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Truncated Weyl Modules as Chari-Venkatesh Modules and Fusion Products

Módulos de Weyl Truncados via Módulos de Chari-Venkatesh e Produtos de Fusão

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

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Coorientador: Ghislain Fourier

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"...To live is to cherish dreams and hopes, making faith our greatest inspiration.
It is to seek in small things, a great reason to be happy..."
(Mário Quintana)

"...Viver é acalentar sonhos e esperanças, fazendo da fé a nossa inspiração maior.
É buscar nas pequenas coisas, um grande motivo para ser feliz..."
(Mário Quintana)

Resumo

Estudamos propriedades estruturais de módulos de Weyl truncados. Dados uma álgebra de Lie simples $\mathfrak{g}$ e um peso integral dominante λ, o módulo de Weyl local graduado $W(\lambda)$ é o objeto universal na categoria dos módulos de dimensão finita graduados de peso máximo para a ágebra de correntes $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$. Para cada inteiro positivo N, o quociente $W_N(\lambda)$ de $W(\lambda)$ pelo submódulo gerado pela ação do ideal $\mathfrak{g} \otimes t^N \mathbb{C}[t]$ sobre o vetor de peso máximo é chamado um módulo de Weyl truncado. Ele satisfaz a mesma propriedade universal de $W(\lambda)$ quando visto como um módulo para a correspondente álgebra de correntes truncada $\mathfrak{g}[t]_N = \mathfrak{g} \otimes \dfrac{\mathbb{C}[t]}{t^N \mathbb{C}[t]}$. Chari-Fourier-Sagaki conjecturaram que se $N \leqslant |\lambda|, W_N(\lambda)$ deve ser isomorfo a um produto de fusão de certos módulos irredutíveis. Nosso principal resultado prova essa conjectura quando λ é um múltiplo de um peso minúsculo e $\mathfrak{g}$ é de tipo ADE. Também damos um passo adiante para provar a conjectura para múltiplos de um peso fundamental "pequeno" que não é minúsculo provando que o módulo de Weyl truncado correspondente é isomorfo ao quociente de um produto de fusão de módulos de Kirillov-Reshetikhin por uma simples relação. Uma parte importante da demonstração de nosso resultado principal é dedicada a provar que qualquer módulo de Weyl truncado é isomorfo a um módulo de Chari-Venkatesh com a correspondente família de partições explicitamente descrita. Este fato é o segundo resultado principal deste trabalho e nos leva a novos resultados no caso $\mathfrak{g} = \mathfrak{sl}_2$ relacionados a bandeiras de Demazure e cadeias de inclusões de Módulos de Weyl truncados.

Palavras-chave: Módulo de Weyl, produto de fusão, álgebra de correntes, teoria de representação, álgebra de Kac-Moody.

Abstract

We study structural properties of truncated Weyl modules. Given a simple Lie algebra $\mathfrak{g}$ and a dominant integral weight λ, the graded local Weyl module $W(\lambda)$ is the universal finite-dimensional graded highest-weight module for the current algebra $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$. For each positive integer N, the quotient $W_N(\lambda)$ of $W(\lambda)$ by the submodule generated by the action of the ideal $\mathfrak{g} \otimes t^N\mathbb{C}[t]$ on the highest-weight vector is called a truncated Weyl module. It satisfies the same universal property as $W(\lambda)$ when regarded as a module for the corresponding truncated current algebra $\mathfrak{g}[t]_N = \mathfrak{g} \otimes \dfrac{\mathbb{C}[t]}{t^N\mathbb{C}[t]}$. Chari-Fourier-Sagaki conjectured that if $N \leqslant |\lambda|$, $W_N(\lambda)$ should be isomorphic to the fusion product of certain irreducible modules. Our main result proves this conjecture when λ is a multiple of a minuscule weight and $\mathfrak{g}$ is simply laced. We also take a further step towards proving the conjecture for multiples of a "small" fundamental weight which is not minuscule by proving that the corresponding truncated Weyl module is isomorphic to the quotient of a fusion product of Kirillov-Reshetikhin modules by a very simple relation. One important part of the proof of the main result, and the second main result of this work, shows that any truncated Weyl module is isomorphic to a Chari-Venkatesh module and explicitly describes the corresponding family of partitions. This leads to further results in the case that $\mathfrak{g} = \mathfrak{sl}_2$ related to Demazure flags and chains of inclusions of truncated Weyl modules.

Keywords: Weyl module, fusion product, current algebra, representation theory, Kac-Moody algebra.

Contents

Introduction

In [14], Chari and Pressley introduced a family of finite dimensional representations called Weyl modules for the affine Kac Moody algebras and their quantized versions. The definition, given via generators and relations, was inspired by the modular representation theory of algebraic groups. Later, others authors gave similar definitions for Weyl modules for others classes of algebras. For example, while Chari and Pressley considered algebras of the form $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$, with $\mathfrak{g}$ a finite-dimensional simple Lie algebra, Feigin and Loktev, in [19], considered the more general class of algebras of the form $\mathfrak{g}\otimes A$, where A is a coordinate ring of an algebraic variety, which was then further generalized in [9] to A being a commutative associative algebra with unit. The Lie bracket on $\mathfrak{g}\otimes A$ is given by $[x\otimes a, y\otimes b] = [x,y]\otimes(ab)$ for $x,y\in\mathfrak{g}, a,b\in A$. For the most general contexts on which Weyl modules are being studied nowadays, see [5, 25, 37] and references therein.

The context of current algebras, i.e., when A is the polynomial ring $\mathbb{C}[t]$ is certainly the most studied for several reasons. On one hand, some structural questions about the structure of quantum Weyl modules and their irreducible quotients can be reduced to similar questions about certain remarkable quotients of the graded Weyl modules for $\mathfrak{g}[t]=\mathfrak{g}\otimes\mathbb{C}[t]$, called Chari-Venkatesh modules [16] (which include the classes of graded Kirillov-Reshetikhin and $\mathfrak{g}$-stable Demazure modules), and their fusion products in the sense of [18]. On the other hand, the study of the category of graded finite-dimensional representations of the current algebra is motivated by applications in mathematical physics, algebraic geometry and geometric Lie theory, as well as combinatorics.

The definition of Weyl modules for current algebras can be explained as follows. Given a triangular decomposition of $\mathfrak{g}$, say $\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+$ where $\mathfrak{h}$ is a Cartan subalgebra and $\mathfrak{n}^\pm$ are choices of positive and negative nilpotent parts, consider the induced decomposition on $\mathfrak{g}[t]$: the sum of the current algebra over each summand (this can be done for general A). Any linear functional λ on $\mathfrak{h}$ can be extended to one on $\mathfrak{h}[t]\oplus\mathfrak{n}^+[t]$ by setting it to be zero on $\mathfrak{h}\otimes t\mathbb{C}[t]\oplus\mathfrak{n}^+[t]$. Then, one can consider the one-dimensional module for $\mathfrak{h}[t]\oplus\mathfrak{n}^+[t]$ determined by λ and the induced module $M(\lambda)$ for $\mathfrak{g}[t]$, which is the Verma type module associated to the above triangular decomposition and λ. The way λ was extended allows us to equip $M(\lambda)$ with a $\mathbb{Z}$-grading by inducing it from that of $\mathfrak{n}^-[t]$. If λ is a dominant integral weight, the irreducible quotient of $M(\lambda)$ is finite-dimensional: it is the corresponding finite-dimensional simple $\mathfrak{g}$-module $V(\lambda)$ with trivial action of $\mathfrak{g}\otimes t\mathbb{C}[t]$. It turns out that $M(\lambda)$ has other finite-dimensional graded quotients and the graded local Weyl module $W(\lambda)$ is the largest of them, i.e., any other finite-dimensional graded quotient of $M(\lambda)$ is also a quotient of $W(\lambda)$. In other words, $W(\lambda)$ is the universal highest-weight module of highest weight λ in the category of finite-dimensional graded

$\mathfrak{g}[t]$-modules. The definition for more general A is similar, but the discussion about the extension of λ to a functional on $\mathfrak{h}\otimes A$ has to be done more carefully. Even for $A=\mathbb{C}[t]$, different extensions will lead to non graded local Weyl modules.

In this work we focus on the study of graded Weyl modules for the truncated current algebras $\mathfrak{g}[t]_N = \mathfrak{g}\otimes \dfrac{\mathbb{C}[t]}{t^N\mathbb{C}[t]}$ with N a positive integer. The description of the definition explained in the previous paragraph applies in this context as well and we denote the associated truncated Weyl module by $W_N(\lambda)$. Moreover, since $\mathfrak{g}[t]_N$ is a graded quotient of $\mathfrak{g}[t]$, every $\mathfrak{g}[t]_N$-module can be regarded as a module for $\mathfrak{g}[t]$. In particular, $W_N(\lambda)$ is a quotient of $W(\lambda)$. The motivation for studying the truncated Weyl modules comes from a conjecture stated in [10] related to Schur positivity which, as seen in [21, 32, 38], can be formulated in terms of conjectural answers for the following question: are the truncated Weyl modules isomorphic to fusion products of irreducible modules and non-truncated Weyl modules? The answer for such question, in particular, produces a way of computing the characters of truncated Weyl modules which is still not known in general.

The notion of fusion products was introduced in [18] as certain operations between cyclic objects in the category of graded finite-dimensional $\mathfrak{g}[t]$-modules related to tensor products. Given a collection of such objects $V_1,\dots,V_l$, a fusion product of these objects is constructed as follows. Choose distinct complex numbers $a_1,\dots,a_l$ and twist the action of $\mathfrak{g}[t]$ on V_j by the automorphism of $\mathfrak{g}[t]$ induced by the automorphism of $\mathbb{C}[t]$ given by $t\mapsto t+a_j$. This produces a family of non-graded modules $V_j^{a_j}$. It is known that, if v_j is a choice of cyclic vector for V_j, then $V_1^{a_1}\otimes\cdots\otimes V_l^{a_l}$ is cyclic on $v=v_1\otimes\cdots\otimes v_l$ which can then be used to define a filtration on this tensor product. The associated graded module is called a fusion product of $V_1,\dots,V_l$ and, conjecturally, for the relevant cases, the construction should not depend on the choices of $a_1,\dots,a_l$. For this reason, the fusion product is simply denoted by $V_1*\cdots*V_l$. Fusion products provide a powerful tool to study several graded finite-dimensional $\mathfrak{g}[t]$-modules. In particular, it was proved in [22] and [35], for simply laced $\mathfrak{g}$ and non-simply laced case, respectively, the following decomposition of the graded local Weyl module as a fusion product:

$$W(\lambda)\cong W(\lambda_1)*\cdots*W(\lambda_k), \quad \text{if} \quad \lambda=\lambda_1+\cdots+\lambda_k.$$

For some structural questions, such as character, this reduces the study to the case of fundamental weights. The question left in the previous paragraph is nothing but the quest for truncated versions of this decomposition.

To explain the conjectural answer, consider the set $P^+(\lambda,N)$ of the elements $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_N)$ such that, λ,λ_i are dominant weights and $\lambda=\sum_{i=1}^{N}\lambda_i$. A partial order on $P^+(\lambda,N)$ was defined in [10] and an algorithm for computing its maximal elements was described in [20]. It turns out that all maximal elements are in the same orbit of the obvious action of the symmetric group and, hence, there exists essentially only one maximal element. The following conjecture was then stated in [32]:

Conjecture: *Suppose* $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$ *is a maximal element of* $P^+(\lambda, N)$*. If* $N \leqslant |\lambda|$, $W_N(\lambda) \cong V(\lambda_1) * \cdots * V(\lambda_N)$.

The conjecture has been proved for certain particular values of λ or N. For instance, it follows from the results of [21] for $N = 2$, $\mathfrak{g} = \mathfrak{sl}_n$, and λ any multiple of a fundamental weight. Other cases for general N, with restrictions on λ or $\mathfrak{g}$ are proved in [32, 38]. The main result of this work extends, in a relevant way, the list of cases on which the conjecture is proved:

Theorem A: *The above conjecture holds if* $\mathfrak{g}$ *is simply laced and* λ *is a multiple of a minuscule weight.*

Note that, if $\mathfrak{g} = \mathfrak{sl}_n$, then the set of nonzero minuscule dominant weights is exactly the set of fundamental weights. Hence, Theorem A expands for general N the case that followed from [21]. In fact, the proof is completely different from that of [21] for $N = 2$. Actually our proof of the conjecture has more sophisticated approach than the other cases. Typically, Demazure modules are used directly in the other cases proved while our proof relies on the theory of Chari-Venkatesh (CV) modules, which, in particular, include the class of $\mathfrak{g}$-stable Demazure modules. The CV modules, introduced in [16], form a family of graded quotients of Weyl modules indexed by $|R^+|$-tuples of partitions where R^+ is the set of positive roots of $\mathfrak{g}$. Given such a partition $\xi = (\xi(\alpha))$, we denote the corresponding CV module by $CV(\xi)$. For instance, when $\xi(\alpha)$ is the the partition with $\lambda(h_\alpha)$ parts all equal to 1 for every $\alpha \in R^+$, where h_α is the associated co-root, then $CV(\xi) \cong W(\lambda)$. For the other extremal case, i.e., when $\xi(\alpha)$ has exactly one part which is $\lambda(h_\alpha)$, for every $\alpha \in R^+$, then $CV(\xi) \cong V(\lambda)$ is the irreducible quotient of $W(\lambda)$. For other possibilities of ξ with each $\xi(\alpha)$ a partition of $\lambda(h_\alpha)$, $CV(\xi)$ will be something in between $W(\lambda)$ and $V(\lambda)$. Although this produces many interesting quotients of $W(\lambda)$, it is not true that all quotients of $W(\lambda)$ are obtained in this way. The second main result of this work, and crucial ingredient in the proof of Theorem A, says that all truncated Weyl modules are obtained in this way:

Theorem B: *For every* λ *and* N, $W_N(\lambda)$ *is isomorphic to* $CV(\xi)$ *for some* ξ.

In fact, we explicitly describe the partition ξ in Theorem B. Note that there is no hypothesis on $\mathfrak{g}$ in this theorem. Theorem B, together with results from [3, 15, 16], gives us tools to obtain further results in the case that $\mathfrak{g} = \mathfrak{sl}_2$ related to Demazure flags and chains of inclusions of truncated Weyl modules. For instance, from the description of ξ and results of [16], one can immediately identify the truncated Weyl modules which are isomorphic to Demazure modules. Otherwise, the results of [3, 15] allows us to study Demazure flags for truncated Weyl modules since every CV module (for $\mathfrak{g} = \mathfrak{sl}_2$) admits a

Demazure flag: a sequence of inclusions of submodules such that the successive quotients are isomorphic to Demazure modules.

The proof of Theorem A also relies on a result from [36] about fusion products of Kirillov-Reshetikhin (KR) modules. The KR modules were originally considered in the quantum setting motivated by mathematical physics [29]. In that setting, a KR module is a minimal irreducible module for the quantum affine algebra, in the sense of [7], having highest weight a multiple of a fundamental weight. The graded KR modules are the so called graded limits of the original KR modules, in the sense of [33]. If i is the node of the Dynkin diagram of $\mathfrak{g}$ associated to this fundamental weight, the graded KR module is isomorphic to $CV(\xi)$ where $\xi(\alpha)$ is exactly that for $W(\lambda)$ except when α is the corresponding simple root α_i, in which case, $\xi(\alpha_i)$ has just one part as in the case of $V(\lambda)$. The minimality property of the KR modules can be interpreted informally by saying that the KR module is very close to being irreducible. Indeed, if $\mathfrak{g}$ is simply laced and the fundamental weight is minuscule, the KR module is indeed irreducible (a fact used crucially in the proof of Theorem A). In [36], it was given a presentation of the fusion product of KR-modules in terms of generators and relations. This is also used crucially in the proof of Theorem A. Moreover, it allows us to take a further step towards proving Theorem A with no hypothesis on $\mathfrak{g}$ and letting λ be a multiple of any fundamental weight. In this work, we consider only the case that the fundamental weight is “minimal” (in some sense) among those which are not minuscule, and prove Proposition 1.6.6 which says that the corresponding truncated Weyl module, under the hypothesis of the above conjecture, is isomorphic to a quotient of a fusion product of KR modules by introducing a very simple relation. We expect to be able to extend Proposition 1.6.6 to other fundamental weights before submitting the results of this Thesis for publication [23].

The text is divided in three chapters. In the first chapter, we give a briefly review on simple Lie algebras, current algebras and their representations. We also define fusion products, CV modules, and KR modules and state our mains results more precisely. The proofs of the main results are given in the second chapter, which also include further results and properties of fusion products and CV modules which are needed in the arguments. In the third chapter, we discuss the aforementioned results related to Demazure flags for $\mathfrak{g} = \mathfrak{sl}_2$, including a discussion about chains of inclusions of truncated Weyl modules.

1 Background and the Main Results

In this chapter we review the basic definitions and background needed to state the main results of this work. Let $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Z}_{\geqslant 0}$ and $\mathbb{Z}_{>0}$ denote the sets of complex numbers, integers, nonnegative and positive integers, respectively.

1.1 Simple Lie Algebras

We start fixing the basic notation and main results about simple Lie algebras that will use in this thesis. For more details about this section see [42, Chapters 1-10], [27, Chapters I-V].

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$, fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ as well as a Borel subalgebra $\mathfrak{b} \supseteq \mathfrak{h}$. Let $R \subseteq \mathfrak{h}^*$ (respectively, R^+) be the set of roots (respectively, the set of positive roots) corresponding to these choices and denote by $\prod = \{\alpha_1, \dots, \alpha_n\}$ the corresponding set of simple roots. Let also $\omega_1, \dots, \omega_n$ denote the corresponding fundamental weights. For convenience, set $I = \{1, \dots, n\}$. If $\alpha = \sum\limits_i n_i\alpha_i \in R$ then $ht\alpha = \sum\limits_i n_i$ is the *height* of α. Let θ be the highest root of R and let ϑ be the highest short root (we use the convention that, for simply laced $\mathfrak{g}$, all roots are short and long at the same time). The root and weight lattices and their positive cones will be denoted by Q, Q^+, P, P^+, respectively, namely, $Q = \sum\limits_i \mathbb{Z}\alpha_i$, $Q^+ = \sum\limits_i \mathbb{Z}_{\geqslant 0}\alpha_i$, $P = \sum\limits_i \mathbb{Z}\omega_i$, $P^+ = \sum\limits_i \mathbb{Z}_{\geqslant 0}\omega_i$.

The restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$ induces an isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$ and a symmetric non-degenerate form $(\ ,\ )$ on $\mathfrak{h}^*$. We normalize this form so that the square length of a long root is 2. Given $\alpha \in R$, let $t_\alpha \in \mathfrak{h}$ be the element that maps to α under the aforementioned isomorphism and set $h_\alpha = \dfrac{2t_\alpha}{(\alpha, \alpha)}$. For $\alpha \in R$, let $\mathfrak{g}_\alpha$ be the corresponding root space of $\mathfrak{g}$ and set $\mathfrak{n}^\pm = \sum\limits_{\alpha \in R^+} \mathfrak{g}_{\pm\alpha}$. We have

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+. \tag{1.1.1}$$

Fix elements $x_\alpha^\pm \in \mathfrak{g}_{\pm\alpha}$, such that $[x_\alpha^+, x_\alpha^-] = h_\alpha$. Also, set $x_i^\pm = x_{\alpha_i}^\pm$. In particular, $h_i = h_{\alpha_i}$. Recall that, for all $\alpha \in R^+$, the vector subspace $\mathfrak{sl}_\alpha$ spanned by $x_\alpha^\pm, h_\alpha$ is isomorphic to $\mathfrak{sl}_2$.

Given any Lie algebra $\mathfrak{a}$ over $\mathbb{C}$, we let $U(\mathfrak{a})$ be the universal enveloping algebra of $\mathfrak{a}$.

Theorem 1.1.1. (Poincaré-Birkhoff-Witt (PBW)) For any basis $\{x_j,\, j \in J\}$ of a Lie algebra $\mathfrak{a}$ with ordered index set J, the monomials

$$x_{j_1} \cdots x_{j_n}, \quad j_1 \leqslant \ldots \leqslant j_n$$

with 1 form a basis for the enveloping algebra $U(\mathfrak{a})$.

We give a brief review on the category $\mathcal{O}^{int}$ of integrable representations of $\mathfrak{g}$ in Bernstein-Gelfand-Gelfand's category $\mathcal{O}$. For more details see [6, 28] for example. Consider the decomposition (1.1.1). We say that a $\mathfrak{g}$-module V is an object in the category $\mathcal{O}$ if the following conditions are satisfied:

(i) $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$, where $V_\mu = \{v \in V : \ hv = \mu(h)v, \ \text{ for all } \ h \in \mathfrak{h}\}$;

(ii) $dim\, V_\mu$ is finite for each $\mu \in \mathfrak{h}^*$;

(iii) there exists a finite set $\{\mu_1, \ldots, \mu_s\} \subset \mathfrak{h}^*$ such that each μ with $V_\mu \neq 0$ satisfies $\mu < \mu_i$ for some $i \in \{1, \ldots, s\}$.

The morphisms in category $\mathcal{O}$ are the homomorphisms of $\mathfrak{g}$-modules.

The space V_μ is said to be the weight space of V of weight μ, and the nonzero vectors of V_μ are called weight vectors of weight μ. Let $wt(V) = \{\mu \in P : \ V_\mu \neq 0\}$ be the set of weights of V. Note that

$$\mathfrak{g}_\alpha V_\mu \subseteq V_{\mu+\alpha} \quad \text{for all} \quad \mu \in \mathfrak{h}^*, \alpha \in R. \tag{1.1.2}$$

A weight vector v is said to be a highest-weight vector if $\mathfrak{n}^+ v = 0$. A module which is generated by a highest-weight vector is said to be a highest-weight module.

If V is a highest-weight module of highest weight λ, it follows from the PBW Theorem together with (1.1.2) that V has a unique maximal proper submodule and, hence, a unique irreducible quotient. In that case, $V = U(\mathfrak{n}^-)v$ and, hence $dim\,(V_\lambda) = 1$ and $V_\mu \neq 0$ only if $\mu \leqslant \lambda$.

Definition 1.1.2. Given $\lambda \in \mathfrak{h}^*$, the *Verma module* $M(\lambda)$ is the $\mathfrak{g}$-module generated by a nonzero vector v with defining relations

$$\mathfrak{n}^+ v = 0, \quad hv = \lambda(h)v \quad \text{for all} \quad h \in \mathfrak{h}.$$

In particular, $M(\lambda)$ is a highest-weight module and all highest-weight modules whose highest weight is λ are quotients of $M(\lambda)$. We denote by $V(\lambda)$ the unique irreducible quotient of $M(\lambda)$. The PBW Theorem together with (1.1.2) implies that $M(\lambda)$ is an object in $\mathcal{O}$ for any λ.

A $\mathfrak{g}$-module V is said to be integrable if the elements x_i^+, x_i^-, $i \in I$, act locally nilpotently, i.e., if for any $v \in V$ and $i \in I$, there exists $m \in \mathbb{Z}_{\geqslant 0}$ such that

$(x_i^+)^m v = 0 = (x_i^-)^m v$. In that case, $x_\alpha^\pm$ also act locally nilpotently for all $\alpha \in R^+$. The category $\mathcal{O}^{int}$ is the full subcategory of $\mathcal{O}$ consisting of integrable modules.

Theorem 1.1.3. Let $\lambda \in \mathfrak{h}^*$ and v be a highest-weight vector of $M(\lambda)$. The module $V(\lambda)$ is integrable if and only if $\lambda \in P^+$. In that case, it is the quotient of $M(\lambda)$ by the submodule generated by $(x_i^-)^{\lambda(h_i)+1}v$ for all $i \in I$ and $dim\, V(\lambda) < \infty$.

Theorem 1.1.4. If V is a simple module in category $\mathcal{O}^{int}$, it is isomorphic to $V(\lambda)$ for some $\lambda \in P^+$. Moreover, every object in $\mathcal{O}^{int}$ is a finite direct sum of simple submodules. In particular, $\mathcal{O}^{int}$ is the category of finite-dimensional $\mathfrak{g}$-modules.

1.2 Current Algebras

In this section we review the basics about finite-dimensional representations of truncated current algebras. For more details see [9, 34, 37]. For any Lie algebra $\mathfrak{a}$ and associative commutative algebra A, the vector space $\mathfrak{a} \otimes A$ can be equipped with a Lie algebra structure by setting

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab \qquad \text{for all} \qquad x, y \in \mathfrak{a}, \quad a, b \in A.$$

If A has an identity element, then the subspace $\mathfrak{a}\otimes 1$ is a Lie subalgebra of $\mathfrak{a}\otimes A$ isomorphic to $\mathfrak{a}$. Hence, we identify $\mathfrak{a}$ with this subalgebra.

In the case that $A = \mathbb{C}[t]$ is the polynomial ring in one variable, this algebra is called the *current algebra* over $\mathfrak{a}$ and will be denote by $\mathfrak{a}[t]$ and if $A = \dfrac{\mathbb{C}[t]}{t^N\mathbb{C}[t]}$ for some $N \in \mathbb{Z}_{\geqslant 0}$, the algebra $\mathfrak{a} \otimes A$ is called the *truncated current Lie algebra* of nilpotence index N and will be denoted by $\mathfrak{a}[t]_N$.

For simplicity, given the goals of this work, assume A is quotient of $\mathbb{C}[t]$, say, $A = \dfrac{\mathbb{C}[t]}{\mathcal{J}}$ for some ideal $\mathcal{J}$. Given a maximal ideal $\mathcal{M}$ of A, let $ev_{\mathcal{M}} : A \to \mathbb{C}$ be the composition of the canonical projection $A \to A/\mathcal{M}$ with the isomorphism

$$\begin{aligned} A/\mathcal{M} &\to \mathbb{C} \\ \bar{1} &\mapsto 1. \end{aligned}$$

Since

$$\mathcal{J} = \prod_{k=1}^{m} \mathcal{J}_k \qquad \text{with} \qquad \mathcal{J}_k = \mathcal{M}_k^{N_k}$$

for some $m \geqslant 0$, $\mathcal{M}_k \in specm(\mathbb{C}[t])$, $N_k > 0$, we have, by the *Chinese Remainder Theorem*,

$$A = \frac{\mathbb{C}[t]}{\mathcal{J}} \cong \bigoplus_{k=1}^{m} \frac{\mathbb{C}[t]}{\mathcal{M}_k^{N_k}}. \tag{1.2.1}$$

Moreover, $\mathcal{M}_k = (t - a_k)\mathbb{C}[t]$ for some $a_k \in \mathbb{C}$, $specm(A) = \{\overline{\mathcal{M}_1}, \ldots, \overline{\mathcal{M}_m}\}$, and

$$ev_{\overline{\mathcal{M}_k}}(\overline{f(t)}) = f(a_k).$$

By abuse of notation, we identify $specm(A)$ with $\{a_1, \ldots, a_m\}$ and write ev_a instead of $ev_{\mathcal{M}}$ if $\mathcal{M} = (t - a)\mathbb{C}[t]$.
Given $a \in specm(A)$ we also denote by ev_a the Lie algebras homomorphism defined by $id_{\mathfrak{a}} \otimes ev_a : \mathfrak{a} \otimes A \to \mathfrak{a}$, i.e.,

$$ev_a(x \otimes \overline{f(t)}) = f(a)x.$$

To simplify notation, set $\mathfrak{a}[t]_{\mathcal{J}} = \mathfrak{a} \otimes A$. Thus, if V is an $\mathfrak{a}$-module, we can consider the $\mathfrak{a}[t]_{\mathcal{J}}$-module V_a obtained by pulling-back the action of $\mathfrak{a}$ to one of $\mathfrak{a}[t]_{\mathcal{J}}$ via ev_a. Modules of this form are called evaluation modules. Notice that V_a is simple if and only if V is simple.

If $\mathfrak{a} = \mathfrak{g}$, given $\lambda \in P^+$, we will denote by $V_a(\lambda)$ the corresponding evaluation module. Since $V(\lambda)$ is generated by a vector v satisfying

$$\mathfrak{n}^+ v = 0 \quad \text{and} \quad hv = \lambda(h)v \quad \text{for all} \quad h \in \mathfrak{h},$$

when v is regarded as an element of $V_a(\lambda)$, we have

$$(\mathfrak{n}^+ \otimes A)v = 0 \qquad \text{and} \qquad (h \otimes t^r)v = a^r \lambda(h)v \quad \text{for all} \quad h \in \mathfrak{h}, r \in \mathbb{Z}_{\geqslant 0}.$$

Given $k \geqslant 0, \lambda_1, \ldots, \lambda_k \in P^+ \backslash \{0\}$, and $a_1, \ldots, a_k \in specm(A)$, it is well-known (see [34] and reference therein) that

$$V_{a_1}(\lambda_1) \otimes \cdots \otimes V_{a_k}(\lambda_k) \quad \text{is irreducible} \quad \Leftrightarrow \quad a_i \neq a_j \quad \text{for} \quad i \neq j.$$

Moreover, every irreducible finite-dimensional $\mathfrak{g}[t]_{\mathcal{J}}$-module is isomorphic to a unique tensor product of this form.

Let $\Xi_{\mathcal{J}}$ be the set of functions from $specm(A)$ to P^+ with finite support, where the support of $\pi \in \Xi_{\mathcal{J}}$ is

$$supp(\pi) = \{a \in specm(A) : \pi(\mathcal{M}) \neq 0\}.$$

We let $\Xi = \Xi_0$ which is identified with the set of functions with finite support from $\mathbb{C}$ to P^+. Since $specm(A)$ can be naturally identified with a subset of $\mathbb{C}$ as mentioned after (1.2.1), we can and will regard an element of $\Xi_{\mathcal{J}}$ as an element of Ξ by extending it to be zero outside $specm(A)$. Given $\pi, \pi' \in \Xi$, let $\pi + \pi'$ be defined by

$$(\pi + \pi')(a) = \pi(a) + \pi'(a), \qquad a \in \mathbb{C}.$$

In particular,

$$supp(\pi + \pi') = supp(\pi) \cup supp(\pi').$$

Given $\pi \in \Xi_{\mathcal{J}}$ and an enumeration $a_1, \ldots, a_k$ of $supp(\pi) \subseteq specm(A)$, let

$$V(\pi) = V_{a_1}(\lambda_1) \otimes \cdots \otimes V_{a_k}(\lambda_k). \tag{1.2.2}$$

Evidently, any other enumeration of $supp(\pi)$ gives rise to an isomorphic tensor product. Thus, we have:

Theorem 1.2.1. The assignment $\pi \mapsto V(\pi)$ induces a bijection between $\Xi_{\mathcal{J}}$ and the set of isomorphism classes of finite-dimensional simple $\mathfrak{g}[t]_{\mathcal{J}}$-modules.

It is also interesting to note that if $a_1, \ldots, a_k$, $\lambda_1, \ldots, \lambda_k$ are as in (1.2.2), $v_j \in V(\lambda_j)_{\lambda_j} \setminus \{0\}$ for $1 \leqslant j \leqslant k$, and $v = v_1 \otimes \cdots \otimes v_k$, then

$$\mathfrak{n}^+[t]v = 0, \quad (h \otimes \overline{f(t)})v = \left(\sum_{j=1}^{k} \lambda_j(h) f(a_j) \right) v \quad \text{and} \quad (x_\alpha^-)^{\lambda h_\alpha)+1} v = 0, \tag{1.2.3}$$

for all $\alpha \in R^+$, where $\lambda = \sum_{j=1}^{k} \lambda_j$. Note that if $k = 1$, i.e., if $\mathcal{J} = \mathcal{M}^N$ for some $\mathcal{M} \in specm(\mathbb{C}[t])$ and $N > 0$, then $\Xi_{\mathcal{J}}$ is naturally identified with P^+. In other words, for each $\lambda \in P^+$, there exists a unique finite dimensional simple $\mathfrak{g}[t]_{\mathcal{J}}$-module generated by a vector v satisfying $\mathfrak{n}^+[t]v = 0$ and $hv = \lambda(h)v$ for all $h \in \mathfrak{h}$.

As a consequence of (1.2.1), we have

$$\mathfrak{g} \otimes \frac{\mathbb{C}[t]}{\mathcal{J}} \cong \mathfrak{g} \otimes \bigoplus_{k=1}^{m} \frac{\mathbb{C}[t]}{\mathcal{M}_k^{N_k}} \cong \bigoplus_{k=1}^{m} \mathfrak{g} \otimes \frac{\mathbb{C}[t]}{\mathcal{M}_k^{N_k}} \cong \bigoplus_{k=1}^{m} \mathfrak{g}[t]_{N_k}.$$

Moreover, for each $a \in \mathbb{C}$, the automorphism ζ_a of $\mathbb{C}[t]$, $t \mapsto t + a$, induces an isomorphism of algebras: $\dfrac{\mathbb{C}[t]}{(t-a)^N \mathbb{C}[t]} \to \dfrac{\mathbb{C}[t]}{t^N \mathbb{C}[t]}$. Because of this, from now on we focus on $A = \dfrac{\mathbb{C}[t]}{t^N \mathbb{C}[t]}$ and write $\mathfrak{g}[t]_N$ as before. It will be convenient to set $\mathfrak{g}[t]_\infty = \mathfrak{g}[t]$.

Note that $\mathfrak{g}[t]_N$ is $\mathbb{Z}$-graded and that the evaluation module $V_0(\lambda)$ is also $\mathbb{Z}$-graded. We shall denote the k-th graded piece of a graded vector space V by $V[k]$. It will be convenient to introduce the notation

$$V_+ = \bigoplus_{k>0} V[k].$$

Denote $\mathcal{G}_N$ the category of graded finite-dimensional $\mathfrak{g}[t]_N$-modules, where the morphisms are those preserving grades. Set $\mathcal{G} = \mathcal{G}_\infty$. Since we have a surjective Lie algebra map $\mathfrak{g}[t] \to \mathfrak{g}[t]_N$, every object from $\mathcal{G}_N$ can be regarded as an object in $\mathcal{G}$. For $m \in \mathbb{Z}$, we consider the grade-shift functor τ_m which does not change the given action on an object and shifts the grades by the rule

$$\tau_m(V)[k] = V[k-m].$$

Set

$$V(\lambda, m) = \tau_m(V_0(\lambda))$$

which can be regarded both as a $\mathfrak{g}[t]$-module as well as a $\mathfrak{g}[t]_N$-module. It follows that the assignment $(\lambda, m) \mapsto V(\lambda, m)$ induces a bijection between $P^+ \times \mathbb{Z}$ and the set of isomorphism classes of simple objects in $\mathcal{G}_N$ (including $N = \infty$).

1.3 Weyl Modules

Chari and Pressley introduced in [14] a family of finite dimensional representations, called Weyl modules, for the affine Kac-Moody algebras and their quantized versions. These modules were introduced inspired by the modular representation theory of algebraic groups. The notion can be generalized for Lie algebras of the form $\mathfrak{g} \otimes A$ and the study of these modules in the case of truncated algebras is the main goal of this work. We proceed with a review of the background we shall need about them. For more details see [9].

Observe that (1.1.1) implies

$$\mathfrak{g} \otimes A = \mathfrak{n}^- \otimes A \oplus \mathfrak{h} \otimes A \oplus \mathfrak{n}^+ \otimes A. \tag{1.3.1}$$

We consider the highest-weight theory associated to the decomposition (1.3.1). For more details see [9, 37, 44].

Definition 1.3.1. A nonzero vector $v \in V$ is a *highest-weight vector* with respect to (1.3.1) if

(i) $(\mathfrak{n}^+ \otimes A)v = 0$;

(ii) there exists $\Lambda \in (\mathfrak{h} \otimes A)^*$ such that $(h \otimes a)v = \Lambda(h \otimes a)v$, for all $(h \otimes a) \in \mathfrak{h} \otimes A$.

The functional $\Lambda \in (\mathfrak{h} \otimes A)^*$ satisfying Definition 1.3.1 is called the *highest weight* of v. A $(\mathfrak{g} \otimes A)$-module V is said to be *highest-weight* module (of highest weight Λ) if it is generated by a highest-weight vector (of highest weight Λ). If $V = U(\mathfrak{g} \otimes A)v$ is a highest-weight module with highest-weight vector v, it follows from the PBW Theorem that $V = U(\mathfrak{n}^- \otimes A)v$.

Let $A = \dfrac{\mathbb{C}[t]}{\mathcal{J}}$, as in the previous section. Given $\Lambda \in (\mathfrak{h}[t]_{\mathcal{J}})^*$, let $M(\Lambda)$ be the $\mathfrak{g}[t]_{\mathcal{J}}$-module generated by a vector v satisfying the defining relations $\mathfrak{n}^+[t]v = 0$ and $xv = \Lambda(h)v$ for all $x \in \mathfrak{h}[t]_{\mathcal{J}}$. Thus, $M(\Lambda)$ is the Verma module of highest weight Λ with respect to the decomposition (1.3.1). Denote by $V(\Lambda)$ the irreducible quotient of $M(\Lambda)$. Note that, by Theorem 1.2.1,

$$V(\Lambda) \quad \text{is finite dimensional} \quad \Leftrightarrow \quad V(\Lambda) \cong V(\pi) \quad \text{for some} \quad \pi \in \Xi_{\mathcal{J}}.$$

It then follows from (1.2.3) that:

Proposition 1.3.2. The module $V(\Lambda)$ is finite-dimensional if and only if there exist $k \geqslant 0$, $a_1, \ldots, a_k \in \mathbb{C}$, $\lambda_1, \ldots, \lambda_k \in P^+$ such that

$$\Lambda(h \otimes \bar{t}^r) = \sum_{j=1}^{k} \lambda_j(h) a_j^r \quad \text{for all} \quad h \in \mathfrak{h}, r \geqslant 0.$$

In other words, we can identify $\Xi_{\mathcal{J}}$ with a subset of $(\mathfrak{h}[t]_{\mathcal{J}})^*$. Let $\mathrm{wt} : \Xi_{\mathcal{J}} \to P^+$ be defined by

$$\mathrm{wt}(\pi) = \sum_{a \in supp(\pi)} \pi(a).$$

Suppose V is a finite-dimensional quotient of $M(\pi)$ and let $v \in V_{\mathrm{wt}(\pi)}$. Then, $U(\mathfrak{sl}_2(\alpha))v$ is finite-dimensional for all $\alpha \in R^+$ and, hence,

$$(x_\alpha^-)^{\lambda(h_\alpha)+1} v = 0, \quad \text{for all} \quad \alpha \in R^+.$$

Definition 1.3.3. Let $\pi \in \Xi_{\mathcal{J}}$. The *Weyl module* $W_{\mathcal{J}}(\pi)$ is the quotient of $M(\pi)$ by the submodule generated by $(x_i^-)^{\mathrm{wt}(\pi)+1} v$, $i \in I$.

For a proof of the next theorem see [9, 34].

Theorem 1.3.4. For every $\pi \in \Xi_{\mathcal{J}}$, $W_{\mathcal{J}}(\pi)$ is finite-dimensional.

Hence, it follows from the comments preceding Definition 1.3.3 that $W_{\mathcal{J}}(\pi)$ is the universal finite-dimensional highest-weight $\mathfrak{g}[t]_{\mathcal{J}}$-module with highest weight π. If $\mathcal{J} = 0$, we denote it simply by $W(\pi)$. It was proved in [9] that

$$W_{\mathcal{J}}(\pi + \pi') \cong W_{\mathcal{J}}(\pi) \otimes W_{\mathcal{J}}(\pi') \quad \text{for all} \quad \pi, \pi' \in \Xi_{\mathcal{J}} \text{ s.t. } supp(\pi) \cap supp(\pi') = \varnothing. \tag{1.3.2}$$

The automorphism ζ_a of $\mathbb{C}[t]$ induces an equivalence between the categories of finite-dimensional $\mathfrak{g}[t]_N$ modules and $\mathfrak{g}[t]_{\mathcal{J}}$-modules with $\mathcal{J} = (t-a)^N \mathbb{C}[t]$. Moreover, Weyl modules are mapped to Weyl modules. This fact, together with (1.3.2), allows us to focus on the case $\mathcal{J} = t^N \mathbb{C}[t]$ which we do henceforth. Set $\Xi_N = \Xi_{\mathcal{J}}$ and $W_N(\pi) = W_{\mathcal{J}}(\pi)$ for $\mathcal{J} = t^N \mathbb{C}[t]$, $\pi \in \Xi_{\mathcal{J}}$. Note that, for all $\pi \in \Xi_N$,

$$\pi(h \otimes t^k) = \mathrm{wt}(\pi)(h)\delta_{k,0}$$

and, hence, $W_N(\pi)$ is $\mathbb{Z}$-graded. Since Ξ_N is naturally identified with P^+ as mentioned before, for $\lambda \in P^+$ we set $W_N(\lambda) := W_N(\pi)$ where π is the unique element of Ξ_N such that $\mathrm{wt}(\pi) = \lambda$. Since every $\mathfrak{g}[t]_N$-module can be naturally regarded as a $\mathfrak{g}[t]$-module, the universal property of $W(\lambda)$ immediately implies that we have an epimorphism of $\mathfrak{g}[t]$-modules:

$$W(\lambda) \twoheadrightarrow W_N(\lambda). \tag{1.3.3}$$

Moreover, set $L_{N,\lambda}$ the submodule of $W(\lambda)$ generated by $(x \otimes t^N)v$ for $x \in \mathfrak{n}^-$, i.e.,

$$L_{N,\lambda} = \{U(\mathfrak{g}[t])(x \otimes t^N)v, \ \forall \, x \in \mathfrak{n}^-\}. \tag{1.3.4}$$

Observe that, when regarded as a $\mathfrak{g}[t]$-module, $W_N(\pi)$ is isomorphic to a module generated by a vector v satisfying the Definition 1.3.3 together with

$$(x \otimes t^r)v = 0 \quad \text{for all} \quad x \in \mathfrak{n}^-, \ r \geqslant N.$$

Hence, $L_{N,\lambda}$ is the kernel of the epimorphism (1.3.3), and we have the following isomorphism of $\mathfrak{g}[t]$-modules

$$\frac{W(\lambda)}{L_{N,\lambda}} \cong W_N(\lambda).$$

Thus, we can define the *truncated Weyl module* $W_N(\lambda)$ directly as a $\mathfrak{g}[t]$-module as the module generated by a nonzero vector v_N with the following defining relations

$$\mathfrak{n}^+[t]v_N = 0; \ \ (h \otimes t^k)v_N = \delta_{0,k}\lambda(h)v_N, \ \forall \, h \in \mathfrak{h};$$

$$(x_\alpha^-)^{\lambda(h_\alpha)+1}v_N = 0 = (x_\alpha^- \otimes t^N)v_N, \ \forall \, \alpha \in R^+.$$

In particular, we have epimorphisms of $\mathfrak{g}[t]$-modules

$$W_N(\lambda) \twoheadrightarrow W_{N'}(\lambda) \quad \text{if} \quad N \geqslant N'. \tag{1.3.5}$$

Also, it follows from the proof of Theorem 1.3.4 that, for all $\alpha \in R^+$,

$$(x_\alpha^- \otimes t^r)v = 0 \quad \text{if} \quad r \geqslant \lambda(h_\alpha). \tag{1.3.6}$$

Hence,

$$W_N(\lambda) \cong W(\lambda) \quad \text{if} \quad N \geqslant \lambda(h_\vartheta). \tag{1.3.7}$$

Note that, if $\mathfrak{g}$ is simply laced, all the roots have the same length and the condition in (1.3.7) becomes $N \geqslant \lambda(h_\theta)$.

1.4 Examples

We will now present some properties and examples of local Weyl modules and truncated Weyl modules when $\mathfrak{g}$ is $\mathfrak{sl}_2$. Let $\{x^-, h, x^+\}$ be a basis of $\mathfrak{g}$ such that $[h, x^-] = -2x^-, [x^+, x^-] = h, [h, x^+] = 2x^+$. Moreover, we are considering $m \in \mathbb{Z}_{\geqslant 0}$, ω the fundamental weight, w a generator of the local Weyl module $W(m\omega)$ and v a generator of the truncated Weyl module $W_N(m\omega)$ for some $N \in \mathbb{Z}_{>0}$.
By (1.3.6),

$$(x^- \otimes t^m)w = 0. \tag{1.4.1}$$

The first property tells us that the submodule of $W(m\omega)$ generated by $(x^- \otimes t^{m-1})w$ can be realized as a quotient of the local Weyl module $W((m-2)\omega)$.

Lemma 1.4.1. If $m > 1$, then $U(\mathfrak{g}[t])(x^- \otimes t^{m-1})w$ is isomorphic to a quotient of the local Weyl module $W((m-2)\omega)$.

Proof. It is enough to show that $(x^- \otimes t^{m-1})w$ satisfies the defining relations of $W((m-2)\omega)$.

(i) Given $k \in \mathbb{Z}_{\geqslant 0}$, since w is a generator of $W(m\omega)$, we have

$$(x^+ \otimes t^k)(x^- \otimes t^{m-1})w = (h \otimes t^{m-1+k})w + (x^- \otimes t^{m-1})(x^+ \otimes t^k)w = 0.$$

Therefore $\mathfrak{n}^+[t](x^- \otimes t^{m-1})w = 0$.

(ii)

$$\begin{aligned}(h \otimes 1)(x^- \otimes t^{m-1})w &= -2(x^- \otimes t^{m-1})w + (x^- \otimes t^{m-1})(h \otimes 1)w \\ &= -2(x^- \otimes t^{m-1})w + (x^- \otimes t^{m-1})mw \\ &= (m-2)(x^- \otimes t^{m-1})w.\end{aligned}$$

(iii) For $k \geqslant 1$,

$$(h \otimes t^k)(x^- \otimes t^{m-1})w = -2(x^- \otimes t^{m-1+k})w + (x^- \otimes t^{m-1})\underbrace{(h \otimes t^k)w}_{0} = -2(x^- \otimes t^{m-1+k})w.$$

By (1.4.1), $(x^- \otimes t^m)w = 0$ and since $m - 1 + k \geqslant m$, it follows

$$(x^- \otimes t^{m-1+k})w = -\frac{1}{2}\left((h \otimes t^{k-1})(x^- \otimes t^m) - (x^- \otimes t^m)(h \otimes t^{k-1})\right) w = 0.$$

Therefore $(h \otimes t^k)(x^- \otimes t^{m-1})w = 0$.

(iv) We need to prove that $(x^- \otimes 1)^{m-1}(x^- \otimes t^{m-1})w = 0$. Note that $(x^- \otimes t^{m-1})w \in W((m)\omega)_{(m-2)\omega}\backslash\{0\}$ and by (i), $\mathfrak{n}^+[t](x^- \otimes t^{m-1})w = 0$, then $(x^- \otimes 1)^{(m-2)+1}(x^- \otimes t^{m-1})w = 0$.

□

Lemma 1.4.2. If $\lambda \in P^+\backslash\{0\}$, then $x^- w_\lambda \neq 0$, where $w_\lambda \in W(\lambda)_\lambda$.

Proof. Let $V_0(\lambda)$ be the evaluation $\mathfrak{g}[t]$-module constructed from $V(\lambda)$, the irreducible highest-weight module of highest weight λ, and $v_0 \in V_0(\lambda)_\lambda$. Clearly v_0 satisfies the defining relations of $W(\lambda)$. Then there exists a surjective homomorphism $W(\lambda) \to V_0(\lambda)$, which maps w_λ to v_0. Since $V_0(\lambda) = U(\mathfrak{n}^-[t])v_0$ implies that $x^- v_0 \neq 0$ and consequently $x^- w_\lambda \neq 0$. □

Remark 1.4.3. Note that (1.4.1) and the Lemmas 1.4.1 and 1.4.2 are valid if instead of the local Weyl modules we consider the truncated Weyl modules.

In the following we give some examples of local Weyl modules and truncated Weyl modules. Each module will be associated to a diagram that we will now describe:
- Each vertex corresponds to a generator of the module. The vertices are distributed in lines and columns.
- Each line corresponds to the degree of the generator according to the gradation of the Lie algebra and each column corresponds to a weight of the generator.

Example 1.4.4.

$$W(\omega) = U(\mathfrak{g}[t])w; \;\; W_N(\omega) = U(\mathfrak{g}[t])v.$$

By definition, w and v satisfy:

$$(x^-)^2 w = 0 = (x^-)^2 v; \;\; (h \otimes t^k)w = 0 = (h \otimes t^k)v, \; \forall\, k \geqslant 1;$$

$$\mathfrak{n}^+[t]w = 0 = \mathfrak{n}^+[t]v; \;\; hw = w \; \text{ and } \; hv = v.$$

Then $W(\omega) = \mathbb{C}w \oplus \mathbb{C}(x^- w) \simeq W_N(\omega)$, as $U(\mathfrak{g}[t])$-module and we have the following diagram:

$$\bullet \quad \bullet$$

$\diamond$

Example 1.4.5.

$$W(2\omega) = U(\mathfrak{g}[t])w; \;\; W_N(2\omega) = U(\mathfrak{g}[t])v.$$

By definition, w and v satisfy:

$$(x^-)^3 w = 0 = (x^-)^3 v; \;\; (h \otimes t^k)w = 0 = (h \otimes t^k)v, \; \forall\, k \geqslant 1;$$

$$\mathfrak{n}^+[t]w = 0 = \mathfrak{n}^+[t]v; \;\; hw = 2w \; \text{ and } \; hv = 2v.$$

By (1.4.1), $(x^- \otimes t^2)w = 0$ and

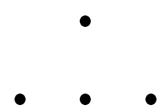
(i) $(x^- \otimes t^3)w = \dfrac{1}{2}[x^- \otimes t^2, h \otimes t]w = 0$

(ii) $(x^- \otimes t)x^- w = 0 = (x^- \otimes t)^2 w$, in fact,
$(x^+ \otimes t)(x^-)^3 w = 0$
$\Rightarrow (h \otimes t + x^-(x^+ \otimes t))(x^-)^2 w = 0$
$\Rightarrow (-2x^- \otimes t + x^-(h \otimes t) + x^-(h \otimes t) + (x^-)^2(x^+ \otimes t))x^- w = 0$
$\Rightarrow (-2(x^- \otimes t)x^- - 4x^-(x^- \otimes t) + 2(x^-)^2(h \otimes t) + (x^-)^2(h \otimes t) + (x^-)^3(x^+ \otimes t))w = 0$
$\Rightarrow -6x^-(x^- \otimes t)w = 0$
$\Rightarrow (x^- \otimes t)x^- w = 0.$
On the other hand, $(x^- \otimes t^2)w = 0$, hence $-\dfrac{1}{2}[(h \otimes t, x^- \otimes t]w = 0$
$\Rightarrow \left(-\dfrac{1}{2}(h \otimes t)(x^- \otimes t) + \dfrac{1}{2}(x^- \otimes t)(h \otimes t)\right) w = 0$

$$\Rightarrow x^- \left(-\frac{1}{2}(h \otimes t)(x^- \otimes t) + \frac{1}{2}(x^- \otimes t)(h \otimes t) \right) w = 0$$
$$\Rightarrow -(x^- \otimes t)^2 w = 0$$
$$\Rightarrow (x^- \otimes t)^2 w = 0.$$

(iii) For $i_1 \geqslant i_0$, $(x^- \otimes t^{i_1})(x^- \otimes t^{i_0})w = (x^- \otimes t^{i_0})(x^- \otimes t^{i_1})w$.
If $i_0 \geqslant 2$ or $i_1 \geqslant 2$, then $(x^- \otimes t^{i_1})(x^- \otimes t^{i_0})w = 0$.
If $i_0 = 0 = i_1 \Rightarrow (x^- \otimes t^{i_1})(x^- \otimes t^{i_o})w = (x^-)^2 w$.
If $i_0 = 1 = i_1 \Rightarrow (x^- \otimes t^{i_1})(x^- \otimes t^{i_0})w = (x^- \otimes t)^2 w = 0$.
If $i_0 = 0$ and $i_1 = 1 \Rightarrow (x^- \otimes t)x^- w = 0$.
By Lemma 1.4.2, $x^- w \neq 0$.

By (1.3.7), for all $N > 1$, the truncated Weyl module $W_N(2\omega)$ is isomorphic to the local Weyl module $W(2\omega)$. Then $W(2\omega) = \mathbb{C}w \oplus \mathbb{C}(x^- w) \oplus \mathbb{C}(x^-)^2 w \oplus \mathbb{C}(x^- \otimes t)w \simeq W_N(2\omega)$, as $U(\mathfrak{g}[t])$-module and we have the following diagram:



For $N = 1$ we have $W_1(2\omega) = \mathbb{C}v \oplus \mathbb{C}(x^- v) \oplus \mathbb{C}(x^-)^2 v$ and the following diagram:

$\diamond$

Example 1.4.6.

$$W(3\omega) = U(\mathfrak{g}[t])w; \;\; W_N(3\omega) = U(\mathfrak{g}[t])v$$

By definition, w and v satisfy:

$$(x^-)^4 w = 0 = (x^-)^4 v; \;\; (h \otimes t^k)w = 0 = (h \otimes t^k)v, \; \forall\, k \geqslant 1;$$

$$\mathfrak{n}^+[t]w = 0 = \mathfrak{n}^+[t]v; \;\; hw = 3w \text{ and } hv = 3v.$$

By (1.4.1), $(x^- \otimes t^3)w = 0$ and for $2 \geqslant i_1 \geqslant i_0 \geqslant 0$,

$$(x^- \otimes t^{i_1})(x^- \otimes t^{i_0})w = (x^- \otimes t^{i_0})(x^- \otimes t^{i_1})w.$$

We have $(x^- \otimes t)(x^-)^2 w = 0 = (x^- \otimes t^2)(x^-)^2 w$. In fact, $(x^+ \otimes t)(x^-)^4 w = 0$
$\Rightarrow (-2(x^- \otimes t)x^- - 4x^-(x^- \otimes t) + 2(x^-)^2(h \otimes t) + (x^-)^2(h \otimes t) + (x^-)^3(x^+ \otimes t))x^- w = 0$
$\Rightarrow (-6(x^-)^2(x^- \otimes t)w + 3(x^-)^2(h \otimes t)x^- w + (x^-)^3(x^+ \otimes t)x^- w = 0$
$\Rightarrow -12(x^- \otimes t)(x^-)^2 w = 0$
$\Rightarrow (x^- \otimes t)(x^-)^2 w = 0.$
On the other hand, $(x^+ \otimes t^2)(x^-)^4 w = 0$, hence
$-6(x^-)^2(x^- \otimes t^2)w + 3(x^-)^2(h \otimes t^2)x^- w + (x^-)^3(x^- \otimes t^2)x^- w = 0$

$\Rightarrow -12(x^- \otimes t^2)(x^-)^2 w = 0$
$\Rightarrow (x^- \otimes t^2)(x^-)^2 w = 0.$
By Lemma 1.4.2, $x^- w \neq 0$. For $N > 2$,

$$W_N(3\omega) \simeq W(3\omega) = \mathbb{C}w \oplus \mathbb{C}(x^- w) \oplus \mathbb{C}(x^-)^2 w \oplus \mathbb{C}(x^-)^3 w \oplus \mathbb{C}(x^- \otimes t)w \oplus \mathbb{C}(x^- \otimes t^2)w \oplus \mathbb{C}(x^- \otimes t)x^- w \oplus \mathbb{C}(x^- \otimes t^2)x^- w$$

and we have the following diagram:

$$\begin{array}{cccc} & \bullet & \bullet & \\ & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \bullet \end{array}$$

However, for $N = 1$ or $N = 2$ we have some differences. If $N = 2$, then $(x^- \otimes t^2)v = 0$, $W_2(3\omega) \simeq \dfrac{W(3\omega)}{U(\mathfrak{g}[t])(x^- \otimes t^2)w}$, therefore $W_2(3\omega) = \mathbb{C}v \oplus \mathbb{C}(x^- v) \oplus \mathbb{C}(x^-)^2 v \oplus \mathbb{C}(x^-)^3 v \oplus \mathbb{C}(x^- \otimes t)v \oplus \mathbb{C}(x^- \otimes t)x^- v$ and we have the diagram:

$$\begin{array}{cccc} & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \bullet \end{array}$$

If $N = 1$ then $(x^- \otimes t)v = 0$, $W_2(3\omega) \simeq \dfrac{W(3\omega)}{U(\mathfrak{g}[t])(x^- \otimes t)w}$, therefore $W_1(3\omega) = \mathbb{C}v \oplus \mathbb{C}(x^- v) \oplus \mathbb{C}(x^-)^2 v \oplus \mathbb{C}(x^-)^3 v$ and we have the diagram:

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \end{array}$$

$\diamond$

Example 1.4.7.

$$W(4\omega) = U(\mathfrak{g}[t])w; \;\; W_N(4\omega) = U(\mathfrak{g}[t])v.$$

By definition, w and v satisfy:

$$(x^-)^5 w = 0 = (x^-)^5 v; \;\; (h \otimes t^k)w = 0 = (h \otimes t^k)v, \; \forall\, k \geqslant 1;$$

$$\mathfrak{n}^+[t]w = 0 = \mathfrak{n}^+[t]v; \;\; hw = 4w \text{ and } hv = 4v.$$

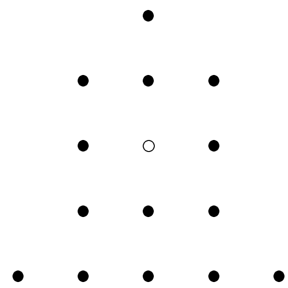
By (1.4.1), $(x^- \otimes t^4)w = 0$ and for $3 \geqslant i_1 \geqslant i_0 \geqslant 0$,

$$(x^- \otimes t^{i_1})(x^- \otimes t^{i_0})w = (x^- \otimes t^{i_0})(x^- \otimes t^{i_1})w.$$

Similarly to the previous examples, we can prove

$$(x^- \otimes t)(x^-)^3 w = 0 = (x^- \otimes t^2)(x^-)^3 w = (x^- \otimes t^3)(x^-)^3 w.$$

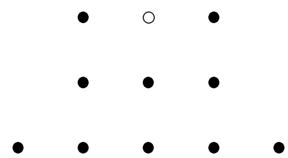
For $N > 3$,
$W_N(4\omega) \simeq W(4\omega) = \mathbb{C}w \oplus \mathbb{C}(x^- w) \oplus \mathbb{C}(x^-)^2 w \oplus \mathbb{C}(x^-)^3 w \oplus \mathbb{C}(x^-)^4 w \oplus \mathbb{C}(x^- \otimes t)w \oplus \mathbb{C}(x^- \otimes t^2)w \oplus \mathbb{C}(x^- \otimes t)x^- w \oplus \mathbb{C}(x^- \otimes t)(x^-)^2 w \oplus \mathbb{C}(x^- \otimes t^3)w \oplus \mathbb{C}(x^- \otimes t^2)(x^-)^2 w \oplus \mathbb{C}(x^- \otimes t^3)(x^-)^2 w \oplus \mathbb{C}(x^- \otimes t^3)x^- w \oplus \mathbb{C}(x^- \otimes t^2)x^- w \oplus \mathbb{C}(x^- \otimes t)^2 w \oplus \mathbb{C}(x^- \otimes t^2)^2 w$ and we have:



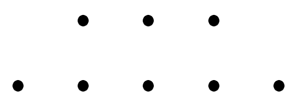
The unique different vertex means that the dimension of this part is two. In this case, we are talking about the submodule generated by $(x^- \otimes t^2)x^- w$ and $(x^- \otimes t)^2 w$.
If $N = 3$ then $(x^- \otimes t^3)v = 0$, $W_3(4\omega) \simeq \dfrac{W(4\omega)}{U(\mathfrak{g}[t])(x^- \otimes t^3)w}$. Moreover, since $(x^- \otimes t^2)(x^-)^3 v = 0$ then

$$
\begin{aligned}
& (x^+ \otimes t)^2 (x^- \otimes t^2)(x^-)^3 v = 0 \\
\Rightarrow\ & (x^+ \otimes t)(h \otimes t^3 + (x^- \otimes t^2)(x^+ \otimes t))(x^-)^3 v = 0 \\
\Rightarrow\ & (x^+ \otimes t)(-2(x^- \otimes t^3)(x^-)^2 + x^-(h \otimes t^3)(x^-)^2 + (x^- \otimes t^2)(h \otimes t + x^-(x^+ \otimes t))(x^-)^2)v = 0 \\
\Rightarrow\ & (x^+ \otimes t)[\underbrace{-2(x^-)^2(x^- \otimes t^3)v}_{0} + \underbrace{x^-(-2x^- \otimes t^3 + x^-(h \otimes t^3))x^- v}_{0} + (x^- \otimes t^2)(-2x^- \otimes t \\
& + x^-(h \otimes t) + x^-(h \otimes t + x^-(x^+ \otimes t)))x^- v] = 0 \\
\Rightarrow\ & (x^+ \otimes t)[-4(x^- \otimes t^2)(x^- \otimes t)x^- v + (x^- \otimes t^2)(-2(x^- \otimes t)x^-]v = 0 \\
\Rightarrow\ & -6(h \otimes t^3 + (x^- \otimes t^2)(x^+ \otimes t))(x^- \otimes t)x^- v = 0 \\
\Rightarrow\ & -6[-2x^- \otimes t^4 + (x^- \otimes t)(h \otimes t^3) + (x^- \otimes t^2)(h \otimes t^2 + (x^- \otimes t)(x^+ \otimes t))]x^- v = 0 \\
\Rightarrow\ & -6x^- \otimes t^2(-2x^- \otimes t^2 + x^-(h \otimes t^2)v = 0 \\
\Rightarrow\ & -12(x^- \otimes t^2)^2 v = 0 \\
\Rightarrow\ & (x^- \otimes t^2)^2 v = 0.
\end{aligned}
$$

Therefore $W_3(4\omega) = \mathbb{C}v \oplus \mathbb{C}(x^- v) \oplus \mathbb{C}(x^-)^2 v \oplus \mathbb{C}(x^-)^3 v \oplus \mathbb{C}(x^-)^4 v \oplus \mathbb{C}(x^- \otimes t)v \oplus \mathbb{C}(x^- \otimes t^2)v \oplus \mathbb{C}(x^- \otimes t)x^- v \oplus \mathbb{C}(x^- \otimes t)(x^-)^2 v \oplus \mathbb{C}(x^- \otimes t^2)(x^-)^2 v \oplus \mathbb{C}(x^- \otimes t^2)x^- v \oplus \mathbb{C}(x^- \otimes t)^2 v$ and we have:



If $N = 2$ then $(x^- \otimes t^2)v = 0$, $W_2(4\omega) \simeq \dfrac{W(4\omega)}{U(\mathfrak{g}[t])(x^- \otimes t^2)w}$, therefore $W_2(4\omega) = \mathbb{C}v \oplus \mathbb{C}(x^- v) \oplus \mathbb{C}(x^-)^2 v \oplus \mathbb{C}(x^-)^3 v \oplus \mathbb{C}(x^-)^4 v \oplus \mathbb{C}(x^- \otimes t)v \oplus \mathbb{C}(x^- \otimes t)x^- v \oplus \mathbb{C}(x^- \otimes t)(x^-)^2 v \oplus \mathbb{C}(x^- \otimes t)^2 v$ and we have:



If $N = 1$ then $(x^- \otimes t)v = 0$, $W_1(4\omega) \simeq \dfrac{W(4\omega)}{U(\mathfrak{g}[t])(x^- \otimes t)w}$, therefore $W_1(4\omega) = \mathbb{C}v \oplus \mathbb{C}(x^- v) \oplus \mathbb{C}(x^-)^2 v \oplus \mathbb{C}(x^-)^3 v \oplus \mathbb{C}(x^-)^4 v$ and we have:

$$\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet$$

◇

1.5 Fusion Products

The notion of fusion products was introduced in [18], and consists of certain operations between cyclic objects in $\mathcal{G}$ closely related to tensor products, providing a very powerful tool to study several objects in $\mathcal{G}$. We now review the definition and a few properties (for more details see [12] and [38]).

Consider the $\mathbb{Z}$-gradation on the universal enveloping algebra $U(\mathfrak{g}[t])$ induced from that of $\mathfrak{g}[t]$. Then, if V is a cyclic $\mathfrak{g}[t]$-module and v generates V, define a filtration on V by

$$F^r V = \sum_{0 \leqslant s \leqslant r} U(\mathfrak{g}[t])[s]v. \tag{1.5.1}$$

For convenience of notation, we set $F^{-1}V$ to be the zero space. The associated graded module $\mathrm{gr}V = \bigoplus_{r \geqslant 0} \dfrac{F^r V}{F^{r-1} V}$ becomes a cyclic $\mathfrak{g}[t]$-module with action given by

$$(x \otimes t^s)(w + F^{r-1}V) = (x \otimes t^s)w + F^{r+s-1}V$$

for all $x \in \mathfrak{g}$, $w \in F^r V$, $r, s \in \mathbb{Z}_{\geqslant 0}$.

Given $a \in \mathbb{C}$, consider the Lie algebra automorphism ζ_a of $\mathfrak{g}[t]$ defined in Section 1.2. Then, given a $\mathfrak{g}[t]$-module V, denote by V_a the pullback of V by ζ_a. Note that, if $V \in \mathcal{G}$ and $a \neq 0$, then V_a is not a graded module. The notation V_a might sound conflicting with that for evaluation modules introduced earlier. However, one easily checks that, if $V = W_0$ for some $\mathfrak{g}$-module W, then the pullback of V by ζ_a is isomorphic to the pullback of W by ev_a, i.e.,

$$V_a \cong W_a. \tag{1.5.2}$$

Thus, the coincidence of notations should not cause confusion. Observe that if $v^a = v$ viewed as an element of V_a, then $x \otimes (t-a)v^a = 0$ if $xv = 0$. Given $k \in \mathbb{Z}_{>0}$, let $a_1, \ldots, a_k$ be a family of distinct complex numbers and let $v_1, \ldots, v_k$ be generators of objects $V^1, \ldots, V^k$ from $\mathcal{G}$, respectively. It was proved in [18] that

$$V^1_{a_1} \otimes \cdots \otimes V^k_{a_k}$$

is generated by $v = v_1^{a_1} \otimes \cdots \otimes v_k^{a_k}$. By abuse of notation, we will often simply write v instead of v^a from now on.

Definition 1.5.1. The *fusion product* of $V^1, \ldots, V^k$ with respect to the parameters $a_1, \ldots, a_k$ is the associated graded module corresponding to the filtration (1.5.1) on $V^1_{a_1} \otimes \cdots \otimes V^k_{a_k}$. It will be denoted by

$$V^1_{a_1} * \cdots * V^k_{a_k}$$

and we notice it that it depends on the choice of cyclic generators too.

Note that we have an isomorphism of $\mathfrak{g}$-modules

$$V^1 \otimes \cdots \otimes V^k \cong_{\mathfrak{g}} V^1_{a_1} * \cdots * V^k_{a_k}.$$

It was conjectured in [18] that, under certain conditions, the fusion product does not actually depend on the choice of the parameters $a_1, \ldots, a_k$. Motivated by this conjecture, it is usual to simplify the notation and write $V^1 * \cdots * V^k$ instead of $V^1_{a_1} * \cdots * V^k_{a_k}$. This conjecture has been proved in some special cases (see [12, 17, 18, 22, 32, 36] and references therein). In all these special cases, each V^j is a quotient of a graded local Weyl module and the cyclic generator v_j is a highest-weight generator. All cases relevant to us are of this form and, hence, we make no further mention about the choice of cyclic generators. In particular, it is known from [22], for simply laced $\mathfrak{g}$, and from [35] for the non-simply laced case, that we have an isomorphism of graded $\mathfrak{g}[t]$-modules:

$$W(\lambda_1) * \cdots * W(\lambda_k) \cong W(\lambda) \quad \text{if} \quad \lambda = \lambda_1 + \cdots + \lambda_k. \tag{1.5.3}$$

Inspired by (1.5.3), for all $\lambda \in P^+$, we have

$$W(\lambda) \cong W(\omega_1)^{*\lambda(h_1)} * \cdots * W(\omega_n)^{*\lambda(h_n)}. \tag{1.5.4}$$

Inspired in (1.5.4), we want to obtain a similar decomposition for the truncated Weyl modules. We now recall a conjectural generalization of (1.5.4) for truncated Weyl modules stated in [32] which is the subject of the main result of this work. This conjecture was motivated by previous work of Chari, Fourier, and Sagaki on Schur positivity [10].

Given $\lambda \in P^+$ and $N \in \mathbb{Z}_{>0}$, let $P^+(\lambda, N)$ be the subset of $(P^+)^N$ consisting of elements $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$ such that

$$\lambda_1 + \cdots + \lambda_N = \lambda.$$

Given $\alpha \in R^+$ and $1 \leqslant k \leqslant N$, define

$$r_{\alpha,k}(\boldsymbol{\lambda}) = \min\{(\lambda_{i_1} + \cdots + \lambda_{i_k})(h_\alpha) : 1 \leqslant i_1 < \cdots < i_k \leqslant N\}.$$

Following [10], equip $P^+(\lambda, N)$ with the partial order defined by

$$\boldsymbol{\mu} \leqslant \boldsymbol{\lambda} \quad \Leftrightarrow \quad r_{\alpha,k}(\boldsymbol{\mu}) \leqslant r_{\alpha,k}(\boldsymbol{\lambda}) \quad \text{for all} \quad \alpha \in R^+,\ 1 \leqslant k \leqslant N.$$

In [20, Lemma 3.1], it was proved that the maximal elements of $P^+(\lambda, N)$ form a unique orbit under the obvious action of the symmetric group S_N on $P^+(\lambda, N)$. Moreover, an algorithm for computing such a maximal element was also obtained and can be described as follows.

If $\lambda = \sum_{i=1}^{n} b_i\omega_i$, let p_i and r_i, $i \in \mathrm{I}$, be the unique nonnegative integers satisfying

$$\sum_{l=i}^{n} b_l = p_iN + r_i \quad \text{and} \quad 0 \leqslant r_i < N.$$

For $i \in \mathrm{I}, 1 \leqslant j \leqslant N$, set

$$m_{i,j} = \begin{cases} p_i + 1 & \text{if } j \leqslant r_i \\ p_i & \text{if } j > r_i. \end{cases}$$

Clearly $m_{i,j} \geqslant m_{i+1,j}$ for any i, j and, hence, $\lambda_j := \sum_{i=1}^{n} m_{i,j}(\omega_i - \omega_{i-1}) \in P^+$, where $\omega_0 = 0$ by convention. Then, the following element is maximal in $P^+(\lambda, N)$

$$\boldsymbol{\lambda}^{max} := (\lambda_1, \dots, \lambda_N).$$

Note that, if $N \geqslant |\lambda| := \sum_{i\in I} \lambda(h_i)$, then an element of $P^+(\lambda, N)$ is maximal if and only if all its nonzero entries are fundamental weights.

Conjecture 1.5.2 ([32, Section 4.1])**.** Let $N \in \mathbb{Z}_{>0}, \lambda \in P^+$, and suppose $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$ is a maximal element of $P^+(\lambda, N)$. If $N \leqslant |\lambda|$, $W_N(\lambda) \cong V(\lambda_1) * \cdots * V(\lambda_N)$ as graded $\mathfrak{g}[t]$-modules.

Conjecture 1.5.2 has been proved in the following special cases:

(i) for $\lambda = N\mu + \nu$ with $\mu \in L^+$ and ν minuscule, where $L^+ = \{\lambda \in P^+ : d_i|\lambda(h_i)\}$ with $d_i = \dfrac{2}{(\alpha_i, \alpha_i)}$, $i \in I$ in [32].

(ii) for $\mathfrak{g}$ of type A, $N = 2$, and $\lambda = m\omega_i$ for some $i \in I$ in [21].

(iii) for simply laced $\mathfrak{g}$, $\lambda = m\theta$ for some $m \geqslant 0$, and $N = |\lambda|$ in [38].

Recall that $\lambda \in P^+$ is said to be minuscule if $\{\mu \in P^+ : \mu < \lambda\} = \varnothing$. The following is the list of non zero minuscule weights.

A_n	$\omega_i,\ 1 \leqslant i \leqslant n$
B_n	ω_n
C_n	ω_1
D_n	$\omega_1, \omega_{n-1}, \omega_n$
E_6	ω_1, ω_6
E_7	ω_7

We follow the numbering of vertices of the Dynkin diagram for $\mathfrak{g}$ as in [4]. Our main result generalizes and provides an alternative proof for item (ii) above.

Theorem 1.5.3. Suppose $\mathfrak{g}$ is simply laced and ω_i is minuscule. Then, Conjecture 1.5.2 holds for $\lambda = m\omega_i$ for all $m \geqslant 0$.

The proof of Theorem 1.5.3 will rely on results about Kirillov-Reshetikhin (KR) and Chari-Venkatesh (CV) modules which we review in the next subsections.

Note that Conjecture 1.5.2 is not a complete generalization of (1.5.4) since we may have $|\lambda| < \lambda(h_\vartheta)$. Regarding the region $|\lambda| \leqslant N \leqslant \lambda(h_\vartheta)$, it was proved in [38] for simply laced $\mathfrak{g}$ that

$$W_N(m\theta) \cong W(\theta)^{*(N-m)} * V(\theta)^{*(2m-N)}. \tag{1.5.5}$$

1.6 Chari-Venkatesh and Kirillov-Reshetikhin Modules

We now recall the definition of certain objects of $\mathcal{G}$ introduced in [16], which are now referred to as Chari-Venkatesh (or CV) modules.

Given a sequence $\mathbf{m} = (m_j)_{j\in\mathbb{Z}_{>0}}$ of nonnegative integers, we let

$$supp(\mathbf{m}) = \{j : m_j \neq 0\}.$$

We denote by $\mathscr{P}$ the set of non-increasing sequences with finite-support and refer to the elements of $\mathscr{P}$ as partitions. For any sequence $\mathbf{m}$ with finite support, we denote by $\underline{\mathbf{m}}$ the partition obtained from $\mathbf{m}$ by re-ordering its elements. Given $\underline{\mathbf{m}} \in \mathscr{P}$, set

$$\ell(\underline{\mathbf{m}}) = \max\{j : m_j \neq 0\} \qquad \text{and} \qquad |\underline{\mathbf{m}}| = \sum_{j\geqslant 1} m_j.$$

If $|\underline{\mathbf{m}}| = m$, then $\underline{\mathbf{m}}$ is said to be a partition of m. We denote by $\mathscr{P}_m$ the set of partition of m. The element m_j of a partition $\underline{\mathbf{m}}$ will be often referred to as the j-th part of $\underline{\mathbf{m}}$. Hence, $\ell(\underline{\mathbf{m}})$, which is often referred to as the length of $\underline{\mathbf{m}}$, is the number of nonzero parts of $\underline{\mathbf{m}}$. Given distinct nonnegative integers $k_1 > k_2 \cdots > k_l$ and $a_1, \ldots, a_l$, we denote by

$$(k_1^{(a_1)}, \ldots, k_l^{(a_l)})$$

the partition where each k_j is repeated a_j times.

Given $\lambda \in P^+$, a family of partitions $\xi = (\xi(\alpha))_{\alpha\in R^+}$ indexed by R^+ is said to be λ-compatible if $\xi(\alpha) \in \mathscr{P}_{\lambda(h_\alpha)}$ for all $\alpha \in R^+$. Namely, ξ is λ-compatible if $\xi(\alpha) = (\xi(\alpha)_1 \geqslant \ldots \geqslant \xi(\alpha)_s \geqslant \ldots \geqslant 0)$ and

$$\lambda(h_\alpha) = \sum_{j\geqslant 1} \xi(\alpha)_j \quad \text{for all} \quad \alpha \in R^+.$$

We will denote by $\mathscr{P}_\lambda$ the set of families of λ-compatible partitions.

Definition 1.6.1. Given $\xi \in \mathscr{P}_\lambda$, the CV-module $CV(\xi)$ is the quotient of $W(\lambda)$ by the submodule generated by

$$\left\{ (x_\alpha^+ \otimes t)^s (x_\alpha^-)^{s+r} w : \alpha \in R^+,\ s, r \neq 0,\ s + r > rk + \sum_{j>k} \xi(\alpha)_j \text{ for some } k \in \mathbb{Z}_{>0} \right\},$$

where $w \in W(\lambda)_\lambda \backslash \{0\}$.

Note that the submodule from Definition 1.6.1 is generated by homogeneous vectors of positive degree and, hence, $CV(\xi) \neq 0$.

Fix $\lambda \in P^+$ and $N \in \mathbb{Z}_{>0} \cup \{\infty\}$. For each $\alpha \in R^+$, if $N < \infty$, let q_α and p_α be the unique non negative integers such that

$$\lambda(h_\alpha) = Nq_\alpha + p_\alpha \quad \text{and} \quad 0 \leqslant p_\alpha < N.$$

If $N = \infty$ set $q_\alpha = 0$ and $p_\alpha = \lambda(h_\alpha)$. Consider the element $\xi_N^\lambda \in \mathscr{P}_\lambda$ given by

$$\xi_N^\lambda(\alpha) = ((q_\alpha + 1)^{(p_\alpha)}, q_\alpha^{(N-p_\alpha)}). \tag{1.6.1}$$

Note that, if $N \geqslant \lambda(h_\alpha)$ for all $\alpha \in R^+$, then $\xi_N^\lambda(\alpha) = (1^{(\lambda(h_\alpha))})$. The second of our main results is:

Theorem 1.6.2. The modules $CV(\xi_N^\lambda)$ and $W_N(\lambda)$ are isomorphic graded $\mathfrak{g}[t]$-modules.

For $N = \infty$ or, more precisely, in the case that $W_N(\lambda) \cong W(\lambda)$, this theorem was already known (see Proposition 2.2.5). The proof of Theorem 1.6.2 will be given in Section 2.3. In Chapter 3, we will obtain further results about truncated Weyl modules in the case $\mathfrak{g} = \mathfrak{sl}_2$ as consequences of Theorem 1.6.2 together with results from [3, 15].

Beside Theorem 1.6.2, the other crucial ingredient for the proof of Theorem 1.5.3 is a result from [36] about fusion products of Kirillov-Reshetikhin (KR) modules which we now review. The graded KR modules are the so called graded limits of the quantum KR modules, which are finite-dimensional irreducible modules for quantum affine algebras (see [13, 33, 36] and references therein).

Definition 1.6.3. Let $w \in W(m\omega_i)_{m\omega_i}$ be nonzero. The graded KR module $KR(m\omega_i)$ is the quotient of $W(m\omega_i)$ by the submodule generated by

$$(x_i^- \otimes t)w.$$

Quite clearly, $KR(m\omega_i) \neq 0$. Moreover, $KR(m\omega_i)$ is isomorphic to $CV(\xi)$ where ξ is exactly that for $W(\lambda)$ except when α is the corresponding simple root α_i, in which case, $\xi(\alpha_i)$ has just one part as in the case of $V(\lambda)$, namely, $\xi(\alpha) = (1^{\lambda(h_\alpha)})$ if $\alpha \neq \alpha_i$ and $\xi(\alpha_i) = (m)$.

Given $N > 0$, $i_1, \ldots, i_N \in I$, $m_1, \ldots, m_N \in \mathbb{Z}_{>0}$, set $\lambda = \sum_{j=1}^{N} m_j \omega_{i_j}$, $\mathbf{i} = (i_1, \ldots, i_N)$, $\mathbf{m} = (m_1, \ldots, m_N)$, $S_i(\mathbf{i}) = \{j : i_j = i\}$, and define the following fusion products

$$KR_{\mathbf{i}}(\mathbf{m}) = KR(m_1\omega_{i_1}) * \cdots * KR(m_N\omega_{i_N}),$$

$$V_{\mathbf{i}}(\mathbf{m}) = V(m_1\omega_{i_1}) * \cdots * V(m_N\omega_{i_N}),$$

for some choice of parameters in the definition of fusion products. If $i_j = i$ for some $i \in I$ and all $1 \leqslant j \leqslant N$, we write $KR_i(\mathbf{m})$ in place of $KR_{\mathbf{i}}(\mathbf{m})$ and we similarly define $V_i(\mathbf{m})$. In [36], it is given a presentation for $KR_{\mathbf{i}}(\mathbf{m})$ in terms of generators and relations which, in particular, proves the independence of $KR_{\mathbf{i}}(\mathbf{m})$ on the choice of parameters for the fusion product.

Theorem 1.6.4 ([36, Theorem B])**.** For every $N > 0$, $\mathbf{i} \in I^N, \mathbf{m} \in \mathbb{Z}^N_{\geqslant 0}$, the module $KR_{\mathbf{i}}(\mathbf{m})$ is isomorphic to the quotient of $W(\lambda)$ by the submodule generated by

$$x_i(r,s)\, w \quad \text{for all} \quad i \in I,\ r > 0,\ s + r > \sum_{j \in S_i(\mathbf{i})} \min\{r, m_j\},$$

where $x_i(r,s)$ is defined as in (2.2.1) below.

Together with Theorem 1.6.2, this presentation will imply the following Corollary, that we will prove in Section 2.4.

Corollary 1.6.5. Let $m, N > 0$, $i \in I$, and $\underline{\mathbf{m}} = \xi_N^{m\omega_i}(\alpha_i)$. Then, there exists an epimorphism of graded $\mathfrak{g}[t]$-modules $KR_i(\mathbf{m}) \to W_N(m\omega_i)$.

Combined with results of [8, 13, 22] on the structure of KR modules (see (2.4.3) below), this corollary will lead to a proof of Theorem 1.5.3.

We also prove a further step towards a proof of Theorem 1.5.3 without any hypothesis on $\mathfrak{g}$ and i. To explain that, introduce the following notation. Given $\eta = \sum_i a_i\alpha_i \in Q$, set

$$\mathrm{ht}_i(\eta) = a_i.$$

For $i \in I$, $k \geqslant 0$, set also

$$R^+_{i,k} = \{\alpha \in R^+ : \mathrm{ht}_i(\alpha) = k\}.$$

By inspecting the root systems, one checks that the set $R^+_{i,>1} = \{\alpha \in R^+ : \mathrm{ht}_i(\alpha) > 1\}$ is empty if ω_i is minuscule and has a unique minimal element otherwise, which of course lies in $R^+_{i,2}$. Let v be a highest-weight generator for $KR_i(\mathbf{m})$, where $\mathbf{m}$ is any sequence $\mathbf{m} = (m_1, \ldots, m_N)$ with $m_j \neq 0$ for all $1 \leqslant j \leqslant N$, and denote by $T_i(\mathbf{m})$ the quotient by the submodule generated by

$$(x^-_\alpha \otimes t^N)v \qquad \text{with} \qquad \alpha = \min R^+_{i,2}.$$

Recall that, if $\mathfrak{g}$ is not of type A, then $\theta = m\omega_i$ for some $1 \leqslant m \leqslant 2$ and $i \in I$. In fact, $m = 1$ unless $\mathfrak{g}$ is of type C in which case $\theta = 2\omega_1$. Let i_θ denote this node. We have the following list, in which we also follow the numbering of vertices of the Dynkin diagram for $\mathfrak{g}$ as in [4].

$$\begin{aligned} i_\theta &= 1 \quad \text{for} \quad \mathfrak{g} \quad \text{of types} \quad CFE_7; \\ i_\theta &= 2 \quad \text{for} \quad \mathfrak{g} \quad \text{of types} \quad BDE_6G; \\ i_\theta &= 8 \quad \text{for} \quad \mathfrak{g} \quad \text{of type} \quad E_8. \end{aligned}$$

Recall that

$$R^+_{i_\theta,2} = \{\theta\}.$$

The third of our main results is:

Proposition 1.6.6. Let $i = i_\theta$ and $\mathbf{m} = \xi_N^{m\omega_i}(\alpha_i)$ for some $m \geqslant 0$. Then, for all $N \leqslant m$, $W_N(m\omega_i) \cong T_i(\mathbf{m})$.

We shall see in Proposition 2.1.3 below that, if $\mathbf{m} \in \mathbb{Z}^N_{\geqslant 0}$, there exists an epimorhism $W_N(m\omega_i) \to V_i(\mathbf{m})$ for all $i \in I$. If $\mathbf{m} = \xi_N^{m\omega_i}(\alpha_i)$ for some $0 \leqslant m \leqslant N$, Corollary 1.6.5 then implies that we have epimorphisms

$$T_i(\mathbf{m}) \to W_N(m\omega_i) \to V_i(\mathbf{m}). \tag{1.6.2}$$

Proposition 1.6.6 says that the first of these is an isomorphism when $i = i_\theta$, while Conjecture 1.5.2 expects that the second is also an isomorphism for all $i \in I$. This motivates:

Conjecture 1.6.7. Let $i \in I, m \geqslant 0$. Then, for every $\mathbf{m} \in \mathscr{P}_m$, $V_i(\mathbf{m})$ is isomorphic to $T_i(\mathbf{m})$.

Note that item (iii) after Conjecture 1.5.2, together with Proposition 1.6.6, proves this Conjecture for types DE, $i = i_\theta$, and $\mathbf{m} = \xi_N^{N\omega_i}$. Evidently, Conjecture 1.6.7, together with (1.6.2), implies Conjecture 1.5.2. We will prove Proposition 1.6.6 in Section 2.5.

2 Proofs

In this chapter we present the proofs of our main results. For this, we will need some extra facts concerning CV modules, fusion products, and KR modules which will be reviewed as required.

2.1 More about Fusion Products

Given a filtered $\mathfrak{g}[t]$-module V, recall that F^rV denotes the corresponding filtered piece. One checks that

$$(x \otimes (t^s - f(t)))\, w \in F^{r+s-1}V, \tag{2.1.1}$$

for all $x \in \mathfrak{g},\ r, s \in \mathbb{Z},\ w \in F^rV,\ f(t) = t^s + b_{s-1}t^{s-1} + \cdots + b_0 \in \mathbb{C}[t]$.

We have the following lemma as a consequence of the definition of fusion product.

Lemma 2.1.1. If $\pi_j : M_j \to V_j, 1 \leqslant j \leqslant N$, is a family of epimorphisms of cyclic graded $\mathfrak{g}[t]$-modules, $\pi_1 \otimes \cdots \otimes \pi_N$ induces an epimorphism of graded $\mathfrak{g}[t]$-modules

$$M_1 * \cdots * M_n \to V_1 * \cdots * V_N.$$

Proof. For each $1 \leqslant j \leqslant N$, let w_j be the chosen generator for M_j and set $v_j = \pi_j(w_j)$. Let also $\Pi = \pi_1 \otimes \cdots \otimes \pi_N$, $w = w_1 \otimes \cdots \otimes w_n$, and $v = v_1 \otimes \cdots \otimes v_N$. Then, if F^rM and F^rV are defined as in (1.5.1) using w and v, respectively, we have $\Pi(F^rM) = F^rV$. Thus, Π induces an epimorphism of the associated graded modules corresponding to these filtrations. □

Lemma 2.1.2. For each $1 \leqslant j \leqslant l$, let V^j be a finite-dimensional cyclic graded $\mathfrak{g}[t]$-module generated by v_j. Suppose $(x \otimes t^{N_j}\mathbb{C}[t])v_j = 0$ for some $x \in \mathfrak{g},\ N_j \geqslant 0$, and set $N = N_1 + \ldots + N_l$. Then

$$(x \otimes t^N)(v_1 * \ldots * v_l) = 0.$$

Proof. For any choice of distinct $a_1, \ldots, a_l \in \mathbb{C}$, let $v_1 \otimes \ldots \otimes v_l$ be the generator of $V^1_{a_1} \otimes \ldots \otimes V^l_{a_l}$ and $v = v_1 * \ldots * v_l$. Let $f(t) = \prod_{j=1}^{l}(t - a_j)^{N_j}$. By (2.1.1), we have

$$(x \otimes t^N)v = (x \otimes f(t))v.$$

Recall that ζ_a is the Lie algebra automorphism of $\mathfrak{g}[t]$ induced by $t \mapsto t + a$ and in V_a, $(x \otimes f(t))v = (x \otimes f(t+a))v = (x \otimes \prod_{j=1}^{l}(t + a - a_j)^{N_j})v$. On the other hand,

$$\begin{aligned}(x \otimes f(t))(v_1 \otimes \cdots \otimes v_l) &= \sum_{j=1}^{l} v_1 \otimes \cdots \otimes \left(x \otimes \prod_{k=1}^{l}(t + a_j - a_k)^{N_k} \right) v_j \otimes \cdots \otimes v_l \\ &= \sum_{j=1}^{l} v_1 \otimes \cdots \otimes \left(x \otimes t^{N_j} \left(\prod_{k=1, k\neq j}^{l} (t + a_j - a_k)^{N_k} \right)\right) v_j \otimes \cdots \otimes v_l \\ &= 0\end{aligned}$$

□

Proposition 2.1.3. Let $l, N \in \mathbb{Z}$, $0 < l \leqslant N$. For each $1 \leqslant j \leqslant l$, suppose V^j is a quotient of $W(\lambda_j)$, for some $\lambda_j \in P^+$. If $(\mathfrak{g} \otimes t^{N_j}\mathbb{C}[t])V^j = 0$, $N_j \in \mathbb{Z}_{>0}$ and $N = N_1 + \ldots + N_l$ then, for any choice of distinct $a_1, \ldots, a_l \in \mathbb{C}$, there exists an epimorphism of graded $\mathfrak{g}[t]$-modules $W_N(\lambda) \to V^1_{a_1} * \cdots * V^l_{a_l}$, where $\lambda = \sum_{j=1}^{l} \lambda_j$.

Proof. Let v_j be a highest-weight vector for $V^j, j = 1, \ldots, l$ and denote by v the image of $v_1 \otimes \cdots \otimes v_l$ in $V^1_{a_1} * \cdots * V^l_{a_l}$. Then, quite clearly, $\mathfrak{n}^+[t]v = \mathfrak{h}[t]_+ v = 0, h(v) = \lambda(h)v$ for all $h \in \mathfrak{h}$, and $(x_i^-)^{\lambda(h_i)+1}v = 0$ for all $i \in I$. Thus, by (1.3.4), it suffices to show that $(x_\alpha^- \otimes t^N)v = 0$ for all $\alpha \in R^+$, but this follows from Lemma 2.1.2. □

2.2 More about CV Modules

Given $r, s \geqslant 0$, consider

$$S(r, s) = \{(b_p)_{0 \leqslant p \leqslant s} :\ b_p \in \mathbb{Z}_{\geqslant 0},\ \sum_{p=0}^{s} b_p = r,\ \sum_{p=0}^{s} p b_p = s\}.$$

Clearly, $S(r, s)$ is finite, $S(0, s)$ is empty if $s > 0$ and it has a unique element if $s = 0$. If $r = 1$, $S(1, s)$ also has a unique element, the sequence with 1 in the s-position and zero elsewhere.

Given $f(u) \in U(\mathfrak{n}^-[t])[[u]]$ and $s \in \mathbb{Z}_{\geqslant 0}$, let $f(u)_s$ be the coefficient of u^s in f. Let also

$$f(u)^{(r)} = \frac{1}{r!} f(u)^r \quad \text{for all} \quad r \geqslant 0.$$

For $\alpha \in R^+$, set

$$\mathbf{x}_\alpha(u) = \sum_{k \geqslant 0} (x_\alpha^- \otimes t^k) u^{k+1} \in U(\mathfrak{n}^-[t])[[u]].$$

and

$$x_\alpha(r, s) = (\mathbf{x}_\alpha(u)^{(r)})_{r+s} \quad \text{for all} \quad r, s > 0.$$

In other words,

$$x_\alpha(r,s) = \sum_{(b_p)\in S(r,s)} (x_\alpha^- \otimes 1)^{(b_0)}(x_\alpha^- \otimes t)^{(b_1)} \ldots (x_\alpha^- \otimes t^s)^{(b_s)}. \tag{2.2.1}$$

The following lemma is a reformulation of a result of Garland [26] (see also [11, 14]).

Lemma 2.2.1. Given $s \in \mathbb{Z}_{>0}$, $r \in \mathbb{Z}_{\geqslant 0}$ and $\alpha \in R^+$, we have

$$(x_\alpha^+ \otimes t)^s (x_\alpha^-)^{s+r} + (-1)^s x_\alpha(r,s) \in U(\mathfrak{g}[t])\mathfrak{n}^+[t] \oplus U(\mathfrak{n}^-[t])(\mathfrak{h} \otimes t\mathbb{C}[t]).$$

Given $k \geqslant 0$, define also

$$\begin{gathered} {}_kS(r,s) = \{(b_p)_{0\leqslant p\leqslant s} \in S(r,s) : \ \ b_p = 0 \ \text{ if } \ p < k\} \\ \text{and} \\ {}_kx_\alpha(r,s) = \sum_{(b_p)\in {}_kS(r,s)} (x_\alpha^- \otimes t^k)^{(b_k)} \ldots (x_\alpha^- \otimes t^s)^{(b_s)}. \end{gathered} \tag{2.2.2}$$

Note that

$${}_kx_\alpha(r,kr) = (x_\alpha^- \otimes t^k)^{(r)}. \tag{2.2.3}$$

If $\alpha = \alpha_i$ for some $i \in I$, we may simplify notation and write $\mathbf{x}_i(u)$, etc.

Lemma 2.2.2. ([16, Proposition 2.6]) Let V be any representation of $\mathfrak{g}[t]$ and let $w \in V$, $x \in \mathfrak{g}$ and $K \in \mathbb{Z}_{\geqslant 0}$. Then, for all $s, r, k \in \mathbb{Z}_{>0}$ with $s + r > rk + K$, we have

$$x_\alpha(r,s)w = 0 \quad \Leftrightarrow \quad {}_kx_\alpha(r,s)w = 0.$$

Let $\xi \in \mathscr{P}_\lambda$ and $w \in W(\lambda)_\lambda \backslash \{0\}$. Denote by $CV'(\xi)$ the quotient of $W(\lambda)$ by the submodule generated by

$$x_\alpha(r,s)w \quad \text{for all} \quad \alpha \in R^+,\ r,s > 0 \quad \text{such that} \quad s + r > rk + \sum_{j>k} \xi(\alpha)_j, \tag{2.2.4}$$

for some $k > 0$. Consider also the quotient $CV''(\xi)$ of $W(\lambda)$ by the submodule generated by

$${}_kx_\alpha(r,s)w \quad \text{for all} \quad \alpha \in R^+,\ s,r,k > 0 \quad \text{such that} \quad s + r > rk + \sum_{j>k} \xi(\alpha)_j. \tag{2.2.5}$$

Proposition 2.2.3. The modules $CV'(\xi)$ and $CV''(\xi)$ are isomorphic to $CV(\xi)$.

Proof. It follows from Lemma 2.2.1 that $CV(\xi) \cong CV'(\xi)$ and from Lemma 2.2.2 that $CV'(\xi) \cong CV''(\xi)$. $\square$

We will often denote by v_ξ a nonzero element of $CV(\xi)_\lambda$ if $\xi \in \mathscr{P}_\lambda$. It follows from the previous proposition and from (2.2.3) that

$$(x_\alpha^- \otimes t^k)^{(r)} v_\xi = 0 \quad \text{for all} \quad \alpha \in R^+, k, r > 0 \text{ s.t. } r > \sum_{j>k} \xi(\alpha)_j. \tag{2.2.6}$$

In particular, since $\sum_{j>k} \xi(\alpha)_j = 0$ for all $k \geqslant \ell(\xi(\alpha))$,

$$(x_\alpha^- \otimes t^k) v_\xi = 0 \quad \text{for all} \quad \alpha \in R^+, k \geqslant \ell(\xi(\alpha)). \tag{2.2.7}$$

Lemma 2.2.4 ([38, Lemma 6])**.** Let $\lambda \in P^+, w \in W(\lambda)_\lambda \backslash \{0\}$, and $\xi \in \mathscr{P}_\lambda$. Then, $x_\alpha(r,s)w = 0$ for all $\alpha \in R^+, r \geqslant \xi(\alpha)_1, s, k > 0$ such that $s + r > rk + \sum_{j>k} \xi(\alpha)_j$.

Proof. Let $\alpha \in R^+$, $s, k \in \mathbb{Z}_{>0}$, be such that $s + r > rk + \sum_{j \geqslant k+1} \xi(\alpha)_j$. Given $r \geqslant \xi(\alpha)_1$, it follows that

$$s + r > rk + \sum_{j \geqslant k+1} \xi(\alpha)_j > \sum_{j \geqslant 1} \xi(\alpha)_j = \lambda(\alpha).$$

Therefore, since $(x_\alpha^- \otimes 1)^{\lambda(h_\alpha)+1} w = 0$, we have $(x_\alpha^+ \otimes t)^s (x_\alpha^-)^{s+r} w = 0$ and it follows from Lemma 2.2.1, $x_\alpha(r,s)w = 0$. ☐

Given $\lambda \in P^+$, consider the two extreme family of λ-compatible partitions:

$$\xi = (\lambda(h_\alpha))_{\alpha \in R^+} \quad \text{and} \quad \xi' = ((1^{\lambda(h_\alpha)}))_{\alpha \in R^+}.$$

The next proposition shows that they correspond to the two extreme nonzero quotients of $W(\lambda)$.

Proposition 2.2.5. Given $\lambda \in P^+$, let ξ and ξ' be defined as above. Then,

$$CV(\xi) \cong ev_0 V(\lambda) \quad \text{and} \quad CV(\xi') \cong W(\lambda).$$

Proof. For the first isomorphism, since $CV(\xi)$ is a quotient of $W(\lambda)$, it is quite clearly that there exists a surjective homorphism

$$CV(\xi) \twoheadrightarrow \mathrm{ev}_0 V(\lambda).$$

We should prove that there exists the opposite surjective homomorphism, i.e.,

$$\mathrm{ev}_0 V(\lambda) \twoheadrightarrow CV(\xi).$$

If $v_\xi \in CV(\xi)_\lambda$, it is enough to show that $(\mathfrak{g} \otimes t\mathbb{C}[t]) v_\xi = 0$, namely, we should show that $(x_\alpha^- \otimes t\mathbb{C}[t]) v_\xi = 0$, for all $\alpha \in R^+$. This is clear from (2.2.7).

For the second isomorphism, by definition, $CV(\xi')$ is a quotient of $W(\lambda)$. Hence, it is enough to show that there exists a surjective homomorphism $CV(\xi') \twoheadrightarrow W(\lambda)$, i.e., we should show that $v \in W(\lambda)_\lambda$, satisfies only one more defining relation of $CV(\xi')$. In fact, since $\xi(\alpha) = ((1)^{\lambda(h_\alpha)})$, by Lemma 2.2.4, since $\xi(\alpha)_1 \leqslant 1$, then $x_\alpha(r,s)v = 0$ in $W(\lambda)$, for all $\alpha \in R^+$, $s,k \in \mathbb{Z}_{>0}$, $s + r > rk + \sum_{j \geqslant k+1} \xi(\alpha)_j$ and we have our statement. □

Assume, for the rest of this section that $\mathfrak{g} = \mathfrak{sl}_2$ and identify P with $\mathbb{Z}$ as usual. We simplify notation and write $x^\pm, h$ in place of $x_1^\pm, h_1$. For $\lambda \in P^+$, the set $\mathscr{P}_\lambda$ is just the set of partitions of λ. We then simplify the notation and write $\xi = (\xi_1 \geqslant \xi_2 \geqslant \cdots \geqslant \xi_l \geqslant 0)$.

Given a partition $\xi = (\xi_1 \geqslant \xi_2 \geqslant \ldots \geqslant \xi_j > 0)$, define partitions $\xi^\pm$ as follows. If $j = 1$, then $\xi^+ = \xi$ and ξ^- is the empty partition. If $j > 1$, then $\xi^- = (\xi_1^- \geqslant \ldots \geqslant \xi_{j-2}^- \geqslant \xi_{j-1}^- \geqslant 0)$ is given by

$$\xi_r^- = \begin{cases} \xi_r, & r < j-1, \\ \xi_{j-1} - \xi_j, & r = j-1, \\ 0, & r \geqslant j. \end{cases}$$

And $\xi^+ = (\xi_1^+ \geqslant \ldots \geqslant \xi_{j-1}^+ \geqslant \xi_j^+ \geqslant 0)$ is the unique partition associated to the sequence $(\xi_1, \ldots, \xi_{j-2}, \xi_{j-1} + 1, \xi_j - 1)$. Note that if $\xi \in \mathscr{P}_\lambda$ then $\xi^+ \in \mathscr{P}_\lambda$ and $\xi^- \in \mathscr{P}_{\lambda - 2\xi_j}$. The following was proved in [16, Theorem 5].

Theorem 2.2.6. Let $\xi \in \mathscr{P}_\lambda$, $l = \ell(\xi)$.

(i) For $l > 1$, there exists a short exact sequence of $\mathfrak{g}[t]$-modules

$$0 \to \tau_{(l-1)\xi_l} CV(\xi^-) \to CV(\xi) \to CV(\xi^+) \to 0.$$

(ii) For any choice of distinct $a_1, \ldots, a_l \in \mathbb{C}$, there exists an isomorphism

$$CV(\xi) \cong V_{a_1}(\xi_1) * \cdots * V_{a_l}(\xi_l)$$

of graded $\mathfrak{g}[t]$-modules. □

This theorem provides us with a presentation in terms of generators and relations for fusion products of irreducible modules. Return to the case $\xi = \xi_N^\lambda = ((q+1)^{(p)}, q^{(N-p)})$ where $\lambda = Nq + p$ with $0 \leqslant p < N$.

Corollary 2.2.7. With the above notation, $V(q+1)^{*p} * V(q)^{*N-p}$ is isomorphic to the quotient of $W(\lambda)$ by the submodule generated by

$$(x^+ \otimes t)^s (x^-)^{s+r} w \quad \text{for all} \quad r,s > 0 \quad \text{s.t.} \quad s + r > rk + q(N-k)_+ + (p-k)_+$$

for some $k > 0$ e $w \in W(\lambda)_\lambda \backslash \{0\}$, where $m_+ = m$ if $m \geqslant 0$ and $m_+ = 0$ if $m < 0$.

Proof. By Theorem 2.2.6, we have $V(q+1)^{*p} * V(q)^{*N-p} \cong CV(\xi_N^\lambda)$. Thus, if v is a highest-weight generator of $CV(\xi_N^\lambda)$, it suffices to show that

$$(x^+ \otimes t)^s (x^-)^{s+r} v = 0$$

for all r, s as in the statement. From (1.6.1) we have

$$(x^+ \otimes t)^s (x^-)^{s+r} v = 0 \quad \text{for all} \quad s, r > 0, s + r > rk + \sum_{j>k} \xi_j \quad \text{for some} \quad k > 0.$$

But if $k \geqslant p$, then

$$s + r > rk + \sum_{j \geqslant k+1} \xi_j \quad \Leftrightarrow \quad s + r > rk + q(N-k)_+$$

and, if $k < p$,

$$s + r > rk + \sum_{j \geqslant k+1} \xi_j \quad \Leftrightarrow \quad s + r > rk + q(N-k) + (b-k).$$

Hence,

$$s + r > rk + \sum_{j \geqslant k+1} \xi_j \quad \Leftrightarrow \quad s + r > rk + q(N-k)_+ + (p-k)_+.$$

□

2.3 Proof of Theorem 1.6.2

In this section we prove that every truncated Weyl module is isomorphic to a CV module for an explicited family of partitions. For this, let $\xi = \xi_N^\lambda$ and $v_\xi \in CV(\xi_N^\lambda)_\lambda \backslash \{0\}$ as defined in (1.6.1). It follows from (2.2.7) that

$$(x_\alpha^- \otimes t^N) v_\xi = 0 \quad \text{for all} \quad \alpha \in R^+,$$

and, hence, there exists a surjective homomorphism of $\mathfrak{g}[t]$-modules

$$W_N(\lambda) \twoheadrightarrow CV(\xi_N^\lambda).$$

To prove the converse, observe that Proposition 2.2.3 implies that it suffices to show

$${}_k x_\alpha(r,s) w = 0 \quad \text{for all} \quad \alpha \in R_+, s, r, k \in \mathbb{Z}_{>0} \quad \text{such that} \quad s + r > rk + \sum_{j>k} \xi_N^\lambda(\alpha)_j,$$

with $w \in W_N(\lambda)_\lambda$. Hence, if we prove Theorem 1.6.2 for $\mathfrak{g} = \mathfrak{sl}_2$, the general case follows by considering the subalgebra $\mathfrak{sl}_\alpha[t]$.

By (1.3.7), for $N \geqslant \lambda$, Theorem 1.6.2 becomes Proposition 2.2.5 hence, we can assume $N < \lambda$. By Proposition 2.1.3, there exists a surjective homomorphism

$$W_N(\lambda) \twoheadrightarrow V(q+1)^{*p} * V(q)^{*(N-p)} \cong CV(\xi_N^\lambda).$$

Thus, by Corollary 2.2.7, it suffices to show that, if $w \in W_N(\lambda)_\lambda$, then

$$(x^+ \otimes t)^s (x^-)^{s+r} w = 0 \quad \text{for all} \quad r, s > 0 \quad \text{s.t.} \quad s + r > rk + q(N-k)_+ + (p-k)_+$$

for some $k > 0$. Thus, assume s, r, k satisfy this condition. If $r > q$, we will see

$$s + r > \lambda$$

hence, $(x^+ \otimes t)^s (x^-)^{s+r} w = (x^+ \otimes t)^s \cdot 0 = 0$ as desired. To prove the above inequality, note that

$$\begin{aligned} k \geqslant p \quad \Rightarrow \quad s + r &> rk + q(N-k)_+ \geqslant (q+1)k + q(N-k)_+ \\ &\geqslant p + q(k + (N-k)_+) \geqslant p + Nq \geqslant \lambda \end{aligned}$$

while

$$k < p \quad \Rightarrow \quad s + r > rk + q(N-k) + (p-k) = rk - k(q+1) + \lambda \geqslant \lambda.$$

It remains to treat the case $r \leqslant q$ which implies

$$s + r > Nr. \tag{2.3.1}$$

Indeed, if $k < N$ then

$$\begin{aligned} s + r > rk + q(N-k)_+ + (p-k)_+ &= rk + q(N-k) + (p-k)_+ \\ \geqslant rk + r(N-k) + (p-k)_+ &= Nr + (p-k)_+ \geqslant Nr. \end{aligned}$$

While if $N \leqslant k$ we have $s + r > rk \geqslant Nr$. By Lemmas 2.2.1 and 2.2.2, we have

$$(x^+ \otimes t)^s (x^-)^{s+r} w = 0 \quad \Leftrightarrow \quad {}_k x(r,s) w = 0,$$

where ${}_k x(r,s)$ is given by (2.2.2). Note that, if $(b_m) \in {}_k S(r,s)$ is such that $b_j > 0$ for some $j \geqslant N$, then $(x^- \otimes t^k)^{(b_k)} \cdots (x^- \otimes t^s)^{(b_s)} w = 0$. Thus, it suffices to show that this is the case for every element of ${}_k S(r,s)$. Fix $(b_m) \in {}_k S(r,s)$ and note that this is obvious when $k \geqslant N$ (take $j = k$). Otherwise, if $k < N$ and we have $b_j = 0$ for all $j \geqslant N$, it would follow that

$$s = \sum_{k \leqslant j < N} j b_j \leqslant (N-1) \sum_{k \leqslant j < N} b_j \leqslant (N-1) r$$

contradicting (2.3.1). This completes the proof of Theorem 1.6.2.

2.4 Proof of Theorem 1.5.3

In this section we use Theorem 1.6.2 as one of the tools to prove Conjecture 1.5.2 when $\mathfrak{g}$ is simply laced and $\lambda = m\omega_i$, for a minuscule weight ω_i and a non negative integer m. Given $\mathbf{i}$ and $\mathbf{m}$ as in Theorem 1.6.4, let

$$\underline{\mathbf{m}}_i = (m_j)_{j \in S_i(\mathbf{i})} \quad \text{for all} \quad i \in I$$

and let $\xi_{\mathbf{i}}^{\mathbf{m}} \in \mathscr{P}_\lambda$ be given by

$$\xi_{\mathbf{i}}^{\mathbf{m}}(\alpha) = \begin{cases} \underline{\mathbf{m}}_i, & \text{if } \alpha = \alpha_i \text{ for some } i \in I, \\ (1^{\lambda(h_\alpha)}), & \text{otherwise.} \end{cases}$$

The following corollary was observed in [36, Remark 3.4(b)].

Corollary 2.4.1. There is an isomorphism $CV(\xi_{\mathbf{i}}^{\mathbf{m}}) \cong KR_{\mathbf{i}}(\mathbf{m})$.

Proof. To simplify notation, let $\xi = \xi_{\mathbf{i}}^{\mathbf{m}}$. We will show that there are surjective maps $CV(\xi) \to KR_{\mathbf{i}}(\mathbf{m})$ and $KR_{\mathbf{i}}(\mathbf{m}) \to CV(\xi)$, thus proving the corollary. Let $v_{KR} \in KR_{\mathbf{i}}(\mathbf{m})_\lambda$ and $v_\xi \in CV(\xi)_\lambda$ be nonzero vectors. By the isomorphism $CV(\xi) \cong CV'(\xi)$ of Proposition 2.2.3, in order to prove the existence of the first map, it suffices to check that

$$x_\alpha(r,s)\, v_{KR} = 0 \quad \text{for all} \quad \alpha \in R^+,\ r,s > 0 \quad \text{s.t.} \quad s + r > rk + \sum_{j>k} \xi(\alpha)_j, \tag{2.4.1}$$

for some $k > 0$. If α is not a simple root, then $r \geqslant 1 = \xi(\alpha)_1$ and this follows from Lemma 2.2.4. Let $i \in I$, assume $\alpha = \alpha_i$, and write

$$\underline{\mathbf{m}}_i = (m_{i,1}, \dots, m_{i,N_i})$$

for some $N_i \geqslant 0$. Then, by definition of ξ, we have $\xi(\alpha)_j = m_{i,j}$ for $1 \leqslant j \leqslant N_i$ and we have

$$rk + \sum_{j>k} \xi(\alpha)_j = rk + m_{i,k+1} + \cdots + m_{i,N_i} \geqslant \sum_{j=1}^{N_i} \min\{r, m_{i,j}\} = \sum_{j \in S_i(\mathbf{i})} \min\{r, m_j\}.$$

Thus, if $s + r > rk + \sum_{j>k} \xi(\alpha)_j$, we also have $s + r > \sum_{j \in S_i(\mathbf{i})} \min\{r, m_j\}$ and Theorem 1.6.4 implies $x_\alpha(r,s)\, v_{KR} = 0$ as desired.

In order to show the existence of the second map, we need to show that

$$x_i(r,s)\, v_\xi = 0 \quad \text{for all} \quad i \in I,\ r > 0,\ s + r > \sum_{j \in S_i(\mathbf{i})} \min\{r, m_j\}. \tag{2.4.2}$$

Fix i, r, s as above. We claim that there must exists $k > 0$ such that $s + r > rk + \sum_{j>k} \xi(\alpha_i)_j$. Assuming this claim, (2.4.2) follows from the isomorphism $CV(\xi) \cong CV'(\xi)$ of Proposition 2.2.3. To prove the claim, assume, by contradiction, that

$$s + r \leqslant rk + \sum_{j>k} \xi(\alpha_i)_j = rk + \sum_{j>k} m_{i,j} \qquad \text{for all} \qquad k > 0.$$

In particular, taking $k = \max\{j : m_{i,j} \geqslant r\}$, we have

$$s + r \leqslant rk + \sum_{j>k} m_{i,j} = \sum_{j=1}^{N_i} \min\{r, m_{i,j}\},$$

contradicting the choice of r, s in (2.4.2). $\square$

Proof of Corollary 1.6.5. Since $W_N(m\omega_i) \cong CV(\xi_N^{m\omega_i})$ by Theorem 1.6.2 and $KR_i(\mathbf{m}) \cong CV(\xi_{\mathbf{i}}^{\mathbf{m}})$ with $\mathbf{i} = (i^{(N)})$ by the previous corollary, we are left to show that there exists an epimorphism

$$CV(\xi_{\mathbf{i}}^{\mathbf{m}}) \to CV(\xi_N^{m\omega_i}).$$

Letting $v \in CV(\xi_N^{m\omega_i})_\lambda$, this is in turn equivalent to showing that

$$x_\alpha(r,s)\, v = 0 \quad \text{for all} \quad \alpha \in R^+,\ r,s > 0 \quad \text{s.t.} \quad s + r > rk + \sum_{j>k} \xi_{\mathbf{i}}^{\mathbf{m}}(\alpha)_j \text{ for some } k > 0.$$

Since $\xi_{\mathbf{i}}^{\mathbf{m}}(\alpha_i) = \xi_N^{m\omega_i}(\alpha_i)$ by definition, we can assume α is not simple, in which case $r \geqslant 1 = \xi_{\mathbf{i}}^{\mathbf{m}}(\alpha)_1$ and we are done by Lemma 2.2.4. □

It is known (see [8, 13, 22]) that, if $\mathfrak{g}$ is simply laced and ω_i is minuscule, then

$$KR(m\omega_i) \cong V(m\omega_i). \tag{2.4.3}$$

Hence, in that case, $KR_i(\mathbf{m}) \cong V_i(\mathbf{m})$. Since we have an epimorphism $W_N(m\omega_i) \to V_i(\mathbf{m})$ by Proposition 2.1.3, Theorem 1.5.3 follows from Corollary 1.6.5.

2.5 Proof of Proposition 1.6.6

In this section we see that the discussion preceding Proposition 1.6.6 implies that the proof of this proposition follows from Lemma 2.5.2 below. We, also, shall need the following simple, however useful lemma.

Lemma 2.5.1. Let V be a $\mathfrak{g}[t]$-module, and suppose $v \in V$ satisfies $\mathfrak{h}[t]_+ v = 0$. Suppose also that $x \in \mathfrak{g}$ satisfies $(x \otimes t^r)v = 0$ and $[h, x] = cx$ for some $r \geqslant 0$, $h \in \mathfrak{h}$ and $c \in \mathbb{C}\backslash\{0\}$. Then, $(x \otimes t^k)v = 0$ for all $k \geqslant r$.

Proof. In fact, if $k = r$ we have nothing to do. Suppose $k > r$, then by hypothesis $(h \otimes t^{k-r})v = 0$ for all $h \in \mathfrak{h}$, hence we have

$$\begin{aligned}(x \otimes t^k)v &= \frac{1}{c}[h \otimes t^{k-r}, x \otimes t^r]v \\ &= \frac{1}{c}((h \otimes t^{k-r})(x \otimes t^r)v - (x \otimes t^r)(h \otimes t^{k-r})v) = 0.\end{aligned}$$

□

Let $\mathbf{m} = \xi_N^{m\omega_i}(\alpha_i)$, $0 < N \leqslant m$ and $i \in I$. Corollary 1.6.5 implies that we have a projection $T_i(\mathbf{m}) \to W_N(m\omega_i)$. Thus, we are left to prove that there exists a projection $W_N(m\omega_i) \to T_i(\mathbf{m})$. Equivalently, we have to prove that

$$v \in KR_i(\mathbf{m})_{m\omega_i}\backslash\{0\} \quad \Rightarrow \quad (x_\alpha^- \otimes t^N)v = 0 \quad \text{for all} \quad \alpha \in R^+.$$

Recall that if $\alpha = \sum_{j \in I} a_j \alpha_j$ then $\mathrm{ht}(\alpha)_i = a_i$ and for $k \geqslant 0$, $R^+_{i,k} = \{\alpha \in R^+ : \mathrm{ht}_i(\alpha) = k\}$. Moreover, if $i = i_\theta$ as in the proposition and $k > 1$ then $R^+_{i,k}$ is empty if ω_i is minuscule and has a unique minimal element otherwise, $\theta \in R^+_{i_\theta,2}$. By definition, $T_i(\mathbf{m})$ is the quotient of $KR_i(\mathbf{m})$ by the submodule generated by

$$(x^-_\alpha \otimes t^N)v \qquad \text{with} \qquad \alpha = \min R^+_{i,2}.$$

Thus, the proof of Proposition 1.6.6 follows from the next lemma.

Lemma 2.5.2. Let $v \in KR_i(\mathbf{m})_{m\omega_i} \backslash \{0\}$. Then, $(x^-_\alpha \otimes t^N)v = 0$ for every α such that $\mathrm{ht}_i(\alpha) \leqslant 1$.

Proof. If $\mathrm{ht}_i(\alpha) = 0$, then

$$(x^-_\alpha \otimes 1)^{\lambda(h_\alpha)+1} v = (x^-_\alpha \otimes 1)v = 0,$$

hence by Lemma 2.5.1, $(x^-_\alpha \otimes t^N)v = 0$.
If $\mathrm{ht}_i(\alpha) = 1$ we proceed by induction on $\mathrm{ht}(\alpha)$. If $\mathrm{ht}(\alpha) = 1$, then $\alpha = \alpha_i$. Let $v_j \in KR(m_j\omega_i)_{m_j\omega_i}$, $1 \leqslant j \leqslant N$, be such that $v = v_1 * \ldots * v_N$. By the definition of $KR(m_j\omega_i)$, $(x^-_i \otimes t)v_j = 0$. Then, by Lemma 2.1.2 $(x^-_i \otimes t^N)v = 0$, showing that induction begins.
If $\mathrm{ht}(\alpha) > 1$, we can write $\alpha = \gamma + \beta$, $\gamma, \beta \in R^+$ and we may assume without loss of generality that $\mathrm{ht}_i(\beta) = 0$ and $\mathrm{ht}_i(\gamma) = 1$. In particular, $x^-_\beta v = 0$. Since

$$x^-_\alpha \otimes t^N = a[x^-_\beta \otimes 1, x^-_\gamma \otimes t^N],$$

for some nonzero complex number a, the result follows by applying the inductive hypothesis to γ. $\square$

3 Further results for $\mathfrak{sl}_2$

In this chapter, we study truncated Weyl modules, for $\mathfrak{g} = \mathfrak{sl}_2$, from the perspective of the theory of Demazure modules. In particular, the truncated Weyl modules which are isomorphic to Demazure modules will be explicitly characterized. For those which are not isomorphic to a Demazure module, we study their Demazure flags of lowest possible level.

3.1 Demazure Flags

First we recall, from [22], the definition of $\mathfrak{g}$-stable Demazure modules which is appropriate to our study and some known results about this class of representations.

Definition 3.1.1. Given $\ell \in \mathbb{Z}_{\geqslant 0}$ and $\lambda \in P^+$, the $\mathfrak{g}$-stable level-ℓ Demazure module $D(\ell, \lambda)$ is the quotient of $W(\lambda)$ by the submodule generated by

$$\{(x_\alpha^- \otimes t^{s_\alpha})v : \alpha \in R^+\} \cup \{(x_\alpha^- \otimes t^{s_\alpha - 1})^{m_\alpha + 1} v : \alpha \in R^+ \text{ such that } m_\alpha < \ell r_\alpha^\vee\},$$

where v is a highest-weight generator of $W(\lambda)$ and $r_\alpha^\vee, s_\alpha$, and m_α are the integers defined by

$$r_\alpha^\vee = \begin{cases} 1, & \text{if } \alpha \text{ is long,} \\ r^\vee, & \text{if } \alpha \text{ is short,} \end{cases} \quad \text{and} \quad \lambda(h_\alpha) = (s_\alpha - 1)\ell r_\alpha^\vee + m_\alpha, \quad 0 < m_\alpha \leqslant \ell r_\alpha^\vee,$$

where $r^\vee$ is the lacing number of $\mathfrak{g}$. Set $D(\ell, \lambda, m) = \tau_m D(\ell, \lambda)$.

Remark 3.1.2. As mentioned before, the definition of Demazure modules above is the more appropriate for our study, but we refer to [22] and [35] for a more traditional definition of a Demazure module. The Demazure modules are integrable ℓ-highest weight modules for the Borel subalgebra $\hat{\mathfrak{b}}$ of the affine Kac Moody algebra $\hat{\mathfrak{g}}$.

In particular, if $\mathfrak{g}$ is simply laced, it follows from (1.3.6) that

$$W(\lambda) \cong D(1, \lambda). \tag{3.1.1}$$

It is well known that there are epimorphisms of graded $\mathfrak{g}[t]$-modules

$$D(\ell, \lambda) \twoheadrightarrow D(\ell', \lambda) \qquad \text{for all} \qquad \lambda \in P^+, \ \ell \leqslant \ell'. \tag{3.1.2}$$

In particular, $D(\ell, \lambda) \cong V_0(\lambda)$ if ℓ is such that $s_\alpha = 1$ for all $\alpha \in R^+$.

Let $\xi_{\ell,\lambda} \in \mathscr{P}_\lambda$ be given by

$$\xi_{\ell,\lambda}(\alpha) = ((\ell r_\alpha^\vee)^{s_\alpha - 1}, m_\alpha) \quad \text{for} \quad \alpha \in R^+. \tag{3.1.3}$$

The following was shown in [16, Theorem 2]:

Theorem 3.1.3. For all $\ell \in \mathbb{Z}_{>0}$ and $\lambda \in P^+$, the modules $D(\ell,\lambda)$ and $CV(\xi_{\ell,\lambda})$ are isomorphic.

It is not always true that a truncated Weyl module is isomorphic to a Demazure module. It is then natural to ask whether truncated Weyl modules can be "approximated" by Demazure modules, which leads us to the concept of Demazure flags. For results about Demazure flags beyond what we will review here, see [3, 15] and references therein.

Definition 3.1.4. A $\mathfrak{g}[t]$-module V admits a *Demazure flag* of level-ℓ if there exist $k > 0, \lambda_j \in P^+, m_j \in \mathbb{Z}, j = 1, \ldots, k$, and a sequence of inclusions

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{k-1} \subset V_k = V \quad \text{with} \quad V_j/V_{j-1} \cong D(\ell, \lambda_j, m_j) \ \forall\ 1 \leqslant j \leqslant k.$$

Let $\mathbb{V}$ be a level-ℓ Demazure flag of V as in Definition 3.1.4 and, for a Demazure module D, define the multiplicity of D in $\mathbb{V}$ by

$$[\mathbb{V} : D] = \#\{1 \leqslant j \leqslant l : V_j/V_{j-1} \cong D\}.$$

As observed in [15, Lemma 2.1], the multiplicity does not depend on the choice of the flag and, hence, by abuse of language, we shift the notation from $[\mathbb{V} : D]$ to $[V : D]$. Also following [15], we consider the generating function

$$[V : D](t) = \sum_{m \in \mathbb{Z}} [V : \tau_m D]\ t^m \ \in\ \mathbb{Z}[t, t^{-1}].$$

It is easy to see that if V is a graded $\mathfrak{g}[t]$-module and U is a graded $\mathfrak{g}[t]$-submodule of V such that U and V/U both admit a Demazure flag of the same level, then V also admits one, and

$$[V : D](t) = [U : D](t) + [V/U : D](t). \tag{3.1.4}$$

In the category of non-graded $\mathfrak{g}[t]$-modules we have $\tau_m D \cong D$ and, hence, one may also be interested in computing the ungraded multiplicity of D in V which is given by

$$[V : D](1) = \sum_{m \in \mathbb{Z}} [V : \tau_m D].$$

3.2 Loewy Series

In the theory of local Weyl modules a problem which is often studied is that of Loewy structure of these modules, especially to determine two standard Loewy series: the radical series and the socle series. We are interested in studying such series for truncated Weyl modules. In this section, we recall these concepts. For more details see [30, 31].

Let A be a $\mathbb{C}$-algebra and M a finite-dimensional A-module. A semisimple filtration of an A-module M is a chain of inclusions of A-modules

$$0 = F^0 M \subseteq \cdots \subseteq F^l M = M \tag{3.2.1}$$

such that each successive quotient is semisimple.

Definition 3.2.1. The *radical* of M, denoted by radM, is the smallest submodule of M such that the corresponding quotient is semisimple.

Equivalently, radM is the intersection of all submodules of M such that the corresponding quotient is semisimple. We call the quotient $\dfrac{M}{\mathrm{r}adM}$ the *head* of the module M. Write r$ad^0M = M$ and, for $k \geqslant 1$, define inductively

$$\mathrm{r}ad^k M = rad(rad^{k-1}M).$$

This defines a semisimple filtration on M called the *radical series*

$$0 \subset \cdots \subset \mathrm{r}ad^2 M \subset \mathrm{r}adM \subset \mathrm{r}ad^0 M = M.$$

Definition 3.2.2. The *socle* of M, denoted by socM, is the largest semisimple submodule of M.

Equivalently, socM is the sum of all simple submodules of M. Consider s$oc^0M = 0$ and for $k \geqslant 1$, let soc^kM be the unique submodule of M such that

$$\mathrm{s}oc\left(\frac{M}{soc^{k-1}M}\right) = \frac{soc^k M}{soc^{k-1}M}.$$

This defines a semisimple filtration on M called the *socle series*

$$0 = \mathrm{s}oc^0 M \subset \mathrm{s}ocM \subset \mathrm{s}oc^2 M \subset \cdots \subset M.$$

If we consider a semisimple filtration as in (3.2.1) then,

$$\mathrm{r}ad^k M \subseteq F^{l-k}M \subseteq \mathrm{s}oc^{l-k}M$$

holds for each k. This implies that the lengths of the radical series and the socle series are equal and that the length of any semisimple filtration of M is greater than or equal to it.

Definition 3.2.3. A *Loewy series* of a finite-dimensional A-module M is a semisimple filtration on M which has the smallest length.

By the comment preceding Definition 3.2.3, the radical series and the socle series are Loewy series. Following [31], we will say that a module is *rigid* if its radical series and socle series coincide. Equivalently, M is rigid if it has a unique Loewy series. Let A be a $\mathbb{Z}$-graded $\mathbb{C}$-algebra. If A is positively graded and M is $\mathbb{Z}$-graded, we can consider the filtration defined by

$$F^k M = \bigoplus_{s \geqslant k} M[s].$$

This filtration is called the *grading filtration* or *grading series*. Observe that $M[s]$ is an $A[0]$-submodule of M for all s. Moreover, if $m = \max\{s : M[s] \neq 0\}$, $M[m]$ is an A-submodule of M. The next lemma is easily established.

Lemma 3.2.4. Assume $A[s] = 0$ for $s < 0$ and let $m = \max\{s : M[s] \neq 0\}$. If N is a simple $A[0]$-submodule of $M[m]$, then N is also a simple A-submodule of M. In particular, the socle of $M[m]$ as an $A[0]$-module is contained in the socle of M.

Lemma 3.2.5. ([31, Lemma 2.3]) Let A be a positively graded $\mathbb{C}$-algebra and M a finite-dimensional A-module such that the grading series on M is semisimple.

(i) If $\dfrac{M}{\mathrm{rad}M}$ is simple, the radical series of M coincides with the grading series.

(ii) If $\mathrm{s}ocM$ is simple, the socle series of M coincides with the grading series.

Recall that if V is a highest weight module, then it has a unique simple quotient, hence, its radical series coincide with the grading series. In particular, $W(\lambda)$ is a graded finite-dimensional highest weight module, thus its radical series and grading series coincide.It is well known that the socles of Demazure modules are simple. Then, by (3.1.1), if $\mathfrak{g}$ is of type ADE, the socle of $W(\lambda)$ is simple and, hence, by Lemma 3.2.5, the socle series of $W(\lambda)$ coincides with its grading series. In particular, $W(\lambda)$ is rigid.

3.3 The $\mathfrak{sl}_2$-Case

In this section, fix $\mathfrak{g} = \mathfrak{sl}_2$ and identify P with $\mathbb{Z}$ as usual. Given $\lambda \in P^+, N \in \mathbb{Z}_{>0}$, set q and p such that $\lambda = Nq + p,\ 0 \leqslant p < N$. Recall that $\xi_N^\lambda = ((q+1)^p, (q)^{N-p})$ and by Theorem 1.6.2, $CV(\xi_N^\lambda) \cong W_N(\lambda)$. We have the following corollary from Theorem 3.1.3.

Corollary 3.3.1. If $\xi = ((a)^{k-1}, b) \in \mathscr{P}_\lambda$, $0 < b \leqslant a$, then $CV(\xi) \cong D(a, \lambda)$.

Proof. It follows from Theorem 3.1.3 noting that $\lambda = (k-1)a + b$. □

Theorem 3.3.2. We have the following isomorphisms of $\mathfrak{g}[t]$-modules

$$W_N(\lambda) \cong \begin{cases} D(q, \lambda), & \text{if } N|\lambda, \\ D(q+1, \lambda), & \text{if } p \in \{N-1, \lambda\}. \end{cases}$$

Note that $p = \lambda$ if and only if $N > \lambda$.

Proof. If $N|\lambda$ then $p = 0$ and $\xi_N^\lambda = (q^{N-1}, q)$. Hence, by Corollary 3.3.1, $CV(\xi_N^\lambda) \cong D(q, \lambda)$. If $p = N-1$, note that $\xi_N^\lambda = ((q+1)^{N-1}, q)$ and, hence, Corollary 3.3.1 implies that $CV(\xi_N^\lambda) \cong D(q+1, \lambda)$. Finally, if $N > \lambda$, then $q = 0$, $\xi_N^\lambda = (1^p)$, and Corollary 3.3.1 implies $CV(\xi_N^\lambda) \cong D(1, \lambda) = D(q+1, \lambda)$. Thus the proof follows from Theorem 1.6.2. □

Recall from Section 2.2 that if $\xi = (\xi_1 \geqslant \ldots \geqslant \xi_j > 0)$ then $\xi^- = (\xi_1, \ldots, \xi_{j-2}, \xi_{j-1} - \xi_j)$ and ξ^+ is the partition associated to the sequence $(\xi_1, \ldots, \xi_{j-2}, \xi_{j-1}+1, \xi_j-1)$. Moreover, by Theorem 2.2.6, there exists a short exact sequence of $\mathfrak{g}[t]$-modules

$$0 \to \tau_{(j-1)\xi_j} CV(\xi^-) \to CV(\xi) \to CV(\xi^+) \to 0. \tag{3.3.1}$$

If $N < \lambda$ and $p = N - 2$, then $(\xi_N^\lambda)^+ = ((q+1)^{N-1}, q-1)$. Hence by Corollary 3.3.1, $CV((\xi_N^\lambda)^+) \cong D(q+1, \lambda)$. Thus, by (3.3.1), we have the following surjective homomorphism

$$W_N(\lambda) \cong CV(\xi_N^\lambda) \twoheadrightarrow CV((\xi_N^\lambda)^+) \cong D\left(q+1, \lambda\right).$$

In general, we have:

Proposition 3.3.3. If $\lambda \in P^+$, $N \in \mathbb{Z}_{>0}$, there exists a surjective homomorphism

$$W_N(\lambda) \twoheadrightarrow D\left(q+1, \lambda\right).$$

Proof. If $N \geqslant \lambda$, by (1.3.7) and (3.1.1)

$$W_N(\lambda) \cong W(\lambda) \cong D(1, \lambda),$$

and the statement follows from (3.1.2). If $N < \lambda$ we have two cases. If $N|\lambda$ the proof follows from Theorem 3.3.2 and (3.1.2). Otherwise, define recursively $(\xi_N^\lambda)^{+j} = ((\xi_N^\lambda)^{+(j-1)})^+$. For example, $(\xi_N^\lambda)^{+2}$ is the partition $((\xi_N^\lambda)^+)^+$. We claim there exists j such that the partition $(\xi_N^\lambda)^{+j}$ has at most one part different from $q+1$, hence by Corollary 3.3.1 yields the following isomorphism

$$CV((\xi_N^\lambda)^{+j}) \cong D(q+1, \lambda)$$

and, by (3.3.1), we have

$$W_N(\lambda) \cong CV(\xi_N^\lambda) \twoheadrightarrow CV((\xi_N^\lambda)^+) \twoheadrightarrow CV((\xi_N^\lambda)^{+2}) \twoheadrightarrow \ldots \twoheadrightarrow CV((\xi_N^\lambda)^{+j}) \cong D\left(q+1, \lambda\right).$$

To prove the above claim we proceed by induction on N, the number of non zero parts of ξ_N^λ. If $N = 1$, we consider $j = 0$ and ξ_N^λ is such partition. Assume the result for all partitions ξ_k^λ with $k < N$. If $q > m := \left\lfloor \dfrac{N-p}{2} \right\rfloor$ then

$$(\xi_N^\lambda)^{+m} = ((q+1)^{p+m}, q^{N-p-2m}, q-m),$$

hence if $N-p$ is even, take $j = m$, otherwise take $j = m+1$
If $q \leqslant m$, then

$$(\xi_N^\lambda)^{+q} = ((q+1)^{p+m}, q^{N-p-2q})$$

has $N-q$ parts. The induction hypothesis applies to $(\xi_N^\lambda)^{+q}$ implies that there exists ℓ such that $((\xi_N^\lambda)^{+q})^{+\ell}$ has at most one part different from $q+1$. Take $j = q+\ell$ and we have our statement. $\square$

Now, we will study the partition $(\xi_N^\lambda)^-$. First, consider $N < \lambda$ and $p = N - 2$ in which case we have $(\xi_N^\lambda)^- = ((q+1)^{N-2}) \in \mathscr{P}_{\lambda-2q}$. In this case, by Corollary 3.3.1 and (3.3.1), we have the following injective homomorphism

$$\tau_{(N-1)q}D\,(q+1, \lambda-2q) \cong \tau_{(N-1)q}CV((\xi_N^\lambda)^-) \hookrightarrow CV((\xi_N^\lambda)) \cong W_N(\lambda).$$

If $p = N - 3$, $(\xi_N^\lambda)^- = ((q+1)^{N-3}, q) \in \mathscr{P}_{\lambda-2q}$ and we get the following injective homomorphism

$$\tau_{(N-1)q}D\,(q+1, \lambda-2q) \cong \tau_{(N-1)q}CV((\xi_N^\lambda)^-) \hookrightarrow CV((\xi_N^\lambda)) \cong W_N(\lambda).$$

In general, we have:

Proposition 3.3.4. If $\lambda \in P^+$, $\lambda \neq N \in \mathbb{Z}_{>0}$, there exists an injective homomorphism

$$\tau_{m_j}D(q+1, \mu) \hookrightarrow W_N(\lambda),$$

for some $m_j \in \mathbb{Z}_{\geqslant 0}$ and $\mu \in P^+$, $\mu \leqslant \lambda$.

Proof. If $N > \lambda$ the proof follows from (1.3.7) and (3.1.1). Suppose $N < \lambda$. If $p = N - 1$ the statement follows from Theorem 3.3.2. Then, assume set $0 < p < N - 1$ and define $(\xi_N^\lambda)^{-j} = ((\xi_N^\lambda)^{-(j-1)})^-$ recursively. Note that, in this case, the first element ξ_1^{-j} of the partition $(\xi_N^\lambda)^{-j}$ is $q+1$, for any j which the corresponding partition has at least two entries q. Moreover, in this case, each time that we add 1 to j, two q entries become 0. Hence, the partition $(\xi_N^\lambda)^{-j}$ is $(\lambda - 2jq)$-compatible. Now, observe that:

(i) if $N - p$ is even, there exists j such that $N - p = 2j$ and $(\xi_N^\lambda)^{-j} = ((q+1)^p)$ and, hence, by Corollary 3.3.1,

$$CV((\xi_N^\lambda)^{-j}) \cong D(q+1, \lambda - 2jq).$$

(ii) if $N - p$ is odd, there exists j such that $N - p = 2j + 1$ and $(\xi_N^\lambda)^{-j} = ((q+1)^p, q)$, hence by Corollary 3.3.1

$$CV((\xi_N^\lambda)^{-j}) \cong D(q+1, \lambda - 2jq).$$

Thus, by (3.3.1), for $j = \left\lfloor \dfrac{N-p}{2} \right\rfloor$ we have

$$\tau_{m_j}D\,(q+1, \lambda-2jq) \cong \tau_{m_j}CV((\xi_N^\lambda)^{-j}) \hookrightarrow \ldots \hookrightarrow \tau_{m_1}CV((\xi_N^\lambda)^-) \hookrightarrow CV(\xi_N^\lambda) \cong W_N(\lambda).$$

This completes the proof. □

We will see below that $W_N(\lambda)$ is not a Demazure module for all other values of p which are not included in the Theorem 3.3.2. The following was proved in [15, Theorem 3.3].

Theorem 3.3.5. Let $\xi \in \mathscr{P}_\lambda$ for some $\lambda \in P^+$. Then, $CV(\xi)$ admits a level-ℓ Demazure flag if and only if $\ell \geqslant \xi_1$. In particular, $D(\ell, \lambda)$ admits a level-ℓ' Demazure flag if and only if $\ell' \geqslant \ell$.

Proof. Set $\xi = (\xi_1 \geqslant \cdots \geqslant \xi_s > 0) \in \mathscr{P}_\lambda$. Consider $\ell \geqslant \xi_1$ and proceed by induction on s. If $s = 1$, then $CV(\xi) \cong V(\lambda)$ and our induction starts. Assume that we have proved the result for all partitions with at most $(s-1)$ parts. The hypothesis applies to $\xi^\pm$, and if we consider $\ell \geqslant \xi_1^+$, we see that $\tau_{s-1}CV(\xi^-)$ and $CV(\xi^+)$ both have a Demazure flag of level ℓ, and by (3.1.4), $CV(\xi)$ has a Demazure flag of level ℓ. Now, if $\ell < \xi_1^+$, i.e., $\ell = \xi_1$ and $\xi_1 = \xi_{s-1}$, hence $\xi = (\xi_1^{s-1}, \xi_s)$, then $CV(\xi) \cong D(\ell, \lambda)$ and there is nothing to prove. On the other hand, if $CV(\xi)$ has a Demazure flag of level ℓ then $D(\ell, \mu)$ is a quotient of $CV(\xi)$ for some $\mu \in P^+$ and since $hv_\xi = \lambda v_\xi$ for $v_\xi \in (CV(\xi))_\lambda$ then $\mu = \lambda$. It remains to show that $\ell \geqslant \xi_1$. By (2.2.6), $(x \otimes t)^{1+\sum_{j\geqslant 2}\xi_j} v_\xi = 0$ then $(x \otimes t)^{1+\sum_{j\geqslant 2}\xi_j} w = 0$, where w is the generator of $D(\ell, \mu)$. From the original definition of Demazure modules which we mentioned in Remark 3.1.2, it follows $\sum_{j\geqslant 2} \xi_j \geqslant \lambda - \ell$ and thus $\ell \geqslant \xi_1$. □

Together with (3.1.1), this theorem implies that $W(\lambda)$ admits a level-ℓ Demazure flag for all $\ell \geqslant 1$. We have the following corollary of Theorems 1.6.2 and 3.3.5.

Corollary 3.3.6. The module $W_N(\lambda)$ admits a level-ℓ Demazure flag if and only if

$$\ell \geqslant \begin{cases} q, & \text{if } \; N|\lambda, \\ q+1, & \text{otherwise.} \end{cases}$$

Proof. By Theorem 3.3.5, $CV(\xi_N^\lambda) \cong W_N(\lambda)$ admits a level-ℓ Demazure flag if and only if $\ell \geqslant \xi_1$. But

(i) if $N|\lambda$ then $p = 0$ and $\xi_N^\lambda = ((q)^N)$;

(ii) otherwise, if N does not divide λ then $p \neq 0$ and $\xi_N^\lambda = ((q+1)^p, (q)^{N-p})$.

□

In light of (3.1.2), it follows that, in order to show that $W_N(\lambda)$ is not a Demazure module for $p \neq 0, N-1, \lambda$, it suffices to show that its level-$(q+1)$ Demazure flag has length bigger than 1. To see this, we will use the short exact sequence (3.3.1). Recall that

$$\xi^+ \in \mathscr{P}_\lambda \qquad \text{and} \qquad \xi^- \in \mathscr{P}_{\lambda - 2\xi_l}. \tag{3.3.2}$$

One also easily checks that

$$\xi = \xi_N^\lambda \quad \text{and} \quad p \neq 0, N-1 \quad \Rightarrow \quad \xi_1^\pm = q+1. \tag{3.3.3}$$

Hence, the length of a level-$(q+1)$ Demazure flag of $CV(\xi)$ is the sum of lengths of level-$(q+1)$ Demazure flags of $CV(\xi^{\pm})$, showing that $W_N(\lambda)$ is not a Demazure module.

Example 3.3.7. If $p = N - 2 \neq \lambda$ we have a length-2 flag:

$$0 \to D(q+1, \lambda - 2q, (N-1)q) \to W_N(\lambda) \to D(q+1, \lambda) \to 0.$$

Consider the case $\lambda = 4$ and $N = 3$, so $p = q = 1$ and the above sequence becomes

$$0 \to V(2,2) \to W_3(4) \to D(2,4) \to 0.$$

One can check, using (3.3.1), that we have exact sequences

$$0 \to V(0,2) \to D(2,4) \to D(3,4) \to 0 \qquad \text{and} \qquad 0 \to V(2,1) \to D(3,4) \to V(4,0) \to 0.$$

The grading series is described by

degree	$D(2,4)$	$V(2,2)$
0	$V(4)$	
1	$V(2)$	
2	$V(0)$	$V(2)$

This implies that $soc(D(2,4)) \cong V(0,2)$ and, using Lemma 3.2.4, we see that

$$soc(W_3(4)) = W_3(4)[2] \cong V(2,2) \oplus V(0,2).$$

This shows that, differently from the non truncated case, truncated Weyl modules may have non simple socle. $\diamond$

Example 3.3.8. If $p = N - 3 \neq \lambda$, the flag has length 2 or 3. To see this, observe that

$$\xi^+ = ((q+1)^{N-2}, q, q-1) \qquad \text{and} \qquad \xi^- = ((q+1)^{N-3}, q).$$

In particular, $CV(\xi^-) \cong D(q+1, \lambda - 2q)$. If $q = 1$ (i.e., $\lambda = 2N - 3$), we have a length-2 flag:

$$0 \to D(2, \lambda - 2, N-1) \to W_N(\lambda) \to D(2, \lambda) \to 0.$$

Otherwise, one easily checks using Theorem 2.2.6 that $CV(\xi^+)$ has a length-2 flag:

$$0 \to D(q+1, \lambda - 2(q-1), (N-1)(q-1)) \to CV(\xi^+) \to D(q+1, \lambda) \to 0.$$

$\diamond$

The following characterization of the truncated Weyl modules having a Demazure flag of length 2 is easily deduced from the computations of the above two examples.

Proposition 3.3.9. Suppose $p \neq 0, N-1$. The level-$(q+1)$ Demazure flag of $W_N(\lambda)$ has length 2 if and only if either $p = N-2 \neq \lambda$ or $p = N-3$ and $q = 1$.

Remark 3.3.10. By Lemma 3.2.5, since $W_N(\lambda)$ obviously has a simple head, its radical series coincides with its grading series. Example 3.3.7 shows that truncated Weyl modules may not have simple socle, hence, Lemma 3.2.5 does not guarantee that the socle series coincides with the grading series, but in this case they coincide.

Given $a, b, \ell \in \mathbb{Z}_{\geqslant 0}$, let

$$\xi^{\ell}_{a,b} = ((\ell+1)^a, (\ell)^b) \in \mathscr{P}_{\lambda^{\ell}_{a,b}}, \qquad \lambda^{\ell}_{a,b} = \ell(a+b) + a,$$

and note that

$$\xi^{\lambda}_{N} = \xi^{q}_{p,N-p} \tag{3.3.4}$$

or, equivalently,

$$CV(\xi^{\ell}_{a,b}) \cong W_{a+b}(\lambda^{\ell}_{a,b}).$$

In particular,

$$CV(\xi^{\ell-1}_{N,0}) = CV(\xi^{\ell}_{0,N}) \cong D(\ell, N\ell) \cong W_N(N\ell).$$

Given $\mu \in P^+$, consider the function

$$\gamma^{\ell}_{a,b}(\mu, t) = [CV(\xi^{\ell}_{a,b}) : D(\ell+1, \mu)](t).$$

Such functions were studied in [3, 15], but a full understanding is still not achieved. For instance, it follows from [15] that

$$\gamma^{1}_{0,\lambda}(\lambda - 2k, t) = [W(\lambda) : D(2, \lambda - 2k)](t) = t^{k\lceil \lambda/2 \rceil} \begin{bmatrix} \lfloor \lambda/2 \rfloor \\ k \end{bmatrix}_t, \tag{3.3.5}$$

for all $0 \leqslant k \leqslant \lambda$, where

$$\begin{bmatrix} m \\ k \end{bmatrix}_t = \prod_{j=0}^{k-1} \frac{1 - t^{m-j}}{1 - t^{k-j}} \quad \text{for} \quad 0 \leqslant k \leqslant m.$$

Our next goal is to study the generating function

$$\gamma^{q}_{p,N-p}(\mu, t) = [W_N(\lambda) : D(q+1, \mu)](t),$$

for $\lambda = \lambda^{q}_{p,N-p} = qN + p, \mu \in P^+$, $N \in \mathbb{Z}_{>0}$. It follows from Propositions 3.3.3 and 3.3.4 that

$$\gamma^{q}_{p,N-p}(\lambda, t) = 1 \qquad \text{and} \qquad \gamma^{q}_{p,N-p}(\mu, t) = 0 \quad \text{if} \quad \lambda - \mu \notin 2\mathbb{Z}_{\geqslant 0}.$$

Also,

$$\gamma^{q}_{p,N-p}(\mu, t) = \delta_{\lambda,\mu} \quad \text{if} \quad N > \lambda \text{ or } p = N-1,$$

because in this case, by Theorem 3.3.2 $W_N(\lambda) \cong D(q+1,\lambda)$. Note that, for $N = \lambda$, we have $q = 1$ and $p = 0$, hence (3.3.5) gives us a formula to compute $\gamma^q_{p,N-p}(\mu,t)$. Let us look at the other cases for which $q = 1$, i.e., assume

$$N < \lambda < 2N - 1.$$

Note that, in this case, we have $\lambda = N + p$, $p \in \{1, \ldots, N-2\}$ and

$$\xi = (2^{(p)}, 1^{(N-p)}).$$

More generally, let $\xi \in \mathscr{P}_\lambda$ be a partition of the form $\xi = ((2)^a, 1^b)$, with $a + b = N$.

Example 3.3.11. If $a = 0$, by Corollary 3.3.1, $CV(\xi) \cong D(1,\lambda) \cong W(\lambda)$, then by (3.3.5),

$$[CV(\xi) : D(2,\lambda)](t) = \begin{bmatrix} \lfloor \lambda/2 \rfloor \\ 0 \end{bmatrix}_t = 1.$$

On the other hand, if $a \neq 0$, we have $\xi^+ = ((2)^{a+1}, 1^{b-2}) \in \mathscr{P}_\lambda$ and $\xi^- = ((2)^a, 1^{b-2}) \in \mathscr{P}_{\lambda-2}$. We have the following equality:

$$[CV(\xi) : D(2,\mu)](t) = [CV(\xi^+) : D(2,\mu)](t) + t^{(N-1)\xi_N}[CV(\xi^-) : D(2,\mu)](t), \quad (3.3.6)$$

equivalently

$$[CV(\xi^+) : D(2,\mu)](t) = [CV(\xi) : D(2,\mu)](t) - t^{(N-1)\xi_N}[CV(\xi^-) : D(2,\mu)](t).$$

◇

For simplicity, we will write $CV((q+1)^a, q^b)$ instead of $CV(\xi) \in \mathscr{P}_\lambda$, where $\xi = ((q+1)^a, q^b)$. Hence, by (3.3.6) we have

$$[CV(2^{a+1}, 1^{b-2}) : D(2,\mu)](t) = [CV(2^a, 1^b) : D(2,\mu)](t) - t^{(N-1)\xi_N}[CV(2^a, 1^{b-2}) : D(2,\mu)](t).$$

Thus, an induction on a shows that $[CV(\xi) : D(2,\mu)](t)$ can be written in terms of the case $a = 0$, namely, in terms of $[D(1,\lambda), D(2,\mu)](t)$.

Example 3.3.12. If $a = 1$, $b \geqslant 2$, using (3.3.6) we have

$$[CV(2, 1^b) : D(2,\mu)](t) = [CV(1^{b+2}) : D(2,\mu)](t) - t^{b+1}[CV(1^b) : D(2,\mu)](t).$$

◇

Example 3.3.13. If $a = 2$, $b \geqslant 2$, using (3.3.6) we have

$$[CV(2^2, 1^b) : D(2,\mu)](t) = [CV(2, 1^{b+2}) : D(2,\mu)](t) - t^{b+2}[CV(2, 1^b) : D(2,\mu)](t).$$

◇

In general, if $a \neq 0$, $b \geqslant 2$ and $N = (a-1)+(b+2) = a+b+1$,

$$[CV(2^a,1^b) : D(2,\mu)](t) = [CV(2^{a-1},1^{b+2}) : D(2,\mu)](t) - t^{N-1}[CV(2^{a-1},1^b) : D(2,\mu)](t).$$

Note that, if $a \neq 0$ and $b = 0$ or $b = 1$, $CV(\xi) \cong D(2,\lambda)$ and the length of level-2 Demazure flag of $CV(\xi)$ is one.
Henceforth, assume $a > 0$. For $b = 0, 1$, we have

$$\gamma^1_{a,b}(\mu,t) = [D(2,2a+b) : D(2,\mu)](t) = \delta_{2a+b,\mu}.$$

Hence, we can assume $b > 2$. In this case, it follows from Theorem 2.2.6

$$\gamma^1_{a,b}(\mu,t) = \gamma^1_{a-1,b+2}(\mu,t) - t^{a+b}\gamma^1_{a-1,b}(\mu,t), \tag{3.3.7}$$

which, combined with (3.3.5), gives a recursive procedure to compute $\gamma^1_{a,b}(\mu,t)$. However, one can use an approach producing a formula without minus signs, as we shall see next.

Given a partition ξ, let ξ^* be the partition obtained from ξ by removing its largest part. In particular, if $\xi \in \mathscr{P}_\lambda$, then

$$\xi^* \in \mathscr{P}_{\lambda-\xi_1}.$$

Note that, if $a > 0$, then

$$(\xi^\ell_{a,b})^* = \xi^\ell_{a-1,b} \tag{3.3.8}$$

The following equality was obtained in [15, Lemma 3.8]

$$[CV(\xi) : D(\xi_1,\mu)](t) = t^{\frac{\lambda-\mu}{2}}\ [CV(\xi^*) : D(\xi_1,\mu-\xi_1)](t). \tag{3.3.9}$$

In particular,

$$[CV(\xi) : D(\xi_1,\mu)](t) = 0 \quad \text{if} \quad \xi_1 > \mu.$$

Using (3.3.8) and iterating (3.3.9) we get

$$\gamma^\ell_{a,b}(\mu,t) = t^{\frac{a}{2}(\lambda^\ell_{a,b}-\mu)}\ \gamma^\ell_{0,b}(\mu - a(\ell+1),t). \tag{3.3.10}$$

In particular, (3.3.4) implies

$$[W_N(\lambda) : D(q+1,\mu)](t) = t^{\frac{p}{2}(\lambda-\mu)}\ [D(q,q(N-p)) : D(q+1,\mu-p(q+1))](t).$$

Example 3.3.14. Given $\lambda = 4$ and $N = 3$, we have by (3.3.10):

$$\gamma^1_{1,2}(\lambda-2k,t) = t^k(\gamma^1_{0,2}(\lambda-2(k+1),t)),$$

where $k \in \{0,1,2\}$.

$$\begin{aligned}
k = 2 \quad &\Rightarrow \quad \gamma^1_{1,2}(0,t) = t^2(\gamma^1_{0,2}(-2,t)) = 0;\\
k = 1 \quad &\Rightarrow \quad \gamma^1_{1,2}(2,t) = t^1(\gamma^1_{0,2}(0,t)) = t^2;\\
k = 0 \quad &\Rightarrow \quad \gamma^1_{1,2}(4,t) = 1,
\end{aligned}$$

where the last equality in the second case is from (3.3.5). $\diamond$

Example 3.3.15. Given $\lambda = 5$ and $N = 4$, we have by (3.3.10):

$$\gamma^1_{1,3}(\lambda - 2k, t) = t^k(\gamma^1_{0,3}(\lambda - 2(k+1), t)),$$

where $k \in \{0, 1, 2\}$.

$$\begin{aligned}
k = 2 \quad &\Rightarrow \quad \gamma^1_{1,3}(1,t) = t^2(\gamma^1_{0,3}(-1,t)) = 0;\\
k = 1 \quad &\Rightarrow \quad \gamma^1_{1,3}(3,t) = t^1(\gamma^1_{0,3}(1,t)) = t^3;\\
k = 0 \quad &\Rightarrow \quad \gamma^1_{1,3}(5,t) = 1,
\end{aligned}$$

where the last equality in the second case is from (3.3.5). $\diamond$

Example 3.3.16. Given $\lambda = 6$ and $N = 4$, we have by (3.3.10):

$$\gamma^1_{2,2}(\lambda - 2k, t) = t^k(\gamma^1_{1,2}(\lambda - 2(k+1), t)),$$

where $k \in \{0, 1, 2, 3\}$.

$$\begin{aligned}
k = 3 \quad &\Rightarrow \quad \gamma^1_{2,2}(0,t) = t^3(\gamma^1_{1,2}(-2,t)) = 0;\\
k = 2 \quad &\Rightarrow \quad \gamma^1_{2,2}(2,t) = t^2(\gamma^1_{1,2}(0,t)) = t^2 \cdot t^2(\gamma^1_{0,2}(-2,t)) = 0;\\
k = 1 \quad &\Rightarrow \quad \gamma^1_{2,2}(4,t) = t^1(\gamma^1_{1,2}(2,t)) = t \cdot t(\gamma^1_{0,2}(0,t)) = t^3;\\
k = 0 \quad &\Rightarrow \quad \gamma^1_{2,2}(6,t) = t^0(\gamma^1_{1,2}(4,t)) = 1,
\end{aligned}$$

where the last two equalities in the cases are from (3.3.5). $\diamond$

In view of (3.3.7), we have another formula for the particular case $q = 1$ using (3.3.10).

Proposition 3.3.17. Let $\lambda, \mu \in P^+, \mu = \lambda - 2k$ for some $k \geqslant 0$ and $\lambda = N + p$, with $0 < p < N - 1$. Then,

$$[W_N(\lambda) : D(2,\mu)](t) = \begin{cases} t^{k\lceil \lambda/2 \rceil} \begin{bmatrix} \lfloor \frac{2N-\lambda}{2} \rfloor \\ k \end{bmatrix}_t, & \text{if } \; k \leqslant \left\lfloor \dfrac{2N-\lambda}{2} \right\rfloor, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By (3.3.10),

$$\gamma^1_{p,N-p}(\mu, t) = t^k(\gamma^1_{p-1,N-p}(\lambda - 2(k+1), t)).$$

The idea of the proof is to iterate this formula. The process stops in two cases: when the weight and the index of the truncation are the same, i.e., when we have $\gamma^1_{0,\lambda}(\mu, t)$ or when the weight at position $\lambda - 2(k+1)$ is less than zero and, by definition, $\gamma^1_{p,N-p}(\mu, t) = 0$. Suppose that after the first step we still do not have either of the above situation. Then

$$\begin{aligned}
\gamma^1_{p,N-p}(\mu, t) \quad &= \quad t^k(\gamma^1_{p-1,N-p}(\lambda - 2(k+1), t))\\
&= \quad t^k(t^k \gamma^1_{p-2,N-p}(\lambda - 2(k+2), t)).
\end{aligned}$$

If we could continue this process indefinitely, we would have

$$\gamma^1_{p,N-p}(\mu,t) = t^{jk}(\gamma^1_{p-j,N-p}(\lambda - 2(k+j)), t)),$$

where j is the number of steps that we iterate the formula. Note that because of our conditions for the process to stop we have two situations to consider:

$$\lambda - 2(k+j) < 0 \Leftrightarrow \frac{\mu}{2} < j \text{ or}$$

$$\lambda - 2j = N - j \Leftrightarrow \lambda - N = j.$$

We can choose j such that one of the cases happens: either $\frac{\mu}{2} < j$ and $\lambda - N \geqslant j$ or we choose $\frac{\mu}{2} \geqslant j = \lambda - N$. If $\frac{\mu}{2} < j$ then $\lambda - 2(k+j) < 0$, hence $\gamma^1_{p,N-p}(\mu,t) = 0$.
On the other hand, if $\frac{\mu}{2} \geqslant j = \lambda - N$ then $\lambda - 2j = N - j$, hence

$$\begin{aligned}
\gamma^1_{p,N-p}(\mu,t) &= t^{(\lambda-N)k}(\gamma^1_{0,2N-\lambda}(\lambda - 2(k+\lambda-N), t)) \\
&= t^{(\lambda-N)k}(\gamma^1_{0,2N-\lambda}(2N-\lambda-2k, t)) \\
&= t^{(\lambda-N)k}\left(t^{k\lceil\frac{2N-\lambda}{2}\rceil}\begin{bmatrix}\lfloor\frac{2N-\lambda}{2}\rfloor \\ k\end{bmatrix}_t\right) \\
&= t^{(\lambda-N+\lceil\frac{2N-\lambda}{2}\rceil)k}\begin{bmatrix}\lfloor\frac{2N-\lambda}{2}\rfloor \\ k\end{bmatrix}_t \\
&= t^{k\lceil\frac{\lambda}{2}\rceil}\begin{bmatrix}\lfloor\frac{2N-\lambda}{2}\rfloor \\ k\end{bmatrix}_t.
\end{aligned}$$

□

In summary, given $\lambda \in P^+, N \in \mathbb{Z}_{>0}$, such that $\lambda = N \cdot q + p$, with $0 \leqslant p < N$ and $\mu = \lambda - 2k$, with $k \in \{0, \ldots, \lfloor\frac{\lambda}{2}\rfloor\}$, we have the following:

p	q	$W_N(\lambda)$	$\gamma^q_{p,N-p}(\mu,t)$
0	q	$\simeq D(q,\lambda)$	
0	1	$\simeq D(1,\lambda)$	$t^{k\lceil\lambda/2\rceil}\begin{bmatrix}\lfloor\frac{\lambda}{2}\rfloor\\ k\end{bmatrix}_t$
λ ($\Rightarrow N>\lambda$)	q	$\simeq D(q+1,\lambda)$	$\delta_{\lambda,\mu}$
$N-1$	q	$\simeq D(q+1,\lambda)$	$\delta_{\lambda,\mu}$
$\{1,\dots,N-2\}^*$	q	$---$	$t^{\frac{\lambda-\mu}{2}}(\gamma^q_{p-1,N-p}(\mu-(q+1),t))$
$\{1,\dots,N-2\}^*$	1	$---$	$\begin{cases} t^{k\lceil\lambda/2\rceil}\begin{bmatrix}\lfloor\frac{2N-\lambda}{2}\rfloor\\ k\end{bmatrix}_t, & \text{if } k\leqslant\lfloor\frac{2N-\lambda}{2}\rfloor,\\ 0, & \text{otherwise.}\end{cases}$

(*) *$W_N(\lambda)$ is not a Demazure module, but it admits a level-$(q+1)$ Demazure flag.*

Proposition 3.3.17 can be used to compute the length of the level-2 Demazure flag. Namely, recall from [15, Section 3.8] that, for every partition ξ,

$$[CV(\xi):D(\ell,\mu)](t)\neq 0 \qquad\Rightarrow\qquad |\xi|-2\mu\in 2\mathbb{Z}_{\geqslant 0}.$$

Then, if we consider the generating function

$$L^\ell_\xi(x,t)=\sum_{k=0}^{\lfloor|\xi|/2\rfloor}[CV(\xi):D(\ell,|\xi|-2k)](t)\;x^k,$$

the length of the level-ℓ Demazure flags of $CV(\xi)$ is

$$L^\ell_\xi=L^\ell_\xi(1,1).$$

In the case $\xi=\xi^\lambda_N$ with λ and N as in Proposition 3.3.17, we get

$$L^2_\xi=\sum_{k=0}^{\lfloor\frac{2N-\lambda}{2}\rfloor}\binom{\lfloor\frac{2N-\lambda}{2}\rfloor}{k}=2^{\lfloor\frac{2N-\lambda}{2}\rfloor}.$$

3.4 Inclusions

In this section we will discuss the inclusions of truncated Weyl modules. In the non truncated case, inclusions of local Weyl modules are almost always related with stability of bases of $W(\lambda)$, see [39, 40, 41]. It is not our propose here discuss about stability of bases for $W_N(\lambda)$. Moreover, there are not so many results about bases for truncated case, see for instance [2, 32]. Our proposal is to verify if there exist chains of inclusions of truncated Weyl modules or in which cases it is possible to guarantee such existence. In general, we obtain negative answers in the most of the cases. In [14], Chari and Pressley produce monomials bases for local Weyl modules when $\mathfrak{g} = \mathfrak{sl}_2$ which the construction was extended for bases to the case $\mathfrak{g} = \mathfrak{sl}_m$ in [12]. The main focus of [39, 40, 41] was to investigate wheter these bases respect inclusions of local Weyl modules. The inclusions in these papers were obtained from the identification of local Weyl modules as Demazure modules. We do not use this approach here. Our main tools are Theorems 1.6.2 and 2.2.6.

Note that, for $\lambda = \lambda^1_{a,b}$ and $N = a + b, b \neq 0$, the exact sequence from Theorem 2.2.6 can be rewritten as

$$0 \to \tau_{N-1} W_{N-1-\delta_{p,N-1}}(\lambda - 2) \to W_N(\lambda) \to W_{N-1}(\lambda) \to 0. \tag{3.4.1}$$

This answers the question about inclusions of truncated Weyl modules when $\ell = 1$.

Let $\lambda \in P^+$. Note that, Theorem 2.2.6 together with Proposition 2.2.5 give us the following inclusion

$$\tau_{\lambda-1} W(\lambda - 2) \hookrightarrow W(\lambda).$$

Combining this with (1.3.3), it follows that we have an exact sequence

$$0 \to \tau_{\lambda-1} W(\lambda - 2) \to W(\lambda) \to W_{\lambda-1}(\lambda) \to 0. \tag{3.4.2}$$

In other words, if $N = \lambda$ and $N' = N - 1$, the kernel of the projection (1.3.5) is, up to grade shift, isomorphic to $W(\lambda - 2) \cong W_{N-2}(\lambda - 2)$ and (3.4.2) can be rewritten as

$$0 \to \tau_{\lambda-1} W_{\lambda-2}(\lambda - 2) \to W_\lambda(\lambda) \to W_{\lambda-1}(\lambda) \to 0. \tag{3.4.3}$$

Our next goal is two-fold. On one hand, we want to study the kernel of (1.3.5) with $N < \lambda$, specially for $N' = N-1$. Note that (3.4.1) gives the answer of this special case when $N < \lambda < 2N$ and it coincides with the case $N = \lambda$ as seen in (3.4.3). Unfortunately, as we shall see in Example 3.4.1, the kernel is not always a truncated Weyl module and, in fact, may not even be a CV module. On the other hand, we want to study possible inclusions of truncated Weyl modules. For instance, if $\lambda = qN + p$ with $0 \leqslant p < N$ as before and, either $p < N - 1$ or $q = 1$, then Theorem 2.2.6 gives rise to the inclusion

$$\tau_{(N-1)q} W_{N-2}(\lambda - 2q) \hookrightarrow W_N(\lambda).$$

The corresponding quotient is a truncated Weyl module if and only if $q = 1$ which is (3.4.1) again. If $q > 1$ and $p = N - 1$, Theorem 2.2.6 does not give rise to an inclusion of truncated Weyl modules, but a second application gives rise to the inclusion

$$\tau_{N-2}\tau_{(N-1)q}W_{N-2}(\lambda - 2(q+1)) \hookrightarrow W_N(\lambda).$$

Denote by $\pi_N^{N'}$ the projection (1.3.5) and, for $N' = N - 1$, simplify the notation and write π_N. Note

$$\pi_N^{N'} = \pi_{N'+1} \circ \cdots \circ \pi_{N-1} \circ \pi_N \qquad \text{for all} \qquad N' < N.$$

Moreover, (1.3.4) implies that

$$\ker(\pi_N) = U(\mathfrak{n}^-[t])(x^- \otimes t^{N-1})w_N$$

where $w_N \in W_N(\lambda)_\lambda \backslash \{0\}$. From Lemma 1.4.1 and Remark 1.4.3 we have a surjective map

$$\varpi_N : \tau_{N-1}W_N(\lambda - 2) \twoheadrightarrow \ker(\pi_N).$$

Let

$$\delta_N(\lambda) = \dim(W_N(\lambda)).$$

It follows from Theorems 1.6.2 and 2.2.6 that

$$\delta_N(\lambda) = (q+2)^p(q+1)^{N-p}.$$

Therefore, ϖ_N is an isomorphism if and only if

$$\delta_N(\lambda) - \delta_N(\lambda - 2) = \delta_{N-1}(\lambda). \tag{3.4.4}$$

Note that

$$\xi_N^{\lambda-2} = \begin{cases} ((q+1)^{(p-2)}, q^{(N-p+2)}), & \text{if } p \geqslant 2, \\ (q^{(N-2+p)}, (q-1)^{(2-p)}), & \text{if } p = 0, 1, \end{cases}$$

while

$$\xi_{N-1}^{\lambda} = ((q+q'+1)^{(p')}, (q+q')^{(N-1-p')}), \quad 0 \leqslant p' < N-1,$$

with $p + q = q'(N-1) + p'$. One can rewrite (3.4.4) in terms of the parameters q and p. In particular, one can easily check that (3.4.4) is always satisfied for $N = 2$, hence, we have exact sequences

$$0 \to W_2(\lambda - 2) \to W_2(\lambda) \to W_1(\lambda) \cong V(\lambda) \to 0.$$

For $q = 1$ the lack of injectivity of ϖ_N follows from (3.4.1). Hence, we may assume $q > 1$.

Example 3.4.1. The smallest example of non injective ϖ_N happens with $N = 3$ and $\lambda = 6$. One can easily check that (3.4.4) is not satisfied. Alternatively, note that $W_3(6) \cong D(2,6)$ has simple socle. If ϖ_3 were injective, then it would contain a submodule isomorphic to $\tau_2 W_3(4)$ which does not have simple socle by Example 3.3.7. In fact, in this case, we see that there exists a short exact sequence

$$0 \to V(0,2) \to W_3(4) \xrightarrow{\varpi_3} \ker(\pi_3) \to 0$$

since

$$\delta_3(6) - \delta_2(6) = 11 \qquad \text{and} \qquad \delta_3(4) = 12.$$

We now explain why $\ker(\pi_3)$ is not a CV module in this case. If it were, then $\ker(\pi_3)$ would be isomorphic to $CV(\xi)$ with ξ being a partition of 4. However, using Theorem 2.2.6, one easily sees that $\dim(CV(\xi)) \neq 11$ for all such partitions.

The only inclusion of a truncated Weyl module in $W_3(6)$ comes from Theorem 2.2.6 which reads

$$0 \to \tau_2 W_1(2) \to W_3(6) \to V(\xi) \to 0 \qquad \text{with} \qquad \xi = (3,2,1).$$

In particular, $soc(W_3(6)) \cong V(2,0) \cong \tau_2 W_1(2)$. ◇

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