Universidade Estadual de Campinas
Instituto de Computação

André Carvalho Silva

Graphs with few crossings and the crossing number of the $K_{p,q}$ in topological surfaces

Grafos com poucos cruzamentos e o número de cruzamentos do $K_{p,q}$ em superfícies topológicas

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2018

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Tese apresentada ao Instituto de Computação da Universidade Estadual de Campinas como parte dos requisitos para a obtenção do título de Doutor em Ciência da Computação.

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Resumo

O número de cruzamentos de um grafo G em uma superfície Σ é o menor número de cruzamentos de arestas dentre todos os possíveis desenhos de G em Σ. Esta tese aborda dois problemas distintos envolvendo número de cruzamentos de grafos: caracterização de grafos com número de cruzamentos igual a um e determinação do número de cruzamentos do $K_{p,q}$ em superfícies topológicas.

Para grafos com número de cruzamentos um, apresentamos uma completa caracterização estrutural. Também desenvolvemos um algoritmo "prático" para reconhecer estes grafos.

Em relação ao número de cruzamentos do $K_{p,q}$ em superfícies, mostramos que para um inteiro positivo p e uma superfície Σ fixos, existe um conjunto finito $\mathcal{D}(p, \Sigma)$ de desenhos "bons" de grafos bipartidos completos $K_{p,r}$ (possivelmente variando o r) tal que, para todo inteiro q e todo desenho D de $K_{p,q}$, existe um desenho bom D' de $K_{p,q}$ obtido através de duplicação de vértices de um desenho D'' em $\mathcal{D}(p, \Sigma)$ tal que o número de cruzamentos de D' é menor ou igual ao número de cruzamentos de D. Em particular, para todo q suficientemente grande, existe algum desenho do $K_{p,q}$ com o menor número de cruzamentos possível que é obtido a partir de algum desenho de $\mathcal{D}(p, \Sigma)$ através da duplicação de vértices do mesmo. Esse resultado é uma extensão de outro obtido por Cristian *et. al.* para esfera.

Abstract

The crossing number of a graph G in a surface Σ is the least amount of edge crossings among all possible drawings of G in Σ. This thesis deals with two problems on crossing number of graphs: characterization of graphs with crossing number one and determining the crossing number of $K_{p,q}$ in topological surfaces.

For graphs with crossing number one, we present a complete structural characterization. We also show a "practical" algorithm for recognition of such graphs.

For the crossing number of $K_{p,q}$ in surfaces, we show that for a fixed positive integer p and a fixed surface Σ, there is a finite set $\mathcal{D}(p, \Sigma)$ of "good" drawings of complete bipartite graphs $K_{p,r}$ (with distinct values of r) such that, for every positive integer q and every good drawing D of $K_{p,q}$, there is a good drawing D' of $K_{p,q}$ obtained from a drawing D'' of $\mathcal{D}(p, \Sigma)$ by duplicating vertices of D'' and such that the crossing number of D' is at most the crossing number of D. In particular, for any large enough q, there exists some drawing of $K_{p,q}$ with fewest crossings which can be obtained from a drawing of $\mathcal{D}(p, \Sigma)$ by duplicating vertices. This extends a result of Christian *et. al.* for the sphere.

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Chapter 1

Introduction

The origin of crossing number problems in graphs is usually attributed to the mathematician Paul Turán [Tur77]. During World War II, Turán worked in a brick factory. The bricks were produced in kilns which were connected to storage yards via rails. The bricks were transported on small wheeled trucks. When the trucks went over crossings of two rails they generally jumped and some bricks fell out of them. This resulted in loss of time and trouble for the workers. This motivated Turán to solve the problem of minimizing the number of crossings on the rails. However, he found it to be very difficult [Tur77]. The problem can be mathematically described as the problem of minimizing the number of crossings in a drawing of the bipartite complete graph $K_{p,q}$ in the plane. Generally speaking, the *crossing number* of a graph G in a surface Σ is the least amount of edge crossings over all possible drawings of G in Σ.

Crossing number of graphs has several practical applications on VLSI (Very Large Scale Integration) and graph drawing problems. Leighton [Lei83] showed that the problem of minimizing the area of a circuit in a circuit board is intrinsically related to the crossing number of the graph that represents the circuit. Purchase [Pur97] concluded that minimizing the number of crossings in drawings of graphs results in drawings which are easier to understand.

Crossing number is also useful in theoretical research. Székely [Szé97] provided several short proofs on what he calls "hard Erdős' problems" in discrete geometry using results from crossing number theory.

We may think of crossing number as a general measure of "non-planarity" of a graph similarly to *genus*, *demigenus*, *thickness* and *skewness* (also known as *removal number*). The (demigenus) genus of a graph is the minimal (demigenus) genus of an (non-orientable) orientable surface in which a graph is embeddable. Graph thickness is the least number of planar graphs that a given graph can be decomposed into. Skewness is the least number of edges that ought to be removed from a graph to make it planar.

This thesis is focused on two selected topics in crossing number: graphs with crossing number one and the crossing number of $K_{p,q}$ on topological surfaces. These problems are discussed in Chapters 3 and 4, respectively. We refer the reader to Schaefer's dynamic survey [Sch13] for a more complete overview on several topics in crossing number.

Our main contributions on this thesis are as follows. We provide a complete structural characterization of graphs with crossing number one and; for a fixed positive integer p

and surface Σ, we prove that there is a finite set $\mathcal{D}(p, \Sigma)$ of "good" drawings of complete bipartite graphs $K_{p,r}$ (with possibly distinct r) such that, for every positive integer q and every good drawing D of $K_{p,q}$, there is a good drawing D' of $K_{p,q}$ obtained from a drawing D'' of $\mathcal{D}(p, \Sigma)$ by duplicating vertices of D'' and such that D' has at most as many crossings as D. In particular, for large enough q, there exists some drawing of $K_{p,q}$ with smallest number of crossings which can be obtained from a drawing of $\mathcal{D}(p, \Sigma)$ by duplicating vertices. This extends a result of Christian, Richter and Salazar [CRS13] for the sphere.

This thesis is organized as follows. The next chapter will provide the necessary notation and knowledge that we use throughout the thesis. Chapter 3 concerns graphs with one crossing. Chapter 4 deals with the crossing number of $K_{p,q}$ in surfaces. Chapter 5 provides some concluding remarks about the topics in this thesis.

Chapter 2

Preliminaries

This section introduces basic notation, definitions and some results that we use throughout the text.

We assume the reader is familiar with a variety of topics. Among them: complexity theory (NP-hardness, big-Oh notation), basic graph theory and set theory.

2.1 Graph theory

This section is mostly composed of conventions and notations while avoiding some basic definitions and folklore results (e.g. graphs, paths, isomorphism, Menger's Theorem, etc.). We refer the reader to a readily available book on graph theory like [Die17] for the missing definitions and concepts.

The graphs in this thesis are always finite and may contain multiple edges and loops, unless otherwise stated. For a graph G, we denote by $V(G)$ and $E(G)$ its vertex and edge sets, respectively.

For a subset S of $V(G)$, let $G - S$ denote the subgraph of G induced by $V(G) - S$. If $S = \{v\}$, then we simply write $G - v$ instead of $G - \{v\}$. Let F denote a set of edges between vertices of G (possibly F is not contained in $E(G)$). Let $G + F$ and $G - F$ to denote the graphs $(V(G), E(G) \cup F)$ and $(V(G), E(G) \setminus F)$, respectively. If $E = \{e\}$, then we simply write $G + e$ or $G - e$ instead.

Let P be a path in G. The *ends* of P are its degree 1 vertices and the *internal vertices* are its degree 2 vertices.

For two vertices u and v of a graph G, an *uv-path* is a path of G whose ends are u and v.

If H is a subgraph or a subset of vertices in G, then we say that P is *H-avoiding* or *avoids* H if no internal vertex of P is in H.

Let P be a path in G and let v and w be vertices of P. We denote by vPw the unique vw-subpath of P. Let Q be a path of G.

A *subdivision* of a graph G is a graph obtained from G by replacing an edge uv by an uv-path whose internal vertices do not belong to G. This operation is called a *subdivision* of an edge e in G. For a vertex v of degree two in G, we say we *suppress* it whenever we remove v and add a new edge joining its neighbors.

A graph that is isomorphic to a subdivision of a $K_{3,3}$ or K_5 is called a *Kuratowski graph*. Given graphs H, K and G, we say that H is a *subdivision* of K in G if H is a subgraph of G isomorphic to a subdivision of K. A *Kuratowski subgraph* of G is a subgraph of G that is a Kuratowski graph.

Let G be a connected graph. A *node* of G is any vertex with degree different from 2. A *branch* of G is any path between nodes of G that does not contain any node as an internal vertex. Two branches are *adjacent* if they have a common end. We will use these terms when we deal with subdivisions in graphs.

2.2 Topology

The aim of this section is to provide a short introduction to some topological concepts used in this text. Most of the definitions and concepts on this section are classical and can be found in any General/Algebraic Topology textbook (e.g. [Mun00]).

2.2.1 General topology

In standard real analysis books, a subset U of $\mathbb{R}^n$ is called *open* if for any $x \in U$, there exists $\varepsilon > 0$ such that the open ball with radius ε centered at x is contained in U.

In topology, we aim for a more abstract concept of open sets which does not rely on any kind of metric, but brings some notion of "neighborhoodness" of points.

Let X be a set and τ a collection of subsets of X. The pair (X, τ) is a *topological space* if:

1. $\emptyset$ and X are in τ,

2. any arbitrary union of sets in τ is also in τ, and

3. any intersection of finitely many sets in τ is also in τ.

In this context, we say that τ is a *topology* in X and its elements are called *open sets*. If the topology is clear from the context, we simply say that X is a topological space. For a given $x \in X$ any open set containing x is denoted a *neighborhood* of x in X. For a topological space, we usually refer its elements as *points*.

As an example, let $\mathcal{I}$ be the collection of all open intervals in $\mathbb{R}$; and let τ be a collection such that $U \in \tau$ if and only if U is the union of a collection of elements of $\mathcal{I}$. We note that $\mathbb{R} = \bigcup_{I \in \mathcal{I}} I$ and that $\emptyset$ is the union of an empty collection of open intervals. The union of a collection of open intervals is also an open interval; and for two sets $U, V \in \tau$, the reader may verify using DeMorgan laws that $U \cap V$ is also the union of open intervals. Thus τ is a topology on $\mathbb{R}$. This topology is considered the *usual topology* of $\mathbb{R}$.

A *basis* $\mathcal{B}$ for a set X is a collection of subsets of X such that:

(a) For any $x \in X$ there exists a $N \in \mathcal{B}$ with $x \in N$;

(b) if x belongs to the intersection of a pair B_1 and B_2 in $\mathcal{B}$, there exists a $B_3 \in \mathcal{B}$ such that $B_3 \subseteq (B_1 \cap B_2)$.

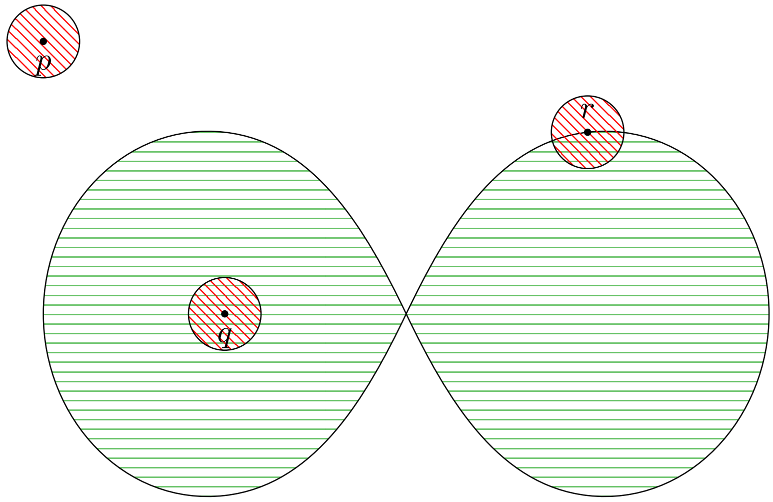


Figure 2.1: The points p,q and r are in the exterior, interior, and boundary, respectively, in the subset of $\mathbb{R}^2$ represented by the horizontal lines.

Let $\mathcal{B}$ be a basis of X. The *topology* $\tau_{\mathcal{B}}$ *generated by* $\mathcal{B}$ is defined as follows: a set U is open in $\tau_{\mathcal{B}}$ if for every $x \in U$ there exists a basis element B such that $x \in B \subseteq U$. An equivalent definition is that the open sets of $\tau_{\mathcal{B}}$ are union of all combinations of basis elements.

Conversely, given a topological space X with topology τ, a collection $\mathcal{B}$ of open sets of X generates τ if for every $x \in X$ and every neighborhood N of x, there exists a $M \in \mathcal{B}$ such that $x \in M \subseteq N$.

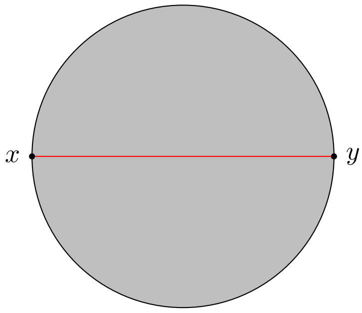
The aforementioned set $\mathcal{I}$ is an example of a basis of $\mathbb{R}$. Conversely, $\mathcal{I}$ also generates the usual topology of $\mathbb{R}$. In contrast, an example of a collection of subsets of $\mathbb{R}$ which satisfies (a) but is not a basis is the collection $\mathcal{S}$ of all semi-infinite intervals of the form $(-\infty, x)$ of (x, ∞), where $x \in \mathbb{R}$. Note that $1/2 \in (0,1) = (-\infty, 1) \cap (0, \infty)$ but no subset of $(0,1)$ can be written as a union of elements of $\mathcal{S}$.

Let X be a topological space with topology τ. Let Y be a topological space with topology τ_Y. The *product topology* on $X \times Y$ is the topology generated from the basis $\tau \times \tau_Y$. Henceforth we assume that any product of spaces mentioned has the product topology.

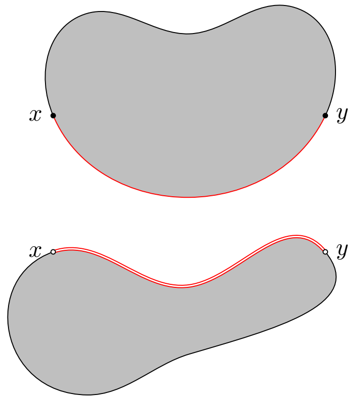
For a given subset A of X, the *interior* of A is composed of all the points of X with at least one neighborhood completely in A. The *exterior* of A is similarly defined but the neighborhoods are completely in $X \setminus A$. The *closure* of A is the subset of points x of X such that every neighborhood of x contains at least one point in A. A point x in is the *boundary* ∂A of A if every neighborhood of x contains a point not in A and another in A. Figure 2.1 illustrates these concepts.

For a subset A of X the *subspace topology* τ_A of A is the collection of sets $\{S \cap A : S \in \tau\}$. For example, a basis for the subspace topology of $[0,1] \subseteq \mathbb{R}$ is composed of all sets (a,b), $[0,b)$ and $(a,1]$, with $0 \leq a < b \leq 1$.

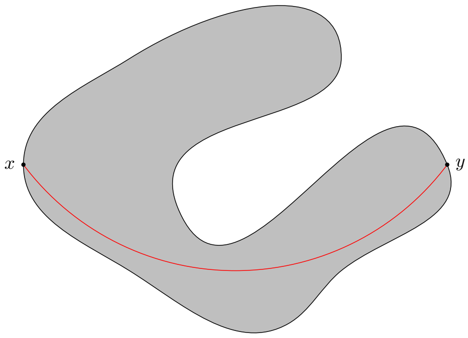
Let X^* be a partition of X. Let $p : X \to X^*$ be the surjective map that maps every element of X to its part in X^*. The *quotient topology* τ_p of the *quotient space* X^* is the set $\{U \subseteq X^* : p^{-1}(U) \text{ is open }\}$. So, one can imagine that the quotient space is obtained



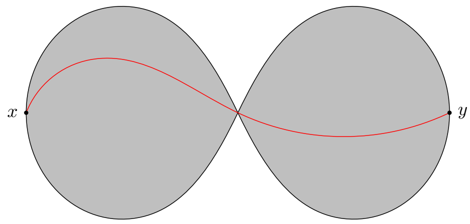
(a) A closed disk Δ in $\mathbb{R}^2$



(b) The image of a non-continuous function on Δ.



(c) The image of a homeomorphism on Δ.



(d) The image of a continuous function that is not a homeomorphism on Δ.

Figure 2.2: Visual representation of a non-continuous, continuous functions, and a homeomorphism of a closed disk in $\mathbb{R}^2$.

by "gluing" or "pasting" the points in the parts of X together. We shall make use of this intuition in some parts of the text. Figure 2.4a provides an example of quotient space.

A function f between two topological spaces X and Y is *continuous* if for any open set U of Y, $f^{-1}(U)$ is also open in X. A *homeomorphism* f between X and Y is a bijection such that f and f^{-1} are both continuous. Refer to Figure 2.2 for some visual examples of continuous functions and homeomorphisms. The next lemma asserts that the composition of continuous functions is also continuous.

Lemma 2.1. *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Then $g \circ f$ is also continuous.*

2.2.2 Surfaces

In this subsection we present a brief overview of concepts and results related to topological surfaces (defined below). In order to give a precise definition of a surface, we need to introduce a few properties that a surface must satisfy as a topological space. We include them only for the sake of completeness. When we deal with surfaces in other chapters we use the more friendly equivalent definition of surfaces given by the Classification of Surfaces Theorem (Theorem 2.5). We present Theorem 2.5 without proof. We refer the

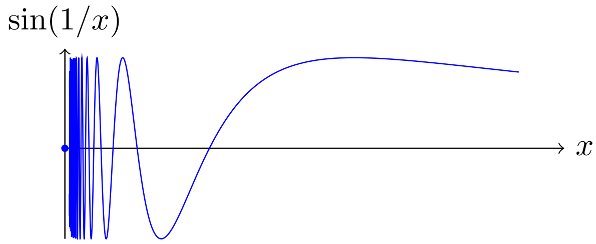


Figure 2.3: The topologist sine curve.

reader to the following books for a proof of Theorem 2.5 along with a more complete introduction to surfaces: [Kin12; MT01; Mun00].

A topological space X is *Hausdorff* if for any two distinct elements x and y of X, there exists disjoint neighborhoods N and M of x and y, respectively.

Let X be a topological space. A collection $\mathcal{A}$ of subsets of X is said to *cover* X if their union is X. It is an *open cover* if $\mathcal{A}$ is a collection of open sets and *finite* if $\mathcal{A}$ is a finite set. We say that X is *compact* if every open cover of X contains a finite cover of X. An example of a space that is not compact is an open disk with the subspace topology in $\mathbb{R}^n$ with the product topology.

An *arc* (or *curve*) α is a continuous mapping from $[0,1]$ to X. If α is injective then, the arc is is *simple*. We sometimes refer to the image of α as the arc itself. We say that α *joins* or *connects* $\alpha(0)$ to $\alpha(1)$ and we say that $\alpha(0)$ and $\alpha(1)$ are the *ends* of α. A *closed curve* is an arc γ such that $\gamma(0) = \gamma(1)$. We say that γ is *simple* if the restriction of γ to $[0,1)$ is injective.

We say that X is *connected* if no two disjoint nonempty open sets A and B of X are such that $A \cup B = X$. We say that X is *arcwise connected* if for every pair of distinct elements x and y in X, there exists an arc that connects them.

Define an equivalence relation $\sim$ on X such that $x \sim y$ if there exists a connected subspace of X containing x and y. The equivalence classes are the *connected components* of X. Similarly, if we define $x \sim y$ to mean that there exists an arc connecting both in X, the equivalence classes are the *arc-components* of X.

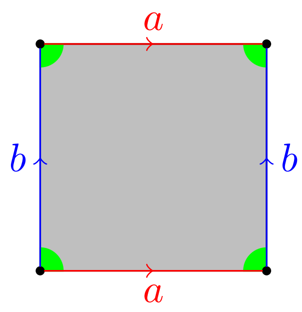
We note that every arcwise connected space is also connected. However, the converse is not true. The topologist sine curve $T = \{(x, \sin(1/z)) : x \in (0,1]\} \cup (0,0)$ (pictured in Figure 2.3) is a classic example of a connected space that is not arcwise-connected.

The following lemmas assert that connectedness and compactness are preserved in continuous maps.

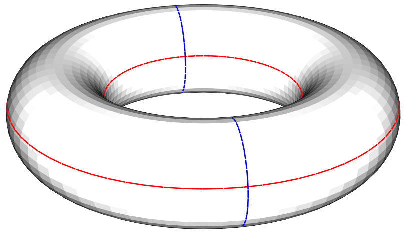
Lemma 2.2. *The image of a continuous map of a connected space is also connected.*

Lemma 2.3. *The image of a continuous map of a compact space is also compact.*

We say that a topological space X is *locally n-euclidean* if for every $x \in X$, there exists a neighborhood of x homeomorphic to $\mathbb{R}^n$. Equivalently, for every point x in an open set U of X, there exists an open set $V \subseteq U$ which is homeomorphic to $\mathbb{R}^n$. A neighborhood homeomorphic to $\mathbb{R}^n$ is called an *euclidean ball.* For locally n-euclidean spaces, connected spaces are also arcwise-connected.

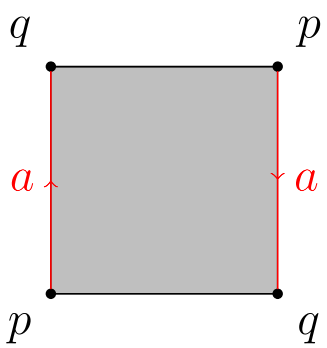


(a) A torus.

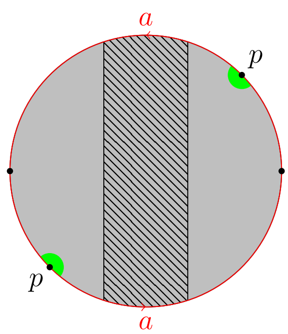


(b) A torus in $\mathbb{R}^3$.

Figure 2.4: Distinct representations of a torus.



(a) A Möbius strip.



(b) A Möbius strip in a projective plane.

Figure 2.5: Fundamental polygons of a Möbius strip and projective plane.

Lemma 2.4. *If X is a locally n-euclidean connected topological space, then it is also arcwise-connected.*

A *n-manifold* is a locally n-euclidean, Hausdorff, nonempty topological space X. A *surface* is a connected compact 2-manifold. A *surface with boundary* is a connected, compact, Hausdorff, nonempty topological space such that every point has a neighborhood homeomorphic to the half-plane or the plane. The *boundary* $\partial(\Sigma)$ of a surface with boundary Σ are all the points with some neighborhood homeomorphic to the half-plane.

Examples of surfaces include the *sphere* $\mathbb{S}_0$, torus and projective plane. Surfaces with boundaries include the closed disk and Möbius strip. Some of these are defined below.

The *torus*[1] $\mathbb{S}_1$ is the quotient space of the unit square $X = [0,1] \times [0,1]$ obtained by the partition: $X^* = \{\{(x,y)\} : x, y \in (0,1)\} \cup \{\{(x,0),(x,1)\} : x \in (0,1)\} \cup \{\{(0,y),(1,y)\} : y \in (0,1)\} \cup \{\{(0,0),(0,1),(1,0),(1,1)\}\}$.

Figures 2.4a and 2.4b depict a torus. The edges with the same labels in Figure 2.4a represent the parts of the quotient space. Note that the corner vertices represent the same point. Together, the arcs around a corner point form a connected neighborhood of that point.

The *Möbius strip* $\mathfrak{M}$ is the quotient space obtained from the unit square by identifying two opposing edges in opposite directions (See Figure 2.5a). We note that the Möbius

[1] Usually defined as the product $S^1 \times S^1$ of two copies of the unit circle S^1. Both these spaces are homeomorphic.

strip has a single continuous closed boundary represented by the edges joining p and q in Figure 2.5a.

The *projective plane* $\mathbb{N}_1$ is the quotient space obtained by identifying the antipodal points of the boundary of a closed disk in the plane. Figure 2.5b depicts the projective plane. The arcs around the point p represents a single neighborhood. Note that the dashed region is a Möbius strip. Indeed, one can obtain a projective plane by gluing a Möbius strip in the boundary of a sphere with an open disk removed.

Given two surfaces Σ_1 and Σ_2, the *connected sum* $\Sigma_1 \# \Sigma_2$ is the surface obtained by removing an open disk from each and identifying its boundaries. The reader may verify that this operation is associative and commutative (up to homeomorphism).

We shall define an *n-torus* $\mathbb{S}_n$ recursively as the connected sum of an $(n-1)$-torus with a torus, where the 0-torus $\mathbb{S}_0$ is the sphere. Similarly, we use $\mathbb{N}_k$ to denote the n-projective plane. Some surfaces have special names. The surface $\mathbb{S}_2$ is usually known as the *double torus* and $\mathbb{N}_2$ is the *Klein bottle*. We note that $\mathbb{S}_1 \# \mathbb{N}_1$ is homeomorphic to $\mathbb{N}_3$.

The following theorem is a folklore result. It classifies all surfaces up to homeomorphism.

Theorem 2.5. *(Classification of Surfaces) Any surface is homeomorphic to $\mathbb{S}_h$, for some $h \geq 0$, or to $\mathbb{N}_k$, for some $k \geq 1$.*

A similar folklore result exists for surfaces with boundary:

Theorem 2.6. *Any surface with boundary Σ is homeomorphic to either a h-torus, for some $h \geq 0$, or to a k-projective plane, for some $k \geq 1$, with a finite number of open disks removed.*

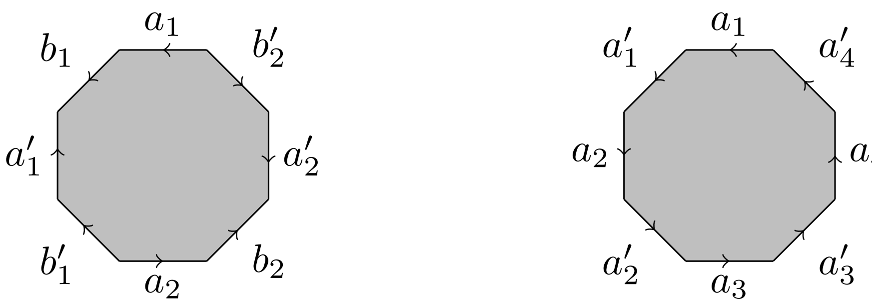
The removed open disks on a surface are oftentimes called *holes*.

A *handle* is a sphere with two holes which is also called a *cylinder*. A *crosscap* is a Möbius strip. By *attaching* a handle to a surface (with or without boundary) Σ we mean that we glue the boundary of the handle to the boundaries of two open disks removed from Σ. Similarly, we attach a crosscap by removing an open disk from Σ and gluing its boundary with the boundary of a Möbius strip. An equivalent statement of Theorem 2.5 is that any surface Σ is equivalent to a sphere with either h handles or k crosscaps attached to it.

Let Σ be a surface. We say that Σ is *orientable*, if it is homeomorphic to an h-torus, for some $h \geq 0$, with a finite number of holes, and *non-orientable*, otherwise.

Every surface without boundary may be represented by a quotient space obtained by identifying the edges of a polygon[2] in the plane. These are called *fundamental polygons*. Let $h \geq 1$ be an integer. Let $a_1, b_1, a'_1, b'_1, ..., a_{2h}, b_{2h}, a'_{2h}, b'_{2h}$ be the edges of a regular $4h$-sided polygon on the plane in counterclockwise order. Identifying a_i with a'_i and b_i with b'_i, for $i = 1, 2, ..., 2h$, such that their paired edges have distinct orientations will result in a space homeomorphic to $\mathbb{S}_h$. Similarly, let $k \geq 0$ be an integer. Let $a_1, a'_1, a_2, a'_2, ..., a_k, a'_k$ be the edges of a $2k$ regular polygon. Identifying the edges a_i with a'_i, for $i = 1, \ldots, k$

[2]Here we consider a polygon as all the points in the bounded region plus its boundary.



(a) (b)

Figure 2.6: Fundamental polygons for $\mathbb{S}_2$ and $\mathbb{N}_4$.

with the same orientation will result in a space homeomorphic to $\mathbb{N}_k$ [3]. Figures 2.6b and 2.6a are examples for the fundamental polygons of $\mathbb{N}_4$ and $\mathbb{S}_2$, respectively.

If Σ is homeomorphic to $\mathbb{S}_h$, for some $h \geq 0$, with a finite number of holes, then its *genus* $g(\Sigma)$ is h. Similarly, if Σ is homeomorphic to $\mathbb{N}_k$ with a finite number of holes, then its *demigenus* $\tilde{g}(\Sigma)$ is k.

Suppose Σ has n holes. We define the *Euler characteristic* $\chi(\Sigma)$ of Σ as:

$$\chi(\Sigma) = \begin{cases} (2 - 2g(\Sigma)) - n & \text{, if } \Sigma \text{ is orientable,} \\ (2 - \tilde{g}(\Sigma)) - n & \text{, otherwise.} \end{cases} \tag{2.1}$$

For example, the sphere, torus and Klein bottle have Euler characteristic $\chi(\mathbb{S}_{\nvdash}) = 2$, $\chi(\mathbb{N}_2) = 0$ and $\chi(\mathbb{S}_1) = 0$, respectively. A projective plane with a hole is homeomorphic to a Möbius strip and hence it has Euler characteristic 0. Thus the Euler characteristic is not enough to distinguish surfaces from each other.

2.2.3 Separation theorems

In this subsection, we state a couple useful classical Theorems about separation in the plane. The proofs of these results are beyond the scope of this text, but the reader can find self-contained proofs in [MT01].

Let X be a connected topological space and let A be a subset of X. We say that A *separates* X if $X \setminus A$ is not connected. The next theorem is the classical Jordan Curve Theorem. It is used implicitly in many parts of the text.

Theorem 2.7. *(Jordan Curve Theorem) Let γ be a simple closed curve in $\mathbb{R}^2$. Then γ separates $\mathbb{R}^2$ into precisely two components W_1 and W_2 such that both have γ as a boundary.*

Schoenflies proved an extension of the Jordan Curve Theorem:

Theorem 2.8. *(Schoenflies Theorem) If f is a homeomorphism of a simple closed curve γ in the plane onto a closed curve γ' in the plane, then f can be extended to a homeomorphism of the entire plane.*

[3] The "polygons" for $\mathbb{S}_0$ and $\mathbb{N}_1$ are degenerate cases with only two edges.

(a) Touching edges.

(b) Crossing edges.

Figure 2.7: Distinct types of intersection of edges.

2.3 Graph drawing

We use $\{s_i\}_{i\in I}$ as a notation for an indexed set with index set I. We often hide the subscript whenever I is clear from the context. A *drawing* D of a graph G in a surface Σ is the union of:

- the image of an injective function $\phi : V(G) \to \Sigma$, and
- the images of the functions $\{\phi_e\}_{e\in E(G)}$ in which ϕ_e is an arc joining the images of the ends of e in ϕ and $\phi_e((0,1))$ is disjoint from $\phi(V)$.

We make no distinction between the images of these functions and the graph objects they represent (edges and vertices). No confusion should arise from this convention.

Let H be a subgraph of G. We use the notation $D[H]$ to denote the drawing of H obtained from D by deleting the corresponding vertices and edges not in H.

Let e and f be edges of G. We say that they *intersect* in D if $\phi_e((0,1))\cap\phi_f((0,1)) \neq \varnothing$[4]. Let x be an intersection point of e and f in D and let B be an euclidean ball in Σ containing x. We may choose B such that: B is disjoint from any edge not containing x, and B contains no other intersection of e and f. We say that e and f *touch* if there exists a separating curve γ in B such that $B \setminus \gamma$ has two components: one disjoint from e and another disjoint from f. Otherwise, we say that e and f *cross*. Figures 2.7a and 2.7b illustrate these concepts. A drawing with no intersection is called an *embedding* of G in Σ.

Let D be a drawing of a graph G in a surface Σ. The *crossing number* $\text{cr}(D)$ of D is the total number of crossings between pairs of edges of G in D. We note a subtle detail in this definition: if three edges have a common intersection in D then we have three crossings, not one. The *crossing number* $\text{cr}_\Sigma(G)$ of G in a surface Σ is the least number of crossings among all possible drawings of G. We say that D is *optimal* if $\text{cr}(D) = \text{cr}_\Sigma(G)$.

We shall adopt a few conventions. For a drawing, if no surface is mentioned, the reader can assume it is a drawing in the plane. We shall use the notation $\text{cr}(G)$ for crossing number on the plane (or sphere).

For a given graph G and a non-negative integer k, computing whether $\text{cr}(G) \le k$ is NP-complete [GJ83]. Kawarabayashi and Reed [KR07] showed that there exists a linear

[4]Note that we do not consider a common end between two edges as an intersection.

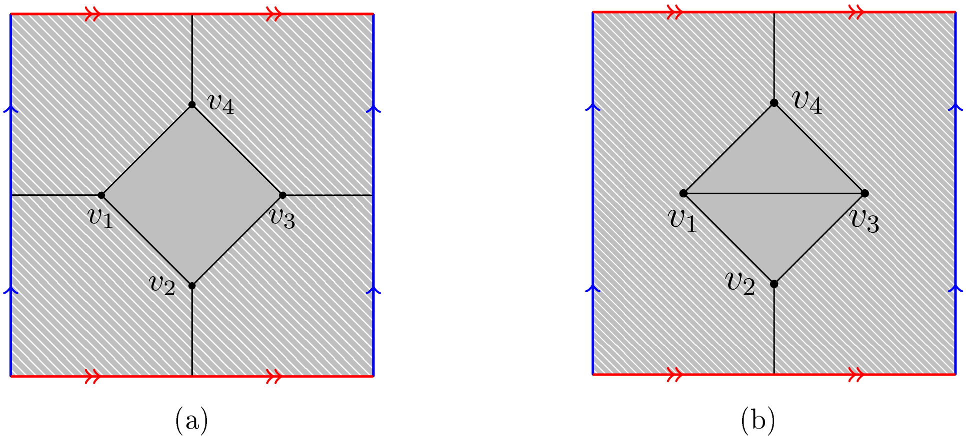


Figure 2.8: Cellular and non-cellular embeddings of K_4 on a torus. The dashed areas compose a single face. On the second picture this face is homeomorphic to a cylinder.

Fixed Parameter Tractable (FPT)[5] algorithm for this problem.

A drawing is *good* if it satisfies the following properties:

1. no pair of edges touch,

2. edges with a common incident vertex do not cross,

3. no pair of edges cross more than once, and

4. no point is the intersection of three edges.

If a drawing is not good, we may eliminate or change the intersection(s) locally in a small enough euclidean ball containing the intersection point. This does not increase the number of crossings. The following lemma is a natural consequence of this fact (we omit the proof):

Lemma 2.9. *Every graph has a good optimal drawing in any surface.*

Henceforth, unless otherwise stated, assume that any drawing mentioned is good. Let D be an embedding of a graph G in a surface Σ. A *face* of D is a connected component of $\Sigma \setminus D$. We say that a vertex v (respectively, an edge e) of G is *incident* with a face F if v (e, respectively) is contained in the boundary of F in D. We say that D is *cellular* if all its faces are homeomorphic to a disk. Figure 2.8 shows examples of cellular and non-cellular embeddings.

The *genus* of G, denoted by $g(G)$, is the smallest genus such that G is embeddable in $\mathbb{S}_{g(G)}$. The *demigenus* of G, denoted by $\tilde{g}(G)$, is similarly defined for non-orientable surfaces. We note that $g(G) = 0$ if and only if G is planar. Determining the genus of a graph is NP-Complete [Tho89]. However, there are closed formulas for the genus and demigenus of the complete graphs [RY68] and bipartite complete graphs [Rin65a; Rin65b]:

[5] A problem is FPT if there exists a parameter k and an algorithm with complexity $f(k)n^c$, where n is the size of the input and c is a constant.

Theorem 2.10. *If $n \geq 3$ then:*

$$g(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil.$$

If $n \geq 3$ and $n \neq 7$, then:

$$\tilde{g}(K_n) = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil.$$

Theorem 2.11. *If $m, n \geq 2$ then:*

$$g(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

If $m \geq 3$ and $n \geq 3$, then:

$$\tilde{g}(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{2} \right\rceil.$$

The classical Kuratowski's Theorem stated below characterizes graphs that can be embedded in the plane (or the sphere equivalently).

Theorem 2.12. *(Kuratowski's Theorem [Kur30]) A graph is planar if and only if it does not contain a subdivision of $K_{3,3}$ or K_5.*

A *Kuratowski subgraph* of G is a subdivision of $K_{3,3}$ or K_5 in G.

2.3.1 Combinatorial embeddings

In this subsection, we present a combinatorial description of cellular embeddings of graphs in surfaces. The concepts discussed here are only used in Chapter 4.

The concepts in this subsection are easier to describe for loopless graphs. Moreover, for this thesis, we shall use these concepts only for loopless graphs. Therefore, for the sake of simplicity, we will assume our graphs are loopless in this section. The reader may find a deeper discussion on combinatorial embeddings in [MT01; GT87; LZ13].

Let v be a vertex of a loopless connected graph G. A *local rotation* π_v around v is a cyclic permutation of the edges incident with v. An *(abstract) embedding scheme* of G is a pair $(\{\pi_v\}_{v\in V(G)}, \lambda)$, where π_v is a local rotation around v and λ is a signal function mapping each edge to $\{0, 1\}$. We say that an edge e of G is *type-0* or *type-1* if $\lambda(e) = 0$ or $\lambda(e) = 1$, respectively.

Let $\Pi = (\{\pi_v\}_{v\in V(G)}, \lambda)$ and $\Pi' = (\{\pi'_v\}_{v\in V(G)}, \lambda')$ be two embedding schemes for a graph G. We say that Π is *equivalent* to Π' if we may obtain Π' from Π by a sequence of *reversal* of the local rotations, that is, for a vertex v of G we invert π_v and subsequently λ for all edges incident with v in Π. Thus if we do this for a sequence of vertices $\{v_i\}$ of G and we obtain Π' at the end, then they are equivalent.

Our goal right now is to show that an embedding scheme of a loopless connected graph G uniquely determines (up to homeomorphism) a cellular embedding of G in some surface Σ. To illustrate that, we show how we can obtain an embedding scheme from a

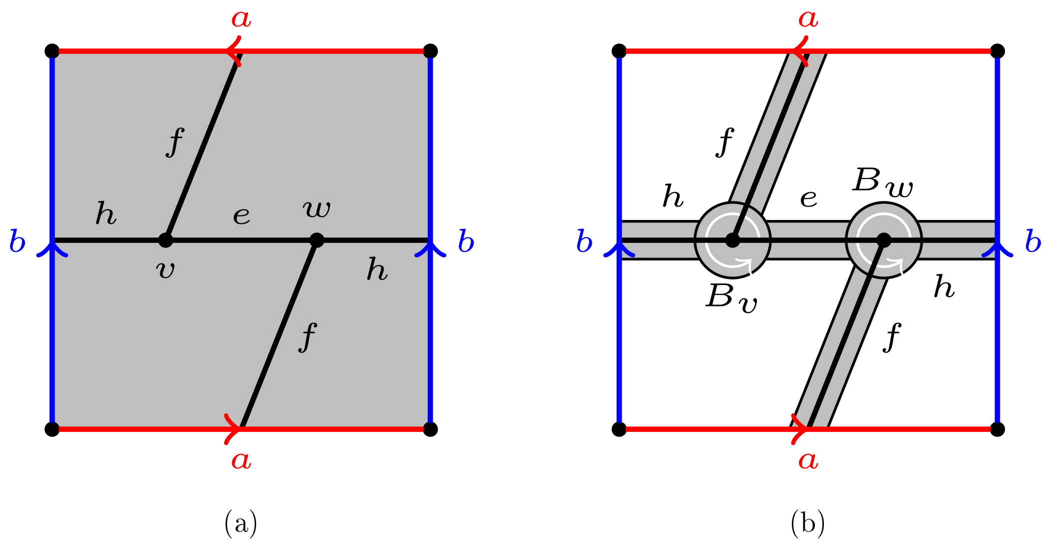


Figure 2.9: A cellular embedding of a graph in the Klein bottle and its ribbon graph.

cellular embedding of G and, subsequently, how to obtain a cellular embedding from an embedding scheme.

Let D be a cellular embedding of a loopless connected graph G in a surface Σ. Each vertex v has an euclidean ball B such that only edges of G incident with v intersect B in D, and they intersect only once. The same applies to an edge e of G and the edges of D with a common end with e. Using the appropriate euclidean balls, one can create a neighborhood of G in D whose shape preserves the graph itself. We describe this process in detail below.

A *1-band* is a homeomorphism $h : [0,1] \times [0,1] \to \Sigma$. The arcs $h([0,1] \times \{j\})$ for $j = 0, 1$ are the *ends* of the band, and the arcs $h(\{j\} \times [0,1])$ are the *sides* of the band. A *0-band* and *2-band* are homeomorphisms of the unit disk in Σ. A *band decomposition* of Σ is a collection of 0-, 1- and 2-bands satisfying:

1. Different bands intersect only along arcs in their boundaries;

2. the union of all bands is Σ;

3. each end of a 1-band is contained in a 0-band;

4. each side of a 1-band is contained in a 2-band;

5. the 0-bands are pairwise disjoint and so are the 2-bands.

The *ribbon graph* of D is the collection of 0- and 1-bands in a band decomposition of Σ that is a neighborhood of G in D such that: every vertex v is in a 0-band and; for every edge e with ends v and w, there exists a 1-band disjoint from every other edge whose ends are 0-bands containing v and w. Figure 2.9 shows an example of an embedding and its ribbon graph. Note that a ribbon graph of an embedding naturally defines a surface with holes (one for each 2-band) and thus is unique up to homeomorphism.

An *orientation* of a 0- or 1-band is an orientation of the points in its boundary (i.e. clockwise or counterclockwise). Note that if we orient a 0-band that is an end of a 1-band, we can induce an orientation of the 1-band based on the direction of its shared arc. We say that a 1-band is *type-0* if the orientations induced by its ends are the same and *type-1* otherwise. As an example, using the orientations of the 0-bands B_v and B_w in Figure 2.9b, the 1-bands containing the edges e and h are type-0 and the one containing f is type-1.

For a vertex v, let B_v be a 0-band containing v in the ribbon graph of D. Note that B_v contains part of the edges incident with v. Let $\{e_i\}$ be the edges incident with v. We may choose a 0-band of v that is small enough such that each edge e_i will intersect only once. Let $\{q_j\}$ be the set of points arising from the intersection of the edges with $\partial(B_v)$. The orientation of B_v naturally induces a cyclic permutation of the points $\{q_j\}$. If G has no loops, then we may associate each e_i to its unique intersection q_i with $\partial(B_v)$. The *cyclic permutation* of the edges around v in D is the natural permutation of $\{e_i\}$ arising from the cyclic permutation of $\{q_j\}$. Thus we define π_v as this cyclic permutation. Moreover, for an edge e we define $\lambda(e) = 0$ if the 1-band associated with e is type-0 and $\lambda(e) = 1$, otherwise. For example, in the band decomposition of Figure 2.9b using the orientations of B_v and B_w in the picture, the local rotations of v and w are $\pi_w = \pi_v = (e\ f\ h)$. Also, from these orientations, we have that $\lambda(e) = \lambda(h) = 0$ and $\lambda(f) = 1$.

The *embedding scheme* of D is the pair $(\{\pi_v\}_{V\in V(G)}, \lambda)$ where λ is a signal function for $E(G)$ such that $\lambda(e) = 1$ if e is type-1 in D or $\lambda(e) = 0$ otherwise. Note that the type of an edge, and thus the embedding scheme of D, depends on how the 0-bands are oriented. Thus we can obtain equivalent embedding schemes by changing the rotation of the 0-bands and subsequently the types of the 1-bands incident with it.

We now describe how to obtain an embedding from an (abstract) embedding scheme. Suppose G is loopless, connected and has no vertex of degree two. Let $(\{\pi_v\}_{v\in V(G)}, \lambda)$ be an abstract embedding scheme of G. A *facial walk* W of Π is a closed walk $v_1e_1v_2e_3v_3...e_kv_1$ of G obtained by the following procedure called the *face traversal procedure*. We start with an arbitrary vertex u and an edge e incident with u whose other end is v. We Traverse the edge from u to v. The next edge e', whose ends we shall name v and w, will depend on whether e is type-0 or type-1. If e is type-0, then $e' = \pi_v(e)$, otherwise, we use $e' = \pi_v^{-1}(e)$. If the latter happens, for the next edge after e' will be $\pi_w^{-1}(e')$, if e' is type-0, and $\pi_w(e')$, otherwise. This will continue until we find another type-1 edge. The procedure stops whenever we find edge e_1 again the next edge is e_2. For example, the embedding scheme obtained from the embedding depicted in Figure 2.9a has only one facial walk $W = v e w \mathbf{f} v e w \mathbf{h} v f w \mathbf{h} v$. Note how at the first time we find e again, the next edge is not f, since we traversed a type-1 edge (f) before, and thus we do not stop there.

If G has a vertex of degree two, we can obtain another graph G' by suppressing the vertices of degree two while making adjustments to the local rotation. If P is an induced path in G, then the resulting edge e of G', obtained from G by suppressing the internal vertices of P, will be type-1 if and only if there is an odd number of type-1 edges in P. We thus apply the face traversal procedure to obtain the facial walks of G' and make the appropriate changes to obtain a facial walk of G.

Now we briefly and informally describe how to obtain an embedding from a set of facial walks. Let $\mathcal{W}$ be the set of all facial walks of G obtained from an abstract embedding scheme of G. We do not distinguish between a facial walk and a cyclic shift of the same.

For each facial walk $W \in \mathcal{W}$, we create a regular polygon P_W with the same number of edges as the length of W and we label each edge of the polygon with an edge in the walk. Note how each edge will appear exactly twice in $\{P_W\}_{W \in \mathcal{W}}$. By gluing the polygons together we obtain a quotient space Γ homeomorphic to a surface. The set of points representing the edges and vertices of the polygon in Γ is an embedding of G.

This subsection discussion can be summarized by the following Theorem.

Theorem 2.13. *Every cellular embedding of a connected graph G is uniquely determined, up to homeomorphism, by its embedding scheme.*

The version of this theorem restricted to orientable surfaces is known as the *Heffter-Edmonds-Ringel rotation principle* [Hef91; Edm60; Rin74]. The general version was made explicit by Ringel [Rin77] in the 50s and the first formal proof of it was published by Stahl [Sta78].

We note that the procedure used to extract an embedding scheme from a cellular embedding can also be used for non-cellular embeddings. Hoffman and Richter [HR84] provided a combinatorial description for non-cellular embeddings. Let D be a good drawing of a loopless connected graph G. The *flattening* of D is the graph P obtained from G by inserting a vertex of degree 4 at each crossing in D. Thus, if e is crossed k times, e is subdivided into $k+1$ edges.

In the remainder of this chapter, we show that there exists only finitely many (up to isomorphism) good drawings of a connected graph G in any surface Σ (Theorem 2.16). This proof works in two steps. We first show that there exists only finitely many flattenings arising from good drawings of G in Σ. Afterwards, for a particular flattening P of G, we show that there exists only finitely many (up to homeomorphism) embeddings of P in a surface Σ.

Let $flat(G, \Sigma)$ be the set of all flattenings (up to graph isomorphism) of G arising from good drawings of G in Σ.

Lemma 2.14. *For any graph G and surface Σ, $flat(G, \Sigma)$ is finite.*

Proof. Let D and D' be good drawings of G in Σ. Let P and P' be their flattening. Let X and X' be the set of pairs of edges that cross in D and D', respectively. Thus, for every element of X we have an associated vertex of P. Similarly for X' and P'.

Let e be an edge of G whose ends are u and v. Suppose f and h are two distinct edges that cross e in G and that e is ordered from u to v in D. Let x be the intersection point of e and f in D. Likewise, let y be the one for e and h. If x precedes y in $D[e]$, then we say that $f \prec_e h$. Likewise, let $\prec'_e$ be the order of the edges crossing e obtained from D'. Note that, for a particular crossing x between edges e and f in D the neighborhood of the vertex arising from x in P depends only on the order of the crossing in both e and f.

Now, suppose that $X = X'$ and that for every edge e of G, the orders $\prec_e$ and $\prec'_e$ are the same. Thus, there exists a natural isomorphism between P and P' such that every vertex x of P, arising from a crossing, is mapped to the vertex x' in P' arising from the crossing of the same pair of edges in D'. Therefore a flattening is characterized by the pairs of edges that cross and the ordering of the crossings on the edges of G in a good drawing.

As good drawings of G have only finitely many possible crossings (at most one for each pair of edges), we conclude that there exist only finitely many (up to graph isomorphism) flattenings arising from good drawings of G. □

We note that any good drawing D of G is also an embedding of its flattening P. Thus D is in some homeomorphism class $\mathcal{E}$ of embeddings of P. Let D' be another good drawing of G, with flattening P'. If P' is isomorphic to P, then D' is also an embedding of P. Thus, if D' is also in $\mathcal{E}$, then D is isomorphic to D'. In short, the isomorphism classes of good drawings of G are homeomorphism classes of elements of $flat(G)$. For our purposes, it suffices to show that for a $P \in flat(G)$ there exists only finitely many homeomorphism classes of embeddings of P.

Lemma 2.15. *For a loopless connected graph P embeddable in a surface Σ, there exists only finitely many (up to homeomorphism) embeddings of P in Σ.*

Proof. Let R be an embedding scheme of an (possibly non-cellular) embedding Π of P in Σ. We note that there can be many distinct non-cellular embeddings of P with R as its embedding scheme.

The embedding scheme uniquely determines (up to homeomorphism) a cellular embedding Π' of P in a surface Γ with R as its embedding scheme (Theorem 2.13). One may see Γ as the surface obtained from Σ by capping off all the faces of Π with disks and thus removing the handles and crosscaps in these faces. Thus $\chi(\Sigma) \leq \chi(\Gamma)$ with equality only if Π is cellular.

Attaching an appropriate number of handles/cross caps to faces of Π' in Γ will result in an embedding of P in a surface Σ' with the same embedding scheme R. If Σ and Σ' have the same number of handle/crosscaps, then Σ' is homeomorphic to Σ. We show that there are only finitely many ways to attach handles and crosscaps to the faces of Π' in Γ to obtain Σ.

Let Q be the set of facial walks of Γ and Q^* a partition on Q. For a given part T in Q^* of size k we can attach any surface Ω with k holes into disks cut from each face of T in Γ. This operation results in a surface of Euler genus $\chi(\Gamma) + \chi(\Omega)$. Thus $\chi(\Omega)$ is bounded as a function of $\chi(\Sigma)$ which implies that there are finitely many possible surfaces we can attach. This, combined with the finiteness of Q, and thus Q^*, shows that there are only finitely many possible embeddings of P in Σ.

It is clear, by the construction above, that any embedding of P in Σ with rotation system R can be obtained from Π' by adding the appropriate number of handles and cross caps to the faces of Π' in Γ. Moreover, we showed that there are only finitely many ways to do that. With this, we conclude that there are only finitely many (up to homeomorphism) embeddings of P with rotation scheme R in Σ. We note that there are only finitely many (up to homeomorphism) possible rotation schemes for P. Thus, overall, there exists only finitely many embedding of P in Σ

□

Theorem 2.16. *For any connected graph G and any surface Σ, there are only finitely many (up to drawing isomorphism) good drawings of G in Σ .*

Chapter 3

Graphs with at most one crossing

In the context of crossing number, we can interpret Kuratowski's classic characterization of planar graphs as: a graph has crossing number at least one if and only if it contains a subdivision of K_5 or $K_{3,3}$. We answer a similar question: when does a graph have crossing number at least 2? The answer is a characterization of graphs with crossing number one (see Theorem 3.8). We also present a practical algorithm to recognize such graphs (see Section 3.6). The results in this chapter were obtained in collaboration with Alan Arroyo and R. Bruce Richter.

Our characterization extends a result of Richter and Arroyo [AR17] (see Theorem 3.1). We characterize the *crossing pairs* of a non-planar graph G. A pair of edges e, f of a graph G is a crossing pair of G if there exists a drawing D of G with $\mathrm{cr}(D) = 1$ (we refer D as a 1-*drawing* of G) in which e and f cross. Clearly, for non-planar graph G, $\mathrm{cr}(G) = 1$ if and only if G has a crossing pair and $\mathrm{cr}(G) \geq 2$ otherwise.

Before announcing the characterization in Theorem 3.8, we briefly review the literature in Section 3.1. Section 3.2 expands on some properties of crossing pairs and details the characterization in Theorem 3.8. In Section 3.3, we expand a bit on crossing pairs of graphs and enunciate Theorem 3.8. Section 3.4 contains the proof of Theorem 3.8. In Section 3.5, we provide a different characterization for crossing pairs. Lastly, in Section 3.6, we detail an algorithm for recognizing graphs with crossing number one based on Theorem 3.8.

3.1 Related works

The problem of characterizing graphs with crossing number at least two was already studied by Arroyo and Richter [AR17] in the context of *peripherally 4-connected graphs.*

A graph G is *peripherally 4-connected* if G is 3-connected and for every vertex 3-cut X of G, and for any partition of the components of $G - X$ into two non-null subgraphs H and K, at least one of H or K has just one vertex. Two edges $e = x_1y_1$ and $f = x_2y_2$ are *linked* if either e, f are incident with a common vertex or there is a 3-cut X in G such that $X \subset \{x_1, y_1, x_2, y_2\}$ and the vertex in $\{x_1, y_1, x_2, y_2\}\backslash X$ induces a trivial component of $G - X$. Otherwise, e, f are *unlinked.* A pair of edges e, f of G is *separated by cycles* if there exists two (vertex-)disjoint cycles C_e and C_f in G with $e \in C_e$ and $f \in C_f$.

Theorem 3.1. *[AR17] A peripherally 4-connected non-planar graph G has crossing number at least two if and only if for any pair of unlinked edges e, f in G they are separated by cycles.*

The *line graph* $L(G)$ of a graph G is a graph with vertex set $E(G)$ and $a, b \in E(G)$ are adjacent in $L(G)$ if and only if a, b share a common vertex in G. Let $\Delta(G)$ denote the maximum degree of a graph G.

We note that the edges incident with a vertex of degree d in a graph G corresponds to a complete graph K_d in $L(G)$. Thus, two vertices of degree 5 in G will create two edge-disjoint copies of K_5 in $L(G)$. Therefore, since $\operatorname{cr}(K_6) = 3$ [Guy72], if $\operatorname{cr}(L(G)) = 1$, then $\Delta(G) \leq 5$. Also, this means that G has at most one vertex with degree 5.

Kulli, Akka and Beineke [KAB79] characterized planar graphs whose line graph has crossing number one, while Jendrol' and Klevusč [JK01] obtained a characterization for non-planar graphs. Their results are detailed in what follows.

Theorem 3.2. *[KAB79] For every planar graph G, we have $\operatorname{cr}(L(G)) = 1$ if and only if:*

(1) $\Delta(G) = 4$ and there is a unique non-cut-vertex of degree 4, or

(2) $\Delta(G) = 5$, every vertex of degree 4 is a cut vertex, and there is a unique vertex of degree 5 with at most 3 edges in any block.

Theorem 3.3. *[JK01] For a non-planar graph G, we have $\operatorname{cr}(L(G)) = 1$ if and only if the following conditions hold:*

(1) $\operatorname{cr}(G) = 1$,

(2) $\Delta(G) \leq 4$, and every vertex of degree 4 is a cut vertex of G, and

(3) there exists a drawing of G in the plane with exactly one crossing in which each crossed edge is incident with a vertex of degree 2.

Akka, Jendrol, Kleščč, and Panshetty [Akk+97] obtained a characterization of planar graphs whose line graph has crossing number two.

A graph G is *k-crossing-critical* if $\operatorname{cr}(G) \geq k$ and every proper subgraph H of G has $\operatorname{cr}(H) < k$. The 1-crossing-critical graphs are exactly the Kuratowski graphs. We note that a graph with crossing number at least 2 contains a 2-crossing-critical graph as a subgraph.

A great deal of attention has been given to 2-crossing-critical graphs [BKQ83; Din+11; Koc87; Ric88; RS09; Šir84; Bok+16].

For an positive integer $n \geq 3$, the *Möbius Ladder* V_{2n} on $2n$ vertices, is the graph obtained from a $2n$-cycle by joining vertices with distance n in the cycle. Bokal, Opporowski, Richter and Salazar [Bok+16] characterized all 3-connected 2-crossing-critical graphs that contains a V_{10} as a minor and all the ones not containing a V_8 as a minor. They also showed how to obtain all the not 3-connected 2-crossing-critical graphs from the 3-connected ones, and showed that there exists only finitely many 3-connected 2-crossing-critical graphs with no V_{10} minor. It remains to characterize or enumerate all the 3-connected 2-crossing-critical graphs with a V_8 but no V_{10} as a minor.

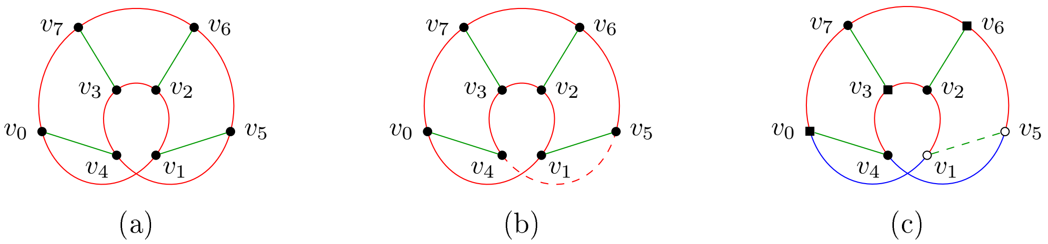


Figure 3.1: Removing v_4v_5 will result in a planar graph, however removing the edge v_1v_5 will result in a subdivision of $K_{3,3}$. The squares and disks represents the parts of the subdivision of $K_{3,3}$.

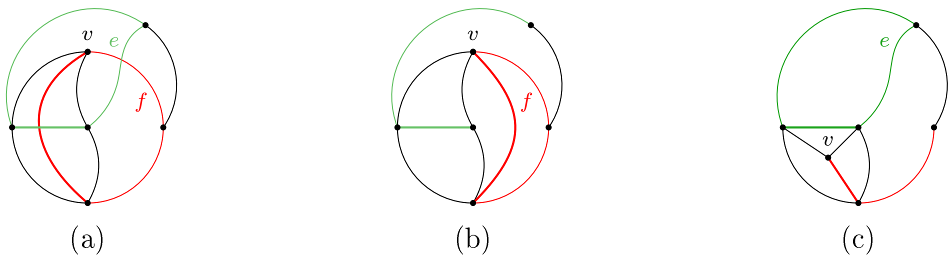


Figure 3.2: The pair of edges e, f is separated by the highlighted cycles, however removing either will result in a planar graph.

3.2 Crossing pairs

In this section we expand on some properties of crossing pairs of graphs. We start by pointing the crossing pairs of Kuratowski graphs.

Lemma 3.4. *A pair of edges e, f is a crossing pair of a Kuratowski graph if and only if they are not in the same branch or adjacent branches.*

The graph in Figure 3.1a is a V_8. The figure shows a 1-drawing of the V_8 where v_0v_1, v_4v_5 is a crossing pair. Removing either v_0v_1 or v_4v_5 will result in a planar graph (Figure 3.1b). However, as shown in Figure 3.1c, removing v_1v_5 will result in a subdivision of $K_{3,3}$. Any drawing of V_8 will also contain a drawing of $V_8 - v_1v_5$ and thus, at least one crossing not involving v_1v_5. This means that v_1v_5 cannot be in a crossing pair of V_8.

In general terms, for a pair of edges e, f of a graph G to be a crossing pair, $G - e$ and $G - f$ must be planar. The next lemma shows this.

Lemma 3.5. *If e, f is a crossing pair of a graph G, then $G - e$ and $G - f$ are planar.*

Proof. Let K be a Kuratowski subgraph of G and let D be a 1-drawing of G in which e and f cross. Since $D[K]$ must have a crossing, it contains e, f, since e, f is the only pair of edges that cross in D. Thus $D[G - e]$ and $D[G - f]$ have no crossing hence $G - e$ and $G - f$ are planar. $\square$

This condition is not sufficient. Let G be the graph depicted in Figure 3.2a. Figures 3.2b and 3.2c show that $G - e$ and $G - f$ are planar, respectively. However, the pair of edges e, f is separated by the highlighted cycles in the figure. Thus, another condition for e, f to be a crossing pair is that e, f is not separated by cycles, as shown on the next lemma.

Lemma 3.6. *Let G be a graph and let $e, f \in E(G)$. If e, f is separated by cycles then e, f is not a crossing pair of G.*

Proof. For a contradiction, assume that there exists a 1-drawing D of G in which e and f cross. By hypothesis, there exist disjoint cycles C_e and C_f of G with $e \in C_e$ and $f \in C_f$. Since D is a 1-drawing, $D[G - f]$ has no crossings and in particular $D[C_e]$ has no self crossing. Note that the ends of the path $C_f - f$ lie in different faces of $D[C_e]$. Thus, this path must cross C_e at some point, contradicting our assumption that D is a 1-drawing. □

One may conjecture that the conditions stated in Lemmas 3.5 and 3.6 would suffice to characterize crossing pairs. However, this is false. Let G be a graph, let e, f be a crossing pair of a graph G and suppose e and f are edges in adjacent branches in a Kuratowski subgraph K of G. Let D be a 1-drawing of G in which e and f cross. Removing either e or f from K will make the graph planar, but since they are in adjacent branches of K they cannot be crossed in any 1-drawing of K. However, $D[K]$ is a 1-drawing of K, a contradiction. This shows that e, f is also a crossing pair of K. This condition is expressed in the next lemma. A subgraph H of G is a 1-*subgraph* if $\text{cr}(H) = 1$.

Lemma 3.7. *If e, f is a crossing pair of a graph G, then it is a crossing pair of every 1-subgraph of G*

Our main result (Theorem 3.8) shows that the conditions expressed in Lemmas 3.5, 3.6 and 3.7 are sufficient. Note that we use a weaker version of Lemma 3.7.

Theorem 3.8. *Let G be a non-planar graph that is not a Kuratowski graph and let $e, f \in E(G)$. Then, e, f is a crossing pair of G if and only if the following conditions hold:*

(i) *$G - e$ and $G - f$ are planar,*

(ii) *e, f are not separated by cycles, and*

(iii) *there exists a proper Kuratowski subgraph H of G such that e, f is a crossing pair of H.*

Theorem 3.8 shows that crossing pairs in a graph with crossing number one have a hereditary property. That is, we start with a list of all crossing pairs of some 1-subgraph H of G (e.g. a Kuratowski subgraph, see Lemma 3.4) and eliminate pairs of edges that are either separated by cycles or not included in some other 1-subgraph. This gives rise to a simple algorithm to recognize these graphs which will be detailed in Section 3.6.

Let G be a non-planar graph and let K be a Kuratowski subgraph of G. Back to our original question, in what conditions does $\text{cr}(G) \geq 2$? As noted before, G must have no crossing pairs. In light of Theorem 3.8, $\text{cr}(G) > 2$ if and only if for every crossing pair e, f of K is either separated by cycles or either $G - f$ or $G - e$ is non-planar.

The proof of Theorem 3.8 is detailed in Section 3.4. But before that, we need a few concepts and theorems detailed in the next section.

3.3 Preliminaries

Let G be a graph and let H be a subgraph of G. An *H-bridge* B of G is either a single edge of $E(G) \setminus E(H)$ with both ends in H, or a component F of $G - V(H)$ together with the edges of G with one end in F and another in H. We note that the set of vertices of H in these edges are part of B and are called *attachments* of B and we denote such set by $Att(B)$. The *nucleus* $Nuc(B)$ of B is defined as $B \setminus Att(B)$. In the case that B is an edge, we say that the B is *trivial*. We note that the definitions of attachment and nucleus depend on the subgraph H and on the graph G, but we omit them in the notation. We make clear from the context to which subgraph we refer.

Let C be a cycle of G. Two distinct C-bridges B_1 and B_2 *overlap* if they have exactly three attachments in common or there exist vertices a, x, b, y occurring in this cyclic order in C such that $a, b \in Att(B_1)$ and $x, y \in Att(B_2)$. If the latter happens, then they *skew overlap*. A useful observation is that if we connect the nuclei of two skew overlapping C-bridges B_1, B_2 through a path avoiding $B_1 \cup B_2 \cup C$, then we obtain a subdivision of $K_{3,3}$. Another useful observation is that if B_1 and B_2 are overlapping C-bridges, then in any embedding D of G, B_1 and B_2 must be drawn in distinct faces of $D[C]$.

For vertices x and y of a graph G, we say that a cycle $C \subseteq G$ *detaches* x from y if there exists two overlapping C-bridges with one containing x in its nucleus and another containing y in its nucleus. We say that vertices x and y are *cofacial* of an embedding D, if x and y are incident with a common face of D. The following theorem by Tutte (and its slight modification in Corollary 3.10) is an important tool in the proof of Theorem 3.8.

Theorem 3.9. *(Tutte [Tut75]) Let G be a planar graph and let $x, y \in V(G)$. Then G has an embedding such that x and y are cofacial unless G contains a cycle C which detaches x from y.*

We need a slightly stronger version of Tutte's result on the existence of embeddings in which a vertex and an edge are incident with a common face. Let G be a graph, let $x \in V(G)$ and let f be an edge not incident with x. Let C be a cycle of G which includes neither x nor f. We say that C *detaches* x from f if there exists two overlapping C-bridges of G with one containing x in its nucleus and another containing f. A vertex x and an edge f are *cofacial* in an embedding D, if x and f are incident with a common face of D.

Corollary 3.10. *Let G be a planar graph, let $x \in V(G)$ and let $f \in E(G)$ not incident with x. Then G has an embedding such that x and f are incident with a common face unless G contains a cycle C which detaches x from f.*

Proof. Let G' be the graph obtained by subdividing f and adding a new vertex y. If we apply Theorem 3.9 to G', x and y, then we obtain the desired result. □

Recall that if H is a subgraph of G then a path P in G is H-avoiding or avoids H if no internal vertex of P is in H. Also recall that the closure of a face F is $F \cup \partial(F)$, that is, the union of the face and its boundary. The following lemma shows how we can extend planar embeddings by drawing bridges one by one. The result is intuitively obvious but the proof is rather technical and relies on the Schoenflies Theorem (Theorem 2.8).

Lemma 3.11. *Let B be a bridge of a cycle C in a planar graph G. Let D be a planar embedding of $G' = G - Nuc(B)$ (or $G' = G - E(B)$ if B is trivial). If no bridge overlapping B is drawn in the closure of a face F of $D[C]$, then we can extend D to a drawing of $G' \cup B$.*

Proof. We claim that there exists a face $F' \subseteq F$ of D incident with all the vertices of $Att(B)$. If $Att(B)$ is a singleton, then this is always true. So we may assume that $|Att(B)| \geq 2$.

If there exists no C-avoiding path joining two distinct vertices of $C - Att(B)$ in the closure of F, then it is easy to see that the vertices in $Att(B)$ are incident with a common face. Thus suppose not and let P be a C-avoiding path in the closure of F with ends x and y in $C - Att(B)$. We note that P is part of some C-bridge B' in G'.

We claim that the vertices in $Att(B)$ are in some component of $C - \{x, y\}$. Indeed, let a and b be vertices of $Att(B)$ and suppose they are in distinct components of $C - \{x, y\}$. We note that, as B is connected, there exists an ab-path Q in B. However, this implies that B' overlaps B in G and is in the closure of F, contradicting our hypothesis.

Let R be the xy-path in C containing the vertices in $Att(B)$. Let $C' := R \cup P$. We note that B is a C'-bridge also, moreover any C'-bridge in F may not overlap B in F. Indeed, since any C'-bridge is either also a C-bridge or is contained in B'. The argument follows inductively on the number of C'-avoiding paths. Thus we conclude that there exists some face F' of D contained in F incident with all the vertices of $Att(B)$.

We show that there exists a simple closed curve $\mathcal{C}$ in the closure of F' with $\mathcal{C} \cap D[C] = D[Att(B)]$. Let C' be the graph corresponding to $\partial(F')$. If C' is a cycle then $\partial(F')$ itself is closed and simple. Otherwise we build $\mathcal{C}$, by circumnavigating $\partial(F')$ while avoiding repeating vertices.

Since $C \cup B$ is planar, there exists an embedding D_B of $C \cup B$. In particular, $D_B[C]$ is a simple closed curve and B is in the closure of some face of D_B. As both $D_B[C]$ and $\mathcal{C}$ are simple closed curves, there exists a mapping f from $D_B[C]$ and $\mathcal{C}$. We can modify this mapping such that for any $v \in Att(B)$, $f(D[v]) = v$. By Schoenflies Theorem (Theorem 2.8), there exists a homeomorphism f' of the plane to itself such that the restriction of f' to C is f. Thus, our desired drawing is $D \cup f'(D_B)$. □

3.4 Proof of Theorem 3.8

Proof. Lemmas 3.5, 3.6 and 3.7 show that the conditions (i), (ii) and (iii) are necessary. We focus on sufficiency.

Let G be a non-planar graph that is not a Kuratowski graph. Let e and f be edges of G such that e, f is not separated by cycles in G. Suppose that e, f is a crossing pair of a proper 1-subgraph H of G and that $G - e$ and $G - f$ are planar. Let u and v be the ends of e.

We consider that a face of a drawing does not include its boundary. For a drawing D of a graph G, a *side* is the closure of one of its faces. For example, if D is an embedding of a cycle C, then D has two sides, each containing $D[C]$. For simplicity, we just say a side of G instead of D, if the drawing is clear from the context.

Claim 1. There exists a cycle C in $G - e$ such that:

1. $f \in E(C)$ and,
2. there exist overlapping C-bridges B_u and B_v in $G - e$ such that u and v belong to the nuclei of B_u and B_v, respectively.

Proof. Let D_H be a 1-drawing of H in which e and f cross. Clearly $H - e$ is planar since $D_H[H - e]$ is an embedding of $H - e$. Since H is not planar, no embedding of $H - e$ can have both u and v incident with the same face. By Theorem 3.9, there exists a cycle C of $H - e$, together with distinct overlapping C-bridges B_u^H and B_v^H containing u and v in their nuclei, respectively.

Since the C-bridges B_u^H and B_v^H overlap, they must be drawn in $D_H[H - e]$ in distinct sides of $D_H[C]$. This shows that u and v are drawn in distinct faces of C with respect to D_H. Therefore e crosses at least one edge of C in D_H, and since f is the only edge that crosses e in D_H, f must be in C.

Let B_u and B_v be the C-bridges in $G - e$ containing u and v in their nuclei, respectively. If $B_u = B_v$, then there is a uv-path P in B_u that is disjoint from C. Then $P + e$ and C are cycles that separate e and f in G, a contradiction. Thus, $B_u \neq B_v$. Moreover, $B_u^H \subseteq B_u$ and $B_v^H \subseteq B_v$, so B_u and B_v overlap on C. $\square$

Our goal now is to find an embedding of $G - e$ in which u and v are on distinct faces incident with f. Such an embedding can be easily extended to a 1-drawing of G where e and f cross.

Let D_e be a planar embedding of $G - e$. Then u and v are drawn on different sides of C, which we call the *u- and v-side of C*, respectively.

Let G_u and G_v denote the subgraphs of $G - e$ embedded on the u-side and v-side of C in D_e, respectively.

First we prove that there exists an embedding D_u of G_u such that u and f are cofacial and C bounds a face of D_u. By an analogous argument applied to v and f in G_v there exists an embedding D_v of G such that v and f are cofacial and C bounds a face of D_v. We can then combine both embeddings to obtain an embedding of $G - e$ where u and v are on distinct faces incident with f.

Claim 2. There exists an embedding D_u of G_u such that C bounds a face of D_u and u and f are cofacial in D_u.

Proof. We invite the reader to follow the proof alongside Figure 3.3. Since f is in C, in D_e there is a face F_u incident with f which is on the u-side of C. We may assume u is not incident with F_u, otherwise $D_e[G_u]$ is our desired embedding. Since B_u overlaps B_v on C, B_u has at least two attachments.

For any distinct vertices x, y of C joined by a C-avoiding path P in G_u, there is a cycle C_P consisting of P and the xy-subpath R of C that contains f. Let G_P denote the subgraph of G that is embedded in the side of C_P that is contained in the u-side of C. These finitely many paths can be partially ordered under inclusion of the G_P subgraphs.

Let P be a minimal path under this order. By minimality of G_P, there is no C_P-bridge which has two attachments such that: both are on P, or one attachment is on P

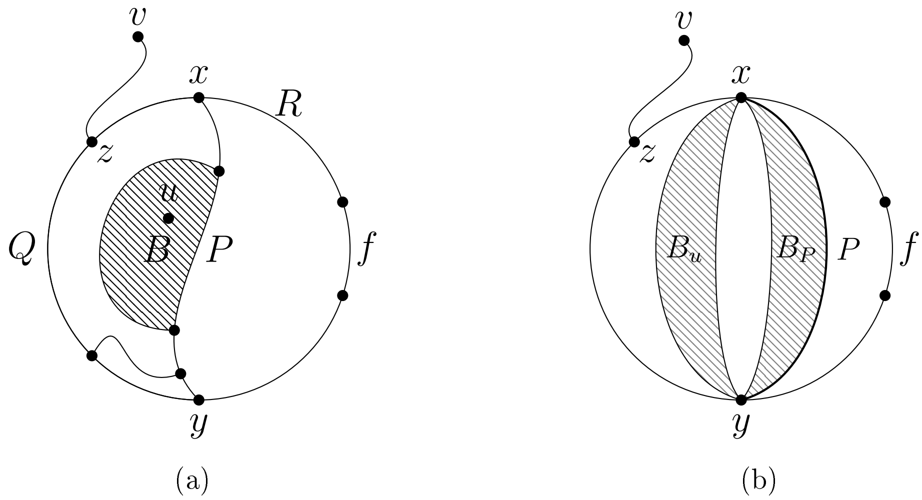


Figure 3.3: An abstract representation of the elements introduced in Claim 3.4

and another is on a component of $R - f$, or one in each component of $R - f$. It follows that every vertex and edge of P is incident with F_u.

We claim there exists some C-avoiding path P such that P is minimal under inclusion of G_P and no path from u to an end of f avoids $V(P)$. In this case, we say that P *separates* u from f. Since B_u overlaps B_v, B_u has at least two distinct attachments. There exists some C-avoiding path W in B_u. Suppose that $u \in V(W)$. If G_W is minimal under inclusion, then u is incident with F_u. Thus, assume that $u \notin V(W)$. In this case, there is some path P that does not contain u such that $G_P \subseteq G_W$ and G_P is minimal under inclusion. It follows that $u \notin V(G_P)$ and P separates u from f.

Thus, suppose no C-avoiding path in G_u contains u. Since $W \subseteq B_u$, some path W' exists that joins u and an internal vertex b of W. No other path from u to another internal vertex of W can be disjoint from W', as otherwise this path together with W' and W contains a C-avoiding path with u as a vertex. Thus some vertex c, closest to b in W', is a cut vertex. It follows that either $b = c$, or c is part of some C-avoiding path W. In any case, either W or W' separates u from f. We then choose P to be a minimal path under inclusion of G_P that is comparable to either W or W', depending on the aforementioned case.

Thus, choose P to be a minimal path under inclusion of G_P that separates u from f. Let x and y be the ends of P in C. Let R and Q to be the xy-subpaths of C that contains and not contains f, respectively. Recall that P separates u from f. As f belongs to R, we have that no attachment of B_u may belong to an internal vertex of R. Thus, we conclude that all attachments of B_u are on Q. Since B_v overlaps B_u in $G - e$, some attachment z of B_v lies in the interior of Q.

Since $Att(B_u) \subseteq V(Q)$, there is a $(P \cup Q)$-bridge B that contains u in its nucleus. We claim that there is no $(P \cup Q)$-avoiding path from u to an internal vertex of Q. Suppose for a contradiction that there is such a path W and let z' be the end of W in Q. Consider the cycle of G obtained from traversing W from u to z', followed by the $z'z$-subpath of Q, and then followed by a C-avoiding zv-path in B_v, and returning back to u by using e. This cycle and $C_P = P \cup R$ separate e and f, which contradicts our hypothesis. So there is no $(P \cup Q)$-avoiding path joining u to an internal vertex of Q. Let B_P be the C-bridge of G_u containing P. This means that either u is in some P-bridge B, and $B_u = B_P$, or $B_u \neq B_P$. The latter means that $Att(B_P) = Att(B_u) = \{x, y\}$.

We now turn our attention to the subgraph $C \cup B_P \cup B_u$ of G and prove that there is an embedding D^* of this subgraph having u and f incident with a common face. Note that we may assume that C bounds a face in D^* as B_P and B_u are the same C-bridge or do not overlap. Suppose this is true. If $B_P = B_u$, then we can simply replace $D[B_u]$ with $D^*[B_u]$ in $D[G_u]$ to get the desired embedding of G_u having C bounding a face and having u and f cofacial. If $B_P \neq B_u$, then we simply replace $D[B_u]$ with $D^*[B_u]$ in $D[G_u]$ (so it is inside the face of D bounded by C_P) to get the desired embedding of G_u having C bounding a face and having u and f cofacial. By contradiction, suppose there is no such embedding D^*. Corollary 3.10 implies there is a cycle C' in $C \cup B_P \cup B_u$ which detaches u from f. Let B'_u and B'_f be the C'-bridges which contain u and f, respectively, with $u \in Nuc(B'_u)$.

Since $f \notin E(C')$ and all internal vertices of R have degree 2 in $C \cup B_P \cup B_u$, R is internally disjoint from C'. We may also assume that x and y are not both attachments of B'_f, as otherwise they would be the only attachments of B'_f and thus B'_u does not overlap B'_f. So assume that, say, $x \in Nuc(B'_f)$. We divide the rest of the proof in two cases depending on whether C' contains an internal vertex of Q or not.

Case 1: *C' contains no internal vertices of Q.*

Let K_v and K_u be the C'-bridges in $G - e$ containing v and u, respectively. Clearly, $K_u \neq K_v$. Note that $Att(B'_u) \subseteq Att(K_u)$. Our immediate aim is to show that the component L_v of $K_v - f$ containing v overlaps K_u on C'. Since the C'-bridge B'_f overlaps B'_u (in $C \cup B_P \cup B_u$), it suffices to show that every attachment of B'_f is an attachment of L_v. As x is in the nucleus of B'_f, then for any attachment w of B'_f we have a C'-avoiding xw-path P_w in B'_f. Recall that there exists a C-avoiding vz-path P_z in B_v. Recall that G_v and G_u have only C in common. As $B_v \subseteq G_v$ and $C' \subseteq G_u$, P_v is also C'-avoiding. The union of P_w, P_v, and the zx-path in Q is a walk from v to w that avoids C'. It follows that w is also an attachment of L_v, as required.

We conclude that $G - e - f$ has two overlapping C'-bridges, one containing u and the other containing v. However, since u and v are the ends of e, this implies that $G - f$ is not planar, a contradiction.

Case 2: *C' contains an internal vertex of Q.*

Recall that (a) every vertex and edge of P is incident with F_u and (b) there is no $(P \cup Q)$-avoiding path joining u to an internal vertex of Q. Also, recall that $C' \subseteq C \cup B_P \cup B_u \subseteq G_u$. Thus both C' and P are in the u-side of C in D_e.

We may assume that $D_e[C']$ is completely contained in the side of $D_e[P \cup Q \cup R]$ disjoint from f. Suppose otherwise. Since $D_e[P]$ separates the u-side of C in D_e into two faces, only one of them is incident with f and thus R. Either C' has a $(P \cup R)$-avoiding subpath in the closure of F_u with distinct ends in $V(P \cup R)$ or C' itself is in the closure of F_u. In the first case, pick one such maximal subpath. Since $V(C')$ is disjoint from the internal vertices of R, both ends of this path are in P. This path contradicts (a). In the second case, C' is contained in a P-bridge B in $C \cup B_P \cup B_u$ with only one attachment.

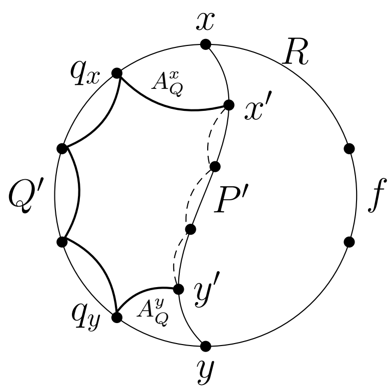


Figure 3.4: The arcs A_P (dashed) and A_Q (bold) composing C'.

We may then redraw B in the face of $D_e[P \cup Q \cup R]$ disjoint from f.

From (b) and the fact that B'_u overlaps B'_f it follows that C' must have at least two distinct vertices in common with P. Among the vertices in $V(C') \cap V(P)$, let x' be the one that is closest to x with respect to P. Similarly define y' for y. Let A_Q and A_P be the internally disjoint $x'y'$-paths whose union is C'.

Since $C' = A_P \cup A_Q$, both paths are drawn in the side of $P \cup Q \cup R$ containing u. Since these paths are internally disjoint, one of them, say A_P, is internally disjoint from the $x'y'$-subpath of $P \cup Q$ containing Q, and symmetrically A_Q is internally disjoint from the $x'y'$-subpath of P.

Another consequence of (b) is that B'_u has all its attachments in A_P. Our next step is to show that all attachments of B'_f are in A_Q. If so, B'_u and B'_f do not overlap and this concludes the proof. Figure 3.4 provides some visual aid for the rest of the proof.

Since $x \in Nuc(B'_f)$, we know that $x \neq x'$. Also, as C' contains an internal vertex of Q, so does A_Q. Let q_x be the vertex $V(Q) \cap V(A_Q)$ closest to x in Q. Let A_Q^x be the $x'q_x$-subpath of A_Q. Similarly define q_y (possibly $q_x = q_y$) and A_Q^y for y'. Let Q' be the q_xq_y-subpath of Q and P' be the $x'y'$-subpath of P.

Let $N = (P \cup Q \cup R) \setminus (V(P' \cup Q'))$. Since the internal vertices of R are in N, we have $V(N) \subseteq Nuc(B'_f)$. Note that $D_e[N]$ and $D_e[A_P]$ are on distinct sides of $D_e[A_Q \cup P']$. Thus, if there is an attachment of B'_f in the interior of A_P, then any path A in B'_f from N to this attachment would have an internal vertex in A_Q or P'. If that happens, A either contradicts (a) or crosses A_Q, depending on whether the N-end of A is in R or not. Thus, we conclude that no such path as A exists and thus B'_f has no attachments in A_P, as desired.

□

The entire argument can be repeated with the v-side of C in D_e, showing that v and f can be made cofacial on the v-side. Putting these embeddings together into one shows that G has a 1-drawing with e crossing f.

□

3.5 Alternative characterization

While working on the proof of Theorem 3.8, we realized that the necessary condition in Lemma 3.7 was in fact a sufficient condition when combined with Lemma 3.6. We say that a pair of edges e, f of a graph G is a *potential crossing pair* if e, f is a crossing pair of every 1-subgraph of G.

Theorem 3.12. *[Sil+18] A pair of edges e, f of a non-planar graph G is a crossing pair if and only if e, f is a crossing pair not separated by cycles.*

Let e, f be a pair of edges of a non-planar graph G. Suppose that e, f is not separated by cycles. Theorem 3.12 implies that if e, f is a crossing pair of some 1-subgraph H of G such that $G - e$ and $G - f$ are planar, then e, f is a potential crossing pair. We were unable to find a graph with crossing number at least two that has a potential crossing pair. Thus, it may be possible that potential crossing pairs only exists in graphs with crossing number one.

We need to introduce a few concepts and a theorem before we proceed with the proof of Theorem 3.12. The proof uses the classic Two Disjoint Paths theorem, proved by many authors [Sey80; Shi80; Tho80; RS90]. Let x, y, a and b be vertices of a graph G. The Two Disjoint Paths theorem gives condition for G to have xy- and ab- paths that are disjoint. In particular, we use a version due to Mohar [Moh94].

We need a few definitions before stating the theorem. A graph G is *nonseparable* if it has no 0- or 1-separation and *separable* otherwise. Let G be a nonseparable graph and suppose G has a 2-separation $\{G_1, G_2\}$ with $V(G_1) \cap V(G_2) = \{x, y\}$. A 2-separation is *elementary* if: either $G_1 - \{x, y\}$ or $G_2 - \{x, y\}$ is nonempty and connected; and either G_1 or G_2 is nonseparable. Graphs without elementary 2-separations are either 3-connected graphs, cycles, parallel edges, or rather small [Tut66].

Suppose $\{G_1, G_2\}$ is an elementary 2-separation of a nonseparable graph G with $\{x, y\} = V(G_1) \cap V(G_2)$. Let G_1' and G_2' be graphs obtained from G_1 and G_2 by adding an extra edge between x and y, respectively. This new edge is called a *virtual edge*. If G_i has no elementary 2-separation itself then G_i' is a *3-connected component* of G, for $i = 1, 2$. Otherwise, the 3-connected components of G are the 3-connected components of G_1' and G_2'. If G is separable, then the 3-connected components of G are the 3-connected components of its blocks. The 3-connected components of G are uniquely determined [Tut66]. They are also called *cleavage units* [Tut66].

A useful observation is that, by construction, each edge of G is in exactly one 3-connected component. Thus, for any 3-connected component H of G, there exists a corresponding subgraph H' of G in which each virtual edge of H is replaced by an H-avoiding path in G. Thus H' is a subdivision of H.

Let C be a cycle of G. Let P_1 and P_2 be a pair of disjoint paths both internally disjoint from C and whose ends are in $V(C)$. They are called a *pair of disjoint crossing paths* if the ends of P_1 and P_2 alternate in C.

A *tripod* in G with respect to a cycle C of G is a subdivision H of a $K_{2,3}$ in G together with three disjoint paths (possibly trivial) joining C with the part of size 3 in H. Both the part of size 2 and the edges of the tripod are disjoint from C. We denote by $Aux(G, C)$ the graph obtained from G by adding a new vertex v and an edge vw for each $w \in V(C)$.

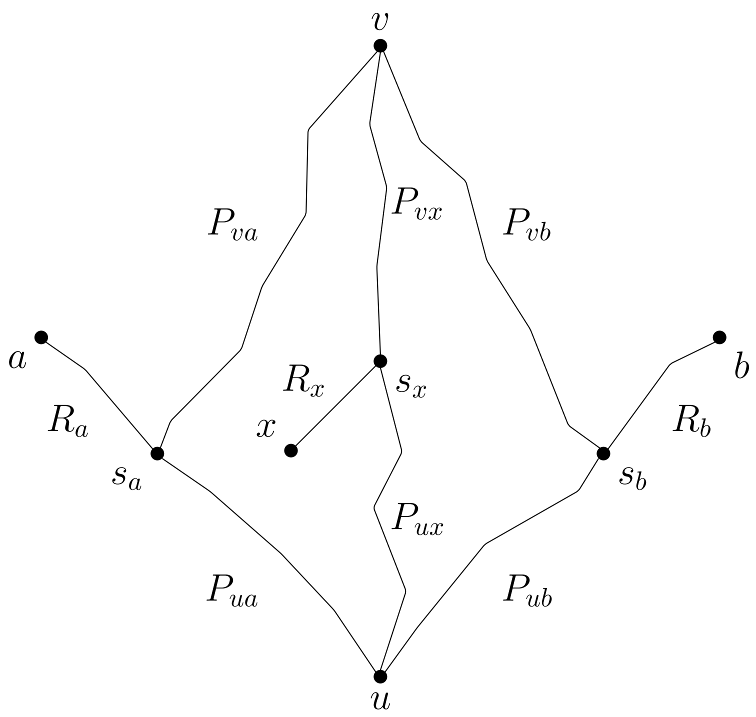


Figure 3.5: The tripod in the proof of Theorem 3.12.

For a disk Δ in the plane, let $\partial(\Delta)$ denote its boundary. We need the following result of Mohar.

Theorem 3.13. *[Moh94] Let G be a graph, let C be a cycle of G and let Δ be a closed disk. Let $\tilde{G} = Aux(G, C)$. There is a linear time algorithm that either finds an embedding of G in Δ with C drawn on $\partial(\Delta)$, or:*

(1) a pair of disjoint crossing paths (w.r.t. C),

(2) a tripod (w.r.t. C) or

(3) a Kuratowski subgraph contained in a 3-connected component of $\tilde{G}$ distinct from the 3-connected component of $\tilde{G}$ containing C.

We are ready to begin the proof of Theorem 3.12.

Proof. (of Theorem 3.12) Lemmas 3.6 and 3.7 show the necessity. We focus on sufficiency.

Let G be a non-planar graph that is not a Kuratowski graph. Let e, f be a pair of edges of G and suppose that e, f is a potential crossing pair not separated by cycles. Let H be a Kuratowski subgraph of G. Let a, b be the ends of e and x, y the ends of f.

Our goal is to show that there exists an embedding of $G - \{e, f\}$ in a closed disk Δ on the plane with $\partial(\Delta)$ containing a, x, b, y in this cyclic order. If so, we can simply draw e and f crossing on the exterior of Δ obtaining a 1-drawing of G.

Let ax, xb, by and ya be new edges and let C be the cycle induced by them. Let $G' = (G - \{e, f\}) \cup C$. If we obtain an embedding of G' on a closed disk Δ with C drawn on $\partial(\Delta)$, we can delete the edges of C to obtain our desired embedding of $G - \{e, f\}$. So suppose there is no such embedding. By Theorem 3.13 one of (1)-(3) holds.

First, suppose (3) holds. Let $\tilde{G} = Aux(G', C)$. Let J be the 3-connected component of $\tilde{G}$ containing C. Let K be a Kuratowski subgraph of $\tilde{G}$ not in J. Let $\{G_1, G_2\}$ be an elementary 2-separation in $\tilde{G}$ with $(G_1 \cap G_2) = \{u, w\}$, such that $J \subseteq G_1 + uw$ and $K \subseteq G_2 + uw$, where uw is a virtual edge. There is a uw-path P in G_1 such that we can exchange the virtual edge uw of K for P to obtain a Kuratowski subgraph in $G_2 \cup P$. We will also call this subgraph K.

Our idea is to obtain a modification of K which contradicts the fact that e and f is a potential crossing pair of G. Note that H is nonseparable and that e, f are disjoint edges in H. Therefore, there are vertex-disjoint paths in H from $\{a, b\}$ to $\{x, y\}$; we may choose a labelling so that these are ax- and by-paths P_{ax} and P_{by}, respectively. At most one of these paths has an edge in G_2.

Suppose that P_{ax} contains an edge of G_2. Then P_{by} is disjoint from $\{u, w\}$ and we may replace P in K with $((P_{ax} \cap G_1) \cup P_{by}) + \{e, f\}$ to get a Kuratowski subgraph of G in which e, f are edges of the same branch. This contradicts the fact that e, f is a potential crossing pair. Therefore, we may assume that P_{ax} and P_{by} are contained in G_1.

Let W be the subgraph of $\tilde{G}$ induced by $V(C) \cup v$. Recall that $E(\tilde{G}) \setminus E(G) \subseteq E(W)$. Thus, if P does not contain any edge of W, then K is also a Kuratowski subgraph of G which does not contain e nor f, a contradiction. If P contains v, then we modify P by shortcutting the neighbors of v in P with some path of C joining them. Thus, since C is a cycle, we assume that P contains at most three edges of C.

If P contains two consecutive edges of C, then we may replaced these with either e or f, in this case K converts to a Kuratowski subgraph of G containing only one of e and f, a contradiction. Thus, we may assume that P contains at most two non-consecutive edges of C in P, they are either ax and by, or ay and bx.

In the first case, we may replace ax with P_{ax} and by with P_{by} to get a uw-walk W in G_2 that uses no edge of C. Thus, W contains a uw-path P' and replacing P with P' in K will result in a Kuratowski subgraph of $G - \{e, f\}$, a contradiction.

In the second case, we replace ay with $P_{ax} + f$ and bx with $P_{by} + e$ to get a uw-walk W in G_2 that uses no edge of C. In this case, G contains a Kuratowski subgraph K' in which any of e and f that are in the same branch of K', again a contradiction.

Now, suppose (1) from Theorem 3.13 holds. If $G - \{e, f\}$ contains a pair of disjoint crossing paths, then these paths joined with e and f show that e and f are separated by cycles, a contradiction.

Finally, suppose (2) from Theorem 3.13 holds. We refer the reader to Figure 3.5 for a visual aid in the following definitions. Let T be a minimal (in the number of vertices) tripod w.r.t. C in G'. As the tripod is edge-disjoint from C, $T \subseteq G - \{e, f\}$. Let K be the subdivision of $K_{2,3}$ in T. We may assume that $\{a, x, b\}$ are the vertices of C connected to K in T. Let $\{u, v\}$ and $S = \{s_a, s_x, s_b\}$ be vertices of K representing the parts of the subdivided $K_{2,3}$. For $i \in S$, let P_{ui} be the ui-subpath in $K \setminus (S \setminus \{i\})$. Similarly, define P_{vi} for v. For $j \in \{a, x, b\}$, let R_j be the js_j-paths connecting C to K in T.

Again, recall that H is a Kuratowski subgraph of G with e and f in different branches. So $H - \{e, f\}$ is connected. It follows that there is a T-avoiding path from y to T in $H - \{e, f\}$ and, therefore, a path in $G - \{e, f\}$. The reader may verify that if P is a $V(T)$-avoiding path from y to $V(T)$ which ends in $R_a - s_a$ or $R_b - s_b$, then $(T \cup P) + e + f$ contains a Kuratowski subgraph in which e and f is not a crossing pair; and, for $i \in \{a, b, c\}$, if y ends in $P_{ui} - \{s_a, s_b\}$ or $P_{vi} - \{s_a, s_b\}$, then e and f are separated by cycles in $(T \cup P) + e + f$.

So any T-avoiding path from y to $V(T)$ in $G - \{e, f\}$ ends in $\{s_a, s_b\}$. Note that by symmetry, the same holds for a in place of y. Let P_y be a path in $G - \{e, f\}$ from y to $\{s_a, s_b\}$, say s_a. By the minimality of T, P_y is nontrivial. If there exists some $z \in (P_y - s_a)$ (respectively, $R_a - s_a$) with a zs_b-path Q that is internally disjoint from P_y (respectively, R_a) then $(K \cup P_y \cup Q) + f$ (respectively, $(K \cup R_a \cup Q) + e$) is a subdivision of $K_{3,3}$ in $G - e$ $(G - f)$, a contradiction.

So, we may assume that s_a separates y (a) from $V(T)$ in $G - f$ $(G - e)$. If R_a is trivial (that is, $s_a = a$), then $\{a, x\}$ separates y from b in H. This implies that either e and f are in the same or adjacent branches of H, a contradiction. If R_a is not trivial, then $(G - \{e, f\}) - s_a$, and consequently $(H - \{e, f\}) - s_a$, has at least three components: one for each vertex in $\{y, a, x\}$. Assuming $H - \{e, f\}$ is connected, since otherwise e and f are in the same branch of H, the only way this can happen is if s_a is a node in H and e and f are in adjacent branches, a contradiction.

□

3.6 Recognizing graphs with crossing number one

In this section, we present an small improvement over a naive practical algorithm for recognizing graphs with at most one crossing. Kawarabayashi and Reed showed that there exists a linear fixed parameter tractable algorithm to check whether a graph has crossing number at most k. The algorithm quite complex and not really practical for implementation. We first describe a more practical algorithm and then improve it using Theorem 3.8.

Let G be a graph and let $k \geq 0$ be an integer. Let $n = |V(G)|$ and $m = |E(G)|$. We first describe a naive $O((m + 2k)^{2k}(n + k))$[1] algorithm to decide if $\text{cr}(G) \leq k$.

We use induction on k. If $k = 0$ we can use an algorithm to decide if G is planar (see [HT74] for the first linear algorithm on n and [BM04] for a simpler version). By induction, for any graph H we know whether $\text{cr}(H) < k$, if so, then we are done. Thus we only need to check if $\text{cr}(G) = k$. We denote as $G_{e,f}$ the graph obtained from G by subdividing e and f once and identifying their subdivision. We note that any drawing of $G_{e,f}$ with $k - 1$ crossings is also a drawing of G with k crossings.

Thus, for every pair of edges e and f of G, we verify if $\text{cr}(G_{e,f}) < k$ by induction. If this is true for at least one pair of edges, this implies that there exists a drawing of $G_{e,f}$ with at most $k - 1$ crossings and thus $\text{cr}(G) \leq k$. If not, we conclude that no drawing D of G with k crossings exists, for otherwise there would be a pair of edges e and f crossing in D and thus $\text{cr}(G_{e,f}) \leq k - 1$.

At each step we generate a quadratic number of subproblems each with size $n + 1$ and $m + 2$. Since we do this at most k times, we get the $O((m + 2k)^{2k}(n + k))$ complexity.

For $k = 1$, we have an $O(m^2 n)$ time algorithm. We note that if $\text{cr}(G) = 1$, then G has a crossing pair, say e, f. Since $G - e$ is planar, this means that $m \leq 3n - 5$ edges. Thus,

[1] We slightly abuse the big-O notation here for didactic purposes

the algorithm is actually runs in $O(n^3)$ time, as we may reject any graph with more than $3n - 5$ edges.

We first check if G is planar with a planarity algorithm. If G is planar, then we are done, otherwise G has a Kuratowski subgraph H. This subgraph can be obtained from the planarity sub-routine at no extra cost [BM04]. As shown by Lemma 3.5, if e, f is a crossing pair of G, then it is also a crossing pair of H. Thus it suffices to check all pairs of edges of H instead of G. The algorithm follows as before.

This modification gives us the complexity $O(m_h^2 n)$ where m_h is the number of edges of the largest Kuratowski subgraph of G. Since $m_h \leq m$, this improves upon the original algorithm, however the time complexity stays the same. Thus, the time complexity of the modified algorithm is also $O(n^3)$.

We can also slightly improve the general algorithm for the case $k \geq 1$ if we use the modified algorithm as a base case.

Chapter 4

Crossing number of $K_{p,q}$ in surfaces

In this chapter we address the general problem of determining the crossing number of $K_{p,q}$ in surfaces. In particular, we address the question about generating optimal drawings of $K_{p,q}$ in surfaces. The work in this chapter was developed in collaboration with R. Bruce Richter.

We prove that for each integer $p \geq 1$ and each surface Σ, we show that there exists a finite set $\mathcal{D}(p, \Sigma) = \{D_1, ..., D_k\}$, where: for each $i \in \{1, .., k\}$, there is an integer r_i such that D_i is a drawing of K_{p,r_i} in Σ; and for each positive integer q, either there is an i such that D_i is an optimal drawing of $K_{p,q}$, or there exists an optimal drawing D of $K_{p,q}$ that is an extension (see definition in the next section) of D_i. As an example, if Zarankiewicz's conjecture (see next section) were true, for any p, a set composed of an embedding of $K_{p,1}$ and $K_{p,2}$ would suffice for the sphere. The proof of existence of this set is one of the main contributions of this chapter.

Theorem 4.1. *[RSL18] Let p be a positive integer and let Σ be a surface. Then, there exists a finite set $\mathcal{D}(p, \Sigma)$ of drawings of bipartite complete graphs in Σ such that, for every positive integer q, either an optimal drawing of $K_{p,q}$ is in $\mathcal{D}(p, \Sigma)$ or there is one that is an extension of a drawing in $\mathcal{D}(p, \Sigma)$.*

We are particularly interested in proving only its finiteness, as the exact cardinality of the set makes it not really practical. This theorem is an extension to higher genus (orientable and non-orientable) surfaces of a result of Christian, Richter and Salazar [CRS13] for the plane/sphere.

We denote the *q-side* and *p-side* of $K_{p,q}$ to be the parts of $K_{p,q}$ of size q and p, respectively. As an intermediate step for the proof of Theorem 4.1, we bound q as a function of Σ and p, as expressed in the next theorem.

For a pair of vertices u and v of a graph G, let $\mathrm{cr}_D(u, v)$ denote the number of crossings between the edges incident with u and v in a drawing D of G. Let $Z(p) = \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p-1}{2} \right\rfloor$.

Theorem 4.2. *[RSL18] Let D be a good drawing of $K_{p,q}$ in a surface Σ such that for any two vertices v and w of the q-side $\mathrm{cr}_D(v, w) < Z(p)$. Then, q is bounded by a function of Σ and p.*

The first section of this chapter presents Zarankiewicz's Conjecture, a classical conjecture about the crossing number of $K_{p,n}$ in the plane, and details some properties of

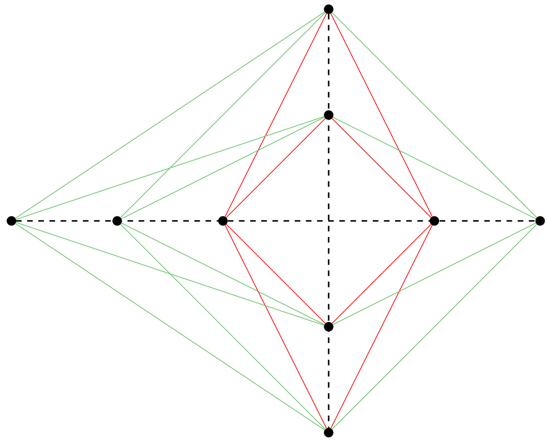


Figure 4.1: Zarankiewicz drawing of $K_{4,5}$.

Zarankiewicz's drawings. In Section 4.2 we state a couple results that we need for the proof of Theorem 4.1. Section 4.3 shows how we can bound the q as a function of Σ and p. We use this fact as an intermediate step for the proof of Theorem 4.1 in Section 4.4.

4.1 Zarankiewicz's drawings

On this section we give an overview of Zarankiewicz's drawings and some of its properties. Due to historical reasons and the fact that the general crossing number problem is NP-hard [Tho89], a lot of attention on the crossing number literature were given to special classes of graphs, especially K_n and $K_{p,q}$ [BW10].

Zarankiewicz [Zar55] proposed a general drawing of $K_{p,q}$ (see Figure 4.1) in the plane. Place p-side vertices along the x-axis in the plane distributing the vertices equally among the negative and positive side of the axis. Do the same with the q-side vertices along the y-axis and draw a segment between every pair of vertices in different axis.

This drawing has the following number of crossings:

$$Z(p,q) = Z(p)Z(q) = \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor . \tag{4.1}$$

This provides the following upper bound: $\mathrm{cr}(K_{p,q}) \leq Z(p,q)$. This gave birth to the famous conjecture by Zarankiewicz [Zar55]:

Conjecture 1. *(Zarankiewicz) For $p, q \geq 3$, $\mathrm{cr}(K_{p,q}) = Z(p,q)$.*

Zarankiewicz proved that the conjecture is true for $p = 3$ and arbitrary q. He also showed that if the conjecture holds for p odd and arbitrary q then it holds for $p + 1$. Kleitman [Kle70] extended the result to $p = 5$ and arbitrary q. Woodall [Woo93] proved that the conjecture is also true for $K_{7,7}$ and $K_{7,9}$. Christian, Richter and Salazar [CRS13] showed that for a fixed p, there exists a function $C(p)$ such that if the Conjecture holds for $q \leq C(p)$ then so does it $q > C(p)$.

Let G be a graph and let u and v be two of its vertices that are distinct. Let D be a drawing of G in the plane/sphere. The next lemma shows a lower bound for $\mathrm{cr}_D(u,v)$.

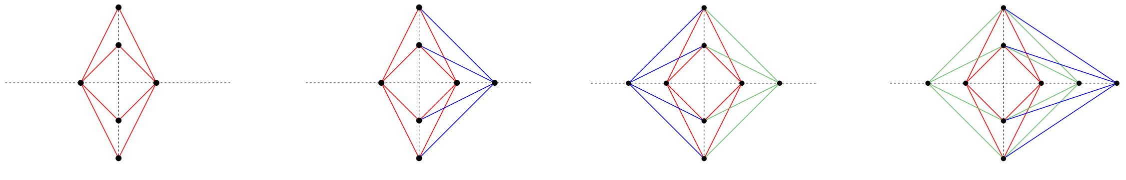


Figure 4.2: For a fixed p, we can obtain Zarankiewicz's drawings throughout duplication from an embedding.

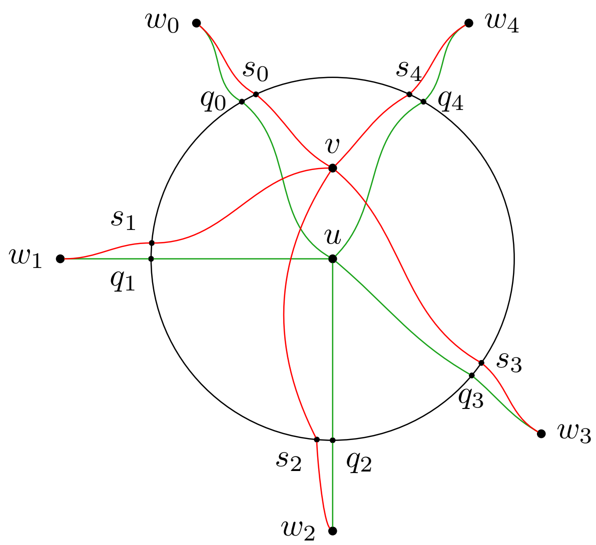


Figure 4.3: A duplicate of v with $Z(p)$ crossings inside the disk.

It was originally used by Kleitman as a tool for proving Zarankiewicz's conjecture for the $K_{5,n}$ family [Kle70]; Woodall [Woo93] proved it in a more general context.

Lemma 4.3. *[Woo93] Let u and v be the vertices of the part of size 2 in a $K_{p,2}$. Let D be a drawing of $K_{p,2}$ in the plane. If the rotations of u and v are the same in D, then* $\mathrm{cr}(D) \geq Z(p)$.

We refer the reader to Figure 4.2. Figure 4.2a is an embedding of $K_{4,2}$. Let u and w be the leftmost and rightmost vertices in the Figure. We note that in any embedding of $K_{p,2}$, the rotations of the vertices in the part of size 2 are each other's inverse. Figure 4.2b shows that we may obtain a Zarankiewicz's drawing of $K_{4,3}$ by adding another vertex in a particular way such that it has the same rotation as v. This adds exactly $Z(4) = 2$ crossings. In Figure 4.2c we do the same, but now we the vertex has the same rotation as u. Lastly, in Figure 4.2d we add another vertex with the same rotation as w. The added edges cross 4 times in total, twice with the edges of each vertex that has the same rotation.

Generally speaking, for a fixed integer p, we may obtain Zarankiewicz's drawings for $K_{p,q}$, with $q \geq 3$, from an embedding of $K_{p,2}$ by adding vertices in a alternating fashion such that each added vertex has the same rotation as one of the vertices in the part of size 2. This particular way of adding vertices is called *duplication* and is defined below.

Let $\mathrm{cr}_D(u)$ be the number of crossings between pairs of edges which include some edge incident with u in a drawing D. The following lemma shows how we can obtain a drawing of $K_{p,q+1}$ from a drawing D of $K_{p,q}$ with exactly $\mathrm{cr}(D) + Z(p) + \mathrm{cr}_D(u)$ crossings.

Lemma 4.4. *Let u and v be vertices of a loopless graph G with the same neighborhood of size p. Let D be a drawing of $G - v$ in a surface Σ. Then there is a drawing D' of G such that $\mathrm{cr}(D') = \mathrm{cr}(D) + Z(p) + \mathrm{cr}_D(u)$.*

Proof. We refer the reader to Figure 4.3 for a visual aid. Let Δ be a sufficiently small closed disk centered in u such that $\partial(\Delta)$ intercepts only edges incident with u, and only once, and their other ends are in its exterior. We place v in a face F of $\Delta \setminus D$ in the interior of Δ. Starting from any point of $\partial(\Delta) \setminus D$ incident with this F, for $i = 0, ..., p-1$, let e_i be the ith edge we meet when we traverse $\partial(\Delta)$ counterclockwise from this point. Let w_i be the end of e_i distinct from u. Let q_i be the intersection of e_i with $\partial(\Delta)$.

For $i = 0, \ldots, p-1$, denote by $q_{i-1}q_i$ the arc joining q_{i-1} and q_i in $\partial(\Delta)$ where $q_p = q_0$.

We split the edges of u in roughly two halves such that for each edge in the first half we draw the edges joining v to each vertex of $\{w_i\}$ incident with the edges on this half on the "left" side of the edges of $\{e_i\}$, and on the "right" side for the other half. This process is described formally below.

For every $i = 0, ..., \left\lfloor \frac{p-1}{2} \right\rfloor$ let s_i be a point in the interior of $q_{i-1}q_i$. We may draw an arc f_i from v to w_i by first drawing an arc from v to s_i crossing exactly i edges incident with u and then following along e_i to w_i. This may be done so that f_i crosses only edges that e_i cross and in the same order.

For every $i = \left\lfloor \frac{p-1}{2} \right\rfloor + 1, ..., p-1$, let s_i be a point in the interior of q_iq_{i+1}. We make sure that s_{p-1} is before s_0 in $q_{p-1}q_0$. We follow the same procedure as before and note that f_i will cross exactly $(p-1) - i$ edges in the interior of Δ.

In total, we have $0 + ... + \left\lfloor \frac{p-1}{2} \right\rfloor$ plus $0 + ... + \left\lfloor \frac{p-2}{2} \right\rfloor$ crossings in the interior of Δ which gives us exactly $Z(p)$ crossings. For $i = 0, ..., p-1$, as each f_i has the same crossings as e_i in the exterior of Δ, we have $\mathrm{cr}_D(u)$ extra crossings overall. This results in a total of $\mathrm{cr}_D(u) + Z(p)$ additional crossings over the crossings of D.

□

The additional vertex v in the proof above is called a *duplicate* of u and the resulting drawing D' is an *extension* of D. We also call this operation *duplication* of a vertex.

Let u and v be two vertices of the q-side of $K_{p,q}$. Let D be a drawing of $K_{p,q}$ in a surface Σ such that π_u and π_v are the rotations of u and v, respectively. Suppose that $\mathrm{cr}_D(u) \leq \mathrm{cr}_D(v)$ and that $\mathrm{cr}_D(u, v) \geq Z(p)$. If we redraw, v as duplicate of u, we obtain a drawing D' with less crossings. This leads to the following lemma,

Lemma 4.5. *[CRS13] If D is a drawing of $K_{p,q}$ in the plane, then there is a drawing D' of $K_{p,q}$ so that $\mathrm{cr}(D') \leq \mathrm{cr}(D)$ and, for any two vertices u and v of the q-side, whose rotations in D' are $\pi_{D'}(u)$ and $\pi_{D'}(v)$, respectively:*

(1) if $\pi_{D'}(u) = \pi_{D'}(v)$, then u and v are duplicates in D'; and

(2) if $\pi_{D'}(u) \neq \pi_{D'}(v)$, then $\mathrm{cr}_{D'}(u, v) < Z(p)$.

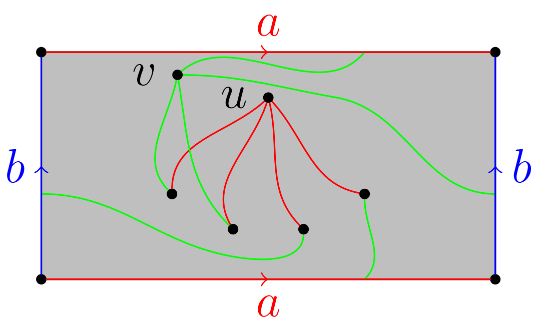


Figure 4.4: A drawing of $K_{2,4}$ on a torus. Note that u and v have the same rotation, however we have only one crossing.

We say that a good drawing D of $K_{p,q}$ is *clean* if the conditions (1) and (2) of the previous lemma holds. Lemma 4.5 shows that for every pair of positive integers p, q, there exists a drawing of $K_{p,q}$ that is clean. Thus, in the plane, we may concern ourselves only with clean drawings. Suppose D is optimal and that D has no duplicates. Thus, by Lemma 4.2, every pair of vertices u, v have distinct rotation and thus $\mathrm{cr}_D(u,v) < Z(p)$. Since there are at most $(p-1)!$ possible rotations, $q < (p-1)!$. This is the conclusion of Theorem 4.2 where Σ is the plane/sphere.

This preliminary work on Lemma 4.5, due by Christian, Richter and Salazar, is the groundwork for the following theorem.

Theorem 4.6. *[CRS13] Let p be a positive integer. If, for every $q \leq ((2Z(p))^{p!}(p!)!)^4$, $\mathrm{cr}(K_{p,q}) = Z(p)Z(q)$, then, for every q, $\mathrm{cr}(K_{p,q}) = Z(p)Z(q)$.*

For general surfaces Lemma 4.3 (and consequently Lemma 4.5) is not true. Figure 4.4 shows a drawing of $K_{4,2}$ in a torus such that, the vertices of the part of size 2 have the same rotation, but the drawing has crossing number $1 < Z(4) = 2$.

There is no known equivalent version of Zarankiewicz conjecture for arbitrary surfaces, although some upper bounds are known. Suppose that $K_{p,h}$ is embeddable in a surface Σ for some $p, h > 0$. Richter and Širáň [RŠ96] obtained the following upper bound:

$$\mathrm{cr}_\Sigma(K_{p,q}) \leq \frac{1}{2}\left\lfloor\frac{q}{h}\right\rfloor\left\{2q - h\left(1 + \left\lfloor\frac{q}{h}\right\rfloor\right)\right\}\left\lfloor\frac{p}{2}\right\rfloor\left\lfloor\frac{p-1}{2}\right\rfloor. \tag{4.2}$$

This bound is achieved by duplicating the vertices of the h-side of $K_{p,h}$ embedding on Σ in a cyclic fashion. For a positive integer q, they showed that the upper bound of Inequality 4.2 is also a lower bound for the crossing number of $K_{3,q}$ graphs in any surface Σ. Similarly, Ho [Ho05; Ho09] also showed that the bound in Inequality 4.2 is also a lower bound for the crossing number of $K_{4,q}$ in Σ, only if Σ is a torus or projective plane.

Just like Zarankiewicz's drawings of $K_{p,q}$, for $q > 2$, may be generated from a single embedding of $K_{p,2}$, our goal, with Theorem 4.1, is to show that for any surface Σ we may generate optimal drawings from a finite set of drawings. The proof is contained in Section 4.4. Before that, we state a couple useful results in the next section and afterwards we prove Theorem 4.2 as a intermediate step for the proof of Theorem 4.1.

4.2 Preliminaries

We need the general graph theoretical version of Ramsey's Theorem [Ram30], stated below.

A *k-edge-coloring* of a graph G is a function ϕ from the edges of G to $\{1, ..., k\}$. A subgraph H of G is *monochromatic* if all the edges of H have the same color.

Theorem 4.7. *(Ramsey's Theorem) Let $k, s_1, s_2, ..., s_k$ be positive integers. Then, there exists an integer $R(s_1, .., s_k)$ such that if $n \geq R(s_1, .., s_k)$, then for every k-edge-coloring of K_n there exists some i in $\{1, ..., k\}$ so that some subgraph K_{s_i} of K_n is colored only with color i.*

Let H be a graph. A graph is *H-free* if it does not contain H as a subgraph. The next theorem, due to Turán [Tur41], limits the number of edges of K_r-free graphs.

Theorem 4.8. *(Turán's Theorem [Tur41]) The number of edges in a K_{r+1}-free graph with n vertices is at most:*

$$\left(1 - \frac{1}{r}\right) \frac{n^2}{2}.$$

4.3 Bounding the q-side

Proof. (of Theorem 4.2) We may assume that $p \geq 3$.

We first note that, for vertices i and j of the p-side, if the edges iv and jw of $K_{p,q}$ cross in D, then there exists a 4-cycle that self-cross in D at least once. Indeed, as the graph is $K_{p,q}$ the edges iw and jv are also in $K_{p,q}$ and together with iv and jw induce a 4-cycle.

For each pair of vertices u and v of the q-side, we define a function f_{uv} on the pair i and j of the p-side such that $f_{uv}(i, j) = 1$ if the 4-cycle of $K_{p,q}$ induced by $\{i, j, u, v\}$ crosses itself in D, and $f_{uv}(i, j) = 0$ otherwise. We note that the set of all possible such functions has size $k = 2^{\binom{p}{2}}$; therefore it is finite.

Let r be an integer such that $K_{3,r}$ is not embeddable in Σ (see Theorem 2.11). Let K_q be a complete graph such that its vertex set is the q-side of $K_{p,q}$. We color each edge uv of K_q with "color" f_{uv}. By Ramsey's Theorem (Theorem 4.7), there exists a function $R := R(s_1, \ldots, s_k)$ such that if $q \geq R$, then every k-edge-coloring of K_q with colors $1, 2, \ldots, k$ contains a monochromatic copy of K_r. Let f be the color of this K_r (so $f = f_{uv}$ for some u, v on the q-side). Note that R is a function on r and k and both depend only on Σ and p, respectively.

Now let us define a graph G whose vertex set is the p-side. We join i and j in G if $f(i, j) = 0$. This means that $ij \in E(G)$ if for any $u, v \in V(K_r)$ the 4-cycle induced by $\{u, v, i, j\}$ in $K_{p,q}$ does not self-cross in D. If there exists a triangle in G, then there exists a drawing of $K_{3,r}$ as a subdrawing of D without crossings, which cannot happen by the choice of r. Thus G is triangle-free. Turán's Theorem (Theorem 4.8) implies that G has at most $(p^2/4)$ edges. Thus, there are at least $\binom{p}{2} - (p^2/4)$ pairs of vertices of the p-side which contributes with at least one crossing in D. Therefore, for any pair of vertices u and v of K_r, we have that $\mathrm{cr}_D(u, v) \geq Z(p)$, a contradiction. $\square$

4.4 Finite number of drawings

Proof. (*of Theorem 4.1*)

Theorem 4.2 implies that there is a number $F(p, \Sigma)$ such that if $q > F(p, \Sigma)$, then there exist distinct vertices u, v such that $\mathrm{cr}_D(u, v) \geq Z(p)$. Let $\mathcal{D}(p, \Sigma)$ consist of all the good drawings in Σ of $K_{p,q}$ with $q \leq F(p, \Sigma)$. Theorem 2.16 implies $\mathcal{D}(p, \Sigma)$ is finite.

For any drawing D of $K_{p,q}$ with $q > F(p, \Sigma)$, we can successively delete $u_1, u_2, \ldots, u_{q-F(p,\Sigma)}$ such that, for each $i = 1, 2, \ldots, q - F(p, \Sigma)$, there is a vertex v_i in $K_{p,q} - \{u_1, \ldots, u_{i-1}\}$ such that $\mathrm{cr}_{D-\{u_1,\ldots,u_{i-1}\}}(u_i, v_i) \geq Z(p)$.

The drawing $D - \{u_1, \ldots, u_{i-1}\}$ is in $\mathcal{D}(p, \Sigma)$. Now reinserting u_i to be a duplicate of v_i (in the order $u_{q-F(p,\Sigma)}, \ldots, u_2, u_1\}$) produces a drawing D' of $K_{p,q}$ such that $\mathrm{cr}(D') \leq \mathrm{cr}(D)$, as required.

$\square$

Chapter 5

Conclusion

The main results of this thesis are detailed in Theorems 3.8,3.12,4.1 and 4.2. In short, we present two distinct characterizations of graphs with crossing number one and a way to obtain optimal drawings of $K_{p,q}$ in surfaces. We now describe a few ways to use or extend these results.

By the time of writing of this thesis. The problem of characterizing 3-connected 2-crossing-critical graphs with a V_8 but no V_{10} as a minor remains open. Theorem 3.8 may help in this regard. Recall that a nonplanar graph has crossing number two if and only if it contains no crossing pair. Thus, we may generate 2-crossing-critical graphs from a V_8 by attaching bridges to it such that every crossing pair of V_8 is either separated by cycles or not contained in a Kuratowski subgraph. This strategy has been used at least a couple times [Arr14; Aus12].

While the famous Zarankiewicz's conjecture has been the topic of much discussion in the crossing number literature, we still lack a good generalization of it for higher genus surfaces. The bound of Equation 4.2 was the first step towards this. However it already fails for $K_{5,5}$ in the projective plane. A straightforward generalization of the bound in 4.2 is to use general drawings instead of embeddings. In this sense, Theorem 4.1 provides the first step in this direction, as it shows that we may need to consider only finitely many drawings as candidates for extensions. However, the bound obtained in the proof is quite large and really not practical.

It may be possible that, for small p, only a handful of drawings are necessary to obtain optimal drawings for the $K_{p,q}$ family. It begs the question: what makes a drawing a good candidate for a set like $\mathcal{D}(p, \Sigma)$ but with minimum size? Ideally we want drawings that minimize the growth of crossings as more and more duplicates are added, even if the drawing itself has a large number of crossings. Answers to these questions may lead to developments of Zarankiewicz's conjecture itself.

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