJ}_n, t, \eta) \to (\mathrm{UJ}_n, t, \eta')$ induces a G_0-graded isomorphism $(\mathrm{UJ}_n, \eta_0) \to (\mathrm{UJ}_n, \eta'_0)$ if and only if $\eta_0 = \eta'_0$ (since $\eta'_0 = \operatorname{rev}\eta'_0$), by Theorem 3.2.9. This proves the first condition.

Now, assume n even and $g_q \neq g'_q$. Let $T = \operatorname{Span}\{X^{\pm}_{i:m} \mid (i, m) \notin \{(q, 1), (q, 0)\}\}$, so T is a graded ideal. Note that T is invariant under all automorphisms of UJ_n, and $\mathrm{UJ}_n/T \simeq \mathrm{UJ}_2$. Since T is invariant under all automorphisms of UJ_n, an isomorphism $\mathrm{UJ}_n \to \mathrm{UJ}_n$

would induce a graded isomorphism $\mathrm{UJ}_n/T \to \mathrm{UJ}_n/T$. But $(\mathrm{UJ}_n, t, \eta)/T \simeq (\mathrm{UJ}_2, (g_q))$, and $(\mathrm{UJ}_n, t, \eta')/T \simeq (\mathrm{UJ}_2, g'_q)$, and $(\mathrm{UJ}_2, g_q) \not\simeq (\mathrm{UJ}_2, g'_q)$ if $g_q \neq g'_q$. □

Lemma 3.3.6. *Let $t \in G$ be of order 2, $\eta = (g_1, \ldots, g_q)$, $\eta' = (g_1, \ldots, g'_q) \in G^q$ where $q = \lceil \frac{n-1}{2} \rceil$. Assume that*

i) *$g_i \equiv g'_i \pmod{\langle t \rangle}$, for $i = 1, 2, \ldots, p$ where $p = \lfloor \frac{n-1}{2} \rfloor$,*
ii) *if n is even then $g_q = g'_q$.*

Then $(\mathrm{UJ}_n, t, \eta) \simeq (\mathrm{UJ}_n, t, \eta')$.

Proof. For every $i = 1, 2, \ldots, p$, let $\epsilon_i = 1$ if $g_i = g'_i$ and $\epsilon_i = -1$ if $g_i \neq g'_i$. Let $\epsilon = \epsilon_1\epsilon_2 \cdots \epsilon_p$ and $A = \mathrm{diag}(\epsilon, \epsilon, \ldots, \epsilon, \epsilon_{p-1}\epsilon_{p-2} \cdots \epsilon_1, \epsilon_{p-2} \cdots \epsilon_1, \ldots, \epsilon_2\epsilon_1, \epsilon_1, 1)$. The map $(\mathrm{UJ}_n, t, \eta) \to (\mathrm{UJ}_n, t, \eta')$ given by $x \mapsto AxA^{-1}$ is a graded isomorphism. □

We summarize the classification of type II gradings on UJ_n.

Theorem 3.3.7. *Every type II grading has commutative support. If G is abelian, then there is 1–1 correspondence between the non-isomorphic type II gradings on UJ_n and the set M where*

(1) *if n is odd, $M = \{(t, \eta) \mid t \in G, o(t) = 2, \eta \in (G/\langle t \rangle)^{\frac{n-1}{2}}\}$,*
(2) *if n is even, $M = \{(t, \eta) \mid t \in G, o(t) = 2, \eta \in (G/\langle t \rangle)^{\frac{n-2}{2}} \times G\}$.*

4. Gradings on UJ_n

In this section, we classify the group gradings on UJ_n. We prove that, up to isomorphism, any grading is either elementary or of type II. It is worth mentioning that the group gradings on UJ_2 were already known [48].

Let K be any field of characteristic not 2, and let G be any group and fix a G-grading on UJ_n. The ideal J of all strictly upper triangular matrices is graded since $J = (\mathrm{UJ}_n, \mathrm{UJ}_n, \mathrm{UJ}_n)$. Also the element e_{1n} is always homogeneous since $\mathrm{Span}\{e_{1n}\} = J^{n-1}$.

As a consequence $B = \mathrm{Ann}_{\mathrm{UJ}_n}\{e_{1n}\} = \{x \in \mathrm{UJ}_n \mid (x)_{(1,1)} + (x)_{(n,n)} = 0\}$ is graded, and $B^2 = B \circ B = \{x \in \mathrm{UJ}_n \mid (x)_{(1,1)} = (x)_{(n,n)}\}$ is as well. It follows $C = B \cap B^2 = \{x \in \mathrm{UJ}_n \mid (x)_{(1,1)} = (x)_{(n,n)} = 0\}$ is graded. Let $U_1 = \mathrm{Ann}_{\mathrm{UJ}_n}(C/J)$ and let $T_1 = U_1^{\circ n} = \{x \in \mathrm{UJ}_n \mid (x)_{(i,j)} = 0,\ \text{for } i \neq 1 \text{ or } (i,j) \neq (i,n)\}$, the n-th power of U_1. It is easy to see that T_1 is an ideal (moreover, a graded ideal). A similar trick in the associative case can be found in the proof of Lemma 2 of [62].

Lemma 3.4.1. *There exists a homogeneous element* $e_2 \in T_1$ *such that* $(\deg e_2)^2 = 1$ *and* $e_2 \equiv e_{11} - e_{nn} \pmod{T_1 \cap J}$.

Proof. Note first that $A = T_1/T_1 \cap J$ is an associative graded algebra whose unit is $\bar{e}_1 = \bar{e}_{11} + \bar{e}_{nn}$. Hence $\bar{e}_1$ is graded and $\deg \bar{e}_1 = 1$. Moreover, we can choose a homogeneous element $x \in T_1$ and we can assume that $\bar{x}$ and $\bar{e}_1$ are linearly independent in A. If $\deg \bar{x} = 1$ then we are done. Otherwise $\deg \bar{x} \neq \deg(\bar{x} \circ \bar{x})$ which implies $\bar{x} \circ \bar{x}$ is a multiple of $\bar{e}_1$, and this proves the lemma. □

Lemma 3.4.2. *Up to a graded isomorphism,* $e_1 = e_{11} + e_{nn}$ *and* $e_2 = e_{11} - e_{nn}$ *are homogeneous and* $\deg e_1 = (\deg e_2)^2 = 1$.

Proof. Let e_2 be as in the previous lemma, and let $e_1 = \frac{1}{2} e_2 \circ e_2$. Note that

(a) $e_1 \equiv e_{11} + e_{nn} \pmod{T_1 \cap J}$.

(b) $(e_1)_{(1,i)} = (e_2)_{(1,i)}$ and $(e_1)_{(i,n)} = -(e_2)_{(i,n)}$, for $i = 2, 3, \ldots, n-1$.

(c) $(e_1)_{(1,n)} = \sum_{i=2}^{n-1} (e_2)_{(1,i)} (e_2)_{(i,n)}$.

As a consequence of the above properties, the associative product $x = e_1(e_1 - 1) = 0$. Indeed,

(a) $(x)_{(1,i)} = (e_1)_{(1,1)}(e_1 - 1)_{(1,i)} + (e_1)_{(1,i)}(e_1 - 1)_{(i,i)} = 0$, for every $i = 1, 2, \ldots, n-1$.

(b) $(x)_{(i,n)} = 0$, for $i = 2, 3, \ldots, n-1$.

(c) Using the above relations one obtains

$$\begin{aligned}(x)_{(1,n)} &= \sum_{i=1}^{n} (e_1)_{(1,i)} (e_1 - 1)_{(i,n)} \\ &= \underbrace{(e_1)_{(1,1)}(e_1 - 1)_{(1,n)}}_{(e_1)_{(1,n)}} + \sum_{i=2}^{n-1} (e_1)_{(1,i)} \underbrace{(e_1 - 1)_{(i,n)}}_{(e_1)_{(i,n)}} = 0.\end{aligned}$$

(d) All remaining entries are evidently zero.

These equalities show that the minimal polynomial of e_1 is $z(z-1)$, hence e_1 is diagonalizable. If $\psi\colon \mathrm{UJ}_n \to \mathrm{UJ}_n$ is the conjugation such that $\psi(e_1) = e_{11} + e_{nn}$, then ψ induces a new G-grading on UJ_n, isomorphic to the original one, such that $e_1 = e_{11} + e_{nn}$ is homogeneous of degree 1.

Consider again the element e_2 from Lemma 3.4.1. Let $r_2 = e_2 \circ e_1 - e_2$. Then $r_2 = e_{11} - e_{nn} + \alpha e_{1n}$ for some $\alpha \in K$, and moreover, r_2 is diagonalizable. Since e_1 and r_2

commute, they are simultaneously diagonalizable, and we can find an inner automorphism ψ' such that $\psi'(e_1) = e_{11} + e_{nn}$ and $\psi'(r_2) = e_{11} - e_{nn}$. This concludes the lemma. $\square$

Now, the following set is homogeneous:

$$\Delta = \mathrm{Ann}_{\mathrm{UJ}_n}(e_{11} + e_{nn}) \simeq \mathrm{UJ}_{n-2}.$$

Thus we write $\mathrm{UJ}_n = T_1 \oplus \Delta$. Note that every inner automorphism (conjugation) of Δ by a matrix M can be extended to an inner automorphism of UJ_n by the matrix

$$M' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore we can repeat the argument above for Δ. Thus we suppose that, up to a graded isomorphism, the elements $u_1 = e_{22} + e_{n-1,n-1}$ and $u_2 = e_{22} - e_{n-1,n-1}$ are homogeneous and $\deg u_1 = (\deg u_2)^2 = 1$. Since $M'e_i(M')^{-1} = e_i$ for $i = 1, 2$, we can also assume the existence of the elements e_1 and e_2 as in Lemma 3.4.2.

Take a homogeneous element $z_1 \in J^{n-2} = \mathrm{Span}\{e_{1,n-1}, e_{2n}, e_{1n}\}$ such that $(z)_{(1,n-1)} \neq 0$. We can change z to $z \circ u_1$, if necessary, in order to obtain $(z)_{(1,n)} = 0$. In this case $u_2 \circ z = -e_2 \circ z$, hence we have

Lemma 3.4.3. *In the notation introduced above,* $\deg e_2 = \deg u_2$.

If $\deg e_2 = 1$ then e_{11} and e_{nn} are homogeneous of degree 1.

Lemma 3.4.4. *If* $\deg e_2 = 1$ *then, up to a graded isomorphism, the grading is elementary.*

Proof. If $n = 2$ then the elements e_{11}, e_{22}, e_{12} are (up to a graded isomorphism) homogeneous hence the grading is elementary. If $n = 3$ we consider the decomposition $\mathrm{UJ}_3 = T_1 \oplus \Delta$. Since $\dim \Delta = 1$ it is easy to prove that the grading is again elementary.

Thus we assume $n > 3$. We decompose $\mathrm{UJ}_n = T_1 \oplus \Delta$. In the notation introduced above, by Lemma 3.4.3 we have $\deg u_2 = \deg e_2 = 1$. We use an induction to conclude that the grading on Δ is elementary. In particular the elements e_{22} and $e_{n-1,n-1}$ are homogeneous. If $z \in J$ is homogeneous with $(z)_{(1,2)} = 1$ then $e_{12} = (z \circ e_{11}) \circ e_{22}$ is homogeneous. In the same way we obtain that $e_{n-1,n}$ is homogeneous. This implies that the elements e_{12}, e_{23}, …, $e_{n-1,n}$ are homogeneous. Therefore the grading is elementary. $\square$

Lemma 3.4.5. *If* $\deg e_2 \neq 1$ *then, up to a graded isomorphism, the grading is of type II.*

Proof. First we assume $n > 3$. Decompose $\mathrm{UJ}_n = T_1 \oplus \Delta$, by Lemma 3.4.3, we assume by induction that up to a graded isomorphism, Δ is equipped with a type II grading.

Let $z'' \in J \cap T_1$ be a homogeneous element with $(z'')_{(1,2)} = 1$, and let $z' = z'' \circ u_1$. The only non-zero entries of z' can be $(1, 2)$, $(n-1, n)$, $(1, n-1)$, $(2, n)$, and z' is homogeneous with $(z')_{(1,2)} = 1$. If $z = \frac{1}{2}(z' \circ u_2 + z' \circ e_2)$ then z is homogeneous and $z = e_{12} + ae_{n-1,n}$ for some a. Since $\deg e_2 \neq 1$ we have $\deg z \neq \deg(z \circ e_2)$, hence $a \neq 0$. Let $A = \mathrm{diag}(1, 1, \dots, 1, a)$. Then $\psi\colon \mathrm{UJ}_n \to \mathrm{UJ}_n$ defined by $x \mapsto AxA^{-1}$ is an isomorphism. The induced grading is such that $\psi(z) = e_{12} + e_{n-1,n}$, $\psi(z \circ e_2) = e_{12} - e_{n-1,n}$ and $\psi(X^{\pm}_{i:1}) = X^{\pm}_{i:1}$, for $i = 2, 3, \dots, \lceil \frac{n-1}{2} \rceil$. As the latter are homogeneous elements the induced grading is of type II.

Now, as in the previous lemma one proves that whenever $\deg e_2 \neq 1$ and $n = 2$ or 3, the grading is of type II. When $n = 3$ and $\deg e_2 \neq 1$, we can find a homogeneous element of type $z = e_{12} + ae_{23}$, where $a \in K$ is non-zero. Thus we can conjugate with the diagonal matrix $\mathrm{diag}(1, 1, a)$ as in the general case in order to obtain a type II grading. □

Thus, we proved:

Theorem 3.4.6. *Every G-grading on UJ_n has commutative support and, up to a graded isomorphism, the grading is either elementary or of type II.*

4.1. **On the graded identities.** We have seen that non-isomorphic elementary gradings satisfy different graded identities. Let G be a group and assume $A_1 = (\mathrm{UJ}_n, t_1, \eta)$ is a type II grading, and either $A_2 = (\mathrm{UJ}_n, t_2, \eta')$ with $t_1 \neq t_2$ or $A_2 = (\mathrm{UJ}_n, \eta'')$ is an elementary grading. Then $f = (x_1^{(t_1)})^{\circ n} = x_1^{(t_1)} \circ x_1^{(t_1)} \circ \cdots \circ x_1^{(t_1)}$ is not a graded identity for A_1, but it is one for A_2.

Now let $A_1 = (\mathrm{UJ}_n, t, \eta)$ and $A_2 = (\mathrm{UJ}_n, t, \eta')$, and assume $A_1 \not\cong A_2$. We shall use the notation of the proof of Lemma 3.3.5, and we will give an alternative proof for it. If $\varphi\colon G \to G_0 = G/\langle t \rangle$ is the canonical projection we denote the elementary gradings induced on A_1 and on A_2 by φ as $\bar{A}_1 = (\mathrm{UJ}_n, \eta_0)$ and $\bar{A}_2 = (\mathrm{UJ}_n, \eta_0')$. Then we have two possibilities: (a) $\eta_0 \neq \eta_0'$, hence we can find a polynomial $f(x_1^{(\bar{g}_1)}, \dots, x_m^{(\bar{g}_m)})$ such that f is a graded identity for $\bar{A}_1$, but not for $\bar{A}_2$ (interchanging $\bar{A}_1$ and $\bar{A}_2$ if necessary). This means that

$$g(x_1^{(g_1)}, x_1^{(g_1 t)}, \dots, x_m^{(g_m)}, x_m^{(g_m t)}) = f(x_1^{(g_1)} + x_1^{(g_1 t)}, \dots, x_m^{(g_m)} + x_m^{(g_m t)})$$

is a graded identity for A_1 but not for A_2.

(b) $\eta_0 = \eta_0'$. In this case, necessarily n is even and, up to a graded isomorphism, $\eta = (g_1, \dots, g_q)$ and $\eta' = (g_1, \dots, g_{q-1}, g_q')$ with $g_q \neq g_q'$. Note that $\operatorname{Supp} J_1/J_1^2 \neq \operatorname{Supp} J_2/J_2^2$, where $J_i = (A_i, A_i, A_i)$, $i = 1$ and $i = 2$. Let $f = z_1^{(1)} \circ z_2^{(2)} \circ \cdots \circ z_q^{(q)} \circ z_{q+1}^{(q-1)} \circ \cdots \circ z_{n-1}^{(1)}$, where $z_j^{(i)} = (x_{3j-2}^{(1)}, x_{3j-1}^{(1)}, x_{3j}^{(g_i)})$. Then f is a graded identity for A_2, but not for A_1.

In this way we have the final result.

Theorem 3.4.7. *Let A_1 and A_2 be two G-gradings on UJ_n. Then $A_1 \simeq A_2$ as graded algebras if and only if $T_G(A_1) = T_G(A_2)$.* □

5. Isomorphism and practical isomorphism of graded Lie algebras

In this section, we discuss about the center of graded Lie algebra. Let $\mathfrak{z}(L)$ be the center of the Lie algebra L. If L is graded then $\mathfrak{z}(L)$ also is (see Proposition 1.1.2).

Definition 3.5.1. Take two G-gradings on L, say L_1 and L_2. The gradings L_1 and L_2 are practically the same if $L_1/\mathfrak{z}(L_1) = L_2/\mathfrak{z}(L_2)$ (equality as G-graded algebras), denoted by $L_1 \overset{G}{=} L_2$.

Another equivalent way to define is the following. $L_1 \overset{G}{=} L_2$ if and only if for each homogeneous non-central $x \in L_1$ there exists a homogeneous non-central element $y \in L_2$ such that $\deg x = \deg y$ and $x - y \in \mathfrak{z}(L)$.

Example 17. Let A be an algebra and denote by $N = \{a \in A \mid aA = Aa = 0\}$ the two-sided annihilator of A. If A is G-graded then N is a graded ideal of A. Every vector subspace of N is an ideal of A. As A^2 is graded then $A^2 \cap N$ is graded as well.

Now choose a vector subspace M such that $N = M \oplus A^2 \cap N$, and consider any (vector space) G-grading on M. This will induce a new G-grading on A, this new grading is in general not G-graded isomorphic to the original one, but it is practically the same grading as the original.

Definition 3.5.2. Let L_1 and L_2 be G-graded Lie algebras. Then L_1 and L_2 are practically G-graded isomorphic if there exists L_1' such that $L_1 \simeq L_1'$ and $L_1' \overset{G}{=} L_2$. We denote $L_1 \overset{G}{\sim} L_2$.

Example 18. Let $G = \mathbb{Z}_2^2 = \{1, g, h, gh\}$, and take $L = UT_2^{(-)} = L_1 \oplus L_g \oplus L_h$. Here $L_1 = K(e_{11} - e_{22})$, $L_g = Ke$, $L_h = Ke_{12}$. It is easy to see that this defines a G-grading on

L. Take also $L'_1 = \mathrm{Span}\{e_{11}, e_{22}\}$, $L'_h = Ke_{12}$. The first and the second gradings are not isomorphic but are practically isomorphic.

Clearly if $L_1 \overset{G}{\sim} L_2$ then L_1 and L_2 satisfy the same graded identities.

The following is an immediate equivalence of the notion of practically isomorphism:

Lemma 3.5.3. *L_1 and L_2 are practically G-graded isomorphic if and only if there exists an isomorphism of (ungraded) algebras $\psi : L_1 \to L_2$ that induces a G-graded isomorphism $L_1/\mathfrak{z}(L_1) \to L_2/\mathfrak{z}(L_2)$.* □

Note that, in this case, for every homogeneous non-central $x \in L_1$, we can find $z \in \mathfrak{z}(L_1)$ such that $y = \psi(x + z)$ is homogeneous in L_2 and $\deg x = \deg y$.

Clearly, if L_1 and L_2 are G-graded isomorphic then they are practically G-graded isomorphic. The converse does not hold, but if L_1 and L_2 are practically G-graded isomorphic then the derived algebras L'_1 and L'_2 are G-graded isomorphic. More precisely:

Lemma 3.5.4. *Assume $\psi : L_1 \to L_2$ is an isomorphism of algebras that induces a G-graded isomorphism $L_1/\mathfrak{z}(L_1) \to L_2/\mathfrak{z}(L_2)$. Then ψ restricts to a G-graded isomorphism $L'_1 \to L'_2$.*

Proof. Let $0 \neq x \in L'_1$ be homogeneous of degree $g \in G$. Then there exist in L_1 nonzero homogeneous x'_i of degree g'_i and x''_i of degree g''_i, $i = 1, \dots, m$, such that $x = \sum_{i=1}^m [x'_i, x''_i]$ and $g'_i g''_i = g$ for all i. Also, there exist $z'_i, z''_i \in \mathfrak{z}(L_1)$ such that $\psi(x'_i + z'_i)$ is homogeneous of degree g'_i and $\psi(x''_i + z''_i)$ is homogeneous of degree g''_i, for all i. Hence,

$$\psi(x) = \psi\left(\sum_{i=1}^m [x'_i + z'_i, x''_i + z''_i]\right) = \sum_{i=1}^m [\psi(x'_i + z'_i), \psi(x''_i + z''_i)]$$

is homogeneous in L_2 of degree g, as desired. □

Now we will see what happens if we strengthen the hypothesis on ψ by assuming, in addition, that it restricts to a G-graded isomorphism $\mathfrak{z}(L_1) \to \mathfrak{z}(L_2)$. This does not yet imply that ψ itself is a G-graded isomorphism, but we have the following:

Theorem 3.5.5. *Let L_1 and L_2 be G-graded Lie algebras, and assume that there exists an isomorphism of (ungraded) algebras $\psi : L_1 \to L_2$ such that both the induced map $L_1/\mathfrak{z}(L_1) \to L_2/\mathfrak{z}(L_2)$ and the restriction $\mathfrak{z}(L_1) \to \mathfrak{z}(L_2)$ are G-graded isomorphisms. Then L_1 and L_2 are isomorphic as G-graded algebras.*

Proof. Let $N_1 \subset \mathfrak{z}(L_1)$ be a graded subspace such that

$$\mathfrak{z}(L_1) = N_1 \oplus (\mathfrak{z}(L_1) \cap L_1').$$

By our hypothesis, $N_2 := \psi(N_1)$ is a graded subspace of $\mathfrak{z}(L_2)$. Since $L_1' \oplus N_1$ is a graded subspace of L_1, there exists a linearly independent set $\mathcal{B}_1 = \{u_i\}_{i\in\mathscr{I}}$ of homogeneous element of L_1 satisfying

$$L_1 = L_1' \oplus N_1 \oplus \operatorname{Span} \mathcal{B}_1.$$

By our hypothesis, we can find $z_i \in \mathfrak{z}(L_1)$ such that $\psi(u_i + z_i)$ is a homogeneous element of L_2 that has the same degree as u_i. Since $\mathfrak{z}(L_1) \subset L_1' \oplus N_1$, the set $\mathcal{B}_2 := \{\psi(u_i + z_i)\}_{i\in\mathscr{I}}$ is linearly independent and satisfies

$$L_2 = L_2' \oplus N_2 \oplus \operatorname{Span} \mathcal{B}_2.$$

Now define a linear map $\theta : L_1 \to L_2$ by setting $\theta|_{L_1'\oplus N_1} = 0$ and $\theta(u_i) = \psi(z_i)$ for all $i \in \mathscr{I}$. This is a "trace-like map" in the sense that its image is contained in $\mathfrak{z}(L_2)$ and its kernel contains L_1'. It follows that $\tilde{\psi} := \psi + \theta$ is an isomorphism of algebras $L_1 \to L_2$. Applying Lemma 3.5.4, we see that ψ, and hence $\tilde{\psi}$, restricts to a G-graded isomorphism $L_1' \oplus N_1 \to L_2' \oplus N_2$. By construction, $\tilde{\psi}(u_i) = \psi(u_i + z_i)$. It follows that $\tilde{\psi}$ is an isomorphism of G-graded algebras. □

Corollary 3.5.6. *Let Γ_1 and Γ_2 be two G-gradings on a Lie algebra L and consider the G-graded algebras $L_1 = (L, \Gamma_1)$ and $L_2 = (L, \Gamma_2)$. If $L_1/\mathfrak{z}(L_1) = L_2/\mathfrak{z}(L_2)$ and $\mathfrak{z}(L_1) = \mathfrak{z}(L_2)$ as G-graded algebras, then $L_1 \simeq L_2$ as G-graded algebras.*

Proof. Apply the previous theorem with ψ being the identity map. □

6. Gradings on $UT_n^{(-)}$

In this section, we classify group gradings on the upper triangular matrices, as a Lie algebra. The classification result is similar to the obtained in the Jordan case, but the calculations are harder. We keep the notations of the Jordan case, but we use

$$X_{i:m} = X_{i:m}^-, X_{i:m}' = X_{i:m}^+.$$

The reason to change from $X^\pm$ to X or X' is simple. Whenever we have a $\mathbb{Z}_2$-grading on $UT_n^{(-)}$ arising from an involution, one has $e_{i:m} - e_{-i:m}$ homogeneous of even degree. So, if we use + to denote an element of even degree, we would need to write $X^+_{i:m} = e_{i:m} - e_{-i:m}$, giving an inconsistence in the sign. Thus, we decided to use a neutral notation, avoiding the use of + and −.

6.1. **Initial considerations on gradings on $UT_n^{(-)}$.** In this subsection, we will give some notions which will be useful in the classification of group gradings on $UT_n^{(-)}$.

Definition 3.6.1. Let $A = \bigoplus_{g\in G} A_g$ be a G-graded algebra. The relevant support of the grading is

$$\mathrm{r}-\mathrm{supp}\, A = \{g \in G \mid A_g \not\subset \mathrm{Ann}(A)\}.$$

Here $\mathrm{Ann}(A) = \{a \in A \mid ab = ba = 0, b \in A\}$ is the annihilator of A.

Let G be a not necessarily abelian group, and let K be an arbitrary field. We fix a G-grading on $UT_n^{(-)}$.

Notation. We denote $J = [UT_n^{(-)}, UT_n^{(-)}]$ the set of all strictly upper triangular matrices.

It is clear that the ideal J^m is graded for any $m \geq 1$. If $m \geq n$ then $J^m = 0$.

Definition 3.6.2. Let $x \in UT_n^{(-)}$ be non-zero. We define the order of x, denoted by $o(x)$, as the least integer m such that $(x)_{(i,i+m)} \neq 0$ for some i (recall that $(x)_{(i,j)}$ stands for the entry (i, j) of x).

Definition 3.6.3. Let $x \in UT_n^{(-)}$ be an element of positive order m. We define $s(x) = \{(i, i + m) \mid (x)_{(i,i+m)} \neq 0\}$, and the weight of x, $w(x) = |s(x)|$.

We say that x, $y \in UT_n^{(-)}$ have the same non-zero entries if $o(x) = o(y)$ and $s(x) = s(y)$.

Definition 3.6.4. We say that $x \in UT_n^{(-)}$ is pure element if it is homogeneous, $m = o(x) > 0$, and there is no pair (y, z) such that:

(i) $y \neq 0$, $z \neq 0$, both y and z are homogeneous, $\deg y = \deg z = \deg x$ and $o(y) = o(z) = m$;

(ii) $w(x) = w(y) + w(z)$ and $x \equiv y + z \pmod{J^{m+1}}$.

Observe that we do not define pure elements of order 0.

As J^m/J^{m+1} is graded then there exist $n - m$ homogeneous elements of order m that are linearly independent modulo J^{m+1}, $0 \leq m \leq n-1$. If $x \in UT_n^{(-)}$ is homogeneous but not pure then we can change x to y and z in the notation above. Since $w(y) < w(x)$ and $w(z) < w(x)$

we can continue this process until obtaining pure elements. Observe that elements of weight 1 are pure, and the process ends in finitely many steps. Therefore, for $1 \leq m \leq n-1$, there exist $n-m$ pure elements of order m that are linearly independent modulo J^{m+1}.

Definition 3.6.5. We call a pair of elements $x,\, y \in UT_n^{(-)}$ a strange pair if both x and y are pure, $o(x) = o(y)$, and $s(x) \cap s(y) \neq \emptyset$, but $s(x) \neq s(y)$. A strange pair is of type 1 if $s(x) \subset s(y)$ or $s(y) \subset s(x)$, and of type 2 otherwise.

We will prove that there are no strange pairs.

Lemma 3.6.6. *Let u be an element of order 0. Then for every $1 \leq m \leq n-1$ and $1 \leq i \leq n-m$ one has $[u, e_{i,i+m}] \equiv \lambda_i^{(m)} e_{i,i+m} \pmod{J^{m+1}}$ for some constant $\lambda_i^{(m)} \in K$.*

Proof. One has $\mathrm{ad}(u)e_{i,i+m} \equiv ((u)_{(i,i)} - (u)_{(i+m,i+m)})e_{i,i+m} \pmod{J^{m+1}}$ and thus the lemma follows. □

Lemma 3.6.7. *Take $x,\, y \in UT_n^{(-)}$ with $o(x) = o(y) = m > 0$ and $s(y) \subset s(x)$. Then there exists u such that $[u, x] \equiv y \pmod{J^{m+1}}$. If, moreover, x and y are homogeneous, then we can choose u homogeneous.*

Proof. Let $u \in UT_n^{(-)}$ be an element of order 0. Write down the conditions on the entries of u needed to obtain $[u, x] \equiv y \pmod{J^{m+1}}$. We have the following equations

$$((u)_{(i,i)} - (u)_{(i+m,i+m)})(x)_{(i,i+m)} = (y)_{(i,i+m)}, \quad i = 1, 2, \ldots, n-m$$

which are independent. There are $w(x)$ equations and at least $w(x)+1$ variables. Hence there is such an element u with $[u, x] \equiv y \pmod{J^{m+1}}$.

Now assume x and y homogeneous. Write $u = u_1 + u_2 + \cdots + u_t$ as a sum of homogeneous elements. Then the expression

$$[u_1, x] + [u_2, x] + \cdots + [u_t, x] \equiv y \pmod{J^{m+1}}$$

involves homogeneous elements only. Hence there exist homogeneous u_l such that $[u_l, x] \equiv y \pmod{J^{m+1}}$ which proves the lemma. □

The following lemma will be important in studying properties of pure elements.

Lemma 3.6.8. *Suppose x and y homogeneous,* $\deg x = \deg y$*,* $m = o(x) = o(y)$ *and* $s(y) \subsetneq s(x)$*. Then x cannot be pure.*

In particular, if x is pure and y homogeneous, with $o(y) = o(x) = m$*,* $\deg x = \deg y$ *and* $s(y) \subset s(x)$*, then* $x \equiv \lambda y \pmod{J^{m+1}}$ *for some* λ*.*

Proof. We induct on $w(y)$.

If $w(y) = 1$ we take $x' = x + \lambda y$ such that $s(x') \cap s(y) = \emptyset$. Then $x \equiv x' - \lambda y \pmod{J^{m+1}}$, moreover $\deg x' = \deg y = \deg x$, $o(x') = o(y) = o(x)$ and $w(x) = w(x') + w(y)$, hence x is not pure.

Now suppose $w(y) > 1$, and let $x' = x + \lambda y$ be such that $w(x') < w(x)$. If $s(x') \cap s(y) = \emptyset$ we have nothing to do. Otherwise, take a homogeneous element u of order 0 such that $[u, x] = x' \pmod{J^{m+1}}$ (such u exists by Lemma 3.6.7), and consider $y' = [u, y]$. We have $\deg y' = \deg y$, $o(y') = o(y)$, and $w(y') < w(y)$. The result follows by induction. □

Now we treat homogeneous elements of order 0.

Lemma 3.6.9. *Let u be homogeneous of order 0,* $u \not\equiv \lambda e \pmod{J}$ *for all* λ*. Then:*

(i) *If* $x \in UT_n^{(-)}$ *with* $m = o(x) > 0$ *and* $[u, x] \not\equiv 0 \pmod{J^{m+1}}$ *then* $s([u, x]) = s(\mathrm{ad}(u)^m x)$ *for all* $m \in \mathbb{N}$*;*

(ii) $\deg u \in G$ *is of finite order;*

(iii) *If x is pure of order m then* $[u, x] \in J^{m+1}$ *or* $s(x) = s(\mathrm{ad}(u)^t x)$ *for every* $t \in \mathbb{N}$*.*

Proof. (i) Follows immediately from Lemma 3.6.6.

(ii) Since $u \not\equiv \lambda e \pmod{J}$ for all $\lambda \in K$ then there exists homogeneous $x \in J$ such that $[u, x] \neq 0$. Then $\mathrm{ad}(u)x$, $\mathrm{ad}(u)^2 x$, $\ldots$, $\mathrm{ad}(u)^m x$ are all non-zero by (i). Hence they are linearly dependent for sufficiently large m. The degrees of some of these elements coincide which implies $\deg u$ is of finite order.

(iii) By (i) we know $s(\mathrm{ad}(u)^t x) = s([u, x]) \subset s(x)$ for all $t \in \mathbb{N}$. Let $s([u, x]) \subsetneq s(x)$. Since by (ii), $\deg u$ is of finite order, there exists $y = \mathrm{ad}(u)^t x$ such that $s(y) \subsetneq s(x)$ and $\deg y = \deg x$, a contradiction to Lemma 3.6.8. □

Corollary 3.6.10. *There is no strange pair of type 1.*

Proof. This follows combining Lemmas 3.6.9 (ii) and (iii), and 3.6.8. □

Lemma 3.6.11. *Let x be pure with $w(x) > 1$. Then there exists y with the same non-zero entries as x (see Definition 3.6.3) such that* $\deg y \neq \deg x$.

Proof. Let u_1, u_2, …, u_n be homogeneous elements of order 0, linearly independent modulo J. Let $m = o(x)$ and take j such that $(x)_{(j,j+m)} \neq 0$. Assume that for each $i = 1, 2, \ldots, n$, either $[u_i, e_{j,j+m}] \equiv 0 \pmod{J^{m+1}}$ or $\deg u_i = 1$. Then there exists $z \in J$ such that $[e_{jj} + z, x] + J^{m+1} = e_{j,j+m} + J^{m+1}$ is homogeneous of the same degree as x, a contradiction.

Thus there is homogeneous u, $\deg u \neq 1$, with $[u, e_{j,j+m}] \notin J^{m+1}$. Hence $y = [u, x] \notin J^{m+1}$. Also $\deg y \neq \deg x$, and by Lemma 3.6.9 (iii), $s(y) = s(x)$. □

Lemma 3.6.12. *The sets* $U_1 = \mathrm{Span}\{e_{11}, e_{nn}, e\} + J$ *and* $T_1 = \mathrm{Span}\{e_{1n}, e_{1i}, e_{in} \mid i = 2, 3, \ldots, n-1\}$ *are graded ideals.*

Proof. $U_1 = \mathrm{Span}\{e_{11}, e_{nn}, e\} + J$ is a graded ideal. Indeed, let $A = \mathrm{Ann}(J^{n-2})$ then e_{12}, $e_{n-1,n} \notin A$ but $e_{i,i+1} \in A$ for all remaining $e_{i,i+1}$. One checks easily that A consists of all matrices whose entries $(1,1)$, $(2,2)$, $(n-1, n-1)$, (n,n), $(1,2)$, and $(n-1,n)$ are zeros, plus the multiples of the identity matrix. As J^{n-2} is graded then A is also graded. Then $B = \mathrm{Ann}_{UT_n^{(-)}}((A + J^2)/J^2)$ is a graded ideal.

We shall prove that $U_1 = B + J$. Since $e_{i,i+1} \in A$ we get $e_{ii} \notin B$ for $i = 2, 3, \ldots, n-1$. Also e_{11}, $e_{nn} \in B$ and we have $U_1 = B + J$.

Since $T_1 = [J, U_1, U_1, \cdots, U_1]$, $n-2$ entries of U_1, then T_1 is also graded. □

Proposition 3.6.13. *There is no strange pair of type 2.*

Proof. We induct on n. If $n \leq 4$ and x is a pure element of order m, then $w(x) \leq 3$. If x, y form a strange pair of type 2, then by Lemma 3.6.11, there exists at least 4 linearly independent elements of order $o(x)$. But $\dim J^m/J^{m+1} \leq 3$, a contradiction.

Now let $n > 4$, and assume the lemma holds for $UT_l^{(-)}$, $l < n$. Let $m \in \{1, 2, \ldots, n-1\}$. Choose pure elements of order m, say x_1, x_2, …, x_{n-m} that are linearly independent modulo J^{m+1}.

Claim 1: There exist pure elements x', y' of order m such that $s(x')$, $s(y') \subset \{(1, 1+m), (n-m, n)\}$.

Indeed, $J^m \cap T_1$ is graded (in the notation of Lemma 3.6.12).

Claim 2: If x is pure then either $s(x) \subset \{(1, 1+m), (n-m, n)\}$ or $|s(x) \cap \{(1, 1+m), (n-m, n)\}| \leq 1$.

If x is pure, $\{(1,1+m),(n-m,n)\} \subset s(x)$, and $w(x) > 2$ then there would be a strange pair of type 1: x with x' or with y' of Claim 1, a contradiction.

Claim 3: If x is pure and $|s(x) \cap \{(1,1+m),(n-m,n)\}| = 1$ then $w(x) = 1$.

Suppose there is a pure x with $\{(1,1+m),(n-m,n)\} \cap s(x) = \{(1,1+m)\}$. Let y be pure with $(y)_{(1+m,n)} \neq 0$, then $[x,y] \neq 0$. If $w(x) > 1$, by Lemma 3.6.11, there would exist pure z with the same non-zero entries as x, and $\deg x \neq \deg z$. But $[y,z] \neq 0$ implies $\deg x = \deg y$, a contradiction. The case of $\{(n-m,n)\}$ is similar.

Now we prove the lemma. As $UT_n^{(-)}/T_1 \cong UT_{n-2}^{(-)}/\mathfrak{z}(UT_{n-2}^{(-)})$ one uses induction and the previous claims. □

6.2. **almost elementary gradings.** In this subsection, we will prove a sufficient condition for a grading on $UT_n^{(-)}$ be practically isomorphic to an elementary grading.

Definition 3.6.14. A G-grading on $UT_n^{(-)}$ is called almost elementary if all pure elements have weight 1 (see Definition 3.6.3 and Definition 3.6.4).

It follows from the above definition that a grading is almost elementary if it is elementary up to entries of larger order. In other words for each e_{ij} there is some $z_{ij} \in J^{j-i+1}$ such that $e_{ij} + z_{ij}$ is homogeneous. We shall prove below that in this case we can "diagonalize" the grading and obtain an elementary grading.

Pay attention that z_{ij} need not be unique. Let $L = UT_3^{(-)}$ be equipped with the trivial grading. If $z_{12} = e_{13}$ then $e_{12} + z_{12}$ is homogeneous. Also $e_{12} + 0$ is homogeneous.

Lemma 3.6.15. *If the grading on $UT_n^{(-)}$ is almost elementary then up to practically the same grading, each homogeneous element of order 0 is of degree 1.*

Proof. We can assume, up to practically the same grading, that $\deg e = 1$. For each $i < j$ the element $e_{ij} + z_{ij}$ is pure for some $z_{ij} \in J^{j-i+1}$. There exists homogeneous element u of degree 1 and order 0 such that $(u)_{(i,i)} \neq (u)_{(j,j)}$. (Otherwise $e_{ij} \in Ann(UT_n^{(-)}/J^{j-i+1}) = eK$). Moreover $\deg u = 1$ since $[u, e_{ij} + z_{ij}] \equiv \lambda e_{ij} \pmod{J^{j-i+1}}$ for some $0 \neq \lambda \in K$. Therefore there exist at least $n-1$ homogeneous, linearly independent (modulo J) elements of order 0, none of which equals e. This concludes the lemma. □

Proposition 3.6.16. *Every almost elementary grading is practically G-graded isomorphic to an elementary grading.*

Proof. According to the previous lemma we can assume, up to practically the same grading, that each element of order 0 is of degree 1.

For every $i = 1, 2, \ldots, n$, we assume that $e_{ii} + z_i$ is homogeneous of degree 1, for some $z_i \in J$. Denote by $I_1 = \mathrm{Span}\{e_{1i} \mid i = 1, 2, \ldots, n\}$ the first row. Then I_1 is an ideal although not necessarily graded. We first show that we can assume $z_1 \in I_1$, and then we prove the proposition, following an idea of [63].

Claim 1: We can assume $z_1 \in I_1$.

Let m be the largest positive integer such that $e_{11} + z_1 \in I_1 + J^m$, and let $i > 1$ be the least integer such that $(z_1)_{(i,i+m)} = a \neq 0$. The element

$$z = [e_{11} + z_1, e_{ii} + z_i] = \underbrace{[e_{11}, e_{ii} + z_i]}_{\in I_1} + \underbrace{[z_1, e_{ii}]}_{\equiv -ae_{i,i+m} \pmod{J^{m+1}}} + \underbrace{[z_1, z_i]}_{\in I_1 + J^{m+1}}$$

is homogeneous of degree 1. Hence we can change $e_{11} + z_1$ to $e_{11} + z_1 + z$ in order to obtain a new element whose $(i, i + m)$ entry is zero. Continuing this process, we obtain the claim.

Claim 2: Up to a practically G-graded isomorphism, the elements $e_{11}, e_{22}, \ldots, e_{nn}$ are homogeneous of degree 1.

This claim is proved by induction on n. By Claim 1, the homogeneous element $e_{11} + z_1$ is such that $z_1 \in I_1$. The matrix $e_{11} + z_1$ is diagonalizable, since it is idempotent. Now up to a G-graded isomorphism, we can assume that e_{11} is homogeneous since the automorphism of $UT_n^{(-)}$ such that $e_{11} + z_1 \mapsto e_{11}$ induces a G-grading on $UT_n^{(-)}$ where e_{11} is homogeneous. Define

$$\psi \colon x \in UT_n^{(-)} \mapsto x - [e_{11}, x] - (x)_{(1,1)} e_{11} \in UT_n^{(-)}.$$

Note that the image of ψ is a graded subalgebra of $UT_n^{(-)}$ which is isomorphic to $UT_{n-1}^{(-)}$ with the induced grading. The claim follows immediately by induction since $[e_{11}, \psi(UT_n^{(-)})] = 0$.

Claim 3: The conclusion of the proposition holds.

By Claim 2, we assume that, up to a practically G-graded isomorphism, $e_{11}, e_{22}, \ldots, e_{nn}$ are homogeneous of degree 1. Let $x_{ij} = e_{ij} + z_{ij}$ be a homogeneous element with $(x_{ij})_{(i,j)} = 1$, $o(x_{ij}) = j - i + 1$. Then $e_{ij} = [x_{ij}, -e_{ii}, e_{jj}]$ is homogeneous, proving the claim and the proposition. □

In particular, if $UT_n^{(-)}$ is endowed with an almost elementary grading then the relevant support of the grading generates an abelian group.

6.3. **almost type II gradings.** In this subsection, we provide a sufficient condition for a grading to be of type II.

Definition 3.6.17. Assume that there exists $t \in G$, $o(t) = 2$, and there are elements $y_1^{(0)\prime}$, $\dots$, $y_q^{(0)\prime}$ that are homogeneous of order 0 and of weight 2, with $\deg y_i^{(0)\prime} = t$ where $q = \lceil \frac{n-1}{2} \rceil$, and

$$(y_i^{(0)\prime})_{(i:0)} = (y_i^{(0)\prime})_{(-i:0)} = 1, \quad i = 1, 2, \dots, q.$$

The grading on $UT_n^{(-)}$ is almost type II if there exist homogeneous $y_1^{(1)}, \dots, y_q^{(1)}$ such that:

(i) Kind 1: $y_i^{(1)} \equiv e_{i:1} - a_i e_{-i:1} \pmod{J^2}$ for some $a_i \neq 0$ and for every $1 \leq i \leq q$;
(ii) Kind 2: $y_i^{(1)} \equiv e_{i:1} - e_{-i:1} \pmod{J^2}$ for every $1 \leq i \leq q$.

As it was done in Lemma 3.6.15, one obtains that for an almost type II grading there exists homogeneous elements of order 0 and weight 2, namely $y_1^{(0)}, y_2^{(0)}, \cdots, y_q^{(0)}$, with $q = \lceil \frac{n-1}{2} \rceil$, with

$$(y_i^{(0)})_{(i:0)} = -(y_i^{(0)})_{(-i:0)} = 1.$$

Let us compare this to the case of almost elementary gradings (see Definition 3.6.14). In an almost type II grading of kind 2 the homogeneous elements are the same as in a type II grading, up to entries of larger degree. Observe that

$$[y_{i_1}^{(j_1)*}, y_{i_2}^{(j_2)*}, \dots, y_{i_m}^{(j_m)*}] \equiv [X_{i_1}^{(j_1)*}, X_{i_2}^{(j_2)*}, \dots, X_{i_m}^{(j_m)*}] \pmod{J^{j_1+\dots+j_m+1}}$$

where all $j_r = 0$ or 1, and $y_{i_r}^{(j_r)*}$ stands for $y_{i_r}^{(j_r)}$or $y_{i_r}^{(j_r)\prime}$, and likewise for $X_{i_r}^{(j_r)*}$.

As before the elements $y_i^{(m)}$, $y_i^{(m)\prime}$ need not be unique.

Proposition 3.6.18. *If a grading is almost type II of kind 1 then it is, up to a G-graded isomorphism, almost type II of kind 2. If a grading is almost type II then it is, up to practically G-graded isomorphism, a type II grading.*

Proof. For the first part, the proof is similar to that of Lemma 3.3.5. Take $a = a_1 a_2 \cdots a_p$ (the a_i were given in Definition 3.6.17), and take A as in Lemma 3.3.5 (replace a for ϵ). Then $x \in UT_n^{(-)} \mapsto AxA^{-1} \in UT_n^{(-)}$ is an isomorphism. It induces an almost type II grading of kind 2 on $UT_n^{(-)}$.

Now, we split the proof of the second part into several steps.

Claim 1: There exists an inner automorphism ψ of $UT_n^{(-)}$ such that $\psi((y_1^{(0)\prime} + y_1^{(0)})/2) = e_{11}$.

We can assume, up to a practically the same grading, that $\deg e = t$. For each $i = 1$, 2, $\ldots$, n, we can write $e_{ii} \equiv (y_i^{(1)\prime} \pm y_i^{(1)})/2 \pmod J$, a sum of an element of degree 1 and an element of degree t. Note that the commutator $[y_i^{(1)\prime} \pm y_i^{(1)}, y_j^{(1)\prime} \pm y_j^{(1)}]$ is also a sum of a homogeneous element of degree 1 and a homogeneous element of degree t. We can repeat the argument in the proof of Proposition 3.6.16, making adequate changes in $y_1^{(0)}$ and $y_1^{(0)\prime}$, in order to diagonalize the element $(y_1^{(0)\prime} + y_1^{(0)})/2$ and conclude the claim.

Let $x_1 = (y_1^{(0)\prime} + y_1^{(0)})/2 = e_{11}$ and $x_n = (y_1^{(0)\prime} - y_1^{(0)})/2$. We shall consider the Jordan canonical form for x_n, but beforehand we establish another claim.

Claim 2: We can assume that $[x_1, x_n] = 0$.

Let $I_1 = \operatorname{Span}\{e_{1i} \mid i = 2, 3, \ldots, n\} = \operatorname{Im}\operatorname{ad}(e_{11})$.

The element $I_1 \ni 4[x_n, e_{11}] = [y_1^{(0)\prime} - y_1^{(0)}, y_1^{(0)\prime} + y_1^{(0)}] = 2[y_1^{(0)}, y_1^{(0)\prime}]$ is homogeneous. Thus $[x_n, e_{11}] = \lambda e_{1n}$ since the unique homogeneous elements in I_1 are scalar multiples of e_{1n} (as the grading is almost type II). If $\lambda = 0$ we are done. If $\lambda \neq 0$ then $\deg e_{1n} = t$, and we can replace $y_1^{(0)\prime}$ by $y_1^{(0)\prime} + \lambda' e_{1n}$, for an adequate λ', in order to obtain $[e_{11} + \lambda' e_{1n}, x_n] = 0$. Note that we substitute $x_1 = e_{11}$ with $x_1 = e_{11} + \lambda' e_{1n}$ and x_n by $x_n + \lambda' e_{1n}$.

Observe that the “new” x_1 is also diagonalizable.

Claim 3: There exists an inner automorphism ψ of $UT_n^{(-)}$ such that $\psi(x_1) = e_{11}$, and $\psi(x_n)$ is in Jordan canonical form.

Assume that $UT_n^{(-)}$ acts on K^n by left multiplication. Since $[x_1, x_n] = 0$ then the eigenspaces of x_1 are invariant under x_n. The linear transformation x_1 has an eigenvector v_1 corresponding to eigenvalue 1. There exists a subspace W, $\dim W = n - 1$, such that $x_1|_W = 0$. Also, note that v_1 is an eigenvector for x_n corresponding to the eigenvalue 0. Moreover $x_n(W) \subset W$. We can consider then $x_n|_W$ in Jordan canonical form and thus obtain the claim.

Claim 4: x_n is diagonal.

By the previous claim, we assume $x_1 = e_{11}$ and x_n is in Jordan canonical form. The condition $[x_1, x_n] = 0$ implies that the first row of x_n is 0. Let $P = J^2 + \operatorname{Span}\{y_1^{(1)}, y_1^{(1)\prime}\}$, then P is a graded ideal in $UT_n^{(-)}$.

The elements $y_1^{(0)} = e_{11} + x_n$ and $y_1^{(0)\prime} = e_{11} - x_n$ are homogeneous, and their non-zero entries are either on the main diagonal or on the first diagonal above it. Let $q = \lceil \frac{n}{2} \rceil$ and

$1 \leq i \leq q$, then

$$[y_i^{(0)}, -y_1^{(0)}] \equiv [y_i^{(0)}, y_1^{(0)\prime}] \pmod{P}, \quad [y_i^{(0)\prime}, -y_1^{(0)}] \equiv [y_i^{(0)\prime}, y_1^{(0)\prime}] \pmod{P}.$$

As $\deg[y_i^{(0)}, -y_1^{(0)}] \neq \deg[y_i^{(0)}, y_1^{(0)\prime}]$ then all these commutators lie in P. In particular $y_1^{(0)}$ and $y_1^{(0)\prime}$ have no non-zero entries at positions $(2,3)$, $(3,4)$, $\dots$, $(n-2, n-1)$. But this implies x_n is diagonal since its entries $(1,2)$ and $(n-1,n)$ are zero.

Claim 5: Up to a practically G-graded isomorphism, we can assume that the elements of order 0 are diagonal.

We know that, up to a practically G-graded isomorphism, the elements $e_{11}+e_{nn}$ and $e_{11}-e_{nn}$ are homogeneous. But $(1-\mathrm{ad}(e_{11}-e_{nn}))UT_n^{(-)}$ is a graded subalgebra isomorphic to $UT_{n-2}^{(-)} \oplus K(e_{11}+e_{nn})$. We apply induction on n and obtain the claim by repeating the previous two steps on $UT_{n-2}^{(-)}$.

Claim 6: The grading is, up to a practically G-graded isomorphism, of type II.

By the previous claim we assume $e_{ii} - e_{n-i+1,n-i+1}$ is homogeneous of degree 1 for each i. Let x be a pure element of positive order, and let m be the largest integer such that $m > o(x)$ and $(x)_{(i,i+m)} \neq 0$ for some i. Then $y = [e_{ii} - e_{n-i+1,n-i+1}, x]$ is homogeneous of the same degree as x, and can be used to vanish the entry $(x)_{(i,i+m)}$. This preserves the weight of x. Doing this, we reach a type II grading. □

6.4. **Conclusion: gradings on $UT_n^{(-)}$.** We already proved that there is no strange pair in any G-grading on $UT_n^{(-)}$ (Corollary 3.6.10 and Proposition 3.6.13). Also e_{1n} is always homogeneous. Let z_1 and z_2 be pure elements of order $n-2$. As there are no strange pairs, necessarily $w(z_1) = w(z_2)$.

Let $T_j = \mathrm{Span}\{e_{1n}, e_{li}, e_{n-i+1,n-l+1} \mid l = 1, 2, \dots, j, i = l+1, l+2, \dots, n-1\}$. We can prove that T_j is a graded ideal in a similar way as T_1 (Lemma 3.6.12). We define as in Lemma 3.6.12 the sets U_2, U_3, $\dots$ and these will be graded. The absence of strange pairs implies that all pure elements have weight at most 2, and if x is pure of weight 2, then $s(x) = \{(i:m), (-i:m)\}$: see which are the pure elements of order m in $T_i \cap J$ and $T_{i+1} \cap J$. This proves

Lemma 3.6.19. *There exist homogeneous elements x_1, x_2, $\dots$, x_p where $p = \lfloor \frac{n}{2} \rfloor$, of weight 1 and $x_i \equiv e_{i,n-i+1} \pmod{J^{n-2i+2}}$.*

The previous lemma says that the element e_{1n} is homogeneous, $e_{2,n-1}$ is homogeneous up to J^{n-2}, etc.

Corollary 3.6.20. *The set* $U = \{x \in UT_n \mid (x)_{(i,i)} = (x)_{(n-i+1,n-i+1)}, 1 \le i \le n\}$ *is graded.*

Proof. We have $U = \cap_i \mathrm{Ann}_{UT_n^{(-)}}((Span\{x_i\} + J^{n-2i+2})/J^{n-2i+2})$, in the notation of the previous lemma. □

The set $U \cap U_1$ (see Lemma 3.6.12) is homogeneous. Let $u_1 \in U \cap U_1$ be homogeneous of order 0 and weight 2 such that $(u_1)_{(1,1)} = (u_1)_{(n,n)} = 1$.

Lemma 3.6.21. *If* $w(z_1) = 2$ *then the grading is almost type II, and if* $w(z_1) = 1$ *then the grading is almost elementary.*

Proof. Assume $w(z_1) = 2$. Then necessarily $\deg u_1$ has order 2. If y is pure element of order 1 with $(y)_{(1,2)} \neq 0$ then, since $[u_1, y] \notin J^2$, y is of weight 2 and $(y)_{(n-1,n)} \neq 0$. If $y \equiv ae_{12} + be_{n-1,n} \pmod{J^2}$ then $[u_1, y] \equiv ae_{12} - be_{n-1,n} \pmod{J^2}$, and this is a multiple of the elements of the form $y_1^{(1)}$ and $y_1^{(1)\prime}$ in the definition of almost type II. Now consider a homogeneous element $u_2 \in U_2 \cap U$ with $(u_2)_{(2,2)} = (u_2)_{(n-1,n-1)} = 1$. Then, looking at appropriate commutators $[u_2, z_1]$ or $[u_2, y]$, we have $\deg u_1 = \deg u_2$, and we can form a linear combination to guarantee $(u_2)_{(1,1)} = (u_2)_{(n,n)} = 0$. Continuing this process, we obtain an almost type II grading.

Now assume $w(z_1) = 1$. This implies $\deg u_1 = 1$. If y is pure of order 1 with $(y)_{(1,2)} \neq 0$, since $s([u_1, y]) = s(y)$, this gives $w(y) = 1$. Similarly if y is pure of order 1 and $(y)_{(n-1,n)} \neq 0$ then $w(y) = 1$. If we look at an element $u_2 \in U_2 \cap U$ with $(u_2)_{(2,2)} = (u_2)_{(n-1,n-1)} = 1$ then, forming adequate commutators, and continuing the process, we obtain that pure elements of order 1 have weight 1. Hence we obtain an almost elementary grading. □

Since almost elementary is elementary (Proposition 3.6.16) and almost type II is type II (Proposition 3.6.18), up to practically graded isomorphisms, Lemma 3.6.21 yield the proof of the following theorem.

Theorem 3.6.22. *Let* $UT_n^{(-)}$ *be* G*-graded. Then the relevant support of the grading is commutative and, up to a practically* G*-graded isomorphism, the grading is either elementary or of type II.*

Chapter 4
Group gradings on block-triangular matrices

In this chapter we study the group gradings on the algebra of upper block-triangular matrices, viewed as associative, Lie and Jordan algebras.

It was proved by Valenti and Zaicev, in 2011, that if G is a finite abelian group and K is an algebraically closed field of characteristic zero, then any G-grading on the algebra of upper block-triangular matrices over K is isomorphic to a tensor product $M_n(K)\otimes UT(n_1, n_2, \ldots, n_s)$, where $UT(n_1, n_2, \ldots, n_s)$ is endowed with an elementary grading and $M_n(K)$ is endowed with a division grading. We prove the same description for arbitrary grading group, and under mild conditions on the base field (its characteristic must be either zero or large enough).

Furthermore we investigate the group gradings using the duality between gradings by a group, and actions by the dual group. To this end we shall assume the grading group abelian, and the base field algebraically closed of characteristic zero. We prove that every group grading on the algebra of block-triangular matrices is induced by a grading on the matrix algebra. Hence we obtain a new approach to the classification of group gradings on the block-triangular matrices, and we are able to determine the isomorphism classes of its gradings. We use the same technique to provide a classification of isomorphism classes of group gradings on the algebra of block-triangular matrices, as a Lie algebra, proving that every grading is induced by some grading on the Lie algebra $\mathfrak{sl}_n$.

Finally, we obtain the classification of group gradings on the upper block-triangular matrices, viewed as a Jordan algebra. It turns out that, under the same restrictions on the grading group and on the base field, the Jordan case is essentially equivalent to the Lie case.

Notations. Given two subalgebras $A_1 \subset M_{n_1}$ and $A_2 \subset M_{n_2}$ of matrix algebras, we canonically identify their tensor product $A_1 \otimes_K A_2$ as a subalgebra of $M_{n_1 n_2}$, via the usual Kronecker product.

Denote by J the Jacobson radical of $U = UT(n_1, n_2, \ldots, n_s)$. Denote also by M_{ij} the block of matrices, so that we can write (as vector spaces) $U = \oplus_{1\le i\le j\le t} M_{ij}$. Formally,

$$M_{ij} = \mathrm{Span}\{e_{k\ell} \mid n_1 + \ldots + n_i < k \le n_1 + \ldots + n_{i+1}, n_1 + \ldots + n_j < \ell \le n_1 + \ldots + n_{j+1}\}.$$

Thus in such notation $J = \bigoplus_{i<j} M_{ij}$. For each k, let $J_k = M_{1,1+k} \oplus \cdots \oplus M_{s-k,s}$.

1. Associative case

Let G be any group and K be any field. Consider any group grading on $UT(n_1, \ldots, n_s)$. We prove that certain subspaces are graded.

Lemma 4.1.1. *If J is graded then all M_{ij} are graded subspaces, up to an isomorphism.*

Proof. Recall that the right annihilator of a graded subset is again graded. As the radical J is graded, one obtains that $R := \mathrm{Ann}^r_U(J) = \bigoplus_{j=1}^t M_{1j}$ (the right annihilator of J) is also graded.

If an associative algebra has a left unit, then there exists a homogeneous left unit in the algebra (see Proposition 1.1.3 of chapter 1). Note that R has a left unit (the identity matrix $E_1 \in M_{11}$), hence it must admit a homogeneous left unit, say u_1. Clearly $u_1^2 = u_1$, hence u_1 is diagonalizable; moreover, the diagonal form of u_1 is exactly E_1. So we can assume E_1 homogeneous, up to a graded isomorphism.

Now, since $(1-E_1)U \simeq UT(n_2, n_3, \ldots, n_s)$ we can proceed by induction. Moreover, if $i < j$ and E_i and E_j are the identity matrices of M_{ii} and M_{jj}, respectively, then $M_{ij} = E_i U E_j$ is a graded subspace. $\square$

Group gradings on matrix algebras are well known, see for instance [35, Chapter 2]. It follows that every $M_{ii} \simeq M_{p_i} \otimes D_i$, where M_{p_i} is a matrix algebra equipped with an elementary grading given by $(g_1, \ldots, g_{p_i})$, and D_i is a graded division algebra. Here the grading on $M_{p_i} \otimes D_i$ is induced by

$$\deg e_{ij} \otimes d = g_i (\deg d) g_j^{-1}.$$

It is well known that every automorphism of a matrix algebra is inner, hence we can find an invertible matrix A_i such that $A_i M_{ii} A_i^{-1} = M_{p_i} \otimes D_i$. Taking the block-diagonal matrix $A' = \mathrm{diag}(A_1, A_2, \ldots, A_s)$, we obtain an automorphism of U such that every $M_{ii} = M_{p_i} \otimes D_i$.

Lemma 4.1.2. *In the notations above, if J is graded, then there exists a graded division ring D, and elements $g_1, g_2, \ldots, g_s \in G$ such that $D_i = {}^{[g_i]}D^{[g_i^{-1}]}$. Moreover, $U \simeq U' \otimes D$, where U' is endowed with an elementary G-grading.*

Proof. For every $i = 1, 2, \ldots, s$, denote by $e_i \in D_i$ the unit element of the graded division algebra D_i, and denote $e_{11}^{(i)} \in M_{p_i}$ the matrix unit with 1 in the entry $(1,1)$ of the i-th matrix block, and 0 elsewhere. For $i < j$, let $X = e_i e_{11}^{(i)} U e_j e_{11}^{(j)}$ (here $e_i e_{11}^{(i)}$ stands for $e_{11} \otimes e_i \in M_{p_i} \otimes D_i = M_{ii}$, and analogously for $e_j e_{11}^{(j)}$).

Note that X is a D_i-left module and a D_j-right module. If D_i consists of $n_i \times n_i$ matrices and D_j of $n_j \times n_j$ matrices then X is identified with $n_i \times n_j$ matrices. From the structure of graded modules over graded division algebras, we obtain $n_i n_j = k_1 n_i^2 = k_2 n_j^2$, for some k_1, $k_2 \in \mathbb{N}$. This is possible only if $n_i = n_j$, hence given a non-zero homogeneous $v \in X$ of degree $h \in G$, we have $X = D_i v = v D_j$. As a consequence, for every $x \in D_i$ there exists $y \in D_j$ such that $xv = vy$. Hence $\deg x = h(\deg y)h^{-1}$. We define the map $T\colon x \in D_i \mapsto y \in D_j$. Clearly T is a linear map with $\deg T(x) = h^{-1}(\deg x)h$, furthermore $vT(x_1x_2) = x_1x_2v = x_1vT(x_2) = vT(x_1)T(x_2)$. Since D_i is a graded division algebra, one obtains $T(x_1x_2) = T(x_1)T(x_2)$, which means T is a weak isomorphism between D_i and D_j. This proves the first part of the Lemma.

Considering now all matrix units $e_{ij}^{(r)} \in M_{p_r}$, $e_{mn}^{(s)} \in M_{p_s}$ we can repeat the argument for $e_{ij}^{(r)} U e_{mn}^{(s)}$, and we conclude that it is a graded (D_r, D_s)-bimodule of dimension $\dim D$. Thus we obtain $U \simeq U' \otimes D$ for some upper block-triangular matrix algebra U' endowed with an elementary grading. □

A very important result is the following statement, due to A. Gordienko.

Lemma 4.1.3 (Corollary 3.3 of [43])**.** *Let A be a finite-dimensional associative algebra over a field K graded by any group G. Suppose that either* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > \dim A$*. Then the Jacobson radical $J(A)$ is a graded ideal of A.*

Combining Gordienko's result with Lemma 4.1.2, we obtain

Theorem 4.1.4. *Let G be any group and consider any G-grading on the upper block-triangular matrix algebra $U = UT(n_1, n_2, \ldots, n_s)$ over a field K. Suppose that either* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > \dim U$*. Then there exists a G-graded division algebra structure on $D = M_n(K)$ and an upper block-triangular matrix algebra $U' = UT(n_1', n_2', \ldots, n_s')$ endowed with an elementary grading, such that $U \simeq U' \otimes D$.*

2. Inducing group gradings

2.1. **Preliminaries.** Let T be a finite abelian group and let $\sigma\colon T\times T\to K^\times$ be a map. We say that σ is a *2-cocycle* if

$$\sigma(u,v)\sigma(uv,w)=\sigma(u,vw)\sigma(v,w),\qquad \text{for every } u,v,w\in T.$$

The twisted group algebra $K^\sigma T$ is constructed as follows: it has $\{X_u\mid u\in T\}$ as a K-vector space basis, and the multiplication is given by $X_uX_v=\sigma(u,v)X_{uv}$. It is readily seen that $K^\sigma T$ is an associative algebra if and only if σ is a 2-cocycle, which we will assume from now on. Note that $A=K^\sigma T$ has a natural T-grading, where each homogeneous component has dimension 1, namely $A_u=\operatorname{Span}\{X_u\}$, for each $u\in T$. This is an example of the so-called graded division algebra. Recall that a graded algebra D is a *graded division algebra* if every non-zero homogeneous element is invertible. We point out that D need not be a division ring: there may be non-invertible elements in D.

Define $\beta\colon T\times T\to K^\times$ by $\beta(u,v)=\sigma(u,v)\sigma(v,u)^{-1}$. Then β is an alternating (also called skew-symmetric) bicharacter of T. Since T is finite $K^\sigma T$ is semisimple as ordinary algebra, as long as $\operatorname{char}K$ does not divide $|T|$. It follows that $K^\sigma T$ is a simple algebra if and only if β is non-degenerate. In particular, the non-degeneracy of β implies that $|T|=\dim K^\sigma T$ is a perfect square. It is known that, if K is algebraically closed, the isomorphism classes of matrix algebras endowed with a division grading by an abelian group are in 1–1 correspondence with the pairs (T,β) where T is a finite subgroup of G and $\beta\colon T\times T\to K^\times$ is a non-degenerate alternating bicharacter (see for example [35, Theorem 2.15]).

For each n-tuple $(g_1,\dots,g_n)$ of elements of G, we can define a G-grading on M_n by declaring that the matrix unit e_{ij} is homogeneous of degree $g_ig_j^{-1}$, for all i and j. Such gradings on M_n are called *elementary*. For any $g\in G$ and any permutation $\sigma\in S_n$, the n-tuple $(g_{\sigma(1)}g,\dots,g_{\sigma(n)}g)$ defines an isomorphic elementary G-grading. Hence an isomorphism class of elementary gradings is described by a function $\kappa\colon G\to\mathbb{Z}_{\geq0}$ where $g\in G$ appears exactly $\kappa(g)$ times in the n-tuple. Moreover G acts on these functions by translation: given $g\in G$, one defines $g\kappa$ as the function $G\to\mathbb{Z}_{\geq0}$ by $g\kappa(x)=\kappa(g^{-1}x)$. For every $\kappa\colon G\to\mathbb{Z}_{\geq0}$ with finite support, we denote $|\kappa|:=\sum_{x\in G}\kappa(x)$.

For a fixed abelian group G, the isomorphism classes of G-gradings on M_n are parametrized by the triples (T,β,κ). Here T is a finite subgroup of G, $\beta\colon T\times T\to K^\times$ is a non-degenerate alternating bicharacter, and $\kappa\colon G/T \to \mathbb{Z}_{\geq 0}$ is such that $|\kappa|\sqrt{|T|} = n$. A grading in the isomorphism class corresponding to (T, β, κ) can be explicitly constructed by making the following two choices:

(i) a k-tuple $\gamma = (g_1, \dots, g_k)$ of elements in G such that each element $x \in G/T$ occurs in γ exactly $\kappa(x)$ times (in particular $k = |\kappa|$), and

(ii) a matrix realization of the graded division algebra D with support T and bicharacter β, that is an isomorphism $D \simeq M_\ell$ where $|T| = \ell^2$. For $n = k\ell$, recall that we identify M_n with $M_k \otimes D$ via the Kronecker product and we define a G-grading by declaring the matrix $e_{ij} \otimes d$, with $1 \leq i, j \leq k$, and d a nonzero homogeneous element of D, to be of degree $g_i \deg(d) g_j^{-1}$.

Finally, two triples (T, β, κ) and (T', β', κ') determine the same isomorphism class if and only if $T' = T$, $\beta' = \beta$, and there exists $g \in G$ such that $\kappa' = g\kappa$ (see e.g. [35, Theorem 2.27]).

2.2. **Associative case revisited.** Let K be an algebraically closed field and let V be a finite-dimensional K-vector space. Denote by $\mathscr{F}$ a flag of subspaces in V, that is

$$0 = V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_s = V.$$

Let $n = \dim V$ and $n_i = \dim V_i/V_{i-1}$, for $i = 1, 2, \dots, s$. We denote by $U(\mathscr{F})$ the set of endomorphisms of V preserving the flag $\mathscr{F}$, which coincides with the upper block-triangular matrices $UT(n_1, \dots, n_s)$ after a choice of basis of V respecting the flag $\mathscr{F}$. We fix such a basis and identify $U(\mathscr{F}) = UT(n_1, \dots, n_s) \subset M_n$.

For each $m \in \mathbb{Z}$, if $|m| < s$, let $J_m \subset M_n$ denote the m-th block-diagonal of matrices. Formally,

$$\begin{aligned} J_m = \operatorname{Span}\{E_{ij} \in M_n \mid\ & \text{there exists } q \in \mathbb{Z}_{\geq 0} \text{ such that} \\ & n_1 + \cdots + n_q < i \leq n_1 + \cdots + n_{q+1}, \text{and} \\ & n_1 + \cdots + n_{q+m} < j \leq n_1 + \cdots + n_{q+m+1}\}. \end{aligned}$$

Setting $J_m = 0$ for $|m| \geq s$, we obtain a $\mathbb{Z}$-grading $M_n = \bigoplus_{m \in \mathbb{Z}} J_m$, which is the elementary grading defined by the n-tuple

$$(\underbrace{-1, \dots, -1}_{n_1 \text{ times}}, \underbrace{-2, \dots, -2}_{n_2 \text{ times}}, \dots, \underbrace{-s, \dots, -s}_{n_s \text{ times}}).$$

This grading restricts to $U(\mathscr{F})$, and we will refer to the resulting grading $U(\mathscr{F}) = \bigoplus_{m\geq 0} J_m$ as the *natural* $\mathbb{Z}$*-grading* of $U(\mathscr{F})$. The associated filtration consists of the powers of the Jacobson radical J of $U(\mathscr{F})$, that is, $J^m = \bigoplus_{i\geq m} J_i$ for all $m \geq 0$.

Let G be any abelian group and denote $G^\# = \mathbb{Z} \times G$. We identify G with the subset $\{0\} \times G \subset G^\#$ and $\mathbb{Z}$ with $\mathbb{Z} \times \{1\} \subset G^\#$. We want to find a relation between $G^\#$-gradings on M_n and G-gradings on $U(\mathscr{F})$.

First, we note that, given any $G^\#$-grading on M_n, we obtain a $\mathbb{Z}$-grading on M_n if we consider the coarsening induced by the projection onto the first component $G^\# \to \mathbb{Z}$.

Definition 4.2.1. A $G^\#$-grading on M_n is said to be *admissible* if $U(\mathscr{F})$ with its natural $\mathbb{Z}$-grading is a graded subalgebra of M_n, where M_n is viewed as a $\mathbb{Z}$-graded algebra induced by the projection $G^\# \to \mathbb{Z}$. We call an isomorphism class of $G^\#$-grading on M_n *admissible* if it contains an admissible grading.

Lemma 4.2.2. *For any admissible $G^\#$-grading on M_n, the induced $\mathbb{Z}$-grading, given by the projection $G^\# \to \mathbb{Z}$, has J_m as its homogeneous component of degree m.*

Proof. From the definition of admissible grading, we know that, for any $m \geq 0$, J_m is contained in the homogeneous component of degree m in the induced $\mathbb{Z}$-grading on M_n. In particular, each E_{ii} is homogeneous of degree 0. It follows that $E_{ii}M_nE_{jj} = KE_{ij}$ is a graded subspace. Hence, all E_{ij} are homogeneous. Moreover, if $E_{ij} \in J_{-m}$, then $E_{ji} \in J_m$ has degree m, so E_{ij} must have degree $-m$, since $E_{ii} = E_{ij}E_{ji}$. The result follows. ☐

Recall from Subsection 2.1 that any isomorphism class of $G^\#$-gradings on M_n is given by a finite subgroup T of $G^\#$ (hence, in fact, $T \subset G$), a non-degenerate bicharacter $\beta : T\times T \to K^\times$ and a function $\kappa : G^\#/T \to \mathbb{Z}_{\geq 0}$ with finite support, where $n = k\ell$, $k = |\kappa|$ and $\ell = \sqrt{|T|}$.

Lemma 4.2.3. *Consider a $G^\#$-grading on M_n with parameters (T, β, κ) and let*

$$\gamma = \big((a_1, g_1), (a_2, g_2), \ldots, (a_k, g_k)\big)$$

be a k-tuple of elements of $G^\#$ associated to κ. Then the $\mathbb{Z}$-grading on M_n induced by the projection $G^\# \to \mathbb{Z}$ is an elementary grading defined by the n-tuple

$$(\underbrace{a_1, \ldots, a_1}_{\ell \text{ times}}, \underbrace{a_2, \ldots, a_2}_{\ell \text{ times}}, \ldots, \underbrace{a_k, \ldots, a_k}_{\ell \text{ times}}).$$

Proof. We have a $G^\#$-graded isomorphism $M_n \simeq M_k \otimes M_\ell$, where M_k has an elementary grading defined by γ and M_ℓ has a division grading with support T. Since T is contained in the kernel of the projection $G^\# \to \mathbb{Z}$, the factor M_ℓ will get the trivial induced $\mathbb{Z}$-grading. The result follows. □

By the previous two lemmas, the isomorphism class of $G^\#$-gradings on M_n with parameters (T, β, κ) is admissible if and only if γ has the following form, up to permutation and translation by an integer:

$$\gamma = \big((-1, g_{11}), \dots, (-1, g_{1k_1}), (-2, g_{21}), \dots, (-2, g_{2k_2}) \dots, (-s, g_{s1}), \dots, (-s, g_{sk_s})\big),$$

where $n_i = k_i\ell$ for all $i = 1, 2, \dots, s$. Equivalently, this condition can be restated directly in terms of κ, regarded as a function $\mathbb{Z} \times G/T \to \mathbb{Z}_{\geq 0}$, as follows: there exist $a \in \mathbb{Z}$ and $\kappa_1, \dots, \kappa_s : G/T \to \mathbb{Z}_{\geq 0}$ with $|\kappa_i|\sqrt{|T|} = n_i$ such that

$$\kappa(a - i, x) = \kappa_i(x), \quad \forall i \in \{1, 2, \dots, s\},\ x \in G/T,$$

and $\kappa(a - i, x) = 0$ if $i \notin \{1, 2, \dots, s\}$.

By Lemma 4.2.2, every admissible $G^\#$-grading $M_n = \bigoplus_{(m,g)\in G^\#} A_{(m,g)}$ restricts to a $G^\#$-grading on $U(\mathscr{F})$, hence the projection onto the second component $G^\# \to G$ induces a G-grading on $U(\mathscr{F})$, namely, $U(\mathscr{F}) = \bigoplus_{g\in G} B_g$ where $B_g = \bigoplus_{m\geq 0} A_{(m,g)}$.

Lemma 4.2.4. *If two admissible $G^\#$-gradings on M_n are isomorphic then they induce isomorphic G-gradings on $U(\mathscr{F})$.*

Proof. Assume that ψ is an isomorphism between two admissible $G^\#$-gradings on M_n. Since ψ preserves degree in $G^\#$, it fixes $U(\mathscr{F})$ as a set and therefore restricts to an automorphism of $U(\mathscr{F})$. This restriction is an isomorphism between the induced G-gradings on $U(\mathscr{F})$. □

Now we want to go back from G-gradings on $U(\mathscr{F})$ to $G^\#$-gradings on M_n. First note that the G-gradings on $U(\mathscr{F})$ obtained as above are not arbitrary, but satisfy the following:

Definition 4.2.5. We say that a G-grading on $U(\mathscr{F})$ is *in canonical form* if, for each $m \in \{0, 1, \dots, s - 1\}$, the subspace J_m is G-graded.

In other words, a G-grading $\Gamma : U(\mathscr{F}) = \bigoplus_{g\in G} B_g$ is in canonical form if and only if it is compatible with the natural $\mathbb{Z}$-grading on $U(\mathscr{F})$. If this is the case, we obtain a $G^\#$-grading on $U(\mathscr{F})$ by taking $J_m \cap B_g$ as the homogeneous component of degree (m, g). We want to show that this $G^{\#}$-grading uniquely extends to M_n.

To this end, let us look more closely at the automorphism group of $U(\mathscr{F})$. We denote by $\mathrm{Int}(x)$ the inner automorphism $y \mapsto xyx^{-1}$ determined by an invertible element x.

Lemma 4.2.6. $\mathrm{Aut}(U(\mathscr{F})) \simeq \{\psi \in \mathrm{Aut}(M_n) \mid \psi(U(\mathscr{F})) = U(\mathscr{F})\}$.

Proof. It is proved in [27, Corollary 5.4.10] that

$$\mathrm{Aut}(U(\mathscr{F})) = \{\mathrm{Int}(x) \mid x \in U(\mathscr{F})^\times\}.$$

On the other hand, every automorphism of the matrix algebra is inner, so let $y \in M_n^\times$ and assume $yU(\mathscr{F})y^{-1} = U(\mathscr{F})$. Then, by the description of $\mathrm{Aut}(U(\mathscr{F}))$ above, we can find $x \in U(\mathscr{F})^\times$ such that

$$\mathrm{Int}(x) \mid_{U(\mathscr{F})} = \mathrm{Int}(y) \mid_{U(\mathscr{F})} .$$

It follows that xy^{-1} commutes with all elements of $U(\mathscr{F})$. Hence $yx^{-1} = \lambda \cdot 1$, for some $\lambda \in K^\times$, and $y = \lambda x \in U(\mathscr{F})^\times$. □

Assume for a moment that $\mathrm{char}\, K = 0$. Since K is algebraically closed and G is abelian, it is well known that G-gradings on a finite-dimensional algebra A are equivalent to actions of the algebraic group $\widehat{G} := \mathrm{Hom}_{\mathbb{Z}}(G, K^\times)$ by automorphisms of A, that is, homomorphisms of algebraic groups $\widehat{G} \to \mathrm{Aut}(A)$ (see, for example, [35, §1.4]). The homomorphism $\eta_\Gamma : \widehat{G} \to \mathrm{Aut}(A)$ corresponding to a grading $\Gamma : A = \bigoplus_{g \in G} A_g$ is defined by $\eta_\Gamma(\chi)(x) = \chi(g)x$ for all $\chi \in \widehat{G}$, $g \in G$ and $x \in A_g$.

By Lemma 4.2.6, we have

$$\mathrm{Aut}\,(U(\mathscr{F})) \simeq \mathrm{Stab}_{\mathrm{Aut}(M_n)}(U(\mathscr{F})) \subset \mathrm{Aut}(M_n),$$

hence, if $\mathrm{char}\, K = 0$, we obtain the desired unique extension of gradings from $U(\mathscr{F})$ to M_n. To extend this result to positive characteristic, we can use group schemes instead of groups. Recall that an *affine group scheme* over a field K is a representable functor from the category Alg_K of unital commutative associative K-algebras to the category of groups (see e.g. [66] or [35, Appendix A]). For example, the *automorphism group scheme* of a finite-dimensional algebra A is defined by

$$\mathbf{Aut}(A)(R) := \mathrm{Aut}_R(A \otimes R), \quad \forall R \in \mathrm{Alg}_K.$$

Another example of relevance to us is $\mathbf{GL}_1(A)$, for a finite-dimensional associative algebra A, defined by $\mathbf{GL}_1(A)(R) := (A \otimes R)^\times$. (In particular, $\mathbf{GL}_1(M_n) = \mathbf{GL}_n$.) Note that we have a homomorphism $\mathrm{Int} : \mathbf{GL}_1(A) \to \mathbf{Aut}(A)$.

If G is an abelian group, then the group algebra KG is a commutative Hopf algebra, so it represents an affine group scheme, which is the scheme version of the character group $\widehat{G}$. It is denoted by G^D and given by $G^D(R) = \mathrm{Hom}_{\mathbb{Z}}(G, R^\times)$. In particular, $G^D(K) = \widehat{G}$. If we have a G-grading Γ on A, then we can define a homomorphism of group schemes $\eta_\Gamma : G^D \to \mathbf{Aut}(A)$ by generalizing the formula in the case of $\widehat{G}$: $(\eta_\Gamma)_R(\chi)(x \otimes r) = x \otimes \chi(g)r$ for all $R \in \mathrm{Alg}_K$, $\chi \in G^D(R)$, $r \in R$, $g \in G$ and $x \in A_g$. In this way, over an arbitrary field, G-gradings on A are equivalent to homomorphisms of group schemes $G^D \to \mathbf{Aut}(A)$.

Lemma 4.2.7. *Over an arbitrary field,* $\mathbf{Aut}(U(\mathscr{F}))$ *is a quotient of* $\mathbf{GL}_1(U(\mathscr{F}))$*, and* $\mathbf{Aut}\,(U(\mathscr{F})) \simeq \mathbf{Stab}_{\mathbf{Aut}(M_n)}(U(\mathscr{F}))$ *via the restriction map.*

Proof. We claim that the homomorphism $\mathrm{Int} : \mathbf{GL}_1(U(\mathscr{F})) \to \mathbf{Aut}(U(\mathscr{F}))$ is a quotient map. Since $\mathbf{GL}_1(U(\mathscr{F}))$ is smooth, it is sufficient to verify that (i) the group homomorphism $\mathrm{Int} : (U(\mathscr{F}) \otimes \overline{K})^\times \to \mathrm{Aut}_{\overline{K}}(U(\mathscr{F}) \otimes \overline{K})$ is surjective, where $\overline{K}$ is the algebraic closure of K, and (ii) the Lie homomorphism $\mathrm{ad} : U(\mathscr{F}) \to \mathrm{Der}(U(\mathscr{F}))$ is surjective (see e.g. [35, Corollary A.49]). But (i) is satisfied by Corollary 5.4.10 in [27], mentioned above, and (ii) is satisfied by Theorem 2.4.2 in the same work.

Since the homomorphism $\mathrm{Int} : \mathbf{GL}_1(U(\mathscr{F})) \to \mathbf{Aut}(U(\mathscr{F}))$ factors through the restriction map $\mathbf{Stab}_{\mathbf{Aut}(M_n)}(U(\mathscr{F})) \to \mathbf{Aut}\,(U(\mathscr{F}))$, it follows that this latter is also a quotient map. But its kernel is trivial, because the corresponding restriction maps for the group $\mathrm{Stab}_{\mathrm{Aut}_{\overline{K}}(M_n(\overline{K}))}(U(\mathscr{F}) \otimes \overline{K})$ and Lie algebra $\mathrm{Stab}_{\mathrm{Der}(M_n)}(U(\mathscr{F}))$ are injective (see e.g. [35, Theorem A.46]). □

Coming back to a G-grading Γ on $U(\mathscr{F})$ in canonical form, we conclude by Lemma 4.2.7 that the corresponding $G^\#$-grading on $U(\mathscr{F})$ extends to a unique $G^\#$-grading $\Gamma^\#$ on M_n. By construction, $\Gamma^\#$ is admissible and induces the original grading Γ on $U(\mathscr{F})$. It is also clear that $\Gamma^\#$ is uniquely determined by these properties. Thus, we have a bijection between admissible $G^\#$-gradings on M_n and G-gradings on $U(\mathscr{F})$ in canonical form.

Lemma 4.2.8. *For any* G*-grading on* $U(\mathscr{F})$*, there exists an isomorphic* G*-grading in canonical form.*

Proof. It follows from Lemma 4.2.7 that the Jacobson radical $J = \bigoplus_{m>0} J_m$ of $U(\mathscr{F})$ is stabilized by $\mathbf{Aut}(U(\mathscr{F}))$. Hence, J is a G-graded ideal. So, by Lemma 4.1.1, there exists an isomorphic grading such that each block is a graded subspace. □

Lemma 4.2.9. *If two G-gradings, Γ_1 and Γ_2, on $U(\mathscr{F})$ are in canonical form and isomorphic to one another, then there exists a block-diagonal matrix $x \in U(\mathscr{F})^\times$ such that $\psi_0 = \mathrm{Int}(x)$ is an isomorphism between Γ_1 and Γ_2.*

Proof. Let $\psi = \mathrm{Int}(y)$ be an isomorphism between Γ_1 and Γ_2. Write $y = (y_{ij})_{1\leq i\leq j\leq s}$ in blocks and let $x = \mathrm{diag}(y_{11}, \dots, y_{ss})$. Then x is invertible, so let $\psi_0 = \mathrm{Int}(x)$.

Fix $m \in \{0, 1, \dots, s-1\}$ and let $a \in J_m$ be G-homogeneous with respect to Γ_1. Since $J^m = J_m \oplus J^{m+1}$, we can uniquely write $\psi(a) = b + c$, where $b \in J_m$ and $c \in J^{m+1}$. Since Γ_2 is in canonical form, J_m and J^{m+1} are G-graded subspaces with respect to Γ_2. Since ψ preserves G-degree, it follows that b and c are G-homogeneous elements with respect to Γ_2 of the same G-degree as a with respect to Γ_1. Finally, note that $\psi_0(a) = b$. Since m and a were arbitrary, we have shown that ψ_0 is an isomorphism between Γ_1 and Γ_2. □

Now we can prove the converse of Lemma 4.2.4.

Lemma 4.2.10. *If two admissible $G^\#$-gradings on M_n induce isomorphic G-gradings on $U(\mathscr{F})$, then they are isomorphic.*

Proof. Let Γ_1 and Γ_2 be two isomorphic G-gradings on $U(\mathscr{F})$ obtained from two $G^\#$-gradings on M_n, $\Gamma_1^\#$ and $\Gamma_2^\#$, respectively. For $i = 1, 2$, let $\eta_i : (G^\#)^D \to \mathbf{Aut}(M_n)$ be the action corresponding to $\Gamma_i^\#$. Consider also the restriction Γ_i' of $\Gamma_i^\#$ to $U(\mathscr{F})$ and the corresponding action $\eta_i' : (G^\#)^D \to \mathbf{Aut}(U(\mathscr{F}))$. By Lemma 4.2.9, we can find an isomorphism $\psi_0 = \mathrm{Int}(x)$ between Γ_1 and Γ_2, where x is block-diagonal. Such ψ_0 preserves the natural $\mathbb{Z}$-grading, so it is actually an isomorphism between the $G^\#$-gradings Γ_1' and Γ_2'. Hence, $\psi_0\eta_1'(\chi) = \eta_2'(\chi)\psi_0$ for all $\chi \in (G^\#)^D(R)$ and all $R \in \mathrm{Alg}_K$. By Lemma 4.2.7, this implies $\psi_0\eta_1(\chi) = \eta_2(\chi)\psi_0$ for all $\chi \in (G^\#)^D(R)$, which means ψ_0 is an isomorphism between $\Gamma_1^\#$ and $\Gamma_2^\#$. □

We summarize the results of this subsection:

Theorem 4.2.11. *The mapping of an admissible $G^\#$-grading on M_n to a G-grading on $U(\mathscr{F})$, given by restriction and coarsening, yields a bijection between the admissible* *isomorphism classes of $G^{\#}$-gradings on M_n and the isomorphism classes of G-gradings on $U(\mathscr{F})$.* □

Admissible isomorphism classes of $G^{\#}$-gradings on M_n can be parametrized by the triples $(T, \beta, (\kappa_1, \ldots, \kappa_s))$, where $T \subset G$ is a finite subgroup, $\beta : T \times T \to K^\times$ is a non-degenerate alternating bicharacter and $\kappa_i : G/T \to \mathbb{Z}_{\geq 0}$ are functions with finite support such that $|\kappa_i| \sqrt{|T|} = n_i$, for each $i = 1, 2, \ldots, s$. Hence, isomorphism classes of G-gradings on $U(\mathscr{F})$ are parametrized by the same triples.

Choosing, for each κ_i, a k_i-tuple γ_i of elements of G, where $k_i = |\kappa_i|$, we reproduce the description of G-gradings on $U(\mathscr{F})$ originally obtained in [64]. Note, however, that we do not need to assume that G is finite, nor $\operatorname{char} K = 0$. Also note that we have a description not only of G-gradings but of their isomorphism classes, which gives an alternative proof of the following result first established in [25, Corollary 4]:

Corollary 4.2.12. *Two G-gradings on $U(\mathscr{F})$, determined by $(T, \beta, (\kappa_1, \ldots, \kappa_s))$ and by $(T', \beta', (\kappa'_1, \ldots, \kappa'_s))$, are isomorphic if and only if $T' = T$, $\beta' = \beta$ and there exists $g \in G$ such that $\kappa'_i = g\kappa_i$, for all $i = 1, 2, \ldots, s$.* □

2.3. **Gradings on the block-triangular matrices as a Lie algebra.** Now we turn our attention to $U(\mathscr{F})^{(-)}$, that is, $U(\mathscr{F})$ viewed as a Lie algebra with respect to the commutator $[x, y] = xy - yx$. We assume that the grading group G is abelian and the ground field K is algebraically closed of characteristic 0, and follow the same approach as in the associative case.

Denote by τ the flip along the secondary diagonal on M_n. Note that $U(\mathscr{F})^\tau = U(\mathscr{F})$ if and only if $n_i = n_{s-i+1}$ for all $i = 1, 2, \ldots, \lfloor \frac{s}{2} \rfloor$. Let

$$U(\mathscr{F})_0 = \{x \in U(\mathscr{F}) \mid \operatorname{tr}(x) = 0\},$$

which is a Lie subalgebra of $U(\mathscr{F})^{(-)}$. Moreover, $U(\mathscr{F})^{(-)} = U(\mathscr{F})_0 \oplus K1$, where $1 \in U(\mathscr{F})$ is the identity matrix. The center $\mathfrak{z}(U(\mathscr{F})^{(-)}) = K1$ is always graded, so 1 is a homogeneous element. If we change its degree arbitrarily, we obtain a new well-defined grading, which is not isomorphic to the original one, but will induce the same grading on $U(\mathscr{F})^{(-)}/K1 \simeq U(\mathscr{F})_0$. It turns out that, up to isomorphism, a G-grading on $U(\mathscr{F})^{(-)}$ is determined by the induced G-grading on $U(\mathscr{F})_0$ and the degree it assigns to the identity matrix (see Corollary 4.2.20).

Conversely, any G-grading on $U(\mathscr{F})_0$ extends to $U(\mathscr{F})^{(-)} = U(\mathscr{F})_0 \oplus K1$ by defining the degree of 1 arbitrarily. Thus, we have a bijection between the isomorphism classes of G-gradings on $U(\mathscr{F})^{(-)}$ and the pairs consisting of an isomorphism class of G-gradings on $U(\mathscr{F})_0$ and an element of G.

We start by computing the automorphism group of $U(\mathscr{F})_0$. To this end, we will use the following description of the automorphisms of $\mathrm{Aut}(U(\mathscr{F})^{(-)})$.

Theorem 4.2.13 ([26, Theorem 4.1.1])**.** *Let ϕ be an automorphism of $U(\mathscr{F})^{(-)}$, and assume $\operatorname{char} K = 0$ or $\operatorname{char} K > 3$. Then there exist $p, d \in U(\mathscr{F})$, with p invertible and d block-diagonal, such that one of the following holds:*

(1) $\phi(x) = pxp^{-1} + \operatorname{tr}(xd)1$, *for all* $x \in U(\mathscr{F})$, *or*

(2) $\phi(x) = -px^{\tau}p^{-1} + \operatorname{tr}(xd)1$, *for all* $x \in U(\mathscr{F})$. □

As a consequence, we obtain the following analog of Lemma 4.2.6:

Lemma 4.2.14. *If $n > 2$ and $n_i = n_{s-i+1}$ for all i, then*

$$\mathrm{Aut}(U(\mathscr{F})_0) \simeq \{\mathrm{Int}(x) \mid x \in U(\mathscr{F})^{\times}\} \rtimes \langle -\tau \rangle;$$

otherwise, $\mathrm{Aut}(U(\mathscr{F})_0) \simeq \{\mathrm{Int}(x) \mid x \in U(\mathscr{F})^{\times}\}$. *In both cases,*

$$\mathrm{Aut}(U(\mathscr{F})_0) \simeq \mathrm{Stab}_{\mathrm{Aut}(\mathfrak{sl}_n)}(U(\mathscr{F})_0).$$

Proof. Let $\psi \in \mathrm{Aut}(U(\mathscr{F})_0)$. We extend ψ to an automorphism ϕ of $U(\mathscr{F})^{(-)}$ by setting $\phi(1) = 1$. By the previous result, ϕ must have one of two possible forms. Assume it is the first one:

$$\phi(x) = pxp^{-1} + \operatorname{tr}(xd)1, \quad \forall x \in U(\mathscr{F}).$$

But as $U(\mathscr{F})_0$ is an invariant subspace for ϕ, we see that, for all $x \in U(\mathscr{F})_0$,

$$0 = \operatorname{tr}(\phi(x)) = \operatorname{tr}(pxp^{-1} + \operatorname{tr}(xd)1) = n\operatorname{tr}(xd).$$

Therefore, $\operatorname{tr}(xd) = 0$ and hence $\psi(x) = \phi(x) = pxp^{-1}$, for all $x \in U(\mathscr{F})_0$, so $\psi = \mathrm{Int}(p)$. The same argument applies if ϕ has the second form. Note that, for $n = 2$, the second form reduces to the first on $UT(1,1)_0$, since $-\tau$ coincides with $\mathrm{Int}(p)$ on $\mathfrak{sl}_2$, where $p = \mathrm{diag}(1, -1)$. On the other hand, for $n > 2$, the two forms do not overlap, since the action of $-\tau$ differs already on the set of zero-trace diagonal matrices from the action of any inner automorphism. We conclude the proof in the same way as for Lemma 4.2.6. □

Let G be an abelian group and define $G^{\#} = \mathbb{Z}\times G$. Similarly to the associative case, we want to relate G-gradings on $U(\mathscr{F})_0$ and $G^{\#}$-gradings on $\mathfrak{sl}_n$, since for the latter a classification of group gradings is known [11] (see also [35, Chapter 3]).

Recall that J_m stands for the m-th block-diagonal of matrices. We consider again the *natural* $\mathbb{Z}$*-grading* on $U(\mathscr{F})_0$: its homogeneous component of degree $m \in \mathbb{Z}$ is $J_m \cap U(\mathscr{F})_0$ if $0 \leq m < s$ and 0 otherwise. We say that a G-grading on $U(\mathscr{F})_0$ is in *canonical form* if, for each $m \in \{0, \ldots, s-1\}$, the subspace $J_m \cap U(\mathscr{F})_0$ is G-graded. A $G^{\#}$-grading on $\mathfrak{sl}_n$ is said to be *admissible* if the coarsening induced by the projection $G^{\#} \to \mathbb{Z}$ has $U(\mathscr{F})_0$, with its natural $\mathbb{Z}$-grading, as a graded subalgebra. An isomorphism class of $G^{\#}$-grading on $\mathfrak{sl}_n$ is called *admissible* if it contains an admissible grading.

Since any $\mathbb{Z}$-grading on $\mathfrak{sl}_n$ is the restriction of a unique $\mathbb{Z}$-grading on the associative algebra M_n, Lemma 4.2.2 still holds if we replace M_n by $\mathfrak{sl}_n$. Therefore, every admissible $G^{\#}$-grading on $\mathfrak{sl}_n$ restricts to $U(\mathscr{F})_0$ and, by means of the projection $G^{\#} \to G$, yields a G-grading on $U(\mathscr{F})_0$, which is clearly in canonical form. Conversely, thanks to Lemma 4.2.14, if a G-grading on $U(\mathscr{F})_0$ is in canonical form then it comes from a unique admissible $G^{\#}$-grading on $\mathfrak{sl}_n$ in this way. Therefore, similarly to the associative case, we obtain a bijection between admissible $G^{\#}$-grading on $\mathfrak{sl}_n$ and G-gradings on $U(\mathscr{F})_0$ in canonical form.

The following result is technical and will be proved in next subsection:

Lemma 4.2.15. *For any G-grading on $U(\mathscr{F})_0$, there exists an isomorphic G-grading in canonical form.*

Clearly, as in Lemma 4.2.4, if two admissible $G^{\#}$-gradings on $\mathfrak{sl}_n$ are isomorphic then they induce isomorphic G-gradings on $U(\mathscr{F})_0$. The converse is established by the same argument as Lemma 4.2.10, using the following analog of Lemma 4.2.9:

Lemma 4.2.16. *If two G-gradings, Γ_1 and Γ_2, on $U(\mathscr{F})_0$ are in canonical form and isomorphic to one another, then there exists an isomorphism ψ_0 between Γ_1 and Γ_2 of the form $\psi_0 = \mathrm{Int}(x)$ or $\psi_0 = -\mathrm{Int}(x)\tau$ where the matrix $x \in U(\mathscr{F})^\times$ is block-diagonal.*

Proof. Let ψ be an isomorphism between Γ_1 and Γ_2. If $\psi = \mathrm{Int}(y)$ then we are in the situation of the proof of Lemma 4.2.9. If $\psi = -\mathrm{Int}(y)\tau$ then the same proof still works because all subspaces J_m are invariant under τ. □

In summary:

Theorem 4.2.17. *The mapping of an admissible $G^\#$-grading on $\mathfrak{sl}_n$ to a G-grading on $U(\mathscr{F})_0$, given by restriction and coarsening, yields a bijection between the admissible isomorphism classes of $G^\#$-gradings on $\mathfrak{sl}_n$ and the isomorphism classes of G-gradings on $U(\mathscr{F})_0$.* □

There are two families of gradings on $\mathfrak{sl}_n$, $n > 2$, namely, Type I and Type II. (Only Type I exists for $n = 2$.) Their isomorphism classes are stated in Theorem 3.53 of [35], but we will use Theorem 45 of [12], which is equivalent but uses more convenient parameters.

By definition, a $G^\#$-grading of Type I is a restriction of a $G^\#$-grading on the associative algebra M_n, so it is parametrized by (T, β, κ), where, as in Subsection 2.2, $T \subset G$ is a finite group, $\beta : T \times T \to K^\times$ is a non-degenerate alternating bicharacter and $\kappa : \mathbb{Z} \times G/T \to \mathbb{Z}_{\geq 0}$ is a function with finite support satisfying $|\kappa|\sqrt{|T|} = n$.

For a Type II grading, there is a unique element $f \in G^\#$ of order 2 (hence, in fact, $f \in G$), called the *distinguished element*, such that the coarsening induced by the natural homomorphism $G^\# \to G^\#/\langle f \rangle$ is a Type I grading. The parametrization of Type II gradings depends on the choice of character χ of $G^\#$ satisfying $\chi(f) = -1$. So, we fix $\chi \in \widehat{G}$ with $\chi(f) = -1$ and extend it trivially to the factor $\mathbb{Z}$. Then, the parameters of a Type II grading are a finite subgroup $T \subset G^\#$ (hence $T \subset G$) containing f, an alternating bicharacter $\beta : T \times T \to K^\times$ with radical $\langle f \rangle$ (so, β determines the distinguished element f), an element $g_0^\# \in G^\#$, and a function $\kappa : \mathbb{Z} \times G/T \to \mathbb{Z}_{\geq 0}$ with finite support satisfying $|\kappa|\sqrt{|T|/2} = n$. These parameters are required to satisfy some additional conditions, as follows.

To begin with, for a Type II grading, T must be 2-elementary. Its Type I coarsening is a grading by $G^\#/\langle f \rangle \simeq \mathbb{Z} \times \overline{G}$ with parameters $(\overline{T}, \bar{\beta}, \kappa)$, where $\overline{T} := T/\langle f \rangle$ is a subgroup of $\overline{G} := G/\langle f \rangle$, $\bar{\beta} : \overline{T} \times \overline{T} \to K^\times$ is the non-degenerate bicharacter induced by β, and κ is now regarded as a function on $\mathbb{Z} \times \overline{G}/\overline{T} \simeq \mathbb{Z} \times G/T$.

Since T is 2-elementary, β can only take values ± 1 and $\ell := \sqrt{|T|/2}$ is a power of 2. If one uses Kronecker products of Pauli matrices (of order 2) to construct a division grading on M_ℓ with support $\overline{T}$ and bicharacter $\bar{\beta}$, then the transposition will preserve degree and thus become an involution on the resulting graded division algebra D. The choice of such an involution is arbitrary, and it will be convenient for our purposes to use τ, which also preserves degree. Since all homogeneous components of D are 1-dimensional, we have

$$(X_{\bar{t}})^\tau = \bar{\eta}(\bar{t})X_{\bar{t}}, \quad \forall \bar{t} \in \overline{T},\ X_{\bar{t}} \in D_{\bar{t}},$$

where $\bar{\eta} : \overline{T} \to \{\pm 1\}$ satisfies $\bar{\eta}(\bar{u}\bar{v}) = \bar{\beta}(\bar{u}, \bar{v})\bar{\eta}(\bar{u})\bar{\eta}(\bar{v})$ for all $\bar{u}, \bar{v} \in \overline{T}$. If we regard $\bar{\eta}$ and $\bar{\beta}$ as maps of vector spaces over the field of two elements, this equation means that $\bar{\eta}$ is a quadratic form with polarization $\bar{\beta}$.

Recall that a concrete $G^\#/\langle f\rangle$-grading with parameters $(\overline{T}, \bar{\beta}, \kappa)$ is constructed by selecting a k-tuple of elements of $G^\#/\langle f\rangle$, as directed by κ, to get an elementary grading on M_k, where $k = |\kappa|$, and identifying $M_n \simeq M_k \otimes D$ via Kronecker product. The remaining parameter $g_0^\#$ can then be used, together with the chosen involution τ on D, to define an anti-automorphism φ on M_n by the formula

$$\varphi(X) = \Phi^{-1}X^\tau\Phi, \quad \forall X \in M_n,$$

where the matrix $\Phi \in M_k \otimes D \simeq M_k(D)$ is constructed in such a way that φ^2 acts on M_n in exactly the same way as χ^2, which acts on M_n because it can be regarded as a character on $G^\#/\langle f\rangle$ (since $\chi^2(f) = 1$) and M_n is a $G^\#/\langle f\rangle$-graded algebra. As a result, we can split each homogeneous component of the $G^\#/\langle f\rangle$-grading on M_n into (at most 2) eigenspaces of φ so that the action of χ on the resulting $G^\#$-graded algebra $M_n^{(-)}$ coincides with the automorphism $-\varphi$. Finally, the restriction of this $G^\#$-grading to $\mathfrak{sl}_n$ is a $G^\#$-grading of Type II with parameters $(T, \beta, g_0^\#, \kappa)$.

In order to construct Φ, two conditions must be met:

(i) κ is $g_0^\#$*-balanced* in the sense that $\kappa(x) = \kappa((g_0^\#)^{-1}x^{-1})$ for all $x \in \mathbb{Z} \times G/T$ (where the inverse in $\mathbb{Z}$ is understood with respect to addition);

(ii) $\kappa(g^\# T)$ is even whenever $g_0^\#(g^\#)^2 \in T$ and $\eta(g_0^\#(g^\#)^2) = -1$ for some $g^\# \in G^\#$.

Such a matrix $\Phi \in M_k(D)$ is given explicitly by Equations (3.29) and (3.30) in [35], but in relation to the usual transposition. Since we are using τ, the order of the k rows has to be reversed and the entries in D chosen in accordance with the above quadratic form $\bar{\eta}$ rather than the quadratic form in [35]. It will also be convenient in our situation to order the k-tuple associated to κ in a different way, as will be described below.

We are only interested in admissible isomorphism classes of $G^\#$-grading on $\mathfrak{sl}_n$. If $n = 2$, the isomorphism condition for (Type I) gradings is the same as in the associative case: all translations of κ determine isomorphic gradings. If $n > 2$, however, one isomorphism class of Type I gradings on $\mathfrak{sl}_n$ can consist of one or two isomorphism classes of gradings on M_n, because (T, β, κ) and $(T, \beta^{-1}, \bar{\kappa})$ determine isomorphic gradings on $\mathfrak{sl}_n$, where the function $\bar{\kappa} : \mathbb{Z} \times G/T \to \mathbb{Z}_{\geq 0}$ is defined by $\bar{\kappa}(i, x) := \kappa(-i, x^{-1})$. Hence, the isomorphism class of $G^{\#}$-gradings of Type I with parameters (T, β, κ) is admissible if and only if at least one of the functions κ and $\bar{\kappa}$ has the form described after Lemma 4.2.3. Assuming it is κ, there must exist $a \in \mathbb{Z}$ and functions $\kappa_1, \ldots, \kappa_s : G/T \to \mathbb{Z}_{\geq 0}$ with $|\kappa_i|\sqrt{|T|} = n_i$, such that

$$\kappa(a - i, x) = \kappa_i(x), \quad \forall i \in \{1, 2, \ldots, s\},\ x \in G/T, \tag{2}$$

and $\kappa(a - i, x) = 0$ if $i \notin \{1, 2, \ldots, s\}$. Then $\bar{\kappa}$ can be expressed in the same form, but with the function $\bar{\kappa}_i(x) := \kappa_i(x^{-1})$ playing the role of κ_{s-i+1} for each i. Thus, the isomorphism classes of G-gradings of Type I on $U(\mathscr{F})_0$ are parametrized by $(T, \beta, (\kappa_1, \ldots, \kappa_s))$, and, if $n_i = n_{s-i+1}$ for all i, then $(T, \beta, (\kappa_1, \ldots, \kappa_s))$ and $(T, \beta^{-1}, (\bar{\kappa}_s, \ldots, \bar{\kappa}_1))$ determine isomorphic G-gradings on $U(\mathscr{F})_0$.

Now consider the isomorphism class of Type II gradings on $\mathfrak{sl}_n$ $(n > 2)$ with parameters $(T, \beta, g_0^{\#}, \kappa)$. Admissibility is a condition on the $\mathbb{Z}$-grading induced by the projection $G^{\#} \to \mathbb{Z}$, which factors through the natural homomorphism $G^{\#} \to G^{\#}/\langle f \rangle$. So, for this isomorphism class to be admissible, it is necessary and sufficient for κ to have the form given by Equation (2), but with $|\kappa_i|\sqrt{|T|/2} = n_i$.

Lemma 4.2.18. *If $g_0^{\#} = (a_0, g_0)$ and κ is given by Equation (2), then κ is $g_0^{\#}$-balanced if and only if $a_0 = s + 1 - 2a$ and $\kappa_i(x) = \kappa_{s-i+1}(g_0^{-1}x^{-1})$ for all $x \in G/T$ and all i.*

Proof. Consider the function $\kappa_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ given by $\kappa_{\mathbb{Z}}(m) = \sum_{g \in G/T} \kappa(m, g)$. Then the support of $\kappa_{\mathbb{Z}}$ is $\{a - s, \ldots, a - 1\}$. On the other hand, if κ is $g_0^{\#}$-balanced, then $\kappa_{\mathbb{Z}}$ is a_0-balanced, which implies $-a_0 - (a - s) = a - 1$. The result follows. □

Therefore, we can replace the parameters $g_0^{\#}$ and κ by g_0 and $(\kappa_1, \ldots, \kappa_s)$. Also, since $g_0^{\#}(g^{\#})^2 \notin T$ for any $g^{\#} = (a - i, g)$ with $s + 1 \neq 2i$, condition (ii) is automatically satisfied if s is even, and affects only $\kappa_{\frac{s+1}{2}}$ if s is odd. Hence, we can restate conditions (i) and (ii) in terms of $\kappa_1, \ldots, \kappa_s$ as follows:

(i') $\kappa_i(x) = \kappa_{s-i+1}(g_0^{-1}x^{-1})$ for all $x \in G/T$ and all i;

(ii') either s is even or s is odd and $\kappa_{\frac{s+1}{2}}(gT)$ is even whenever $g_0 g^2 \in T$ and $\eta(g_0 g^2) = -1$ for some $g \in G$.

Note that condition (i') implies that $n_i = |\kappa_i|\ell = |\kappa_{s-i+1}|\ell = n_{s-i+1}$, so Type II gradings on $U(\mathscr{F})_0$ can exist only if $n_i = n_{s-i+1}$ for all i, as expected from the structure of the automorphism group (see Lemma 4.2.14).

Let us describe explicitly a Type II grading on $U(\mathscr{F})_0$ in the isomorphism class parametrized by $(T, \beta, g_0, (\kappa_1, \dots, \kappa_s))$. For each $1 \le i < \frac{s+1}{2}$, we fill two $|\kappa_i|$-tuples, γ_i and γ_{s-i+1}, simultaneously as follows, going from left to right in γ_i and from right to left in γ_{s-i+1}. For each coset $x \in G/T$ that lies in the support of κ_i, we choose an element $g \in x$ and place $\kappa_i(x)$ copies of g into γ_i and as many copies of $g_0^{-1}g^{-1}$ into γ_{s-i+1}. If s is odd, we fill the middle $|\kappa_i|$-tuple γ_i, with $i = \frac{s+1}{2}$, in the following manner: γ_i will be the concatenation of (possibly empty) tuples $\gamma^{\triangleleft}$, γ^{+}, γ^{0}, γ^{-} and $\gamma^{\triangleright}$ (in this order), where $\gamma^{\triangleleft}$ and γ^{+} are to be filled from left to right, γ^{-} and $\gamma^{\triangleright}$ from right to left, and γ^{0} in any order. For each x in the support of κ_i, we choose an element $g \in x$. If $g_0 g^2 \notin T$, we place $\kappa_i(x)$ copies of g into $\gamma^{\triangleleft}$ and as many copies of $g_0^{-1}g^{-1}$ into $\gamma^{\triangleright}$. If $g_0 g^2 \in T$ and $\eta(g_0 g^2) = -1$, we place $\frac{1}{2}\kappa_i(x)$ copies of g in each of γ^{+} and γ^{-}. Finally, if $g_0 g^2 \in T$ and $\eta(g_0 g^2) = 1$, we place $\kappa_i(x)$ copies of g into γ^{0}. Concatenating these $\gamma_1, \dots, \gamma_s$ results in a k-tuple $\gamma = (g_1, \dots, g_k)$ of elements of G. Taking them modulo $\langle f \rangle$, we define a $\overline{G}$-grading on M_k and, consequently, on $M_n \simeq M_k \otimes D$, so $M_n = \bigoplus_{\bar{g} \in \overline{G}} R_{\bar{g}}$. Then we construct a matrix $\Phi \in M_k(D) \simeq M_k \otimes D$ as follows:

$$
\begin{aligned}
\Phi = {} & \operatorname{diag}(\chi(g_1^{-1})I_\ell, \dots, \chi(g_p^{-1})I_\ell) \oplus \operatorname{diag}(X_{\bar{g}_0\bar{g}_{p+1}^2}, \dots, X_{\bar{g}_0\bar{g}_{p+q}^2}) \\
& \oplus \widetilde{\operatorname{diag}}(X_{\bar{g}_0\bar{g}_{p+q+1}^2}, \dots, X_{\bar{g}_0\bar{g}_{k-p-q}^2}) \\
& \oplus \operatorname{diag}(-X_{\bar{g}_0\bar{g}_{k-p-q+1}^2}, \dots, -X_{\bar{g}_0\bar{g}_{k-p}^2}) \oplus \operatorname{diag}(\chi(g_{k-p+1}^{-1})I_\ell, \dots, \chi(g_k^{-1})I_\ell),
\end{aligned} \tag{3}
$$

where p is the sum of the lengths of $\gamma_1, \dots, \gamma_{\lfloor \frac{s}{2} \rfloor}$, and $\gamma^{\triangleleft}$, q is the length of γ^{+}, and $\widetilde{\operatorname{diag}}$ denotes arrangement of entries along the secondary diagonal (from left to right). Finally, we use Φ to define a G-grading on $M_n^{(-)}$:

$$
M_n^{(-)} = \bigoplus_{g \in G} R_g \text{ where } R_g = \{X \in R_{\bar{g}} \mid \Phi^{-1}X^{\tau}\Phi = -\chi(g)X\}, \tag{4}
$$

which restricts to the desired grading on $U(\mathscr{F})_0$.

Thus we obtain the following classification of G-gradings on $U(\mathscr{F})_0$ from our Theorem 4.2.17 and the known classification for $\mathfrak{sl}_n$ (as stated in [12, Theorem 45] and [35, Theorem 3.53]).

Corollary 4.2.19. *Every grading on $U(\mathscr{F})_0$ by an abelian group G is isomorphic either to a Type I grading with parameters $(T, \beta, (\kappa_1, \ldots, \kappa_s))$, where $|\kappa_i| = n_i\sqrt{|T|}$, or to a Type II grading with parameters $(T, \beta, g_0, (\kappa_1, \ldots, \kappa_s))$, where $|\kappa_i|\sqrt{|T|/2} = n_i$ and T is 2-elementary. Type II gradings can occur only if $n > 2$ and $n_i = n_{s-i+1}$ for all i, and their parameters are subject to the conditions (i') and (ii') above. Moreover, gradings of Type I are not isomorphic to gradings of Type II, and within each type we have the following:*

(I) *$(T, \beta, (\kappa_1, \ldots, \kappa_s))$ and $(T', \beta', (\kappa'_1, \ldots, \kappa'_s))$ determine the same isomorphism class if and only if $T' = T$ and there exists $g \in G$ such that either $\beta' = \beta$ and $\kappa'_i = g\kappa_i$ for all i, or $n > 2$, $\beta' = \beta^{-1}$ and $\kappa'_i = g\bar{\kappa}_{s-i+1}$ for all i, where $\bar{\kappa}(x) := \kappa(x^{-1})$ for all $x \in G/T$.*

(II) *$(T, \beta, g_0, (\kappa_1, \ldots, \kappa_s))$ and $(T', \beta', g'_0, (\kappa'_1, \ldots, \kappa'_s))$ determine the same isomorphism class if and only if $T' = T$, $\beta' = \beta$, and there exists $g \in G$ such that $g'_0 = g^{-2}g_0$ and $\kappa'_i = g\kappa_i$ for all i.* □

Finally, we can use Theorem 3.5.5 to pass from $U(\mathscr{F})_0$ to $U(\mathscr{F})^{(-)}$.

Corollary 4.2.20. *Let Γ_1 and Γ_2 be two G-gradings on $U(\mathscr{F})^{(-)}$. Then Γ_1 and Γ_2 are isomorphic if and only if they assign the same degree to the identity matrix 1 and induce isomorphic gradings on $U(\mathscr{F})^{(-)}/K1 \simeq U(\mathscr{F})_0$.*

Proof. The "only if" part is clear. For the "if" part, take an automorphism ψ_0 of $U(\mathscr{F})_0$ that sends the grading induced by Γ_1 to the one induced by Γ_2, extend ψ_0 to an automorphism ψ of $U(\mathscr{F})^{(-)} = U(\mathscr{F})_0 \oplus K1$ by setting $\psi(1) = 1$, and apply the Theorem 3.5.5. □

2.4. **Commutativity of the grading group.** Our immediate goal is to prove Lemma 4.2.15. The arguments will work without assuming a priori that the grading group is abelian, and, in fact, our second goal will be to prove that the elements of the support of any group grading on $U(\mathscr{F})_0$ must commute with each other. It will be more convenient to make computations in $U(\mathscr{F})^{(-)}$. So, suppose $U(\mathscr{F})^{(-)}$ is graded by an arbitrary group G. We still assume that $\operatorname{char} K = 0$, but K need not be algebraically closed.

Write $U(\mathscr{F}) = \bigoplus_{1\le i\le j\le s} B_{ij}$, where each B_{ij} is the set of matrices with non-zero entries only in the (i,j)-th block. Thus, $J_m = B_{1,m+1} \oplus B_{2,m+2} \oplus \cdots \oplus B_{s-m,s}$ for all $m \in \{0, 1, \ldots, s-1\}$. It is important to note that $[J_1, J_m] = J_{m+1}$ and hence the Lie powers of the Jacobson radical $J = \bigoplus_{m>0} J_m$ coincide with its associative powers.

Let $e_i \in B_{ii}$ be the identity matrix of each diagonal block and let

$$\mathfrak{d} = \mathrm{Span}\{e_1, e_2, \ldots, e_s\}.$$

We can write $B_{ii} = \mathfrak{s}_i \oplus Ke_i$, where $\mathfrak{s}_i = [B_{ii}, B_{ii}] \simeq \mathfrak{sl}_{n_i}$. Let $S = \bigoplus_{i=1}^{s} \mathfrak{s}_i$ and $R = \mathfrak{d} \oplus J$. Then $U(\mathscr{F})^{(-)} = S \oplus R$ is a Levi decomposition.

We will need the following graded version of Levi decomposition, which was established in [53] and then improved in [43] by weakening the conditions on the ground field:

Theorem 4.2.21 ([43, Corollaries 4.2 and 4.3])**.** *Let L be a finite-dimensional Lie algebra over a field K of characteristic 0, graded by an arbitrary group G. Then the radical R of L is graded and there exists a maximal semisimple subalgebra B such that $L = B \oplus R$ (direct sum of graded subspaces).* □

Corollary 4.2.22. *Consider any G-grading on $U(\mathscr{F})^{(-)}$. Then the ideal R is graded. Moreover, there exists an isomorphic G-grading on $U(\mathscr{F})^{(-)}$ such that S is also graded.*

Proof. By Theorem 4.2.21, there exists a graded Levi decomposition $U(\mathscr{F})^{(-)} = B \oplus R$. But $U(\mathscr{F})^{(-)} = S \oplus R$ is another Levi decomposition, so, by Malcev's Theorem (see e.g. [44, Corollary 2 on p. 93]), there exists an (inner) automorphism ψ of $U(\mathscr{F})^{(-)}$ such that $\psi(B) = S$. Applying ψ to the given G-grading on $U(\mathscr{F})^{(-)}$, we obtain a new G-grading on $U(\mathscr{F})^{(-)}$ with respect to which S is graded. □

Lemma 4.2.23. *For any G-grading on $U(\mathscr{F})^{(-)}$, there exists an isomorphic G-grading such that the subalgebras $\mathfrak{d}$ and S are graded.*

Proof. We partition $\{1, \ldots, s\} = \{i_1, \ldots, i_r\} \cup \{j_1, \ldots, j_{s-r}\}$ so that $n_{i_k} = 1$ and $n_{j_k} > 1$. Denote $e_\triangle = \sum_{k=1}^{r} e_{i_k}$, then $e_\triangle U(\mathscr{F}) e_\triangle \simeq UT_r$, the algebra of upper triangular matrices (if $r > 0$).

By Corollary 4.2.22, we may assume that S is graded. Then its centralizer in R, $N := \mathrm{C}_R(S)$, is a graded subalgebra. It coincides with $\mathrm{Span}\{e_{j_1}, \ldots, e_{j_t}\} \oplus e_\triangle U(\mathscr{F}) e_\triangle$, and its center (which is also graded) coincides with $\mathrm{Span}\{e_{j_1}, \ldots, e_{j_t}, e_\triangle\}$. If $r = 0$, then $N = \mathfrak{d}$ and we are done. Assume $r > 0$. Then we obtain a G-grading on $N/\mathfrak{z}(N) \simeq UT_r^{(-)}/K1 \simeq (UT_r)_0$. By Theorem 3.6.22, after applying an automorphism of $UT_r^{(-)}$, the subalgebra of diagonal matrices in $UT_r^{(-)}$ is graded. Since $-\tau$ preserves this subalgebra, we may assume that the automorphism in question is inner. But an inner automorphism of $e_\triangle U(\mathscr{F})e_\triangle$ can be extended to an inner automorphism of $U(\mathscr{F})$. Indeed, let $y \in e_\triangle U(\mathscr{F})e_\triangle$ be invertible in $e_\triangle U(\mathscr{F})e_\triangle$. Then $x = \sum_{k=1}^{s-r} e_{j_k} + y \in U(\mathscr{F})^\times$ and $\mathrm{Int}(x)$ extends $\mathrm{Int}(y)$. Moreover, $\mathrm{Int}(x)$ preserves S. Therefore, we may assume that the subalgebra of diagonal matrices in $N/\mathfrak{z}(N)$ is graded. But the inverse image of this subalgebra in N is precisely $\mathfrak{d}$, so $\mathfrak{d}$ is graded. □

It will be convenient to use the following technical concept:

Definition 4.2.24. Let L be a G-graded Lie algebra. We call $x \in L$ *semihomogeneous* if $x = x_h + x_z$, with x_h homogeneous and $x_z \in \mathfrak{z}(L)$. If $x_h \notin \mathfrak{z}(L)$, we define the *degree* of x as $\deg x_h$ and denoted it by $\deg x$.

An important observation is that if x and y are semihomogeneous and $[x, y] \neq 0$, then $[x, y]$ is homogeneous of degree $\deg x \deg y$ (as $[x, y]$ will coincide with $[x_h, y_h]$).

Proposition 4.2.25. *For any G-grading on $U(\mathscr{F})^{(-)}$, there exists an isomorphic G-grading with the following properties:*

(i) *the subalgebras $\mathfrak{s}_k + \mathfrak{s}_{s-k+1}$ are graded,*

(ii) *the elements $e_k - e_{s-k+1}$ ($k \neq \frac{s+1}{2}$) are semihomogeneous of degree 1, and*

(iii) *the elements $e_k + e_{s-k+1}$ are semihomogeneous of degree f (if $s > 2$), where $f \in G$ is an element of order at most 2.*

Proof. By Lemma 4.2.23, we may assume that S and $\mathfrak{d}$ are graded subalgebras. Also note that $J = [R, R]$ and all of its powers are graded ideals. We proceed by induction on s. If $s = 1$, then $\mathfrak{s}_1 = S$ is graded and there is nothing more to prove. If $s = 2$, then $\mathfrak{s}_1 \oplus \mathfrak{s}_2 = S$ is graded. Also, $\mathrm{Span}\{e_1, e_2\} = \mathfrak{d}$ and $e_1 + e_2 = 1$ is central, so $e_1 - e_2$ is a semihomogeneous element. Its degree must be equal to 1, because $[e_1 - e_2, x] = 2x$ for any $x \in J = B_{12}$. Now assume $s > 2$.

Claim 1: $N := B_{11} \oplus B_{ss} \oplus K1 \oplus J$ is graded.

First suppose $s \geq 4$. Consider $J^{s-2} = J_{s-2} \oplus J_{s-1}$ (the three blocks in the top right corner) and the graded ideal $C := \mathrm{C}_R(J^{s-2}) = R \cap \mathrm{C}_{U(\mathscr{F})^{(-)}}(J^{s-2})$. It is easy to see that

$$C = \mathrm{Span}\{e_2, \ldots, e_{s-1}\} \oplus K1 \oplus B_{23} \oplus \cdots \oplus B_{s-2,s-1} \oplus J^2.$$

Now, the adjoint action induces on C/J^2 a natural structure of a graded $U(\mathscr{F})^{(-)}$-module, and one checks that $N = \mathrm{Ann}_{U(\mathscr{F})^{(-)}}(C/J^2) + J$, so N is graded.

If $s=3$, then consider $J^2 = J_2 = B_{13}$ and the graded ideal $\tilde{C} := \mathrm{C}_{U(\mathscr{F})^{(-)}}(J^2)$. One checks that

$$\tilde{C} = B_{22} \oplus K1 \oplus J,$$

and hence $N = \mathrm{Ann}_{U(\mathscr{F})^{(-)}}(\tilde{C}/J)$. This completes the proof of Claim 1.

It follows that $S \cap N = \mathfrak{s}_1 \oplus \mathfrak{s}_s$ is a graded subalgebra, and

$$I_1 := \mathfrak{d} \cap N = \mathrm{Span}\{e_1, e_s, 1\}$$

is graded as well. Hence, $\mathrm{C}_{I_1}(J^{s-1}) = \mathrm{Span}\{e_1 + e_s, 1\}$ is graded, so we conclude that $e_1 + e_s$ is semihomogeneous. Denote its degree by f.

Claim 2: $f^2 = 1$ and $e_1 - e_s$ is semihomogeneous of degree 1.

Since $I_1/K1$ is spanned by the images of e_1 and e_s, there must exists a semihomogeneous linear combination $\tilde{e}$ of e_1 and e_s that is not a scalar multiple of $e_1 + e_s$. Consider the graded I_1-module J^{s-2}/J^{s-1}. As a module, it is isomorphic to $B_{1,s-1} \oplus B_{2,s}$, where 1 acts as 0, e_1 as the identity on the first summand and 0 on the second, and e_s as 0 on the first and the negative identity on the second. Using this isomorphism, we will write the elements $x \in J^{s-2}/J^{s-1}$ as $x = x_1 + x_2$ with $x_1 \in B_{1,s-1}$ and $x_2 \in B_{2,s}$. Since the situation is symmetric in e_1 and e_s, we may assume without loss of generality that $\tilde{e} = e_1 + \alpha e_s$, $\alpha \neq 1$. Pick a homogeneous element $x = x_1 + x_2$ with $x_1 \neq 0$. First, we observe that $(e_1 + e_s) \cdot ((e_1 + e_s) \cdot x) = x$, which implies $f^2 = 1$. If $x_2 = 0$, then $\tilde{e} \cdot x = (e_1 + e_2) \cdot x = x$, and this implies that the semihomogeneous elements $\tilde{e}$ and $e_1 + e_2$ both have degree 1, which proves the claim. If $\alpha = 0$, then $\tilde{e} \cdot x = x_1 - \alpha x_2 = x_1$ is homogeneous and we can apply the previous argument. So, we may assume that $\alpha \neq 0$.

Suppose for a moment that we have $\deg \tilde{e} = 1$. If $\alpha = -1$, we are done. Otherwise, we can consider the homogeneus element $0 \neq x + \alpha^{-1} \tilde{e} \cdot x \in B_{1,s-1}$ and apply the previous argument again.

It remains to prove that $\deg \tilde{e} = 1$. Denote this degree by g and assume $g \neq 1$. Considering

$$D := \mathrm{Span}\{x, \tilde{e} \cdot x, \tilde{e} \cdot (\tilde{e} \cdot x), \ldots\},$$

we see, on the one hand, that $\dim D \leq 2$, because $D \subset \mathrm{Span}\{x_1, x_2\}$. On the other hand, non-zero homogeneous elements of distinct degrees are linearly independent, so the order of g does not exceed 2. By our assumption, it must be equal to 2. Then x and $\tilde{e} \cdot x$ form a basis of D and $y := \tilde{e} \cdot (\tilde{e} \cdot x)$ has the same degree as x. Therefore, $y = \lambda x$ for some $\lambda \neq 0$. On the other hand, $y = x_1 + \alpha^2 x_2$, hence $\alpha = \pm 1$. The case $\alpha = 1$ is excluded, whereas $\alpha = -1$ implies $\tilde{e} \cdot x = x$, which contradicts $g \neq 1$. The proof of Claim 2 is complete.

We have established all assertions of the proposition for $k = 1$. We are going to use the induction hypothesis for $k > 1$. To this end, consider the graded space

$$\begin{aligned} T_1 :&= \Big(\mathrm{id} - \frac{1}{2}\mathrm{ad}(e_1 - e_s)\Big)\Big(\mathrm{id} - \mathrm{ad}(e_1 - e_s)\Big)U(\mathscr{F})^{(-)} \\ &= B_{11} \oplus eU(\mathscr{F})e \oplus B_{ss}, \end{aligned}$$

where $e := 1 - (e_1 + e_s)$ and $eU(\mathscr{F})e \simeq UT(n_2, \dots, n_{s-1})$. Then $L_1 := \mathrm{C}_{T_1}(J^{s-1}) = K(e_1 + e_s) \oplus eU(\mathscr{F})e$ is graded, and we can apply the induction hypothesis to $L_1/K(e_1 + e_s) \simeq UT(n_2, \dots, n_{s-1})$. Therefore, for $1 < k \leq \frac{s+1}{2}$, the subalgebras $K(e_1+e_s) \oplus (\mathfrak{s}_k + \mathfrak{s}_{s-k+1}) \subset L_1$ are graded, the elements $e_k + e_{s-k+1}$ are semihomogeneous of degree f' in L_1 (if $s > 4$), and the elements $e_k - e_{s-k+1}$ ($k \neq \frac{s+1}{2}$) are semihomogeneous of degree 1 in L_1. For the subalgebras, we get rid of the unwanted term $K(e_1 + e_s)$ by passing to the derived algebra. The elements require more care.

Claim 3: $e_k + e_{s-k+1}$ are semihomogeneous of degree f in $U(\mathscr{F})^{(-)}$.

If $s = 3$, then $e_2 = 1 - (e_1 + e_3)$ is semihomogeneous of degree f. If $s = 4$, then $e_2 + e_{s-1} = 1 - (e_1 + e_s)$ is semihomogeneous of degree f. So, assume $s > 4$. We know there exist α_k such that $\alpha_k(e_1 + e_s) + e_k + e_{s-k+1}$ are semihomogeneous of degree f' in $U(\mathscr{F})^{(-)}$. If $\alpha_2 = 0$, then pick a non-zero homogeneous element $x \in J^{s-2}/J^{s-1}$. Since $(e_1 + e_s) \cdot x = -(e_2 + e_{s-1}) \cdot x \neq 0$, we conclude that $f = f'$ and the claim follows, because we can subtract the scalar multiples of $e_1 + e_s$ from the elements $\alpha_k(e_1 + e_s) + e_k + e_{s-k+1}$. If $\alpha_2 \neq 0$, consider instead the graded $U(\mathscr{F})^{(-)}$-module $([e_1 - e_s, J^2] + J^3)/J^3$. As a module, it is isomorphic to $B_{13} \oplus B_{s-2,s}$, so $e_2 + e_{s-1}$ annihilates it. Picking a non-zero homogeneous element x, we get

$$(\alpha_2(e_1 + e_s) + e_2 + e_{s-1}) \cdot x = \alpha_2(e_1 + e_s) \cdot x \neq 0,$$

so again $f = f'$ and the claim follows.

Claim 4: $e_k - e_{s-k+1}$ are semihomogeneous of degree 1 in $U(\mathscr{F})^{(-)}$.

We know there exist α'_k such that $\alpha'_k(e_1 + e_s) + e_k - e_{s-k+1}$ are semihomogeneous of degree 1 in $U(\mathscr{F})^{(-)}$. If $f = 1$, then we can subtract the scalar multiples of $e_1 + e_s$, so we are done. If $f \neq 1$, we want to prove that $\alpha'_k = 0$. By way of contradiction, assume $\alpha'_k \neq 0$. If $k < \frac{s}{2}$, then $e_k - e_{s-k+1}$ annihilates the graded module $([e_1 - e_s, J^k] + J^{k+1})/J^{k+1}$, so, using the argument in the proof of Claim 3, we conclude that $\deg(e_1 + e_s) = 1$, a contradiction. It remains to consider the case $s = 2k$. If $s > 4$, then $e_{s/2} - e_{s/2+1}$ annihilates the graded module $([e_1 - e_s, J] + J^2)/J^2$, which is isomorphic to $B_{12} \oplus B_{s-1,s}$, so the same argument works. If $s = 4$, then $e_2 - e_3$ does not annihilate this module, but acts on it as the negative identity. Picking a non-zero homogeneous element x, we get

$$x + (\alpha'_2(e_1 + e_s) + e_2 - e_3) \cdot x = \alpha'_2(e_1 + e_s) \cdot x \neq 0,$$

so again $\deg(e_1 + e_s) = 1$, a contradiction.

The proof of the proposition is complete. □

Proof of Lemma 4.2.15. We extend a given G-grading on $U(\mathscr{F})_0$ to $U(\mathscr{F})^{(-)}$ by defining the degree of 1 an arbitrarily. Then $U(\mathscr{F})_0 \simeq U(\mathscr{F})^{(-)}/K1$ as a graded algebra. By Lemma 4.2.23, we may assume that $\mathfrak{d}$ and S are graded, hence the subalgebra $J_0 = \mathfrak{d} \oplus S$ and its homomorphic image $J_0/K1 \simeq J_0 \cap U(\mathscr{F})_0$ in $U(\mathscr{F})_0$ are graded. (In fact, by Proposition 4.2.25, we can say more: every subalgebra $B_{ii} + B_{s-i+1} + K1$ is graded.) To deal with J_m for $m > 0$, we will use the semihomogeneous elements $d_i := e_i - e_{s-i+1}$ of degree 1 ($i \neq \frac{s+1}{2}$). Fix $i < j$. If $i + j \neq s + 1$, then

$$B_{ij} \oplus B_{s-j+1,s-i+1} = \mathrm{ad}(d_i - d_j)\mathrm{ad}(d_i)\mathrm{ad}(d_j)U(\mathscr{F})^{(-)},$$

which is a graded subspace. If $i + j = s + 1$, then

$$B_{ij} = (\mathrm{id} - \mathrm{ad}(d_i))\mathrm{ad}(d_i)J^{s-i+1}$$

is graded. Thus, $B_{ij} + B_{s-j+1,s-i+1}$ is graded for all $i < j$, hence so is J_m. □

Now, we proceed to prove that the support of any G-grading on $U(\mathscr{F})_0$ is a *commutative subset* of G in the sense that its elements commute with each other. The key observation is that, if x and y are homogeneous elements in any G-graded Lie algebra and $[x, y] \neq 0$, then $\deg x$ must commute with $\deg y$. By induction, one can generalize this as follows: if $x_1, \ldots, x_k$ are homogeneous and $[\ldots[x_1, x_2], \ldots, x_k] \neq 0$ then the degrees of x_i must commute pair-wise. This fact was used to show that the support of any graded-simple Lie algebra is commutative (see e.g. [53, Proposition 2.3] or the proof of Proposition 1.12 in [35]). We will need the following two lemmas.

Lemma 4.2.26. *Suppose a semidirect product of Lie algebras $V \rtimes L$ is graded by a group G in such a way that both the ideal V and the subalgebra L are graded. Assume that the support of L is commutative and, as an L-module, V is faithful and generated by a single homogeneous element. Then the support of $V \rtimes L$ is commutative.*

Proof. Let v be a homogeneous generator of V as an L-module and let $g = \deg v$. Denote by H the abelian subgroup generated by $\operatorname{Supp} L$. Then $\operatorname{Supp} V$ is contained in the coset Hg. In particular, the subgroup generated by $\operatorname{Supp}(V \rtimes L)$ is also generated by H and g, so it is sufficient to prove that g commutes with all elements of $\operatorname{Supp} L$. Let $a \neq 0$ be a homogeneous element of L. Since V is faithful, there exists a homogeneous element $w \in V$ such that $a \cdot w \neq 0$. But, in the semidirect product, $a \cdot w = [a, w]$, hence $\deg a$ and $\deg w$ commute. Since $\deg a \in H$, $\deg w \in Hg$, and H is abelian, we conclude that $\deg a$ commutes with g. □

Lemma 4.2.27. *Suppose the Lie algebra $V \rtimes (L_1 \times L_2)$ is graded by a group G in such a way that V, L_1 and L_2 are graded. Assume that each $\operatorname{Supp} L_i$ is commutative, V is faithful as an L_i-module ($i = 1, 2$) and graded-simple as an $(L_1 \times L_2)$-module. Then the support of $V \rtimes (L_1 \times L_2)$ is commutative.*

Proof. One checks that, if we redefine the bracket on the ideal V to be zero while keeping the same bracket on the subalgebra $L_1 \times L_2$ and the same $(L_1 \times L_2)$-module structure on V, the resulting semidirect product is still G-graded, so we may suppose $[V, V] = 0$. Let v be any non-zero homogeneous element of V (hence a generator of V as an $(L_1 \times L_2)$-module). Let W_i be the L_i-submodule generated by v. Since the actions of L_1 and L_2 on V commute with each other, W_i must be a faithful L_i-module, so we can apply Lemma 4.2.26 to the graded subalgebra $W_i \rtimes L_i$ and conclude that $\deg v$ commutes with the elements of $\operatorname{Supp} L_i$. It remains to prove that the elements of $\operatorname{Supp} L_1$ commute with the elements of $\operatorname{Supp} L_2$. Let $a_1 \neq 0$ be a homogeneous element of L_1. Pick a homogeneous $v \in V$ such that $v_1 := a_1 \cdot v \neq 0$ and denote $g = \deg v$ and $g_1 = \deg v_1$. By the previous argument, both g and g_1 commute with every element of $\operatorname{Supp} L_2$. But this implies that $\deg a_1$ commutes with every element of $\operatorname{Supp} L_2$. □

Theorem 4.2.28. *The support of any group grading on $U(\mathscr{F})_0$ over a field of characteristic* 0 *generates an abelian subgroup.*

Proof. The result is known for simple Lie algebras, so we assume $s > 1$. We extend the grading to $U(\mathscr{F})^{(-)}$ and bring it to the form described in Proposition 4.2.25. Then, as in the proof of Lemma 4.2.15 just above, we can break J into the direct sum of graded subspaces of the form $B_{ij} \oplus B_{s-j+1,s-i+1}$ $(i + j \neq s + 1)$ or B_{ij} $(i + j = s + 1)$, for all $1 \leq i < j \leq s$. Also, $\tilde{\mathfrak{s}}_i := \mathfrak{s}_i + \mathfrak{s}_{s-i+1}$ are graded subalgebras (possibly zero). Note that any non-zero $\tilde{\mathfrak{s}}_i$ is graded-simple and, therefore, its support is commutative, except in the following situation: $i \neq \frac{s+1}{2}$ and one of the ideals $\mathfrak{s}_i$ and $\mathfrak{s}_{s-i+1}$ is graded. In this case, the other ideal is graded, too, being the centralizer of the first in $\tilde{\mathfrak{s}}_i$, and we can apply Lemma 4.2.27 to the graded algebra $B_{i,s-i+1} \oplus \tilde{\mathfrak{s}}_i \simeq B_{i,s-i+1} \rtimes (\mathfrak{s}_i \times \mathfrak{s}_{s-i+1})$ to conclude that the support of $\tilde{\mathfrak{s}}_i$ is still commutative. Moreover, its elements commute with those of $\operatorname{Supp} B_{i,s-i+1}$, so we are done in the case $s = 2$. From now on, assume $s > 2$.

Case 1: $f = 1$.

Here each block B_{ij} and each subalgebra $\mathfrak{s}_i$ is graded. Indeed, each element e_i is semihomogeneous of degree 1. If $i + j = s + 1$, then we already know that B_{ij} is graded, and otherwise $B_{ij} = \operatorname{ad}(e_i)(B_{ij} \oplus B_{s-j+1,s-i+1})$, so it is still graded. For $\tilde{\mathfrak{s}}_i$, it is sufficient to consider $i \leq \frac{s+1}{2}$. If $i = \frac{s+1}{2}$, then we already know that $\mathfrak{s}_i$ is graded, and otherwise we can find $j > i$ such that $j \neq s - i + 1$, which implies that $\mathfrak{s}_i = \mathrm{C}_{\tilde{\mathfrak{s}}_i}(B_{ij})$ is still graded.

Applying Lemma 4.2.27 to $B_{ij} \rtimes (\mathfrak{s}_i \times \mathfrak{s}_j)$, we conclude that the supports of non-zero $\mathfrak{s}_i$ and $\mathfrak{s}_j$ commute element-wise with one another and also with $\operatorname{Supp} B_{ij}$. (This works even if one of $\mathfrak{s}_i$ and $\mathfrak{s}_j$ is zero.) It follows that $\operatorname{Supp} S$ generates an abelian subgroup H in G. It also commutes element-wise with $\operatorname{Supp} J$. Indeed, since $\operatorname{Supp} B_{ij}$ is contained in a coset of H, it is sufficient to prove that the degree of one non-zero homogeneous element of B_{ij} commutes with the elements of $\operatorname{Supp} \mathfrak{s}_k$. We already know this if $k = i$ or $k = j$. Otherwise, we will have $k < i < j$, $i < k < j$ or $i < j < k$. In the last case, we have $[B_{ij}, B_{jk}] = B_{ik}$, so we can find homogeneous elements $x \in B_{ij}$ and $y \in B_{jk}$ such that $0 \neq [x, y] \in B_{ik}$. Since the elements of $\operatorname{Supp} \mathfrak{s}_k$ commute with $\deg y$ and with $\deg x \deg y$, they must commute with $\deg x$ as well. The other two cases are treated similarly.

It remains to prove that $\operatorname{Supp} J$ is commutative. Since J_1 generates J as a Lie algebra, it is sufficient to prove that, for any $1 \leq i \leq j \leq s - 1$, the sets $\operatorname{Supp} B_{i,i+1}$ and $\operatorname{Supp} B_{j,j+1}$

commute with one another element-wise. But we can find homogeneous elements $x_1 \in B_{12}$, $x_2 \in B_{23}, \dots, x_{s-1} \in B_{s-1,s}$ such that $[\dots[x_1, x_2], \dots, x_{s-1}] \neq 0$, so the degrees of $x_1, x_2, \dots, x_{s-1}$ must commute pair-wise. The coset argument completes the proof of Case 1.

Case 2: $f \neq 1$.

Here we work with $\tilde{B}_{ij} := B_{ij} + B_{s-j+1,s-i+1}$. If $\tilde{\mathfrak{s}}_i$ and $\tilde{\mathfrak{s}}_j$ are distinct (that is, $i+j \neq s+1$) and non-zero, then $\tilde{B}_{ij}$ is a direct sum of two non-isomorphic simple $(\tilde{\mathfrak{s}}_i \times \tilde{\mathfrak{s}}_j)$-submodules. We claim that it is a graded-simple $(\tilde{\mathfrak{s}}_i \times \tilde{\mathfrak{s}}_j)$-module. Indeed, otherwise one of the submodules B_{ij} and $B_{s-j+1,s-i+1}$ would be graded. But there exist scalars λ_i such that $\tilde{e}_i := e_i + e_{s-i+1} + \lambda_i 1$ are homogeneous of degree f, and $\mathrm{ad}(\tilde{e}_i)$ acts as the identity on B_{ij} and the negative identity on $B_{s-j+1,s-i+1}$, which forces $f = 1$, a contradiction.

Therefore, we can apply Lemma 4.2.27 to $\tilde{B}_{ij} \rtimes (\tilde{\mathfrak{s}}_i \times \tilde{\mathfrak{s}}_j)$ and conclude that the supports of non-zero $\tilde{\mathfrak{s}}_i$ and $\tilde{\mathfrak{s}}_j$ commute element-wise with one another, hence $\mathrm{Supp}\, S$ is commutative.

Now consider $\tilde{B}_{ij}$, with $i+j \neq s+1$, as an $((\tilde{\mathfrak{s}}_i \times \tilde{\mathfrak{s}}_j) \times K\tilde{e}_i)$-module, where one of $\tilde{\mathfrak{s}}_i$ and $\tilde{\mathfrak{s}}_j$ is allowed to be zero. The simple submodules B_{ij} and $B_{s-j+1,s-i+1}$ are non-isomorphic, because they are distinguished by the action of $\tilde{e}_i$. Hence, our argument in the first paragraph shows that $\tilde{B}_{ij}$ is a graded-simple module, so we can apply Lemma 4.2.27 to $\tilde{B}_{ij} \rtimes ((\tilde{\mathfrak{s}}_i \times \tilde{\mathfrak{s}}_j) \times K\tilde{e}_i)$ and conclude that the supports of $\tilde{\mathfrak{s}}_i$ and $\tilde{\mathfrak{s}}_j$ commute element-wise with f and also with $\mathrm{Supp}\, \tilde{B}_{ij}$. Moreover, f commutes with $\mathrm{Supp}\, \tilde{B}_{ij}$. If $i+j = s+1$, then $\tilde{B}_{ij} = B_{ij}$ and we can apply Lemma 4.2.26 to $B_{ij} \rtimes \tilde{\mathfrak{s}}_i$.

Therefore, the elements of $\mathrm{Supp}\, S$ commute with f and together generate an abelian subgroup H in G. Then, by the same argument as in Case 1 (but using $\tilde{B}_{ij}$ instead of B_{ij}), we show that $\mathrm{Supp}\, S$ commutes element-wise with $\mathrm{Supp}\, J$. In order to prove that f commutes with $\mathrm{Supp}\, J$, it is sufficient to consider J_1. As we have seen, f commutes with $\mathrm{Supp}\, \tilde{B}_{ij}$ where $i+j \neq s+1$. The only case that is not covered in J_1 is $\tilde{B}_{s/2,s/2+1} = B_{s/2,s/2+1}$ for even s. Since $s > 2$, we have $[\tilde{B}_{s/2-1,s/2}, B_{s/2,s/2+1}] = \tilde{B}_{s/2-1,s/2+1}$. Since f commutes with $\mathrm{Supp}\, \tilde{B}_{s/2-1,s/2}$ and with $\mathrm{Supp}\, \tilde{B}_{s/2-1,s/2+1}$, we conclude that f commutes with $\mathrm{Supp}\, B_{s/2,s/2+1}$ as well. The commutativity of $\mathrm{Supp}\, J$ is proved by the same argument as in Case 1. □

3. The Jordan case

Every Jordan isomorphism from the algebra $U(\mathscr{F})$, $s > 1$, to an arbitrary associative algebra R is either an associative isomorphism or anti-isomorphism [24, Corollary 3.3]. As we saw in Subsection 2.3, $U(\mathscr{F})$ admits an anti-automorphism if and only if $n_i = n_{s-i+1}$ for all i. So, taking into account the structure of the automorphism group of $U(\mathscr{F})$ (see Lemma 4.2.6), we obtain that the automorphism group of $U(\mathscr{F})^{(+)}$, that is, the algebra $U(\mathscr{F})$ viewed as a Jordan algebra with respect to the symmetrized product $x \circ y = xy + yx$, is either $\{\mathrm{Int}(x) \mid x \in U(\mathscr{F})^\times\}$ or $\{\mathrm{Int}(x) \mid x \in U(\mathscr{F})^\times\} \rtimes \langle\tau\rangle$. In both cases, the following holds:

Lemma 4.3.1. $\mathrm{Aut}(U(\mathscr{F})^{(+)}) \simeq \mathrm{Aut}(U(\mathscr{F})_0)$. □

Hence, if K is algebraically closed of characteristic 0 and the grading group G is abelian, then the classification of G-gradings on the Jordan algebra $U(\mathscr{F})^{(+)}$ is equivalent to the classification of G-gradings on the Lie algebra $U(\mathscr{F})_0$ (see also [35, §5.6] for the simple case, $s = 1$). Thus, we get the same parametrization of the isomorphism classes of gradings as in Corollary 4.2.19. The only difference is the sign in the construction of Type II gradings on $M_n^{(+)}$ (compare with Equation (4) and recall that Φ is given by Equation (3)):

$$M_n^{(+)} = \bigoplus_{g \in G} R_g \text{ where } R_g = \{X \in R_{\bar{g}} \mid \Phi^{-1} X^\tau \Phi = \chi(g) X\},$$

which are then restricted to $U(\mathscr{F})^{(+)}$.

We note, however, that this result does not exclude the existence of group gradings on $U(\mathscr{F})^{(+)}$ with non-commutative support. In view of Theorem 4.2.28, these gradings, if they exist, are not analogous to gradings on $U(\mathscr{F})_0$.

Chapter 5
Asymptotics of the graded codimensions of upper triangular matrices

1. Introduction

Let A be an associative PI algebra. As pointed out in the Introduction of this thesis, describing the ideal of identities $T(A)$ of A is a very difficult problem. Even for the matrix algebras $M_n(K)$, the description of their T-ideals is known only for $n \leq 2$. For $M_2(K)$, the problem was resolved in characteristic zero [56, 32], positive characteristic different from 2 [47], finite fields [52]. On the other hand, there are instances where the T-ideal has a tight description: the Grassmann algebras, the upper triangular matrices, see for example the monograph [33, Chapter 5].

Let P_m be the vector space of the multilinear polynomials of degree m in $x_1, \ldots, x_m$ in the free associative algebra. It is well known that in characteristic 0, the sets $P_m \cap T(A)$ describe completely $T(A)$. As P_m is a (left) module over the symmetric group $\mathcal{S}_m$ then $P_m \cap T(A)$ is a submodule, and one considers the quotient $\mathcal{S}_m$-module $P_m(A) = P_m/(P_m \cap T(A))$. By applying the well developed representation theory of $\mathcal{S}_m$ one obtains deep and profound results about PI algebras, see for example the monograph [38].

One can study $T(A)$ indirectly. Let $c_m = c_m(A) = \dim P_m(A)$ be the codimension sequence of A. A celebrated theorem of Regev [59] states that the sequence $(c_m)_{m\in\mathbb{N}}$ is exponentially bounded for any associative PI algebra A, that is, there exists $d > 0$ such that $c_m \leq d^m$ for every m. Computing the exact codimensions of a given algebra is difficult and unsolved in most of the cases.

As $c_m \leq d^m$ for some $d > 0$ then $\limsup_{m\to\infty} \sqrt[m]{c_m} \leq d < \infty$. Hence one can consider the limits $0 \leq \liminf \sqrt[m]{c_m} \leq \limsup \sqrt[m]{c_m} < \infty$. Amitsur conjectured that these always coincide, that is $\sqrt[m]{c_m(A)}$ converges, and the limit is a non-negative integer. This conjecture was confirmed by Giambruno and Zaicev, see for example [38, Chapter 6]. This limit is called the *PI exponent* of A, and is denoted by $\exp(A)$. The PI exponent is of significant interest in PI theory, and led to new directions of research. These include the study and classification of minimal varieties [38, Chapter 8 and references therein], of varieties with polynomial growth [38, Chapter 7 and references therein], and so on.

One can extend all of the above to varieties of non-associative algebras. Codimensions and PI exponent of such algebras are defined in an analogous way. In such general context the exponent need not be an integer; moreover it may not exist. In fact the codimension sequence need not be exponentially bounded. On the other hand, for large classes of important algebras the exponent is well behaved (see [38, Chapter 12] and the references therein for the Lie case, as an example). For finite dimensional Jordan and alternative algebras the Amitsur's conjecture holds [37].

One can generalize this theory to the case of graded algebras. It was proved in [2] that if A is an associative G-graded algebra in characteristic 0, and G is a finite group then the graded PI exponent, $\exp^G(A)$ exists and is an integer. Also, in [42, 41] the author proved that the conjecture of Amitsur holds for H-codimensions for associative and Lie algebras where H is a Hopf algebra. We remark that the graded exponent for any grading is known for a lot of algebras, like matrix algebras [9], simple Lie algebras [42], and so on.

It is natural to study the asymptotic behaviour of the codimension sequence c_m. One may look for a function $f\colon \mathbb{N} \to \mathbb{R}_+$ such that $\lim_{x\to\infty} f(n)/c_n = 1$. In this case, we denote $c_m \sim f(m)$. Regev conjectured that for any PI algebra A, there exist constants c and a half integer g such that $c_m(A) \sim cm^g d^m$ where $d = \exp(A)$. This is indeed the case for unitary PI-algebras, as prove by Berele and Regev for finitely generated algebras satisfying some Capelli identity in [21], and by Berele in [20] for any unitary PI-algebra. In [40], the authors proved a weaker version of the previous result for any PI-algebra.

In this section we study the graded exponent for gradings on the algebra of upper triangular matrices UT_n. In the associative case, the asymptotic behaviour of the (ordinary) codimensions of UT_n is known in characteristic zero: $c_m \sim \frac{1}{n^{n-1}} m^{n-1} n^m$, see for example [38, Chapter 8]. For the Lie case, in [54], Petrogradskii computed the exact codimension growth for the Lie algebra $UT_n^{(-)}$ and the asymptotic behaviour of its codimension sequence: $c_m(UT_n^{(-)}) \sim m^{n-1} \frac{1}{(n-1)^n} (n-1)^m$.

We prove that the asymptotic behaviour of any grading on UT_n coincides with the ordinary case above. If we view UT_n as a Lie algebra or as a Jordan algebra, and fix any group grading, then the graded exponent is $n-1$ or n, respectively.

In [39], the authors study the exponent of upper-block triangular matrix algebras, in the non-graded associative, Lie, Jordan case and the case of proper polynomials. They obtain relations among these exponents. Here we prove that, for the upper triangular matrices, the results from [39] hold for the graded case for the Jordan, Lie and associative products.

2. Preliminaries

We fix an infinite field K. We do not use representation theory of the symmetric group thus most of our arguments are characteristic-free. On the other hand we use results about the ordinary codimensions holding in characteristic 0, thus in stating some of the results we will require K of characteristic 0.

In [49] the authors computed the codimensions of the canonical $\mathbb{Z}_n$-grading on UT_n where $\deg e_{ij} = j - i \in \mathbb{Z}_n$.

Theorem 5.2.1 ([49])**.** *The graded codimensions for the canonical $\mathbb{Z}_n$-grading on UT_n satisfy*

$$c_m^{\mathbb{Z}_n}(UT_n) = \sum_{q=0}^{\min\{m,n-1\}} \binom{m}{q}\binom{n-1}{q} q\,!\,(q+1)^{m-q}.$$

The asymptotics of the graded codimensions is $c_m^{\mathbb{Z}_n}(UT_n) \sim \frac{1}{n^{n-1}} m^{n-1} n^m$. In particular, the graded exponent is $\exp^{\mathbb{Z}_n}(UT_n) = n$.

We shall obtain once again this asymptotic behaviour here, as a particular case of a more general result.

Let A be G-graded, then the inequalities $c_m^1(A) \le c_m^G(A) \le |G|^m c_m^1(A)$ hold, see for example [10]. Here $c_m^1(A)$ stands for the ordinary codimension (that is the graded codimension for the trivial grading by the trivial group $G = \{1\}$). The right-hand inequality makes sense only for finite groups.

A natural question is whether this inequality is “the best possible". In other words are there G-graded algebras whose codimensions equal the lower bound, and G-graded algebras that attain the upper bound?

The trivial grading gives the lower bound. In [6, 7] it was proved that the Grassmann algebra under appropriate gradings, attains the upper bound. Therefore for finite groups the inequality is the best possible.

We start with the elementary gradings on the algebra of upper triangular matrices. We shall prove a similar inequality which does not depend on the cardinality of the group, and hence we can also consider infinite groups.

Let $A = UT_n$, and in the associative case, let F denote the free group of rank $n-1$. In the Lie and Jordan cases we consider the free abelian group of rank $n-1$, that is $F = \mathbb{Z}^{n-1}$. Assume further that $L = \{l_1, l_2, \ldots, l_{n-1}\}$ are the free generators of F.

Definition 5.2.2. The universal elementary F-grading on A is the elementary grading defined by $\eta = (l_1, l_2, \ldots, l_{n-1})$, that is $\deg_F e_{i,i+1} = l_i$ for $i = 1, 2, \ldots, n-1$.

Let A be endowed with an elementary G-grading where G is an arbitrary group. In the Lie or Jordan case, we require G abelian. Assume G is generated by the support of the grading, that is $\deg_G e_{12}, \ldots, \deg_G e_{n-1,n}$ generate G. The map $\bar{\psi} : L \to G$ given by $\bar{\psi}(l_i) = \deg_G e_{i,i+1}$ extends uniquely to a group homomorphism $\psi \colon F \to G$. We identify $G = F/\ker \psi = \psi(F)$.

Lemma 5.2.3. *The $\psi(F)$-grading on A induced by the F-grading and by the homomorphism ψ, coincides with the original G-grading.*

Proof. Since $\deg_{\psi(F)} e_{i,i+1} = \deg_G e_{i,i+1}$, for all i the lemma follows. $\square$

When $\dim A < \infty$ every grading is finite. Take the free F-graded algebra $K\langle X_F\rangle$ freely generated by $\{x_i^{(l)} \mid i \in \mathbb{N}, l \in F\}$. We proceed with the following identification. For $t \in G$ and $i \in \mathbb{N}$ we define

$$y_i^{(t)} = \begin{cases} \sum_{l \in \operatorname{Supp}\eta \cap \psi^{-1}(t)} x_i^{(l)}, & \text{if } \operatorname{Supp}\eta \cap \psi^{-1}(t) \neq \emptyset, \\ x_i^{(l)}, & \text{for a choice of } l \text{ with } \psi(l) = t, \text{ otherwise.} \end{cases}$$

The algebra generated by $\{y_i^{(t)} : i \in \mathbb{N}, t \in G\}$ is isomorphic to the free G-graded algebra; we identify these two algebras. Under this identification $T_G(A) \subset T_F(A)$. Also $T_1(A) \subset T_G(A)$ where $T_1(A)$ is the ideal of ordinary polynomial identities of A (that is the graded identities for the trivial grading). Hence as in [10] one proves the inequality $c_m^G(A) \le c_m^F(A)$. Thus we obtain

Theorem 5.2.4. *Let G be any group and take an elementary G-grading on $A = UT_n$. Then for every $m \in \mathbb{N}$, the following inequality holds*

$$c_m^1(A) \le c_m^G(A) \le c_m^F(A).$$

Similar inequalities can be obtained for the Lie and Jordan case. We shall use these for the type II gradings, in the Lie and Jordan cases. We will see that in the associative case $c^1_m(A)$ and $c^F_m(A)$ are asymptotically equal. We shall use these inequalities to capture the asymptotic behaviour of $c^G_m(A)$.

3. The associative case

Let P^F_m denote the vector space of F-graded associative multilinear polynomials in m variables. Call η the elementary universal F-grading on UT_n. The graded polynomial identities follow from the η-bad sequences, see [29]. We start computing the F-graded codimensions of UT_n.

Lemma 5.3.1. *Let $\mu = (t_1, t_2, \dots, t_c)$ be an η-good sequence and define $z_i = x_i^{(t_i)}$, $x_j = x_j^{(0)}$, for $1 \le i \le c$, $1 \le j \le c'$. Consider the monomials*

$$x_{i_{01}} x_{i_{02}} \cdots x_{i_{0l_0}} z_1 x_{i_{11}} x_{i_{12}} \cdots x_{i_{1l_1}} z_2 \cdots z_c x_{i_{c1}} x_{i_{c2}} \cdots x_{i_{cl_c}} \tag{5}$$

where for each $0 \le m \le c$ we have $l_m \ge 0$ and $i_{m1} < i_{m2} < \cdots < i_{ml_m}$. Then these monomials are linearly independent modulo $T_F(A)$.

Proof. Let $\{\xi^{(i)}_m \mid i \in \mathbb{N}, m = 1, 2, \dots, n\}$ be commuting variables. The substitution $x_h = \sum_{l=1}^n \xi^{(h)}_l e_{ll}$ and $z_i = e_i$ where $\deg e_i = t_i = \deg z_i$ shows that these monomials are linearly independent. □

The polynomials (5) span the vector space of the relatively free F-graded algebra $K\langle X_F\rangle / T_F(UT_n)$.

Lemma 5.3.2. *One has* $c^F_m(A) = 1 + \sum\limits_{i=1}^{\min\{n-1,m\}} \binom{n}{i+1}\binom{m}{i} i!\,(i+1)^{m-i}$.

Proof. Let $a = (a_1, a_2, \dots, a_m) \in (F)^m$, and form the subsequence $\mu_a = (a_{j_1}, a_{j_2}, \dots, a_{j_i})$ obtained from a after removing all trivial a_j (and keeping the original order of the remaining entries). We compute $\dim P^a_m(UT_n)$. If $\sigma\mu_a$ is η-bad for every $\sigma \in S_i$ then $P^a_m(UT_n) = 0$. (Recall η stands for the elementary universal F-grading on UT_n.) Also if $\mu_a = \emptyset$, that is $a = (1, 1, \dots, 1)$ then $\dim P^a_m(UT_n) = 1$. Thus we assume $\mu_a \ne \emptyset$.
Claim 1. $\dim P^a_m(UT_n) = (i+1)^{m-i}$.

In the monomials of $P^a_m(UT_n)$ the variables of non-trivial degree can be ordered in a unique way, and the variables of trivial degree appear anywhere among the variables of non-trivial degree. Consecutive variables of trivial degree can be ordered. Hence $\dim P_m^a(UT_n)$ coincides with the number of ways we can put $m-i$ different variables into $i+1$ places.

Claim 2. There are $\binom{m}{i}$ sequences $a' \in (F)^m$ such that $\mu_{a'} = \mu_a$.

There are $\binom{m}{m-i} = \binom{m}{i}$ ways to insert $m-i$ elements all equal to 1 in the sequence a' and obtain different sequences of length m.

Claim 3. There are $\binom{n}{i+1}$ good sequences of length i.

The e_{kl}, $k < l$ determine $\deg e_{kl}$. Hence there is 1–1 correspondence between the good sequences and the non-zero products $e_{i_1j_1}e_{i_2j_2}\cdots e_{i_ij_i}$. This product is non-zero if and only if $1 \le i_1 < j_1 = i_2 < j_2 = i_3 < \cdots < j_{i-1} = i_i < j_i \le n$. Hence there are as many non-zero products as ways of choosing $i+1$ elements in a set of n elements $\{i_1, j_1, j_2, \ldots, j_i\} \subset \{1, 2, \ldots, n\}$. This equals $\binom{n}{i+1}$.

Claim 4. Let $a' \in G^m$ be such that $\sigma\mu_{a'} = \mu_a$ for some $\sigma \in S_i$. Then $\dim P_m^{a'}(UT_n) = \dim P_m^a(UT_n)$. In particular, we can permute the elements a_{j_1}, a_{j_2}, $\ldots$, a_{j_i} thus obtaining the factor $i!$.

The above claims conclude the proof. $\square$

Corollary 5.3.3. $c_m^F(A) \sim \dfrac{1}{n^{n-1}} m^{n-1} n^m$.

Proof. The proof is the same as that of [49, Corollary 4.1]. $\square$

Using the classification of the gradings on UT_n, Theorem 5.2.4, and the well known asymptotic behaviour of the codimensions of UT_n in the associative case, we obtain

Theorem 5.3.4. *Let* $\operatorname{char} K = 0$ *and let* UT_n *be* G*-graded where* G *is an arbitrary group. Then* $c_m^G(UT_n) \sim \dfrac{1}{n^{n-1}} m^{n-1} n^n$. *In particular,* $\exp^G(UT_n) = n$.

4. The Lie algebra case: elementary gradings

In this subsection η will denote the universal $\mathbb{Z}^{n-1}$-grading on $UT_n^{(-)}$. Write $UT_n^{(-)} = \oplus_{g \in \mathbb{Z}^{n-1}} (UT_n^{(-)})_g$. A classification of the graded identities for elementary gradings on $UT_n^{(-)}$ is not known. Therefore we first determine the graded identities for $(UT_n^{(-)}, \eta)$, and then compute the graded codimensions.

Definition 5.4.1. A variable $x_1^{(g)}$ is a null variable if $(UT_n^{(-)})_g = 0$.

Lemma 5.4.2. *Let* $\mu \in (\mathbb{Z}^{n-1})^c$ *be* η*-bad sequence then* f_μ *follows from some null variable or from* $[x_1^{(0)}, x_2^{(0)}] = 0$.

Proof. There is at most one linearly independent element of each non-trivial degree. □

We remark that, if $\mu = (a_1, \ldots, a_m) \in (\mathbb{Z}^{n-1})^m$ is an associative η-good sequence, then the unique associative η-good sequences obtained by using the elements $a_1, \ldots, a_m$ are of the form $(a_i, a_{i+1}, \ldots, a_j)$, for some $1 \le i \le j \le m$.

Any sequence $\mu \in (\mathbb{Z}^{n-1})^c$ with repetitions is η-bad since in order to form the non-zero product we use only strictly upper triangular matrix units. Denote by T the $T_{\mathbb{Z}^{n-1}}$-ideal generated by all null variables and $[x_1^{(0)}, x_2^{(0)}]$.

Let $\mu = (a_1, \ldots, a_m) \in G^m$ be a sequence, and, using the notation of last remark of section 2, let $z_i = f_i^{(a_i)}$, and $f_\mu = [z_1, \ldots, z_m]$. Given a permutation $\sigma \in S_m$, we denote $f_\mu^{(\sigma)} = [z_{\sigma^{-1}(1)}, z_{\sigma^{-1}(2)}, \cdots, z_{\sigma^{-1}(c)}]$. Recall the definition of the set $\mathscr{T}_m$ given in chapter 2.

Lemma 5.4.3. *Let $\mu \in (\mathbb{Z}^{n-1})^m$ be an associative η-good sequence. Then one has that $f_\mu \equiv (-1)^{m+1} f_\mu^{(\tau_m)} \pmod T$.*

Proof. Write $f_\mu = [z_1, z_2, \ldots, z_m]$. All equalities will be considered modulo T. We induct on m. If $m = 2$, then the result follows by antisymmetry. By the induction hypothesis, $f_\mu^{(\tau_m)} = [z_m, \ldots, z_1] = (-1)^m [z_2, \ldots, z_m, z_1]$.

Since $[z_m, z_1] = 0$, by applying the Jacobi identity we obtain $f_\mu^{(\tau_m)} = [z_2, \ldots, z_{m-1}, z_1, z_m]$. Applying the Jacobi identity several times, and the antisymmetry, we get

$$f_\mu^{(\tau_m)} = (-1)^m [z_2, z_1, z_3, \ldots, z_m] = (-1)^{m+1} [z_1, z_2, \ldots, z_m] = (-1)^{m+1} f_\mu,$$

and this concludes the proof. □

The previous lemma together with Lemma 3.2.7 yield the following corollary.

Corollary 5.4.4. *If $\mu \in (\mathbb{Z}^{n-1})^m$ is associative η-good then $f_\mu \equiv \pm f_\mu^{(\sigma)} \pmod T$ for any $\sigma \in \mathscr{T}_m$. Moreover $f_\mu^{(\sigma)} \equiv 0 \pmod T$ for any $\sigma \in S_m \setminus \mathscr{T}_m$.*

Definition 5.4.5. A Lie monomial g is suitable if $g = [g_1, g_2, \ldots, g_c]$ where

(i) $g_i = [x_1^{(t_i)}, x_{j_{i1}}^{(0)}, x_{j_{i2}}^{(0)}, \ldots, x_{j_{is_i}}^{(0)}]$, for some $s_i \ge 0$, $j_{i1} \le j_{i2} \le \cdots \le j_{is_i}$, for $i = 1, 2, \ldots, c$;

(ii) The sequence $(\deg g_1, \deg g_2, \ldots, \deg g_c)$ is associative η-good.

Lemma 5.4.6. *The suitable monomials span the vector space $L\langle X_{\mathbb{Z}^{n-1}} \rangle$ modulo T.*

Proof. It suffices to prove that $g' = [x_{j_1}^{(i_1)}, x_{j_2}^{(i_2)}, \ldots, x_{j_{c'}}^{(i_{c'})}]$ is a linear combination of suitable monomials.

Applying the Jacobi identity several times and the graded identity $[x_1^{(0)}, x_2^{(0)}]$ (which lies in T) we conclude that g is a linear combination of $g = [g_1, g_2, \ldots, g_c]$ where each g_i is as in (i) of the previous definition, modulo T.

Denote $\mu = (\deg_{\mathbb{Z}^{n-1}} g_1, \ldots, \deg_{\mathbb{Z}^{n-1}} g_c)$. If μ is Lie η-bad then the corresponding g must be 0: otherwise, repeating the argument of the previous lemma, we see $g = \sigma g \pmod{T}$ where $\sigma\mu$ is associative η-good. This completes the proof. □

Corollary 5.4.7. *The suitable monomials form a basis of the vector space $L\langle X_{\mathbb{Z}^{n-1}}\rangle$ modulo $T_{\mathbb{Z}^{n-1}}(UT_n^{(-)})$.*

Proof. By the previous lemma the suitable monomials form a spanning set. An appropriate substitution by generic matrices (note that every variable of non-zero degree appears at most once) shows that the suitable monomials are linearly independent. □

In this way we have proved the following theorem.

Theorem 5.4.8. *The $\mathbb{Z}^{n-1}$-graded identities of $(UT_n^{(-)}, \eta)$ follow from:*

$$[x_1^{(0)}, x_2^{(0)}] = 0, \qquad x_1^{(g)} = 0, \qquad (UT_n^{(-)})_g = 0.$$

Corollary 5.4.9. *The codimensions of the elementary universal grading satisfy*

$$c_m^{\mathbb{Z}^{n-1}}(UT_n^{(-)}) = \sum_{i=1}^{\min\{n-1,m\}} \binom{n}{i+1}\binom{m}{i} i^{m-i} i\,!$$

In particular $\exp^F(UT_n^{(-)}) = n-1$.

Proof. The proof does not differ from that in the associative case. □

As a consequence we obtain the graded exponent of any elementary grading on $UT_n^{(-)}$.

Theorem 5.4.10. *Let* $\operatorname{char} K = 0$. *For any abelian group G and any elementary G-grading on the Lie algebra $UT_n^{(-)}$, we have*

$$m^{n-1}(n-1)^{m-n} \preceq c_m^G(UT_n^{(-)}) \preceq m^{n-1}(n-1)^{m-n+1},$$

where $\preceq$ indicates asymptotically less than or equal. In particular $\exp^G(UT_n^{(-)}) = n-1$.

5. Type II gradings, and elementary gradings on UJ_n

Here we obtain an upper bound for the codimensions for the elementary gradings on UJ_n, and for the type II gradings in the Lie and Jordan cases.

Let η be the universal elementary $\mathbb{Z}^{n-1}$-grading on UJ_n. The next lemma follows the idea of [9, Proposition 2.3(b)].

Lemma 5.5.1. *Let A be a not necessarily associative algebra graded by an abelian group G, $A = \oplus_{g\in G} A_g$. Let $a \in G^m$ be such that $g_k \in G$ appears $n_k > 0$ times in a, $1 \le k \le l$, and let $g = g_1^{n_1} g_2^{n_2} \cdots g_l^{n_l}$. Denote by P_m^a the multilinear G-graded polynomials in m variables whose degrees respect the sequence a. Then*

$$\dim P_m^a(A) \le (\dim A_{g_1})^{n_1} (\dim A_{g_2})^{n_2} \cdots (\dim A_{g_l})^{n_l} \cdot \dim A_g.$$

Proof. Each $f \in P_m^a(A)$ can be viewed, by evaluation, as a multilinear map

$$f\colon A_{g_1}^{n_1} \times A_{g_2}^{n_2} \times \cdots \times A_{g_l}^{n_l} \to A_g.$$

This identification is well defined and injective. This completes the proof. □

Lemma 5.5.2. *In the notation of the previous lemma, let $L = \oplus_{g\in G} L_g$ be a G-graded Lie algebra and let $\bar{L} = L/\mathfrak{z}(L)$ be the factor by its center. Let $\bar{L} = \oplus_{g\in G} \bar{L}_g$ be the induced G-grading, then*

$$\dim P_m^a(L) \le (\dim \bar{L}_{g_1})^{n_1} (\dim \bar{L}_{g_2})^{n_2} \cdots (\dim \bar{L}_{g_l})^{n_l} \cdot \dim L_g.$$

Proof. The ideal $\mathfrak{z}(L)$ is graded. Each $f \in P_m^a(L)$ can be viewed as

$$f\colon L_{g_1}^{n_1} \times L_{g_2}^{n_2} \times \cdots \times L_{g_l}^{n_l} \to L_g.$$

Since $f(\mathfrak{z}(L)) = 0$ we can ignore the center and consider f as a map

$$f\colon \bar{L}_{g_1}^{n_1} \times \bar{L}_{g_2}^{n_2} \times \cdots \times \bar{L}_{g_l}^{n_l} \to L_g.$$

This completes the proof. □

For the universal grading one has $\dim(\mathrm{UJ}_n)_0 = n$ and $\dim(\mathrm{UJ}_n)_l \le 1$ for any $0 \ne l \in \mathbb{Z}^{n-1}$. As a consequence of Lemma 5.5.1 we obtain

Lemma 5.5.3. *The $\mathbb{Z}^{n-1}$-graded codimensions satisfy*

$$c_m^{\mathbb{Z}^{n-1}}(\mathrm{UJ}_n) \leq \sum_{j=0}^{\min\{n-1,m\}} \binom{m}{j} n^{m-j} \binom{n}{j+1} j\,!$$

Proof. The factor n^{m-j} comes from Lemma 5.5.1. The remaining factors are analogous to those of the computation in the associative case, see Lemma 5.3.2. Also $\dim P_m^a(\mathrm{UJ}_n) = 1$ for the constant sequence $a = (0, 0, \ldots, 0)$. □

Using similar argument we obtain an upper bound for the codimensions of the type II gradings, in the Lie and Jordan cases. First we consider the Jordan algebra UJ_n.

Definition 5.5.4. Let $d = \lceil \frac{n}{2} \rceil$ and $M = \mathbb{Z}_2 \times \mathbb{Z}^d$. The universal type II M-grading on UJ_n is defined by the element $(1, 0) \in M$ and by the sequence $(l_1, l_2, \ldots, l_d) \in M^d$. Here $l_i = (0, l_i)$ and $\{l_1, l_2, \ldots, l_d\}$ is a basis of the free abelian group $\mathbb{Z}^d$.

Given a group G and a type II G-grading $(\mathrm{UJ}_n, t, \eta')$, there is a map $f\colon M \to G$ such that the induced grading on UJ_n, by f and by the universal M-grading, coincides with the G-grading $(\mathrm{UJ}_n, t, \eta')$. Hence we can apply an analog of Theorem 5.2.4, that is the graded codimensions of the universal type II grading give an upper bound for the graded codimensiosn of any type II grading.

We proceed with a constructions analogous to the elementary case.

Definition 5.5.5. A sequence $\mu = (a_1, a_2, \ldots, a_m) \in M^m$ is M-good Jordan sequence if there exist r_1, r_2, …, $r_m \in \{X_{i:l}^{\pm} \mid l > 0\}$ such that $r_1 \circ r_2 \circ \cdots \circ r_m \neq 0$ and $\deg r_i = a_i$.

Let $J \subset \mathrm{UJ}_n$ be the set of strictly upper triangular matrices with the induced grading and take any total ordering on $H = \operatorname{Supp} J$.

Lemma 5.5.6. *There exist at most $2^i \binom{n}{i+1}$ M-good ordered Jordan sequences of i elements of H.*

Proof. It is sufficient to count the subsets $\{r_1, r_2, \ldots, r_i\} \subset \{X_{j:l}^{\pm} \mid l > 0\}$ having the property that $r_{\sigma(1)} \circ r_{\sigma(2)} \circ \cdots \circ r_{\sigma(i)} \neq 0$ for some $\sigma \in S_i$.

If we consider only the subsets of $\{X_{j:l}^{+} : l > 0\}$ this gives the upper bound $\binom{n}{i+1}$. But for each M-good sequence $\mu = (a_1, a_2, \ldots, a_i)$, we can obtain new sequences $(b_1, b_2, \ldots, b_i)$ where, for each $j = 1$, 2, …, i we have $b_j = a_j$ or $b_j = a_j + (1, 0)$. Some of these sequences might be not M-good. But these contain all M-good ordered sequences of i elements. There are $2^i\binom{n}{i+1}$ such sequences. □

Lemma 5.5.7. $c_m^M(\mathrm{UJ}_n) \leq \sum_{i=0}^{\min\{n-1,m\}} 2^i\binom{n}{i+1}(i\,!)^2\binom{m}{i}n^{m-i}$.

Proof. Let a be a sequence of i elements in $\mathbb{Z}^d$ and $m-i$ elements belonging to $\{(1,0),(0,0)\}$. We obtain an upper bound for the quantity of such sequences a and compute $\dim P_m^a(\mathrm{UJ}_n)$.

The term $2^i\binom{n}{i+1}$ comes from the previous lemma. The first multiple $i!$ is to ignore the “ordered" condition of the previous lemma. The second $i!$ and $\binom{m}{i}$ are analogous to the associative case (Lemma 5.3.2).

Assume a contains p_1 elements $(0,0)$ and p_2 elements $(1,0)$. By Lemma 5.5.1 we obtain the upper bound $\dim P_m^a(\mathrm{UJ}_n) \leq \lceil \frac{n}{2} \rceil^{p_1} \lfloor \frac{n}{2} \rfloor^{p_2}$. Considering all possibilities for p_1 and p_2, and using the binomial formula, we obtain the factor

$$\sum_{p_1+p_2=m-i}\binom{m-i}{p_1}\left\lceil\frac{n}{2}\right\rceil^{p_1}\left\lfloor\frac{n}{2}\right\rfloor^{p_2} = \left(\left\lceil\frac{n}{2}\right\rceil+\left\lfloor\frac{n}{2}\right\rfloor\right)^{m-i} = n^{m-i},$$

and this proves the lemma. □

We obtain a similar upper bound for the Lie case. We define M-good Lie sequences in the same way as in the Jordan case.

Lemma 5.5.8. *For every M-good Lie sequence $a = (a_1, a_2, \ldots, a_m)$, there exists an M-good sequence $b = (b_1, b_2, \ldots, b_m)$ in the Jordan sense such that for every $i = 1, 2, \ldots, m$, it holds $b_i = a_i$ or $b_i = a_i + (1,0)$.*

Proof. The proof is an easy induction. □

Lemma 5.5.9. *The graded Lie codimensions for the universal type II grading on $UT_n^{(-)}$ satisfy $c_m^M(UT_n^{(-)}) \leq \sum_{i=0}^{\min\{n-1,m\}} 2^{2i}\binom{n}{i+1}(i\,!)^2(n-1)^{m-i}$*

Proof. The proof is analogous to the Jordan case, by using Lemma 5.5.2 instead of Lemma 5.5.1 (in order to obtain the rightmost $(n-1)^{m-i}$ instead of n^{m-i}). By using the previous lemma we obtain the extra factor 2^i. □

Adapting Theorem 5.2.4, we obtain the graded exponent for any type II grading on $UT_n^{(-)}$. It coincides with the ordinary one and with that of each elementary grading.

Theorem 5.5.10. *Let* $\operatorname{char} K = 0$ *and let* G *be any abelian group and consider any type II* G*-grading on* $UT_n^{(-)}$*. Then* $\exp^G(UT_n^{(-)}) = n - 1$.

6. Lower bound for the Jordan case

Here we apply techniques based on generic matrices in order to obtain a lower bound for the codimensions of the ordinary identities for the Jordan algebra UJ_n. Recal that the ordinary identities of this Jordan algebra are not known, apart from the cases $n = 1$ and 2. We do not need a tight lower bound, so our computations can be improved significantly. But as we are interested in the asymptotic such an improvement is irrelevant for the final result.

Let $m > n$ and consider the set of monomials

$$x_{\sigma(1)} \circ x_{\sigma(2)} \circ \cdots \circ x_{\sigma(m)} \tag{6}$$

such that $1 = \sigma^{-1}(n) < \sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(n-1)$. The variables preceding x_1 are ordered; the ones between x_i and x_{i+1}, for $i = 1, 2, \ldots, n-2$, are ordered; the variables succeeding x_{n-1} are also ordered.

We make the following evaluation by generic matrices

$$x_1 = e_{12}, x_2 = e_{23}, \ldots, x_{n-1} = e_{n-1,n}$$
$$x_i = \textstyle\sum_{j=1}^n \xi_j^{(i)} e_{jj}, i = n, n+1, \ldots, m.$$

This evaluation shows that the monomials (6) are linearly independent modulo $T(\mathrm{UJ}_n)$. Such an evaluation gives $p(\xi_j^{(i)})e_{1n}$ where $p(\xi_j^{(i)})$ is a polynomial in the variables $\xi_j^{(i)}$, and the variables that appear in p uniquely determine the positions of $x_{n+1}, x_{n+2}, \ldots, x_m$ relative to $x_1, x_2, \ldots, x_{n-1}$.

Lemma 5.6.1. *The ordinary codimensions of the Jordan algebra* UJ_n *satisfy, for* $m > n$

$$c_m(\mathrm{UJ}_n) \geq \sum_{\substack{n_1+n_2+\cdots+n_n=m-n \\ n_1,n_2,\ldots,n_n\geq 0}} \binom{m-n}{n_1, n_2, \ldots, n_n}.$$

Proof. This counts how many monomials of the form (6) there are. □

One proves by induction

$$\sum_{\substack{n_1+n_2+\cdots+n_n=m-n \\ n_1,n_2,\ldots,n_n\geq 0}} \binom{m-n}{n_1, n_2, \ldots, n_n} \geq n^{m-n}.$$

Hence we obtain the lower bound $c_m(\mathrm{UJ}_n) \geq n^{m-n}$.

Using the upper bound given in the previous section, the classification of the gradings on UJ_n and the analog of Theorem 5.2.4, we finally obtain

Theorem 5.6.2. *Let* UJ_n *be endowed with any* G*-grading. Then the graded exponent satisfies*

$$\exp^G(\mathrm{UJ}_n) = n.$$

Remark. The above lower bound can be obtained also directly. According to [39], the equality $\exp(\mathrm{UJ}_n) = n$ holds. Then one obtains $\liminf \sqrt[m]{c_m^G(\mathrm{UJ}_n)} \geq n$ for any G-grading on UJ_n, and our theorem follows.

Chapter 6
Graded algebras as universal algebras

"Usefulness comes before cleanness"
Y. Bahturin

1. Introduction: The Problem and some cases

In this chapter we consider the following problem. Given two algebras A and B over the same field, suppose that A and B satisfy the same polynomial identities, is it true that A is isomorphic to B?

Naturally, this question stated in all its generality admits easy counter-examples.

- (Non-simple algebras:) Every algebra A satisfies the same polynomial identities as $A \oplus A$. However, in general, A does not need to be isomorphic to $A \oplus A$; if $\dim A < \infty$ this is clear. Another example that does not depend on the dimension of A is A and $A \otimes B$ where B is an arbitrary associative and commutative algebra.
- (Simple but not central simple algebras:) The non-isomorphic commutative $\mathbb{R}$-algebras $\mathbb{C}$ and $\mathbb{R}$ satisfy the same polynomial identities. Note that both these algebras are simple but $\mathbb{C}$ is not central simple.
- (infinite-dimensional algebras:) All the algebras K, $K[x]$ (polynomial ring) and $K(x)$ (field of fractions of $K[x]$) satisfy the same identities, but they are not isomorphic.
- (Central simple algebras over a non-algebraically closed field:) If A is an algebra over an infinite field k and K is a field extension of k, then every polynomial identity of A holds in $A \otimes_k K$ (see, for example, [38, Lemma 1.4.2, p. 10]). Thus every identity of the real quaternion algebra $\mathbb{H}$ is satisfied by $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq M_2(\mathbb{C})$. Since $M_2(\mathbb{C}) \cong M_2(\mathbb{R}) \otimes \mathbb{C}$, it follows that this identity is satisfied by $M_2(\mathbb{R})$. This argument can be reversed and so $\mathbb{H}$ and $M_2(\mathbb{R})$ satisfy the same polynomial identities. Now $\mathbb{H}$ is a division algebra and $M_2(\mathbb{H})$ is not. So they are not isomorphic.

As a result, it is natural to impose certain restrictions and ask the following.

Question. Let A and B be two finite-dimensional simple algebras over the same algebraically closed field. If they satisfy the same polynomial identities, is it true that $A \simeq B$?

We present below some known cases of this question.

The celebrated Theorem of Amitsur–Levitzki states that the matrix algebra M_n satisfies an identity of degree $2n$, but it does not satisfy any identity of degree less than $2n$ (see, for example, [38]). From the Wedderburn–Artin theorem, one concludes that any finite-dimensional central-simple associative algebra over an algebraically closed field K is a matrix algebra $M_n(K)$. So we deduce the following.

Proposition 6.1.1. *Let A and B be finite-dimensional simple associative algebras over an algebraically closed field. Then $A \simeq B$ if and only if they satisfy the same polynomial identities.*

So this closes the ordinary associative case.

The same question was solved for other important classes of algebras. Kushkulei and Razmyslov solved the question for simple Lie algebras [51]. In the context of simple Jordan algebras, Drensky and Racine [34] gave an affirmative answer. Shestakov and Zaicev also investigated the question for arbitrary simple algebras [60]. Despite the already proved case, Razmyslov in his book [58] proves one much stronger result in the context of Universal algebras. It is clear that Razmyslov's Theorem includes all ungraded simple cases.

The same question can be stated in the context of graded algebras: if two algebras satisfy the same graded polynomial identities, then is it true that they are graded isomorphic?

In [50], Koshlukov and Zaicev studied this problem in the context of abelian gradings on the associative simple algebras. In [3], Aljadeff and Haile have proved this result for non-abelian groups. In both cases, the authors have explored the structure of the G-gradings on the matrix algebras, using graded polynomials to recover information about the graded algebras. Finally, Bianchi and Diniz closed the question for finite-dimensional graded-simple algebras, where the grading group is abelian [22]. They proved the following.

Proposition 6.1.2. *Let A and B be finite-dimensional G-graded (not necessarily associative) algebras over an algebraically closed field, where G is any abelian group. Assume A and B graded-simple. Then $A \simeq_G B$ if and only if they satisfy the same G-graded polynomial identities.*

In this chapter, we prove using a simple and unnoticed argument that Razmsylov's theory also implies the graded cases cited above.

We mention that there are some studies of the same problem in the non-simple case as well. The authors of the paper [63] prove that every group grading on the upper triangular matrices UT_n is isomorphic to the so-called elementary grading. Moreover, in [29], it was proved that two elementary gradings on UT_n are isomorphic if and only if they satisfy the same G-graded polynomial identities. Combining both results, we obtain the following.

Proposition 6.1.3. *Let A and B be two G-gradings on the associative algebra UT_n where G is any group. Then $A \simeq_G B$ if and only if A and B satisfy the same G-graded polynomial identities.*

Also, the same questions have been discussed for some cases of gradings on the upper-block triangular matrices (see [30]). Our Theorem 3.4.7 solves the question for group gradings on UJ_n.

2. Preliminaries: Universal algebras

In this chapter we deal with the so called Ω-algebras, where Ω is a set, called *signature*. One has $\Omega = \bigcup_{m=0}^{\infty} \Omega_m$. An algebra A is called an Ω-algebra, if A is a vector space such that every $\omega \in \Omega_m$ defines an m-ary operation on A, that is, a linear map $\omega : \underbrace{A \otimes \cdots \otimes A}_{m \text{ times}} \to A$. In a natural way, one can define the standard notions of subalgebras, homomorphisms of algebras, ideals, and so on.

Given a non-empty set X, one defines the free Ω-algebra $F = F_\Omega(X)$ as follows. First we build the set $W = W_\Omega(X)$ of Ω-monomials in X as the union of subsets W_n, $n = 0, 1, 2, \ldots$ given by $W_0 = \Omega_0 \cup X$ and for $n > 0$,

$$W_n = \bigcup_{m=1}^{\infty} \bigcup_{\omega \in \Omega_m} \bigcup_{i_1+\cdots+i_m+1 \leq n} \omega(W_{i_1}, \ldots, W_{i_m}).$$

From this definition, it follows that for any $\omega \in \Omega_m$ and any $a_1, \ldots, a_m \in W$ the expression $\omega(a_1, \ldots, a_m)$ is a well-defined element of W. The elements of W_n are called monomials of degree n.

Then we consider the linear span $F = F_\Omega(X)$ of $W = W_\Omega(X)$. If $F_n = \mathrm{Span}\{W_n\}$ then $F = \bigoplus_{n=0}^{\infty} F_n$. The elements of F_n are called homogeneous polynomials of degree n. In a usual way, one defines the degree of an arbitrary nonzero polynomial. By linearity, every $\omega \in \Omega_m$ defines an m-linear operation on F. Also, it follows that for any Ω-algebra A, each map $\varphi\colon X \to A$ uniquely extends to a homomorphism $\bar{\varphi}\colon F_\Omega(X) \to A$. The Ω-algebra $F = F_\Omega(X)$ is called the free Ω-algebra with the basis (set of free generators) X.

The equation of the form $f(x_1, \ldots, x_n) = 0$ where $f(x_1, \ldots, x_n) \in F_\Omega(X)$ is called a (polynomial) identity in an Ω-algebra A if under any map $\varphi\colon X \to A$ one has $\bar{\varphi}(f(x_1, \ldots, x_n)) = 0$. In other way, $f(a_1, \ldots, a_n) = 0$, for any $a_1, \ldots, a_n \in A$. If $A = F_\Omega(X)$ then I is a T-ideal if with every $f(x_1, x_2, \ldots, x_n) \in I$ also $f(a_1, a_2, \ldots, a_n) \in I$, for any $a_1, \ldots, a_n \in F_\Omega(X)$. Given a T-ideal I, the algebra $F_\Omega(X)/I$ is called a *relatively free* algebra. The set of all (left hand sides) of the identical relations in an algebra A, depending on the variables in X is a T-ideal $T(A)$ of $F_\Omega(X)$. We denote the relatively free algebra $F_\Omega(X)/T(A)$ by $F_\Omega^A(X)$. If A and B satisfy the same identities then $F_\Omega^A(X) = F_\Omega^B(X)$. Finally, we denote by $\operatorname{var}_\Omega A$ the variety of Ω-algebras generated by A.

From now on, for each set Ω we assume that $\cup_{m=2}^{\infty}\Omega_m \neq \emptyset$. In his book [58] Yuri Pitirimovich Razmyslov proved the following remarkable result.

Theorem 6.2.1. [58, Corollary 1 of Theorem 5.3] *Two simple finite-dimensional Ω-algebras over an algebraically closed field, satisfying the same polynomial identities, are isomorphic. In other words, in the variety $\operatorname{var}_\Omega A$ generated by a simple algebra A there are no other finite-dimensional simple Ω-algebras.*

It is clear that this Theorem generalizes previous results for simple ungraded algebras. Moreover, Corollary 2 of Theorem 5.3 of [58] states the same result for prime algebras.

In this chapter we show that any two algebras graded by the same group can be regarded as ungraded Ω-algebras of the same signature Ω, and their graded identities are identities of Ω-algebras. This allows us to settle the main problem in the case of finite-dimensional graded-simple (and graded-prime) algebras.

3. Graded algebras as Universal algebras

Let K be an arbitrary field, G any semigroup and A an algebra over K (not necessarily associative, nor finite-dimensional). Consider a G-grading on A. Then we have the natural projections

$$\pi_g\left(\sum_{g\in G} a_g\right) = a_g, \quad g\in G. \tag{7}$$

Consider the set $\Omega_G = \{\mu, \pi_g \mid g \in G\}$. We define an algebra with signature Ω_G, where the underlying vector space is A itself, μ can induce the original product on A, and each π_g induces the unary operation given by (7). Then to any G-graded algebra, we can associate an algebra with signature Ω_G. The following two propositions are immediate:

Proposition 6.3.1. *Let $I \subset A$. Then I is a G-graded ideal of A if and only if I is an ideal of A as an Ω_G-algebra.*

In particular, A is simple as a G-graded algebra if and only if A is simple as an Ω_G-algebra.

Proof. Let $I \subset A$ be such that aI, $Ia \subset I$ for every $a \in A$. Let $b \in I$ and write $b = \sum_{g\in G} b_g$. Then I is a G-graded ideal if and only if $b_g \in I$, for every $g \in G$. Also I is an Ω_G-ideal if and only if $b_g = \pi_g(b) \in I$, for each $g \in G$. Hence I is a G-graded ideal if and only if I is an ideal in the sense of Ω_G-algebras.

The second statement follows immediate from the first. □

Proposition 6.3.2. *Let A and B be two G-graded algebras and let $f\colon A \to B$ be any map. Then f is a homomorphism of G-graded algebras if and only if f is a homomorphism of Ω_G-algebras.*

In particular, A is isomorphic to B as a G-graded algebra if and only if A is isomorphic to B as Ω_G-algebras.

Proof. Let $f\colon A \to B$ be a linear map such that $f(a_1a_2) = f(a_1)f(a_2)$, for all $a_1, a_2 \in A$. Then f is G-graded if and only if for every $a \in A$, homogeneous of degree g, $f(a)$ is homogeneous of degree g (or zero). Also, f is an Ω_G-homomorphism if and only if $\pi_g(f(a)) = f(\pi_g(a))$, for all $a \in A$ and all $g \in G$.

Assume first f is a G-graded map and let $a \in A$, $g \in G$. Write $a = \sum_{g\in G} a_g$. Then $\pi_g(a) = a_g$ and, since f is G-graded, the decomposition of $f(a)$ into homogeneous elements is $f(a) = \sum_{g\in G} f(a_g)$. Hence $\pi_g(f(a)) = f(a_g) = f(\pi_g(a))$. As a conclusion, f is an Ω_G-homomorphism.

Conversely, assume f is a homomorphism of Ω_G-algebras. Let $a \in G$ be homogeneous of degree g. Then $a = \pi_g(a)$, hence $f(a) = f(\pi_g(a)) = \pi_g(f(a))$, that is $f(a)$ is homogeneous of degree g. This concludes that f is G-graded.

The second assertion follows immediately from the first. □

Let F_{Ω_G} be the free Ω_G-algebra with free generators $x_1, x_2, \ldots$ Since F_{Ω_G} admits a unique binary operation, we can omit μ and denote the binary operation of F_{Ω_G} by juxtaposition together with the parentheses. It is easy to prove the following two identities, for any G-graded algebra for the given operations above:

(i) If $g, h \in G$ then

$$\pi_{gh}(\pi_g(x_1)\pi_h(x_2)) = \pi_g(x_1)\pi_h(x_2).$$

(ii) Let $g, h \in G$, then

$$\pi_g(\pi_h(x)) = \begin{cases} \pi_g(x), \text{ if } g = h, \\ 0, \text{ if } g \neq h. \end{cases}$$

From now on, fix a finite G-grading on A, that is, suppose that $\operatorname{Supp} A$ is finite (for instance, this is trivially achieved if A is finite-dimensional or if G is finite). In this case, if $S = \operatorname{Supp} A$, then we have the identities

$$x = \sum_{g \in S} \pi_g(x), \quad \pi_h(x) = 0, h \notin S. \tag{8}$$

Hence we can assume that the variables of F_{Ω_G} always appear with at least one unary operation applied to them.

Let M be a set of monomials of F_{Ω_G} defined inductively by the following rule. A monomial $m \in F_{\Omega_G}$ belongs to M if and only if one of the following holds:

(1) $m = \pi_g(x_i)$, for some $g \in G$ and some $i \in \mathbb{N}$, or

(2) $m = m_1 m_2$, where $m_1, m_2 \in M$.

Let T be the T-ideal generated by the identities (i) and (ii).

Lemma 6.3.3. *M spans F_{Ω_G} as vector space, modulo T.*

Proof. It is sufficient to prove that every monomial $m \in F_{\Omega_G}$ is a linear combination of elements of M modulo T. We work modulo T and we will prove by induction that:

(a) $m \in M$,

(b) there exists $g \in G$ such that $\pi_g(m) = m$.

Assume first

$$m = u_1 \cdots u_n(x_i), \quad n \geq 1$$

where $u_1, \ldots, u_n$ are unary operations. In view of identity (ii), it is necessary that modulo the above identities either $m = 0$ or $n = 1$, hence we obtain $m = \pi_g(x_i) \in M$, proving (a) and (b).

Now assume that m contains more than 1 variable. Then, using the induction hypothesis, we can suppose

$$m = u_1 \cdots u_n(m_1 m_2),$$

where $m_i \in M$. Moreover, there exists $g_i \in G$ such that $m_i = \pi_{g_i}(m)$. Let $g = g_1 g_2$, then identity (i) says that $\pi_{g_1}(m_1)\pi_{g_2}(m_2) = \pi_g(\pi_{g_1}(m_1)\pi_{g_2}(m_2))$. Hence we can write $m = u_1 \cdots u_n \pi_g(m_1 m_2)$. Applying identity (ii) again, we see that either $m = 0$ or $m = \pi_g(m_1 m_2) = m_1 m_2$. This proves (a) and (b), concluding the Lemma. □

Denote by $K\langle X^G \rangle$ the free non-associative G-graded algebra where the free generators are $x_i^{(g)}$, $g \in G$, $i \in \mathbb{N}$. Define the map $\psi\colon K\langle X^G \rangle \to F_{\Omega_G}/T$ where

$$\begin{aligned} \psi(x_i^{(g)}) &= \pi_g(x_i), \\ \psi(m_1 m_2) &= \psi(m_1)\psi(m_2), \end{aligned}$$

and extend ψ by linearity to $K\langle X^G \rangle$. So ψ preserves the product and it is surjective, by the previous lemma.

Lemma 6.3.4. *Let $f \in K\langle X^G \rangle$. Then f is a graded identity of A if and only if $\psi(f)$ is a polynomial identity of A as an Ω_G-algebra.*

Proof. Assume f is not a G-graded identity. Then there exists an evaluation

$$e\colon x_i^{(g)} \mapsto a_i^{(g)} \in A$$

such that $e(f) \neq 0$. Define the evaluation $e'\colon x_i \mapsto \sum_{g \in G} a_i^{(g)}$ (since the grading is finite, the sum is well defined). Let $m \in K\langle X^G \rangle$ be a monomial. An easy induction on the length of m proves that $e'(\psi(m)) = e(m)$, hence we obtain $e'(\psi(f)) = e(f) \neq 0$, proving that $\psi(f)$ is not a polynomial identity of A, as an Ω_G-algebra.

Conversely, if $\psi(f)$ is not a polynomial identity of A, as an Ω_G-algebra, then there exists an evaluation $e_2'\colon x_i \mapsto \sum b_i^{(g)}$ such that $e_2'(\psi(f)) \neq 0$. So we can define the evaluation $e_2 = e_2' \circ \psi$. Hence $e_2(f) = e_2'(\psi(f)) \neq 0$ concluding that f is not a G-graded polynomial identity of A. □

An immediate consequence of Lemma 6.3.4 is the following

Corollary 6.3.5. *Let A and B be two algebras endowed with finite G-gradings. Then A and B satisfy the same graded polynomial identities if and only if A and B satisfy the same polynomial identities as Ω_G-algebras.*

Now using Proposition 6.3.1 and Corollary 6.3.5, applying Razmyslov's Theorem (Corollary 6.2.1), and concluding with Proposition 6.3.2, we obtain our main result.

Theorem 6.3.6. *Let A and B be finite-dimensional G-graded algebras which are graded simple over an algebraically closed field K where G is any semigroup. Then A and B are isomorphic as G-graded algebras if and only if they satisfy the same G-graded polynomial identities.*

3.1. **Further generalizations.** Let A be an Ω-algebra and let G be any semigroup. Consider a vector space G-grading on A, that is, we fix a vector space decomposition $A = \bigoplus_{g \in G} A_g$. For every $m \in \mathbb{N}$, one obtains naturally a G-grading on

$$\otimes^m A := \underbrace{A \otimes \cdots \otimes A}_{m \text{ times}}$$

defining the homogeneous component of degree g by

$$\left(\otimes^m A\right)_g = \sum_{g_1 g_2 \cdots g_m = g} A_{g_1} \otimes \cdots \otimes A_{g_m},$$

see [35, chapter 1, p. 11]. We say that A is a G-graded Ω-algebra if every m-ary operation $\omega : \otimes^m A \to A$ is G-homogeneous, that is, $\omega\left(\left(\otimes^m A\right)_g\right) \subset A_g$. This notion generalizes the common notion of semigroup grading on an algebra.

We can turn the G-graded Ω-algebra A into an algebra with signature $\Omega_G = \Omega \cup \{\pi_g \mid g \in G\}$, as we did before. This gives a full faithful functor from the category of G-graded Ω-algebras to the category of Ω_G-algebras.

Similarly to the previous case, we obtain

Lemma 6.3.7. *Let A and B be two G-graded Ω-algebras. Then*

(i) *A is simple as Ω_G-algebra if and only if A is simple as G-graded Ω-algebra.*

(ii) *A is isomorphic to B as Ω_G-algebras if and only if A is isomorphic to B as G-graded Ω-algebras.*

Denote by F_{Ω_G} the free Ω_G-algebra with countable number of free generators. The following are polynomial identities of a given G-graded Ω-algebra:

(i) $\pi_{g_1}\pi_{g_2}x_1 = 0$, for $g_1 \neq g_2$,

(ii) $\pi_g\pi_g(x_1) = \pi_g x_1$,

(iii) $\omega(\pi_{g_1}x_1, \ldots, \pi_{g_n}x_n) = \pi_g\omega(\pi_{g_1}x_1, \ldots, \pi_{g_n}x_n)$ where $g = g_1 \cdots g_n$, $\omega \in \Omega$.

Let $Y = \{\pi_g(x_i) \mid i \in \mathbb{N}, g \in G\}$. Consider the set of monomials $M = \operatorname{mon}_\Omega Y$, which consists of all monomials generated by Y with operations of Ω. Assume the G-grading finite, e.g., A is finite-dimensional. Let $\mathfrak{G}$ be the variety of Ω_G-algebras generated by identities (i)–(iii) above. Denote by $F_{\Omega_G}^{\mathfrak{G}}$ the relatively free Ω_G-algebra in $\mathfrak{G}$. As in the previous section, we can assume that every variable has at least one unary operation of kind π_g applied on it.

Lemma 6.3.8. *M spans $F_{\Omega_G}^{\mathfrak{G}}$.*

Proof. Let $w \in F_{\Omega_G}$ be a monomial. We work modulo T. We prove by induction on the degree of w that

(a) w is a linear combination of monomials in M,

(b) if $w \in M$ then there exists $g \in G$ such that $\pi_g(w) = w$.

If $\deg w = 0$, then (a) and (b) follow from our assumption on the grading being finite. So assume $\deg w > 0$. Then we can write $w = \omega(w_1, \ldots, w_m)$ where $m \geq 1$, and $\deg w_i < \deg w$ for all i; or $w = \pi_g(u)$ for some $g \in G$, and $\deg u = \deg w - 1$. For the first case, by the induction step, every w_i is a linear combination of monomials in M. Hence w is a linear combination of monomials in M. In addition, suppose $w_i \in M$ and $\pi_{g_i}w_i = w_i$, for some $g_i \in G$, for each i. Then identity (iii) implies that w satisfies (b).

For the last case $w = \pi_g(u)$, we can apply the induction step on u. Thus u is a linear combination of monomials in M, say $u = \sum u_i$. The induction hypothesis says that $\pi_{h_i}u_i = u_i$ for some $h_i \in G$. By identities (i)–(ii), one has $\pi_g\pi_{h_i}u_i = \delta_{gh_i}u_i$. Hence w satisfies (a). If it happens that $w \in M$ then $u \in M$, and the last computation implies that w satisfies (b) as well. $\square$

Let $X^G = \{x^{(g)} \mid x \in X, g \in G\}$. For each $\omega_0 \in \Omega_0$, associate an arbitrary homogeneous degree to it, namely $\deg_G \omega_0 \in G$. Also, define $\deg_G x^{(g)} = g$. Then F_Ω^{gr}, the free Ω-algebra with the basis X^G, induces a G-grading as follows. We already defined the homogeneous degree for the elements of degree 0. Now given $\omega(w_1, \ldots, w_n)$ where $\omega \in \Omega$, $n \geq 1$, we set $\deg_G \omega(w_1, \ldots, w_n) = \deg_G w_1 \cdots \deg_G w_n$. This is a well defined G-grading on F_Ω^{gr}.

There is no doubt that we should not exclude 0-ary operations in the free Ω-algebra since the polynomial identities with 1 or without 1 play an essential role in the theory. However, it is interesting to mention one example. In the context of associative algebras A with a unit (where 1 is a 0-ary operation), it is not possible to find a graded homomorphism $F_\Omega^{\mathrm{gr}} \to A$, unless we impose $\deg_G 1 = 1 \in G$ in Ω_0.

We can then consider the G-graded evaluations and speak about the G-graded polynomial identities of G-graded Ω-algebras. The same argument as in the previous section can be used to conclude that

Lemma 6.3.9. *The map $\psi_G \colon F_\Omega^{\mathrm{gr}} \to F_{\Omega_G}^{\mathfrak{G}}$, given by $\psi_G(x_i^{(g)}) = \pi_g(x_i)$ is a bijective homomorphism of Ω-algebras. Moreover*

$$\psi_G(\mathrm{Id}_\Omega^{\mathrm{gr}}(A)) = \mathrm{Id}_{\Omega_G}(A).$$

Hence A and B satisfy the same G-graded polynomial identities as Ω-algebras if and only if they satisfy the same polynomial identities as Ω_G-algebras. As a consequence, we can apply Razmyslov's Theorem in the setting of G-graded Ω-algebras.

Theorem 6.3.10. *Let A and B be two finite-dimensional G-graded Ω-algebras over an algebraically closed field which are simple as G-graded Ω-algebra, where G is any semigroup. Then A is isomorphic to B, as a G-graded Ω-algebra, if and only if A and B satisfy the same G-graded polynomial identities as G-graded Ω-algebras.*

4. Further examples

A significant number of recent research has been devoted to polynomial identities in algebras with additional structure. It turns out that many of these algebras can be viewed and dealt as Ω-algebras, for an appropriate signature Ω, as we are going to present below.

4.1. **Algebras with involution.** Let $(A, *)$ be an algebra with involution. Consider the signature $\Omega^* = \{\mu, \nu\}$, where μ is of arity 2 and ν of arity 1. Then A is an Ω^*-algebra with

$\mu(a,b)=ab$, and $\nu(a)=a^*$. Actually, A belongs to the variety $\mathfrak{U}$ of Ω^*-algebras given by the laws

$$\nu(\mu(x,y))=\mu(\nu(y),\nu(x)) \text{ and } \nu(\nu(x))=x. \tag{9}$$

Conversely, any algebra in $\mathfrak{U}$ is an algebra with involution. An ideal I of A as an Ω^*-algebra is a vector subspace closed under $\mu(\cdot,a)$, $\mu(a,\cdot)$, for every $a\in A$ and $\nu(\cdot)$. If we make A to be an algebra with involution then if A is simple as Ω^*-algebras it becomes involution simple, that is, does not have proper nonzero ideals closed under the involution. Since the condition $A\cdot A\neq\{0\}$ holds for Ω^*-algebras simultaneously with the same condition for algebras with involution, it follows that simple Ω^* algebras are exactly the same as simple algebras with involution.

Identities of Ω^*-algebras are the elements of the relatively free Ω^*-algebra $F_{\mathfrak{U}}^{\Omega^*}(X)$ of $\mathfrak{U}$ or, as we described them, Ω^*-polynomials in the countable set of free generators X. This algebra is also an algebra with involution, as explained above. Identities of algebras with involution are usually understood as the elements of the free algebra $F^*(X\cup Y)$, where X is in bijective correspondence θ with Y. An involution is defined on this algebra by induction if one sets $x^*=\theta(x)$, $y^*=\theta^{-1}(y)$ for the monomials of degree 1 and if $w=uv$ where $\deg u,\deg v<\deg w$ then one sets $w^*=v^*u^*$. One replaces each $y\in Y$ by x^* where $\theta(y)=x$ and then the elements of $F^*(X\cup Y)$ are the polynomials in the variables $X\cup X^*$. As any algebra with involution, $F^*(X\cup X^*)$ is an algebra in the variety $\mathfrak{U}$, so the identity map $\mathrm{id}_X\colon X\to X$ extends to the onto homomorphism $F_{\Omega^*}^{\mathfrak{U}}(X)\to F(X\cup X^*)$. The converse is also true. The isomorphism obtained transforms the identities with involution to the identities of Ω^*-algebras, and vice versa. Applying Razmyslov's Theorem, we get the following.

Theorem 6.4.1. *Two finite-dimensional algebras with involution over an algebraically closed field, which are involution-simple and satisfy the same polynomial identities with involution are isomorphic as algebras with involution.*

4.2. **Superalgebras with involution and superinvolution.** A superalgebra is a $\mathbb{Z}_2$-graded algebra with some additional properties. To be precise, let $\mathcal{V}$ be a variety of algebras (not necessarily associative), and let $A=A_0\oplus A_1$ be a $\mathbb{Z}_2$-graded algebra. Then A is a superalgebra in the variety $\mathcal{V}$ whenever its Grassmann envelope $E(A)=A_0\otimes E_0\oplus A_1\otimes E_1$ lies in $\mathcal{V}$ as an ordinary algebra. Recall that one does not require $A\in\mathcal{V}$. In fact if $\mathcal{V}$ is the variety of all associative algebras then an A is a superalgebra in $\mathcal{V}$ if and only if it is associative and $\mathbb{Z}_2$-graded. But if A is a Lie or a Jordan superalgebra then it need not be a Lie or Jordan algebra, and in all interesting instances it is not.

Given $(A = A_0 \oplus A_1, *)$ a superalgebra with involution (or superinvolution), we can consider the signature $\Omega^* = \{\mu, \nu\}$ where μ is binary, while ν is unary. Now A becomes an Ω^*-algebra if one sets $\mu(a,b) = ab$, the product in A and $\nu(a) = a^*$. Moreover, A is a $\mathbb{Z}_2$-graded Ω^*-algebra. We consider the variety $\mathfrak{V}$ of Ω^*-algebras given by the law (9), if $*$ is an involution; otherwise, if $*$ is a superinvolution, we let $\mathfrak{V}$ be defined by the following $\mathbb{Z}_2$-graded polynomial identities:

$$\begin{aligned} \nu(\nu(x_1^{(0)} + x_2^{(1)})) &= x_1^{(0)} + x_2^{(1)}, \\ \nu(\mu(x_1^{(a)}, x_2^{(b)})) &= (-1)^{ab} \mu(\nu(x_2^{(b)}), \nu(x_1^{(a)})). \end{aligned}$$

Every superalgebra with involution (or superinvolution) can be viewed as a $\mathbb{Z}_2$-graded Ω^*-algebra, moreover, an algebra in the variety $\mathfrak{V}$; and conversely. Using the same arguments as in the previous example, one concludes that the graded polynomial identities of superalgebras with involution (superinvolution) of A are the same when viewed as the polynomial identities as Ω^*-algebras. Hence, Theorem 6.3.10 implies the following.

Theorem 6.4.2. *Two finite-dimensional superalgebras with involution (or superinvolution) over an algebraically closed field, which are simple as superalgebras with involution (or superinvolution), satisfying the same graded polynomial identities with involution (or superinvolution) are isomorphic as graded algebras with involution (or superinvolution).*

4.3. **Colour Lie superalgebra.** Let $L = \bigoplus_{g \in G} L_g$ be a G-graded algebra with product $[\cdot,\cdot]$ such that there exists an alternating bicharacter $\varepsilon \colon G \times G \to K^*$ satisfying, for $x_i \in L_{g_i}$, $i = 1, 2, 3$:

$$\begin{aligned} &[x_1, x_2] = -\varepsilon(g_1, g_2)[x_2, x_1], \\ &[x_1, [x_2, x_3]] = [[x_1, x_2], x_3] + \varepsilon(g_1, g_2)[x_2, [x_1, x_3]]. \end{aligned} \tag{10}$$

Then (L, ε) is called a colour Lie superalgebra (see, for instance, [14]). For a fixed ε, the variety of colour Lie superalgebras is the variety defined by the identities (10). We can view colour Lie superalgebras as non-associative graded algebras. Hence, we can apply the theory developed in Section 3.

Another related example, given that $\operatorname{char} K = p > 0$, is a colour Lie p-superalgebra. A colour Lie p-superalgebra is a colour Lie superalgebra L with an additional partial map $x \mapsto x^{[p]}$, defined on some homogeneous components, satisfying the following:

$$(\alpha x)^{[p]} = \alpha^p x^{[p]},$$

$$(\operatorname{ad} x^{[p]})(z) = [x^{[p]}, z] = (\operatorname{ad} x)^p(z),$$

$$(x+y)^{[p]} = x^{[p]} + y^{[p]} + \textstyle\sum_i s_i(x,y),$$

where s_i is some polynomial. Note that $x \mapsto x^{[p]}$ is not always linear. Hence we cannot always see the "raising to p-th power" as an unary operation. So it is not obvious how we can describe a colour Lie p-superalgebra as an Ω-algebra. However, in the context of simple algebras, the second identity completely defines $\operatorname{ad} x^{[p]}$. Since ad is a linear isomorphism in the case of simple finite-dimensional algebras we conclude that the p-th power map is completely defined by the product of the algebra. In this way an isomorphism of colour Lie p-superalgebras preserving the product will preserve the p-th power as well. Hence Theorem 6.3.6 (or Theorem 6.3.10) implies the following.

Theorem 6.4.3. *Two finite-dimensional colour Lie (p-)superalgebras over an algebraically closed field, which are simple as colour Lie superalgebras, satisfying the same graded polynomial identities are isomorphic as colour Lie (p-)superalgebras.*

4.4. **Trace identities.** Another important example to consider is *trace identities.* We consider the signature $\Omega^* = \{\nu, \mu\}$, where μ is a binary operation and ν is an unary operation. A matrix algebra M_n becomes an Ω^*-algebra with $\mu(a,b) = ab$, the usual matrix multiplication, and $\nu(A) = a \cdot I$, where a is the usual trace of A and I is the $n \times n$ identity matrix.

Let $\mathfrak{T}$ be the variety of algebras satisfying the following polynomial identities (see [57]):

- $\mu(\nu(x), y) = \mu(y, \nu(x))$,
- $\nu(\mu(x,y)) = \nu(\mu(y,x))$,
- $\nu(\mu(x, \nu(y))) = \mu(\nu(x), \nu(y))$.

Then $M_n \in \mathfrak{T}$. Any Ω^*-algebra in $\mathfrak{T}$ is called an algebra with trace. We say that an algebra A with trace is trace simple if $A \cdot A \neq 0$ and A does not contain a non-trivial ideal invariant by

ν. A homomorphism between algebras with trace must preserve the map ν. So Razmyslov's Theorem 6.2.1 (or Theorem 6.3.10) translates as:

Theorem 6.4.4. *Two finite-dimensional algebras over an algebraically closed field, which are simple as algebras with trace, and satisfy the same trace polynomial identities are isomorphic as algebras with trace.*

Traces of generic matrices are related to the invariants of matrix algebras. It is also known that trace polynomial identities of matrix algebras are consequences of the Cayley-Hamilton identity [57, 55]. It is worth mentioning that *ordinary* polynomial identities for the matrix algebra M_n are known only when $n \leq 2$, for infinite fields, and in the case $n = 2$, in characteristic different from 2. If one considers matrices over finite fields then the identities are known for $n \leq 4$.

Let $A_1 = M_{n_1} \oplus \cdots \oplus M_{n_r}$ and $A_2 = M_{n'_1} \oplus \cdots \oplus M_{n'_s}$, and assume $n_1 \geq n_2 \geq \ldots \geq n_r$, $n'_1 \geq \ldots \geq n'_s$, $r > 1$ and $s > 1$. It is clear that $A_1 \cong A_2$ if and only if $r = s$ and $n_i = n'_i$ for all i. Moreover, assume $n_1 = n'_1$. In this case, A_1 and A_2 satisfy the same polynomial identities, namely, the polynomial identities of the matrix algebra M_{n_1}. Both algebras are not simple as ordinary algebras, but they are tr-simple, if we define trace as the induced trace from M_n, where $n = n_1 + \ldots + n_r$. Our results say that we can find a trace identity satisfied by one algebra, but not by the other.

4.5. **Algebras with the action of Hopf algebras.** Let A be an algebra (with a unique binary operation) and H a Hopf algebra. We say that A is a left H-algebra, or a left H-module algebra, if A is an unital left H-module and for any $a, b \in A$ and $g, h \in H$ the following hold (see [13]):

- $(gh) * a = g * (h * a)$,
- $h * (ab) = \sum (h_{(1)} * a)(h_{(2)} * b)$.

Algebras with Hopf actions include important examples. We cite two of them.

Action by automorphisms: Let G be a subgroup of the group of automorphisms of the algebra A. It is well-known that the group algebra KG is a Hopf algebra. Then the action of G on A by automorphisms is a particular case of Hopf action by the group algebra KG.

Action by derivations: Let D be a Lie subalgebra of the algebra of derivations of A. Then the universal enveloping algebra $U(D)$ is a Hopf algebra. Thus the action of D on A by derivations can be viewed as a Hopf action of $U(D)$.

Now we present the classical construction of the free Hopf module algebras (see [13]). Fix a Hopf algebra H. Let $T = T(H) = \sum_{n\geq 1} \otimes^n H$ be the tensor algebra of the vector space H, not containing the field. Each $\otimes^n H$ is a H-module, by means of

$$h * (h_1 \otimes \cdots \otimes h_n) = \sum (h_{(1)}h_1) \otimes \cdots \otimes (h_{(n)}h_n).$$

Hence T is an H-module as well. Let X be a set of variables and let $T(X)$ be the vector space generated by all tw, where $t \in T$ and w is a non-associative word. Then $T(X)$ is a H-module if we define $h * tw = (h * t)w$, for $h \in H$. Now let $\mathscr{H}(X)$ be the vector subspace of $T(X)$ generated by all tw, with $|t| = |w|$. By [13, Proposition 1], $\mathscr{H}(X)$ has the following Universal property. If A is any H-algebra and $\varphi\colon X \to A$ is any map, then there exists unique homomorphism of H-algebras $\bar{\varphi}\colon \mathscr{H}(X) \to A$ extending φ. Hence one naturally defines polynomial identities of H-algebras, using elements of $\mathscr{H}(X)$.

Now let $\Omega_H = \{\mu\} \cup \{\rho_h \mid h \in H\}$. If A is an H-algebra, then the operations in Ω_H are defined on A in the following way: μ defines the original product and for each $h \in H$, $\rho_h(a) := h * a$. Consider the relatively free Ω_H-algebra defined by the following polynomial identities:

i. $\rho_h(x) + \rho_g(x) = \rho_{h+g}(x)$,

ii. $\rho_{\alpha h}(x) = \alpha\rho_h(x)$, for $\alpha \in K$,

iii. $\rho_1(x) = x$ (where $1 \in H$ is the unit),

iv. $\rho_h(\rho_g(x)) = \rho_{hg}(x)$,

v. $\rho_h(xy) = \sum(\rho_{h_{(1)}}x)(\rho_{h_{(2)}}y)$.

An argument similar to the one given in Lemma 6.3.4, can be used to translate H-polynomial identities into polynomial identities of the relatively free Ω_H-algebra satisfying identities i–v above. It is not hard to see that two H-algebras are isomorphic if and only if they are isomorphic as Ω_H-algebras, under the operations defined above. Hence, Razmyslov's Theorem can be applied.

Theorem 6.4.5. *Two finite-dimensional H-algebras over an algebraically closed field, which are simple as H-algebras and satisfy the same H-polynomial identities, are isomorphic as H-algebras.*

4.6. **Algebras with generalized action.** Let $\mathcal{H} = (H, \Delta^{(1)}, \Delta^{(2)})$ be a triple where H is a unital associative algebra and $\Delta^{(1)}$, $\Delta^{(2)}$ are two linear maps, called coproducts $\Delta^{(1)}$, $\Delta^{(2)}\colon H \to H \otimes H$. Using Sweedler's notation, we can write $\Delta^{(i)}(h) = h^{(i)}_{(1)} \otimes h^{(i)}_{(2)}$, meaning that $\Delta^{(i)}(h)$ are arbitrary tensors of degree 2. In contrast with Hopf algebras, we impose no restrictions on the coproducts.

An algebra A is called an $\mathcal{H}$-algebra if A is a left H-module via $(h, a) \to h * a$, for any $h \in H$ and $a \in A$ and for any a, $b \in A$, one has

$$h * (ab) = (h^{(1)}_{(1)} * a)(h^{(1)}_{(2)} * b) + (h^{(2)}_{(1)} * b)(h^{(2)}_{(2)} * a)$$

Such algebras, with a minor modification, have appeared in [19, 41]. In a natural way one can define the notions of the homomorphisms of $\mathcal{H}$-algebras, simple $\mathcal{H}$-algebras and so on. The construction of a free $\mathcal{H}$-algebra also does not create any problems (see algebras with H-action above) and so one can speak about $\mathcal{H}$-identities. As earlier, one can define the set $\Omega_{\mathcal{H}}$ consisting of one binary operation μ and the set of unary operations ρ_h, for each $h \in H$. As earlier, if A is a $\mathcal{H}$-algebra then $\rho_h(a) = h * a$. The variety of $\Omega_{\mathcal{H}}$-algebras is distinguished by the family of identical relations, one for each $h \in H$:

$$\rho_h(\mu(x, y)) = \mu(\rho_{h^{(1)}_{(1)}}(x), \rho_{h^{(1)}_{(2)}}(y)) + \mu(\rho_{h^{(2)}_{(1)}}(y), \rho_{h^{(2)}_{(2)}}(x))$$

The $\mathcal{H}$-identities can be rewritten as $\Omega_{\mathcal{H}}$-identities, following the approach of Lemma 6.3.4. Skipping obvious details, we obtain one more consequence of Razmyslov's Theorem.

Theorem 6.4.6. *Two finite-dimensional simple $\mathcal{H}$-algebras over an algebraically closed field, satisfying the same $\mathcal{H}$-polynomial identities, are isomorphic as $\mathcal{H}$-algebras.*

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