UNIVERSIDADE ESTADUAL DE CAMPINAS
INSTITUTO DE FILOSOFIA E CIÊNCIAS HUMANAS

BRUNO RAMOS MENDONÇA

TRADITIONAL THEORY OF SEMANTIC INFORMATION WITHOUT SCANDAL OF DEDUCTION: A moderately externalist reassessment of the topic based on urn semantics and a paraconsistent application

TEORIA TRADICIONAL DA INFORMAÇÃO SEMÂNTICA SEM ESCÂNDALO DA DEDUÇÃO: Uma reavaliação moderadamente externalista do tópico baseada em semântica urna e uma aplicação paraconsistente

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Orientador: Dr. WALTER ALEXANDRE CARNIELLI

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Prof. Dr. Marcelo Esteban Coniglio

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RESUMO

A presente tese mostra que é possível reestabelecer a *teoria tradicional da informação semântica* (no que segue apenas TSI, originalmente proposta por Bar-Hillel e Carnap (1952, 1953)) a partir de uma descrição adequada das condições epistemológicas de nossa competência semântica. Uma consequência clássica de TSI é o assim chamado *escândalo da dedução* (no que segue SoD), tese segundo a qual verdades lógicas têm quantidade nula de informação. SoD é problemático dado que conflita com o caráter ampliativo do conhecimento formal. Baseado nisso, trabalhos recentes (e.g., Floridi (2004)) rejeitam TSI apesar de suas boas intuições sobre a natureza da informação semântica. Por outro lado, esta tese reconsidera a estratégia de assumir a *semântica urna* (RANTALA, 1979) como o pano de fundo metateórico privilegiado para o reestabelecimento de TSI sem SoD. A presente tese tem o seguinte plano. O capítulo 1 introduz o plano geral da tese. No capítulo 2, valendo-se fortemente de trabalhos clássicos sobre o *externalismo semântico*, eu apresento algum suporte filosófico para essa estratégia ao mostrar que a semântica urna corretamente caracteriza as condições epistemológicas de nossa competência semântica no uso de quantificadores. O capítulo 3 oferece uma descrição precisa da semântica urna a partir da apresentação de suas definições básicas e alguns de seus teoremas mais fundamentais. No capítulo 4, eu me concentro mais uma vez no tema da informação semântica ao formalizar TSI em semântica urna e provar que nesse contexto SoD não vale. Finalmente, nos capítulos 5 e 6 eu considero resultados modelo-teóricos mais avançados sobre semântica urna e exploro uma possível aplicação paraconsistente das ideias principais dessa tese, respectivamente.

Palavras-chave. Informação semântica; Teoria de modelos; Externalismo semântico; Escândalo da dedução; Semântica urna; Teorema de Fraïssé-Hintikka; Categoricidade; Paraconsistência; Lógicas da inconsistência formal.

ABSTRACT

This thesis shows that it is possible to reestablish the *traditional theory of semantic information* (TSI, originally proposed by Bar-Hillel and Carnap (1952, 1953)) by providing an adequate account of the epistemological conditions of our semantic competence. A classical consequence of TSI is the so-called *scandal of deduction* (hereafter SoD) according to which logical truths have null amount of information. SoD is problematic since it does not make room for the ampliative character of formal knowledge. Based on this, recent work on the subject (e.g., Floridi (2004)) rejects TSI despite its good insights on the nature of semantic information. On the other hand, this work reconsiders the strategy of taking *urn semantics* (RANTALA, 1979) as a privileged metatheoretic framework for the formalization of TSI without SoD. The present thesis is planned in the following way. Chapter 1 introduces the thesis' overall plan. In chapter 2, relying heavily on classical works on *semantic externalism*, I present some philosophical support for this strategy by showing that urn semantics correctly characterizes the epistemological conditions of our semantic competence in the use of quantifiers. Chapter 3 offers a precise description of urn semantics by characterizing its basic definitions and some of its most fundamental theorems. In chapter 4, turning the focus once again to semantic information, I formalize TSI in urn semantics and show that in this context SoD does not hold. Finally, in chapter 5 and 6 I consider more advanced model-theoretic results on urn semantics and explore a paraconsistent possible application of the present idea, respectively.

Keywords. Semantic information; Model theory; Semantic externalism; Scandal of deduction; Urn semantics; Fraïssé-Hintikka theorem; Categoricity; Paraconsistency; Logics of formal inconsistency.

Contents

Chapter 1

Introduction

This thesis has as its central goal to provide a defense of the *traditional theory of semantic information* (hereafter TSI, originally proposed by Bar-Hillel and Carnap (1952, 1953)). The general purpose of TSI is to provide criteria for the measurement of the amount of information that is carried by sentences solely in function of their propositional content. Broadly speaking, in model-theoretic terms, TSI measures the amount of information of a sentence by counting how many models it has: the more models it has, the less informative it is. Now it is well-known that, in its original setup, TSI implies the *scandal of deduction* (SoD) (Such nomenclature is due to Hintikka (1970a)), a problematic thesis according to which logical truths carry null amount of information. In this sense, the main job here is to show that TSI can be subtly modified in order to block SoD.[1]

Preliminarily, let me restrict the scope of the present work in two ways. First of all, this work's scope is limited by the consideration of first order languages only. This limitation of scope is reasonable, given that my direct motivation is to provide an account of the semantic information carried by logical truths. Furthermore, let me note that "information" is a polysemic expression (see Adriaans (2013) for a presentation of different possible interpretations of the concept) in a so pervasive sense that, even focusing on the subcategory "semantic information", we still face difficulties in delimiting an univocal notion. Although I do not intend to face this challenge here, according to Bar-Hillel and Carnap's own original motivation, I would like to make progress in a *minimum* characterization of semantic information associated with the mere propositional content of sentences (in disregard of their actual truth-values). In other words, the minimum semantic information of a sentence is all the information that is grasped by anyone[2]

[1]In an earlier version of this project I have also intended to face the problem that TSI implies the so-called *Bar-Hillel and Carnap's paradox* of semantic information (hereafter BCP) according to which contradictions carry maximum amount of information. In fact, BCP and SoD are dual facts about TSI and the modifications of the traditional theory that I am proposing here are sufficient for blocking BCP as well. However, the philosophical argument that justifies this work is better-suited as a treatment of the problem of SoD. Hence, I will focus on this problem here and will ignore the problem of BCP.

[2]The use of the expression "grasp" may mislead the reader in thinking that "minimum semantic information" denotes some kind of psychological entity. This is not the case: by this I mean a purely semantic entity grasped by anyone with any mental configuration who understands the considered sentence.

who simply understands the sentence.[3]

Naturally, it is reasonable to ask: why should we bother with re-establishing TSI without SoD?[4] Roughly speaking, the motivation for such a work is the recognition that mathematical knowledge, in general, and logical knowledge, in particular, have an *ampliative character* in the sense that when we learn of, for instance, the validity of a mathematical theorem we discover something *new* about the world as it is.[5] As an evidence of the ampliative character of logical knowledge, note that the question of whether a certain first order sentence expresses a logical truth is in general just *semi-decidable* (Cf. BOOLOS; BURGESS; JEFFREY, 2002, pp. 126-132). Moreover, even though the propositional fragment of first order logic is, by its turn, decidable, the problem of determining whether a quantifier-free sentence expresses a logical truth usually has huge computational complexity. Now, the ampliative character of logical knowledge is incompatible with SoD: if a logical truth does not carry any minimum semantic information, nothing new can be learned from such a sentence, what contradicts the ampliative character of logical knowledge.

Despite the evidence for the ampliative character of logical knowledge, we inherit from philosophical tradition the opposite, widely shared idea that logical knowledge is non-ampliative. So, such preliminary justification of the present thesis needs greater development. According to philosophical tradition, we can say (not without some controversy) that at least logical knowledge is analytic knowledge.[6] However, philosophers have often equated analyticity with non-ampliative knowledge. For an important example, this equation is crystallized in Wittgenstein's highly influent "picture theory" of meaning in *Tractatus*. Details apart, his picture theory characterizes propositions similarly to pictures of *state of affairs*, the basic components of reality (WITTGENSTEIN, 1963, §1.1). According to Wittgenstein (1963, §§2.224,2.225), state

[3]I do not discard that a sentence can carry even more information than this. In this sense, the theory of semantic information that I am advancing here does not intend to capture the complete informativeness of a sentence, but just what I am calling its minimum informativeness. Someone could sustain, as Floridi (2004), that sentences carry also an *additional* amount of information associated with their actual truth-values. However, for present purposes it is sufficient to consider only a theory of minimum semantic information.

[4]Note that Bar-Hillel and Carnap acknowledge that SoD is a consequence of TSI but do not evaluate it as a problematic feature of the theory. According to the authors, logical truths in fact do carry a non-null amount of information. However, in their opinion, the non-null informativeness of logical truths has nothing to do with semantic content, but just with a "psychological content" that people associate with sentences (BAR-HILLEL; CARNAP, 1952, p. 3).

[5]To make it very clear, by recognizing that mathematical and logical knowledge have an ampliative character I surely do not intend to equate those kinds of knowledge with knowledge generated by either inductive or abductive reasoning. The body of knowledge of Mathematics and logic flourishes strictly through deductive reasoning. However, this state of affairs is pretty compatible with the acknowledgement that we learn something new, something that we did not know previously, every time that a new logico-mathematical theorem is proved.

[6]I say "not without some controversy" obviously because of Quine's skeptical arguments against the notion of analyticity. Unfortunately, I cannot consider this issue here, limiting myself to dogmatically rely on a reply by Grice and Strawson (1956): if we read Quine's arguments as claims that analyticity is an incoherent concept, it is reasonable to reply that a respectable philosophical tradition commits itself with a widely shared and well-behaved analytic-synthetic distinction, even though we cannot provide an adequate, explicit definition of it. Further, if we read Quine's arguments as claims that there are no analytic sentences, it is reasonable to reply that there is a wide agreement in philosophical tradition that at least logical truths are analytic (For more on Grice and Strawson's reply to Quines's arguments, see Juhl and Loomis (2009 , pp. 102-104)).

of affairs are essentially *bipolar*, that is, it is an essential property of a state of affairs that it is possible for it to be the case as well as it is possible for it to not be the case. Now, since analytic sentences are *necessarily* true, it follows that those sentence do not picture any state of affairs. In sum, analytic sentences are *senseless* (Ibid., §4.462).[7] In turn, in early Wittgenstein's opinion analytic sentences *show* the structure of reality, something that is always *presupposed* in the use of language and in its intended match with the world as it is (Ibid., §2.172). There are no new facts about logic and Mathematics. Actually, there are no facts at all in formal sciences, a claim that influenced Carnap as well as other members of Vienna's circle in sustaining that logical and mathematical knowledge consist just in (stipulated) linguistic rules which are always presupposed in every meaningful use of language (Cf. JUHL; LOOMIS, 2009, pp. 20–ff.).

I agree with Wittgenstein's premise that bipolarity is an essential property of propositions, but not exactly in the same Wittgensteinian terms.[8] Propositions are the contents of possible *claims*, that is, possible entries in the public space of belief commitment and reasoning. Now, any such entry is in principle subject to doubt, that is, it needs to be possible *for us* to wonder whether such a claim is in fact true. But this is only possible *for us* if *we can* picture a situation in which that claim is not true. So, maybe we should read the bipolarity of propositions through the lenses of a conceptual distinction between semantic and epistemic modalities associated with our linguistic competence. Our ability to understand sentences that say something about a certain object depends on certain epistemic possibilities of us (namely, the possibility of doubting the truthfulness of a given claim) which perhaps do not reduce themselves to the possibilities of the object itself. More precisely, even though logical truths just provide knowledge about logical concepts, concepts that form the *necessary* structure of language, they have a *contingent* aspect associated with our knowledge of the actual character of that structure, and in this sense they are contents of possible claims: they do have a sense, namely, they *mean* the limits of sense. In fact, one of the central claims of the present thesis is that, although logic is presupposed in any meaningful use of language, this does not mean that we know what those bounds are. We can be surprised, and normally we do, in discovering how narrow the logical limits of sense are. However a logical truth is an analytical sentence, and, consequently, it is necessarily true. So, I still maintain that logical truths do not express contingent facts. The *prima facie* contingency of logic lies just in our knowledge about the subject. We can usually mistake what are the real logical conditions of our linguistic practice. Still, this is a relevant sense in which logic can augment our knowledge of the world: more specifically, it can augment our knowledge of the logical structure of our language and reality.

The remarks above on an epistemic reading of the bipolarity of propositions are pretty much

[7] There is a long debate about Wittgenstein's claim that analytic sentences are senseless, a debate that is inserted in a wider discussion on early Wittgenstein's notion of "lacking of sense" (and the place of Tractarian phrases themselves in this philosophical framework), see Moyal-Sharrock (2007) for an overview. For questions of space, I will entirely skip this exegetical discussion here.

[8] In fact, early Wittgenstein had not a precise argument for the essential bipolarity of propositions, being an unquestioned preconception of his work. About *Tractatus*' dogmatism on the nature of propositions, see McGinn (2006, pp. 21-27).

general: how can we give flesh and blood to these ideas? Consider that ϕ is a sentence expressing a logical truth. If we do not know this, then, given the analytic character of any logical truth, we do not know the real *truth-conditions* of this sentence. Hence, this observation shows that (i) *understanding a sentence does not require full knowledge of its truth-conditions*. Of course, some *partial* knowledge of truth-conditions is still a necessary condition for semantic competence. So, secondly, we can argue that (ii) *understanding a sentence requires some partial knowledge of its truth-conditions*. Let me call the concept of semantic competence defined by claims (i) and (ii) the *epistemically weak account of semantic competence*. Now, if this account is correct, anyone who simply understands a sentence has a partial knowledge of its truth-conditions. In this sense, the minimum semantic information of a sentence is associated with some partial knowledge of its truth-conditions.

This work shows that, in certain special cases, we can formalize this partial knowledge of the truth-conditions of a sentence ϕ as the set of models of ϕ in a non-classical logic known as *urn semantics* (originally proposed by Rantala (1979)). Moreover, if we substitute classical logic by urn semantics as our underlying logic, we can formalize TSI without SoD. So, I hope to show that, based on a more adequate epistemological conception of semantic competence, TSI provides accurate criteria for measuring the amount of minimum semantic information carried by sentences of first order languages.

This is enough to give a general description of the thesis' central claim. Still, several questions remain in need of development. I list some of them below.

A. What kind of partial knowledge of truth-conditions is associated with semantic competence?

B. How does urn semantics formalize this partial knowledge of truth-conditions?

C. What happens with TSI when we replace classical logic by urn semantics as its underlying semantic framework?

D. Finally, does this project allow further generalizations (e.g., the consideration of minimum semantic information in the context of non-classical logics)?

In the rest of this work I consider all of these questions. In what follows, I summarize the way in which I tackle each one of them. First, question A is too general and hugely escapes the scope of this work. In fact, the partial knowledge of truth-conditions associated with semantic competence depends on semantic aspects of both logical and non-logical vocabulary. However, the consideration of non-logical vocabulary is relatively uninteresting in this context of discussion. Indeed, even considering just logical vocabulary, to offer an account of the epistemological conditions for the competent use of this vocabulary is still a huge task. For methodological reasons, in this work I focus particularly on the examination of the epistemological basis of our semantic competence in the use of quantifiers. Let me note, however, that the choice of restricting my focus in such a way is based on methodological reasons: whether this proposed solution can be generalized is an open question to be left for further investigation.[9]

Chapter 2 aims to present an answer to question A (under such qualifications). First I show that classical works on *semantic externalism* give support for the epistemically weak account of semantic competence. Semantic externalism says that the semantic contents of linguistic expressions are not constituted by facts about either our mental or physiological history, but are constituted by facts about the external world. To cite a recurring example in this literature, from the perspective of semantic externalism, the meaning of "water" does not depend on the way that English speakers' minds and bodies are constituted but just on the real nature of the natural kind water (for a broad exposition on externalism in Philosophy of mind and language, see Lau and Deutsch (2016)).

Classic arguments on semantic externalism such as Putnam's *Twin Earth argument* (PUTNAM, 1973) as well as Kripke's *puzzle about belief* (KRIPKE, 1979) show evidence that semantic competence is accompanied by a partial knowledge of truth-conditions. Moreover, Loar (1996), Stalnaker (1990) and Jackson (2004) offer classic accounts of the partial knowledge of truth-conditions considered in Kripke's and Putnam's mental experiments. Now, I claim two things: first, even though Putnam's and Kripke's mental experiments consider cases in which a person partially ignores the correction criteria for the use of non-logical vocabulary, they suggest analogous cases in which a person partially ignores the correction criteria for the use of quantifiers. Secondly, I argue that some improvement of Loar's, Stalnaker's and Jackson's accounts generates an adequate analysis of those analogous cases. More specifically, as an improvement of these accounts, I suggest that the partial knowledge of truth-conditions associated with our semantic competence in the use of quantifiers reduces itself to a partial knowledge of the possible domains of the quantifiers occurring in a given sentence.

In chapter 3, I show that we can formalize this general idea on the semantics of quantifiers in urn semantics. In recent literature there is a renewed interest in urn semantics (for a nice example, see French (2015)), but a comprehensive presentation of its formal properties and most basic theorems is still missing. So, in chapter 3 I define urn semantics and introduce some of its most important properties. Rantala (1979) originally introduced urn semantics in *game-theoretic* terms. In fact it is still an open question whether this semantics accepts a *Tarskian* characterization. Loosely speaking, by a Tarskian semantic framework I mean any framework of semantics that describes the concepts of *satisfaction* and, consequently, *truth* in truth-functional terms. In chapter 3 I affirmatively solve this question by providing a Tarskian framework for urn semantics. The Tarskian characterization of urn semantics is an important achievement since it enables us to present easier proofs of already known meta-theorems as well as to prove some new results.

As Rantala soon recognized, urn semantics can be defined in two different ways, that is, it

[9]In fact, there is in recent literature some interesting proposals on the topic which focus on classical propositional logic. For a nice entry in this direction, see D'Agostino and Floridi (2009). The eventual comparison of the present work with such proposals is a topic of further research.

can be defined in terms of either *perfect* or *imperfect* urn semantics.[10] Based on the Tarskian framework for urn semantics that is offered in this thesis, I precisely define these systems and show some common properties of them. Thus, I show that urn semantics, in both of its versions, determines *decidable* systems of logic. Further, I prove that these semantics accept quite liberal *characterization theorems*. By a characterization theorem for a logical system I mean any result that states necessary and sufficient conditions for some formula to be a theorem of the given logic (This terminology is suggested by Shoenfield (1967, p. 41)). These characterization theorems show that both systems of urn semantics yield easily reachable satisfiability conditions; consequently, they satisfy formulas that are unsatisfiable in classical logic. This is a crucial result for this thesis' central argument: since TSI measures semantic information by counting how many models a formula has, such characterization theorems have the consequence that some validities of classical logic do not have null information when TSI is formalized in urn semantics.

The proof of these characterization theorems depends on some auxiliary results. In particular, it depends on the fact that urn semantics validates a weaker version of the so-called *Fraisse-Hintikka theorem* (HODGES, 1997, pp. 84-85). I also prove that this auxiliary result holds. Note that this is an important result on its own for the present investigation since the Fraïssé-Hintikka theorem plays a decisive role in the formalization of TSI an in the proof of more advanced results about urn semantics.

Chapter 4 considers question C. This chapter is divided in two parts. In the first part of the chapter, after a general outline of the main aspects of a formal characterization of TSI, I present this theory with urn semantics as its semantic background and I prove that, in this context, SoD does not hold. Further, in the second part of the chapter, I draw some comparison between the present formalization of TSI and some related alternative solutions of the problem, in particular I compare my proposal with suggestions by Hintikka (1970a, 1970b) and Jago (2009, 2013).

Whilst chapters 2-4 present the thesis' central core, the remainder chapters show more advanced results on the subject. As I have said in the beginning of this introduction, TSI's main goal is to measure the amount of minimum semantic information carried by a given sentence. Since TSI does this by counting the number of models of a sentence, in chapter 5 I consider the question of how many models (up to *isomorphism*) a given first order theory has in this semantics. The classic counterpart of this question is associated with important problems of classic model theory: for instance, in classic model theory it is interesting to examine whether and at what conditions a theory is *categorical*, that is, has only one model up to isomorphism.

Olin (1978) started to investigate this questions and obtained remarkable results on perfect urn semantics. Now, Olin's results rely on particular properties of this system that are not satisfied in imperfect urn semantics. My interest here is to consider what happens with categoricity in imperfect urn semantics. In this sense, I present some results on categoricity in finite models

[10]As it will be more clear in the rest of this work, this terminology is influenced by Rantala's game-theoretic approach on the subject.

for imperfect urn semantics. Further, I prove that imperfect urn semantics validates *compactness* and *Löwenheim-Skolem* theorems.

In chapter 6 I change the focus from classical to non-classical logics and start to investigate what happens with a similar reassessment of TSI when the original semantic background is not classical logic but some system of the family of *logics of formal inconsistency* (hereafter LFIs) (CARNIELLI; CONIGLIO; MARCOS, 2007). LFIs are paraconsistent logics that relativize the classical *principle of explosion* only for *consistent* formulas. That is, in these systems a contradiction $\phi \wedge \neg\phi$ implies anything else if and only if ϕ is a consistent formula. In LFIs not every formula is assumed to be consistent. In effect, the assumption of consistency of ϕ is made explicit by the assumption of $\circ\phi$, in which "$\circ$" is a primitive logical symbol denoting consistency. In sum, LFIs are logics whose technical framework enables us to talk about the consistency and inconsistency of formulas within object-language itself, and in doing so they provide a way of restricting the validity of the classical principle of explosion.

The results on TSI and LFIs that I obtain here have a still very initial character: I show here that QmbC, a minimal first order system of LFIs, validates a weaker version of Fraïssé-Hintikka theorem. In recent literature there is an increasing interest in LFIs as a promising source of new approaches to traditional problems in philosophical logic. Now, with the present work I hope to achieve (in further studies) two new applications of TSI. First, since TSI associates semantic information with consistency, given LFIs' internalization of consistency, I hope to achieve a further internalization of *informativeness* in this non-standard context. Secondly, since TSI stratifies informativeness in terms of a degree of consistency, an internalization of TSI in the context of LFIs may provide a way for defining a class of finer $\circ$-operators, characterizing the consistency of stricter sets of formulas. In this sense, the results obtained in chapter 6 are the initial steps in this research project.

By proposing a defense of TSI I follow here a *conservative* approach to the problem of semantic information. This approach to the subject is in clearly opposition to the mainstream strategy in recent literature. So, we can restate, now in a stronger version, the question with which I have started this introduction: why should we bother with re-establishing TSI *anyway*? This is a tough question that greatly escapes the scope of this thesis. A proper answer to it depends, first, on a comprehensive review of the so many different strategies of solution of the problem of semantic information (a task that I do not pursue here: for a nice work in this direction, see Bremer (2003)), and, secondly, on a detailed comparison between these strategies and my own. However, at least I offer a negative and a positive remark as a partial answer.

As a negative remark on the problem we should note that, on the contrary to the most part of the recent work on the subject (for an influent example, see Floridi (2004, 2005)) the present thesis is not committed with the claim that only true sentences carry non-null information. Let us call this claim the *veridicality principle*. As Fetzer (2004, pp. 224-225) argues, the veridicality principle seems to be untenable in face of simple counter-examples, which call for the definition of a notion akin to minimum semantic information, the object of TSI. So, whether TSI is an adequate theory of information is a question to be analyzed in the context of investigation of the notion of minimum semantic information, in disregard of theories that embrace the veridicality principle. As a positive remark on the problem, we should note that TSI coheres with strong insights on the nature of semantic information. More specifically, since in TSI the amount of information of a sentence is in inverse proportion to the number of models of it, TSI is compatible with the intuition that semantic information is directly proportional to the preciseness of propositional content (ADRIAANS, 2013, sec. 3). Hence, the preservation of TSI is in large part the preservation of such insights as well.

In the development of this work, I explore some general set-theoretical terminology and notation, just as we can find it in a classical textbook such as Suppes (1960). In particular, I denote the concatenation of ordered sequences by means of the symbol "$\frown$". In order to simplify exposition, I use some metatheoretical notation such as "$\Leftrightarrow$" for "if and only if". Further, I avoid using quotation marks, generally appealing to context for distinguishing use and mention.

Some parts of this work have already been published (in co-authorship) elsewhere. In particular, chapter 6 is based on a paper authored by me and prof. Walter A. Carnielli that has been accepted for publication in IGPL's special issue "Recovery Operators and Logics of Formal Consistency & Inconsistencies".

Part I

Some results on the logic and philosophy of semantic information

Chapter 2

An account of the epistemological conditions of the semantic competence in the use of quantifiers

2.1 Introduction

In chapter 1, in opposition to a *naïve* view on TSI, I argued that the minimum semantic information of a sentence is associated with some partial knowledge of its truth-conditions. Informally speaking, this thesis' general motive is the following: when an ordinary speaker understands a sentence, this individual does not grasp a full characterization of the set of situations in which it is true; rather, she grasps just a partial description of this set. Therefore, a full development of this thesis demands a preliminary account of what kind of thing this partial knowledge of truth-conditions is. My central concern in this chapter is to provide a proper answer to this question focusing particularly on an analysis of our semantic competence in the use of quantifiers.

As I anticipated in thesis' introduction, my strategy here is the following. In section 2.2, based on a revision of the literature on semantic externalism, we analyze classical mental experiments such as Putnam's Twin Earth argument (PUTNAM, 1973) and Kripke's puzzle about belief (KRIPKE, 1979). These mental experiments indicate that often semantic competence is associated with a partial knowledge of truth-conditions. Now, although these mental experiments consider particularly the semantics of non-logical vocabulary, based on them, I suggest two analogous cases focused on the semantics of quantifiers. In section 2.3, I revise some classical accounts of Putnam's and Kripke's mental experiments. First, I analyze the concept of *realization-conditions* proposed by Loar (1996); secondly, I examine some further developments of this concept suggested by Stalnaker (1990) and Jackson (2004) from the perspective of *2-dimensional semantics*. Finally, in section 2.4, I argue that, with some improvements, these 2-dimensional accounts offer an adequate account of the epistemological conditions of our use of quantifiers.

2.2 Mental experiments on semantic externalism

Putnam's and Kripke's mental experiments on semantic externalism display some evidence that semantic competence does not demand full knowledge of truth-conditions. Let us consider Putnam's Twin Earth argument first. Basically speaking, this mental experiment aims to show that language speakers' mental and physiological properties are irrelevant for the determination of the semantic contents of *natural kinds* (linguistic expressions like "water", "gold" etc.). We can formulate this mental experiment as follows.

Consider an identical copy of Earth, let me call it Twin Earth, with the exception that "water" in Twin Earth denotes a compound other than H_2O, let me call it XYZ. Further, consider an Earthling and her "twin" in Twin Earth. Putnam observes that, even though both individuals share the same life history and are physiologically and psychologically indiscernibles, when they understand the sentence "water quenches thirst," they understand distinct propositional contents: in Earth, the truth-conditions of "water quenches thirst" is the class of models that satisfy the fact that H_2O quenches thirst, whereas in Twin Earth its set of truth-conditions is the class of models that satisfy the fact that XYZ quenches thirst. Then, Putnam asks us to consider an extraordinary situation in which those individuals interchange worlds. Since this contextual exchange is unrecognizable for them, they preserve in their new contexts the linguistic competence in using the sentence "water quenches thirst" but they do not know that the sentence's relevant truth-conditions have changed. Now, this happens because already in the original context they did not fully know what the sentence's truth-conditions actually were. Hence, Putnam concludes, the linguistic competence of an ordinary speaker is relatively independent of her knowledge of truth-conditions.

Of course, this does not mean that semantic competence is fully independent of *some* knowledge of truth-conditions. In this sense, Putnam (1973, p. 706) suggests the so-called hypothesis of *division of linguistic labor* according to which every linguistic community is divided in, on one hand, a group of experts holding the correction criteria for the use of language and, on the other hand, a group of laymen which, despite the fact that they do not master that correction criteria, use language thanks to a cooperative relation with the former group. So, even from an externalist perspective on semantics, semantic competence can be seen to rely on some knowledge of truth-conditions possessed by (some) language speakers.

In what follows, by *transparent* belief ascriptions I mean ascriptions of the form "S believes that ϕ", in which ϕ is any sentence and S is an ordinary language speaker. Further, by *opaque* belief ascriptions I mean ascriptions of the form "S believes 'ϕ'", in which 'ϕ' is the name of the sentence ϕ. The fundamental difference between transparent and opaque belief ascriptions is that the occurrence of ϕ in transparent ascriptions can be substituted by any equivalent sentence preserving the ascription's truth-value.

By its turn, in Kripke's puzzle we are asked to consider Pierre, a french person who does not *know* that the French word "Londres" and the English word "London" name the same city. Now, since Londres is London, the transparent ascriptions "Pierre believes that London is pretty" and "Pierre believes that Londres is pretty" are equivalent. However, given that, by assumption, Pierre does not know that "London" and "Londres" are both names for the same city, the opaque ascriptions "Pierre believes 'Londres is pretty'" and "Pierre believes 'London is pretty' " are not equivalent: in fact, Pierre behaves differently with respect to each one of these beliefs, that is, Pierre attributes distinct inferential roles to each one of them. Moreover, we could conceive of cases in which just one of the opaque ascriptions is true. Now, this difference between transparent and opaque belief ascriptions shows that Pierre has just a partial knowledge of the truth-conditions of "London is pretty" and "Londres is pretty".

Putnam's and Kripke's mental experiments accept a great number of variations (see Loar (1996) for a comprehensive list of these variations). By their turn, the following examples present new variations focused on the semantics of quantifiers. The formulation of these examples involves the consideration of some logical results whose technical details are ommited (for more technical details, see Shoenfield (1967) and Boolos, Burgess and Jeffrey (2002). The following example 2.1 is a variation of Twin Earth's scenario.

Example 2.1 *Consider two worlds (for simplicity, Earth and Twin Earth) and a commuter between them (in what follows, let us call him Paul).*[1] *Consider that (by an eccentricity) Paul's knowledge of arithmetic was obtained in terms of a first order formalization of* Peano's arithmetic. *In this sense, Paul's arithmetic knowledge is limited to the first order theorems of Peano's arithmetic, nothing else. Further, let us consider that Paul ignores any metatheoretic results about first order logic, that is, he has practice in working with some deductive system for first order logic but has never studied any of its metatheoretical properties.*

Assume that earthian mathematicians, differently from their twin earthian counterparts, accept the validity of the ω-rule, that is, they accept that $\forall x\phi \Leftrightarrow \phi(0), \ldots, \phi(n), \ldots$, for any arithmetic sentence ϕ. Consequently, it is arguably the case that universal arithmetic sentences have different truth-conditions in Earth and Twin Earth.

Now, consider the following statement of the axiom of successor*:* "for any x, there is a successor of x." *Since this is a universal arithmetic sentence, it has different truth-conditions in Earth and Twin Earth. However, given the assumed limitation of Paul's arithmetic knowledge, he ignores such differences of truth-conditions. On the other hand, in either context, Paul has some partial knowledge of the truth-conditions of the axiom of successor, namely, he at least knows that if the considered axiom is true, then, every number n whose existence is proved from Peano's axioms (that is, every natural number n) has successor.*

Note the analogy between example 2.1 and Putnam's mental experiment: in Putnam's case, an individual is unable to differentiate between two readings of "water quenches thirst". Anal-

[1] In honor of Paul Benacerraf, who proposed a similar mental experiment in Benacerraf (1965).

ogously, Paul is unable to distinguish between two readings of the axiom of successor. Now, someone could object that this analogy is untenable given the specificities of *a priori* knowledge. The objection would go more or less like this. Differently from a posteriori knowledge, a priori knowledge is independent of any particular configuration of the external world. There is no fact about the world that can change the truth-value of an arithmetic sentence. Consequently, it makes no sense to suppose that the truth-conditions of a mathematical or logical sentence are sensitive to contextual changes.

This objection relies on the idea that a priori knowledge is an *environmentally independent* kind of knowledge (Cf. HAWTHORNE, 2007, pp. 201–ff.). Despite the apparently plausibility of this idea, there are serious problems with it. A particularly important problem is that the environmentally independent account disregards the social nature of a priori knowledge (Ibid., pp. 206–207).[2] There are facts about our mathematical community that determine the truth-conditions of arithmetic sentences. (Societal) things being otherwise, those truth-conditions would be different as well. In this sense, insofar as the earthian mathematicians accept the validity of the ω-rule, then their epistemic state fixes the truth-conditions of universal arithmetic statements in a certain way.

Kripke's puzzle about belief suggests the following analogous example.

Example 2.2 *Consider Alonzo*[3] *is a person who, as any beginner logic student, knows the basics of* recursive functions *and* Turing machines. *On the other hand, in general, Alonzo ignores metatheoretical questions such as to whether Church's thesis is correct or as to whether recursivity is equivalent to Turing-computability. In this sense assume that, in particular, Alonzo believes "for any computable function f, there is a Turing machine F that is equivalent to f" but he does not believe "for any computable function f, there is a recursive function g that is equivalent to it". Let us denote these sentences as ϕ_1 and ϕ_2, respectively. Of course, these sentences have the same truth-conditions. So, Alonzo's epistemic situation does not exclude as logically impossible epistemically possible scenarios in which some computable function f is Turing-computable but* there is not *a recursive function g equivalent to it.*

Again, observe the parallel between example 2.2 and Kripke's puzzle. In Kripke's, Pierre wrongly differentiates the equivalent sentences "London is pretty" and "Londres is pretty". Analogously, in example 2.2, Alonzo wrongly distinguishes the sentences ϕ_1 and ψ_2. In the former case, Pierre's epistemic situation can be redirected to different and independent partial characterizations of their truth-conditions. Analogously, in example 2.2 Alonzo has some ignorance about the domains of the quantifiers occurring in ϕ_1 and ϕ_2. In the rest of this chapter we give a precise account of what is happening in these examples.

[2]Note that this is not an exclusive feature of a priori knowledge: a posteriori knowledge depends on social facts as well (Cf. BURGE, 1979).

[3]In honor of Alonzo Church: the equivalence between Turing machines and recursive functions is an important evidence in support of Church's thesis.

2.3 Some classical accounts of Putnam's and Kripke's scenarios

Any examination of examples 2.1 and 2.2 needs to begin by the consideration of some classical analyses of Putnam's and Kripke's mental experiments. In this section I consider the account by Loar (1996) and, next, consider some further developments of Loar's ideas by Stalnaker (1990) and Jackson (2004).

The central claim in Loar (1996) is that transparent and opaque belief ascriptions have different identity criteria. In order to show this, first, Loar explores variations of the Twin Earth argument (similar to example 2.1) in which, for two non-equivalent sentences ϕ and ψ and for an individual S, although the opaque ascriptions "S believes 'ϕ' " and "S believes 'ψ' " are equivalent, their corresponding transparent ascriptions are not equivalent. That is, such examples show that (i) *a difference between transparent belief ascriptions does not imply a difference between their corresponding opaque ascriptions.* Secondly, based on variations of Kripke's puzzle (similar to example 2.2), Loar builds examples showing that, for two equivalent sentences ϕ and ψ, even though the transparent ascriptions "S believes that ϕ" and "S believes that ψ" are equivalent, the corresponding opaque ascriptions are not. So, such examples show that (ii) *identity between transparent belief ascriptions does not imply identity between the corresponding opaque ascriptions.*

According to Loar's diagnosis, the difference between transparent and opaque belief ascriptions can be reduced to a difference of content: whilst transparent belief ascriptions are associated with the truth-conditions of believed sentences, opaque ascriptions have to do with an epistemologically nuanced perspective on such truth-conditions. For instance, considering a variation of Twin Earth argument:

> Similarly, we may imagine a commuter between Earth and Twin Earth who is bi-worldly in his language without knowing of the systematic referential differences between English and Twin English. He would assert "water quenches thirst." Again it seems that two belief ascriptions are in order, but that they should be seen as merely different extrinsic descriptions of what is, as regards psychological explanation, *the commuter's one way of conceiving things* (Ibid., p. 187, our italics).

However, Loar's diagnosis faces at least two problems (Ibid., pp. 187–ff.). First, the reader could ask what is the content of opaque belief ascriptions in contrast with the content of transparent ascriptions. A transparent belief ascription "S believes that ϕ" says that an individual S believes in a sentence with definite truth-conditions ϕ. How can we describe the content of "S believes 'ϕ' " and, at the same time, differentiate it from its corresponding transparent ascription? Secondly, we must be careful to ensure that any answer to this question preserves the *intentionality* of belief attitudes. By saying that belief attitudes have an intentional character I mean the fact that someone's belief in a sentence ϕ can be evaluated as either a correct or incorrect belief. The intentionality of a belief in ϕ seems to be associated with the truth-conditions of ϕ. Thus, a characterization of opaque belief ascriptions that disregards truth-conditions runs the risk of not contemplating the intentionality of belief attitudes.

According to Loar, the content of opaque belief ascriptions is associated with what he calls their *realization-conditions*. Loar loosely describes the realization-conditions of a sentence ϕ as the set of possible worlds that make ϕ true in case that someone's conceptions of the propositional content of ϕ "are or were not misconceptions" (Ibid., p. 188). In order to clarify this idea, Loar appeals to a comparison with *indexical* sentences:

> Suppose I find a diary with the entry "Hot and sunny today; phoned Maria to invite her to the beach." Now, the date has been torn off the page. Still I appear to understand the diarist's explanation of his/her phoning Maria, despite not knowing the truth-conditions (in one sense) of the thought expressed by "hot and sunny today." Is there not, however, a sense in which I do know the truth-conditions? Suppose on Tuesday one thinks, "it is hot and sunny today" and on Friday one thinks, "it is hot and sunny today." They have the same truth-conditions in the sense of conditions of truth in abstraction from context. Call them *context-indeterminate*, by contrast with the context-determinate truth-conditions [...] To put this together with the former point, we may say this: if psychological explanation involves intentionality, the *context-indeterminate realization-conditions* are all the intentionality required (Ibid., p. 188).

In the passage above, Loar brings to the fore the property of indexical sentences of having different truth-conditions in different contexts of use. Kaplan (1989) proposed a particularly important account of this property in terms of a *2-dimensional semantics*. Exploring a similar approach, Stalnaker (1990) suggested that the notion of realization-conditions accepts a 2-dimensional formalization.

Kaplan's analysis of indexicals provides a prototypical example of 2-dimensional approach to semantics. Kaplan acknowledges that the meaning of indexicals is the product of two complementing factors, namely, the *character* and the *content* of the indexical. The content of an expression is its semantic content: so, for example, the content of a name like "Kaplan" is the person named Kaplan, the content of "water" is the substance water etc. The character of an expression, by its turn, is a function mapping from contexts of use of a given expression to its possible contents. For instance, the character of the expression "I" determines that used by me "I" means the author of this work, used by Kaplan it means Kaplan etc. Derivatively, the character of a sentence depends on the character of its component expressions, and the content of a sentence in a certain context depends on the contents of its component expressions in the same context. For instance, the content of "I am in Earth" in the present context is the fact of the author of this work being in Earth right now.[4]

Kaplan's treatment of indexical expressions is an important influence for more general 2-dimensional semantics such as Stalnaker's 2-dimensional account of assertions. The *locus classicus* of his proposal is Stalnaker (1978). In this paper, Stalnaker develops the idea that the truth-conditions of *any* sentence depends always on its context of assertion. Therefore, for Stalnaker, as in Kaplan's analysis of indexicals – but now considering all types of sentences – contexts of assertion play a double semantic role: first, they attribute propositional contents to sentences; secondly, they assess propositional contents as true or false.

Stalnaker's 2-dimensional semantics faces the following problem: how is conversation even possible between people who are in contexts of linguistic use that attribute different propositional contents to one and the same sentence? In order to introduce Stalnaker's answer to this problem, consider an expansion of Putnam's Twin Earth argument such that, in addition to Earth and Twin Earth, there is also a pair of *doppelgängers* of Earth and Twin Earth to be denoted as W_e and W_t, respectively. In W_e, just as in Earth, "water" means H_2O but, different from Earth, "water quenches thirst" is a false sentence. By its turn, in W_t "water" means XYZ and, again, "water quenches thirst" is false. Following closely Stalnaker's own way of presenting such ideas, the matrix below summarizes the different readings of "water quenches thirst" in Earth, Twin Earth, W_e and W_t as well as the different ways that those worlds evaluate as true or false each one of these interpretations.

	Earth	Twin Earth	W_e	W_t
Earth	T	F	F	F
Twin Earth	F	T	F	F
W_e	T	F	F	F
W_t	F	T	F	F

Matrix 2.1

In Matrix 2.1, each horizontal line denotes the interpretation of "water quenches thirst" in one of the considered worlds. In this sense, the first horizontal line shows that "water quenches thirst" in Earth means that H_2O quenches thirst. Further, each vertical line of the matrix represents the different evaluations of the different interpretations of " water quenches thirst". In this sense, the second row of the first vertical line of the matrix shows that "water quenches thirst" in its Twin Earthian reading is false in Earth.

For Stalnaker (1978, pp. 153–ff.), one of the basic purposes of a conversational act is to minimize uncertainty of the speakers' set of assumptions about the differents ways reality can be. Since an individual S in Earth and her doppelgänger SS in Twin Earth understand different things by "water quenches thirst", S's assertion of this sentence in a conversation with SS is

[4]Kaplan (1989) formalizes such very general ideas on the semantics of indexical terms and sentences in terms of a specific modal logic whose technical details we skip in this presentation. We adopt the same methodology in the proceeding discussion about Stalnaker's contribution to the debate.

unable to minimize uncertainty by preserving just the cases in which H_2O quenches thirst. However, S's assertion can at least force a weaker minimization by excluding cases in which "water quenches thirst" in either its Earthian or in its Twin Earthian interpretation is false. So, if we consider that Earth, Twin Earth, W_e and W_t form S's and SS's set of assumptions about the different ways reality can be, S's assertion minimizes the uncertainty of this set by excluding W_e and W_t (Ibid., pp. 155–ff.). Consequently, in a conversation between S and SS, the assertion of "water quenches thirst" communicates a propositional content whose truth-conditions are represented by the diagonal line of the Matrix 2.1. Due to this property of the matrix, Stalnaker calls this propositional content as the *diagonal proposition* of the considered sentence (Ibid., p. 149). In more general terms, assume that $S_1, \ldots, S_n$ are speakers, $\{W_1, \ldots, W_m\}$ is their set of pressupositions of the ways in which reality can be and, for every $1 \leq i \leq m$, $\sigma(W_i)$ is the propositional content of ϕ in W_i: if any S_j asserts a sentence ϕ, by this he communicates something whose truth-value in W_i is equal to the truth-value of $\sigma(W_i)$ in W_i.

Stalnaker (1990, pp. 133–ff.) argues that the notion of diagonal proposition precisely captures the concept of realization-conditions.[5] However, as Stalnaker himself acknowledges, the notion of diagonal proposition does not seem to provide a partial description of the truth-conditions of a sentence.[6] A partial knowledge of the truth-conditions of a sentence ϕ should point to something common to all of its different readings in different linguistic contexts. However, we can build cases in which it is at least doubtful whether the diagonal proposition of a sentence is describing any common aspect of its different truth-conditions in different contexts of interpretation. For instance, we can conceive of alternative generalizations of Twin Earth argument with additional worlds in which "water quenches thirst" means radically different things such as "what time does the next bus leaves for the zoo?" (Ibid., pp. 135–136). Although it is doubtful whether there is anything in common between so different interpretations of "water quenches thirst", it is still possible to define the diagonal proposition of this sentence in this new situation. So, it seems that Stalnaker's notion of diagonal proposition falls short from characterizing the partial knowledge of truth-conditions associated with the act of understanding a given sentence.

However, note that among 2-dimensional semanticists there is a plurality of points of view on what is captured by diagonal propositions. Hence, Stalnaker's claim on the subject is far from

[5]Note that Loar himself expresses reserve about reconstructions of his idea in terms of Kaplanian characters. In this sense, Loar (1996, p. 191, fn. 13) rejects a comparison of his work with Fodor's account of opaque belief ascriptions (FODOR, 1987). Despite the question of whether such 2-dimensional semantics are legitimate reconstructions of Loar's ideas, I claim in the following that they suggest a nice formal characterization of the epistemological ground of our semantic competence.

[6]In fact, from a strongly externalist point of view, Stalnaker (1990, p. 132) argues that the act of understanding a sentence is not associated with any such perspectival grasping of truth-conditions. On the other hand, Stalnaker (1990, pp. 138–141) argues that the concept of diagonal proposition contemplates the philosophical motivations behind Loar's work. In this sense, Stalnaker claims that Putnam's and Kripke's cases advance a puzzle about belief ascription, but do not provide enough evidence for drawing an essential difference between two kinds of belief ascriptions. In his opinion, the diagonal proposition of "water quenches thirst" represents the different contents that a belief in it might have, but does not represent any context-invariant partial description of its truth-conditions.

being canonical. Against Stalnaker, philosophers like Jackson (2004) and Chalmers (2004) sustain that the diagonal proposition of a given sentence expresses a partial knowledge of its truth-conditions ascribable to anyone who understands the sentence. In what follows, let us consider with more attention Jackson's ideas on the subject.

Jackson (2004, p. 275) claims that the diagonal proposition[7] of a sentence like "water quenches thirst" formalizes a very limited characterization of its truth-conditions, something that, roughly speaking, could be stated as *watery stuff* quenches thirst. By "watery stuff" we should understand a phenomenical property that, like a Kaplanian character, indirectly fixes the denotation of "water" in every possible context of use, something that Jackson (2004, p. 270) calls a *representational property*. Now, the extension of the property watery stuff varies between contexts of use. Consequently, the truth-conditions of "water quenches thirst" vary as well from context to context. However, in any situation of linguistic performance, by using the expression "water" someone is referring to something that at least satisfies the property of being watery stuff. Hence, the partial knowledge of truth-conditions associated with our semantic competence consists in the mastering of some identificatory properties that, nevertheless, do not exhaust the correction criteria for the use of language. Note that, in this way, Jackson blocks anomalous variations of Twin Earth argument with additional worlds attributing completely arbitrary interpretations to "water quenches thirst".

The success of Jackson's proposal depends on the precise specification of adequate representational properties. This requirement imposes a huge onus of proof on Jackson's philosophical theory. In fact, the existence of representational properties does not seem plausible. Schroeter and Bigelow (2009) nicely describes this apparent implausibility:

> When considering what would count as water if the actual world were like Putnam's Twin Earth, you may have an immediate and straightforward conviction about the correct answer: XYZ. This immediacy seems an indication that you're relying on an implicit criterion for identifying the reference of 'water' [...] however, this impression is due entirely to a poverty of examples [...] Twin Earth is specifically designed to vindicate all of your current beliefs about water except for one. [...] Things are different, however, when you start to consider contexts that diverge more sharply from your current assumptions about your actual environment. What should you say about the reference of 'water' if it turns out that you've been interacting with a number of different chemical substances in different contexts? Could 'water' turn out to be ambiguous? Or could it refer to a disjunctive kind? What if there hadn't been any coherent chemical kind here at all? Could water have turned out to be an ecological kind? Or perhaps a culinary kind? How can you adjudicate among these many possible different interpretations? (Ibid., pp. 95–96)

[7] In fact, Jackson refers more specifically to the similar notion of *A-intension*. The comparison between such concepts is an important subject of investigation (see Jackson (2004, p. 276, fn. 5)), but this does not need to concern us here.

Of course, this is not a fatal criticism: maybe we can still find good reason for believing in the existence of representational properties and perhaps we might hope to provide reasonable identification criteria for such kind of properties. However, Schroeter and Bigelow's criticism correctly shows how difficult such an enterprise promises to be.

To sum up, I started this section presenting Loar's account of Putnam's and Kripke's mental experiments. Then, we saw that Loar's concept of realization-conditions suggests a promising description of the partial knowledge of truth-conditions associated with our semantic competence by Putnam's and Kripke's scenarios. However, since "realization-conditions" is a rather obscure concept, I considered Stalnaker's formalization of this notion in terms of diagonal proposition. Stalnaker's account has an important limitation, namely, the diagonal proposition of a sentence does not seem to capture any partial description of its truth-conditions. In order to solve this problem, we need to impose further constraints on the construction of diagonal propositions. Following this strategy, Jackson associates the diagonal proposition of a sentence with the grasping of representational properties. Jackson's solution is very interesting but, since there are good reasons for doubting that representational properties exist, the eventual success of his solution depends on the availability of further evidence that there are representational properties. In next section, focusing particularly on examples 2.1 and 2.2, we claim that there is a particular kind of representational properties that plays a role in our use of quantified formulas.

2.4 Representational properties and the use of quantifiers

Although Jackson's philosophical theory faces serious (and, perhaps, insurmountable) problems, his idea that semantic competence depends on the grasping of representational properties seems to be useful in the analysis of particular cases. In this section we claim that the epistemological conditions of our semantic competence in the use of quantifiers can be reduced to the mastering of the use of some representational properties.

According to Jackson, when someone understands the sentence "water quenches thirst", this person accesses the state of affairs that there is something that is watery stuff and it quenches thirst. Maybe this person is unable to completely identify which substance actually forms the extension of "watery stuff" (whether it is H_2O or XYZ), but she knows at least that it satisfies the identificatory property of being watery stuff. Analogously, considering example 2.1, it is possible to argue that when Paul understands the axiom of successor he accesses the state of affairs that there is some domain that *includes the natural numbers* and in which it holds that every number has successor. By the conditions of the mental experiment, we know that Paul is unable to completely identify the domain of the quantifiers occurring in the axiom of successor, but he knows at least that this domain contains the natural numbers. Therefore, in analogy with Jackson's claim, it is possible to argue that the epistemological conditions of our semantic competence in the use of quantifiers can be redirected to the mastering of representational properties such as "including natural numbers", properties that fix some domain of quantification in every context of use.

So, my central claim here is that our semantic competence in using quantifiers is mediated by some representational properties that we will call *quantificational properties*. Quantificational properties identify, within a context of use, what are the domains of the quantifiers occurring in a given formula. Since these properties always characterize some feature that is shared by different domains of quantification, they generate an adequate representation of the partial knowledge of truth-conditions associated with our semantic competence in the use of quantifiers.

Now, a quantificational property such as "including natural numbers" describes some part of the extension of a quantificational domain: it says that, whatever the real domain of a quantifier is, at least it includes a set of elements of a certain kind. We can formalize quantificational properties by using some non-standard semantics whose technical framework enables us to *relativize* quantifiers for specific parts of a given model. The parts of the model that play the role of quantificational domains represent the known parts of the domain and these are exactly the parts that characterize the relevant quantificational properties. The entire domain of the model (that, in classical logic, is the full domain of all quantifiers) represents the real domain of any quantifier. Any semantics that nicely formalizes such relativization of quantifiers must fulfill at least two general requirements: first, it must draw a precise distinction between the real domains of the quantifiers occurring in a given sentence and the parts of these domains that characterize the relevant quantificational properties; secondly, it must be sensitive to the fact that someone's knowledge about the domain of different quantifiers is possibly variable.

Urn semantics (a non-standard semantics for first order languages originally proposed by Rantala (1979)) fulfills both requirements and promises to offer a nice formalization of a theory of quantificational properties. Roughly speaking, urn semantics preserves the classical, model-theoretic definition of *structure* but slightly changes the definition of *satisfaction*. In particular, it modifies the semantic behavior of quantifiers. Fixed any given structure $\mathcal{M}$, instead of taking it, as in classical logic, as *the* domain of all quantifiers, in urn semantics parts of $\mathcal{M}$ are designated as domains of quantification. The actual domain of a given quantifier varies depending on its position in a given sentence.

For a generic example, consider any formula $Q_1x_1 \ldots Q_nx_n\ \psi$ such that each symbol Q_i is either $\exists$ or $\forall$ and ψ is a quantifier-free formula. In urn semantics, the quantificational domain of each Q_i is a part P_i of $\mathcal{M}$: note that there are no general constraints on how P_i needs to be metatheoretically defined. Q_i quantifies on P_i because the quantifier occupies the i-th position in the nesting of quantifiers $Q_1x_1 \ldots Q_nx_n$, i.e., if P_i is different from P_j, then Q_i and Q_j quantify in distinct parts of the model. This possibility of varying the domains of quantifiers occurring in different positions of a nesting of quantifiers satisfies the second requirement stated above. In next chapter, I introduce a precise account of this semantic system.

For more interesting examples, let me consider example 2.1 again: urn semantics formalizes the quantificational property of "including natural numbers" by relativizing the quantifiers occurring in the axiom of sucessor to the subset of natural numbers in any model of Peano's arithmetic. Moreover, it is possible to give an account of example 2.2 based on urn semantics in the following way. From a descriptivist point of view, Jackson (1998) argues that Pierre in Kripke's puzzle associates some representational property with "London" that he is not disposed to attribute to "Londres." Analogously, it is possible to formalize Alonzo's epistemic situation in urn semantics by considering a relativization of the existential quantifiers occurring in ϕ_1 and ϕ_2 that collects a Turing machine for every computable function f but does not collect any recursive function for some computable function f.

At this point it is possible to see, in broad strokes, how this semantic account describes the ampliative character of logical knowledge. In this sense, consider the following very simple example of logical truth $\neg(\forall x \exists y(P(x) \wedge \neg P(y)))$, in which P is an arbitrary unary predicate. A person who does not recognize the validity of this formula does not see that the domain of $\exists y$, *whatever it is*, provides a counter-example to the formula $\forall x \exists y(P(x) \wedge \neg P(y))$. Based on urn semantics, we can say that this person does not know that, whatever the domain of $\forall x$ is, it includes the counter-example provided by $\exists y$. Note that an adequate formalization of this situation of ignorance of the validity of $\neg(\forall x \exists y(P(x) \wedge \neg P(y)))$ can be achieved by just relativizing the quantifier $\forall x$ for a part of a given model that excludes any counter-example that can be provided by $\exists y$.

The remarks above point to the existence of quantificational properties, kinds of representational properties that determine the epistemological conditions of our semantic competence in the use of quantifiers. Now, we should ask about a minimum description of the essential aspects of quantificational properties. How can we minimally describe the quantificational properties on which our semantics competence relies? In each linguistic context, which elements of a given model necessarily compose the parts referred by quantificational properties? This question does not accept any easy answer. In fact, we should decide between at least the following two general kinds of answer here.

Let us consider a sentence of the form $\forall x\phi$. On one hand, in parallel to the presently suggested analysis of example 2.1, maybe we should say that the quantificational property that enables us to read this sentence determines that, for any model that satisfies $\forall x\phi$, the domain of the model includes at least one element for each closed term of the considered language. In this sense, by understanding $\forall x\phi$ we know at least that, for it to be true, $\phi x[t]$ should be true as well, for any closed term t of the language.

On the other hand, perhaps this is too permissible. In this case, we could try a more parsimonious approach according to which quantificational properties refer just to generic elements. Fine (1985) proposed a well-known theory of generic objects in order to deal with the semantics of quantifiers: according to Fine, when we read $\forall x\phi$ we understand that the relevant generic object satisfies ϕ. In very general terms, a generic object is an object whose properties are all and just the ones that are satisfied by every element of any given structure (let us call such properties as generic properties). Of course, the availability of a concrete list of generic properties depends on the considered range of structures Ibid., p. 16: for present concerns, I would like to say that at least all logical properties are generic. Fine's theory of generic objects faces several ontological as well as logical difficulties and proposes some ingenious solutions to each one of these challenges. Consequently, it promises to be a strong philosophical ground for a description of the essential aspects of quantificational properties. Now, an adequate examination of these issues is a theme for further studies.

Chapter 3

New semantic foundations for TSI

3.1 Introduction

Before starting this chapter, let me give a brief summary of what I have done so far. My central goal in this thesis is to show that if we replace TSI's semantic framework by some deviant, better-suited semantic system, we can adequately formalize TSI without SoD. I have claimed that this more adequate semantic framework can be obtained by exploring the basic idea that our semantic competence is accompanied by some partial knowledge of the truth-conditions of sentences.

In the previous chapter, based on an analysis of classical works on semantic externalism, I suggested a general account of the epistemological conditions of our semantic competence in the use of quantifiers. Briefly speaking, I argued that in general our semantic competence in the use of these expressions is associated with a partial knowledge of the extensions of quantificational domains.

Further, I anticipated that the non-standard system of urn semantics (RANTALA, 1979) adequately formalizes this general account. Urn semantics diverges from classical logic particularly in its definition of the semantics of quantifiers, i.e., urn semantics relativizes quantifiers for parts of a given model in function of their positions in a considered formula. In the rest of this chapter I provide a detailed characterization of this non-classical semantics, by showing its main concepts as well as by proving some of its most fundamental meta-theorems. In this sense, this exposition introduces the formal conditions for a subsequent reestablishment of TSI without SoD.

This chapter is planned in the following way. I start section 3.2 by presenting Rantala's original way of defining urn semantics. As we will see, Rantala introduced this semantic system following a game-theoretic approach of the subject and, for a lot of time, it was an open question whether it was possible to design a Tarskian framework for urn semantics. In subsection 3.2.1 I solve this question by presenting a Tarskian framework equivalent to Rantala's game-theoretic system. Further, in subsection 3.2.2 I note that the literature on the subject distinguishes at least two different systems of urn semantics that I call, by suggestion of Rantala's game-theoretic approach, perfect and imperfect urn semantics, respectively. Consequently, in this subsection I also formally differentiate these systems of urn semantics.

In his seminal work, Rantala proved the *decidability* of the system (Ibid., pp. 468–469). Further, Rantala showed that some classically unsatisfiable formulas are satisfiable in urn semantics (Ibid., pp. 470–472). In section 3.3 my main goal is to present Rantala's decidability theorem and to generalize his theorem on satisfiability in urn semantics as a characterization theorem for the system. These theorems are fundamental results on urn semantics: in particular, the latter one consists in a very important property on which this thesis relies. Now, the proof of this theorem appeals to a version of the Fraïssé-Hintikka theorem, a well-known model-theoretic result on the existence of Hintikka-normal forms. This result is also obtained in this section. Note that, in the context of this work, the Fraïssé-Hintikka theorem plays a central role both in the obtainment of additional properties of urn semantics as well as in the precise characterization of TSI. So, the obtainment of this theorem is crucial for this thesis' main argument. Finally, I conclude the chapter with some general remarks on its most important results.

First, let me introduce some terminology. Urn semantics preserves the standard concept of *structure*: in fact urn semantics is defined only by changing the classical notion of *satisfaction*, particularly concerning the satisfiability of quantified formulas. Let me denote structures by the letters $\mathcal{M}, \mathcal{N}, \mathcal{M}', \mathcal{N}'$ etc. If $\mathcal{M}$ is a structure for a first order language $\mathcal{L}$, sometimes I say that $\mathcal{M}$ is an $\mathcal{L}$-structure. In general I consider here *full* first order languages, that is, languages equipped with predicates (including identity), functions and constants. I always assume that the signature of $\mathcal{L}$ is $\{\wedge, \vee, \neg, \forall, \exists\}$.

Terms and formulas of $\mathcal{L}$ are defined in the usual way. By closed terms and sentences I mean terms and formulas without free-variables, respectively. In the following exposition, sometimes I need to consider the *complexity* of terms occurring in a given formula. The function complexity of terms cp respects the following clauses:

- If t is either a constant or a variable, then $cp(t) = 0$;
- If t is $f(s_1, \ldots, s_n)$, in which f is an n-ary function, then $cp(t) = \max\{cp(s_1), \ldots, cp(s_n)\} + 1$.

Particularly for the presentation of the Fraïssé-Hintikka theorem for urn semantics, I need to consider *unnested formulas* (terminology by Hodges (1997, p. 51)), that is, formulas whose terms have complexity at most 1.

The interpretation of any symbol $\alpha \in \mathcal{L}$ in a given structure $\mathcal{M}$ is denoted by $\alpha^{\mathcal{M}}$. For any first order language $\mathcal{L}^+ \supseteq \mathcal{L}$ and for any $\mathcal{L}$-structure $\mathcal{M}$, let me say that an $\mathcal{L}^+$-structure $\mathcal{M}^+$ is an $\mathcal{L}^+$*-expansion* of $\mathcal{M}$, in symbols $\mathcal{M}^+ \supseteq \mathcal{M}$, if both structures share the same domain and, for any symbol $\alpha \in \mathcal{L}$, $\alpha^{\mathcal{M}} = \alpha^{\mathcal{M}^+}$. In this case, I also say that $\mathcal{M}$ is an $\mathcal{L}$*-restriction* of $\mathcal{M}^+$.

I make use of some further notation for denoting complex formulas. For a finite set of formulas $\Psi = \{\psi_1, \ldots, \psi_n\}$, let $\bigwedge \Psi$ denote $\bigwedge\limits_{1 \le i \le n} \psi_i$ and $\bigvee \Psi$ denote $\bigvee\limits_{1 \le i \le n} \psi_i$. Moreover, let

$\neg\Psi = \{\neg\psi : \psi \in \Psi\}$ and $\exists x\Psi = \{\exists x\psi : \psi \in \Psi\}$. I say that a term t *occurs free* in a formula ϕ if all variables occurring in t are free in ϕ. If t does not occur free in ϕ, I say that t is *bounded* in ϕ. By $\phi t_1[t_2]$ I denote the formula generated by *substitution* of all free-occurrences of a term t_1 in ϕ by a term t_2. I say that a term t_1 is *substitution-free* for a term t_2 in ϕ if and only if there is no subformula ψ of ϕ in which t_2 is bounded and t_1 occurs free.

In classical logic, the satisfaction of a formula of the form $\exists x\phi$ in a structure $\mathcal{M}$ can be characterized, for convenience, as the satisfaction of the formula $\phi x[a]$, in which a is a constant denoting the element a in $\mathcal{M}$. The same can be done for universal quantifiers. From this point of view, the satisfaction of formulas of a given language $\mathcal{L}$ is first defined for sentences of an expansion of $\mathcal{L}$ with new constants, one for each element of $\mathcal{M}$, and, subsequently, this definition of satisfaction is easily generalized for open formulas of $\mathcal{L}$. Hereafter, for the sake of simplicity, I follow the same methodological orientation.

When examining Fraïssé-Hintikka theorem I also need to consider the *quantifier rank* of a formula ϕ, in symbols $qr(\phi)$. The function qr is defined in the following way:

- For every atomic formula ϕ, $qr(\phi) = 0$;
- If ϕ is $\neg\psi$, then $qr(\phi) = qr(\psi)$;
- If ϕ is either $\psi \wedge \gamma$ or $\psi \vee \gamma$, then $qr(\phi) = \max\{qr(\psi), qr(\gamma)\}$;
- If ϕ is either $\forall x\psi$ or $\exists x\psi$, then $qr(\phi) = qr(\psi) + 1$.

3.2 Urn semantics

3.2.1 Rantala's game semantics and its Tarskian reconstruction

Informally speaking, the general idea behind the definition of urn semantics is to design a way of relativizing quantifiers for parts of a given model in function of their positions in a formula. In this sense, this semantics provides a procedure for building cases in which, given a nesting of quantifiers occurring in a formula, different quantifiers could *elect* witnesses in different parts of a model. Now, in order to design such a procedure, first of all we need to define the notion of eligibility that plays a role in this informal description.

Definition 3.1 *Let* M *be a non-empty set.* $\mathfrak{B} = \{\mathfrak{B}_m : m < \omega\}$ *is an* eligibility set *of* M *if and only if it is an infinitely countable collection of non-empty parts of M.*

For every $n < \omega$*, a* set of eligible n-tuples $\mathfrak{M}_n \subseteq \mathrm{M}^{n+1}$ *is such that:*

- $\mathfrak{M}_0 = \mathfrak{B}_0$*;*
- $\mathfrak{M}_{n+1} = \{\langle a_0, \ldots, a_n, b\rangle : \langle a_0, \ldots, a_n\rangle \in \mathfrak{M}_n$ *and* $b \in \mathfrak{B}_{n+1}\}$.

Let $\mathfrak{M} = \bigcup\limits_{n<\omega} \mathfrak{M}_n$. $\mathfrak{M}$ *is the* set of eligible sequences *of* M *over* $\mathfrak{B}$.

In the following, when $\mathfrak{B}$ is an eligibility set of M and M is the domain of a model $\mathcal{M}$, I say that $\mathfrak{B}$ is an eligibility set of $\mathcal{M}$ as well. I adopt the same terminology with respect to sets of eligible sequences. Further, I call a pair $\langle \mathcal{M}, \mathfrak{B} \rangle$ a *urn structure*, and I denote it as $\mathcal{M}_{\mathfrak{B}}$. When $\mathcal{M}$ is an $\mathcal{L}$-structure, for some first order language $\mathcal{L}$, I also say that $\mathcal{M}_{\mathfrak{B}}$ is an $\mathcal{L}$-urn structure.

As I have mentioned before, Rantala defines urn semantics from a game-theoretic perspective. This means that, in Rantala's work, urn semantics is presented in terms of a *game semantics*. Roughly speaking, a game semantics for a given logic is a finitary two-player game such that the winning conditions of one of the players in the game characterizes the notion of satisfaction defining the logic.[1] The next definition introduces Rantala's game-theoretic urn semantics.

In what follows, for players A and E, for some fixed formula ϕ and some urn structure $\mathcal{M}_{\mathfrak{B}}$, by a *round* in a urn game semantics for ϕ and $\mathcal{M}_{\mathfrak{B}}$ I mean a quartet $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ such that α_1 and α_2 are either A or E, α_3 is some subformula of ϕ and α_4 is either $\emptyset$ or some eligible sequence of $\mathcal{M}$ over $\mathfrak{B}$. Such a notation means that, in the considered round, α_1 *demanded* α_2 *to hold the subformula* α_3 *under condition* α_4. Moreover, by a *match* of a urn game semantics for ϕ and $\mathcal{M}_{\mathfrak{B}}$ I mean some finite, well-ordered sequence of rounds.

For abbreviation I denote the i-th round of a match p in a urn game semantics as p(i). For any $0 < j \leq 4$, by p(i)^j I mean the j-th coordinate of p(i). Further, by $\overline{\text{p(i)}^1}$ and $\overline{\text{p(i)}^2}$ I refer to the player that is not p(i)^1 or p(i)^2, respectively: for instance, if p(i)^1 is A, then $\overline{\text{p(i)}^1}$ is E. The notation works in the same way in the case of p(i)^2.

Definition 3.2 *Let $\mathcal{L}$ be a first order language,* A *and* E *be players, and let ϕ be a formula of $\mathcal{L}$. Further, let $\mathcal{M}$ be some classic $\mathcal{L}$-structure and $\mathfrak{B}$ be an eligibility set of $\mathcal{M}$. A* urn game semantics *for ϕ and $\mathcal{M}_{\mathfrak{B}}$, in symbols $\mathfrak{G}(\phi, \mathcal{M}_{\mathfrak{B}})$, is a set of matches recursively defined as follows:*

1. *For any match $p \in \mathfrak{G}(\phi, \mathcal{M}_{\mathfrak{B}})$, $p(0) = \langle \mathrm{E}, \mathrm{E}, \phi, \emptyset \rangle$, that is, at the start of any match of $\mathfrak{G}(\phi, \mathcal{M}_{\mathfrak{B}})$,* E *holds ϕ;*

Now, assume that, for some $i < \omega$, the rounds $p(0), \ldots, p(i)$ have been already defined:

1' *Assume that the subformula ϕ' held in p(i) is atomic. If this formula is classically satisfied in $\mathcal{M}$, then player p(i)^2 wins the game. Otherwise, p(i)^2 loses it.*

2' *If the formula p(i)^3 is $\psi \wedge \gamma$, p(i+1) is either $\langle \overline{p(i)^2}, p(i)^2, \psi, p(i)^4 \rangle$ or $\langle \overline{p(i)^2}, p(i)^2, \gamma, p(i)^4 \rangle$;*

3' *If the formula p(i)^3 is $\psi \vee \gamma$, p(i+1) is either $\langle p(i)^2, p(i)^2, \psi, p(i)^4 \rangle$ or $\langle p(i)^2, p(i)^2, \gamma, p(i)^4 \rangle$;*

4' *If the formula p(i)^3 is $\neg\psi$, p(i+1) is $\langle p(i)^2, \overline{p(i)^2}, \psi, p(i)^4 \rangle$;*

5' *If the formula p(i)^3 is $\forall x \psi$, p(i+1) is $\langle \overline{p(i)^2}, p(i)^2, \psi x[a], p(i)^4 \frown (a) \rangle$, for some $a \in \mathfrak{B}_j$, in which j is the length of p(i)^4;*

[1] For a primer on game semantics for classical logic, see Väänänen (2011, pp. 93–97).

6' *If the formula p(i)^3 is $\exists x\psi$, p(i+1) is $\langle p(i)^2, p(i)^2, \psi x[a], p(i)^4 \frown (a)\rangle$, for some $a \in \mathfrak{B}_j$, in which j is the length of p(i)^4.*

Based on Definition 3.2, it is possible to define the satisfiability of a formula ϕ in a urn structure $\mathcal{M}_{\mathfrak{B}}$ in terms of the existence of a *winning strategy* for player E in the game $\mathfrak{G}(\phi, \mathcal{M}_{\mathfrak{B}})$. The current literature presents some attempt of offering a Tarskian reconstruction of this semantics (an important proposal in this direction is Cresswell (1982)). Since it is still an open question whether there is in fact such a Tarskian framework for urn semantics,[2] I introduce in the following such a system and prove its equivalence with Rantala's game-theoretic semantics.

The main challenge in developing a Tarskian system of urn semantics is the non-compositionality that quantifiers have in this logic: informally speaking, when searching for a "witness" for a formula of the form $\exists x\psi$ in a urn structure $\mathcal{M}_{\mathfrak{B}}$, generally we need to consider a sequence of "witnesses" that were *previously* picked up. Although such "witnesses" do not need to occur in $\exists x\psi$, still they determine the interpretation of this formula. It is possible to emulate the non-compositionality of quantifiers exploring the following strategy. First, we define a *relative* notion of satisfaction that emulates the eligibility conditions for the interpretation of quantifiers. Secondly, we define an *absolute* notion of satisfaction characterizing the truth-conditions of a formula in terms of the relative satisfaction of its subformulas.

Definition 3.3 *Let $\mathcal{L}$ be a first order language and $\mathcal{M}_{\mathfrak{B}}$ be an $\mathcal{L}$-urn structure. Let $\mathcal{S} = \{a : a \in \mathcal{M}\}$ be a set of new constants, $\mathcal{L}^+ = \mathcal{L} \cup \mathcal{S}$ and $\mathcal{F}or_{\mathcal{L}^+}$ be the set of sentences of $\mathcal{L}^+$. A function of* relative urn satisfaction *$e : \mathcal{F}or_{\mathcal{L}^+} \times \omega \to 2$ is such that, for every $n < \omega$:*

- *For every atomic formula ϕ, then $e(\phi, n) = 1 \Leftrightarrow \mathcal{M}$ classically satisfies ϕ;*
- *If ϕ is $\neg\psi$, then $e(\phi, n) = 1 \Leftrightarrow e(\psi, n) = 0$;*
- *If ϕ is $\psi \wedge \gamma$, then $e(\phi, n) = 1 \Leftrightarrow e(\psi, n) = e(\gamma, n) = 1$;*
- *If ϕ is $\psi \vee \gamma$, then $e(\phi, n) = 1 \Leftrightarrow$ either $e(\psi, n) = 1$ or $e(\gamma, n) = 1$;*
- *If ϕ is $\forall x\psi$, then $e(\phi, n) = 1 \Leftrightarrow e(\psi x[a], n+1) = 1$, for every $a \in \mathfrak{B}_n$;*
- *If ϕ is $\exists x\psi$, then $e(\phi, n) = 1 \Leftrightarrow e(\psi x[a], n+1) = 1$, for some $a \in \mathfrak{B}_n$.*

Moreover, a function of absolute urn satisfaction *$u : \mathcal{F}or_{\mathcal{L}^+} \to 2$ is such that $u(\phi) = e(\phi, 0)$, for every $\phi \in \mathcal{F}or_{\mathcal{L}^+}$.*

Theorem 3.4 *Let $\mathcal{L}$, $\mathcal{M}_{\mathfrak{B}}$ and $\mathcal{L}^+$ be as above. Let ϕ be a formula of $\mathcal{L}^+$ and let u be the function of absolute urn satisfaction defined by $\mathcal{M}_{\mathfrak{B}}$. The following are equivalent:*

[2] Although Cresswell (1982) is an important attempt of solving this problem, the results obtained in his paper are not fully successful. More recently, French (2015) correctly defined a Tarskian framework for urn semantics but, in this work, he did not prove the equivalence of this system with Rantala's proposal. Moreover, French's work ignores that there are two different systems of urn semantics (what I called in the beginning of this chapter perfect and imperfect urn semantics).

1. $u(\phi) = 1$*;*

2. Player E has a winning strategy in urn game semantics $\mathfrak{G}(\phi, \mathcal{M}_{\mathfrak{B}})$*.*

Proof. ($1 \Rightarrow 2$) Assuming $u(\phi) = 1$, we recursively define a winning strategy σ for E in $\mathfrak{G}(\phi, \mathcal{M}_{\mathfrak{B}})$ as follows. Assume that, for some $i < \omega$, the strategy σ has already been defined for rounds $p(0), \ldots, p(i)$. Suppose that $p(i)^4$ has length j. The following directives define σ in round $p(i+1)$:

- Assume A holds a formula of the form $\psi \wedge \gamma$ in $p(i)$. If $e(\psi, j) = 0$, then E demands A to hold ψ in $p(i+1)$; otherwise, E demands A to hold γ in $p(i+1)$;
- Assume E holds a formula of the form $\psi \vee \gamma$ in $p(i)$. If $e(\psi, j) = 1$, then E demands herself to hold ψ in $p(i+1)$; otherwise, E demands herself to hold γ in $p(i+1)$;
- Assume A holds a formula of the form $\forall x\psi$ in $p(i)$. If there is some $a \in \mathfrak{B}_j$ such that $e(\psi x[a], j+1) = 0$, then E demands A to hold $\psi x[a]$; otherwise, E demands A to hold $\psi x[b]$, for some arbitrary b in $\mathfrak{B}_j$;
- Assume E holds a formula of the form $\exists x\psi$ in $p(i)$. If there is some $a \in \mathfrak{B}_j$ such that $e(\psi x[a], j+1) = 1$, then E demands herself to hold $\psi x[a]$; otherwise, E demands herself to hold $\psi x[b]$, for some b in $\mathfrak{B}_j$;

By induction on the indexes of the rounds of any match p in which E follows the strategy σ, the reader may verify that, for any subformula ϕ' of ϕ held at some round $p(i)$, E holds ϕ' in $p(i)$ if and only if $e(\phi', j) = 1$, in which j is the length of $p(i)^4$. So, σ is a winning strategy for E.

($2 \Rightarrow 1$) Assuming $u(\phi) = 0$, we recursively define a winning strategy σ' for A in $\mathfrak{G}(\phi, \mathcal{M}_{\mathfrak{B}})$ considering a set of directives dual to the ones defining σ. Therefore, by a similar argument it is possible to prove that, in any match in which A follows σ', for any subformula ϕ' of ϕ that is held at some round $p(i)$, E holds ϕ' in $p(i)$ if and only if $e(\phi', j) = 0$, in which j is the length of $p(i)^4$. So, σ' is a winning strategy for A. ■

3.2.2 Perfect and imperfect urn semantics

Note that in order to define urn semantics I did not make extensive use of the concept of eligible sequence (Definition 3.1). Instead, I appealed directly to the notion of eligibility set. As Rantala (1979, pp. 464–465) observes, this is connected with a game-theoretic property of urn semantics, namely, urn game semantics is a game of *imperfect information* in the sense that, in general, when considering the relative urn satisfaction of a formula $\exists x\phi$ in a given urn structure, we do not need to know the “witnesses” that were previously picked up, we only need to know *how many* “witnesses” were picked up so far.

It is possible to define a special version of urn semantics that characterizes a game of perfect information. Let me call this system as perfect urn semantics (or, for simplicity, just *p*-urn semantics). In contrast, I call the previously defined standard system of urn semantics as imperfect urn semantics (or, for simplicity, just *i*-urn semantics).

Essentially, p-urn semantics impose some additional constraints on the definition of a set of eligible sequences: whilst in i-urn semantics, for a fixed urn structure $\mathcal{M}_{\mathfrak{B}}$, an eligible sequence $\langle a_0, \ldots, a_i \rangle$ of $\mathcal{M}$ over $\mathfrak{B}$ can be extended with *any* element of $\mathfrak{B}_{i+1}$, in p-urn semantics we can refine this process. In this sense, in p-urn semantics it is possible to build cases in which two eligible sequences of $\mathcal{M}$ over $\mathfrak{B}$ are extendable by different sets of elements of $\mathcal{M}$. In what follows, let me make more precise how this refinement of the concept of eligibility can be formally achieved.

Definition 3.5 *Let* M *be a non-empty set. For every* $n < \omega$*, a* set of perfectly eligible *(for simplicity, just p-eligible)* n-tuples $\widehat{\mathfrak{M}}_n \subseteq \mathrm{M}^{n+1}$ *is such that:*

- $\widehat{\mathfrak{M}}_0$ *is some part* $\mathfrak{C}_0$ *of* M*;*

Now, assume that $\mid \widehat{\mathfrak{M}}_n \mid= \kappa$ *and let* $\mathfrak{C}_{n+1}$ *be a collection of non-empty subsets* $\mathfrak{C}_{n+1,\lambda}$ *of* M*, for each ordinal* $\lambda < \kappa$*:*

- $\widehat{\mathfrak{M}}_{n+1,\lambda} = \{\alpha \frown (b) : \alpha$ *is the* λ*-th sequence in* $\widehat{\mathfrak{M}}_n$ *and* $b \in \mathfrak{C}_{n+1,\lambda}\}$*;*
- $\widehat{\mathfrak{M}}_{n+1} = \bigcup\limits_{\lambda<\kappa} \widehat{\mathfrak{M}}_{n+1,\lambda}$.

Let $\mathfrak{C} = \{\mathfrak{C}_n : n < \omega\}$*.* $\widehat{\mathfrak{M}} = \bigcup\limits_{n<\omega} \widehat{\mathfrak{M}}_n$ *is the set of p-eligible sequences of* M *over* $\mathfrak{C}$*.*

Generalizing the terminology introduced above, let me say that the set $\mathfrak{C}$ above is a *p-eligibility set* of M and, when M is the domain of $\mathcal{M}$, I say that $\mathfrak{C}$ is a p-eligibility set of $\mathcal{M}$ as well. Also, I call $\widehat{\mathfrak{M}}$ the set of p-eligible sequences of $\mathcal{M}$ over $\mathfrak{C}$ when M is the domain of this structure. Further, I say that the pair $\langle \mathcal{M}, \mathfrak{C} \rangle$ is a *p-urn structure* and I denote it as $\widehat{\mathcal{M}_{\mathfrak{C}}}$. When $\mathcal{M}$ is an $\mathcal{L}$-structure, $\widehat{\mathcal{M}_{\mathfrak{C}}}$ is an $\mathcal{L}$-p-urn structure. In order to avoid confusion, I also use the corresponding terminology i-eligibility set, i-eligible sequence and ($\mathcal{L}$-)i-urn structure when dealing with i-urn semantics.

The next definition game-theoretically characterizes p-urn semantics.

Definition 3.6 *Let* $\mathcal{L}$ *be a first order language,* A *and* E *be players, and let* ϕ *be a formula of* $\mathcal{L}$*. Let* $\widehat{\mathcal{M}_{\mathfrak{C}}}$ *be an* $\mathcal{L}$*-p-urn structure and let* $\widehat{\mathfrak{M}}$ *be the set of p-eligible sequences of* $\mathcal{M}$ *over* $\mathfrak{C}$*. A* p-urn game semantics *for* ϕ *and* $\widehat{\mathcal{M}_{\mathfrak{C}}}$*, in symbols* $\mathfrak{F}(\phi, \widehat{\mathcal{M}_{\mathfrak{C}}})$ *is a set of matches recursively defined as follows:*

1. *For any match* $p \in \mathfrak{F}(\phi, \widehat{\mathcal{M}_{\mathfrak{C}}})$*,* $p(0) = \langle E, E, \phi, \emptyset \rangle$*;*

Now, assume that for some $i < \omega$*, the rounds* $p(0), \ldots, p(i)$ *have been already defined:*

1” $\mathfrak{F}(\phi, \widehat{\mathcal{M}_{\mathfrak{C}}})$ satisfies clauses 1’–4’ of definition 3.2.

2” If the formula p(i)^3 is $\forall x\psi$, then p(i+1) is the quartet $\langle \overline{p(i)^2}, p(i)^2, \psi x[a], p(i)^4 \frown (a)\rangle$, for a such that: if $p(i)^4 = \emptyset$, then $a \in \mathfrak{C}_0$; otherwise, there are ordinals j and λ such that $p(i)^4$ is the λ-th sequence in $\widehat{\mathfrak{M}}_{j-1}$ and $a \in \mathfrak{C}_{j,\lambda}$;

3” If the formula p(i)^3 is $\exists x\psi$, then p(i+1) is the quartet $\langle p(i)^2, p(i)^2, \psi x[a], p(i)^4 \frown (a)\rangle$, for a such that: if $p(i)^4 = \emptyset$, then $a \in \mathfrak{C}_0$; otherwise, there are ordinals j and λ such that $p(i)^4$ is the λ-th element in $\widehat{\mathfrak{M}}_{j-1}$ and $a \in \mathfrak{C}_{j,\lambda}$.

Again, relying on some variation of what was done above, it is possible to develop a Tarskian counterpart of p-urn game semantics.

Definition 3.7 *Let $\mathcal{L}$ be a first order language and $\widehat{\mathcal{M}_{\mathfrak{C}}}$ be an $\mathcal{L}$-p-urn structure. Let $\mathcal{S} = \{a : a \in \mathcal{M}\}$ be a set of new constants, $\mathcal{L}^+ = \mathcal{L} \cup \mathcal{S}$ and $\mathcal{F}or_{\mathcal{L}^+}$ be the set of sentences of $\mathcal{L}^+$. Let $\widehat{\mathfrak{M}}$ be the set of p-eligible sequences of $\mathcal{M}$ over $\mathfrak{C}$ and $\kappa =$ max $\{\mid \widehat{\mathfrak{M}}_i \mid : i < \omega\}$. A function of* relative p-urn satisfaction *$\widehat{e} : \mathcal{F}or_{\mathcal{L}^+} \times \omega \times \kappa \to 2$ is such that, for every $n < \omega$ and every ordinal $\lambda < \kappa$:*

- *For every atomic formula ϕ, then $\widehat{e}(\phi, n, \lambda) = 1 \Leftrightarrow \mathcal{M}$ classically satisfies ϕ;*
- *If ϕ is $\neg\psi$, then $\widehat{e}(\phi, n, \lambda) = 1 \Leftrightarrow \widehat{e}(\psi, n, \lambda) = 0$;*
- *If ϕ is $\psi \wedge \gamma$, then $\widehat{e}(\phi, n, \lambda) = 1 \Leftrightarrow \widehat{e}(\psi, n, \lambda) = \widehat{e}(\gamma, n, \lambda) = 1$;*
- *If ϕ is $\psi \vee \gamma$, then $\widehat{e}(\phi, n, \lambda) = 1 \Leftrightarrow$ either $\widehat{e}(\psi, n, \lambda) = 1$ or $\widehat{e}(\gamma, n, \lambda) = 1$;*
- *Assuming ϕ is $\forall x\psi$, consider the following cases:*
  - *If $n = 0$, then $\widehat{e}(\phi, n, \lambda) = 1 \Leftrightarrow \widehat{e}(\psi x[a], n+1, \rho) = 1$, for every $a \in \mathfrak{C}_0$, in which ρ is the index of a in $\mathfrak{C}_0$;*
  - *If $n \neq 0$ and $\lambda <\mid \widehat{\mathfrak{M}}_{n-1} \mid$, then $\widehat{e}(\phi, n, \lambda) = 1 \Leftrightarrow$ $\widehat{e}(\psi x[a], n+1, \rho) = 1$, for every $a \in \mathfrak{C}_{n,\lambda}$, in which ρ is the index of $\beta \frown (a)$ in $\widehat{\mathfrak{M}}_n$, for β the λ-th element of $\widehat{\mathfrak{M}}_{n-1}$;*
  - *If $n \neq 0$ and $\lambda \geq\mid \widehat{\mathfrak{M}}_{n-1} \mid$, then $\widehat{e}(\phi, n, \lambda) = 1 \Leftrightarrow$ $\widehat{e}(\psi x[a], n+1, \rho) = 1$, for every $a \in \mathfrak{C}_{n,\xi}$, in which ξ is the index of the maximum element β of $\widehat{\mathfrak{M}}_{n-1}$ and ρ is the index of $\beta \frown (a)$ in $\widehat{\mathfrak{M}}_n$;*
- *Assuming ϕ is $\exists x\psi$, consider the following cases:*
  - *If $n = 0$, then $\widehat{e}(\phi, n, \lambda) = 1 \Leftrightarrow \widehat{e}(\psi x[a], n+1, \rho) = 1$, for some $a \in \mathfrak{C}_0$, in which ρ is the index of a in $\mathfrak{C}_0$;*

- *If $n \neq 0$ and $\lambda <| \widehat{\mathfrak{M}}_{n-1} |$, then $\widehat{e}(\phi, n, \lambda) = 1 \Leftrightarrow$ $\widehat{e}(\psi x[a], n+1, \rho) = 1$, for some $a \in \mathfrak{C}_{n,\lambda}$, in which ρ is the index of $\beta \frown (a)$ in $\widehat{\mathfrak{M}}_n$, for β the λ-th element of $\widehat{\mathfrak{M}}_{n-1}$;*
- *If $n \neq 0$ and $\lambda \geq| \widehat{\mathfrak{M}}_{n-1} |$, then $\widehat{e}(\phi, n, \lambda) = 1 \Leftrightarrow$ $\widehat{e}(\psi x[a], n+1, \rho) = 1$, for some $a \in \mathfrak{C}_{n,\xi}$, in which ξ is the index of the maximum element β of $\widehat{\mathfrak{M}}_{n-1}$ and ρ is the index of $\beta \frown (a)$ in $\widehat{\mathfrak{M}}_n$.*

Moreover, a function of absolute p-urn satisfaction *$\widehat{u} : \mathcal{F}or_{\mathcal{L}^+} \to 2$ is such that $\widehat{u}(\phi) = \widehat{e}(\phi, 0, 0)$, for every $\phi \in \mathcal{F}or_{\mathcal{L}^+}$.*

Theorem 3.8 *Let $\mathcal{L}$, $\widehat{\mathcal{M}_{\mathfrak{C}}}$, $\widehat{\mathfrak{M}}$ and $\mathcal{L}^+$ be as above. Let ϕ be a formula of $\mathcal{L}^+$ and let $\widehat{u}$ be the function of absolute p-urn satisfaction defined by $\widehat{\mathcal{M}_{\mathfrak{C}}}$. The following are equivalent:*

1. *$\widehat{u}(\phi) = 1$;*
2. *Player E has a winning strategy in p-urn game semantics $\mathfrak{F}(\phi, \widehat{\mathcal{M}_{\mathfrak{C}}})$.*

Proof. $(1 \Rightarrow 2)$ As in the proof of Theorem 3.4, assuming $\widehat{u}(\phi) = 1$, it is possible to define a winning strategy $\widehat{\sigma}$ for E in $\mathfrak{F}(\phi, \mathcal{M}_{\mathfrak{C}})$.

Suppose that for some $i < \omega$, $\widehat{\sigma}$ has been already defined for rounds $p(0), \ldots, p(i)$. Assume that $p(i)^4$ is the λ-th element of $\widehat{\mathfrak{M}}_{j-1}$ (if $j = 0$, $p(i)^4$ is $\emptyset$). The following directives define $\widehat{\sigma}$ for round $p(i+1)$:

- The directives of $\widehat{\sigma}$ for the case of conjunction and disjunction are the same as the ones defining strategy σ in the proof of Theorem 3.4.
- If A holds a formula of the form $\forall x \psi$ in $p(i)$, then consider two possible cases:
  - Assuming $j = 0$, if there is some $a \in \mathfrak{C}_0$ such that $\widehat{e}(\psi x[a], j+1, \rho) = 0$, in which ρ is the index of a in $\mathfrak{C}_0$, then E demands A to hold $\psi x[a]$; otherwise, E demands A to hold $\psi x[b]$, for some arbitrary b in $\mathfrak{C}_0$;
  - Assuming $j \neq 0$, if there is some $a \in \mathfrak{C}_{j,\lambda}$ such that $\widehat{e}(\psi x[a], j+1, \rho) = 0$, in which ρ is the index of $p(i)^4 \frown (a)$ in $\widehat{\mathfrak{M}}_j$, then E demands A to hold $\psi x[a]$; otherwise, E demands A to hold $\psi x[b]$, for some arbitrary b in $\mathfrak{C}_{j,\lambda}$;
- If E holds a formula of the form $\exists x \psi$ in $p(i)$, then consider two cases:
  - Assuming $j = 0$, if there is some $a \in \mathfrak{C}_0$ such that $\widehat{e}(\psi x[a], j+1, \rho) = 1$, in which ρ is the index of a in $\mathfrak{C}_0$, then E demands herself to hold $\psi x[a]$; otherwise, E demands herself to hold $\psi x[b]$, for some arbitrary b in $\mathfrak{C}_0$;
  - Assuming $j \neq 0$, if there is some $a \in \mathfrak{C}_{j,\lambda}$ such that $\widehat{e}(\psi x[a], j+1, \rho) = 1$, in which ρ is the index of $p(i)^4 \frown (a)$ in $\widehat{\mathfrak{M}}_j$, then E demands herself to hold $\psi x[a]$; otherwise, E demands herself to hold $\psi x[b]$, for some arbitrary b in $\mathfrak{C}_{j,\lambda}$.

By induction on the rounds of any match $p \in \mathfrak{F}(\phi, \widehat{\mathcal{M}_{\mathfrak{C}}})$ in which E followed the strategy $\widehat{\sigma}$ the reader might verify that, for any subformula ϕ' of ϕ held at some round $p(i)$, E holds ϕ' in $p(i)$ if and only if $\widehat{e}(\phi', j, \lambda) = 1$, in which $p(i)^4$ is the λ-th element of $\widehat{\mathfrak{M}}_{j-1}$. Therefore, $\widehat{\sigma}$ is a winning strategy for E in $\mathfrak{F}(\phi, \widehat{\mathcal{M}_{\mathfrak{C}}})$.

($2 \Rightarrow 1$) In the other direction, assuming $\widehat{u}(\phi) = 0$, it is possible to define a winning strategy $\widehat{\sigma}'$ for A in $\mathfrak{F}(\phi, \widehat{\mathcal{M}_{\mathfrak{C}}})$ by considering a dual of $\widehat{\sigma}$, as in the proof of Theorem 3.4. Again, based on an argument by induction it is easy to verify that, for any match p of $\mathfrak{F}(\phi, \widehat{\mathcal{M}_{\mathfrak{C}}})$ in which A has followed $\widehat{\sigma}'$, for any subformula ϕ' of ϕ that is held at some round $p(i)$, E holds ϕ' in $p(i)$ if and only if $\widehat{e}(\phi', j, \lambda) = 0$, in which $p(i)^4$ is the λ-th element of $\widehat{\mathfrak{M}}_{j-1}$. ■

Notation 3.9 *In the rest of this work, for a formula ϕ and an i-urn structure $\mathcal{M}_{\mathfrak{B}}$ with a function of relative urn satisfaction e, for any $n < \omega$, I denote $e(\phi, n) = 1$ by the symbols $\mathcal{M}_{\mathfrak{B}} \models_n \phi$. Sometimes, when $\mathcal{M}_{\mathfrak{B}} \models_0 \phi$, I simplify the notation by writing $\mathcal{M}_{\mathfrak{B}} \models \phi$. Moreover, when considering a p-urn structure $\widehat{\mathcal{M}_{\mathfrak{C}}}$ with a function of relative p-urn satisfaction $\widehat{e}$, I denote $\widehat{e}(\phi, n, \lambda) = 1$ by the symbols $\widehat{\mathcal{M}_{\mathfrak{C}}} \models_{n,\lambda} \phi$ and when $n = 0$, I also simplify the notation by writing $\widehat{\mathcal{M}_{\mathfrak{C}}} \models \phi$.*

Although perfect and imperfect urn semantics define quite different game-theoretic systems of semantics, it is important to note that the class of i-urn structures is properly included in the class of p-urn structures. In fact, since any classical structure $\mathcal{M}$ can be seen as an i-urn structure $\mathcal{M}_{\mathfrak{B}}$ whose set of i-eligible sequences is just the set of sequences of $\mathcal{M}$ itself, the following chain of inclusions between classical, i- and p-urn structures holds:

$$\text{Classical structures} \subset \text{i-urn structures} \subset \text{p-urn structures.}$$

However, this does not mean that differences between these semantics may be overlooked: in subsequent chapters, looking for more advanced model-theoretic results on urn semantics, I show that i- and p-urn semantics generate in fact quite different formalizations of TSI.

3.3 Some fundamental results on urn semantics

Since this thesis' main motivation is to apply logic in the development of a theory of semantic information, we should ask ourselves which one of the above introduced systems of urn semantics better captures the informativeness of sentences and, in particular, the informativeness of logical truths. I still do not have enough data to analyze this question here, but at this point I can at least advance some initial thoughts on the subject.

First of all, in contrast with classical logic, imperfect as well as perfect urn semantics describe *decidable* logical systems, that is, in these non-standard semantics, for any formula ϕ, there is an effective procedure for determining whether ϕ is or is not a validity of the system. Hence, independently of which one of these systems better captures semantic information, we know from the start that urn semantics characterizes an effectively computable notion of information. In the framework of urn semantics it is possible to effectively compute the minimum semantic information carried by a given sentence, something that cannot be done in classical TSI.

Secondly, there are characterization theorems showing that satisfiability in perfect or imperfect urn semantics is much more permissible than in classical logic. This theorem is very important for this work: in particular, it means that a lot of logical truths of classical logic have non-maximum classes of models in urn semantics and, by TSI, carry non-null amount of information. Hence, despite whether i-urn semantics is in fact a better system than p-urn semantics or vice versa, we know for sure that both systems provide a formalization of TSI without SoD.

The decidability of urn semantics and a partial result about characterization theorems for these systems were originally obtained by Rantala (1979). Note that Rantala's proofs of these results are heavily grounded in his game-theoretic approach of the subject. In this section, based on the previously introduced Tarskian framework for urn semantics, I systematize Rantala's proof of the decidability of urn semantics. Moreover, I generalize Rantala's theorem on satisfiability in urn semantics in terms of full-blooded characterization theorems for p- and i-urn semantics. Especially for the proof of the characterization theorems, I need to rely on a version of the Fraïssé-Hintikka theorem for urn semantics, a result that I also present in this section.

3.3.1 Decidability of the system

In the rest of this subsection, I focus exclusively on p-urn semantics since the present results are easily generalizable for i-urn semantics. First, I stipulate some notation and terminology. Let me say that a formula ϕ is *p-equivalent* to some formula ψ if and only if, for any p-urn structure $\widehat{\mathcal{M}_{\mathfrak{C}}}$, $\widehat{\mathcal{M}_{\mathfrak{C}}} \models_{n,\lambda} \phi \Leftrightarrow \widehat{\mathcal{M}_{\mathfrak{C}}} \models_{n,\lambda} \psi$, for every $n < \omega$ and for every ordinal λ. I denote p-equivalence by the symbol $\equiv_p$. Further, I say that ϕ is *p-equisatisfiable* with ψ if and only if the existence of some p-urn structure $\widehat{\mathcal{M}_{\mathfrak{C}}}$ such that $\widehat{\mathcal{M}_{\mathfrak{C}}} \models_{n,\lambda} \phi$ mutually implies the existence of some p-urn structure $\widehat{\mathcal{N}_{\mathfrak{C}'}}$ such that $\widehat{\mathcal{N}_{\mathfrak{C}'}} \models_{n,\rho} \psi$. I denote p-equisatisfiability by the symbol $\approx_p$. The concepts of i-equivalence and i-equisatisfiability are analogously defined and are symbolized by $\equiv_i$ and $\approx_i$, respectively.

The next lemmas are necessary for proving the decidability of p-urn semantics.

Lemma 3.10 *Let ϕ and ψ be formulas, and let it be that the variables x_1 and x_2 are not free in ψ and ϕ, respectively. The following holds:*

1. $\forall x\phi \equiv_p \neg\exists x\neg\phi$;
2. $\exists x\phi \equiv_p \neg\forall x\neg\phi$;
3. $\exists x(\phi \vee \psi) \equiv_p (\exists x\phi \vee \exists x\psi)$;
4. $\forall x(\phi \wedge \psi) \equiv_p (\forall x\phi \wedge \forall x\psi)$;

5. $\exists x_1\phi \wedge \exists x_2\psi \approx_p \exists x_1\exists x_2(\phi \wedge \psi)$;

6. $\forall x_1\phi \vee \forall x_2\psi \approx_p \forall x_1\forall x_2(\phi \vee \psi)$;

7. $\exists x_0 \dots \exists x_m\phi \approx_p \forall x_0 \dots \forall x_m\phi$;

8. $\exists x\phi \wedge \forall x\psi \approx_p \exists x(\phi \wedge \psi)$;

Proof. 1. $\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{n,\lambda} \forall x\phi \Leftrightarrow \widehat{\mathcal{M}}_{\mathfrak{C}} \models_{n+1,\rho} \phi x[a]$, for every $a \in \mathfrak{C}_{n,\lambda}$. This holds if and only if there is no $a \in \mathfrak{C}_{n,\lambda}$ such that $\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{n+1,\rho} \neg\phi x[a]$. So, $\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{n,\lambda} \neg\exists x\neg\phi$. The proof of 2 is very similar to this one.

3. $\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{n,\lambda} \exists x(\phi\vee\psi) \Leftrightarrow$ either $\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{n+1,\rho} \phi x[a]$ or $\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{n+1,\rho} \psi x[a]$, for some $a \in \mathfrak{C}_{n,\lambda}$. This holds if and only if either $\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{n,\lambda} \exists x\phi$ or $\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{n,\lambda} \exists x\psi$. So, $\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{n,\lambda} \exists x\phi \vee \exists x\psi$. The proof of 4 works analogously.

5. $\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{n,\lambda} \exists x_1\phi \wedge \exists x_2\psi \Leftrightarrow \widehat{\mathcal{M}}_{\mathfrak{C}} \models_{n+1,\rho} \phi x_1[a]$ and $\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{n+1,\rho'} \psi x_2[b]$, for some $a, b \in \mathfrak{C}_{n,\lambda}$. This holds if and only if $\widehat{\mathcal{M}}_{\mathfrak{C}'} \models_{n,\lambda} \exists x_1\exists x_2(\phi \wedge \psi)$, for a p-eligibility set $\mathfrak{C}'$ of $\mathcal{M}$ such that, for every $i \leq n$, $\mathfrak{C}'_i = \mathfrak{C}_i$ and, for every $j > n$, $\mathfrak{C}'_j = \mathfrak{C}_{j-1}$. The proof of 6 works in the same way.

7. Assume $\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{n,\lambda} \exists x_0 \dots \exists x_m\phi$. Let α be the λ-th element in $\widehat{\mathfrak{M}}_{n-1}$. So, there are $a_0 \in \mathfrak{C}_{n,\lambda_0}, \dots, a_m \in \mathfrak{C}_{n+m,\lambda_m}$ such that $\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{n+m+1,\rho} \phi x_0 \dots x_m[a_0 \dots a_m]$, in which, for every $0 \leq i \leq m$, $\alpha \frown (a_0, \dots, a_i)$ is the λ_i-th element in $\widehat{\mathfrak{M}}_{n+i}$. Consider a variation $\mathfrak{C}'$ of $\mathfrak{C}$ such that $\mathfrak{C}'_{n+i,\lambda_i} = \{a_i\}$. Therefore, $\widehat{\mathcal{M}}_{\mathfrak{C}'} \models_{n,\lambda} \forall x_0 \dots \forall x_m\phi$. Item 8 follows from 7. ∎

Lemma 3.11 *Suppose y is substitution-free for a variable x in a formula ϕ. Then, $Qy\phi \equiv_p Qx\psi y[x]$, in which Q is either $\forall$ or $\exists$.*

Proof. Consider some p-urn structure $\widehat{\mathcal{M}}_{\mathfrak{C}}$ such that $\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{n,\lambda} \exists y\phi$ (in the case of $\forall y\phi$, assume $\widehat{\mathcal{M}}_{\mathfrak{C}} \not\models_{n,\lambda} \forall y\phi$). Then, $\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{n+1,\rho} \phi y[a]$, for some $a \in \mathfrak{C}_{n,\lambda}$. So, $\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{n,\lambda} \exists x\phi a[x]$. ∎

Based on these lemmas we can finally prove the decidability of urn semantics.

Theorem 3.12 *Let ϕ be a formula. The following are equivalent:*

1. *ϕ is unsatisfiable in p-urn semantics;*

2. *For some formula ψ that is the existential closure of a contradiction of classical propositional logic, $\phi \approx_p \psi$;*

Proof. ($1 \Rightarrow 2$) Assuming 1, by induction on the quantifier rank of ϕ we verify that 2 holds as well. First, considering $qr(\phi) = 0$, 2 holds by classical logic, since urn semantics only changes the satisfiability conditions of quantified formulas.

Now, consider $qr(\phi) = q + 1$. We need to consider a set of possible cases:

- Case a: ϕ is $\exists x\psi$. By 1, ψ is unsatisfiable and, by inductive hypothesis, there is ψ' that is the existential closure of a contradiction of classical propositional logic and $\psi \approx_p \psi'$. So, $\phi \approx_p \exists x\psi'$;

- Case b: ϕ is $\forall x\psi$. Given that, by item 7 of Lemma 3.10, $\phi \approx_p \exists x\psi$, case b reduces itself to case a;

- Case c: ϕ is $\neg\forall x\psi$. By item 1 of Lemma 3.10, $\phi \equiv_p \exists x\neg\phi$. Then, case c reduces itself to case a. By a similar argument based on item 2 of Lemma 3.10, the case in which ϕ is $\neg\exists x\psi$ reduces itself to case b;

- Case d: ϕ is $\exists x_1\psi \vee \exists x_2\gamma$. By Lemma 3.11, for some variable x that does occur neither in ψ nor in γ, $\phi \equiv_p \exists x\psi \vee \exists x\gamma$. By item 3 of Lemma 3.10, $\phi \equiv_p \exists x(\psi \vee \gamma)$, therefore case d reduces itself to case a. By a similar argument based on item 4 of Lemma 3.10, the case in which ϕ is $\forall x_1\psi \wedge \forall x_2\gamma$ reduces itself to case b;

- Case e: ϕ is $\exists x_1\psi \wedge \exists x_2\gamma$. By Lemma 3.11, for some variable that does not occur neither in ψ not in γ, $\phi \equiv_p \exists x_1\psi \wedge \exists x\gamma$. By item 5 of Lemma 3.10, $\phi \approx_p \exists x_1\exists x(\psi \wedge \gamma)$, so case e reduces itself to case a. By a similar argument based on item 6 of Lemma 3.10, the case in which ϕ is $\forall x_1\psi \vee \forall x_2\gamma$ reduces itself to case b;

- Case f: ϕ is $\exists x_1\psi \wedge \forall x_2\gamma$. By Lemma 3.11, for some variable x that occurs neither in ψ nor in γ, $\phi \equiv_p \exists x\psi \wedge \forall x\gamma$. By item 8 of Lemma 3.10, $\phi \approx_p \exists x(\psi \wedge \gamma)$. Hence, case f reduces itself to case a.

($2 \Rightarrow 1$) Assuming the negation of 1, again by induction on the quantifier rank of ϕ we prove that 2 is also not the case. The proof of this direction works analogously. Therefore, I skip it for space reasons. ■

Corollary 3.13 *p- and i-urn semantics characterize decidable systems of logic.*

Proof. The decidability of p-urn semantics follows immediately from Theorem 3.12. Now, the proof of this theorem does not depend on any specific property of p-urn semantics and can be easily generalized for i-urn semantics. Hence, Theorem 3.12 holds as well for this system. ■

3.3.2 Fraïssé-Hintikka theorem in urn semantics

Note that Theorem 3.12 does not imply that *only* truths of classical propositional logic are logically valid in urn semantics. It means just that all logical truths of urn semantics are, *in urn semantics*, equisatisfiable with truths of classical propositional logic. However, we still might ask *which* are the logical truths of urn semantics. In this subsection I start to answer this question by introducing a version of the Fraïssé-Hintikka theorem for urn semantics.

Given that every i-urn structure is also a p-urn structure, I can conveniently maintain the focus here mainly on p-urn semantics. A further generalization of the present results for i-urn semantics is subsequently achieved. Preliminarily, let me define the notions of *game-normal formula* (terminology by Hodges (1997, p. 85)) and *Hintikka-normal form*. For a language $\mathcal{L}$, let $\Phi_{\mathcal{L}}(x_1, \ldots, x_n)$ be the set of *unnested* atomic formulas of $\mathcal{L}$ with variables within $x_1, \ldots, x_n$.

Definition 3.14 *Let $\mathcal{L}$ be a finite first order language. For every $n, m < \omega$, a formula $\theta(m, x_1, \ldots, x_n)$ of $\mathcal{L}$ is a* game-normal formula *if and only if the following holds:*

- *If $m = 0$, $\theta(m, x_1, \ldots, x_n)$ is $\bigwedge \Sigma \wedge \bigwedge \neg(\Phi_{\mathcal{L}}(x_1, \ldots, x_n) - \Sigma)$, for some $\Sigma \subseteq \Phi_{\mathcal{L}}(x_1, \ldots, x_n)$;*

Now, assume that $m > 0$ and the set $\Theta(m-1, x_1, \ldots, x_n, y_m)$ of game-normal formulas of $\mathcal{L}$ with quantifier rank $m-1$ and free-variables $x_1, \ldots, x_n, y_m$ has already been defined:

- *$\theta(m, x_1, \ldots, x_n)$ is $(\bigwedge \exists y_m \Gamma) \wedge (\forall y_m \bigvee \Gamma)$, for some $\Sigma \subseteq \Theta(m-1, x_1, \ldots, x_n, y_m)$.*

For any $m, n < \omega$, by $\Theta(m, x_1, \ldots, x_n)$ I denote the set of game-normal formulas of $\mathcal{L}$ with quantifier rank m and free-variables $x_1, \ldots, x_n$.

Definition 3.15 *Let $\mathcal{L}$ be a finite first order language. I say that a formula ϕ of $\mathcal{L}$ is a* Hintikka-normal form *(for simplicity, just H-normal form) if and only if ϕ is $\bigvee \Gamma$, for some $m, n < \omega$ and for some $\Gamma \subseteq \Theta(m, x_1, \ldots, x_n)$.*

In classical model theory, game-normal formulas are quite relevant since they define *back and forth* equivalence classes. Informally speaking, two structures are said to be back and forth equivalent if it is possible both to construct isomorphisms between finite parts of them and to finitely extend such isomorphisms ω-times. Let me precisely define these concepts in the context of urn semantics and prove the definability of back and forth equivalence by game-normal formulas.

Definition 3.16 *Let $\mathcal{L}$ be a first order language, and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. An injective function $\sigma : \mathcal{M} \to \mathcal{N}$ is an* embedding *if and only if the following holds:*

- *For any constant $c \in \mathcal{L}$, $\sigma(c^{\mathcal{M}})$ is $c^{\mathcal{N}}$;*
- *For any n-ary function $f \in \mathcal{L}$, $\sigma(f^{\mathcal{M}}(a_1 \ldots a_n))$ is $f^{\mathcal{N}}(\sigma(a_1) \ldots \sigma(a_n))$;*
- *For any n-ary predicate $R \in \mathcal{L}$, $\sigma(R^{\mathcal{M}}) = \{\langle \sigma(a_1), \ldots, \sigma(a_n) \rangle : \langle a_1, \ldots, a_n \rangle \in R^{\mathcal{M}}\} \subseteq R^{\mathcal{N}}$;*

In case the domain of $\mathcal{M}$ is included in the domain of $\mathcal{N}$, $\mathcal{M}$ is said to be a substructure *of $\mathcal{N}$, in symbols, $\mathcal{M} \subseteq \mathcal{N}$. If σ is bijective, usually we say that there is an* isomorphism *between $\mathcal{M}$ and $\mathcal{N}$, in symbols $\mathcal{M} \cong \mathcal{N}$.*

In the next definition, for a given p-urn structure $\widehat{\mathcal{M}_{\mathfrak{C}}}$, for a fixed subset $\mathfrak{D} \subseteq \mathfrak{C}$ and for a substructure $\mathcal{M}^* \subseteq \mathcal{M}$, let a restriction $\widehat{\mathfrak{M}}\lceil(\mathcal{M}^*, \mathfrak{D})$ of the set $\widehat{\mathfrak{M}}$ of p-eligible sequences of $\mathcal{M}$ over $\mathfrak{C}$ be as follows:

- $\widehat{\mathfrak{M}}\lceil(\mathcal{M}^*, \mathfrak{D})_0 = \{a \in \mathfrak{C}_0 \cap \mathcal{M}^* : \mathfrak{C}_0 \in \mathfrak{D}\}$;
- $\widehat{\mathfrak{M}}\lceil(\mathcal{M}^*, \mathfrak{D})_{n+1} = \{\alpha \frown (a) : a \in \mathfrak{C}_{n+1,\lambda} \cap \mathcal{M}^*,\ \alpha$ is the λ-th element in $\widehat{\mathfrak{M}}_n,\ \alpha \in \widehat{\mathfrak{M}}\lceil(\mathcal{M}^*, \mathfrak{D})_n$ and $\mathfrak{C}_{n+1,\lambda} \in \mathfrak{D}\}$;
- $\widehat{\mathfrak{M}}\lceil(\mathcal{M}^*, \mathfrak{D}) = \bigcup\limits_{n<\omega} \widehat{\mathfrak{M}}\lceil(\mathcal{M}^*, \mathfrak{D})_n$.

The next definition characterizes isomorphism between parts of p-urn structures.

Definition 3.17 *Let $\mathcal{L}$ be a first order language and let $\widehat{\mathcal{M}_{\mathfrak{C}}}$ and $\widehat{\mathcal{N}_{\mathfrak{C}'}}$ be $\mathcal{L}$-p-urn structures. Let $\widehat{\mathfrak{M}}$ and $\widehat{\mathfrak{N}}$ be the sets of p-eligible sequences of $\mathcal{M}$ over $\mathfrak{C}$ and of $\mathcal{N}$ over $\mathfrak{C}'$, respectively. Consider substructures $\mathcal{M}^* \subseteq \mathcal{M}$, $\mathcal{N}^* \subseteq \mathcal{N}$, subsets $\mathfrak{D} \subseteq \mathfrak{C}$, $\mathfrak{D}' \subseteq \mathfrak{C}'$ and, for a set of new predicates $P = \{P_n : n < \omega,\ P_n$ is n-ary$\}$, $\mathcal{L}^+ = \mathcal{L} \cup P$. There is an isomorphism between $\langle \mathcal{M}^*, \mathfrak{D}\rangle$ and $\langle \mathcal{N}^*, \mathfrak{D}'\rangle$, in symbols $\langle \mathcal{M}^*, \mathfrak{D}\rangle \cong \langle \mathcal{N}^*, \mathfrak{D}'\rangle$, if and only if the following holds:*

- *There are $\mathcal{L}^+$-expansions $\mathcal{M}^+ \supset \mathcal{M}^*$ and $\mathcal{N}^+ \supset \mathcal{N}^*$ such that, for every $n < \omega$, $P_n^{\mathcal{M}^+}$ is $\widehat{\mathfrak{M}}\lceil(\mathcal{M}^*, \mathfrak{D})_n$, and $P_n^{\mathcal{N}^+}$ is $\widehat{\mathfrak{N}}\lceil(\mathcal{N}^*, \mathfrak{D}')_n$;*
- *$\mathcal{M}^+ \cong \mathcal{N}^+$.*

The next definition characterizes back and forth equivalence between p-urn structures. In what follows, for any structure $\mathcal{M}$, by $\{a_0, \ldots, a_n\}_{\mathcal{M}}$ I denote the substructure of $\mathcal{M}$ with domain $\{a_0, \ldots, a_n\}$.

Definition 3.18 *Let $\mathcal{L}$ be a first order language, $\widehat{\mathcal{M}_{\mathfrak{C}}}$ and $\widehat{\mathcal{N}_{\mathfrak{C}'}}$ be $\mathcal{L}$-p-urn structures. Let $\widehat{\mathfrak{M}}$ and $\widehat{\mathfrak{N}}$ be the sets of p-eligible sequences of $\mathcal{M}$ over $\mathfrak{C}$ and of $\mathcal{N}$ over $\mathfrak{C}'$, respectively. $\widehat{\mathcal{M}_{\mathfrak{C}}}$ and $\widehat{\mathcal{N}_{\mathfrak{C}'}}$ are* back and forth equivalent, *in symbols $\widehat{\mathcal{M}_{\mathfrak{C}}} \simeq \widehat{\mathcal{N}_{\mathfrak{C}'}}$, if and only if there is a poset H of isomorphisms between countable parts of $\widehat{\mathcal{M}_{\mathfrak{C}}}$ and $\widehat{\mathcal{N}_{\mathfrak{C}'}}$ such that the following holds:*

- *The first element $\sigma_0 \in H$ is $\emptyset$;*

Now, assume that, for some $n < \omega$, there are substructures $\{a_0, \ldots, a_n\}_{\mathcal{M}}$ and $\{b_0, \ldots, b_n\}_{\mathcal{N}}$ such that $a_0 \in \mathfrak{C}_0$, $b_0 \in \mathfrak{C}'_0$ and, for any $0 < i \leq n$, $a_i \in \mathfrak{C}_{i,\lambda_i}$ and $b_i \in \mathfrak{C}'_{i,\rho_i}$, in which λ_i and ρ_i are the indexes of $\langle a_0, \ldots, a_{i-1}\rangle$ and $\langle b_0, \ldots, b_{i-1}\rangle$ in $\widehat{\mathfrak{M}}_{i-1}$ and $\widehat{\mathfrak{N}}_{i-1}$, respectively. Let $\mathfrak{D} = \{\mathfrak{C}_0\} \cup \{\mathfrak{C}_{i+1,\lambda_{i+1}} : i < n\}$, $\mathfrak{D}' = \{\mathfrak{C}'_0\} \cup \{\mathfrak{C}'_{i+1,\rho_{i+1}} : i < n\}$ and consider that λ' and ρ' are the indexes of $\langle a_0, \ldots, a_n\rangle \in \widehat{\mathfrak{M}}_n$ and $\langle b_0, \ldots, b_n\rangle \in \widehat{\mathfrak{N}}_n$, respectively. Further, consider there is an isormorphism $\sigma_n : \langle\{a_0, \ldots, a_n\}_{\mathcal{M}}, \mathfrak{D}\rangle \to \langle\{b_0, \ldots, b_n\}_{\mathcal{N}}, \mathfrak{D}'\rangle$, $\sigma_n \in H$:

- *For any $a \in \mathfrak{C}_{n+1,\lambda'}$, there is $b \in \mathfrak{C}'_{n+1,\rho'}$ such that there is isomorphism $\sigma_{n+1} : \langle\{a_0, \ldots, a_n,\ a\}_{\mathcal{M}}, \mathfrak{D} \cup \{\mathfrak{C}_{n+1,\lambda'}\}\rangle \to \langle\{b_0, \ldots, b_n,\ b\}_{\mathcal{N}}, \mathfrak{D}' \cup \{\mathfrak{C}_{n+1,\rho'}\}\rangle$, for which $\sigma_n \subseteq \sigma_{n+1} \in H$;*

- *For any $b \in \mathfrak{C}'_{n+1,\rho'}$, there is $a \in \mathfrak{C}_{n+1,\lambda'}$ such that there is isomorphism $\sigma_{n+1} : \langle \{a_0, \ldots, a_n,\ a\}_{\mathcal{M}}, \mathfrak{D} \cup \{\mathfrak{C}_{n+1,\lambda'}\}\rangle \to \langle \{b_0, \ldots, b_n,\ b\}_{\mathcal{N}}, \mathfrak{D}' \cup \{\mathfrak{C}_{n+1,\rho'}\}\rangle$, for which $\sigma_n \subseteq \sigma_{n+1} \in H$;*

- *H is the smallest poset satisfying clauses 1–3 above.*

$\widehat{\mathcal{M}}_{\mathfrak{C}}$ and $\widehat{\mathcal{N}}_{\mathfrak{C}'}$ are k-back and forth equivalent, in symbols $\widehat{\mathcal{M}}_{\mathfrak{C}} \simeq_k \widehat{\mathcal{N}}_{\mathfrak{C}'}$, if and only if the condition above holds for at least every $n < k$.

Lemma 3.19 *Let $\mathcal{L}$ be a finite first order language and $\widehat{\mathcal{M}}_{\mathfrak{C}}$ be an $\mathcal{L}$-p-urn structure. For every $m, n, k < \omega$, for any elements $a_1, \ldots, a_n$ of $\mathcal{M}$ and for any ordinal λ, there is only one game-normal formula $\theta \in \Theta(m, x_1, \ldots, x_n)$ such that $\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{k,\lambda} \theta x_1 \ldots x_n[a_1 \ldots a_n]$.*

Proof. Proceed by induction on $m < \omega$. Consider $m = 0$ and fix some $k, n < \omega$, elements $a_1, \ldots, a_n$ of $\mathcal{M}$ as well as some ordinal λ. The relevant θ is the game-normal formula $\bigwedge \Sigma \wedge \bigwedge \neg(\Phi_{\mathcal{L}}(x_1, \ldots, x_n) - \Sigma)$, in which $\Sigma = \{\phi \in \Phi_{\mathcal{L}}(x_1, \ldots, x_n) : \widehat{\mathcal{M}}_{\mathfrak{C}} \models_{k,\lambda} \phi x_1 \ldots x_n[a_1 \ldots a_n]\}$.

In the inductive step, assume that for some $m < \omega$ the property was already obtained. Let $\Gamma = \{\theta \in \Theta(m, x_1, \ldots, x_n, y) : \widehat{\mathcal{M}}_{\mathfrak{C}} \models_{k+1,\rho} \theta x_1 \ldots x_n y[a_1 \ldots a_n\ b]$, for some element $b \in \mathfrak{C}_{k,\lambda}\}$. Since $\mathcal{L}$ is finite, Γ is finite as well. Therefore,$(\bigwedge \exists y \Gamma) \wedge (\forall y \bigvee \Gamma)$ is our desired game-normal formula. ■

Theorem 3.20 *Let $\mathcal{L}$ be a finite first order language, $\widehat{\mathcal{M}}_{\mathfrak{C}}$ and $\widehat{\mathcal{N}}_{\mathfrak{C}'}$ be $\mathcal{L}$-p-urn structures. Let $\widehat{\mathfrak{M}}$ and $\widehat{\mathfrak{N}}$ be the sets of p-eligible sequences of $\mathcal{M}$ over $\mathfrak{C}$ and of $\mathcal{N}$ over $\mathfrak{C}'$, respectively. For every $k < \omega$, the following are equivalent:*

1. *$\widehat{\mathcal{M}}_{\mathfrak{C}} \simeq_k \widehat{\mathcal{N}}_{\mathfrak{C}'}$;*
2. *For every $m \leq k$, there is $\theta \in \Theta(m, \emptyset)$ that is satisfied both in $\widehat{\mathcal{M}}_{\mathfrak{C}}$ as well as in $\widehat{\mathcal{N}}_{\mathfrak{C}'}$.*

Proof. ($1 \Rightarrow 2$) Assume 1. Item 2 is a particular case of the following more general result:

($*$) For any m and n such that $m + n \leq k$, for any $\theta \in \Theta(m, x_0, \ldots, x_{n-1})$, for any ordinal $\lambda <| \widehat{\mathfrak{M}}_{n-1} |$ and for $\langle a_0, \ldots, a_{n-1}\rangle$ the λ-th sequence in $\widehat{\mathfrak{M}}_{n-1}$,

$$\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{n,\lambda} \theta x_0 \ldots x_{n-1}[a_0 \ldots a_{n-1}] \Leftrightarrow \widehat{\mathcal{N}}_{\mathfrak{C}'} \models_{n,\lambda'} \theta x_0 \ldots x_{n-1}[\sigma_n(a_0) \ldots \sigma_n(a_{n-1})],$$

in which σ_n is in the poset of isomorphisms H defining $\widehat{\mathcal{M}}_{\mathfrak{C}} \simeq_k \widehat{\mathcal{N}}_{\mathfrak{C}'}$, $\{a_0, \ldots, a_{n-1}\}_{\mathcal{M}}$ defines σ_n's domain and $\langle \sigma_n(a_0), \ldots, \sigma_n(a_{n-1})\rangle$ is the λ'-th sequence in $\widehat{\mathfrak{N}}_{n-1}$.

The proof of ($*$) follows by an induction on m. First, let $m = 0$. By 1, for any p-eligible sequence $\langle a_0, \ldots, a_{n-1}\rangle$ of $\widehat{\mathfrak{M}}_{n-1}$, there is $\sigma_n \in H$ such that $\{a_0, \ldots, a_{n-1}\}_{\mathcal{M}}$ defines σ_n's domain. By isomorphism, it is the case that $\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{n,\lambda} \phi x_0 \ldots x_{n-1}[a_0 \ldots a_{n-1}]$, for every ϕ in the set $\Sigma = \{\phi \in \Phi_{\mathcal{L}}(x_0, \ldots, x_{n-1}) \cup \neg\Phi_{\mathcal{L}}(x_0, \ldots, x_{n-1}) : \widehat{\mathcal{N}}_{\mathfrak{C}'} \models_{n,\lambda'} \phi x_0 \ldots x_{n-1}[\sigma_n(a_0) \ldots \sigma_n(a_{n-1})]\}$. Therefore, the game-normal formula $\bigwedge \Sigma$ is satisfied both by $\widehat{\mathcal{M}}_{\mathfrak{C}}$ and $\widehat{\mathcal{N}}_{\mathfrak{C}'}$.

For the inductive step, assume the property holds for some $m < k$. For any n such that $m + n < k$, consider an arbitrary game-normal formula $\theta \in \Theta(m + 1, x_0, \ldots, x_{n-1})$. Giving the general syntactic form of θ, there are two possible cases:

- $\exists$-case: $\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{n,\lambda} \exists y \theta' x_0 \ldots x_{n-1}[a_0 \ldots a_{n-1}]$, for some $\theta' \in \Theta(m, x_0, \ldots, x_{n-1}, y)$. So, $\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{n+1,\rho} \theta' x_0 \ldots x_{n-1} y[a_0 \ldots a_{n-1}\ a]$, for some $a \in \mathfrak{C}_{n,\lambda}$. By inductive hypothesis and 1, there is $\sigma_{n+1} \in H$ whose domain is defined by $\{a_0, \ldots, a_{n-1}, a\}_{\mathcal{M}}$. Then, since $\sigma_{n+1}(a) \in \mathfrak{C}'_{n,\lambda'}$, it is the case that $\widehat{\mathcal{N}}_{\mathfrak{C}'} \models_{n,\lambda'} \exists y \theta' x_0 \ldots x_{n-1}[\sigma_{n+1}(a_0) \ldots \sigma_{n+1}(a_{n-1})]$.

- $\forall$-case: $\widehat{\mathcal{M}}_{\mathfrak{C}} \not\models_{n,\lambda} \forall y \bigvee \Gamma x_0 \ldots x_{n-1}[a_0 \ldots a_{n-1}]$, for some $\Gamma \subset \Theta(m, x_0, \ldots, x_{n-1}, y_0)$. This holds if and only if $\widehat{\mathcal{M}}_{\mathfrak{C}} \not\models_{n+1,\rho} \theta' x_0 \ldots x_{n-1} y[a_0 \ldots a_{n-1}\ a]$, for some $a \in \mathfrak{C}_{n,\lambda}$ and for every $\theta' \in \Gamma$. By inductive hypothesis and 1, there is $\sigma_{n+1} \in H$ whose domain is defined by $\{a_0, \ldots, a_{n-1}, a\}_{\mathcal{M}}$. Then, since $\sigma_{n+1}(a) \in \mathfrak{C}'_{n,\lambda'}$, it is the case that $\widehat{\mathcal{N}}_{\mathfrak{C}'} \not\models_{n,\lambda}$ $\forall y \bigvee \Gamma x_0 \ldots x_{n-1}[\sigma_{n+1}(a_0) \ldots \sigma_{n+1}(a_{n-1})]$.

This completes the proof of $(*)$, 2 being a subcase of it.

($2 \Rightarrow 1$) Assuming 2, by recursion on $m < k$, define a poset H that characterizes k-back and forth equivalence between the considered p-urn structures.

For some $m < k$, suppose a poset H characterizing $\widehat{\mathcal{M}}_{\mathfrak{C}} \simeq_m \widehat{\mathcal{N}}_{\mathfrak{C}'}$ was already defined. Let $\langle a_0, \ldots, a_{m-1} \rangle \in \widehat{\mathfrak{M}}_{m-1}$ such that, for every $0 \leq i < m$, λ_i is the index of $\langle a_0, \ldots, a_i \rangle \in \widehat{\mathfrak{M}}_i$. Consider $\mathfrak{D} = \{\mathfrak{C}_0\} \cup \{\mathfrak{C}_{i,\lambda_{i-1}} : 0 < i < m\}$ and fix some $\sigma_m \in H$ with domain $\langle \{a_0, \ldots, a_{m-1}\}_{\mathcal{M}}, \mathfrak{D} \rangle$. Consider $\langle \sigma_m(a_0), \ldots, \sigma_m(a_{m-1}) \rangle$ is the ρ-th sequence in $\widehat{\mathfrak{N}}_{m-1}$. I will show that for any $a \in \mathfrak{C}_{m,\lambda_{m-1}}$, there is some $b \in \mathfrak{C}'_{m,\rho}$ such that there is an isomorphism $\sigma^* \supset \sigma_{m-1}$ with domain $\langle \{a_0, \ldots, a_{m-1}, a\}_{\mathcal{M},} \mathfrak{D} \cup \{\mathfrak{C}_{m,\lambda_{m-1}}\} \rangle$ and such that $\sigma^*(a) = b$.

Let ξ be the index of $\langle a_0, \ldots, a_{m-1}, a \rangle$ in $\widehat{\mathfrak{M}}_m$ and let $\theta \in \Theta(k-m-1, x_0, \ldots, x_{m-1}, y)$ such that $\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{m+1,\xi} \theta x_0 \ldots x_{m-1} y[a_0 \ldots a_{m-1}\ a]$. So, $\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{m,\lambda_{m-1}} \exists y \theta x_0 \ldots x_{m-1}[a_0 \ldots a_{m-1}]$. By 2 and inductive hypothesis, $\widehat{\mathcal{N}}_{\mathfrak{C}'} \models_{m,\rho} \exists y \theta x_0 \ldots x_{m-1}[\sigma_m(a_0) \ldots \sigma_m(a_{m-1})]$. Let $b \in \mathfrak{C}'_{m,\rho_{m-1}}$ be such that, $\widehat{\mathcal{N}}_{\mathfrak{C}'} \models_{m,\xi'} \theta x_0 \ldots x_{m-1} y[\sigma_m(a_0) \ldots \sigma_m(a_{m-1})\ b]$, in which ξ' is the index of $\langle \sigma_m(a_0), \ldots, \sigma_m(a_{m-1})\ b \rangle$ in $\widehat{\mathfrak{N}}_m$. Consider an extension $\sigma^* \supset \sigma_m$ such that $\sigma^*(a) = b$. I will prove that σ^* is our intended isomorphism.

For any closed term $t \in \mathcal{L}$, $t^{\mathcal{M}} = a \Leftrightarrow t = y$ occurs in $\theta \Leftrightarrow t^{\mathcal{N}} = b$.

For any n-ary relation $R \in \mathcal{L}$, for every $\alpha \in \{a_0, \ldots, a_{m-1}, a\}^n$ and $\bar{x} \in \{x_0, \ldots, x_{m-1}, y\}^n$, $\alpha \in R^{\mathcal{M}} \Leftrightarrow R(\bar{x})$ occurs in $\theta \Leftrightarrow \sigma_m(\alpha) \in R^{\mathcal{N}}$. Finally, the case of functions can be reduced to the case of relations. So, σ^* is isomorphism. ■

Note that, given the already mentioned inclusions of the classes of classical and i-urn structures in the class of p-urn structures, Lemma 3.19 and Theorem 3.20 are *conservative* generalizations of similar results for classical logic and i-urn semantics. Based on these results it is possible to present a version of the Fraïssé-Hintikka theorem for urn semantics.

Theorem 3.21 *Let ϕ be an unnested formula of a finite first order language $\mathcal{L}$ with quantifier rank k and free-variables $x_1, \ldots, x_n$. Then, for every $q \geq k$, there is a H-normal form ψ with quantifier rank q and free-variables $x_1 \ldots, x_n$ such that $\phi \equiv_p \psi$ and $\phi \equiv_i \psi$.*

Proof. The theorem follows by induction on ϕ. Consider ϕ is an atomic formula. Fix some $q \geq 0$ and $\Gamma = \{\theta \in \Theta(q, x_1, \ldots, x_n) : \widehat{\mathcal{M}_\mathfrak{C}} \models \theta \wedge \phi x_1 \ldots x_n[a_1 \ldots a_n],$ for some elements $a_1, \ldots, a_n$ of $\mathcal{M}\}$. By Lemma 3.19 and the satisfiability of ϕ, Γ is not empty. Further, by the finiteness of $\mathcal{L}$, Γ is a finite set. I will prove that the H-normal form $\bigvee \Gamma$ is equivalent to ϕ.

First, assume $\widehat{\mathcal{M}_\mathfrak{C}} \models \phi$. By Lemma 3.19, there is some $\theta \in \Theta(q, x_1, \ldots, x_n)$ such that $\widehat{\mathcal{M}_\mathfrak{C}} \models \theta$, and, by definition of Γ, θ is in this set. So, $\widehat{\mathcal{M}_\mathfrak{C}} \models \bigvee \Gamma$. Moreover, for the proof of necessity, suppose that $\widehat{\mathcal{M}_\mathfrak{C}} \not\models \phi$ and fix some $\theta \in \Gamma$. Suppose that $\widehat{\mathcal{M}_\mathfrak{C}} \models \theta$. By Theorem 3.20, $\widehat{\mathcal{M}_\mathfrak{C}} \models \phi$, what contradicts our original assumption. So, $\widehat{\mathcal{M}_\mathfrak{C}} \not\models \bigvee \Gamma$. Therefore, ϕ is equivalent to $\bigvee \Gamma$.

For the inductive step, fix some $q \geq k$. Assume ϕ is $\psi \wedge \gamma$. By inductive hypothesis, there are H-normal forms $\bigvee \Psi$ and $\bigvee \Gamma$ of ψ and γ, respectively, $q = \max\{qr(\bigvee \Psi), qr(\bigvee \Gamma)\}$. Hence, $\bigvee(\Psi \cap \Gamma)$ is the H-normal form of ϕ. Similarly, it is easy to verify that $\bigvee(\Psi \cup \Gamma)$ and $\bigvee(\Theta(q, x_1, \ldots, x_n) - \Gamma)$ are the H-normal forms of $\psi \vee \gamma$ and $\neg\psi$, respectively.

Now, assume ϕ is $\exists y \psi$. By inductive hypothesis, ψ has a H-normal form $\bigvee \Gamma$, in which $\Gamma \subseteq \Theta(q-1, x_1, \ldots, x_n, y)$. Then, $\exists y \psi$ is equivalent to $\bigvee \exists y \Gamma$. Consider $\Delta = \mathcal{P}(\Gamma)$ (i.e., Δ is the power set of Γ), and let $\Delta_1, \ldots, \Delta_m$ be all its elements. For every $0 < i \leq m$, the formula $\bigwedge \exists y \Delta_i \wedge \forall y \bigvee \Delta_i$ is a game-normal formula in $\Theta(q, x_1, \ldots, x_n)$. Further, $\bigvee \exists y \Gamma \equiv_p \bigvee_{0<i\leq m} (\bigwedge \exists y \Delta_i \wedge \forall y \bigvee \Delta_i)$. So, this formula is the H-normal form of ϕ. The case in which ϕ is $\forall x \psi$ can be reduced to the previously considered ones by exploring the fact that $\forall x \psi \equiv_p \neg\exists x \psi$.

This is enough for proving that there is a relevant H-normal form ψ such that $\phi \equiv_p \psi$. Given that every i-urn structure is also a p-urn structure, it follows that $\phi \equiv_i \psi$. ■

Now I generalize Theorem 3.21 by showing that every (unnested or not) formula is p-equisatisfiable with a H-normal form. In order to simplify the notation, let *Hnf*(ϕ) denote the set of H-normal forms of an unnested formula ϕ. *Hnf*(ϕ) can be defined as follows:

- If ϕ is atomic formula of $\mathcal{L}$ with free-variables $x_1, \ldots, x_n$, then *Hnf*(ϕ) is the set of formulas of the form $\bigvee \Gamma_q$ such that, for every $q \geq 0$, $\Gamma_q = \{\theta \in \Theta(q, x_1, \ldots, x_n) :$ for some p-urn structure $\widehat{\mathcal{M}_\mathfrak{C}}$ and for some elements $a_1, \ldots, a_n$ of $\mathcal{M}$, $\widehat{\mathcal{M}_\mathfrak{C}} \models \theta \wedge \phi x_1 \ldots x_n[a_1 \ldots a_n]\}$;

- If ϕ is $\psi \wedge \gamma$, then *Hnf*(ϕ)=$\{\bigvee(\Psi_q \cap \Gamma_q) : \bigvee \Psi_q \in$ *Hnf*(ψ) and $\bigvee \Gamma_q \in$ *Hnf*(γ), $qr(\phi) \leq qr(\bigvee \Psi_q) = qr(\bigvee \Gamma_q) = q\}$;

- If ϕ is $\psi \vee \gamma$, then *Hnf*(ϕ)=$\{\bigvee(\Psi_q \cup \Gamma_q) : \bigvee \Psi_q \in$ *Hnf*(ψ) and $\bigvee \Gamma_q \in$ *Hnf*(γ), $qr(\phi) \leq qr(\bigvee \Psi_q) = qr(\bigvee \Gamma_q) = q\}$;

- If ϕ is $\neg\psi$, then *Hnf*(ϕ)=$\{\bigvee(\Theta(q, x_1, \ldots, x_n) - \Gamma) : qr(\phi) \leq q, \bigvee \Gamma \in$ *Hnf*(ψ)$\}$;

- If ϕ is $\exists y\psi$, then *Hnf*(ϕ)=$\{ \bigvee_{0<i\leq m} (\bigwedge \exists y \Delta_{i,q} \wedge \forall y \bigvee \Delta_{i,q}) : qr(\phi) \leq q$, $\{\Delta_{1,q}, \ldots, \Delta_{m,q}\} = \mathcal{P}(\Gamma),$ for $\Gamma \subseteq \Theta(q-1, x_1, \ldots, x_n, y)$ such that $\bigvee \Gamma \in$ *Hnf*$(\psi)\}$.

Further, consider any formula ϕ and let $t_1, \ldots, t_m$ be the terms occurring in ϕ that do not occur as arguments of any other term occurring in the same formula (in what follows, for simplicity, I call such kind of terms as *maximal terms* of a formula). Let † be the following mapping generator of unnested formulas:

- If $t_1, \ldots, t_m$ have complexity at most 1 (that is, if ϕ is an unnested formula), then $\phi^\dagger$ is ϕ;
- Assume $t'_1, \ldots, t'_o$ are all terms in $\{t_1, \ldots, t_m\}$ such that, for every $0 < i \leq o$, t'_i is $f_i(s_{i,1} \ldots s_{i,k_i})$ and $s_{i,1}, \ldots, s_{i,k_i}$ are not variables. Suppose that the mapping † was already defined for any formula whose maximal terms have complexity at most $q = \max\{cp(s_{i,l}) :: 0 < i \leq o,\ 0 < l \leq k_i\}$. Let $\{s'_1, \ldots, s'_j\} = \{s_{i,l} : 0 < i \leq o,\ 0 < l \leq k_i\}$. Consider a list $\phi_0, \ldots, \phi_j$ of formulas generated from ϕ in the following way: suppose that, in step k, a formula ϕ_k was constructed from ϕ. Let ψ be the greatest subformula of ϕ_k in which s'_{k+1} occurs free. So, let ϕ_{k+1} be the formula generated from ϕ_k by replacing the subformula ψ by $\exists z_{k+1}(\psi\ z_{k+1}[s'_{k+1}] \wedge (z_{k+1} = s'_{k+1}))$, for some variable z_{k+1} that does not occur in ϕ_k. Note that, in this way, all terms occurring in ϕ_j have complexity at most q. Then $\phi^\dagger$ is $\phi_j^\dagger$.

In classical logic, a formula ϕ is always equivalent to some unnested formula γ. The next proposition shows that, in urn semantics, ϕ and γ are at least equisatisfiable.

Proposition 3.22 *Let $\mathcal{L}$ be a first order language and ϕ be a formula of $\mathcal{L}$. Then, there is some unnested formula γ such that $\phi \approx_p \gamma$ and $\phi \approx_i \gamma$.*

Proof. In what follows, I focus particularly on i-urn semantics: given that every i-urn structure is also a p- one, the obtained result is immediately generalizable for p-urn semantics as well. By Theorem 3.12, there is an existential closure of a satisfiable formula of classical propositional logic, let it be ψ, such that $\phi \approx_i \psi$. Without loss of generality, it is convenient to consider that ψ is of the form $\exists x_1 \ldots \exists x_n \psi'$ such that ψ' is quantifier-free formula.

I will prove that $\psi \approx_i \psi^\dagger$ by repetition of the following procedure. Let $t_1, \ldots, t_m$ be all the maximal terms of ψ. Assume $t'_1, \ldots, t'_o$ are all terms in $\{t_1, \ldots, t_m\}$ such that, for every $0 < i \leq o$, t'_i is $f_i(s_{i,1} \ldots s_{i,k_i})$ and $s_{i,1}, \ldots, s_{i,k_i}$ are not variables. Let $\{s'_1, \ldots, s'_j\} = \{s_{i,l} : 0 < i \leq o,\ 0 < l \leq k_i\}$.

Assume there is an i-urn structure $\mathcal{M}_{\mathfrak{B}}$ such that $\mathcal{M}_{\mathfrak{B}} \models_n \psi$. Let $\mathfrak{B}'$ be a variation of $\mathfrak{B}$ such that, for every $i < n$, $\mathfrak{B}'_i = \mathfrak{B}_i$ and, for every $n \leq l < n+j$, $\mathfrak{B}'_l = \{s'_{l-n+1}\}$. Consider a formula ψ_j generated from ψ exactly as in the second clause of the characterization of † above. Then, $\mathcal{M}_{\mathfrak{B}'} \models_n \psi_j$. Now, by repeating this procedure a finite number of times, $\psi^\dagger$ and a variation $\mathfrak{B}''$ of $\mathfrak{B}$ such that $\mathcal{M}_{\mathfrak{B}''} \models_n \psi^\dagger$ are generated. Therefore $\psi \approx_i \psi^\dagger$ and, consequently, $\phi \approx_i \psi^\dagger$. Since $\psi^\dagger$ is an unnested formula, it is the desired formula γ. ■

Corollary 3.23 *For every formula ϕ of a finite first order language $\mathcal{L}$, there is an unnested formula γ such that, for every $\psi \in$ Hnf(γ), $\phi \approx_p \psi$ and $\phi \approx_i \psi$.*

Based on the above result, for any formula ϕ, let *Hnf*$^*(\phi)$ be the set of H-normal forms Hnf(γ), for an unnested formula γ that is equivalent with ϕ in urn semantics. Corollary 3.23 shows two divergences between classical logic and urn semantics. First, note that it is strictly weaker than a similar consequence of classical Fraïssé-Hintikka theorem: in classical logic, every formula is *equivalent* to a bunch of H-normal forms. Furthermore, observe that, in the proof of proposition 3.22, the desired formula γ is obtained by exploring relations of equisatisfiability that do not hold in classical logic. Consequently, this state of affairs means that an arbitrary formula ϕ has different sets of H-normal forms in classical logic and in urn semantics.

3.3.3 A characterization theorem

Corollary 3.23 enables to formulate a characterization theorem for urn semantics, a result that shows which truths of classical logic are preserved in urn semantics.

Definition 3.24 *For a finite first order language $\mathcal{L}$ and for any $m, n < \omega$, $\theta \in \Theta(m, x_1, \ldots, x_n)$ is a* perfectly-consistent-game-normal formula *(hereafter I say simply that θ is a pc-game-normal formula) if and only if the following holds:*

- *For any term t of $\mathcal{L}$ occurring in θ, $(t = t)$ occurs in θ;*
- *For every $r < \omega$, for any terms $t_1, \ldots, t_r$ and $t'_1, \ldots, t'_r$ with complexity at most 1, variables within $x_1, \ldots, x_n$ and such that $(t_i = t'_i)$ occurs in θ, for every $0 < i \leq r$ and for every atomic formula ϕ with variables within $y_1, \ldots, y_r$, $\phi y_1, \ldots, y_r[t_1, \ldots, t_r]$ occurs in θ if and only if $\phi y_1, \ldots, y_r[t'_1, \ldots, t'_r]$ occurs in θ;*
- *For every atomic formula ϕ of $\mathcal{L}$ with variables within $x_1, \ldots, x_n$, ϕ occurs in θ if and only if $\neg\phi$ does not occur in θ.*

Moreover, I say that a formula ψ is a pc-H-normal form *if and only if ψ is a H-normal form and every game-normal formula occurring in ψ is a pc-game-normal formula.*

In this subsection I prove that the satisfiability of a formula in p-urn semantics is based on the existence of pc-H-normal forms. Later, I generalize this result for i-urn semantics by considering a similar set of formulas. My strategy is to construct canonical models for pc-H-normal forms. In classical logic, it is possible to build canonical models of consistent sets of formulas by extending them to *Hintikka sets* (HODGES, 1997, pp. 40–42). Similarly, given a relevant H-normal form ψ I define something akin to a Hintikka set for which surely there is a p-urn structure.

Let a *subformula-chain* be a sequence of formulas $\langle \phi_0, \ldots, \phi_n \rangle$ of a first order language $\mathcal{L}$ such that ϕ_0 is any formula and, for any $0 \leq i < n$, ϕ_{i+1} is one of the following alternatives:

- If ϕ_i is $\neg\neg\psi$, then ϕ_{i+1} is ψ;
- If ϕ_i is either $\psi \wedge \gamma$ or $\psi \vee \gamma$, then ϕ_{i+1} is either ψ or γ;
- If ϕ_i is either $\neg(\psi \wedge \gamma)$ or $\neg(\psi \vee \gamma)$, then ϕ_{i+1} is either $\neg\psi$ or $\neg\gamma$;
- If ϕ_i is either $\exists x\psi$ or $\forall x\psi$, then ϕ_{i+1} is $\psi x[t]$, for some term t of $\mathcal{L}$;
- If ϕ_i is either $\neg\exists x\psi$ or $\neg\forall x\psi$, then ϕ_{i+1} is $\neg\psi x[t]$, for some term t of $\mathcal{L}$.

Let an *identity-chain* be a sequence $\langle\phi_0, \dots, \phi_{m+1}\rangle$ such that, for terms $t_1, \dots, t_m, t'_1, \dots, t'_m$, ϕ_0 is an atomic formula $\psi y_1, \dots, y_m[t_1, \dots, t_m]$, for any $0 < i \leq m$ ϕ_i is $t_i = t'_i$ and ϕ_{m+1} is $\psi y_1, \dots, y_m[t'_1, \dots, t'_m]$.

Let me say that a sequence of terms $\langle t_1, \dots, t_m\rangle$ is a *sequence of witnesses* of a subformula-chain T if and only if there are formulas $\phi_{k_1}, \dots, \phi_{k_m}$ in T such that, for every $0 < i \leq m$ and for some formula ψ_i of $\mathcal{L}$, ϕ_{k_i} is either $\exists x\psi_i$ or $\forall x\psi_i$ and ϕ_{k_i+1} is $\psi_i x[t_i]$. By the *quantifier rank* of a subformula-chain T, in symbols $QR(T)$, I mean the total number of quantified formulas that T has. Finally, for every subformula-chain T whose last element is either $\forall x\psi$ or $\neg\exists x\psi$, for some formula ψ, let the *existential dual* $ed(T)$ of T be a subformula-chain that diverges from T exactly in the fact that either $\exists x\psi$ or $\exists x\neg\psi$ is its last element, respectively. T and $ed(T)$ share the same sequence of witnesses and same quantifier rank.

Definition 3.25 *Let $\mathcal{L}$ be a first order language. Consider a list of sets Δ_n, for every $n < \omega$, such that the following holds:*

1. *For every term t of $\mathcal{L}$, $(t = t) \in \Delta_0$;*
2. *For every atomic formula ϕ of $\mathcal{L}$ with free-variables $y_1, \dots, y_m$, if there are terms $t_1, \dots, t_m, t'_1, \dots, t'_m$ of $\mathcal{L}$ such that, for every $0 < i \leq m$, $(t_i = t'_i) \in \Delta_0$, then $\phi y_1 \dots y_m[t_1 \dots t_m] \in \Delta_0 \Leftrightarrow \phi y_1 \dots y_m[t'_1 \dots t'_m] \in \Delta_0$;*
3. *For every atomic formula ϕ of $\mathcal{L}$, either $\phi \notin \Delta_0$ or $\neg\phi \notin \Delta_0$;*
4. *If $\neg\neg\phi \in \Delta_0$, then $\phi \in \Delta_0$;*
5. *If $\phi \wedge \psi \in \Delta_0$, then $\phi, \psi \in \Delta_0$; if $\neg(\phi \wedge \psi) \in \Delta_0$, then either $\neg\phi$ or $\neg\psi$ are in Δ_0;*
6. *If $\phi \vee \psi \in \Delta_0$, then either ϕ or ψ are in Δ_0; if $\neg(\phi \vee \psi) \in \Delta_0$, then $\neg\phi, \neg\psi \in \Delta_0$;*
7. *For every $\phi \in \Delta_0$, for every subformula-chain T in the set of subformulas of ϕ such that, for some formula ψ, its last element is either $\forall x\psi$ or $\neg\exists x\psi$, there is a formula $\phi' \in \Delta_0$ such that $ed(T)$ is a subformula-chain in the set of subformulas of ϕ'.*

Now, assume $\Delta_0, \dots, \Delta_n$ have already been defined. Then, Δ_{n+1} is the smallest *set satisfying the following clauses:*

1' *For every atomic formula ϕ of $\mathcal{L}$ with free-variables $y_1, \ldots, y_m$, if there are terms $t_1, \ldots, t_m$, $t'_1, \ldots, t'_m$ of $\mathcal{L}$ such that, for every $0 < i \leq m$, $(t_i = t'_i) \in \bigcup_{j \leq n+1} \Delta_j$, then $\phi y_1 \ldots y_m[t_1 \ldots t_m] \in \Delta_{n+1} \Leftrightarrow \phi y_1 \ldots y_m[t'_1 \ldots t'_m] \in \Delta_{n+1}$;*

2' *For every formula ϕ of $\mathcal{L}$ that is either atomic or the negation of an atomic formula, if $\phi \in \Delta_n$, then $\phi \in \Delta_{n+1}$;*

3' *Δ_{n+1} satisfies clauses 3–7 above;*

4' *Suppose $\exists x\phi \in \Delta_n$ and, for some ordinal $\lambda \leq$ max $\{\aleph_0, \mid \mathcal{L} \mid\}$, $\{T_i : i < \lambda\}$ is the set of all subformula-chains in $\bigcup_{j \leq n} \Delta_j$ such that $\exists x\phi$ is the last element in T_i and $QR(T_i) = n + 1$. Then, for every $i < \lambda$, there is a term $t_{\phi,i}$ of $\mathcal{L}$ such that $\phi x[t_{\phi,i}] \in \Delta_{n+1}$;*

5' *Suppose $\neg\exists x\phi \in \Delta_n$. For every formula ψ of $\mathcal{L}$ such that $\exists x\psi \in \Delta_n$ and, for some ordinal $\lambda \leq$ max $\{\aleph_0, \mid \mathcal{L} \mid\}$, $\{T_i : i < \lambda\}$ is the set of all subformula-chains in $\bigcup_{j \leq n} \Delta_j$ such that $\exists x\psi$ is the last element in T_i and $QR(T_i) = n+1$, if there is some T_i with the same sequence of witnesses as that of some subformula-chain T in $\bigcup_{j \leq n} \Delta_j$ that has $\neg\exists x\phi$ as its last element, then $\neg\phi x[t_{\psi,i}] \in \Delta_{n+1}$, for a term $t_{\psi,i}$ defined as in clause 4' above;*

6' *Suppose $\forall x\phi \in \Delta_n$. For every formula ψ of $\mathcal{L}$ such that $\exists x\psi \in \Delta_n$ and, for some ordinal $\lambda \leq$ max $\{\aleph_0, \mid \mathcal{L} \mid\}$, $\{T_i : i < \lambda\}$ is the set of all subformula-chains in $\bigcup_{j \leq n} \Delta_j$ such that $\exists x\psi$ is the last element in T_i and $QR(T_i) = n+1$, if there is some T_i with the same sequence of witnesses as that of some subformula-chain T in $\bigcup_{j \leq n} \Delta_j$ that has $\forall x\phi$ as its last element, then $\phi x[t_{\psi,i}] \in \Delta_{n+1}$, for a term $t_{\psi,i}$ defined as in clause 4' above;*

7' *Suppose $\neg\forall x\phi \in \Delta_n$ and, for some ordinal $\lambda \leq$ max $\{\aleph_0, \mid \mathcal{L} \mid\}$, $\{T_i : i < \lambda\}$ is the set of all subformula-chains in $\bigcup_{j \leq n} \Delta_j$ such that $\neg\forall x\phi$ is the last element in T_i and $QR(T_i) = n+1$. Then, for every $i < \lambda$, there is a term $t_{\phi,i}$ of $\mathcal{L}$ such that $\neg\phi x[t_{\phi,i}] \in \Delta_{n+1}$.*

We say that $\Delta = \bigcup_{n<\omega} \Delta_n$ is a perfect-Hintikka set *(hereafter, just p-H set) of formulas of $\mathcal{L}$.*

Lemma 3.26 *Let Δ be a p-H set of sentences of a first order language $\mathcal{L}$. Then, there is a p-urn structure $\widehat{\mathcal{M}_\mathfrak{C}}$ such that, for every $n < \omega$, for every $\phi \in \Delta_n$ and for some ordinal $\lambda <$ max $\{\aleph_0, \mid \mathcal{L} \mid\}$, $\widehat{\mathcal{M}_\mathfrak{C}} \models_{n,\lambda} \phi$.*

Proof. Fixed Δ, I will define a canonical p-urn structure for it. Let M be the set of closed terms of $\mathcal{L}$ and let $\Pi(\mathrm{M}) = \{[\![t]\!] : t \in \mathrm{M}\}$ be a partition such that, for any terms $t, s \in \mathrm{M}$, $s \in [\![t]\!]$ if and only if $t = s \in \Delta$. $\Pi(\mathrm{M})$ is the domain of our urn structure. Let $^{\mathcal{M}}$ be the following interpretation function:

- For any constant $c \in \mathcal{L}$, $c^{\mathcal{M}}$ is $[\![c]\!]$;

- For any n-ary function $f \in \mathcal{L}$, $f^{\mathcal{M}} : \Pi(\mathbf{M})^n \to \Pi(\mathbf{M})$ is such that $f^{\mathcal{M}}([\![t_1]\!] \dots [\![t_n]\!]) = [\![s]\!]$ if and only if $f(t_1 \dots t_n) = s \in \Delta$;

- For any n-ary relation $R \in \mathcal{L}$, $R^{\mathcal{M}} = \{\langle [\![t_1]\!], \dots, [\![t_n]\!] \rangle : \; R(t_1 \dots t_n) \in \Delta\}$.

Let $\mathcal{M} = \langle \Pi(\mathbf{M}), \,^{\mathcal{M}} \rangle$. An adequate p-eligibility set $\mathfrak{C}$ and a set of p-eligible sequences $\widehat{\mathfrak{M}}$ for $\mathcal{M}$ can be defined as follows:

- $\mathfrak{C}_0 = \widehat{\mathfrak{M}}_0 = \{[\![t]\!]$: for some formula ϕ of $\mathcal{L}, \exists x\phi \in \Delta_0$ and $\phi x[t] \in \Delta_1\}$;

Assume that, for some $n < \omega$, $\mathfrak{C}_0, \dots, \mathfrak{C}_n$ and $\widehat{\mathfrak{M}}_0, \dots, \widehat{\mathfrak{M}}_n$ have been defined. Let α be the λ-th sequence in $\widehat{\mathfrak{M}}_n$, for some ordinal $\lambda <\mid \widehat{\mathfrak{M}}_n \mid$. For every formula ϕ of $\mathcal{L}$ such that $\exists x\phi \in \Delta_n$, for some ordinal $\rho \leq$ max $\{\aleph_0, \mid \mathcal{L} \mid\}$, let $\{T(\phi)_i : i < \rho\}$ be the set of all subformula-chains in $\bigcup_{j \leq n} \Delta_j$ such that $\exists x\phi$ is the last element in $T(\phi)_i$ and $QR(T(\phi)_i) = n + 1$. By definition of p-H set, there is a set of terms $\{t_{\phi,i} : i < \rho\}$ such that $\phi x[t_{\phi,i}] \in \Delta_{n+1}$.

- If there is some formula $\exists x\phi \in \Delta_n$ such that some $T(\phi)_i$ has α as its sequence of witnesses, then $\mathfrak{C}_{n+1,\lambda} = \{[\![t_{\phi,i}]\!]$: α is the sequence of witnesses of $T(\phi)_i\}$. Otherwise, for an arbitrary $t \in$ M, let $\mathfrak{C}_{n+1,\lambda} = \{[\![t]\!]\}$. Based on this, $\mathfrak{C}_{n+1}$ and $\widehat{\mathfrak{M}}_{n+1}$ are straightforwardly defined.

Let $\mathfrak{C} = \bigcup_{n<\omega} \mathfrak{C}_n$ and $\widehat{\mathfrak{M}} = \bigcup_{m<\omega} \widehat{\mathfrak{M}}_m$. Finally, I will prove the following claim:

$(**)$ For every $n < \omega$, for any formula ϕ of $\mathcal{L}$ with free-variables $x_1, \dots, x_m$ and for any closed terms $t_1, \dots, t_m$ of $\mathcal{L}$, the following holds:

- If $\phi x_1 \dots x_m[t_1 \dots t_m] \in \Delta_n$, then $\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{n,\lambda} \phi x_1 \dots x_m[[\![t_1]\!] \dots [\![t_m]\!]]$, for every $\lambda <\mid \widehat{\mathfrak{M}}_{n-1} \mid$ such that, for some subformula-chain T in $\bigcup_{j \leq n} \Delta_j$ such that $\phi x_1 \dots x_m[t_1 \dots t_m]$ is its last element and $QR(T) \leq n+1$, the sequence of witnesses of T is the λ-th sequence in $\widehat{\mathfrak{M}}_{n-1}$;

- If $\neg\phi x_1 \dots x_m[t_1 \dots t_m] \in \Delta_n$, then $\widehat{\mathcal{M}}_{\mathfrak{C}} \not\models_{n,\lambda} \phi x_1 \dots x_m[[\![t_1]\!] \dots [\![t_m]\!]]$, for every $\lambda <\mid \widehat{\mathfrak{M}}_{n-1} \mid$ such that, for some subformula-chain T in $\bigcup_{j \leq n} \Delta_j$ with $\neg\phi x_1 \dots x_m[t_1 \dots t_m]$ as its last element and $QR(T) \leq n + 1$, the sequence of witnesses of T is the λ-th sequence in $\widehat{\mathfrak{M}}_{n-1}$.

The proof of this claim follows by induction on ϕ. If ϕ is atomic and $\phi \in \Delta_n$, it follows by definition of $\mathcal{M}$. If $\neg\phi \in \Delta_n$, by definition of p-H set, $\phi \notin \Delta$ and, again, the property follows by definition of $\mathcal{M}$.

Consider ϕ is $\psi \wedge \gamma$ and assume $\phi \in \Delta_n$. Fix some $\lambda <\mid \widehat{\mathfrak{M}}_{n-1} \mid$ such that, for some subformula-chain T in $\bigcup_{j \leq n} \Delta_j$ with ϕ as its last element and $QR(T) \leq n + 1$, the sequence of witnesses of T is the λ-th sequence $\alpha \in \widehat{\mathfrak{M}}_{n-1}$. By definition of p-H set, $\psi, \gamma \in \Delta_n$ and these formulas are the last elements in subformula-chains T' and T'' in $\bigcup_{j\leq n} \Delta_j$ whose sequence of witnesses is α. So, by inductive hypothesis, $\widehat{\mathcal{M}_{\mathfrak{C}}} \models_{n,\lambda} \psi \wedge \gamma$. On the other hand, assume $\neg\phi \in \Delta_n$. Fix some T and α such as above. By definition of p-H set, either $\neg\psi \in \Delta_n$ with some subformula-chain T' in $\bigcup_{j\leq n} \Delta_j$ such that $\neg\psi$ is its last element and α is its sequence of witnesses, or $\neg\gamma \in \Delta_n$ with some subformula-chain T'' in $\bigcup_{j\leq n} \Delta_j$ such that $\neg\gamma$ is its last element and α is its sequence of witnesses. Without loss of generality, consider the first case only. By inductive hypothesis, $\widehat{\mathcal{M}_{\mathfrak{C}}} \not\models_{n,\lambda} \psi$. Therefore, $\widehat{\mathcal{M}_{\mathfrak{C}}} \not\models_{n,\lambda} \phi$. By a dual reasoning, it is possible to see that the property holds for the case in which ϕ is $\psi \vee \gamma$. For convenience, I omit the proof of this case here.

Consider that ϕ is $\neg\psi$ and assume $\phi \in \Delta_n$. By inductive hypothesis, $\widehat{\mathcal{M}_{\mathfrak{C}}} \not\models_{n,\lambda} \psi$. So, $\widehat{\mathcal{M}_{\mathfrak{C}}} \models_{n,\lambda} \phi$. On the other hand, if $\neg\phi \in \Delta_n$, by definition of p-H set, $\psi \in \Delta_n$. By inductive hypothesis, $\widehat{\mathcal{M}_{\mathfrak{C}}} \models_{n,\lambda} \psi$. So, $\widehat{\mathcal{M}_{\mathfrak{C}}} \not\models_{n,\lambda} \phi$.

Assume ϕ is $\exists x\psi$ and $\phi \in \Delta_n$. As above, fix some relevant $T(\psi)_i$ and α. By definition of p-H set, $\psi x[t_{\psi,i}] \in \Delta_{n+1}$. By inductive hypothesis, $\widehat{\mathcal{M}_{\mathfrak{C}}} \models_{n+1,\rho} \psi x[t_{\psi,i}]$ in which ρ is the index of $\alpha \frown [\![t_{\psi,i}]\!] \in \widehat{\mathfrak{M}}_n$. By definition of $\mathfrak{C}$, $[\![t_{\psi,i}]\!]$ is in $\mathfrak{C}_{\lambda,n}$. So, $\widehat{\mathcal{M}_{\mathfrak{C}}} \models_{n,\lambda} \exists x\psi$.

Finally, assume $\neg\exists x\psi \in \Delta_n$ and fix some relevant T and α as before. By definition of p-H set, for every $\exists x\gamma \in \Delta_n$ such that some $T(\gamma)_i$ has α as its sequence of witnesses, $\neg\psi x[t_{\gamma,i}] \in \Delta_{n+1}$. By inductive hypothesis, $\widehat{\mathcal{M}_{\mathfrak{C}}} \not\models_{n+1,\rho} \psi x[t_{\gamma,i}]$ in which ρ is the index of $\alpha \frown [\![t_{\gamma,i}]\!] \in \widehat{\mathfrak{M}}_n$. By definition of $\mathfrak{C}$, every such $[\![t_{\gamma,i}]\!]$ is in $\mathfrak{C}_{\lambda,n}$, and, by clause 7 in definition of p-H set, there is such a term $[\![t_{\gamma,i}]\!]$. So, $\widehat{\mathcal{M}_{\mathfrak{C}}} \not\models_{n,\lambda} \exists x\psi$. By a dual reasoning, the property also holds in case ϕ is $\forall x\psi$. ■

The next lemma shows that from any pc-H-normal form it is possible to generate a p-H set. In what follows, for any terms $s_n, \ldots, s_0$, let $\Theta(m, s_n, \ldots, s_0) = \{\theta\ y_n \ldots y_0[s_n \ldots s_0] : \theta \in \Theta(m, y_n, \ldots, y_0)\}$, and let $\Phi_{\mathcal{L}}(s_n, \ldots, s_0) = \{\theta y_n \ldots y_0[s_n \ldots s_0] : \theta \in \Phi_{\mathcal{L}}(y_n, \ldots, y_0)\}$.

Lemma 3.27 *For a finite first order language $\mathcal{L}$, let $\theta \in \Theta(m, \emptyset)$ be a pc-H-normal form. Then, there is a p-H set Δ such that $\theta \in \Delta_0$.*

Proof. By definition, θ is of the form $(\bigwedge \exists y_m \Gamma) \wedge (\forall y_m \bigvee \Gamma)$, for some $\Gamma \subseteq \Theta(m-1, y_m)$. Let $\Delta_0 = \{\theta\} \cup \exists y_m \Gamma \cup \{\forall y_m \bigvee \Gamma,\ \exists y_m \bigvee \Gamma\} \cup \{t = t : t \text{ is closed term of } \mathcal{L}\}$. Δ_0 satisfies clauses 1 and 5 of p-H set by definition, and since Δ_0 has neither atomic formulas nor negations of atomic formulas besides trivial identities, it vacuously satisfies clauses 2–3 as well. For a similar reason, Δ_0 satisfies clauses 4 and 6. Further, given the syntactic structure of game-normal formulas, Δ_0 satisfies clause 7. So, Δ_0 is adequately defined.

Assume now that $\Delta_0, \ldots, \Delta_n$ have already been defined. By the syntactic structure of θ, for some $r < \omega$, for every formula ϕ of the form $\forall x\psi$ or $\exists x\psi$ that is in Δ_n, $\phi \in \bigcup_{l<r} (\exists y_{m-n} \Gamma_l \cup \{\forall y_{m-n} \bigvee \Gamma_l,\ \exists y_{m-n} \bigvee \Gamma_l\})$, for some $\Gamma_l \subseteq \Theta(m-n-1, s_m, \ldots, s_{m-n+1}, y_{m-n})$, for some terms $s_m, \ldots, s_{m-n+1}$ of $\mathcal{L}$. Furthermore, note that, for every $\phi, \psi \in (\exists y_{m-n} \Gamma_l \cup$

$\{\forall y_{m-n} \bigvee \Gamma_l,\ \exists y_{m-n} \bigvee \Gamma_l\}$), for any subformula-chains T and T' in $\bigcup_{j \leq n} \Delta_n$ that have ϕ and ψ as last elements, respectively, and such that $QR(T) = QR(T') = n+1$, T and T' share the sequence of witnesses $\langle s_m, \ldots, s_{m-n+1} \rangle$.

For every $l < r$ and for $k_l =\mid \Gamma_l \mid$, consider new terms $t_{l,1}, \ldots, t_{l,k_l}$ of $\mathcal{L}$ and let $\Sigma_l = \{\theta_i y_{m-n}[t_{l,i}] : 0 < i \leq k_l,\ \theta_i \in \Gamma_l\}$. Consider two cases:

- For every $0 < i \leq k_l$, if $0 < m-n$, then $\theta_i y_{m-n}[t_{l,i}]$ is of the form $(\bigwedge \exists y_{m-n-1} \Lambda_i) \wedge (\forall y_{m-n-1} \bigvee \Lambda_i)$, for some $\Lambda_i \subseteq \Theta(m-n-2, s_m, \ldots, s_{m-n+1}, t_{l,i})$. In this case, consider $\Sigma_{l,i} = \exists y_{m-n-1} \Lambda_i \cup \{\forall y_{m-n-1} \bigvee \Lambda_i,\ \exists y_{m-n-1} \bigvee \Lambda_i\}$;

- Otherwise, θ_i is of the form $\bigwedge \Lambda'_i \wedge \bigwedge \neg(\Phi_{\mathcal{L}}(s_m, \ldots, s_{m-n+1}, t_{l,i}) - \Lambda'_i)$, for some $\Lambda'_i \subseteq \Phi_{\mathcal{L}}(s_m, \ldots, s_{m-n+1}, t_{l,i})$. In this case, consider $\Sigma_{l,i} = \Lambda'_i \cup \neg(\Phi_{\mathcal{L}}(s_m, \ldots, s_{m-n+1}, t_{l,i}) - \Lambda'_i)$.

In either case, let $\Delta_{n+1} = \bigcup_{l<r} (\Sigma_l \cup \bigcup_{0<i\leq k_l} (\{\bigvee \Gamma_l\, y_{m-n}[t_{l,i}]\} \cup \Sigma_{l,i})) \cup \{t = t : t$ is closed term of $\mathcal{L}\}$. I will show that Δ_{n+1} satisfies clauses 1'–7' in definition of p-H set.

First of all, since Δ_{n+1} only inherits trivial identities, which are all the atomic formulas in Δ_n, Δ_{n+1} satisfies clause 2' of the definition.

Given the syntactic structure of game-normal formulas, Δ_{n+1} vacuously satisfies clause 4 of the definition. For every $l < r$ and for every $0 < i \leq k_l$, $\bigvee \Gamma_l\, y_{m-n}[t_{l,i}]$ are the only disjunctions in Δ_n and $\theta_i y_{m-n}[t_{l,i}]$ are in Δ_{n+1}, for some $\theta_i y_{m-n}[t_{l,i}] \in \Sigma_l$. So, Δ_{n+1} satisfies clause 6 as well. Further, given that θ is a pc-H-normal form, Δ_{n+1} satisfies clause 3 of definition. So, Δ_{n+1} satisfies clause 3'.

Again, based on the assumption that θ is a pc-H-normal form, we verify that Δ_{n+1} satisfies clause 1'. In the definition of Δ_{n+1} new constants were added for every existential formula of Δ_n, so Δ_{n+1} satisfies 4'. Further, based on the syntactic structure of game-normal formulas, Δ_{n+1} vacuously satisfies 5' and 7'. Finally, for every $l < r$ and for every $0 < i \leq k_l$, $\forall y_{m-n} \bigvee \Gamma_l$ are the only universal formulas in Δ_n and $\bigvee \Gamma_l\, y_{m-n}[t_{l,i}] \in \Delta_{n+1}$, for every such term $t_{l,i}$. By construction of $\Delta_0, \ldots, \Delta_{n+1}$, these are the only terms that need to be considered. So, Δ_{n+1} satisfies clause 6'. Hence, Δ_{n+1} is well-defined. ■

Based on this lemma, we can finally prove the satisfiability of pc-H-normal forms in p-urn semantics.

Theorem 3.28 *Let ϕ be a sentence of a finite first order language $\mathcal{L}$. ϕ is satisfiable in p-urn semantics if and only if, for some $\bigvee \Gamma \in Hnf^*(\phi)$ and for some $\theta \in \Gamma$, θ is pc-H-normal form.*

Proof. The proof of sufficiency follows immediately from Lemma 3.26, Theorem 3.27 and Corollary 3.23. For the proof of necessity, consider that for some $\bigvee \Gamma \in \text{Hnf}^*(\phi)$, for every $\theta \in \Gamma$, θ is not pc-H-normal form. In this case, for some game-normal formula θ' occurring as subformula of θ, either $(t \neq t)$ occurs in θ' or γ and $\neg\gamma$ occur in θ', for some atomic formula

γ whose variables are free in θ'. So, given the syntactic structure of game-normal formulas, for any p-urn game semantics $\mathfrak{F}(\psi, \widehat{\mathcal{M}_{\mathfrak{C}}})$, it is possible to define a winning strategy for player A in which A demands E either to hold $t \neq t$, or to hold γ, if $\widehat{\mathcal{M}_{\mathfrak{C}}} \not\models \gamma$, or $\neg\gamma$, if $\widehat{\mathcal{M}_{\mathfrak{C}}} \models \gamma$. ■

Note that Theorem 3.28 characterizes a decision method for satisfiability in p-urn semantics. So, given the undecidability of classical logic, there are classically unsatisfiable formulas ϕ such that, for some pc-H-normal form $\bigvee \Gamma \in$ Hnf $^*(\phi)$, there is a pc-H-normal form $\theta \in \Gamma$. Moreover, Theorem 3.28 guarantees that p-urn semantics has a property of *satisfiability in finite models*.

Corollary 3.29 *Let $\mathcal{L}$ be a finite first order language and ϕ be a formula of $\mathcal{L}$ that is satisfiable in p-urn semantics. So, ϕ is satisfied by a finite p-urn structure.*

Proof. Given that ϕ is satisfiable in p-urn semantics, by Theorem 3.28, for some $\bigvee \Gamma \in$ Hnf $^*(\phi)$ and for some $\theta \in \Gamma$, θ is pc-H-normal form. Since in the proof of Lemma 3.27 the construction of a p-H set from a given pc-H-normal form only needs to consider a finite number of terms, as a particular case of Lemma 3.26 we can construct a finite canonical model $\widehat{\mathcal{M}_{\mathfrak{C}}}$ of θ. Now, for the unnested formula γ such that Hnf $^*(\phi) =$ Hnf(γ), by Theorem 3.21, $\bigvee \Gamma \equiv_p \gamma$. Therefore, $\widehat{\mathcal{M}_{\mathfrak{C}}} \models \gamma$. By the contructive proofs of Theorem 3.12 and proposition 3.22, for a variation $\mathfrak{C}'$ of $\mathfrak{C}$, $\widehat{\mathcal{M}_{\mathfrak{C}'}} \models \phi$. Therefore, ϕ is satisfiable in finite p-urn structures. ■

3.3.4 Generalizing for i-urn semantics

In this subsection I show how it is possible to generalize the above results for i-urn semantics. In order to do this, instead of considering pc-game-normal formulas, I consider the stronger notion of *ic*-game-normal formula. In what follows I precisely define this set of game-normal formulas, but, first of all, I need to introduce some additional concepts about their syntactic structures. For any $\theta \in \Theta(m, x_1, \ldots, x_n)$, for $0 < k_r < \ldots < k_1 \leq m$ and for $\{y_{q_1}, \ldots, y_{q_s}\} = \{y_1, \ldots, y_m\} - \{y_{k_1}, \ldots, y_{k_r}\}$, let $\theta\ \overline{(y_{k_1}, \ldots, y_{k_r})}$ be as follow:

- If θ is $\bigwedge \Sigma \wedge \bigwedge \neg(\Phi_{\mathcal{L}}(x_1, \ldots, x_n, y_1, \ldots, y_m) - \Sigma)$, then, for $\Sigma' = \{\phi : \phi \in \Sigma \cap \Phi_{\mathcal{L}}(x_1, \ldots, x_n, y_{q_1}, \ldots, y_{q_s})\}$, $\theta\ \overline{(y_{k_1}, \ldots, y_{k_r})}$ is the formula $\bigwedge \Sigma' \wedge \bigwedge \neg(\Phi_{\mathcal{L}}(x_1, \ldots, x_n, y_{q_1}, \ldots, y_{q_s}) - \Sigma')$;
- Assume that, for some $0 < i < r$, $\theta\ \overline{(y_{k_1}, \ldots, y_{k_i})}$ has already been defined. The formula $\theta\ \overline{(y_{k_1}, \ldots, y_{k_{i+1}})}$ is defined by substituting all of its subformulas $\theta' \in \Theta(0, x_1, \ldots, x_n, y_{q_1}, \ldots, y_{q_s}, y_{k_{i+1}}, \ldots, y_{k_r})$ by $\theta'\ \overline{(y_{k_{i+1}})}$.

Informally speaking, $\theta\ \overline{(y_{k_1}, \ldots, y_{k_r})}$ preserves all the propositional content of θ except as regards its discourse about $y_{k_1}, \ldots, y_{k_r}$. Moreover, let me say that a game-normal formula $\theta \in \Theta(m, x_1, \ldots, x_n)$ *has enough copies of game-normal formulas* if the following holds:

- For any $\theta', \theta'' \in \Theta(m-1, x_1, \ldots, x_n, y_m)$ such that θ' and θ'' are different subformulas of θ, for every $\theta^* \in \Theta(m-2, x_1, \ldots, x_n, y_m, y_{m-1})$ subformula of θ', there is $\theta^{**} \in \Theta(m-2, x_1, \ldots, x_n, y_m, y_{m-1})$ subformula of θ'' such that $\theta^* \overline{(y_m)}$ is $\theta^{**} \overline{(y_m)}$ (I say that θ^{**} is a *copy* of θ^* in θ'');

- For any $2 < i < m$, consider any $\theta', \theta'' \in \Theta(i-1, x_1, \ldots, x_n, y_m, \ldots, y_i)$ such that θ' and θ'' are different subformulas of θ. Consider two cases:

  - For some $i \leq j < m$, θ' and θ'' are subformulas of some $\theta^+ \in \Theta(j, x_1, \ldots, x_n, y_m, \ldots, y_{j+1})$ (without loss of generality, consider θ^+ with minimum quantifier rank). For every $\theta^* \in \Theta(i-2, x_1, \ldots, x_n, y_m, y_{i-1})$ that is subformula of θ', there is $\theta^{**} \in \Theta(i-2, x_1, \ldots, x_n, y_m, y_{i-1})$ subformula of θ'' such that $\theta^* \overline{(y_{k_1}, \ldots, y_{k_r})}$ is $\theta^{**} \overline{(y_{k_1}, \ldots, y_{k_r})}$, for all $i \leq k_1 < \ldots < k_r \leq j$ such that, for every $0 < l \leq r$, the subformulas $\theta^{++}, \theta^{+++} \in \Theta(k_l - 1, x_1, \ldots, x_n, y_m, \ldots, y_{k_l})$ of θ^+ which are superformulas of θ^* and θ^{**}, respectively, are not copies between themselves.
  - If there is no such j, for every $\theta^* \in \Theta(i-2, x_1, \ldots, x_n, y_m, y_{i-1})$ subformula of θ', there is $\theta^{**} \in \Theta(i-2, x_1, \ldots, x_n, y_m, y_{i-1})$ subformula of θ'' such that $\theta^* \overline{(y_{k_1}, \ldots, y_{k_r})}$ is $\theta^{**} \overline{(y_{k_1}, \ldots, y_{k_r})}$, for all $i \leq k_1 < \ldots < k_r \leq m$ such that, for every $0 < l \leq r$, the subformulas $\theta^{++}, \theta^{+++} \in \Theta(k_l - 1, x_1, \ldots, x_n, y_m, \ldots, y_{k_l})$ of θ^+ which are superformulas of θ^* and θ^{**}, respectively, are not copies between themselves.

Definition 3.30 *$\theta \in \Theta(m, x_1, \ldots, x_n)$ is an* imperfectly-consistent-game-normal formula *(hereafter, I say simply that θ is an ic-game-normal formula) if and only if the following holds:*

- *θ is pc-H-normal form;*
- *θ has enough copies of its game-normal formulas;*

My strategy here is the same as before, i.e., I prove that every ic-game-normal formula can be extended to something similar to Hintikka sets: more precisely, I show that given an ic-game-normal formula it is possible to generate from it an *i-H set*.

Definition 3.31 *Let $\mathcal{L}$ be a first order language. Consider a list of sets ∇_n, for every $n < \omega$, such that the following holds:*

1. *∇_0 satisfies clauses 1–7 of Definition 3.25.*

Assume $\nabla_0, \ldots, \nabla_n$ have already been defined. Then, ∇_{n+1} is the smallest *set satisfying the following clauses:*

1' ∇_{n+1} satisfies clauses 1'–4' and 7' of Definition 3.25;

2' *Suppose $\forall x\phi \in \nabla_n$. For every formula ψ of $\mathcal{L}$ such that $\exists x\psi \in \nabla_n$ and, for some ordinal $\lambda \leq$ max $\{\aleph_0, \mid \mathcal{L} \mid\}$, $\{T_i : i < \lambda\}$ is the set of all subformula-chains in $\bigcup_{j\leq n} \nabla_j$ such that $\exists x\psi$ is the last element in T_i and $QR(T_i) = n+1$, $\phi x[t_{\psi,i}] \in \nabla_{n+1}$, for a term $t_{\psi,i}$ defined as in clause 4' of Definition 3.25;*

3' *Suppose $\neg\exists x\phi \in \nabla_n$. For every formula ψ of $\mathcal{L}$ such that $\exists x\psi \in \nabla_n$ and, for some ordinal $\lambda \leq$ max $\{\aleph_0, \mid \mathcal{L} \mid\}$, $\{T_i : i < \lambda\}$ is the set of all subformula-chains in $\bigcup_{j\leq n} \nabla_j$ such that $\exists x\psi$ is the last element in T_i and $QR(T_i) = n+1$, $\neg\phi x[t_{\psi,i}] \in \nabla_{n+1}$, for a term $t_{\psi,i}$ defined as in clause 4' of Definition 3.25.*

$\nabla = \bigcup_{n<\omega} \nabla_n$ *is an* imperfect-Hintikka set *(hereafter, just i-H set) of formulas of $\mathcal{L}$.*

Lemma 3.32 *Let ∇ be an i-H set of sentences of a first order language $\mathcal{L}$. Then, there is an i-urn structure $\mathcal{M}_{\mathfrak{B}}$ such that, for every $n < \omega$, for every $\phi \in \nabla_n$, $\mathcal{M}_{\mathfrak{B}} \models_n \phi$.*

Proof. The result is obtained by construction of a canonical i-urn structure for ∇ just like in the proof of Theorem 3.28, with the difference that in this case I need to consider less constraints for characterizing i-eligibility than it was necessary there. ■

Lemma 3.33 *For a finite first order language $\mathcal{L}$, let $\theta^* \in \Theta(m, \emptyset)$ be an ic-game-normal formula. Then, there is an i-H set ∇ such that $\theta^* \in \nabla_0$.*

Proof. I will define ∇ by recursion on $n < \omega$. ∇_0 is defined just like Δ_0 in the proof of Lemma 3.27.

Assume now that $\nabla_0, \ldots, \nabla_n$ have already been defined. Consider, for any $m-n < i \leq m$, finite sets of terms $N_i = \{s_{i,1}, \ldots, s_{i,n_i}\}$, and consider the finite set of sequences of witnesses $N = \{\langle s_{m,\, l_m}, \ldots, s_{m-n+1,\, l_{m-n+1}}\rangle :$ for every $m-n < i \leq m,\ 0 < l_i \leq n_i\}$, $n^* =\mid N \mid$. For any $0 < l \leq n^*$, let α_l be the l-th sequence in N.

Assume that, by construction, for every $0 < l \leq n^*$, for some $\Gamma_l \subseteq \Theta(m-n-1, x_1, \ldots, x_n, \alpha_l, y_{m-n})$, the set $\bigcup_{0<l\leq n^*} (\exists y_{m-n}\Gamma_l \cup \{\forall y_{m-n} \bigvee \Gamma_l,\ \exists y_{m-n} \bigvee \Gamma_l\})$ contains all formulas of the form $\exists x\psi$ or $\forall x\psi$ which are in ∇_n. Note that, for all formulas $\phi \in \exists y_{m-n}\Gamma_l \cup \{\forall y_{m-n} \bigvee \Gamma_l,$ $\exists y_{m-n} \bigvee \Gamma_l\}$, for any subformula-chain T in $\bigcup_{j\leq n} \nabla_n$ such that $QR(T) = n+1$ and which have ϕ as its last element, α_l is the sequence of witnesses of T.

For every $0 < l \leq n^*$ and for $\mid \Gamma_l \mid= k_l$, consider new terms $t_{l,1}, \ldots, t_{l,k_l}$ of $\mathcal{L}$ and let $\Sigma_l = \{\theta_i\ y_{m-n}[t_{l,i}] : 0 < i \leq k_l, \theta_i \in \Gamma_l\}$. Furthermore, consider $\Sigma'_l = \{\theta\ y_{m-n}[t_{l,i}] : 0 < i \leq k_l,\ \theta \in \Gamma_j,$ for some $0 < j \leq n^*$ different from l such that $\theta\ \alpha_j[y_m, \ldots, y_{m-n+1}]$ is a copy of $\theta_i\ \alpha_l[y_m, \ldots, y_{m-n+1}]\}$. Given that θ^* is an ic-game-normal formula, for every $0 < j \neq l \leq n^*$ and for every $0 < i \leq k_l$ there is a corresponding θ: in fact, I will force Σ'_l to have only one such θ to each j, l and i.

Fix an arbitrary $0 < i \leq k_l$. Let $\theta_1, \ldots \theta_q$ be all formulas in $\Sigma_l \cup \Sigma'_l$ of the form $\theta\ y_{m-n}[t_{l,i}]$. Now, consider two cases:

- If $0 < m - n$, then, for every $0 < h \leq q$, θ_h is of the form $(\bigwedge \exists y_{m-n-1}\Lambda_h) \wedge (\forall y_{m-n-1} \bigvee \Lambda_h)$, for some $\Lambda_h \subseteq \Theta(m-n-2, \alpha_{h'}, t_{l,i}, y_{m-n-1})$ and h' such that $\theta_h \in \Gamma_{h'}$. In this case, consider that $\Sigma_{l,i}$ is $\bigcup_{0<h\leq q} (\exists y_{m-n-1}\Lambda_h \cup \{\forall y_{m-n-1} \bigvee \Lambda_h,\ \exists y_{m-n-1} \bigvee \Lambda_h\})$;

- Otherwise, θ_h is of the form $\bigwedge \Lambda'_h \wedge \bigwedge \neg(\Phi_{\mathcal{L}}(\alpha_{h'}, t_{l,i}) - \Lambda'_h)$, for some $\Lambda'_h \subseteq \Phi_{\mathcal{L}}(\alpha_{h'}, t_{l,i})$ and h' such that $\theta_h\ t_{l,i}[y_{m-n}] \in \Gamma_{h'}$. In this case, consider that $\Sigma_{l,i} = \bigcup_{0<h\leq q} (\Lambda'_h \cup \neg(\Phi_{\mathcal{L}}(\alpha_{h'}, t_{l,i}) - \Lambda'_h))$.

Let $\Sigma'_{l,i} = \{\bigvee \Gamma_{h'}\ y_{m-n}[t_{l,i}] : \theta_h\ t_{l,i}[y_{m-n}] \in \Gamma_{h'},\ 0 < h \leq q\}$. Finally, let $\nabla_{n+1} = \bigcup_{0<l\leq n^*} (\Sigma_l \cup \Sigma'_l \cup \bigcup_{0<i\leq k_l} (\Sigma_{l,i} \cup \Sigma'_{l,i})) \cup \{t = t : t \text{ is closed term of } \mathcal{L}\}$.

Now, I will show that ∇_{n+1} is well-defined. Given the similarities between ∇_{n+1} and Δ_{n+1}, clauses 1, 4–7, 4', 5' and 7' of Definition 3.25 are satisfied as well by ∇_{n+1}. It is possible to immediately verify also that, for every $0 < j \neq l \leq n^*$, for any $0 < i \leq k_l$ and for $\forall y_{m-n} \bigvee \Gamma_j \in \nabla_n$, $\theta\ y_{m-n}[t_{l,i}] \in \nabla_{n+1}$. So, ∇_{n+1} satisfies clause 2' of Definition 3.31. Finally, since θ^* is a pc-H-normal form, if there is some atomic formula ϕ such that ϕ and $\neg\phi$ are in ∇_{n+1}, then these formulas say conflicting things about some terms $s_1, \ldots, s_n$ not occurring in θ^*. However, this contradicts the fact that θ^* is an ic-game-normal formula jointly with the fact that $s_1, \ldots, s_n$ were introduced by means of the copies of game-normal formulas occurring in θ^*. So, ∇_{n+1} satisfies clause 1' of Definition 3.31 and, therefore, is well-defined. ■

Based on these lemmas, a generalized version of the previously obtained characterization theorem (Theorem 3.28) holds for i-urn semantics.

Theorem 3.34 *Let ϕ be a sentence of a finite first order language $\mathcal{L}$. ϕ is satisfiable in i-urn semantics if and only if, for some $\bigvee \Gamma \in Hnf^*(\phi)$ and for some $\theta \in \Gamma$, θ is an ic-game-normal formula.*

Just as in the case of p-urn semantics, Theorem 3.34 describes a decision procedure for satisfiability in i-urn semantics. Furthermore, the following corollary shows that, like p-urn semantics, i-urn semantics also satisfies a property of finite satisfiability.

Corollary 3.35 *Let $\mathcal{L}$ be a finite first order language and ϕ be a formula of $\mathcal{L}$ that is satisfiable in i-urn semantics. Then, ϕ is satisfied by some finite i-urn structure.*

3.4 Conclusion

In this chapter, I presented some fundamental results on urn semantics. First, I showed that, in addition to Rantala's original game-theoretic approach to the subject, it is possible to define Tarskian semantic frameworks for urn semantics. Further, I showed that there are two independent systems of urn semantics, what I called p- and i-urn semantics. Then, a precise description of these systems enabled the obtainment of some fundamental metatheorems on urn semantics. First, we have seen that p- and i-urn semantics determine decidable systems of logic. Secondly, we have proved interesting characterization theorems for them. In order to prove these results, I relied on the validity of a weaker version of the Fraïssé-Hintikka theorem according to which every formula is equisatisfiable with a bunch of Hintikka-normal forms in urn semantics. The proof of this auxiliary result was also presented in this chapter. In addition to the intrinsic interest of these results for the development of the model theory of urn semantics, they are also crucially important for a characterization of TSI. In the next chapter, I turn the focus once again to TSI: in this sense, I formalize this theory based on the framework of urn semantics and show that, in this metatheoretic context, SoD does not hold.

Chapter 4

A formalization of TSI in urn semantics

4.1 Introduction

In this chapter, based on the framework of urn semantics introduced in chapter 3, I finally reestablish TSI and show that, through the lenses of urn semantics, TSI does not imply SoD. This work is divided into two parts: in the first part, I present an outline of the main aspects of a formal characterization of TSI and show that, in urn semantics, SoD does not hold. In the second part, I compare this proposal of reestablishment of TSI with similar solutions in the literature. In particular, I compare the present result with proposals by Hintikka (1970a, 1970b) and Jago (2007, 2009, 2013). In this way, I argue that my proposal is immune to some problems faced by these alternative solutions. Moreover, I claim that this reestablishment of TSI nicely deals with some objections raised by Jago against theories of semantic information based on urn semantics.

4.2 Formalizing TSI in urn semantics

4.2.1 General outline of the theory

What is semantic information? Any examination of this question depends on answering the following more fundamental one: what is information (of any kind)? As I already mentioned in the introduction of this thesis, there is no straightforward answer to this question: information is a polysemic expression that means different things in different contexts of use and it is still not so clear whether such multiplicity of meanings is reducible to a common ground. An important attempt of providing such common core of multiple notions of information is the idea that information, of whatever kind, is *meaningful* and *well-formed data* (this thesis is called the *general definition of information* by Floridi (2016, sec. 1.2), hereafter *GDI*).

Let me make this more clear by way of an example: consider again Loar's case in which someone opens a diary and finds the entry "Hot and sunny today; phoned Maria to invite her to the beach." By reading this sentence, the person in question receives some information.

According to GDI this sentence is informative because the set of linguistic expressions composing it is data, the underlying syntax of the sentence characterizes the well-formedness of the data and the sentence is meaningful. Note that the propositional content itself of a sentence is also information because the meanings of all the symbols composing the sentence are data, the semantic relations connecting these meanings is the well-formedness of these data and the propositional content is the proper meaning of the sentence. Furthermore, in similar terms, the partial representation of the meaning of a sentence associated with our semantic competence is also information (for simplicity, hereafter let us call this variety of the information of a sentence as *epistemically nuanced semantic information*). There is no incoherence in this fact: a structured object such as a sentence may simultaneously carry different varieties of information.[1]

This remark shows that there is no rivalry between Bar-Hillel and Carnap's theory of semantic information and my revision of TSI. These two versions of TSI aim to characterize different levels of semantic information. Bar-Hillel and Carnap's TSI arguably captures the informativeness associated with the propositional content of sentences. On the other hand, my revised version of TSI aims to capture the epistemically nuanced semantic information of sentences. Similarly it is possible to verify that SoD is not an immediate problem for TSI. SoD is a problem for TSI if this theory aims to characterize the epistemically nuanced semantic information of sentences, but arguably it is not a problem for supporters of TSI who want to give an account of the information associated with the propositional content of sentences.

Although GDI provides a very reasonable description of basic features shared by any notion of information, against this theory it is sometimes argued that some of its primitive notions are unclear, in particular the notion of data. What is data? In very general terms, datum is just the apparent fact of a lack of uniformity within some context. Floridi (Ibid., sec. 1.3) calls this very general definition as the *diaphoric definition of data* (hereafter DDD). For a more vivid example, consider the ringing of a telephone bell: according to DDD, the ringing is a datum simply because it cuts the silence.

DDD is a very general definition of datum. In fact, DDD is designed to satisfy at least four *neutrality* conditions, namely, DDD is supposed to be neutral with respect to taxonomy, typology, ontology and geneticity. DDD is taxonomically neutral since it characterizes datum relationally, that is, in terms of a relation of difference within a context, without specifying what is the relevant kind of difference in each case. DDD is typologically neutral since it does not suggest any substantial catalog of different kinds of data. DDD is ontologically neutral since it demands data representation (Ibid., sec. 1.6) without specifying what kind of representation is required in each case.[2] Finally, DDD is genetically neutral since it characterizes data as entities whose nature is independent of their access by any informee. In this sense, whether the ringing of a telephone bell is datum is independent of whether someone can hear it.

[1] Of course, I do not intend this list to be an exhaustive selection of the different varieties of information carried by a sentence.

[2] The literature on DDD generally identify representation with physical implementation (Ibid., sec. 1.6). I disregard this issue here since my claim is independent of such discussions.

However, perhaps the reader might be unsatisfied with this formalist account and might ask for a more substantial characterization of information based on GDI+DDD. Now, as I see this issue, the fact that this is a formalist account is not so much an objection to its validity but simply the recognition that it does not answer all questions about the concept of information (in particular, it does not solve any substantial questions on the nature of information). In fact, I believe that all these open questions are more convincingly addressed through a concrete, case by case investigation. In this sense, instead of continuing with this generalist agenda on the nature of information, it seems more fruitful to focus specifically on the notion of semantic information in order to further articulate GDI+DDD in more substantial terms.

So, let me go back to the question of what is semantic information. A fundamental insight on the nature of information is the idea that it varies in function of the degree of *precision* of a given message (Cf. ADRIAANS, 2013, sec. 3). For instance, the sentence "water quenches thirst and is good for your health" is more informative than "water quenches thirst." This fundamental idea is formalized in terms of *probability* in the pioneer work by Shannon and Weaver (1949): the greater the informativeness of some data, the less probable it is.

Bar-Hillel and Carnap's theory of semantic information preserves this fundamental insight about the concept of information as well as its formalization in terms of probability as Shannon and Weaver propose. This is a nice move since, as Popper has classically shown (Cf. CURD; COVER, 1998, pp. 3–10), the semantic information of a sentence varies in function of its potential *falsifiability*, a quality that is inversely linked to its degree of probability.[3]

The standard axiomatization of probability is due to Kolmogorov (1950). This system is characterized by Hájek (2012, sec. 1) as follows:

Definition 4.1 *Let* Ω *be a set and* Δ *a set of parts of* Ω *that has* Ω *as an element. Further, consider that* Δ *is closed for complementation and union. Let* $P : \Delta \to [0, 1]$ *be a function such that:*

- $P(\Gamma) \geq 0$*, for all* $\Gamma \in \Delta$*;*
- $P(\Omega) = 1$*;*
- $P(\Gamma \cup \Sigma) = P(\Gamma) + P(\Sigma)$*, for any mutually exclusive sets* $\Gamma, \Sigma \in \Delta$.

P *is a* probability function *and* (Ω, Δ, P) *a* probability space.

What is the idea behind Kolmogorov's formalism? Although, similarly to the case of "information," there is no univocal interpretation of the notion of probability, Bar-Hillel and Carnap (1952, 1953) proposed a semantic account of this notion particularly relevant for the development of TSI. According to Bar-Hillel and Carnap's interpretation, for a finite language $\mathcal{L}$, Ω

[3] This is independent of whether Popper's falsificationism provides a good demarcation criterion between science and pseudo-science. Despite this question Popper is right in claiming that the greater the informativeness of a theory, the greater is its potential to exclude cases.

in the definition above is the class of all the *different* classical $\mathcal{L}$-structures. In this sense, for a formula ϕ of $\mathcal{L}$, the probability of ϕ being the case is $P(\Gamma)$, in which Γ is the subset of Ω composed by all the different $\mathcal{L}$-models of ϕ. In other words, the probability of a sentence is determined by measuring how many different models it has.[4]

Moreover, similarly to Shannon and Weaver's theory of information, in TSI the information of a sentence ϕ, $I(\phi)$, is defined as the inverse of the probability P of ϕ (in Carnap's sense) by the equation $I(\phi) = 1 - P(\phi)$. Hence, TSI formalizes the idea that the semantic information carried by a sentence is the collection of the different models in which the given sentence is false.[5]

Note that, if we want to measure, by means of TSI, the amount of information carried by a sentence, then we need to count how many different models it has. So, the precise construction of TSI depends on a preliminary account of the criteria of *qualitative* difference (and identity) between models that is particularly relevant here. Two objects a and b are qualitatively the same if, for every relevant property Q, a is Q if and only if b is Q. In this sense, it is necessary to investigate which are the properties Q characterizing the qualitative identity (difference) between models that is relevant for TSI. This is a technical issue, therefore I postpone it to the next subsection.

Besides this particularly important issue on identity criteria for models, the reader could also ask whether the semantic notion of probability is in any sense useful for characterizing the epistemically nuanced semantic information. If we want to describe the information associated with the propositional content of a sentence, it surely makes a lot of sense to consider the different scenarios in which the given sentence is true, its degree of falsifiability etc. Does the same happen with the epistemically nuanced semantic information?

In my opinion, the consideration of probability in the formalization of the epistemically nuanced semantic information is in fact meaningful. The partial representation of truth-conditions associated with our semantic competence presents different scenarios that satisfy a given sentence. The only difference here is that this partial representation is imperfect and, consequently, defeasible. The difference mentioned in the introduction of this thesis between ontological and epistemological modalities is of help here. The partial representation of truth-conditions associated with our semantic competence describes *possible* scenarios that satisfy a given sentence. However, some of these scenarios are epistemically but not ontologically possible.

TSI's probabilistic conception (this terminology is suggested by Floridi (2016, sec. 4)) of (epistemically nuanced) semantic information satisfies and further explores GDI+DDD. Note in particular that it satisfies the genetic neutrality condition, since, even in the case of the epis-

[4]This is not exactly Bar-Hillel and Carnap's own way of presenting their ideas on probability: as I present next, they follow a more syntactic approach on the subject. Despite such differences, my presentation is equivalent to their approach. Bar-Hillel and Carnap (1952, p. 14) call this notion of probability as *absolute logical probability*.

[5]In fact Bar-Hillel and Carnap (1952) describe two different formal concepts of semantic information. The authors argue that there is no *unique* concept that captures all aspects of the notion of semantic information. In this thesis I consider only the definition of semantic information of a sentence in terms of the complementary of its probability.

temically nuanced semantic information, the class of different models of a sentence can be objectively presented, as I have done so far in the framework of urn semantics.

Now, even though TSI is based on strong insights, in principle this theory is not the only possible way of describing the (epistemically nuanced) semantic information. Alternatives to TSI might be motivated by at least two general kinds of criticism to that theory. First, one could object the semantic presuppositions behind TSI. Secondly, it is possible to object to the conception of information itself underlying this theory.

An important example of the first way of criticism is the *inferential approach* to semantic information that describes the semantic information of a sentence in terms of the space of inferences in which it might play a role (Ibid., sec. 4). This approach to the subject is motivated by an *inferentialist* conception of semantics that radically diverges from the model-theoretic approach that is favoured here (the evaluative comparison of these approaches is a theme that completely escapes the scope of this work).

The latter kind of criticism is championed by Floridi (2004, 2005) who argues that GDI generates an excessively weak conception of semantic information that recognizes misinformation as a kind of information. Floridi famously claims that semantic information satisfies the veridicality principle according to which only true content is in fact information. As I anticipated in the introduction of this thesis, there are strong reasons for recognizing misinformation as a kind of information (Cf. FETZER, 2004) and, consequently, for blocking the principle of truthfulness.

4.2.2 Technical aspects

In the previous subsection I showed that TSI's measurement of information depends on counting how many *different* models a given sentence has. So, in order to precisely define this theory, first I provide a relevant criteria of qualitative identity for structures.

Working particularly in the context of classical propositional logic, in order to measure the probability of a sentence ϕ, instead of looking directly to its class of models, Bar-Hillel and Carnap (1952 , sec. 4) consider the *disjunctive normal form* of ϕ. For a finite propositional language $\mathcal{L}$ (that is, for a finite language whose non-logical signature is a set of 0-ary predicates), a formula ψ of $\mathcal{L}$ is said to be in disjunctive normal form if and only if ψ is $\bigvee \Gamma$, for some set Γ of game-normal formulas with quantifier rank 0 (see Definition 3.14 above). It is well-known that in classic propositional logic, every formula ϕ is equivalent to a disjunctive normal form $\bigvee \Gamma$ (Cf. SHOENFIELD, 1967, p. 40). Based on this, Carnap defines the probability of ϕ as the sum of the probabilities of all game-normal formulas in Γ.

Bar-Hillel and Carnap follow a syntactic approach. However, from a semantic point of view note that, in classic propositional logic, each $\gamma \in \Gamma$ defines a class of isomorphic models of ϕ. Consequently, Γ defines a partition by isomorphism of the class of models of ϕ. So, Bar-Hillel and Carnap's TSI measures probability (and information) of ϕ by counting how many non-isomorphic models it has.

Is isomorphism a relevant criterion of identity for the construction of TSI when we go first order? First, there is a "pragmatic" reason why isomorphism is *not* a good criterion of identity in this context of investigation. In first order logic, due to upward Löwenheim-Skolem theorem, there are formulas whose classes of non-isomorphic models are proper classes. So, the measurement of the probability of such a formula based on the counting of its non-isomorphic models attributes maximum information to it, a quite undesirable result.

Secondly, there are at least two more conceptual reasons for rejecting isomorphism as a good criterion of identity between structures. Continuing in a model-theoretic perspective, Hodges (1997, pp. 73–74) rejects isomorphism as a relevant criterion of identity based on its non-absoluteness. In Hodges' words:

> [. . .] the existence of an isomorphism can depend on some subtle questions about the surrounding universe of sets. We can sharpen this second point with the help of some set theory. If M is a transitive model of set theory containing vector spaces A and B of dimensions ω and ω_1 over the same countable field, then A and B are not isomorphic in M, but they are isomorphic in an extension of M got by 'collapsing' the cardinal ω_1 down to ω (Ibid., pp. 73–74).

In this passage, Hodges calls attention to the well-known fact that the theory of vector spaces is κ-categorical (that is, all models of this theory with cardinality κ are isomorphic between themselves) if and only if κ is an uncountable cardinal (Ibid., p. 42). In order to see why the non-absoluteness of categoricity is problematic in the present context of investigation, note that, motivated by the "pragmatic" reason mentioned above, someone could propose to define the probability of a sentence in terms of the number of its non-isomorphic models with cardinality at most κ, for some infinite cardinal κ. The problem with this approach is that the resulting measure of probability (and, consequently, of information) can vary depending on whether κ is countable or not.[6]

Moreover, the search for some relevant criteria of qualitative identity between structures is a central issue of *structuralism* in philosophy of Mathematics. Resnik (1981, p. 535) in particular advanced an interesting argument against the idea that isomorphism is such a criterion. According to Resnik, isomorphism wrongly distinguishes structures that are treated *as the same* in standard mathematical practice. For instance, although the structure of the natural numbers with successor function s, in symbols ($\mathbb{N}$, s), is not isomorphic with the structure of the naturals with the "less than" relation, in symbols ($\mathbb{N}$, $<$), mathematicians widely agree that these structures are identical.

[6]In classical logic, given Morley's well-known *categoricity theorem* (MARKER, 2006, pp. 207–ff.) these are the only cases that I need to consider. Even though it is true that, in classical logic, there are first order theories that have 2^{κ} non-isomorphic models with cardinality κ, for every uncountable κ (Ibid., pp. 189–ff.), this is not relevant in the present context of investigation given that TSI provides a measure of semantic information that is blind for differences between infinite cardinals. Whether Morley's theorem holds in urn semantics is an open question and a very advanced issue that fully escapes the scope of this work.

Hodges introduced his above mentioned remark on the non-absoluteness of isomorphism in a context of exposition of basic results on back and forth equivalence. In fact, unlike isomorphism, back and forth consists in an absolute equivalence relation between structures. Furthermore, in the context of finite first order languages, the class of back and forth non-equivalent models of any formula is a finite class. So, back and forth equivalence is immune to the "pragmatic" argument advanced above. Note also that the Hintikka normal forms (see Definition 3.15 above), the syntactic counterpart of back and forth equivalence, are the first order generalization of disjunctive normal forms. Therefore, the focus on back and forth equivalence is a natural path for the first order formalization of TSI.[7]

Now I start to present, more technically, the formal apparatus of TSI. In what follows, unless I explicitly say otherwise, my presentation of TSI is neutral about whether the underlying logic is either classical logic or urn semantics. Let $\mathcal{L}$ be a finite first order language and consider ϕ a formula of $\mathcal{L}$. For any $k < \omega$, let $\Gamma(k, \phi)$ be the set of formulas of $\mathcal{L}$ such that $\bigvee \Gamma(k, \phi) \in Hnf^*(\phi)$ and $qr(\bigvee \Gamma(k, \phi)) = k$.[8] Remember that $\Theta(k, x_1, \ldots, x_n)$ is the set of game-normal formulas of $\mathcal{L}$ with quantifier rank k and free-variables $x_1, \ldots, x_n$. Let $\Theta^c(k, x_1, \ldots, x_n)$ be its subset of consistent formulas.[9]

Consider a list of functions $(P_k : k < \omega)$ such that $P_k : \mathcal{P}(\Theta^c(k, x_1, \ldots, x_n)) \to [0, 1]$, for every $k < \omega$, and the triple $(\Theta^c(k, x_1, \ldots, x_n), \mathcal{P}(\Theta^c(k, x_1, \ldots, x_n)), P_k)$ is a *probability space* as in Definition 4.1 above. P_k satisfies two additional clauses:

A. First, $P_k(\Gamma) = 0$ if and only if Γ is empty set;

B. Secondly, for any $\theta \in \Theta^c(k, x_1, \ldots, x_n)$, for every $q > k$, $P_k(\{\theta\}) = P_q(\Gamma(q, \theta))$.

Let $\mathcal{F}or_{\mathcal{L}}(k)$ be the set of formulas of $\mathcal{L}$ with variables within $x_1 \ldots, x_n$ such that there is some $\psi \in Hnf^*(\phi)$ with quantifier rank k. Each P_k can be extended to a function of k-*probability* $P'_k : \mathcal{F}or_{\mathcal{L}}(k) \to [0, 1]$ such that, for any $\phi \in \mathcal{F}or_{\mathcal{L}}(k)$, $P'_k(\phi) = P_k(\Gamma(k, \phi) \cap \Theta^c(k, x_1, \ldots, x_n))$.

Based on this notion of probability, it is possible to define a measure of semantic information.

Definition 4.2 *Consider a list of functions* $(I_k : k < \omega)$ *such that, for every* $k < \omega$, $I_k : \mathcal{F}or_{\mathcal{L}}(k) \to [0, 1]$ *and, for every* $\phi \in \mathcal{F}or_{\mathcal{L}}(k)$, $I_k(\phi) = 1 - P'_k(\phi)$. I_k *is a* k-information function.

[7]Resnik (1981, pp. 535–536) proposes a different criterion of equivalence between structures based on bi-interpretability conditions. However, the model theoretic aspects of such criterion are not so clear and it does not present itself clearly as a way of generalizing Bar-Hillel and Carnap's TSI. Anyway, the relationship between back and forth equivalence and Resnik's criterion of structural equivalence is a promising topic of future research.

[8]Note that $\Gamma(k, \phi)$ denotes possibly different sets of formulas in classical logic and urn semantics, as I showed in the previous chapter.

[9]As in the previous footnote, in general $\Theta^c(k, x_1, \ldots, x_n)$ denotes different subsets in classical logic, p- and i-urn semantics.

Given that P_k is a measure, then it is easy to verify that I_k is a measure as well. TSI is the theory according to which the amount of information carried by a formula $\phi \in \mathcal{F}or_{\mathcal{L}}(k)$ is $I_k(\phi)$, for any $k < \omega$. The notion of k-information satisfies two basic theorems. First, every formula that is unsatisfiable *in the underlying logic* carries maximum information. In dual terms, every formula valid in the underlying logic carries null information. Secondly, the measure of semantic information remains *stable* through ranks.

Theorem 4.3 *Let $\phi \in \mathcal{F}or_{\mathcal{L}}(k)$. Then $I_k(\phi) = 1$ if and only if ϕ is unsatisfiable in the underlying logic. Further, $I_k(\phi) = 0$ if and only if ϕ is valid in the underlying logic.*

Proof. If ϕ is unsatisfiable, then $\Gamma(k, \phi) \cap \Theta^c(k, x_1, \ldots, x_n) = \emptyset$. So, by definition P'_k and requirement A, $P'_k(\phi) = P_k(\Gamma(k, \phi) \cap \Theta^c(k, x_1, \ldots, x_n)) = 0$. Hence, $I_k(\phi) = 1 - 0 = 1$. For the second part of the theorem, observe that if ϕ is valid in the underlying logic, then $\Gamma(k, \phi) \cap \Theta^c(k, x_1, \ldots, x_n) = \Theta^c(k, x_1, \ldots, x_n)$. By definition of probability space, $P_k(\Theta^c(k, x_1, \ldots, x_n)) = 1$. So, $I_k(\phi) = 1 - 1 = 0$ ■

Theorem 4.4 *Let ϕ be as in Theorem 4.3. For any $q > r \geq k$, $P'_q(\phi) = P'_r(\phi)$.*

Proof. Let $\theta_1, \ldots, \theta_m$ be all the consistent formulas in $\Gamma(r, \phi)$. By definition, $P'_r(\phi) = P_r(\{\theta_1\}) + \ldots + P_r(\{\theta_m\})$. By requirement B, for every $0 < i \leq m$, $P_r(\theta_i) = P_q(\Gamma(q, \theta_i))$. Now, $\bigcup_{0<i\leq m} \Gamma(q, \theta_i)$ collects all the consistent game-normal formulas in $\Gamma(q, \phi)$. So, $P'_q(\phi) = P_q(\bigcup_{0<i\leq m} \Gamma(q, \theta_i)) = P'_r(\phi)$. ■

Theorem 4.4 shows that I_k describes an absolute notion of semantic information. In order to determine what is the semantic information of a given formula in $\mathcal{F}or_{\mathcal{L}}(k)$, it is enough to consider one I_q, $q \geq k$. Theorem 4.3 is not equivalent to BCP and SoD. BCP means that every contradiction of classical logic carries maximum information. However, as I showed in chapter 3, there are contradictions of classical logic that are satisfied in urn semantics. Similarly, SoD means that validities of classical logic carry null information. On the other hand, I also showed in chapter 3 that some validities of classical logic have counter-models in urn semantics.

Now, there is at least one important difference between the behaviors of TSI in classical logic and in urn semantics. Given the undecidability of classical logic, in general $\Theta^c(k, x_1, \ldots, x_n)$ is not a recursively enumerable set. Therefore, $P'_k(\phi)$ and, consequently, $I_k(\phi)$ are generally not computable. Hence, in the framework of classical logic, TSI does not provide an effectively computable notion of semantic information. Based on this observation, Hintikka (1970b) suggested an alternative notion known as *surface* semantic information that is effectively computable and blocks SoD (in next section I compare Hintikka's proposal with mine).

On the other hand, urn semantics is a decidable system of logic (Corollary 3.13). So, $\Theta^c(k, x_1, \ldots, x_n)$ denotes a recursive set in urn semantics. Therefore, $P'_k(\phi)$ and $I_k(\phi)$ are computable and TSI characterizes an effectively notion of semantic information in urn semantics.

Finally, I prove that urn semantics formalizes TSI without SoD.

Corollary 4.5 (Failure of SoD in urn semantics) *Urn semantics does not satisfy SoD, that is, there is some validity of classical logic ϕ in $\mathcal{F}or_{\mathcal{L}}(k)$ such that $I_k(\phi) > 0$.*

Proof. By Theorems 3.28 and 3.34, there is some classically unsatisfiable formula ψ in $\mathcal{F}or_{\mathcal{L}}(k)$ that is satisfiable in urn semantics. So, $\Gamma(k, \psi) \cap \Theta^c(k, x_1, \ldots, x_n) \neq \emptyset$ and, by requirement A, $P_k(\Gamma(k, \psi) \cap \Theta^c(k, x_1, \ldots, x_n)) > 0$. Note that $\Gamma(k, \neg\psi) \cap \Theta^c(k, x_1, \ldots, x_n) = \Theta^c(k, x_1, \ldots, x_n) - (\Gamma(k, \psi) \cap \Theta^c(k, x_1, \ldots, x_n))$. So, $P'_k(\neg\psi) = 1 - P_k((\Gamma(k, \psi) \cap \Theta^c(k, x_1, \ldots, x_n))) < 1$. Hence, $I_k(\neg\psi) = 1 - P'_k(\neg\psi) > 0$. $\neg\psi$ is our desired ϕ. ■

4.3 A comparison with similar approaches

4.3.1 Hintikka's depth and surface information

In this section I compare the presently suggested reestablishment of TSI with similar proposals in the literature. First, I compare this work with the theory of semantic information by Hintikka (1970a, 1970b).

Hintikka's work has several points of contact with this thesis. First, Hintikka (1970b, p. 264) accepts the probabilistic conception of information favoured by TSI: in the same steps of Bar-Hillel and Carnap, Hintikka equates semantic information with degree of uncertainty. Furthermore, Hintikka supports the formal characterization of the semantic information of a formula in terms of the complementary of its probability. Secondly, Hintikka (Ibid., pp. 288–289) is a pioneer in acknowledging that SoD is problematic and must be rejected by a sound theory of semantic information. Finally, Hintikka characterizes the probability of a formula ϕ in terms of the number of *adequate* game-normal formulas in $\Gamma(k, \phi)$.

The fundamental difference between my proposal and Hintikka's can be redirected to the following question: what is the relevant notion of adequacy that plays a role in the characterization of the probability of a formula? As I showed in the previous section, TSI characterizes the probability of ϕ in terms of the number of consistent formulas in $\Gamma(k, \phi)$. Hintikka's proposal, on the other hand, is not so simple.

I mentioned previously that $\Theta^c(k, x_1, \ldots, x_n)$ does not denote a recursively enumerable set in classical logic. However, by completeness of classical logic (Cf. SHOENFIELD, 1967, pp. 43–48), the complementary of $\Theta^c(k, x_1, \ldots, x_n)$ is semi-decidable. Hintikka (1965b) showed that the following is a semi-decision procedure for determining the extension of $\Theta(k, x_1, \ldots, x_n) - \Theta^c(k, x_1, \ldots, x_n)$. In what follows, for a formula ϕ with free-variables x and y, let the *inverse* of ϕ over x and y be the formula generated from ϕ by interchanging the occurrences of x and y in it (for instance, if ϕ is an atomic formula $P(x, y)$, its inverse over x and y is $P(y, x)$).

Definition 4.6 *Let θ be a game-normal formula with quantifier rank k and free-variables $x_1, \ldots, x_n$. θ is* trivially inconsistent *if and only if one of the following conditions holds:*

- *For some game-normal formula* $\gamma \in \Theta(k-2, x_1, \ldots x_n, y_k, y_{k-1})$ *occurring in* θ*, the inverse of* γ *over* y_k *and* y_{k-1} *does not occur in* θ*;*

- *For two game-normal formulas* θ_1 *and* θ_2 *occurring in* θ *with quantifier rank* $k-1$*, for any game-normal formula* $\gamma \in \Theta(k-2, x_1, \ldots x_n, y_k, y_{k-1})$ *occurring in* θ_1*, the inverse of* γ *over* y_k *and* y_{k-1} *does not occur in* θ_2*;*

- *For any game-normal formula* $\gamma \in \Theta(k-1, x_1, \ldots x_n, y_k)$ *occurring in* θ*, it is not valid that* $\theta \to \gamma\ y_k[x_i]$*, for some* $0 < i \leq n$*.*

It is easy to verify that every trivially inconsistent game-normal formula is in fact inconsistent in classical logic. Hintikka (Ibid.) also proved that, for every inconsistent game-normal formula θ with quantifier rank $k < \omega$, there is some $q \geq k$ such that all game-normal formulas in $\Gamma(q, \theta)$ are trivially inconsistent. Hence, as a semi-decision procedure for determining whether some formula θ is in $\Theta(k, x_1, \ldots, x_n) - \Theta^c(k, x_1, \ldots, x_n)$, we simply need to consider whether there is some H-normal form $\bigvee \Gamma(q, \theta)$ composed only by trivially inconsistent game-normal formulas. For simplicity, let me call this procedure *Hintikka's expansion method*.

Let $\Theta^{ti}(k, x_1, \ldots, x_n)$ be the subset of trivially inconsistent formulas in $\Theta(k, x_1, \ldots, x_n)$ and let $\overline{\Theta^{ti}}(k, x_1, \ldots, x_n) = \Theta(k, x_1, \ldots, x_n) - \Theta^{ti}(k, x_1, \ldots, x_n)$. Clearly, $\Theta^{ti}(k, x_1, \ldots, x_n)$ is recursive and, by Hintikka's expansion method, is included (properly, when $k < 1$) in $\Theta(k, x_1, \ldots, x_n) - \Theta^c(k, x_1, \ldots, x_n)$.

Now, Hintikka defines two different notions of semantic information. On one hand, Hintikka (1970b, pp. 274–275) defines the notion of *depth* information that is exactly the concept of k-information (Definition 4.2). On the other hand, Hintikka (Ibid., pp. 276–281) defines the notion of *surface* information based on an alternative notion of probability that he calls *pre-logical probability*.

Hintikka's functions of pre-logical probability Q_k are closely similar to functions P_k with two differences: first, whilst functions P_k are defined on parts of $\Theta^c(k, x_1, \ldots, x_n)$, the functions Q_k are defined on parts of $\overline{\Theta^{ti}}(k, x_1, \ldots, x_n)$; secondly, whilst functions P_k satisfy requirement B, functions Q_k replace this requirement by the following one:

B'. For any $\theta \in \overline{\Theta^{ti}}(k, x_1, \ldots, x_n)$, for every $q > k$, $Q_k(\{\theta\}) = Q_q(\Gamma(q, \theta) \cap \overline{\Theta^{ti}}(q, x_1, \ldots, x_n))$, if $\Gamma(q, \theta) \cap \overline{\Theta^{ti}}(q, x_1, \ldots, x_n) \neq \emptyset$.

Based on the concept of pre-logical probability, similarly to what I have done above, it is possible to define a probability function Q'_k on $\mathcal{F}or_{\mathcal{L}}(k)$ such that $Q'_k(\phi) = Q_k(\Gamma(k, \phi) \cap \overline{\Theta^{ti}}(k, x_1, \ldots, x_n))$. Finally, the surface information of ϕ in rank k (or simply, the k-surface information of ϕ) is defined as $J_k(\phi) = 1 - Q'_k(\phi)$.

There are important differences between depth and surface information. First, SoD fails for surface information, that is, there are logical truths ϕ such that, for some $k < \omega$, $J_k(\phi) > 0$. The reason behind this failure of SoD is that, for some logical truth ϕ, $\Gamma(k, \phi) \not\supseteq \overline{\Theta^{ti}}(k, x_1, \ldots, x_n)$.

Further, surface information is not a stable measure of semantic information since, for some contradiction ϕ such that $\Gamma(k, \phi)$ has non-trivially inconsistent game-normal formulas, for some $q > k$, $J_k(\phi) < J_q(\phi)$ (Ibid., pp. 279–281). In fact, depth information can be seen as the limit of variability of surface information through ranks (Ibid., pp. 283–284). For instance, based on Hintikka's expansion method it is possible to verify that the surface information of a contradiction ϕ gradually increases when we go upwards in the hierarchy of ranks. In the limit, the surface of information of ϕ collapses with its depth information.

Does surface information provide a good strategy for the reestablishment of TSI? In principle, there are reasons to think it does. First, considering that non-trivial inconsistencies have an *appearance* of consistency, surface information could be seen as representing the initial epistemic situation of an ordinary cognitive agent towards the status of logical truths and contradictions (Ibid., pp. 291–292). Further, Hintikka (Ibid., p. 142) claims that his expansion method nicely describes the concrete process of discovery of the status of logical truths and contradictions by real cognitive agents. In this sense, surface information supposedly formalizes the process of accessing the actual information (i.e., depth information) carried by formulas. Note that, according to Hintikka (1965a), his expansion method gives flesh and blood for a precise conception of synthetic a priori knowledge. Hence, if this expansion method is in fact a good formalization of the real process of discovery of the status of logical truths, it offers a strong account of logical knowledge as synthetic a priori.

However, Hintikka's proposal faces important challenges. First, Sequoiah-Grayson (2008, pp. 88–ff.) objects that surface information still satisfies a restricted version of SoD for monadic and propositional logic: surely there are really informative logical truths in both fragments of classical logic, but Hintikka's surface information is completely blind for anyone of these cases. This is not a strong criticism against Hintikka's proposal (in fact, this is an instance of a fallacious reasoning strategy whose use is unfortunately very common in current philosophical discussion) but, since it may also be seen as a challenge for my proposal (Cf. JAGO, 2007, pp. 331–338), in the following I advance a proper reply to it.

Generally speaking, philosophical investigation of any kind is always motivated by a set of problems, questions or simply curious phenomena that present themselves to us pre-philosophically. For a simple example, the whole contemporary debate on the nature and definition of "knowledge" is motivated by the pre-philosophical acknowledgement that we are living creatures who have the capability of producing and communicating knowledge. This state of affairs seems curious to us and, because of this, we start to reflect on what knowledge is. In fact, one of the central dangers for any philosophical enterprise is exactly its alienation from its motivating pre-philosophical questions and phenomena (Cf. HANSSON, 2000, pp. 168–ff.).

Now, the philosophical investigation on the ampliative character of logical knowledge is motivated by the pre-philosophical observation that sometimes we ampliate our knowledge by discovering that some sentence expresses a logical truth. For simplicity, let me call this observation as the *experience of the ampliativeness of logic*. Can we say that all the different situations in which we have experienced the ampliativeness of logic are all instances of the same kind of phenomenon? Given our poverty of empirical evidence, we cannot be so sure: maybe there are different types of ways in which logical knowledge can be ampliative. In this sense, perhaps the case of knowledge of propositional logic is drastically different from its corresponding quantificational case.

Therefore, the above stated Sequoiah-Grayson's criticism against Hintikka's proposal is not adequate. Hintikka's proposal needs to be evaluated by what it proposes itself to do, namely, to give an account of the ampliative character of the knowledge of logical truths whose validity depends on the semantics of quantifiers. At this point of the philosophical enterprise, we cannot say whether the ampliativeness of logic is an univocal phenomenon or rather a web of different but interconnected facts. Perhaps there is only one way in which logical knowledge can be ampliative. However, in face of the poverty of empirical evidence, it is conceptually more parsimonious to assume the exactly opposite hypothesis that there are varieties of logical knowledge.

Besides that, Sequoiah-Grayson offers a stronger criticism against Hintikka's proposal:

> [...] it is arguable that Hintikka understands his disproof method [i.e., Hintikka's expansion method] as an auxiliary process that allows us to identify and measure the informativeness of deductive inferences and logical truths irrespectively of the actual proof procedure used in their derivation. The complexity of Hintikka's disproof method apparently brings with it the exposure of the deep structure of deductions. [...] However, Hintikka is not necessarily in a good position simply because of this. In fact, what good reasons do we have for believing that measures of surface information do capture some of the informativeness of deductive inferences and logical truths? (Ibid., pp. 87-88)

What are the reasons for thinking that the non-trivially inconsistent game-normal formulas in $\Gamma(k, \phi)$ jointly describe the initial epistemic scenario of a cognitive agent who simply understands the propositional content of ϕ? Further, what are the reasons for thinking that Hintikka's expansion method adequately formalizes the mechanism underlying ordinary cognitive agent's process of discovery of the status of logical truths and contradictions? Hintikka does not advance any reason in support of these claims, on the contrary to what I have done in the first chapter of this thesis in order to support my proposal.[10]

4.3.2 Jago's account of semantic information

More recently, Jago (2009, 2013) proposed an account of semantic information and the ampliative character of logical knowledge that is conceptually very similar to the one proposed in

[10]However, Hintikka's insight that an adequate account of semantic information should be associated with a nice characterization of rational agents' epistemic dynamics should not be overlooked. Even though we might have some reservation with respect to Hintikka's theory of semantic information, the theoretic goals of Hintikka's work on epistemology of logic still do not have a good resolution in the literature's state-of-art.

this thesis. In the spirit of what is done here, one of Jago's central premises is that the semantic information of a sentence ϕ must be conceived in terms of the multiplicity of possibilities that are excluded by the truth of ϕ (JAGO, 2013, pp. 317–319). Now, in classical logic, this fundamental idea immediately implies SoD: given that logical truths do not exclude any possibility, then logical truths convey null information.

In order to solve this problem, Jago explores the same strategy followed in this thesis: he enriches the semantic framework that he is working with with a class of epistemically possible worlds. These worlds represent a class of apparent possibilities which are excluded as ontologically impossible by the validity of a logical truth.

What kinds of ontologically impossible worlds are epistemically possible? Jago's answer to this question relies on the following framework. Jago (Ibid., pp. 327–330) considers a modal frame $\mathcal{F} = \langle W, R, V^+, V^- \rangle$ in which W is a set of worlds, R is an accessibility relation, $V^+(w)$ and $V^-(w)$ are the sets of formulas verified and falsified in the world w, respectively. Considering a generic multi-conclusion sequent calculus, each $w \in W$ represents a valid sequent $V^+(w) \vdash V^-(w)$. Informally speaking, w is a *prima facie* considerable world not excluded by the initial epistemic situation of a cognitive agent who does not know that $V^+(w) \vdash V^-(w)$ is valid.

Jago defines R as a set of pairs and triples such that $\langle w, u \rangle \in R$ if and only if

$$\frac{V^+(u) \vdash V^-(u)}{V^+(w) \vdash V^-(w)}$$

is a valid transformation, and $\langle w, u, t \rangle \in R$ if and only if

$$\frac{V^+(u) \vdash V^-(u) \quad V^+(t) \vdash V^-(t)}{V^+(w) \vdash V^-(w)}$$

is a valid transformation.

Further, Jago considers rooted, directed and acyclical graphs in $\mathcal{F}$ such that, for any w, u in the domain of the graph, $\langle w, u \rangle$ is an edge of the graph if and only if there is a subset $X \subset W$ such that $\langle w, X \rangle \in R$, $u \in X$ and any $x \in X$ if and only if $\langle w, x \rangle$ is an edge of the graph. In other words, each such graph with root w represents a possible proof of $V^+(w) \vdash V^-(w)$. For simplicity, Jago (Ibid., p. 329) calls such graphs as *point-graphs*.

Now, Jago defines the class of epistemically possible worlds in the following terms: for a *previously fixed* $f < \omega$, a world w is epistemically possible if and only if the smaller point-graph in $\mathcal{F}$ with root w has size greater than f. That is, in the characterization of the initial epistemic situation of an ordinary cognitive agent, Jago considers as epistemically possible only those worlds whose characteristic sequent calculus has a minimum degree of proof complexity.

What is the value of f in this definition of epistemically possible world? In this point, there is a fundamental difference between Jago's work and mine: Jago thinks that it is not possible to fix a priori the value of f, because it changes from context to context. More precisely, Jago

(Ibid., pp. 329–ff.) prefers to say that the notion of "epistemically possible" is *vague*.

Even though there is surely a difficulty in differentiating epistemically possible worlds from mere impossibilities, Jago's claim is untenable: as I argued in the first chapter of this thesis, an ordinary cognitive agent, in her initial epistemic situation, has some partial knowledge of truth-conditions associated with her semantic competence. Now, since two individuals who understand a given sentence access the same propositional content, the minimum knowledge of truth-conditions associated with their semantic competence needs to be the same. So, this thesis advances a claim that drastically diverges from Jago's work: I believe that the presently suggested account of the knowledge of truth-conditions connected with our semantic competence and the formalization of this account based on urn semantics provide a *discrete* definition of epistemically possible world.

On the other hand, Jago criticizes the identification of epistemically possible worlds with urn structures:

> There are, however, several serious problems with using urn models in this way [i.e., for characterizing epistemically possible worlds]. [...] In using these models, Hintikka works with a notion of an agent's logical competence, measured in terms of quantifier depth d. The rough idea is that greater competence correlates with the ability to reason correctly with sentences involving higher numbers of embedded quantifiers. But quantifier depth does not seem to be a very good measure of [...] logical competence [...] Suppose Anna completes a (correct) proof, in which no sentence has a quantifier depth greater than d, of a mathematical statement A. If she completed the proof through skill and not random luck, her achievement reflects her competence, and so we must assign her a competence of at least d. Then, no d'-invariant model was ever an epistemic possibility for Anna, for any $d' \leq d$. But since A appears in the proof, it must have a quantifier depth no greater than d, and so is true in all d-invariant models. Consequently, in a Hintikka-style model of knowledge, Anna is modelled as having known A all along; and in a model of content, her proof is modelled as being contentless and uninformative. This is just what we want to avoid (Ibid., pp. 324–325).

Jago's criticism confuses two different *moments* of epistemic competence. On one hand, given the completeness of first order logic, of course any cognitive agent has the *potential* of having full epistemic competence in discovering the validity of any given logical truth. However, the full knowledge of the truth-conditions of a logical truth is not available from the start. In general, the process of discovering the validity of logical truths involves great complexity. So, in the initial moment in which someone merely understands a sentence expressing a logical truth, this person does not know completely the truth-conditions of the sentence. This initial epistemic scenario is what is nicely formalized through urn semantics, not the final one in which the cognitive agent in question already knows that the considered sentence is a logical truth. However, the former scenario is exactly what needs to be described in order to give an account of the ampliative character of logical knowledge and the semantic information of logical truths.

Finally, it is possible to point out a further problematic feature of Jago's proposal. The predicate "does not have proofs with complexity smaller than f" is not partially recursive. Therefore, the notion of probability playing a role in Jago's proposal and, consequently, its derivative notion of semantic information are not computable. This is not necessarily a bad feature of Jago's proposal, however note that some alternative proposals (including mine) do not present this feature.

4.4 Conclusion

In this chapter I presented a reassessment of TSI based on urn semantics. I showed that, based on this framework, TSI is a stable measure of semantic information that does not imply SoD. Furthermore, I considered two similar proposals on the subject by Jago (2009, 2013) and by Hintikka (1970a, 1970b).

Despite the similarities between these works, contrary to my proposal, Jago's and Hintikka's face some important challenges: first, Jago and Hintikka propose non-effectively computable notions of information. This is in fact a minor problem of these theories. The biggest problem of Hintikka's proposal is the lack of evidence of its soundness. By its turn, the most pressing difficulty of Jago's proposal is the fact that his account conceives the notion of epistemically possible world as a vague concept. Finally, I also presented that Jago has some concerns against the use of urn semantics as a framework for reestablishing TSI, and *en passant* I replied to these objections.

This exposition ends the main part of this thesis. In the second part I consider some more advanced results on urn semantics and semantic information. In the next chapter I consider some questions about categoricity in urn semantics. In the present context of investigation, this is a relevant research agenda since, in the first order case, TSI measures the information of a sentence by counting how many non-equivalent models it has, and the relevant notion of equivalence here (i.e., back and forth equivalence) is conceptually dependent on the notion of isomorphism.

Part II

More advanced studies

Chapter 5

Measuring semantic information: categoricity in urn semantics

5.1 Introduction

In what follows, for a first order language $\mathcal{L}$, by a theory T of $\mathcal{L}$ (an $\mathcal{L}$-theory) I mean any set of sentences (i.e., closed formulas) of $\mathcal{L}$. I showed in the last chapter that TSI measures the semantic information of a formula by counting how many non-equivalent models it has. The notion of equivalence that is relevant here is back and forth equivalence, a notion that is conceptually dependent on the notion of isomorphism. So, in order to understand the range of variability of information of sentences it is necessary to consider the following question first: for a first order language $\mathcal{L}$ and for an $\mathcal{L}$-theory T, how many models T has up to isomorphism in urn semantics?

Olin (1978) presented a seminal investigation on this question with interesting results. However, he focused on p-urn semantics without considering its i- counterpart. In this chapter I expand the scope of Olin's research for i-urn semantics. Preliminarily let me introduce some further terminology. Following common usage, I say that two (urn) structures are *elementary equivalent* if and only if they agree in all sentences. I denote elementary equivalence by the symbol "$\equiv$". Moreover, I generalize the model-theoretic notion of *categoricity* for urn semantics in the following way.

Definition 5.1 *Let T be an $\mathcal{L}$-theory. In p-urn semantics, T is* categorical *if and only if any two $\mathcal{L}$-p-urn models $\widehat{\mathcal{M}}_{\mathfrak{C}}$ and $\widehat{\mathcal{N}}_{\mathfrak{C}'}$ of T are isomorphic. T is κ-*categorical *if and only if any two $\mathcal{L}$-p-urn models $\widehat{\mathcal{M}}_{\mathfrak{C}}$ and $\widehat{\mathcal{N}}_{\mathfrak{C}'}$ of T of cardinality κ are isomorphic.*

Sometimes in the following I consider the stronger notion of *strict* categoricity as well as the auxiliary concept of *full eligibility*.

Definition 5.2 *Let $\widehat{\mathcal{M}}_{\mathfrak{C}}$ be an $\mathcal{L}$-p-urn structure and $\widehat{\mathfrak{M}}$ be the set of p-eligible sequences of $\mathcal{M}$ over $\mathfrak{C}$. $\widehat{\mathcal{M}}_{\mathfrak{C}}$ is* fully eligible *if and only if for any element a of $\mathcal{M}$, a occurs in some $\bar{a} \in \widehat{\mathfrak{M}}$.*

Definition 5.3 *Let T be an $\mathcal{L}$-theory. In p-urn semantics, T is* strictly categorical *if and only if any two fully eligible $\mathcal{L}$-p-urn models $\widehat{\mathcal{M}}_{\mathfrak{C}}$ and $\widehat{\mathcal{N}}_{\mathfrak{C}'}$ of T are isomorphic. T is* strictly κ-categorical *if and only if any two fully eligible $\mathcal{L}$-p-urn models $\widehat{\mathcal{M}}_{\mathfrak{C}}$ and $\widehat{\mathcal{N}}_{\mathfrak{C}'}$ of T of cardinality κ are isomorphic.*

Usually I say of a p-urn structure $\widehat{\mathcal{M}}_{\mathfrak{C}}$ that it is (strictly) (κ-)categorical in order to mean that its theory (i.e., the set of sentences that are satisfied by $\widehat{\mathcal{M}}_{\mathfrak{C}}$) has the relevant property. Given the inclusion of i-urn structures in the class of p- ones, Definitions 5.1–5.3 are immediately generalizable to i-urn semantics.

Olin's work considers the conditions for (strict) (κ-)categoricity in urn semantics. It is easy to see that in urn semantics there are neither finite categorical nor infinite κ-categorical models, in clear opposition to classical logic. For proving that there are no finite categorical models in urn semantics, consider any finite p-urn structure $\widehat{\mathcal{M}}_{\mathfrak{C}}$: for any structure $\mathcal{N}$ that extends $\mathcal{M}$ with a new non-eligible element a, $\widehat{\mathcal{N}}_{\mathfrak{C}}$ is elementary equivalent to $\widehat{\mathcal{M}}_{\mathfrak{C}}$ but non-isomorphic to it. By a similar argument, we show that any infinite p-urn structure $\widehat{\mathcal{M}}_{\mathfrak{C}}$ is extendable to an elementary equivalent but non-isomorphic $\widehat{\mathcal{N}}_{\mathfrak{C}}$ with same cardinality. Note that these results hold also in i-urn semantics given the inclusion of its class of structures in the class of p-urn structures.

However, there are also some non-trivial questions about categoricity in urn semantics. In particular, strict categoricity presents itself as a particularly complex subject matter. Olin (1978) obtained some important results on this topic, focusing attention on p-urn semantics. Whether there are finite or infinite strict categorical models in i-urn semantics is still an open issue. In this chapter I consider this problem in particular.

This chapter is organized as follows: in section 5.2, I present three major results by Olin (1978) on strict (κ-)categoricity in p-urn semantics. In subsection 5.3.1, changing the focus to i-urn semantics I investigate whether there are infinite strictly categorical i-urn structures. Finally, in subsection 5.3.2, I present some initial results on finite categoricity in i-urn semantics.

5.2 Olin's results on p-urn semantics

Olin (1978) presented limitative results about categoricity and p-urn semantics. First, Olin showed that there are no finite strictly categorical p-urn structures. Secondly, he showed that, given any infinite, non-fully eligible p-urn structure $\widehat{\mathcal{M}}_{\mathfrak{C}}$, there is a fully eligible, elementary equivalent but non-isomorphic $\widehat{\mathcal{N}}_{\mathfrak{C}'}$ of same cardinality. Finally, Olin showed that infinite strict κ-categoricity holds under special conditions in p-urn semantics. In what follows, I present an overview of Olin's results.[1]

[1] Olin (1978, pp. 339–ff.) also obtained some results on finite κ-categoricity in p-urn semantics: Olin showed that fully eligible p-urn structures of cardinality 2 are 2-categorical and also proved that, particularly for monadic languages, this result is generalizable for any cardinality $\kappa < \omega$. Olin left as an open question whether it is possible to achieve a similar result on the behavior of finite κ-categoricity in p-urn semantics for all first order languages. These questions as well as their corresponding analogs in i-urn semantics are quite interesting, I did not consider them here for methodological reasons only.

Olin's proofs of these results essentially rely on what we might call *isomorphic sequences*. I precisely define this concept below. For a p-urn structure $\widehat{\mathcal{M}_{\mathfrak{C}}}$ and for any p-eligible sequence $\bar{a}$ of $\mathcal{M}$ over $\mathfrak{C}$, let me say that an element b of $\mathcal{M}$ is an *i-th eligible extension* of $\bar{a}$ over $\mathfrak{C}$ if and only if b is the i-th element in some p-eligible sequence $\bar{b}$ of $\mathcal{M}$ over $\mathfrak{C}$ whose initial segment is $\bar{a}$. I call $I(\bar{a}, \mathfrak{C}) = \{c \in \mathcal{M} : c$ is i-th eligible extension of the p-eligible sequence $\bar{a}$ of $\mathcal{M}$ over $\mathfrak{C}$, for some $i < \omega\}$ the set of *eligible extensions* of $\bar{a}$ over $\mathfrak{C}$.

Definition 5.4 *Let $\widehat{\mathcal{M}_{\mathfrak{C}}}$ be an $\mathcal{L}$-p-urn structure, $\bar{a}$ and $\bar{b}$ be two p-eligible sequences of $\mathcal{M}$ over $\mathfrak{C}$ with length n. A bijection $\iota : I(\bar{a}, \mathfrak{C}) \to I(\bar{b}, \mathfrak{C})$ is* isomorphism *between the eligible extensions of $\bar{a}$ and $\bar{b}$ if and only if the following holds:*

- $\iota(a_i) = b_i$*, for any $0 < i \leq n$;*
- *For any sequence $\bar{c}$ of $I(a, \mathfrak{C})$ such that $\bar{a}$ is its initial segment, $\bar{c}$ is p-eligible sequence of $\mathcal{M}$ over $\mathfrak{C}$ if and only if $\iota(\bar{c})$ is it as well;*
- *For any atomic formula $P(x_1, \ldots, x_m)$ and for any sequence $\bar{c}$ of $I(a, \mathfrak{C})$ with length m, $\widehat{\mathcal{M}_{\mathfrak{C}}} \models P(\bar{c}) \Leftrightarrow \widehat{\mathcal{M}_{\mathfrak{C}}} \models P(\iota(\bar{c}))$.*

If ι is a bijection between the eligible extensions of p-eligible sequences $\bar{a}$ and $\bar{b}$, for any p-eligible sequence $\bar{c}$ that extends $\bar{a}$, $\iota(\bar{c})$ is the *isomorphic copy* of $\bar{c}$. Accordingly, $I(\bar{a}, \mathfrak{C})$ and $I(\bar{b}, \mathfrak{C})$ are isomorphic copies of each other.

Theorem 5.5 *There are no finite, strictly categorical p-urn structures.*

Proof. Consider an arbitrary p-urn structure $\widehat{\mathcal{M}_{\mathfrak{C}}}$ and let $\widehat{\mathfrak{M}}$ be the set of p-eligible sequences of $\mathcal{M}$ over $\mathfrak{C}$. Consider also that the domain M of $\mathcal{M}$ has cardinality $n < \omega$ and $\widehat{\mathcal{M}_{\mathfrak{C}}}$ is fully eligible. Let $a \in \mathfrak{C}_0$.

For an arbitrary natural number $m > n$ and for each $0 < i \leq m$, consider non-empty sets $B(i)$ with same cardinality as $I(a, \mathfrak{C})$ and such that $[\bigcup_{0<i\leq m} B(i)] \cap M = B(i) \cap B(j) = \emptyset$, for each $0 < j \leq m$ different from i. Further, consider bijections $\iota_i : I(a, \mathfrak{C}) \to B(i)$.

Let $N = M \cup [\bigcup_{0<i\leq m} B(i)]$ and $\widehat{\mathfrak{N}} \supset \widehat{\mathfrak{M}}$ be a set of p-eligible sequences of N such that a sequence $\bar{b}$ is in $\widehat{\mathfrak{N}} - \widehat{\mathfrak{M}}$ if and only if $\bar{b} \in N - M$ and, for some $0 < i \leq m$, $\bar{b} = \iota_i(\bar{a})$, for some $\bar{a} \in \widehat{\mathfrak{M}}$ with a in its first position.

Consider the extension $\mathcal{N} \supset \mathcal{M}$ with domain N such that, for every $n < \omega$ and for every n-ary predicate P of $\mathcal{L}$:

- For each $0 < i \leq m$, for any $\bar{b} \in B(i)^n$, $\mathcal{M}$ (classically) satisfies $P(\iota_i^{-1}(\bar{b}))$ if and only if $\mathcal{N}$ (classically) satisfies $P(\bar{b})$;
- For any $\bar{b} \in N^n - [\bigcup_{0<i\leq m} B(i)^n]$, $\mathcal{N}$ (classically) satisfies $P(\bar{b})$.

So, for each $0 < i \leq m$, ι_i is isomorphism between the eligible extensions of a and $\iota_i(a)$. Let $\mathfrak{C}'$ be the p-eligibility set of $\mathcal{N}$ that generates $\widehat{\mathfrak{N}}$. Now, I will prove that $\widehat{\mathcal{M}_{\mathfrak{C}}} \equiv \widehat{\mathcal{N}_{\mathfrak{C}'}}$. In fact, we only need to consider whether the new p-eligible sequences of $\widehat{\mathcal{N}_{\mathfrak{C}'}}$ change the satisfaction of quantified formulas. In this sense, note that the mappings ι_i describe a back and forth equivalence between $\widehat{\mathcal{M}_{\mathfrak{C}}}$ and $\widehat{\mathcal{N}_{\mathfrak{C}'}}$. Therefore, given Theorems 3.20 and 3.21, these p-urn structures are elementary equivalent. However, they are not isomorphic, since the cardinality of N is greater than the cardinality of M. ■

As a general remark on Olin's strategy, let me note that this proof works by fixing an eligible element a of the given urn structure and extending this model with isomorphic copies of the set of eligible extensions of a. This extension preserves the full eligibility of the original model. Moreover, given that it only adds isomorphic copies, by back and forth equivalence we verify that the extended model is elementary equivalent to the originally given one. However, we can guarantee this elementary equivalence only because p-urn semantics enables us to reject as ineligible any sequence that mixes elements of the original model with newly added ones. This is not possible in i-urn semantics. Therefore, it is an interesting question to ask whether it is possible to preserve Theorem 5.5 in i-urn semantics (in subsection 5.3.2 I present some partial results on this issue).

Olin's proof of the next theorem follows a similar strategy.

Theorem 5.6 *Let $\widehat{\mathcal{M}_{\mathfrak{C}}}$ be a non-fully eligible $\mathcal{L}$-p-urn structure of cardinality $\kappa \geq \aleph_0$. Then, there is some $\mathcal{L}$-p-urn structure $\widehat{\mathcal{M}'_{\mathfrak{C}'}}$ of same cardinality that is elementary equivalent to $\widehat{\mathcal{M}_{\mathfrak{C}}}$ but non-isomorphic to it. In particular, $\widehat{\mathcal{M}'_{\mathfrak{C}'}}$ is fully eligible.*

Proof. Let $\mathfrak{C}^*$ be the set of elements occurring in p-eligible sequences of $\mathcal{M}$ over $\mathfrak{C}$. Consider the following two cases:

- First, assume that the cardinality of $\mathfrak{C}^*$ is κ. In this case, it is enough to consider the substructure $\mathcal{N} \subset \mathcal{M}$ with $\mathfrak{C}^*$ as its domain. Note that $\widehat{\mathcal{N}_{\mathfrak{C}}} \equiv \widehat{\mathcal{M}_{\mathfrak{C}}}$ but $\widehat{\mathcal{N}_{\mathfrak{C}}} \not\cong \widehat{\mathcal{M}_{\mathfrak{C}}}$;
- Secondly, assume that the cardinality of $\mathfrak{C}^*$ is less than κ. Consider again a substructure $\mathcal{N}$ as above. As in the proof of Theorem 5.5, for a fixed $a \in \mathfrak{C}_0$, generate an extension $\mathcal{M}' \supset \mathcal{N}$ of cardinality κ by adding κ isomorphic copies of the set of eligible extensions of a. Let $\mathfrak{C}'$ be the extension of $\mathfrak{C}$ with such isomorphic copies. $\widehat{\mathcal{M}'_{\mathfrak{C}'}} \equiv \widehat{\mathcal{M}_{\mathfrak{C}}}$ but, since $\widehat{\mathcal{M}'_{\mathfrak{C}'}}$ is fully eligible, then $\widehat{\mathcal{M}'_{\mathfrak{C}'}} \not\cong \widehat{\mathcal{M}_{\mathfrak{C}}}$.

■

Theorem 5.6 depends on the fact that $\widehat{\mathcal{M}_{\mathfrak{C}}}$ is not fully eligible. The next theorem shows that, under special conditions, there are infinite, strictly κ-categorical p-urn structures.

Theorem 5.7 *Let $\widehat{\mathcal{M}_{\mathfrak{C}}}$ be a fully eligible $\mathcal{L}$-p-urn structure of cardinality $\kappa \geq \aleph_0$. The following are equivalent:*

1. *$\widehat{\mathcal{M}_{\mathfrak{C}}}$ is strictly κ-categorical;*

2. *The following holds: i) $\mathcal{L}$ is a monadic language, ii) for any two elements $a, b \in \mathfrak{C}_0$, $I(a, \mathfrak{C})$ is isomorphic with $I(b, \mathfrak{C})$ and iii) for any $a \in \mathfrak{C}_0$, $I(a, \mathfrak{C}) = \{a\}$.*

The proof of this theorem uses the following lemmas.

Lemma 5.8 *For $\kappa \geq \aleph_0$, let $\widehat{\mathcal{M}_{\mathfrak{C}}}$ be a fully eligible and strictly κ-categorical $\mathcal{L}$-p-urn structure. Then, for any element $a \in \mathfrak{C}_0$, the cardinality of the set $Iso(a) = \{c \in \mathfrak{C}_0 :$ there is an isomorphism between the eligible extensions of c and $a\}$ is κ. Further, for any two elements $a, b \in \mathfrak{C}_0$, there is an isomorphism between their eligible extensions.*

Proof. Let M be the domain of $\mathcal{M}$. For the first part of the lemma, assume that, for some $a \in \mathfrak{C}_0$, the cardinality of $Iso(a)$ is less than κ. Then, by adding κ isomorphic copies of $I(a, \mathfrak{C})$ it is possible to generate extensions $\mathcal{N} \supset \mathcal{M}$ and $\mathfrak{C}' \supset \mathfrak{C}$ of cardinality κ such that $\widehat{\mathcal{M}_{\mathfrak{C}}} \equiv \widehat{\mathcal{N}_{\mathfrak{C}'}}$ but $\widehat{\mathcal{M}_{\mathfrak{C}}} \not\cong \widehat{\mathcal{N}_{\mathfrak{C}'}}$, as in the proofs of Theorems 5.5 and 5.6.

For the second part, assume that there are $a, b \in \mathfrak{C}_0$ such that there is no isomorphism in their eligible extensions. By the first part of the lemma, $Iso(a)$ and $Iso(b)$ have both cardinality κ. Then, consider the substructure $\mathcal{N} \subset \mathcal{M}$ with domain $N = (M - \bigcup\{I(c, \mathfrak{C}) : c \in Iso(b)\}) \cup I(b, \mathfrak{C})$. Given that $Iso(a) \subset N$, then $\mathcal{N}$ has cardinality κ. Consider that $\mathfrak{C}'$ is the subset of $\mathfrak{C}$ obtained by eliminating $\bigcup\{I(c, \mathfrak{C}) : c \in Iso(b)\}$. As in the proof of Theorem 5.5, it is possible to show that $\widehat{\mathcal{M}_{\mathfrak{C}}}$ is back and forth equivalent to $\widehat{\mathcal{N}_{\mathfrak{C}'}}$ and, consequently, $\widehat{\mathcal{M}_{\mathfrak{C}}} \equiv \widehat{\mathcal{N}_{\mathfrak{C}'}}$. However, since $Iso(b)$ is smaller in $\mathcal{N}$ than in $\mathcal{M}$, then $\widehat{\mathcal{M}_{\mathfrak{C}}} \not\cong \widehat{\mathcal{N}_{\mathfrak{C}'}}$. ■

Lemma 5.9 *Let $\widehat{\mathcal{M}_{\mathfrak{C}}}$ be a fully eligible and strictly κ-categorical $\mathcal{L}$-p-urn structure, for some $\kappa \geq \aleph_0$. Then, for any two elements $a, b \in \mathfrak{C}_0$ of $\mathcal{M}$, $I(a, \mathfrak{C}) \cap I(b, \mathfrak{C}) = \emptyset$. Further, $\mathcal{L}$ is monadic language.*

Proof. Let M be the domain of $\mathcal{M}$. For the first part of the lemma, suppose that there are elements $a, b \in \mathfrak{C}_0$ such that $I(a, \mathfrak{C}) \cap I(b, \mathfrak{C}) \neq \emptyset$. So, consider a variation $\mathfrak{C}'$ of $\mathfrak{C}$ generated by replacing each $c \in I(a, \mathfrak{C}) \cap I(b, \mathfrak{C})$ by two different elements c_a and c_b. Add all such elements c_a and c_b to $I(a, \mathfrak{C}')$ and $I(b, \mathfrak{C}')$, respectively. Now, let N be the set generated from M after such replacements.

Consider the variation $\mathcal{N}$ of $\mathcal{M}$ such that, for any n-ary predicate $P \in \mathcal{L}$ and for any sequence of elements $a'_1, \ldots, a'_n$ of $\mathcal{N}$, $\mathcal{N}$ satisfies $P(a'_1 \ldots a'_n)$ if and only if $\mathcal{M}$ satisfies $P(a_1 \ldots a_n)$, in which, for any $0 < i \leq n$, if a'_i is in M, $a'_i = a_i$; otherwise, a_i is some $c \in I(a, \mathfrak{C}) \cap I(b, \mathfrak{C})$ and a'_i is either c_a or c_b.

$\widehat{\mathcal{N}_{\mathfrak{C}'}}$ has cardinality κ as well. Since there are isomorphisms $\iota : I(a, \mathfrak{C}) \rightarrow I(a, \mathfrak{C}')$ and $\iota' : I(b, \mathfrak{C}) \rightarrow I(b, \mathfrak{C}')$, there is a back and forth equivalence between $\widehat{\mathcal{M}_{\mathfrak{C}}}$ and $\widehat{\mathcal{N}_{\mathfrak{C}'}}$. Hence, $\widehat{\mathcal{M}_{\mathfrak{C}}} \equiv \widehat{\mathcal{N}_{\mathfrak{C}'}}$. However, $\widehat{\mathcal{M}_{\mathfrak{C}}} \not\cong \widehat{\mathcal{N}_{\mathfrak{C}'}}$, what contradicts the strict κ-categoricity of $\widehat{\mathcal{M}_{\mathfrak{C}}}$.

For the second part, assume that $\mathcal{L}$ is not monadic. So, fix a n-ary predicate P, for some $n \geq 2$. Consider the following two cases:

- First, assume that there are $a_1, \ldots, a_n \in \mathfrak{C}_0$ such that $a_i \neq a_j$ for each $0 < i < j \leq n$ and $\mathcal{M}$ satisfies $P(a_1 \ldots a_n)$ (we know that there are such number of elements in $\mathfrak{C}_0$ by lemma 5.8). So, consider a variant $\mathcal{N}$ of $\mathcal{M}$ with same domain and same set of p-eligible sequences such that, for every atomic formula ψ different from $P(x_1 \ldots x_n)$ and for any sequence $\bar{c}$, $\mathcal{N}$ satisfies $\psi\ \bar{x}[\bar{c}]$ if and only if $\mathcal{M}$ satisfies it as well, but $\mathcal{N}$ does not satisfy $P(b_1 \ldots b_n)$, for every sequence of elements $b_1, \ldots, b_n$ of $\mathfrak{C}_0$ such that $b_i \neq b_j$ for each $0 < i < j \leq n$. By the first part of the lemma, the sets of eligible extensions of any two elements in $\mathfrak{C}_0$ are exclusively disjoint. So, there is a back and forth equivalence between $\widehat{\mathcal{M}}_{\mathfrak{C}}$ and $\widehat{\mathcal{N}}_{\mathfrak{C}}$. Consequently, $\widehat{\mathcal{M}}_{\mathfrak{C}} \equiv \widehat{\mathcal{N}}_{\mathfrak{C}}$ but clearly $\widehat{\mathcal{M}}_{\mathfrak{C}} \not\cong \widehat{\mathcal{N}}_{\mathfrak{C}}$;

- Secondly, assume that for every sequence of elements $a_1 \ldots a_n \in \mathfrak{C}_0$, such that $a_i \neq a_j$ for each $0 < i < j \leq n$, $\mathcal{M}$ does not satisfy $P(a_1 \ldots a_n)$. So, it is enough to consider a variant $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{N}$ satisfies $P(b_1 \ldots b_n)$, for some sequence of elements $b_1, \ldots, b_n \in \mathfrak{C}_0$ such that $b_i \neq b_j$ for each $0 < i < j \leq n$.

■

Lemma 5.10 *Let $\widehat{\mathcal{M}}_{\mathfrak{C}}$ be a fully eligible and strictly κ-categorical $\mathcal{L}$-p-urn structure, for some $\kappa \geq \aleph_0$. Then, for every element $a \in \mathfrak{C}_0$, $I(a, \mathfrak{C})$ has cardinality less than κ. In particular, $I(a, \mathfrak{C}) = \{a\}$.*

Proof. For the first part of the lemma, suppose that there is some $a \in \mathfrak{C}_0$ such that $I(a, \mathfrak{C})$ has cardinality κ. By lemma 5.8, for every $b \in \mathfrak{C}_0$, there is isomorphism between $I(a, \mathfrak{C})$ and $I(b, \mathfrak{C})$. So, consider the substructure $\mathcal{N} \subset \mathcal{M}$ such that the domain of $\mathcal{N}$ is $I(a, \mathfrak{C})$ and consider $\mathfrak{C}'$ is the subset of $\mathfrak{C}$ involving only the elements in $I(a, \mathfrak{C})$. The cardinality of $\mathcal{N}$ is κ and $\widehat{\mathcal{M}}_{\mathfrak{C}} \equiv \widehat{\mathcal{N}}_{\mathfrak{C}'}$. However, $\widehat{\mathcal{M}}_{\mathfrak{C}} \not\cong \widehat{\mathcal{N}}_{\mathfrak{C}'}$.

For the second part, assume that, for some $a \in \mathfrak{C}_0$, there is $b \in I(a, \mathfrak{C})$ such that $a \neq b$. So, there is a p-eligible sequence $\bar{a} \frown (b)$ of $\mathcal{M}$ over $\mathfrak{C}$. Now, given that, by the first part of the lemma, the cardinality of $I(a, \mathfrak{C})$ is less than κ, there is an extension $\mathcal{N} \supset \mathcal{M}$ of cardinality κ such that $\mathcal{N}$'s domain is $M \cup Iso(\bar{a} \frown (b))$, in which the set $Iso(\bar{a} \frown (b)) = \{c :$ there is isomorphism between the eligible extensions of $\bar{a} \frown (c)$ and $\bar{a} \frown (b)\}$ has cardinality κ. Consider $\mathfrak{C}'$ is the extension of $\mathfrak{C}$ based on $Iso(\bar{a} \frown (b))$. $\widehat{\mathcal{N}}_{\mathfrak{C}'}$ is elementary equivalent to $\widehat{\mathcal{M}}_{\mathfrak{C}}$. However, $I(a, \mathfrak{C}')$ has cardinality κ. So, $\widehat{\mathcal{M}}_{\mathfrak{C}} \not\cong \widehat{\mathcal{N}}_{\mathfrak{C}'}$. ■

Finally, it is possible to prove Theorem 5.7.

Proof of Theorem 5.7. ($1 \Rightarrow 2$) This direction of the theorem is a consequence of Lemmas 5.8–5.10.

($2 \Rightarrow 1$) Assume that $\mathcal{L}$ and $\widehat{\mathcal{M}}_{\mathfrak{C}}$ satisfy conditions i)–iii). Consider a fully eligible $\mathcal{L}$-p-urn structure $\widehat{\mathcal{N}}_{\mathfrak{C}'}$ of cardinality κ and such that $\widehat{\mathcal{M}}_{\mathfrak{C}} \equiv \widehat{\mathcal{N}}_{\mathfrak{C}'}$. Let M and N be the domains of $\mathcal{M}$ and $\mathcal{N}$, respectively.

Fix an arbitrary bijection $\iota : M \to N$. Suppose that ι is not isomorphism. Then, since $\mathcal{L}$ is monadic, for some $a \in M$, for some atomic formula $P(x)$, $\mathcal{M}$ satisfies $P(a)$ but $\mathcal{N}$ does not satisfy $P(\iota(a))$. Given that $\widehat{\mathcal{N}}_{\mathfrak{C}'}$ is fully eligible, $\iota(a)$ occurs in the i-th position of some p-eligible sequence of $\mathcal{N}$ over $\mathfrak{C}'$. Therefore, $\widehat{\mathcal{N}}_{\mathfrak{C}'} \models \exists x_0 \ldots \exists x_{i-1} \exists x \neg P(x)$. By elementary equivalence, $\widehat{\mathcal{M}}_{\mathfrak{C}} \models \exists x_0 \ldots \exists x_{i-1} \exists x \neg P(x)$. By conditions ii) and iii), $\widehat{\mathcal{M}}_{\mathfrak{C}} \models_{i+1,\lambda} \neg P(a)$. Hence, $\mathcal{M}$ does not satisfy $P(a)$, what is a contradiction. ■

Theorem 5.7 shows how difficult it is to inhibit p-urn semantics' powerful tool for generating non-isomorphic extensions by the addition of isomorphic copies: only some special monadic structures are strictly κ-categorical. Now, this powerful tool is not available in i-urn semantics. Thus, in what follows I examine whether Olin's results can be preserved in i-urn semantics.

5.3 Categoricity in i-urn semantics

5.3.1 Infinite strict categoricity: compactness and Löwenheim-Skolem theorems for i-urn semantics

I turn the focus now to i-urn semantics and consider whether there are strict categoricity in this semantics. First, I show that, as a strengthening of the classical case, there are no infinite strictly categorical i-urn structures. Then, in next subsection, I show some results about finite categoricity in this semantics.

The comparison with classical logic offers some methodological advice here. In classical logic, Löwenheim-Skolem theorems block categoricity in the infinite case: first, *upward* Löwenheim-Skolem says that for any infinite model there is greater, elementary equivalent models. Moreover, *downward* Löwenheim-Skolem says that for any non-enumerable model there are also smaller, elementary equivalent ones. In this sense, it is interesting to consider whether these theorems hold in i-urn semantics.

Upward Löwenheim-Skolem relies on compactness theorem, a fundamental property of classical logic. So it is necessary to consider whether compactness holds in i-urn semantics. I start this subsection by considering this issue. In classical logic, compactness theorem can be obtained by exploring Henkin's strategy of extending *finitely satisfiable* theories to *maximal*, finitely satisfiable ones that have *witness property* (Cf. MARKER, 2006, pp. 35–40). In what follows, I explore a similar strategy. However, this will not work straightforwardly because i-urn semantics relativizes the notion of satisfaction with respect to eligibility sets. Fortunately, it is possible to improve Henkin's strategy in order to obtain compactness theorem for i-urn semantics. In this sense, let me define some basic terminology.

Definition 5.11 *An $\mathcal{L}$-theory T is* maximal *if and only if, for every sentence ϕ of $\mathcal{L}$, either $\phi \in$* T *or $\neg\phi \in$* T.

Definition 5.12 *Let $\{\mathcal{L}_n : n < \omega\}$ be a set of languages such that $\mathcal{L}_{n+1} = \mathcal{L}_n \cup \mathcal{C}_n$ for sets of new constants $\mathcal{C}_n = \{c_{m,\psi} : m < \omega,\ \exists x\psi$ is $\mathcal{L}_n$-sentence and x is free in $\psi\}$. For a set of theories $\mathcal{T} = \{T_n : n < \omega,\ T_n$ is $\mathcal{L}_n$-theory$\}$, $\bigcup\mathcal{T}$ has* witness property *if and only if the following holds:*

- *T_0 is an $\mathcal{L}_0$-theory;*
- *T_{n+1} is $\mathcal{L}_{n+1}$-theory collecting*
  - *formulas $\neg\exists x\psi \vee\ \forall y\psi\ x[c_{m,\psi}]$ of $\mathcal{L}_{n+1}$, for every $c_{m,\psi} \in \mathcal{C}_n$ and for some variable y not occurring in ψ,*
  - *and formulas $\neg\forall x\psi \vee \forall y\psi\ x[c_{m,\gamma}]$ of $\mathcal{L}_{n+1}$, for all $c_{m,\gamma} \in \mathcal{C}_n$ and for some variable y not occurring in ψ.*

Let $\mathcal{C} = \bigcup\limits_{n<\omega} \mathcal{C}_n$. For every $c_{m,\psi} \in \mathcal{C}$, $\neg\exists x\psi \vee\ \forall y\psi\ x[c_{m,\psi}]$ is called a m-witness axiom. *Further, $\neg\forall x\psi \vee \forall y\psi\ x[c_{m,\gamma}]$ is called a* universal *m-witness axiom.*

For any $m < \omega$, let me say that a formula ϕ is *m-satisfiable* if and only if for some i-urn structure $\mathcal{M}_{\mathfrak{B}}$, $\mathcal{M}_{\mathfrak{B}} \models_m \phi$. Further, I say that T is m-satisfiable if and only if there is $\mathcal{M}_{\mathfrak{B}}$ such that, for every sentence $\phi \in T$, $\mathcal{M}_{\mathfrak{B}} \models_m \phi$.

Definition 5.13 *For any $\mathcal{L}$-theory T and for any $m < \omega$, T is* finitely m-satisfiable *if and only if, for any finite part $S \subseteq T$, S is m-satisfiable. If a theory T is finitely 0-satisfiable, I simply say that T is finitely satisfiable.*

The next lemma presents a generalized version of *Lindenbaum's construction* of maximal and finitely satisfiable theories (Cf. MARKER, 2006, p. 38).

Lemma 5.14 *For some $m < \omega$, let T be a finitely m-satisfiable $\mathcal{L}$-theory. Then, there is a maximal finitely m-satisfiable $\mathcal{L}$-theory $T' \supseteq T$.*

Proof. Consider an enumeration $(\phi_n : n < \omega)$ of all formulas of $\mathcal{L}$. Let T' be as follows:

- Let $T_0 = T$
- Let $T_{n+1} = T_n \cup \{\phi_n\}$, if this theory is finitely m-satisfiable; otherwise, $T_{n+1} = T_n \cup \{\neg\phi_n\}$;
- $T' = \bigcup\limits_{n<\omega} T_n$.

Clearly, T' is maximal and includes T. Fix some finite $S \subset T'$. $S \subseteq T_n$, for some $n < \omega$. Since T_n is finitely m-satisfiable, S is m-satisfiable. Therefore, T' is finitely m-satisfiable. ■

The following lemma shows an interesting construction of extensions of theories with witness property.

Lemma 5.15 *Let T be some finitely satisfiable $\mathcal{L}$-theory. There is an extension $\mathcal{L}' \supset \mathcal{L}$ and an $\mathcal{L}'$-theory $T' \supset T$ for which the following holds:*

- *T' has witness property;*
- *For any $n, m < \omega$ and any finite parts $S \subset T$, $S' = \{\neg\exists x\psi \vee \forall y\psi\ x[c_{k_i,\psi}] : i \leq n\} \cup \{\neg\forall x\psi \vee \forall y\psi\ x[c_{k_j,\gamma}] : j \leq m\} \subset T' - T$, there is an i-urn model $\mathcal{M}_{\mathfrak{B}}$ of S such that $\mathcal{M}_{\mathfrak{B}} \models_{k_i-1} \neg\exists x\psi \vee \forall y\psi\ x[c_{k_i,\psi}]$, for every $i \leq n$, and $\mathcal{M}_{\mathfrak{B}} \models_{k_j-1} \neg\forall x\psi \vee \forall y\psi\ x[c_{k_j,\gamma}]$, for every $j \leq m$.*

Proof. Generate $\mathcal{L}'$ and T' recursively as in Definition 5.12, considering $\mathcal{L} = \mathcal{L}_0$ and $T = T_0$. For the second part of the lemma, fix any finite parts S, S' of T and $T' - T$, respectively. By hypothesis of the lemma, there is an i-urn model $\mathcal{M}_{\mathfrak{B}} \models \bigwedge S$. Fix any $(\neg\exists x\psi \vee \forall y\psi\ x[c_{k_i,\psi}]) \in S'$. Then, consider two cases:

- If $\mathcal{M}_{\mathfrak{B}} \models_{k_i-1} \exists x\psi$, then, for some $a \in \mathfrak{B}_{k_i}$, $\mathcal{M}_{\mathfrak{B}} \models_{k_i} \psi\ x[a]$. So, consider an $\mathcal{L}'$-expansion $\mathcal{N}$ of $\mathcal{M}$ such that $a = c^{\mathcal{N}}_{k_i,\psi}$;
- If $\mathcal{M}_{\mathfrak{B}} \not\models_{k_i-1} \exists x\psi$, then $\mathcal{N}_{\mathfrak{B}} \models_{k_i-1} \neg\exists x\psi \vee \forall y\psi\ x[c_{k_i,\psi}]$, for any $\mathcal{L}'$-expansion $\mathcal{N}$ of $\mathcal{M}$.

Now, fix any $(\neg\forall x\psi \vee \forall y\psi\ x[c_{k_j,\gamma}]) \in S'$. Consider again two cases:

- If $\mathcal{M}_{\mathfrak{B}} \models_{k_j-1} \forall x\psi$, then, for every $a \in \mathfrak{B}_{k_j}$, $\mathcal{M}_{\mathfrak{B}} \models_{k_j} \psi\ x[a]$. So, consider an $\mathcal{L}'$-expansion $\mathcal{N}$ of $\mathcal{M}$ such that for some such a, $a = c^{\mathcal{N}}_{k_j,\gamma}$;
- If $\mathcal{M}_{\mathfrak{B}} \not\models_{k_j-1} \forall x\psi$, then $\mathcal{N}_{\mathfrak{B}} \models_{k_j-1} \neg\forall x\psi \vee \forall y\psi\ x[c_{k_j,\gamma}]$, for any $\mathcal{L}'$-expansion $\mathcal{N}$ of $\mathcal{M}$.

■

Based on Lemmas 5.14 and 5.15, it is possible to prove the compactness theorem for i-urn semantics.

Theorem 5.16 (Compactness) *For any $\mathcal{L}$-theory T, T is finitely satisfiable if and only it is 0-satisfiable.*

Proof. By lemma 5.15, there are $\mathcal{L}' \supset \mathcal{L}$ and an $\mathcal{L}'$-theory $T' \supset T$ with witness property and such that for any $n, m < \omega$ and any finite parts $S \subset T$, $S' = \{\neg\exists x\psi \vee \forall y\psi\ x[c_{k_i,\psi}] : i \leq n\} \cup \{\neg\forall x\psi \vee \forall y\psi\ x[c_{k_j,\gamma}] : j \leq m\} \subset T' - T$, there is an i-urn model $\mathcal{M}_{\mathfrak{B}}$ of S such that $\mathcal{M}_{\mathfrak{B}} \models_{k_i-1} \neg\exists x\psi \vee \forall y\psi\ x[c_{k_i,\psi}]$, for every $i \leq n$, and $\mathcal{M}_{\mathfrak{B}} \models_{k_j-1} \neg\forall x\psi \vee \forall y\psi\ x[c_{k_j,\gamma}]$, for every $j \leq m$.

For each $0 < m < \omega$, let U_m be the set of (universal) m-witness axioms in T'. Now, we simultaneously build, for each $m < \omega$, a maximal, finitely m-satisfiable theory $T(m)$ as follows. For every $m < \omega$, consider an enumeration $(\phi_{n,m} : n < \omega)$. These enumerations generate the following matrix:

$$\begin{array}{l|llllll}
0: & \phi_{0,0} & \phi_{1,0} & \phi_{2,0} & \dots & \phi_{n,0} & \dots \\
1: & \phi_{0,1} & \phi_{1,1} & \phi_{2,1} & \dots & \phi_{n,1} & \dots \\
2: & \phi_{0,2} & \phi_{1,2} & \phi_{2,2} & \dots & \phi_{n,2} & \dots \\
\dots & & & \dots & & & \\
m: & \phi_{0,m} & \phi_{1,m} & \phi_{2,m} & \dots & \phi_{n,m} & \dots \\
\dots & & & \dots & & &
\end{array}$$

Matrix 5.1

Let $T(0)_0 = T \cup U_1$ and, for every $0 < m < \omega$, $T(m)_0 = U_{m+1}$. Let g be the function that formalizes the Cantorian enumeration of Matrix 5.1 and consider $g'(m, s) = r$ such that $T(m)_r$ is the last step in the recursive definition of $T(m)$ before $g(s)$.

For some $s < \omega$, let $\phi_{n,m} = g(s)$. $T(m)_{n+1}$ is defined by the following clause:

- $T(m)_{n+1} = T(m)_n \cup \{\phi_{n,m}\}$, if, for any $p < \omega$, for any $k_0 < \ldots < k_p < \omega$ and any finite parts $S_0 \subset T(k_0)_{g'(k_0,s)}, \ldots, S_p \subset T(k_p)_{g'(k_p,s)}$ there is an i-urn model $\mathcal{M}_{\mathfrak{B}}$ such that $\mathcal{M}_{\mathfrak{B}} \models_{k_i} \bigwedge S_i$, for every $i \leq p$, and $\mathcal{M}_{\mathfrak{B}} \models_m \phi_{n,m}$; otherwise, $T(m)_{n+1} = T(m)_n \cup \{\neg\phi_{n,m}\}$.

For every $m < \omega$, let $T(m) = \bigcup\limits_{n<\omega} T(m)_n$. The following facts hold about such sets:

i) By Lemma 5.14, every $T(m)$ is maximal and finitely m-satisfiable;

ii) Further, for every $p < \omega$, for any $k_0 < \ldots < k_p < \omega$ and any finite parts $S_0 \subset T(k_0), \ldots, S_p \subset T(k_p)$, there is an i-urn model $\mathcal{M}_{\mathfrak{B}}$ such that $\mathcal{M}_{\mathfrak{B}} \models_{k_i} \bigwedge S_i$, for every $i \leq p$;

Let $T^+ = \bigcup\limits_{m<\omega} T(m)$. Now, I will build a canonical model $\mathcal{M}_{\mathfrak{B}}$ such that, for every $m < \omega$, $\mathcal{M}_{\mathfrak{B}} \models_m T(m)$. Let $\mathcal{C}$ be the set of constants in $\mathcal{L}'$. So, the partition $\Pi(\mathcal{C}) = \{[c] :$ for any $d \in \mathcal{C},\ d \in [c]$ if and only if $(d = c) \in T^+\}$ will be the domain of the model. For any n-ary predicate P and function f in $\mathcal{L}'$, consider that $P^{\mathcal{M}} = \{\langle [c_1], \ldots, [c_n] \rangle \in \Pi(\mathcal{C})^n : P(c_1, \ldots, c_n) \in T^+\}$ and $f([c_1], \ldots, [c_n])^{\mathcal{M}} = [d]$ if and only if $(f(c_1, \ldots, c_n) = d) \in T^+$. For every constant $c \in \mathcal{C}$ and any $m < \omega$, let $[c] \in \mathfrak{B}_m$ if and only if $c \in [c_{m,\psi}]$, for some $c_{m,\psi} \in \mathcal{C}$. This is enough in order to characterize $\mathcal{M}$ and its eligibility set $\mathfrak{B}$.

Finally, I will prove by induction that, for any formula ϕ of $\mathcal{L}'$ with free-variables $x_1, \ldots, x_n$, $\mathcal{M}_{\mathfrak{B}} \models_m \phi\, x_1 \ldots x_n[[c_1] \ldots [c_n]] \Leftrightarrow \phi\, x_1 \ldots x_n[c_1 \ldots c_n] \in T(m)$. If ϕ is atomic formula, the property follows immediately from the definition of $\mathcal{M}$.

If ϕ is $\psi \wedge \gamma$, then $\mathcal{M}_{\mathfrak{B}} \models_m \phi$ if and only if $\mathcal{M}_{\mathfrak{B}} \models_m \psi$ and $\mathcal{M}_{\mathfrak{B}} \models_m \gamma$. By inductive hypothesis, that holds if and only if $\psi, \gamma \in T(m)$. By maximality and finite m-satisfiability, that is the case if and only if $\psi \wedge \gamma \in T(m)$.

If ϕ is $\psi \vee \gamma$, then $\mathcal{M}_{\mathfrak{B}} \models_m \phi$ if and only if either $\mathcal{M}_{\mathfrak{B}} \models_m \psi$ or $\mathcal{M}_{\mathfrak{B}} \models_m \gamma$. Without loss of generality, assume that $\mathcal{M}_{\mathfrak{B}} \models_m \psi$. So, by inductive hypothesis, that holds if and only if $\psi \in T(m)$. By maximality and finite m-satisfiability, that is the case if and only if $\psi \vee \gamma \in T(m)$.

If ϕ is $\neg\psi$, then $\mathcal{M}_{\mathfrak{B}} \models_m \neg\psi$ if and only if $\mathcal{M}_{\mathfrak{B}} \not\models_m \psi$. By inductive hypothesis, that holds if and only if $\psi \notin T(m)$. By maximality and finite m-satisfiability, that is the case if and only if $\neg\psi \in T(m)$.

Finally, assume ϕ is $\exists x\psi$. For the proof of sufficiency, assume that $\exists x\psi \in T(m)$. By the fact ii) above and witness property, $\psi\ x[c_{m,\psi}] \in T(m+1)$. By inductive hypothesis, $\mathcal{M}_{\mathfrak{B}} \models_{m+1} \psi\ x[[c_{m,\psi}]]$. Then $\mathcal{M}_{\mathfrak{B}} \models_m \exists x\psi$, since $[c_{m,\psi}] \in \mathfrak{B}_m$.

For the proof of necessity, assume that $\mathcal{M}_{\mathfrak{B}} \models_m \exists x\psi$. Then, $\mathcal{M}_{\mathfrak{B}} \models_{m+1} \psi\ x[[c_{m,\gamma}]]$, for some $[c_{m,\gamma}] \in \mathfrak{B}_m$. By inductive hypothesis, $\psi\ x[c_{m,\gamma}] \in T(m+1)$. By ii) and witness property, $\forall x\neg\psi \notin T(m)$. By maximality and finite m-satisfiability, $\exists x\psi \in T(m)$. The case in which ϕ is $\forall x\psi$ is a dual of the existential case. Hence, this is enough for proving the claim and, as a particular case of it, $\mathcal{M}_{\mathfrak{B}} \models T$. ■

In the proof of compactness theorem for i-urn semantics, witness constants $c_{m,\psi}$ play a double role by denoting instances of existential sentences and by characterizing eligibility in the canonical model as well. Based on this it is also possible to prove the following corollary about the construction of fully eligible canonical models for finitely satisfiable theories.

Corollary 5.17 *Let T be a finitely satisfiable $\mathcal{L}$-theory such that, for every constant $c \in \mathcal{L}$, $\forall x_0 \ldots \forall x_{i-1} \exists x(x = c) \in T$, for some $i < \omega$. Then, there is a fully eligible $\mathcal{L}$-i-urn structure $\mathcal{M}_{\mathfrak{B}}$ such that $\mathcal{M}_{\mathfrak{B}} \models T$.*

Just as in classical logic, upward Löwenheim-Skolem for i-urn semantics is a corollary of compactness theorem.

Corollary 5.18 (Upward Löwenheim-Skolem) *Let $\mathcal{M}_{\mathfrak{B}}$ be an $\mathcal{L}$-i-urn structure with infinite cardinality κ. For any cardinality $\lambda \geq \kappa$, there is an $\mathcal{L}$-i-urn structure $\mathcal{N}_{\mathfrak{B}'}$ of cardinality λ such that $\mathcal{M} \equiv \mathcal{N}$.*

Proof. Let $T = \{\phi : \ \phi$ is $\mathcal{L}$-sentence and $\mathcal{M}_{\mathfrak{B}} \models \phi\}$ and, for a set $\mathcal{C} = \{c_i : i < \lambda\}$ of new constants, let $S = \{c_i \neq c_j : i < j < \lambda\}$. Let $\mathcal{L}' = \mathcal{L} \cup \mathcal{C}$ and fix finite parts $T' \subset T$ and $S' = \{c_1, \ldots, c_n\} \subset S$, for some arbitrary $n < \omega$. Consider $\mathcal{M}'_{\mathfrak{B}}$ is an $\mathcal{L}'$-expansion of $\mathcal{M}_{\mathfrak{B}}$ such that $c_1^{\mathcal{M}'} \neq c_2^{\mathcal{M}'} \neq \ldots \neq c_n^{\mathcal{M}'}$. $\mathcal{M}'_{\mathfrak{B}} \models \bigwedge T' \wedge \bigwedge S'$. By compactness, there is an $\mathcal{L}'$-i-urn structure $\mathcal{N}'_{\mathfrak{B}'} \models T \cup S$. Consider $\mathcal{N}_{\mathfrak{B}'}$ is the $\mathcal{L}$-reduct of $\mathcal{N}'_{\mathfrak{B}'}$. $\mathcal{N}_{\mathfrak{B}'}$ is elementary equivalent to $\mathcal{M}_{\mathfrak{B}}$ and has cardinality λ. ■

Further, based on Corollary 5.17 it is possible to prove a stronger version of upward Löwenheim-Skolem in which the generated bigger model is fully eligible.

Corollary 5.19 *Let $\mathcal{M}_{\mathfrak{B}}$ be an $\mathcal{L}$-i-urn structure with infinite cardinality κ. Moreover, consider that, for any constant c in $\mathcal{L}$, for some $i < \omega$, $\mathcal{M}_{\mathfrak{B}} \models \forall x_0 \ldots \forall x_{i-1} \exists x(x = c)$. Then, for any* *cardinality $\lambda \geq \kappa$, there is a fully eligible $\mathcal{L}$-i-urn structure $\mathcal{N}_{\mathfrak{B}'}$ of cardinality λ such that $\mathcal{M}_{\mathfrak{B}} \equiv \mathcal{N}_{\mathfrak{B}'}$.*

Proof. Consider T, $\mathcal{C}$, $\mathcal{L}'$ and S such as in the proof of Corollary 5.18 above. Let $U = \{\forall x_0 \ldots \forall x_{i-1} \exists x(x = c) : c \in \mathcal{C}$, for some $i < \omega\}$. $T \cup S \cup U$ is finitely satisfiable. Therefore, by compactness, there is an $\mathcal{L}'$-i-urn structure $\mathcal{N}'_{\mathfrak{B}'}$ of cardinality λ such that $\mathcal{N}'_{\mathfrak{B}'} \models T \cup S \cup U$. So, the $\mathcal{L}$-reduct $\mathcal{N}_{\mathfrak{B}'}$ of $\mathcal{N}'_{\mathfrak{B}'}$ is elementary equivalent to $\mathcal{M}_{\mathfrak{B}}$ and, by Corollary 5.17, it is fully eligible. ■

Hence, there are no infinite, strictly categorical urn structures in i-urn semantics. In classical logic, upward Löwenheim-Skolem implies a stronger result according to which every infinite model is *elementary embedded* in models of any greater cardinality. In the following, I prove a similar result for i-urn semantics.

First, based on Definition 3.16, for i-urn structures $\mathcal{M}_{\mathfrak{B}}$ and $\mathcal{N}_{\mathfrak{B}'}$, let me say that there is an *embedding* from $\mathcal{M}_{\mathfrak{B}}$ into $\mathcal{N}_{\mathfrak{B}'}$, in symbols $\mathcal{M}_{\mathfrak{B}} \subseteq \mathcal{N}_{\mathfrak{B}'}$, if and only if $\mathcal{M} \subseteq \mathcal{N}$ and, for every $m < \omega$, $\mathfrak{B}_m \subseteq \mathfrak{B}'_m$. If $\mathcal{M}_{\mathfrak{B}} \subseteq \mathcal{N}_{\mathfrak{B}'}$, for any atomic formula ϕ with free-variables $x_1, \ldots, x_n$, $\mathcal{M}_{\mathfrak{B}} \models_m \phi\, x_1 \ldots x_n[a_1 \ldots a_n] \Leftrightarrow \mathcal{N}_{\mathfrak{B}'} \models_m \phi\, x_1 \ldots x_n[h(a_1) \ldots h(a_n)]$, in which h is the mapping that characterizes the embedding. The following concept of elementary embedding extends this agreement to all formulas.

Definition 5.20 *Let $\mathcal{M}_{\mathfrak{B}}$ and $\mathcal{N}_{\mathfrak{B}'}$ be $\mathcal{L}$-i-urn structures. $\mathcal{M}_{\mathfrak{B}}$ is* elementary embedded *in $\mathcal{N}_{\mathfrak{B}'}$, in symbols $\mathcal{M}_{\mathfrak{B}} \preceq \mathcal{N}_{\mathfrak{B}'}$, if and only if there is an injective function h from the domain of $\mathcal{M}$ to the domain of $\mathcal{N}$ such that:*

- *For any $m, n < \omega$, for every $\mathcal{L}$-formula ϕ with free-variables $x_1, \ldots, x_n$, for any sequence $\bar{a} = \langle a_1, \ldots, a_n \rangle$ of $\mathcal{M}$, $\mathcal{M}_{\mathfrak{B}} \models_m \phi\, x_1 \ldots x_n[\bar{a}] \Leftrightarrow \mathcal{N}_{\mathfrak{B}'} \models_m \phi\, x_1 \ldots x_n[h(a_1) \ldots h(a_n)]$;*
- *For any $m < \omega$ and for every element a of $\mathcal{M}$, $a \in \mathfrak{B}_m \Leftrightarrow h(a) \in \mathfrak{B}'_m$.*

Whenever $\mathcal{M}_{\mathfrak{B}} \preceq \mathcal{N}_{\mathfrak{B}'}$, $\mathcal{N}_{\mathfrak{B}'}$ is said to be an elementary extension *of $\mathcal{M}_{\mathfrak{B}}$.*

For an $\mathcal{L}$-i-urn structure $\mathcal{M}_{\mathfrak{B}}$ of cardinality κ, for a set of new constants $\mathcal{D}(\mathcal{M}_{\mathfrak{B}}) = \{d_i : i < \kappa\}$, for $\mathcal{L}^+ = \mathcal{L} \cup \mathcal{D}(\mathcal{M}_{\mathfrak{B}})$, consider an $\mathcal{L}^+$-expansion $\mathcal{M}^+_{\mathfrak{B}}$ of $\mathcal{M}_{\mathfrak{B}}$ such that $(\cdot)^{\mathcal{M}^+} \restriction \mathcal{D}(\mathcal{M}_{\mathfrak{B}})$ is a bijection onto the domain of $\mathcal{M}$. For every $m < \omega$, the *m-elementary diagram* of $\mathcal{M}_{\mathfrak{B}}$ is the $\mathcal{L}^+$-theory $Diagel(\mathcal{M}_{\mathfrak{B}})_m = \{\phi\, x_1 \ldots x_n[d_1 \ldots d_n] : \phi$ is $\mathcal{L}$-formula with free-variables $x_1, \ldots x_n$, $d_1, \ldots, d_n \in \mathcal{D}(\mathcal{M}_{\mathfrak{B}})$ and $\mathcal{M}^+_{\mathfrak{B}} \models_m \phi\, x_1 \ldots x_n[d_1 \ldots d_n]\}$.

Lemma 5.21 *Let $\mathcal{M}_{\mathfrak{B}}$ and $\mathcal{N}_{\mathfrak{B}'}$ be $\mathcal{L}$-i-urn structures. In i-urn semantics, the following are equivalent:*

1. *$\mathcal{M}_{\mathfrak{B}} \preceq \mathcal{N}_{\mathfrak{B}'}$;*
2. *For some $\mathcal{L}^+$-expansion $\mathcal{N}^+_{\mathfrak{B}'}$ of $\mathcal{N}_{\mathfrak{B}'}$ and for every $m < \omega$, $\mathcal{N}^+_{\mathfrak{B}'} \models_m Diagel(\mathcal{M}_{\mathfrak{B}})_m$.*

Proof. (1 $\Rightarrow$ 2) Fix any formula $\phi\, x_1 \ldots x_n[d_1 \ldots d_n] \in Diagel(\mathcal{M}_{\mathfrak{B}})_m$ and consider $\mathcal{M}^+_{\mathfrak{B}}$ in the definition of $Diagel(\mathcal{M}_{\mathfrak{B}})_m$. Let h be the mapping that defines $\mathcal{M}_{\mathfrak{B}} \preceq \mathcal{N}_{\mathfrak{B}'}$. Let $\mathcal{N}^+_{\mathfrak{B}'}$ be such that, for any $d \in \mathcal{D}(\mathcal{M}_{\mathfrak{B}})$, $d^{\mathcal{N}^+} = h(d^{\mathcal{M}^+})$. Therefore, by 1, $\mathcal{N}^+_{\mathfrak{B}'} \models_m \phi\, x_1 \ldots x_n[d_1 \ldots d_n]$.

(2 $\Rightarrow$ 1) Based on 2, construct a mapping h characterizing elementary embedding. For any element a of $\mathcal{M}$, let $d_a \in \mathcal{D}(\mathcal{M}_{\mathfrak{B}})$ be such that $a = d_a^{\mathcal{M}^+}$. For any element a of $\mathcal{M}$, consider $h(a) = d_a^{\mathcal{N}^+}$. Now, for any elements $a_1, \ldots, a_n$ of $\mathcal{M}$, $\mathcal{M}_{\mathfrak{B}} \models_m \phi\, x_1 \ldots x_n[a_1 \ldots a_n]$ if and only if $\mathcal{M}^+_{\mathfrak{B}} \models_m \phi\, x_1 \ldots x_n[d_{a_1} \ldots d_{a_n}]$. By 2, that is the case if and only if $\mathcal{N}^+_{\mathfrak{B}'} \models_m \phi\, x_1 \ldots x_n[d_{a_1} \ldots d_{a_n}] \Leftrightarrow \mathcal{N}_{\mathfrak{B}'} \models_m \phi\, x_1 \ldots x_n[h(a_1) \ldots h(a_n)]$. ■

Corollary 5.22 *Let $\mathcal{M}_{\mathfrak{B}}$ be an $\mathcal{L}$-i-urn structure with infinite cardinality κ. For any cardinality $\lambda \geq \kappa$, there is an $\mathcal{L}$-i-urn structure $\mathcal{N}_{\mathfrak{B}'}$ of cardinality λ such that $\mathcal{M}_{\mathfrak{B}} \preceq \mathcal{N}_{\mathfrak{B}'}$.*

Proof. Consider $\mathcal{C}$, $\mathcal{L}'$ and S such as in the proof of Corollary 5.18 above. Note that $Diagel(\mathcal{M}_{\mathfrak{B}})_0 \cup S$ is finitely satisfiable. So, by compactness, there is an $\mathcal{L}'$-i-urn structure $\mathcal{N}'_{\mathfrak{B}'}$ of cardinality λ such that $\mathcal{N}'_{\mathfrak{B}'} \models Diagel(\mathcal{M}_{\mathfrak{B}})_0 \cup S$. Further, it is possible to build $\mathcal{N}'_{\mathfrak{B}'}$ such that, for any $m < \omega$, $\mathcal{N}'_{\mathfrak{B}'} \models Diagel(\mathcal{M}_{\mathfrak{B}})_m$. By Lemma 5.21, $\mathcal{M}_{\mathfrak{B}} \preceq \mathcal{N}_{\mathfrak{B}'}$, in which $\mathcal{N}_{\mathfrak{B}'}$ is the $\mathcal{L}$-reduct of $\mathcal{N}'_{\mathfrak{B}'}$. ■

Finally, I show that downward Löwenheim-Skolem holds as well in i-urn semantics. The classical proof of this theorem is naturally generalizable for this non-standard semantics. In order to prove this result, first I need to recover the *Tarski-Vaught test* of elementary embedding in the context of i-urn semantics.

Lemma 5.23 (Tarski-Vaught test) *Let $\mathcal{M}_{\mathfrak{B}}$ and $\mathcal{N}_{\mathfrak{B}'}$ be $\mathcal{L}$-i-urn structures, $\mathcal{M}_{\mathfrak{B}} \subseteq \mathcal{N}_{\mathfrak{B}'}$. In i-urn semantics, the following are equivalent:*

1. *$\mathcal{M}_{\mathfrak{B}} \preceq \mathcal{N}_{\mathfrak{B}'}$;*
2. *For any formula ϕ of $\mathcal{L}$ with free-variables $x_1, \ldots, x_n, y$ and for all elements $a_1, \ldots, a_n$ of $\mathcal{M}$, if $\mathcal{N}_{\mathfrak{B}'} \models_{m+1} \phi\, x_1 \ldots x_n, y[a_1 \ldots a_n, b]$, for some $b \in \mathfrak{B}'_m$, then there is $c \in \mathfrak{B}_m$ such that $\mathcal{N}_{\mathfrak{B}'} \models_{m+1} \phi\, x_1 \ldots x_n, y[a_1 \ldots a_n, c]$.*

Proof. (1 $\Rightarrow$ 2) Assume that $\mathcal{N}_{\mathfrak{B}'} \models_{m+1} \phi\, x_1 \ldots x_n, y[a_1 \ldots a_n, b]$, for some $b \in \mathfrak{B}'_m$. Then, $\mathcal{N}_{\mathfrak{B}'} \models_m \exists y\phi\, x_1 \ldots x_n[a_1 \ldots a_n]$. By 1, $\mathcal{M}_{\mathfrak{B}} \models_m \exists y\phi\, x_1 \ldots x_n[a_1 \ldots a_n]$ and, consequently, $\mathcal{M}_{\mathfrak{B}} \models_{m+1} \phi\, x_1 \ldots x_n, y[a_1 \ldots a_n, c]$, for some $c \in \mathfrak{B}_m$. By 1 again, $\mathcal{N}_{\mathfrak{B}'} \models_{m+1} \phi\, x_1 \ldots x_n, y$ $[a_1 \ldots a_n, c]$.

(2 $\Rightarrow$ 1) Assuming 2, 1 is obtained by induction on ϕ. The atomic case and the cases in which ϕ is either $\neg\psi$, $\psi \wedge \gamma$ or $\psi \vee \gamma$ depend on $\mathcal{M}_{\mathfrak{B}} \subseteq \mathcal{N}_{\mathfrak{B}'}$.

If ϕ is $\exists x\psi$, then $\mathcal{M}_{\mathfrak{B}} \models_m \exists x\psi \Rightarrow \mathcal{N}_{\mathfrak{B}'} \models_m \exists x\psi$, by inductive hypothesis. Further, $\mathcal{N}_{\mathfrak{B}'} \models_m \exists x\psi \Rightarrow \mathcal{M}_{\mathfrak{B}} \models_m \exists x\psi$, by 2.

The case in which ϕ is $\forall x\psi$ is a dual: $\mathcal{N}_{\mathfrak{B}'} \models_m \forall x\psi \Rightarrow \mathcal{M}_{\mathfrak{B}} \models_m \forall x\psi$, by inductive hypothesis. Further, $\mathcal{M}_{\mathfrak{B}} \models_m \forall x\psi \Rightarrow \mathcal{N}_{\mathfrak{B}'} \models_m \forall x\psi$, by 2. ■

Definition 5.24 *Let T be a set of formulas of $\mathcal{L}$. T has* Skolem functions *if and only if, for any formula ϕ of $\mathcal{L}$ with free-variables $x_1, \ldots, x_n, y$, there is an n-ary function $f_\phi \in \mathcal{L}$ such that $(\neg\exists y\phi \vee \forall z\phi\, y[f_\phi(\bar{x})]) \in T$, for some variable z that does not occur in ϕ.*

Theorem 5.25 *Let $\mathcal{M}_{\mathfrak{B}}$ be an $\mathcal{L}$-i-urn structure with infinite cardinality κ. Then, for any infinite cardinal $\lambda \leq \kappa$, there is an $\mathcal{L}$-i-urn structure $\mathcal{N}_{\mathfrak{B}'}$ of cardinality λ and such that $\mathcal{N}_{\mathfrak{B}'} \preceq \mathcal{M}_{\mathfrak{B}}$.*

Proof. First, recursively define an extension $\mathcal{L}' \supset \mathcal{L}$ as follows:

- $\mathcal{L}_0 = \mathcal{L}$;
- $\mathcal{L}_{i+1} = \mathcal{L}_i \cup \{f_{m,\phi} : m < \omega,\ f_{m,\phi}$ is a n-ary Skolem function, ϕ is $\mathcal{L}_i$-formula with free-variables $x_1, \ldots, x_n, y$, for some $n < \omega\}$;
- $\mathcal{L}' = \bigcup\limits_{i<\omega} \mathcal{L}_i$.

For each $m < \omega$, let $T(m) = \{\phi : \phi$ is $\mathcal{L}$-formula and $\mathcal{M}_{\mathfrak{B}} \models_m \phi\}$. Secondly, define sets of $\mathcal{L}'$-formulas $T(m)' \supset T(m)$ that have Skolem functions in the following way:

- $T(m)_0 = T(m)$;
- For $\bar{x} = \langle x_1, \ldots, x_n\rangle$, let $T(m)_{i+1} = T(m)_i \cup \{(\neg\exists y\phi \vee \forall z\phi\, y[f_{m,\phi}(\bar{x})]) :\ \phi$ is $\mathcal{L}_i$-formula with free-variables $x_1, \ldots, x_n, y$, for some variable z that does not occur in $\phi\}$;
- $T(m)' = \bigcup\limits_{i<\omega} T(m)_i$.

Consider $\mathcal{M}'_{\mathfrak{B}}$ is an $\mathcal{L}'$-expansion of $\mathcal{M}_{\mathfrak{B}}$ such that, for every Skolem function $f_{m,\phi}$ and for any elements $a_1, \ldots, a_n$ of $\mathcal{M}$, if, for some $b \in \mathfrak{B}_m$, $\mathcal{M}'_{\mathfrak{B}} \models_{m+1} \phi\, x_1 \ldots x_n y[a_1 \ldots a_n, b]$, then $(f_{m,\phi}(a_1, \ldots, a_n))^{\mathcal{M}'} = b$; otherwise, $(f_{m,\phi}(a_1, \ldots, a_n))^{\mathcal{M}'}$ is any $c \in \mathfrak{B}_m$. Clearly, for any $m < \omega$, $\mathcal{M}'_{\mathfrak{B}} \models_m T(m)'$.

Consider any set of elements X such that $X \cap \mathfrak{B}_0 \neq \emptyset$. Define by recursion an extension $X' \supseteq X$ as follows:

- Let $X_0 = X$;
- Let $X_{i+1} = X_i \cup \{f_{m,\phi}(a_1, \ldots, a_n)^{\mathcal{M}'} : m, n < \omega,\ a_1, \ldots, a_n \in X_i\}$;
- $X' = \bigcup\limits_{i<\omega} X_i$.

For any $m < \omega$, let $\mathfrak{B}'_m = \mathfrak{B}_m \cap X'$. Let $\mathfrak{B}'$ be the collection of such sets $\mathfrak{B}'_m$. Let $\mathcal{N}_{\mathfrak{B}'} \subseteq \mathcal{M}_{\mathfrak{B}}$ be the substructure generated by X'. Based on Lemma 5.23, I will prove that $\mathcal{N}_{\mathfrak{B}'} \preceq \mathcal{M}_{\mathfrak{B}}$.

Fix some $\mathcal{L}$-formula ϕ with free-variables $x_1, \ldots, x_n, y$ and elements $a_1, \ldots, a_n$ of $\mathcal{N}$. Assume that, for some $b \in \mathfrak{B}_m$, $\mathcal{M}_{\mathfrak{B}} \models_{m+1} \phi\, x_1 \ldots x_n, y[a_1 \ldots a_n, b]$. So, $b = f_{m,\phi}(a_1, \ldots, a_n)^{\mathcal{M}'}$ and, therefore, $b \in X' \cap \mathfrak{B}'_m$. Hence, by Lemma 5.23, $\mathcal{N}_{\mathfrak{B}'} \preceq \mathcal{M}_{\mathfrak{B}}$. ■

As a final remark, note that if $\mathcal{M}_{\mathfrak{B}}$ is fully eligible, downward Löwenheim-Skolem always produce fully eligible elementary substructures, on the contrary to the upward case.

5.3.2 Finite categoricity in i-urn semantics

Theorem 5.5 above shows that there is not finite strict categoricity in p-urn semantics. In this sense, p-urn semantics radically diverges from classical logic since all classical finite structures are categorical. I show in this section that, in a certain sense, i-urn semantics offers an intermediate scenario between these logical systems.

As in the previous subsection, the comparison with classical logic is again quite fruitful. In classical logic, finite categoricity follows from the fact that the sentence "there are exactly n elements", for some $n < \omega$, is first order expressible. I explore here a similar strategy in order to obtain some initial result about finite categoricity in i-urn semantics: I show that, under special conditions, a class of finite i-urn structures which agree in all unnested sentences have sets of eligible sequences of same cardinality. In the rest of this section, I always assume that $\mathcal{L}$ is a finite language. Furthermore, I explore once again the set of notions employed in subsections 3.3.2–3.3.4 and sometimes I rely on the framework of urn game semantics (Definition 3.2).

The following theorem shows that, for any $n, m < \omega$, the satisfaction of the property "there are *at least* n elements occurring in the m-th position of some eligible sequence" by i-urn structures is associated with some syntactic properties of the game-normal sentences that they verify.

Theorem 5.26 *Let $\mathcal{M}_{\mathfrak{B}}$ be an $\mathcal{L}$-i-urn structure and assume that, for some $m < \omega$, the cardinality of $\mathfrak{B}_m$ is greater or equal than n. Let $\theta \in \Theta(m+1, \emptyset)$ be the game-normal sentence such that $\mathcal{M}_{\mathfrak{B}} \models \theta$. In i-urn semantics, the following are equivalent:*

1. *For some $\theta' \in \Theta(1, y_{m+1}, \ldots, y_2)$ subformula of θ, θ' has r existential subformulas, for some $r \geq n$;*
2. *For every $\mathcal{N}_{\mathfrak{B}'}$ that agrees with $\mathcal{M}_{\mathfrak{B}}$ in all unnested sentences, $\mathfrak{B}'_m$ has cardinality greater or equal than n.*

Proof. ($1 \Rightarrow 2$) Fix some $\mathcal{N}_{\mathfrak{B}'}$ that agrees with $\mathcal{M}_{\mathfrak{B}}$ in all unnested sentences. Then, $\mathcal{N}_{\mathfrak{B}'} \models \theta$. So, based on the syntactic structure of θ we know that, for an eligible sequence $\langle a_{m+1}, \ldots, a_2 \rangle$ of $\mathcal{N}$ over $\mathfrak{B}'$, $\mathcal{N}_{\mathfrak{B}'} \models_m \theta' \; y_{m+1} \ldots y_2[a_{m+1} \ldots a_2]$.

θ' is of the form $\exists y \theta''_1 \wedge \ldots \wedge \exists y \theta''_r \wedge \forall y \bigvee_{0<j\leq r} \theta''_j$. Given that any two game normal formulas with same quantifier rank and same set of free-variables are jointly inconsistent (Lemma 3.19), there are at least $b_1, \ldots, b_r \in \mathfrak{B}'_m$ such that $\mathcal{N}_{\mathfrak{B}'} \models_{m+1} \theta''_j \; y_{m+1} \ldots y_1[a_{m+1} \ldots a_2, b_j]$, for $0 < j \leq r$.

($2 \Rightarrow 1$) Assume 1 is not the case. It is possible to generate an $\mathcal{L}$-i-urn structure $\mathcal{N}_{\mathfrak{B}'}$ that agrees with $\mathcal{M}_{\mathfrak{B}}$ in all unnested sentences such that $\mathfrak{B}'_m$ has cardinality smaller than n in the following way.

Consider the set $\Lambda = \{\theta_k$ is game-normal sentence $: k < \omega, qr(\theta_k) = k+1, \mathcal{M}_{\mathfrak{B}} \models \theta_k\}$. For any $k < q < \omega$, θ_k is equivalent to $\theta_q\overline{(y_{k+2}, \ldots, y_{q+1})}$. So, by Theorem 3.34, it is possible to construct a "canonical" model $\mathcal{N}_{\mathfrak{B}'}$ satisfying Λ. By the negation of condition 1, any game-normal formula occurring in θ_m has less than n existential subformulas. Therefore, we need to consider only less than n "witnesses" in the definition of $\mathfrak{B}'_m$. The situation does not change when we consider the remainder sentences in $\{\theta_k : k < \omega\}$, given the equivalence between θ_m and $\theta_q\overline{(y_{m+2}, \ldots, y_{q+1})}$, for any $m < q < \omega$. By Theorem 3.21, $\mathcal{N}_{\mathfrak{B}'}$ and $\mathcal{M}_{\mathfrak{B}}$ agree in all unnested sentences. ■

Theorem 5.26 shows how it is possible to enforce in i-urn semantics the minimum finite cardinality of the elements of an eligibility set. In turn, the following results show that in i-urn semantics it is also possible to enforce the satisfaction of the property "there are *exactly* n elements occurring in the m-th position of some eligible sequence" by finite i-urn structures.

First, let me define a notion of *indiscernibility* for i-urn semantics. As I have done in section 5.2, for any eligible sequence $\bar{a}$ of a structure $\mathcal{M}$ over $\mathfrak{B}$, I consider below the set $I(\bar{a}, \mathfrak{B})$ of eligible extensions of $\bar{a}$ over $\mathfrak{B}$.

Definition 5.27 *Let $\mathcal{M}_{\mathfrak{B}}$ be an $\mathcal{L}$-i-urn structure, $\mathfrak{M}$ be the set of eligible sequences of $\mathcal{M}$ over $\mathfrak{B}$ and $\bar{b} = \langle b_0, \ldots, b_{m-1}\rangle \in \mathfrak{M}$. A set A is* m-indiscernible over $\bar{b}$ *if and only if the following holds:*

- *$A \cap \mathfrak{B}_m \neq \emptyset$;*
- *For all elements $a, b \in A$ and for every $q > m$, $a \in \mathfrak{B}_q \Leftrightarrow b \in \mathfrak{B}_q$;*
- *For every $a \in A \cap \mathfrak{B}_m$, for all $n < r < \omega$, for all eligible sequences $\bar{d} = \langle d_0, \ldots, d_r\rangle \in I(\bar{b} \frown (a), \mathfrak{B})$, for any segment $\langle c_0, \ldots, c_n\rangle$ of $\bar{d}$ and for every n-ary predicate $P \in \mathcal{L}$, $\mathcal{M}$ satisfies $P(c_0 \ldots c_n)$ if and only if it satisfies $P(\bar{b}')$, for any sequence $\bar{b}'$ generated from $\langle c_0, \ldots, c_n\rangle$ by some permutation of A.*

If, for every $\bar{a} \in \mathfrak{M}_{m-1}$, A is m-indiscernible over $\bar{a}$, then I say simply that A is m-indiscernible.

In i-urn semantics, agreement in all unnested sentences does not necessarily preserve m-indiscernibility. In order to see this consider the following simple example.

Example 5.28 *Let $\mathcal{M}_{\mathfrak{B}}$ and $\mathcal{N}_{\mathfrak{B}'}$ be $\mathcal{L}$-i-urn structures with domains $\{b_1, \ldots, b_6\}$ and $\{c_1, \ldots, c_6\}$, respectively. Assume that $\mathfrak{B}_0 = \{b_1, b_2, b_3\}$, $\mathfrak{B}_1 = \{b_4\}$ and $\mathfrak{B}_m = \{b_4, b_5, b_6\}$, for every $m > 1$. Analogously, let $\mathfrak{B}'_0 = \{c_1, c_2, c_3\}$, $\mathfrak{B}'_1 = \{c_4\}$ and $\mathfrak{B}'_m = \{c_4, c_5, c_6\}$, for every $m > 1$. Further, let $\mathcal{L} = \{P\}$, for a binary predicate P. These urn structures satisfy the following formulas:*

- *For $i = 1, 2$ or 3, $\mathcal{M}$ satisfies $P(b_i\, b_4)$ $P(b_i\, b_5)$, but $\mathcal{M}$ satisfies $\neg P(b_i\, b_6)$ as well. So, $\{b_4, b_5\}$ is m-indiscernible.*

- *On the other hand, $\mathcal{N}$ satisfies $P(b_i\, b_4)$, $P(b_1\, b_5)$, $P(b_2\, b_6)$, $P(b_3\, b_6)$, but it safisfies as well $\neg P(b_2\, b_5)$, $\neg P(b_3\, b_5)$, $\neg P(b_1\, b_6)$. So, there is not m-indiscernibility in $\mathcal{N}_{\mathfrak{B}'}$ but there is an m-indiscernible set over each element in $\mathfrak{B}'_0$.*

However, it is possible to prove a slightly weaker result.

Lemma 5.29 *Assume that $\mathcal{M}_{\mathfrak{B}}$ and $\mathcal{N}_{\mathfrak{B}'}$ are finite $\mathcal{L}$-i-urn structures that agree in all unnested sentences, and $\mathfrak{M}$ and $\mathfrak{N}$ are sets of eligible sequences of $\mathcal{M}$ over $\mathfrak{B}$ and of $\mathcal{N}$ over $\mathfrak{B}'$, respectively. For some $m < \omega$, let $\mathfrak{B}_m$ be an m-indiscernible set such that, for every $q < m$, $\mathfrak{B}_m \cap \mathfrak{B}_q = \emptyset$ and, for some $r > m$, $\mathfrak{B}_m \subseteq \mathfrak{B}_r$. Then, for any $r > m$ such that $\mathfrak{B}_m \subseteq \mathfrak{B}_r$, for every eligible sequence $\bar{b} \in \mathfrak{N}_{m-1}$, for every $a \in \mathfrak{B}'_m$ and for each eligible sequence $\bar{b}' \in \mathfrak{N}_{r-1}$ that extends $\bar{b} \frown (a)$, there is an element $a' \in \mathfrak{B}'_r$ such that $\{a, a'\}$ is r-indiscernible over $\bar{b}'$.*

Proof. Assume that, for some $r > m$ such that $\mathfrak{B}_m \subseteq \mathfrak{B}_r$, for some $\bar{b} \in \mathfrak{N}_{m-1}$, for some $a \in \mathfrak{B}'_m$, for some eligible sequence $\bar{b}' \in \mathfrak{N}_{r-1}$ that extends $\bar{b} \frown (a)$, there is no element $a' \in \mathfrak{B}'_r$ such that $\{a, a'\}$ is r-indiscernible over $\bar{b}'$.

So, consider the set $\Lambda = \{\theta_k \in \Theta(k+1, x_0, \ldots, x_{r-1}) : k < \omega, \mathcal{N}_{\mathfrak{B}'} \models_r \theta_k\, x_0 \ldots x_{r-1}[\bar{b}']\}$. Given that $\mathcal{M}_{\mathfrak{B}}$ and $\mathcal{N}_{\mathfrak{B}'}$ agree in all unnested sentences, $\mathcal{M}_{\mathfrak{B}}$ is finite and, for any $k < q < \omega$, θ_k is equivalent to $\theta_q\overline{(y_{k+2}, \ldots, y_{q+1})}$, then there is some eligible sequence $\bar{c}' \in \mathfrak{B}_{r-1}$ such that $\mathcal{M}_{\mathfrak{B}} \models_r \theta_k\, x_0 \ldots x_{r-1}[\bar{c}']$, for every $k < \omega$.

Consider the initial segment $\bar{c} \frown (d) \in \mathfrak{M}_m$ of $\bar{c}'$. By the m-indiscernibility of $\mathfrak{B}_m$, there is some $d' \in \mathfrak{B}_r$ that is m-indiscernible from d over $\bar{c}$. So, consider the set $\Lambda' = \{\sigma_k \in \Theta(k+1, x_0, \ldots, x_r) : k < \omega, \mathcal{M}_{\mathfrak{B}} \models_{r+1} \sigma_k\, x_0 \ldots x_r[\bar{c}' \frown (d')]\}$. Since $\mathcal{M}_{\mathfrak{B}}$ and $\mathcal{N}_{\mathfrak{B}'}$ agree in Λ, $\mathcal{N}_{\mathfrak{B}'}$ is finite and, for any $k < q < \omega$, σ_k is equivalent to $\sigma_q\overline{(y_{k+2}, \ldots, y_{q+1})}$, then there is some $a' \in \mathfrak{B}'_r$ such that $\mathcal{N}_{\mathfrak{B}'} \models_{r+1} \sigma_k\, x_0 \ldots x_r[\bar{b}' \frown (a')]$, for every $k < \omega$. Therefore, $\{a, a'\}$ is r-indiscernible over $\bar{b}'$, what contradicts the original assumption. ■

The following theorem presents sufficient and necessary conditions for the satisfaction of the property "there is *only* 1 element occurring in the m-th position of eligible sequences" by finite i-urn structures. In what follows, for an eligibility set $\mathfrak{B}$, let $R(\mathfrak{B}_m) = \{\mathfrak{B}_r : r > m, \mathfrak{B}_m \subseteq \mathfrak{B}_r\}$.

Theorem 5.30 *Let $\mathcal{M}_{\mathfrak{B}}$ be a finite $\mathcal{L}$-i-urn structure such that $\mathfrak{B}_m = \{a\}$. Let $\mathfrak{M}$ be the set of eligible sequences of $\mathcal{M}$ over $\mathfrak{B}$. In i-urn semantics, the following are equivalent:*

1. *At least one of the following facts is the case:*

   i) *For some $n < m$, $\mathfrak{B}_m \subseteq \mathfrak{B}_n$;*

   ii) *For some $\mathfrak{B}_r \in R(\mathfrak{B}_m)$, for some eligible sequence $\bar{b} \in \mathfrak{M}_{m-1}$ and for some eligible sequence $\bar{b}' \in \mathfrak{M}_{r-1}$ that extends $\bar{b} \frown (a)$ there is no element $a' \in \mathfrak{B}_r$ such that $\{a, a'\}$ is r-indiscernible over $\bar{b}'$;*

2. *For every finite $\mathcal{L}$-i-urn structure $\mathcal{N}_{\mathfrak{B}'}$ that agrees with $\mathcal{M}_{\mathfrak{B}}$ in all unnested sentences, $\mathfrak{B}'_m$ is a singleton.*

Proof. ($1 \Rightarrow 2$) Assume 1 and consider the following cases:

- Assume that fact i) is the case. Then, just consider that $\mathcal{M}_{\mathfrak{B}} \models \forall x_0 \ldots \forall x_{n-1} \exists x_n \forall x_{n+1} \ldots \forall x_m \, (x_n = x_m)$;

- On the other hand, assume that ii) is the case. Suppose that there is a finite $\mathcal{L}$-i-urn structure $\mathcal{N}_{\mathfrak{B}'}$ that agrees with $\mathcal{M}_{\mathfrak{B}}$ in all unnested sentences and such that the cardinality of $\mathfrak{B}'_m$ is greater than 1. Then, $\mathfrak{B}'_m$ is m-indiscernible, otherwise $\mathcal{N}_{\mathfrak{B}'}$ and $\mathcal{M}_{\mathfrak{B}}$ would disagree in some game-normal sentence, contradicting their agreement in all unnested sentences. Therefore, by Lemma 5.29, for any $\mathfrak{B}'_r \in R(\mathfrak{B}'_m)$, for every eligible sequence $\bar{b} \in \mathfrak{M}_{m-1}$ and for each eligible sequence $\bar{b}' \in \mathfrak{M}_{r-1}$ that extends $\bar{b} \frown (a)$, there is an element $a' \in \mathfrak{B}_r$ such that $\{a, a'\}$ is r-indiscernible over $\bar{b}'$. However, this contradicts condition ii).

($2 \Rightarrow 1$) Assume 2 and suppose that 1 is not the case. Then, it is the case that:

i') For every $n < m$, $\mathfrak{B}_n \cap \mathfrak{B}_m = \emptyset$;

ii') For every $\mathfrak{B}_r \in R(\mathfrak{B}_m)$, for every eligible sequence $\bar{b} \in \mathfrak{M}_{m-1}$ and for every eligible sequence $\bar{b}' \in \mathfrak{M}_{r-1}$ that extends $\bar{b} \frown (a)$ there is some element $a' \in \mathfrak{B}_r$ such that $\{a, a'\}$ is r-indiscernible over $\bar{b}'$.

Adding some new element c to the domain of $\mathcal{M}$, it is possible to define extensions $\mathcal{N} \supset \mathcal{M}$ and $\mathfrak{B}' \supset \mathfrak{B}$ such that:

- For every $n < \omega$, if $a \in \mathfrak{B}_n$, then $\mathfrak{B}'_n = \mathfrak{B}_n \cup \{c\}$; otherwise, $\mathfrak{B}'_n = \mathfrak{B}_n$;

- For any sequence $\langle a_1, \ldots, a_n \rangle$ of $\mathcal{N}$ in which c occurs, for any n-ary predicate $P \in \mathcal{L}$, $\mathcal{N}$ satisfies $P(a_1 \ldots a_n)$ if and only if $\mathcal{N}$ satisfies $P(\bar{a}')$, in which $\bar{a}'$ is the sequence generated from $\langle a_1, \ldots, a_n \rangle$ by permutation of a with c in it.

Since $\mathcal{M}_{\mathfrak{B}}$ is finite, $\mathcal{N}_{\mathfrak{B}'}$ is it as well. Moreover, $\{a, c\}$ is m-indiscernible. For each $m < \omega$, consider the game-normal sentence $\theta_m \in \Theta(m+1, \emptyset)$ such that $\mathcal{N}_{\mathfrak{B}'} \models \theta_m$. Using urn game semantics, I will show that $\mathcal{N}_{\mathfrak{B}'}$ and $\mathcal{M}_{\mathfrak{B}}$ agree in all such sentences and consequently, by Theorem 3.21, agree in all unnested sentences.

Fix any $k \geq m$. By Theorem 3.4, player E has winning strategy δ in $\mathfrak{G}(\theta_k, \mathcal{N}_{\mathfrak{B}'})$. I will show that player E has winning strategy in $\mathfrak{G}(\theta_k, \mathcal{M}_{\mathfrak{B}})$ as well. Let p be a match induced by δ in $\mathfrak{G}(\theta_k, \mathcal{N}_{\mathfrak{B}'})$. Consider the following three cases:

- First, if c is never considered in p, then p is a winning match for E in $\mathfrak{G}(\theta_k, \mathcal{M}_{\mathfrak{B}})$ as well;

- Secondly, assume that c is considered in p but a is not. Then, let p' be the variation of p such that all uses of c in p are substituted by uses of a. Given that $\{a, c\}$ is m-indiscernible, p' is a winning match for E in $\mathfrak{G}(\theta_k, \mathcal{M}_{\mathfrak{B}})$;

- Finally, assume that both a and c are considered in p and let $\bar{d}$ be the whole eligible sequence that is considered in p. By i') and ii'), there is a' such that, for the first r-th position in which c was elected in p, $\{a, a'\}$ is r-indiscernible over $\bar{d}'$, in which $\bar{d}'$ is the initial segment with length r of the eligible sequence generated from $\bar{d}$ by substituting c by a' in it. Let p' be the variation of p in which all uses of c in p are substituted by uses of a'. The match p' is a winning for E in $\mathfrak{G}(\theta_k, \mathcal{M}_{\mathfrak{B}})$.

So, $\mathcal{N}_{\mathfrak{B}'}$ and $\mathcal{M}_{\mathfrak{B}}$ agree in all unnested sentences but $\mathfrak{B}'_m$ is not a singleton, what contradicts 2. ■

Finally, I consider the more challenging question of whether it is possible to enforce the satisfaction of the property "there is *exactly* n elements occurring in the m-th position of some eligible sequence" by finite i-urn structures, for $n > 1$. As the reader may see, the following propositions represent partial results on this difficult issue which demand further generalization.

Proposition 5.31 *Let $\mathcal{M}_{\mathfrak{B}}$ be a finite $\mathcal{L}$-i-urn structure and, for some $m < \omega$, $\mathfrak{B}_m = \{a_1, \ldots, a_n\}$. Let the game-normal sentence $\theta \in \Theta(m+1, \emptyset)$ be such that $\mathcal{M}_{\mathfrak{B}} \models \theta$ and, for some $\theta^* \in \Theta(1, y_1, \ldots, y_m)$ subformula of θ, consider that θ^* has n existential subformulas. Then, the item 1 below implies item 2:*

1. *There are $k_1 < \ldots < k_n < m$ such that $a_i \in \mathfrak{B}_{k_i}$;*

2. *For every finite $\mathcal{L}$-i-urn structure $\mathcal{N}_{\mathfrak{B}'}$ that agrees with $\mathcal{M}_{\mathfrak{B}}$ in all unnested sentences, the cardinality of $\mathfrak{B}'_m$ is n.*

Proof. By Theorem 5.26, for every such $\mathcal{N}_{\mathfrak{B}'}$, the cardinality of $\mathfrak{B}'_m$ is at least n. Suppose that there is one such model such that the cardinality of $\mathfrak{B}'_m$ is greater than n. So, note that

$$\mathcal{M}_{\mathfrak{B}} \models Qx_0 \ldots Qx_{m-1}[\forall x_m \bigvee_{0<i\leq n} x_m = x_{k_i}],$$

in which, for every $j < m$, if $j = k_i$ for some $0 < i \leq n$, then Qx_j is $\exists x_j$; otherwise, Qx_j is $\forall x_j$. However, $\mathcal{N}_{\mathfrak{B}'}$ does not satisfy this sentence, what contradicts the agreement between these models in all unnested sentences. ■

In what follows, for any i-eligibility set $\mathfrak{B}$ and $m < \omega$, let $R(a, m) = \{\mathfrak{B}_r : m < r, a \in \mathfrak{B}_r\}$.

Notation 5.32 *Let $\mathcal{M}_{\mathfrak{B}}$ be an i-urn structure such that $\mathfrak{B}_m = \{a_1, \ldots, a_n\}$, and let $\mathfrak{M}$ the set of eligible sequences of $\mathcal{M}$ over $\mathfrak{B}$. In what follows, I will make use of the following notions:*

- *For some $0 < i \leq n$ and for any $\mathfrak{B}_r \in R(a_i, m)$, if there is some $d \in \mathfrak{B}_r$ such that, for some eligible sequence $\bar{b} \in \mathfrak{M}_{r-1}$, $\{a_i, d\}$ is r-indiscernible over $\bar{b}$, then I denote d as $d(i, \bar{b})$. Moreover, let $D(a_i)$ be the set of such elements;*

- *For some $\bar{b} \in \mathfrak{M}_{m-1}$, I say that $D(a_i)$ includes a* chain of indiscernibles starting in $\bar{b}$ *if and only if there are $d(i, \bar{b}_0), \ldots, d(i, \bar{b}_q)$ such that $\bar{b}_0$ extends $\bar{b} \frown (a_i)$ and, for every $0 < j \leq q$, $\bar{b}_j$ extends $\bar{b}_{j-1} \frown (d(i, \bar{b}_{j-1}))$.*

Proposition 5.33 *Let $\mathcal{M}_{\mathfrak{B}}$, θ and θ^* be as in Proposition 5.31 above and let $\mathfrak{M}$ be the set of eligible sequences of $\mathcal{M}$ over $\mathfrak{B}$. Then, the item 1 below implies item 2:*

1. *For some eligible sequence $\bar{b} \in \mathfrak{M}_{m-1}$, for every $0 < i \leq n$, the greatest chain of indiscernibles in $D(a_i)$ that start in $\bar{b}$ is smaller then the cardinality of $R(a_i, m)$;*

2. *For every finite $\mathcal{L}$-i-urn structure $\mathcal{N}_{\mathfrak{B}'}$ that agrees with $\mathcal{M}_{\mathfrak{B}}$ in all unnested sentences, the cardinality of $\mathfrak{B}'_m$ is n.*

Proof.

By Theorem 5.26, for every such $\mathcal{N}_{\mathfrak{B}'}$, the cardinality of $\mathfrak{B}'_m$ is at least n. As in the proof of Proposition 5.31, suppose that there is one such model such that the cardinality of $\mathfrak{B}'_m$ is greater than n. Let $\mathfrak{N}$ be the set of eligible sequences of $\mathcal{N}$ over $\mathfrak{B}'$. So, the following fact holds:

($*$) For every eligible sequence $\bar{c} \in \mathfrak{N}_{m-1}$, there are $a, a' \in \mathfrak{B}'_m$ such that $\{a, a'\}$ is m-indiscernible over $\bar{c}$.

Consider $\Lambda = \{\theta_k \in \Theta(k+1, x_0, \ldots, x_{m-1}) : k < \omega, \mathcal{M}_{\mathfrak{B}} \models_m \theta_k\, x_0 \ldots x_{m-1}[\bar{b}]\}$. given that $\mathcal{M}_{\mathfrak{B}}$ and $\mathcal{N}_{\mathfrak{B}'}$ agree in all unnested sentences, $\mathcal{N}_{\mathfrak{B}'}$ is finite and, for any $k < q < \omega$, θ_k is equivalent to $\theta_q\overline{(y_{k+2}, \ldots, y_{q+1})}$, then there is an eligible sequence $\bar{c} \in \mathfrak{N}_{m-1}$ such that $\mathcal{N}_{\mathfrak{B}'} \models_m \theta_k\, x_0 \ldots x_{m-1}[\bar{c}]$, for every $k < \omega$.

Now, based on fact ($*$), consider elements $a, a' \in \mathfrak{B}'_m$ which are m-indiscernible over $\bar{c}$. Consider $\Lambda' = \{\theta'_k \in \Theta(k+1, x_0, \ldots, x_m) : k < \omega, \mathcal{N}_{\mathfrak{B}'} \models_{m+1} \theta'_k\, x_0 \ldots x_m[\bar{c} \frown (a)]\}$. Again, given that $\mathcal{M}_{\mathfrak{B}}$ and $\mathcal{N}_{\mathfrak{B}'}$ agree in Λ, $\mathcal{M}_{\mathfrak{B}}$ is finite and, for any $k < q < \omega$, θ'_k is equivalent to $\theta'_q\overline{(y_{k+2}, \ldots, y_{q+1})}$, then there is some $a_i \in \mathfrak{B}_m$ such that $\mathcal{M}_{\mathfrak{B}} \models_{m+1} \theta'_k\, x_0 \ldots x_m[\bar{b} \frown (a_i)]$.

Let $d(i, \bar{b}_0), \ldots, d(i, \bar{b}_q)$ be the greatest chain of indiscernibles in $D(a_i)$ that start in $\bar{b}$ and let $r > m$ be such that $\bar{b}_q \in \mathfrak{M}_{r-1}$. Since the cardinality of $R(a_i, m)$ is greater than $q+1$, there is some $s > r$ such that $\mathfrak{B}_s \in R(a_i, m)$. So, consider some eligible sequence $\bar{b}' \in \mathfrak{M}_{s-1}$ that extends $\bar{b}_q \frown (d(i, \bar{b}_q))$. Further, consider $\Lambda'' = \{\theta''_k \in \Theta(k+1, x_0, \ldots, x_{s-1}) : k < \omega, \mathcal{M}_{\mathfrak{B}} \models_s \theta''_k\, x_0 \ldots x_{s-1}[\bar{b}']\}$. Given that $\mathcal{M}_{\mathfrak{B}}$ and $\mathcal{N}_{\mathfrak{B}'}$ agree in Λ', $\mathcal{N}_{\mathfrak{B}'}$ is finite and, for any $k < q < \omega$, θ''_k is equivalent to $\theta''_q\overline{(y_{k+2}, \ldots, y_{q+1})}$, then there is some eligible sequence $\bar{c}' \in \mathfrak{N}_{s-1}$ that extends $\bar{c} \frown (a)$ such that $\mathcal{N}_{\mathfrak{B}'} \models_s \theta''_k\, x_0 \ldots x_{s-1}[\bar{c}']$.

Consider $\Lambda^+ = \{\theta^+_k \in \Theta(k+1, x_0, \ldots, x_s : k < \omega, \mathcal{N}_{\mathfrak{B}'} \models_{s+1} \theta^+_k\, x_0 \ldots x_s[\bar{c}' \frown (a')]\}$. Given that $\mathcal{M}_{\mathfrak{B}}$ and $\mathcal{N}_{\mathfrak{B}'}$ agree in Λ'', $\mathcal{M}_{\mathfrak{B}}$ is finite and, for any $k < q < \omega$, θ^+_k is equivalent to

$\theta_q^+\overline{(y_{k+2},\ldots,y_{q+1})}$, then there is some $d \in \mathfrak{B}_r$ such that $\mathcal{M}_{\mathfrak{B}} \models_{s+1} \theta_k^+\, x_0 \ldots x_s[\bar{b}' \frown (d)]$. But then $\{a_i, d\}$ is s-indiscernible over $\bar{b}'$, what contradicts the fact that $d(i, \bar{b}_0), \ldots, d(i, \bar{b}_q)$ is the greatest chain of indiscernibles in $D(a_i)$ that start in $\bar{b}$. ■

Propositions 5.31–5.33 give sufficient conditions for the satisfaction of "there are exactly n (for $n > 1$) elements occurring in the m-th position of some eligible sequence" by finite i-urn structures. It is still an open question whether it is possible to give a complete set of sufficient and necessary conditions for the satisfaction of this property. In fact, I conjecture that we can obtain such a set by adding one more condition comparing the behaviors of all elements $a, b \in \mathfrak{B}_m$ with respect to $I(\bar{c}, \mathfrak{B})$ and $I(\bar{d}, \mathfrak{B})$, for different eligible sequences $\bar{c}, \bar{d} \in \mathfrak{M}_{m-1}$. However, the obtainment of such a set of sufficient and necessary conditions for the satisfaction of that property depends on a more thorough study of back and forth equivalence and definability in i-urn semantics.

Anyway, based on the results obtained in this subsection, it is already possible to present an interesting conclusion about finite categoricity in i-urn semantics.

Corollary 5.34 *For some fully eligible finite $\mathcal{L}$-i-urn structure $\mathcal{M}_{\mathfrak{B}}$, for any fully eligible finite $\mathcal{L}$-i-urn structure $\mathcal{N}_{\mathfrak{B}'}$, if $\mathcal{M}_{\mathfrak{B}}$ and $\mathcal{N}_{\mathfrak{B}'}$ agree in all unnested sentences, then there is an isomorphism between $\mathcal{M}_{\mathfrak{B}}$ and $\mathcal{N}_{\mathfrak{B}'}$.*

Note that this does not mean that there are strictly categorical finite i-urn structures. Informally speaking, corollary 5.34 considers a somewhat weaker notion of being strictly categorical *with respect to* finite i-urn structures. In other words, i-urn semantics represents a weakening of classical logic's already weak capacity to measure the cardinality of sets. In classical logic, the cardinality of finite sets is first order expressible: in this sense, a classical structure $\mathcal{M}$ has cardinality n (for $n < \omega$) if and only if $\mathcal{M}$ satisfies the sentence $\exists x_1 \ldots \exists x_n[\bigwedge_{0<i<j\leq n} x_i \neq x_j \wedge \forall y \bigvee_{0<i\leq n} y = x_i]$. On the other hand, in i-urn semantics cardinality n is not first order expressible *tout court*. In fact, the results obtained in this section show that in i-urn semantics cardinality n is first order expressible only with respect to the subclass of finite i-urn structures.

Interestingly, note that in classical logic back and forth equivalence cannot enforce generical finiteness, that is, the property "there is a finite number of elements" is not first order expressible in classical logic. In turn, in i-urn semantics, as a weakening of classical logic, even the property "there are exactly n elements" is not first order expressible (that is, it is expressible only with respect to finite models). So, in a sense, this fact radicalizes the situation in classical logic: classical logic cannot distinguish infinity from generic finiteness; in turn, *under normal conditions*, i-urn semantics cannot distinguish infinity from finite cardinality.

5.4 Conclusion

In this chapter I presented some results on categoricity in urn semantics, further exploring some initial results by Olin (1978) on the subject. In this seminal paper, Olin showed that, contrary to classical logic, in p-urn semantics there are no strictly categorical finite models and for any infinite non-fully eligible model it is always possible to identify an elementary equivalent, non-isomorphic one, what represents some failure of κ-categoricity. Moreover, Olin proved that strict κ-categoricity can only be obtained for very special kinds of p-urn structures. In sum, Olin's theorems show that, in p-urn semantics, satisfiable theories have in general a great number of models.

However, in my presentation of Olin's results, I have noted that his proof strategies rely on the possibility of adding isomorphic copies of eligible sequences to an originally given model. This is not possible in i-urn semantics, what immediately raises the question of whether it is possible to obtain Olin's results in this system. In this sense, I examined here whether some of the most fundamental theorems on categoricity in classical logic hold in i-urn semantics.

Thus, first, I proved that compactness and strong versions of Löwenheim-Skolem theorems hold in i-urn semantics. These results show that in i-urn semantics, there are not strictly categorical infinite models, what, consequently, shows that i-urn semantics inherits classical logic's blindness for differences between infinite cardinals. Secondly, I showed that, in i-urn semantics, under special conditions it is possible to enforce finite models to have a certain cardinality $n < \omega$, a result that I have informally labeled as "finite strict categoricity with respect to finite models." These results on finite categoricity give just a partial picture of the issue and need to be generalized in future work.

Finally, based on the presents results on categoricity in i-urn semantics, I would like to make a general comment on the formalization of TSI in urn semantics. In chapter 3, I have anticipated that, given the important differences between p- and i-urn semantics, one could propose the following question: which urn semantics offers the best formalization of TSI? I will consider this question in more detail in thesis' conclusion but I can already anticipate that, in my opinion, p-urn semantics offers the best general framework for the formalization of TSI. The reason for that is simple: p-urn semantics provides a more fine-grained description of the class of urn structures and offers more flexible conditions for the satisfiability of formulas that are not satisfiable in classical logic. Now, as I will argue in the conclusion of this thesis, I think that both systems of urn semantics present excessively liberal accounts of epistemically possible models and in this sense should be, in future research, constrained in order to generate more precise descriptions of the epistemic conditions of semantic competence and, consequently, semantic information. However, not necessarily the adequate constraints needed here will reject only p-urn structures, that is, there is the possibility that, in an improved framework obtained by applying contraints in urn semantics, some of the structures that in fact characterize epistemic possibility are p-urn structures. I will argue with more detail in this direction in the final chapter of this thesis, but this general remark already anticipates my opinion on this issue.

Chapter 6

Internalizing semantic information: preliminary studies on *Logics of formal inconsistency* and TSI

6.1 Introduction

Throughout this thesis I proposed a revision of the traditional theory of semantic information in order to give an account of the real informativeness of logical truths. Of course, the work that was actually developed here has an obvious limitation of scope: by logical truths and validities I always meant *classical* logical truths and validities. Thus, one could propose the following question: $(*)$ what would happen with TSI if the logic that characterizes our reasoning and discursive practice was not classical logic but some other non-classical system? This is the question that I would like to explore in this chapter.

Question $(*)$ is too complex. In fact, it can be analyzed at least in terms of the following simpler ones:

A. Given a non-classical system of logic **L**, does **L** provide the technical resources for the formalization of TSI (Fraïssé-Hintikka theorem, existence of Hintikka normal forms etc.)?

B. Is it possible to generate a urn semantics based on **L**? Moreover, in case it is possible, does this urn semantics have all the good properties required for the rejection of SoD?

In this chapter I would like to start the study of $(*)$ by considering question A for a particular system of LFIs known in the literature as QmbC. In the introduction of this thesis I already presented a general description of LFIs (and QmbC, in particular) as interesting systems of paraconsistent logic that control explosion by attributing consistency and inconsistency to first order formulas in first order language itself. QmbC is a very minimal system of such logics. In this sense, I would like to see whether QmbC validates the Fraïssé-Hintikka theorem, has

Hintikka-normal forms and so on. As I have said above, any result on this issue represents a first step in the formalization of TSI in LFIs.

The study of the relation between LFIs and TSI finds justification both in a general and in a particular motivation. As a more general motivation, this investigation is part of a research agenda on the semantics of non-classical logical vocabulary. In order to provide an account of the meanings of a particular set of non-classical logical connectives, it is necessary to give an account of the informativeness of sentences generated by such vocabulary.

Moreover, in particular, this study is driven by the idea of exploring LFIs' "internalization" of consistency in order to further internalize information. Roughly speaking, TSI formalizes information in terms of "degree of consistency", that is, the amount of information carried by a formula varies with respect to the numbers of models it has in a fixed semantic system. Now, given LFIs' internalization of consistency, based on it we can hope to achieve a subsequent internalization of information. The search for internalizations of metatheoretic notions is a widespread topic of research that shades new light on traditional fundamental notions. For a good example in this direction, consider the results obtained by the internalization of "proof" based on provability logics (Cf. VERBRUGGE, 2017). Now, since the amount of information of formulas varies in terms of degrees of consistency, we can hope to achieve a way of defining finer-grained operators "$\circ$" that verify the consistency of some formulas based on the amount of information that they carry.

There are also some further motivations for considering whether Fraïssé-Hintikka theorem holds in QmbC which are independent from informational issues. In recent literature, it is possible to find an increasing interest in LFIs as a promising source of new approaches to traditional problems in philosophical logic. Such philosophical agenda on LFIs requires a previous elucidation of the semantic counterpart of these systems. However, we still do not have a complete understanding of their semantic aspects, either in a philosophical or in a purely logical sense.

From a philosophical perspective, there are important questions about the meaning of "$\circ$" (Cf. BARRIO, 2017). Moreover, from a model-theoretic perspective, we still do not have an adequate understanding of the semantic relations holding between LFIs and classical logic. In particular, given that any LFI is generated by imposing constraints on the validity of the classical principle of explosion, one could ask which classical model-theoretic properties are satisfied by these logics. Following this purely logical agenda on the semantics of LFIs, Carnielli et al. (2014) started the study of model theory for the minimal system QmbC. In this paper, they define a Tarskian semantics for QmbC and prove completeness, compactness and Löwenheim-Skolem theorems for it. Such results suggest the agenda of searching for more advanced classical model-theoretic properties which could be at least partially preserved in QmbC. Furthermore, note that it would be especially relevant to find theorems which could be *only* partially preserved in QmbC.

Fraïssé-Hintikka theorem is a very fruitful theorem of classical model theory. So, the establishment of this theorem in QmbC can suggest that a variety of classical semantic properties are also preserved in this logic. Furthermore, note that Fraïssé-Hintikka theorem relies heavily on the truth-functionality of classical logic (in what follows, I treat the expressions "truth-functionality" and "compositionality" as synonyms): in classical logic, the determination of the truth-values of all atomic formulas of a given language is a sufficient condition for determining the truth-values of all formulas of the language. This is not the case in QmbC as well as in various other LFIs. Such systems are essentially non-compositional in the sense that, in their contexts, there are formulas whose truth-values are partially independent from the truth-values of the atomic formulas occurring in them. So, it is not trivial to ask whether Fraïssé-Hintikka theorem can even be partially preserved in QmbC, what makes the obtainment of this theorem in the context of LFIs an even more valuable achievement. In fact, as I show next, non-compositionality is a fundamental reason of divergence between classical Fraïssé-Hintikka theorem and the weaker version of this theorem that I prove here for QmbC.

This chapter is organized in the following way. In section 2 I present a brief overview of the basic results introduced by Carnielli et al. (2014) about model theory for LFIs. Then, in section 3, I present the version of Fraïssé-Hintikka theorem for QmbC. In the conclusion, I compare the classical and non-classical versions of Fraïssé-Hintikka theorem with each other, considering whether it is possible to generate a strengthening of the latter in LFI-extensions of QmbC.

6.2 Model theory for QmbC

In this section I present an overview of the basic model theory for QmbC. To begin, let me introduce a few basic notions. Let a *QmbC language* be a language whose logical signature is $\{\vee, \wedge, \rightarrow, \neg, \circ, \exists, \forall\}$. By $\mathcal{L}$ I understand a *relational* QmbC language. Consequently, $\mathcal{L}$ only has variables as terms. Further, consider that $\mathcal{L}$ has no identity relation either.[1] The set of formulas of $\mathcal{L}$ is recursively defined in the usual way.

Whereas classical Fraïssé-Hintikka theorem states a property relating back and forth equivalence and common satisfaction of formulas with a fixed maximum quantifier rank, the weaker version of Fraïssé-Hintikka theorem that holds in QmbC concentrates on the following slight variation of the notion of "quantifier rank". Let $\widehat{qr}$ denote the function $\neg\circ$*-free quantifier rank* such that $\widehat{qr}(\phi)$ is the length of the greatest nesting of quantifiers occurring in a formula ϕ which are neither in the scope of "$\neg$" nor of "$\circ$". The function $\widehat{qr}$ works in the following way:

- If ϕ is either atomic formula, $\neg\psi$ or $\circ\psi$, then $\widehat{qr}(\phi) = 0$;
- If ϕ is $\psi * \gamma$, such that "$*$" is either "$\wedge$", "$\vee$" or "$\rightarrow$", then $\widehat{qr}(\phi) = max\{\widehat{qr}(\psi), \widehat{qr}(\gamma)\}$;
- If ϕ is either $\forall x\psi$ or $\exists x\psi$, then $\widehat{qr}(\phi) = \widehat{qr}(\psi) + 1$;

[1] I focus here on relational languages without identity in order to concentrate on a simpler case first. In the conclusion I consider whether the present result can be generalized for "full" QmbC languages.

In the following, sometimes I need to consider the *complexity* of formulas. In this sense, the function that computes the complexity of terms introduced in section 3.1 is naturally extended for formulas in the following way:

- If ϕ is atomic formula, then $cp(\phi) = 0$;
- $cp(\neg\psi) = cp(\circ\psi) = cp(\forall x\psi) = cp(\exists x\psi) = cp(\psi) + 1$;
- $cp(\bigwedge \Psi) = cp(\bigvee \Psi) = \sum_{0<i\leq n} cp(\psi_i)$, for $\Psi = \{\psi_1, \ldots, \psi_n\}$.

The following is an axiomatization of QmbC.

Axiomatic system for QmbC

S1. $\phi \to (\psi \to \phi)$;

S2. $(\phi \to \psi) \to ((\phi \to (\psi \to \gamma)) \to (\phi \to \gamma))$;

S3. $\phi \to (\psi \to (\phi \wedge \psi))$

S4. a. $(\phi \wedge \psi) \to \phi$;

b. $(\phi \wedge \psi) \to \gamma$;

S5. a. $\phi \to (\phi \vee \psi)$;

b. $\psi \to (\phi \vee \psi)$;

S6. $(\phi \to \gamma) \to ((\psi \to \gamma) \to ((\phi \vee \psi) \to \gamma))$;

S7. $\phi \vee (\phi \to \psi)$;

S8. $\phi \vee \neg\phi$

S9. $\circ\phi \to (\phi \to (\neg\phi \to \psi))$;

S10. $\phi_t[x] \to \exists x\phi$, t is substitution-free for x in ϕ;

S11. $\forall x\phi \to \phi_x[t]$, t is substitution-free for x in ϕ;

Inference rules:

R1. Modus ponens;

R2. $\phi \to \psi \setminus \phi \to \forall x\psi$, x does not occur free in ϕ;

R3. $\phi \to \psi \setminus \exists x\phi \to \psi$, x does not occur free in ψ.

Definition 6.1 *For a sequence of formulas $\langle\phi_1, \ldots, \phi_n\rangle$ of $\mathcal{L}$, I say that* $\{\phi_1, \ldots, \phi_{n-1}\}$ deduces ϕ_n*, in symbols* $\{\phi_1, \ldots, \phi_{n-1}\} \vdash \phi_n$*, if and only if the following holds:*

- *ϕ_0 is an instance of axiom scheme Sj, for some $j = 1, \ldots, 11$;*
- *For every $0 < m \leq n$, either ϕ_m is an instance of axiom scheme Sj, for some $j = 1, \ldots, 11$, or ϕ_m is obtained from application of inference rule $R1$ to ϕ_k and ϕ_q, for some $k < q < m$, or ϕ_m is obtained from application of inference rule Rj' to ϕ_k, for some $j' = 2, 3$ and for some $k < m$.*

For any set of formulas $\Gamma \cup \{\phi\}$, $\Gamma \vdash \phi$ means that there is a finite set $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash \phi$.

Axiom schemes S1-S8 plus modus ponens describe *minimal positive* classical logic, the fragment of classical logic that contains its positive theorems only. The expansion of this system with axiom S9 generates the propositional fragment of QmbC (denoted in the literature as mbC (CARNIELLI et al., 2014, p. 3)). This axiomatic system for QmbC has some important properties. First, it is a *Tarskian* deductive system, that is, the above axioms and rules formalize a reflexive, monotonic and transitive deducibility relation "$\vdash$" (Ibid., pp. 18–19). Further, it validates the *deduction theorem* (Ibid., p. 7).

An important difference between QmbC and classical logic is the fact that *replacement property* fails for QmbC (CARNIELLI; CONIGLIO; MARCOS, 2007, p. 26). The failure of this property means that in QmbC, in general, it is not possible to replace equivalent subformulas occurring in the scope of the operators "$\neg$" or "$\circ$". This failure is also inherited by stronger LFIs and can be seen as an essential feature of this family of systems. The failure of replacement property in QmbC and other LFIs is a consequence of the non-compositional character of these systems that I mentioned above. As I show in the following, given the failure of replacement property in QmbC, it is necessary to include an additional clause to the definition of satisfaction for QmbC in order to guarantee the validity of the *substitution lemma* for this system.

Finally, based on this axiomatics, it is possible to define a strong negation "$\sim$" as follows: for a fixed formula ψ, for any formula ϕ, $\sim \phi$ is defined as $\phi \rightarrow (\circ\psi \wedge \psi \wedge \neg\psi)$. Defined in this way, the connective "$\sim$" interprets classical negation in QmbC (CARNIELLI et al., 2014, p. 5). The existence of a strong negation "$\sim$" is important for the achievement of this chapter's main result, because with "$\sim$" it is possible to prove interdefinability of "$\forall$" and "$\exists$" in QmbC, that is, it is possible to prove that $\vdash\sim \forall x\phi \rightarrow \exists x \sim \phi$ and $\vdash\sim \forall x \sim \phi \rightarrow \exists x\phi$ (Ibid., pp. 8–9).

Now, the notions of *$\mathcal{L}$-structure* and QmbC-*satisfiability* can be defined in the following way (Ibid., pp. 10–11).

Definition 6.2 (QmbC-structure) *An* $\mathcal{L}$-structure *$\mathcal{M}$ is a pair $\langle M, (\cdot)^{\mathcal{M}} \rangle$, in which $M \neq \emptyset$ and $(\cdot)^{\mathcal{M}}$ is an interpretation function such that, for any n-ary predicate symbol P, $P^{\mathcal{M}}$ is some part of M^n.*

Given an $\mathcal{L}$-structure $\mathcal{M}$, for any $\mathcal{L}$-formula ϕ with free-variables $\bar{x} = x_1, \ldots, x_n$ and for $\bar{a} = \langle a_1, \ldots, a_n \rangle \in M^n$, I denote that $\langle a_1, \ldots, a_n \rangle$ *satisfies* ϕ in $\mathcal{M}$ by the usual notation $\mathcal{M} \models \phi\, \bar{x}[\bar{a}]$.

Definition 6.3 (QmbC-satisfiability) *For any $\mathcal{L}$-structure $\mathcal{M}$, for any $n < \omega$, for every formula ϕ with free-variables $\bar{x} = x_1, \ldots, x_n$, for any $\bar{a} \in M^n$, the* satisfaction *of ϕ by $\bar{a}$ in $\mathcal{M}$ is characterized by the following set of clauses:*

- *If ϕ is atomic formula $P(\bar{x})$, then $\mathcal{M} \models \phi\ \bar{x}[\bar{a}] \Leftrightarrow \bar{a} \in P^{\mathcal{M}}$;*
- *If ϕ is $\psi \wedge \gamma$, then $\mathcal{M} \models \phi\ \bar{x}[\bar{a}] \Leftrightarrow \mathcal{M} \models \psi\ \bar{y}[\bar{b}]$ and $\mathcal{M} \models \gamma\ \bar{z}[\bar{c}]$, such that $\bar{y}$ and $\bar{z}$ are the (possibly empty) lists of variables within $\bar{x}$ that occur in ψ and γ, respectively, and $\bar{b}$ and $\bar{c}$ are the corresponding segments of $\bar{a}$;*
- *If ϕ is $\psi \vee \gamma$, then $\mathcal{M} \models \phi\ \bar{x}[\bar{a}] \Leftrightarrow$ either $\mathcal{M} \models \psi\ \bar{y}[\bar{b}]$ or $\mathcal{M} \models \gamma\ \bar{z}[\bar{c}]$, in which $\bar{y}$, $\bar{z}$, $\bar{b}$ and $\bar{c}$ should be understood as above;*
- *If ϕ is $\psi \rightarrow \gamma$, then $\mathcal{M} \models \phi\ \bar{x}[\bar{a}] \Leftrightarrow$ either $\mathcal{M} \not\models \psi\ \bar{y}[\bar{b}]$ or $\mathcal{M} \models \gamma\ \bar{z}[\bar{c}]$, in which $\bar{y}$, $\bar{z}$, $\bar{b}$ and $\bar{c}$ should be understood as above;*
- *If ϕ is $\exists y \psi$, then $\mathcal{M} \models \phi\ \bar{x}[\bar{a}] \Leftrightarrow \mathcal{M} \models \psi\ \bar{x}\ y[\bar{a} \frown (b)]$, for some element b of $\mathcal{M}$;*
- *If ϕ is $\forall y \psi$, then $\mathcal{M} \models \phi\ \bar{x}[\bar{a}] \Leftrightarrow \mathcal{M} \models \psi\ \bar{x}\ y[\bar{a} \frown (b)]$, for every element b of $\mathcal{M}$;*
- *If ϕ is $\neg\psi$, then $\mathcal{M} \not\models \phi\ \bar{x}[\bar{a}] \Rightarrow \mathcal{M} \models \psi\ \bar{x}[\bar{a}]$;*
- *If ϕ is $\circ\psi$, then $\mathcal{M} \models \phi\ \bar{x}[\bar{a}] \Rightarrow$ either $\mathcal{M} \not\models \neg\psi\ \bar{x}[\bar{a}]$ or $\mathcal{M} \not\models \psi\ \bar{x}[\bar{a}]$.*
- *For any $0 < i \leq n$, for any variable y such that x_i is substitution-free for y in ϕ, if $\mathcal{M} \models \phi\ \bar{x}[\bar{a}] \Leftrightarrow \mathcal{M} \models \phi\ x_1 \ldots x_i[y]\ x_{i+1} \ldots x_n[\bar{a}]$, then:*
  - *$\mathcal{M} \models \neg\phi\ \bar{x}[\bar{a}] \Leftrightarrow \mathcal{M} \models \neg\phi\ x_1 \ldots x_i[y]\ x_{i+1} \ldots x_n[\bar{a}]$, and*
  - *$\mathcal{M} \models \circ\phi\ \bar{x}[\bar{a}] \Leftrightarrow \mathcal{M} \models \circ\phi\ x_1 \ldots x_i[y]\ x_{i+1} \ldots x_n[\bar{a}]$.*

The last clause in Definition 6.3 forces the preservation of truth-values by substitution of terms with same interpretation. Consequently, it supports a substitution lemma for QmbC (Ibid., pp. 13–15). In fact, that clause is a necessary condition of the lemma given the already mentioned failure of replacement property in QmbC.

Definitions 6.2 and 6.3 provide a Tarskian semantics for QmbC. Note that, by this semantics, "$\neg$" is in fact a paraconsistent negation since, for some formulas ϕ, it is possible to devise an $\mathcal{L}$-structure $\mathcal{M}$ such that $\mathcal{M} \models \phi$, $\mathcal{M} \models \neg\phi$ but, by the semantics of "$\circ$", $\mathcal{M} \not\models \circ\phi$. So, $\mathcal{M}$ has a non-*trivial* theory.[2] With this semantics it is possible also to define *semantic consequence* in standard terms.

[2] Of course the reader could ask a more fundamental question: is "$\neg$" really a negation? This question is a special case of Slater's challenge to paraconsistency (SLATER, 1995). This is not the proper context for addressing this important question. However, note that, for any negation "$*$", "$*$" is a paraconsistent negation if and only if there are cases in which, for some formula ϕ, it is possible to assume ϕ and $*\phi$ without explosion. So, assuming that the connective "$\neg$" of QmbC is a negation, given that there are QmbC-structures that, for some formula ϕ, verify ϕ and $\neg\phi$ without trivialization, "$\neg$" passes the test of paraconsistency.

Definition 6.4 *A set* Γ *of formulas of* $\mathcal{L}$ semantically implies *a formula* ϕ*, in symbols* $\Gamma \models \phi$*, if and only if, for any* $\mathcal{L}$*-structure* $\mathcal{M}$*, either there is some* $\psi \in \Gamma$ *such that* $\mathcal{M} \not\models \psi$ *or* $\mathcal{M} \models \phi$*.*

Hence, QmbC validates the following result: $\Gamma \models \phi$ if and only if $\Gamma \cup \{\sim \phi\}$ is unsatisfiable. Carnielli et al. (2014) proved that QmbC is a *sound* and *complete* system of logic.

Theorem 6.5 (Soundness and completeness for QmbC) *Let* $\Gamma \cup \{\phi\}$ *be a set of formulas of* $\mathcal{L}$*. Then,* $\Gamma \vdash \phi \Leftrightarrow \Gamma \models \phi$*.*

Proof. (Ibid., pp. 15–22). ■

Compactness and downward Löwenheim-Skolem for QmbC are corollaries of Theorem 6.5 (Ibid., p. 23).

Corollary 6.6 (Compactness for QmbC) *Let* $\Gamma \cup \{\phi\}$ *be a set of formulas of* $\mathcal{L}$*.* $\Gamma \models \phi$ *if and only if, for some finite set of formulas* $\Gamma^* \subseteq \Gamma$*,* $\Gamma^* \models \phi$*.*

Proof. Assume $\Gamma \models \phi$. By completeness of QmbC, $\Gamma \vdash \phi$. Since deduction in QmbC is finitary, there is a finite $\Gamma^* \subseteq \Gamma$, $\Gamma^* \vdash \phi$. By soundness of QmbC, $\Gamma^* \models \phi$. ■

Considering a variation of Henkin's classic strategy, the proof of completeness of QmbC in Carnielli et al. (2014) builds a "canonical" model for a given non-trivial theory. Now, the existence of such "canonical" models guarantees downward Löwenheim-Skolem for QmbC.

Corollary 6.7 (Downward Löwenheim-Skolem for QmbC) *Let* Γ *be a non-trivial set of formulas of* $\mathcal{L}$*. There is an* $\mathcal{L}$*-structure* $\mathcal{M}$ *that satisfies* Γ *and the cardinality of* $\mathcal{M}$ *is equal to the cardinality of* $\mathcal{L}$*.*

Proof. (Ibid., p. 23). ■

The proof of upward Löwenheim-Skolem for QmbC in Carnielli et al. (2014, pp. 23–24) works by extending a previously given model with copies of its elements. Here, I explore a more standard strategy based on compactness.

Corollary 6.8 (Upward Löwenheim-Skolem for QmbC) *Let* $\mathcal{M}$ *be an* $\mathcal{L}$*-structure of the same cardinality of* $\mathcal{L}$ *(let it be* δ*), and let* Γ *be a set of formulas of* $\mathcal{L}$ *such that* $\mathcal{M} \models \Gamma$*. Then, for every cardinal* $\kappa \geq \delta$*, there is an* $\mathcal{L}$*-structure* $\mathcal{N}$ *of cardinality* κ *such that* $\mathcal{N} \models \Gamma$*.*

Proof. Let $Th(\mathcal{M})$ be the theory of $\mathcal{M}$ and consider a set $\mathcal{C} = \{c_i : i < \kappa\}$ of new constants. Let $\mathcal{L}' = \mathcal{L} \cup \mathcal{C}$ and consider an $\mathcal{L}'$-theory $\Delta = \{c_i \neq c_j : i \neq j, c_i, c_j \in \mathcal{C}\}$. Consider a finite part Δ' of Δ and assume that $c_1, \ldots, c_n$ are all the constants occurring in Δ'. Let $\mathcal{M}'$ be the $\mathcal{L}'$-expansion of $\mathcal{M}$ such that, for every $0 < i < j \leq n$, $c_i^{\mathcal{M}'} \neq c_j^{\mathcal{M}'}$. So, $\mathcal{M}' \models \Delta'$. By compactness, there is an $\mathcal{L}'$-structure $\mathcal{N}'$ that satisfies $Th(\mathcal{M}) \cup \Delta$. Consider the $\mathcal{L}$-reduct of $\mathcal{N}'$, let it be $\mathcal{N}$. So, $\mathcal{N}$ satisfies Γ and is of cardinality κ. ■

The next definition characterizes embedding and elementary embedding for QmbC.

Definition 6.9 *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures.*

- *$\mathcal{M}$ is* embedded *in $\mathcal{N}$ if and only if there is an injection $h : \mathcal{M} \to \mathcal{N}$ such that, for every formula ϕ of $\mathcal{L}$ with free-variables $\bar{x} = x_1, \ldots, x_n$ and such that $\widehat{qr}(\phi) = 0$, for every sequence $\bar{a} = \langle a_1, \ldots, a_n \rangle$ of $\mathcal{M}$, $\mathcal{M} \models \phi\, \bar{x}[\bar{a}] \Leftrightarrow \mathcal{N} \models \phi\, \bar{x}[h(a_1) \ldots h(a_n)]$;*

- *$\mathcal{M}$ is* elementary embedded *in $\mathcal{N}$ if and only if there is an injection $h : \mathcal{M} \to \mathcal{N}$ such that, for every formula ϕ of $\mathcal{L}$ with free-variables $\bar{x} = x_1, \ldots, x_n$, for every sequence $\bar{a} = \langle a_1, \ldots, a_n \rangle$ of $\mathcal{M}$, $\mathcal{M} \models \phi\, \bar{x}[\bar{a}] \Leftrightarrow \mathcal{N} \models \phi\, \bar{x}[h(a_1) \ldots h(a_n)]$;*

If the domain of $\mathcal{M}$ is some part of the domain of $\mathcal{N}$ and $\mathcal{M}$ is (elementary) embedded in $\mathcal{N}$, then $\mathcal{M}$ is called a (elementary) substructure *of $\mathcal{N}$ and $\mathcal{N}$ is called an* (elementary) extension *of $\mathcal{M}$, in symbols $\mathcal{M} \subseteq \mathcal{N}$ ($\mathcal{M} \preceq \mathcal{N}$, respectively).*

Analogous to what I have done in subsection 5.3.1, for any $\mathcal{L}$-structure $\mathcal{M}$ of cardinality κ, for a set of new constants $\mathcal{D}(\mathcal{M}) = \{d_i : i < \kappa\}$, for $\mathcal{L}^+ = \mathcal{L} \cup \mathcal{D}(\mathcal{M})$, for an $\mathcal{L}^+$-expansion $\mathcal{M}^+$ of $\mathcal{M}$ such that $(\cdot)^{\mathcal{M}^+} \restriction \mathcal{D}(\mathcal{M})$ is a bijection onto the domain of $\mathcal{M}$, let the elementary diagram of $\mathcal{M}$ be the theory $Diagel(\mathcal{M}) = \{\phi\, x_1 \ldots x_n[d_1 \ldots d_n] : \phi$ is $\mathcal{L}$-formula with free-variables $x_1, \ldots, x_n, d_1, \ldots, d_n \in \mathcal{D}(\mathcal{M})$ and $\mathcal{M}^+ \models \phi\, x_1 \ldots x_n[d_1 \ldots d_n]\}$. It is also possible to prove a diagram lemma in QmbC.

Lemma 6.10 *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. The following are equivalent:*

1. *$\mathcal{M} \preceq \mathcal{N}$;*

2. *For some $\mathcal{L}^+$-expansion $\mathcal{N}^+$ of $\mathcal{N}$, $\mathcal{N}^+ \models Diagel(\mathcal{M})$.*

Proof. ($1 \Rightarrow 2$) Let h be the mapping that defines $\mathcal{M} \preceq \mathcal{N}$ and consider $\mathcal{N}^+$ such that, for any $d \in \mathcal{D}(\mathcal{M})$, $d^{\mathcal{N}^+} = h(d^{\mathcal{M}^+})$. So, by 1, $\mathcal{N}^+ \models \phi\, x_1 \ldots x_n[d_1 \ldots d_n]$, for any sentence $\phi\, x_1 \ldots x_n[d_1 \ldots d_n] \in Diagel(\mathcal{M})$.

($2 \Rightarrow 1$) For every $d \in \mathcal{D}(\mathcal{M})$, denote d as d_a, for the element a of $\mathcal{M}$ such that $d^{\mathcal{M}^+} = a$. Let h be a mapping from $\mathcal{M}$ into $\mathcal{N}$ such that, for any element a of $\mathcal{M}$, $h(a) = d_a^{\mathcal{N}^+}$. So, for any sequence $\langle a_1, \ldots, a_n \rangle$ of $\mathcal{M}$, $\mathcal{M} \models \phi\, x_1 \ldots x_n[a_1 \ldots a_n] \Leftrightarrow \mathcal{M}^+ \models \phi\, x_1 \ldots x_n[d_{a_1} \ldots d_{a_n}] \Leftrightarrow$ (By 2) $\mathcal{N}^+ \models \phi\, x_1 \ldots x_n[d_{a_1} \ldots d_{a_n}] \Leftrightarrow \mathcal{N} \models \phi\, x_1 \ldots x_n[h(a_1) \ldots h(a_n)]$. ∎

Corollary 6.11 *Let $\mathcal{M}$ be an $\mathcal{L}$-structure of the same cardinality of $\mathcal{L}$ (let it be δ). Then, for every cardinal $\kappa \geq \delta$, there is an $\mathcal{L}$-structure $\mathcal{N}$ of cardinality κ such that $\mathcal{M} \preceq \mathcal{N}$.*

Proof. Let Δ be a theory such as in the proof of Corollary 6.8. Note that, for every finite part Δ' of $Diagel(\mathcal{M}) \cup \Delta$, there is an $\mathcal{L}^+$-expansion of $\mathcal{M}$ that satisfies Δ'. So, by compactness, there is an $\mathcal{L}^+$-structure $\mathcal{N}^+$ of cardinality κ such that $\mathcal{N}^+ \models Diagel(\mathcal{M})$. By Lemma 6.10, for the $\mathcal{L}$-reduct $\mathcal{N}$ of $\mathcal{N}^+$, $\mathcal{M} \preceq \mathcal{N}$. ∎

6.3 Fraïssé-Hintikka theorem for QmbC

6.3.1 Preliminary notions

Based on these model-theoretic results for QmbC, now I state and prove a weaker version of Fraïssé-Hintikka theorem for QmbC. First, let me introduce some important auxiliary concepts. Below I define isomorphism in the context of QmbC.

Definition 6.12 (Isomorphism in QmbC) *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures with domains M and N, respectively. $\mathcal{M}$ is* isomorphic *with $\mathcal{N}$, in symbols $\mathcal{M} \cong \mathcal{N}$, if and only if there is a bijection $h : M \to N$ such that, for any $n < \omega$ and any sequence $\bar{a} = \langle a_1, \dots, a_n \rangle$ in M^n, the following holds:*

- *For any atomic formula ϕ of $\mathcal{L}$ with free-variables $\bar{x} = x_1, \dots, x_n$, $\mathcal{M} \models \phi\ \bar{x}[\bar{a}] \Leftrightarrow \mathcal{N} \models \phi\ \bar{x}[h(a_1) \dots h(a_n)]$;*
- *For any formula ϕ of $\mathcal{L}$ with free-variables $\bar{x}$,*
  - $\mathcal{M} \models \neg\phi\ \bar{x}[\bar{a}] \Leftrightarrow \mathcal{N} \models \neg\phi\ \bar{x}[h(a_1) \dots h(a_n)]$;
  - $\mathcal{M} \models \circ\phi\ \bar{x}[\bar{a}] \Leftrightarrow \mathcal{N} \models \circ\phi\ \bar{x}[h(a_1) \dots h(a_n)]$.

Given the failure of replacement property in QmbC, the last clause in the definition above is required in order to validate *isomorphism property*, i.e., the fact that isomorphism implies elementary equivalence (terminology by Ebbinghaus (1985, p. 28)).

Theorem 6.13 (Isomorphism property) *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. Then $\mathcal{M} \cong \mathcal{N} \Rightarrow \mathcal{M} \equiv \mathcal{N}$.*

Proof. Proof by induction on formulas. For atomic formulas as well as for the cases of "$\circ$" and "$\neg$" it follows from definition of isomorphism. ■

Finally, based on Definition 6.12 it is possible to define a variation of back and forth equivalence for QmbC – let me call it *back and forth equivalence in elementary extensions*.

Definition 6.14 (Back and forth equivalence in elementary extensions) *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. $\mathcal{M}$ and $\mathcal{N}$ are* back and forth equivalent in elementary extensions*, in symbols $\mathcal{M} \asymp \mathcal{N}$, if and only if there is a poset H of isomorphisms between substructures of elementary extensions of $\mathcal{M}$ and $\mathcal{N}$ such that the following holds:*

- *The first element $h_0 \in H$ is isomorphism between empty substructures of $\mathcal{M}$ and $\mathcal{N}$;*
- *Secondly, assume that, for some $i < \omega$, there are elementary extensions $\mathcal{M}_i$ and $\mathcal{N}_i$ of $\mathcal{M}$ and $\mathcal{N}$, respectively, such that, for some finite structures $\mathcal{M}'_i \subseteq \mathcal{M}_i$ and $\mathcal{N}'_i \subseteq \mathcal{N}_i$, there is an isomorphism $h_i \in H$ such that $h_i : \mathcal{M}'_i \to \mathcal{N}'_i$. The following holds:*

- *For any element a of $\mathcal{M}_i$, there is some elementary extension $\mathcal{N}_{i+1}$ of $\mathcal{N}_i$ such that, for some element b of $\mathcal{N}_{i+1}$, there is an isomorphism $h_{i+1} \in H$ such that $h_{i+1} : \mathcal{M}'_i \cup \{a\} \rightarrow \mathcal{N}'_i \cup \{b\}$, $h_i \subseteq h_{i+1}$;*
- *For any element b of $\mathcal{N}_i$, there is some elementary extension $\mathcal{M}_{i+1}$ of $\mathcal{M}_i$ such that, for some element a of $\mathcal{M}_{i+1}$, there is an isomorphism $h_{i+1} \in H$ such that $h_{i+1} : \mathcal{M}'_i \cup \{a\} \rightarrow \mathcal{N}'_i \cup \{b\}$, $h_i \subseteq h_{i+1}$.*

$\mathcal{M}$ and $\mathcal{N}$ are k-back and forth equivalent in elementary extensions (in symbols $\mathcal{M} \asymp_k \mathcal{N}$) if and only if the condition above holds at least for $i < k$.

In usual terminology, a formula of the form $Q_1x_1 \ldots Q_nx_n\psi$ in which, for every $0 < i \leq n$, "Q_i" is either "$\forall$" or "$\exists$" and ψ is a quantifier-free formula is called a *prenex normal form.* In classical logic there is a series of *prenex operations* (terminology by Shoenfield (1967, p. 37)) which together characterize a procedure for generating, from any given formula ϕ, a prenex normal form that is equivalent to ϕ. Let me call such a procedure a *prenexification* of formulas. Now, a further consequence of the non-compositionality of QmbC is that, in this system, there is no procedure of prenexification.

Theorem 6.15 *For some formula ϕ of $\mathcal{L}$, there is no formula γ that is equivalent to $\circ\phi$ and is a prenex normal form. Furthermore, there is no formula γ that is equivalent to $\neg\phi$ and is a prenex normal form.*

Proof. I will show that, particularly in the case that ϕ is $\forall xP(x)$, for a monadic predicate "P" of $\mathcal{L}$, there is no prenexification either of $\circ\phi$ or $\neg\phi$.

Consider in particular the case of "$\neg$" (The case of "$\circ$" works in an analogous way). Assume that there is some prenex normal form γ that is equivalent to $\neg\forall xP(x)$. Since $P(x)$ is the only considered atomic formula, for some $0 < m < \omega$, γ is of the form $Q_1y_1 \ldots Q_my_m\psi$ in which ψ is a quantifier-free formula whose atomic subformulas are $P(y_1), \ldots, P(y_m)$ and, for any $0 < i \leq m$, "Q_i" is either "$\forall$" or "$\exists$".

Consider an $\mathcal{L}$-structure $\mathcal{M}$ such that $\mathcal{M} \models P(a)$, for every element a of $\mathcal{M}$. Then $\mathcal{M} \models \forall xP(x)$. Consider two cases:

a) $\mathcal{M} \models Q_1y_1 \ldots Q_my_m\psi$. In this case, stipulate that $\mathcal{M} \not\models \neg\forall xP(x)$.

b) $\mathcal{M} \not\models Q_1y_1 \ldots Q_my_m\psi$. In this case, let it be that $\mathcal{M} \models \neg\forall xP(x)$.

Both cases are possible given that $\mathcal{M} \models \forall xP(x)$ and, by the non-compositionality of QmbC, the evaluation of $\neg\forall xP(x)$ is independent of the evaluations of ψ and $\forall xP(x)$. Cases a) and b) contradict the fact that γ and $\neg\phi$ are equivalent formulas. Hence, $\neg\phi$ does not accept prenexification. ■

Corollary 6.16 *QmbC does not have prenexification.*

Let me say that a formula ψ is a $\neg\circ$*-free* formula if and only if any quantifier occurring in it is either in the scope of "$\neg$" or "$\circ$". Further, I say that a formula ϕ is a *quasi*-prenex normal form if ϕ is of the form $Q_1x_1 \ldots Q_nx_n\psi$ in which any "Q_i" is either "$\forall$" or "$\exists$" and ψ is $\neg\circ$-free formula. Even though QmbC does not have a prenexification procedure, it has what we can call a *quasi*-prenexification procedure, i.e., a procedure of taking any formula ϕ and generating from it an equivalent one in quasi-prenex normal form. The fact that QmbC has only quasi-prenexification is directly linked with the fact that only a weaker version of Fraïssé-Hintikka theorem holds in QmbC.

Lemma 6.17 *For every formula ϕ of $\mathcal{L}$, there is a formula $\phi^\dagger$ that is equivalent to ϕ and is in quasi-prenex normal form.*

Proof. Let ϕ, ψ and γ be formulas of $\mathcal{L}$. Further, consider the following mapping $(\cdot)^\dagger$ from the set of formulas of $\mathcal{L}$ into itself:

- If ϕ is either atomic formula, $\neg\psi$ or $\circ\psi$, then $\phi^\dagger$ is ϕ;
- Assuming that "Q" is either "$\exists$" or "$\forall$", $(Qx\phi)^\dagger$ is $Qx(\phi)^\dagger$;
- Assuming that "Q" and "K" are either "$\exists$" or "$\forall$", "Q" is different from "K" and "$+$" is either "$\vee$" or "$\wedge$", then the following holds:
  - If ϕ is $Qx\gamma$, in which x does not occur free in ψ, and $\widehat{qr}(\phi) \geq \widehat{qr}(\psi)$, then:
    * $(\phi + \psi)^\dagger$ is $Qx((\gamma + \psi)^\dagger)$;
    * $(\psi + \phi)^\dagger$ is $Qx((\psi + \gamma)^\dagger)$;
    * $(\phi \to \psi)^\dagger$ is $Kx((\gamma \to \psi)^\dagger)$;
    * $(\psi \to \gamma)^\dagger$ is $Qx((\psi \to \gamma)^\dagger)$;
  - If $\widehat{qr}(\phi) = \widehat{qr}(\psi) = 0$, then
    * $(\phi + \psi)^\dagger$ is $(\phi^\dagger + \psi^\dagger)$;
    * $(\phi \to \psi)^\dagger$ is $(\phi^\dagger \to \psi^\dagger)$.

Clearly, $\phi^\dagger$ is a quasi-prenex normal form. I will prove by induction on the $\neg\circ$-free quantifier rank of ϕ that $\phi^\dagger$ is equivalent to ϕ. Given that $\phi^\dagger$ is ϕ when ϕ is either an atomic formula, $\neg\psi$ or $\circ\psi$, I only need to consider the inductive cases, that is, the cases in which ϕ is either a conjunction, a disjunction or an implication. In these cases, induction works exactly as in classical logic. ■

As a final preliminary point, the following lemma on the cardinality of sets of formulas of a given maximum complexity is required for proving this chapter's main theorem.

Lemma 6.18 *Consider that $\mathcal{L}$ is a finite language. For $i < \omega$, let Δ_i be such that $\Delta_i = \{\phi :$ free-variables of ϕ are within $\bar{x} = x_1, \ldots, x_m$ and $cp(\phi) \leq i\}$. Then, Δ_i is a finite set.*

Proof. I prove this lemma by induction on $i < \omega$. Δ_0 is the set of atomic formulas with variables within $\bar{x}$. This is a finite set, since $\mathcal{L}$ is a finite language.

Now assume Δ_i has cardinality n, for some $n < \omega$. Consider that $\Lambda_{\neg\circ}$ is the set of formulas of $\mathcal{L}$ such that $\phi \in \Lambda_{\neg\circ}$ if and only if ϕ is $\#\psi$, in which "$\#$" is either "$\neg$" or "$\circ$" and $\psi \in \Delta_i$. So, the cardinality of $\Lambda_{\neg\circ}$ is n. Moreover, consider that $\Lambda_{\vee\wedge}$ is the set of formulas of $\mathcal{L}$ such that $\phi \in \Lambda_{\vee\wedge}$ if and only if ϕ is $\psi + \gamma$, in which "$+$" is either "$\wedge$" or "$\vee$" and $\psi, \gamma \in \Delta_i$. Consequently, the cardinality of $\Lambda_{\vee\wedge}$ is $2 \times n^2$. Finally, consider that $\Lambda_{\forall\exists}$ is the set of formulas of $\mathcal{L}$ such that $\phi \in \Lambda_{\forall\exists}$ if and only if ϕ is $Qx\psi$, in which "Q" is either "$\forall$" or "$\exists$", x is either x_j, for some $0 < j \leq m$, or a new variable, and $\psi \in \Delta_i$. So, the cardinality of $\Lambda_{\forall\exists}$ is $\leq (m+1) \times n$.

Let $\Delta_{i+1} = \Delta_i \cup \Lambda_{\neg\circ} \cup \Lambda_{\vee\wedge} \cup \Lambda_{\forall\exists}$. Δ_{i+1} is a finite set and, for any ϕ in Δ_{i+1}, the complexity of ϕ is at most $i+1$. ■

6.3.2 Main result

Based on the results obtained in subsection 6.3.1, it is possible to prove the main result of this chapter.

Theorem 6.19 *Let $\mathcal{L}$ be a finite language, and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. Then, for any $k < \omega$, the following are equivalent:*

1. *$\mathcal{M} \asymp_k \mathcal{N}$;*

2. *For any sentence ϕ of $\mathcal{L}$ such that $\widehat{qr}(\phi) \leq k$, $\mathcal{M} \models \phi \Leftrightarrow \mathcal{N} \models \phi$.*

Proof. ($1 \Rightarrow 2$) Assume 1. The result is a consequence of the following claim:

($*$) Let H be the set of isomorphisms defining $\mathcal{M} \asymp_k \mathcal{N}$. For any $m, n < \omega$ such that $m + n \leq k$, for any elementary extensions $\mathcal{M}_m \succeq \mathcal{M}$ and $\mathcal{N}_m \succeq \mathcal{N}$, for every $h_m \in H$ with domain $\{a_1, \ldots, a_m\}$, for some sequence $\bar{a} = \langle a_1, \ldots, a_m \rangle$ of $\mathcal{M}_m$, and whose image is some finite substructure of $\mathcal{N}_m$, and for every quasi-prenex normal form ϕ of $\mathcal{L}$ whose free-variables are within $\bar{x} = x_1, \ldots, x_m$ and $\widehat{qr}(\phi) = n$, $\mathcal{M}_m \models \phi\ \bar{x}[\bar{a}] \Leftrightarrow \mathcal{N}_m \models \phi\ \bar{x}[h_m(a_1) \ldots h_m(a_m)]$.

In order to prove this claim, fix some $\mathcal{M}_m$, $\mathcal{N}_m$ and h_m such as in its condition. Claim ($*$) follows by induction on n.

First, assume that $n = 0$. By definition of isomorphism, for every ϕ such that $\widehat{qr}(\phi) = n = 0$, $\mathcal{M}_m \models \phi\ \bar{x}[\bar{a}] \Leftrightarrow \mathcal{N}_m \models \phi\ \bar{x}[h_m(a_1) \ldots h_m(a_m)]$.

Now assume $n > 0$. So, ϕ is of the form $Q_1 y_1 \ldots Q_n y_n \psi$ such that, for every $0 < i \leq n$, "Q_i" is either "$\forall$" or "$\exists$". Consider the following two cases:

a) Assume "$Q_1 y_1$" is "$\exists y_1$" and $\mathcal{M}_m \models \phi\ \bar{x}[\bar{a}]$. Then, $\mathcal{M}_m \models Q_2 y_2 \ldots Q_n y_n \psi\ \bar{x} y_1[\bar{a} \frown (b)]$, for some element b of $\mathcal{M}_m$. Since $\mathcal{M} \asymp_k \mathcal{N}$, there is some $\mathcal{N}_{m+1} \succeq \mathcal{N}_m$ and an isomorphism $h_{m+1} \in H$ with domain $\{a_1, \ldots, a_m, b\}$ and whose image is some finite substructure of

$\mathcal{N}_{m+1}$, $h_m \subseteq h_{m+1}$. By inductive hypothesis, $\mathcal{N}_{m+1} \models Q_2 y_2 \dots Q_n y_n \psi\ \bar{x} y_1 [h_{m+1}(a_1) \dots h_{m+1}(a_m)\ h_{m+1}(b)]$. By elementary embedding, $\mathcal{N}_m \models \phi\ \bar{x}[h_m(a_1) \dots h_m(a_m)]$. Proof in the other direction works in exactly the same way.

b) Assume "$Q_1 y_1$" is "$\forall y_1$". Exploring the definability of "$\forall$" in terms of "$\sim$" and "$\exists$", assume $\mathcal{M}_m \models \exists y_1 \sim Q_2 y_2 \dots Q_n y_n \psi\ \bar{x}[\bar{a}]$. Using the quasi-prenex normal form of the formula $\exists y_1 \sim Q_2 y_2 \dots Q_n y_n \psi$, the rest follows exactly as in case a). I omit the details for space reason.

This completes the proof of $(*)$, condition 2 being a subcase of it.

$(2 \Rightarrow 1)$ Assume 2. The proof goes by induction on $i \leq k$. First, assume $i = 0$. By 2, $\mathcal{M}$ and $\mathcal{N}$ agree in all $\neg\circ$-free sentences. So, $\emptyset$ is isomorphism between (empty) substructures of $\mathcal{M}$ and $\mathcal{N}$. Hence, $\mathcal{M} \asymp_0 \mathcal{N}$.

For the inductive case, assume that, for elementary extensions $\mathcal{M}_i \succeq \mathcal{M}$ and $\mathcal{N}_i \succeq \mathcal{N}$, there is a sequence $\bar{a} = \langle a_1, \dots, a_i \rangle$ of $\mathcal{M}_i$ such that there is an isomorphism $h_i \in H$ with domain $\{a_1, \dots, a_i\}$ and whose image is some finite substructure of $\mathcal{N}_i$.

In order to obtain the result, it is necessary to consider the following tasks:

- For any element c of $\mathcal{M}_i$, for some elementary extension $\mathcal{N}_{i+1} \succeq \mathcal{N}_i$, look for some element d of $\mathcal{N}_{i+1}$ such that there is an isomorphism $h_{i+1} \in H$, $h_{i+1} = h_i \cup \{\langle c, d \rangle\}$;

- For any element d of $\mathcal{N}_i$, for some elementary extension $\mathcal{M}_{i+1} \succeq \mathcal{M}_i$, look for an element c of $\mathcal{M}_{i+1}$ such that there is an isomorphism $h_{i+1} \in H$, $h_{i+1} = h_i \cup \{\langle c, d \rangle\}$.

Without loss of generality, let me consider here the former task. Fix some element c of $\mathcal{M}_i$. Now consider the chain $(\Delta^r : r < \omega)$ of sets of formulas of $\mathcal{L}$ such that $\Delta^r = \{\phi :$ for $n + i < k, \phi$ is quasi-prenex normal form $Q_1 y_1 \dots Q_n y_n \psi$ with free-variables within $\bar{x} = x_1, \dots, x_i$ and z, $cp(\psi) \leq r, \mathcal{M}_i \models \phi\ \bar{x} z[\bar{a} \frown (c)]\}$. Note that, for any $r < q < \omega$, $\Delta^r \subset \Delta^q$.

Fix some Δ^r. By Lemma 6.18, Δ^r is a finite set. Then, $\bigwedge \Delta^r$ is a well formed formula with $\widehat{qr}(\bigwedge \Delta^r) = k - i - 1$. By definition, $\mathcal{M}_i \models \bigwedge \Delta^r\ \bar{x} z[\bar{a} \frown (c)]$. By Lemma 6.17, there is a quasi-prenex normal form $Q z_1 \dots Q z_{k-i-1} \bigwedge \Sigma^r$ that is equivalent to the formula $\bigwedge \Delta^r$. Then, $\mathcal{M}_i \models Q z_1 \dots Q z_{k-i-1} \bigwedge \Sigma^r\ \bar{x} z[\bar{a} \frown (c)]$. So, $\mathcal{M}_i \models \exists z Q z_1 \dots Q z_{k-i-1} \bigwedge \Sigma^r\ \bar{x}[\bar{a}]$.

Since $\widehat{qr}(\exists z Q z_1 \dots Q z_{k-i-1} \bigwedge \Sigma^r) + i = k$, by the claim $(*)$ demonstrated in the first part of this proof, $\mathcal{N}_i \models \exists z Q z_1 \dots Q z_{k-i-1} \bigwedge \Sigma^r\ \bar{x}[h_i(a_1) \dots h_i(a_i)]$, for any $r < \omega$. So, $\bigcup(\Delta^r : r < \omega)$ is finitely satisfied by $\mathcal{N}_i$. By Corollary 6.11, there is some elementary extension $\mathcal{N}_{i+1} \succeq \mathcal{N}_i$ such that, for some element d of $\mathcal{N}_{i+1}$, $\mathcal{N}_{i+1} \models Q z_1 \dots Q z_{k-i-1} \bigwedge \Sigma^r\ \bar{x} z[h_i(a_1) \dots h_i(a_i)\ d]$, for every $r < \omega$.

Define $h_{i+1} = h_i \cup \{\langle c, d \rangle\}$. I will prove that h_{i+1} is an isomorphism. First, note that, for any atomic formula ϕ with free-variables within $x_1, \dots, x_i, z$, if $\phi \in \Delta^0$, then $\mathcal{M}_i \models \phi\ \bar{x} z[\bar{a} \frown (c)]$ and $\mathcal{N}_{i+1} \models \phi\ \bar{x} z[h_{i+1}(a_1) \dots h_{i+1}(a_i)\ d]$; otherwise, $\mathcal{M}_i \not\models \phi\ \bar{x} z[\bar{a} \frown (c)]$ and $\mathcal{N}_{i+1} \not\models \phi\ \bar{x} z[h_{i+1}(a_1) \dots h_{i+1}(a_i)\ d]$. So, h_{i+1} preserves satisfaction of atomic formulas.

Secondly, note that, for any formula of the form $\neg\psi$ with free-variables within $x_1, \ldots, x_i, z$, if $\phi \in \Delta^{cp(\psi)+1}$, then $\mathcal{M}_i \models \phi\ \bar{x}z[\bar{a} \frown (c)]$ and $\mathcal{N}_{i+1} \models \phi\ \bar{x}, z[h_{i+1}(a_1) \ldots h_{i+1}(a_i)\ d]$; otherwise, $\mathcal{M}_i \not\models \phi\ \bar{x}z[\bar{a} \frown (c)]$ and $\mathcal{N}_{i+1} \not\models \phi\ \bar{x}z[h_{i+1}(a_1) \ldots h_{i+1}(a_i)\ d]$. So, h_{i+1} preserves satisfaction of all formulas of the form $\neg\psi$. Hence, by a similar argument involving formulas of the form $\circ\psi$, it is possible to conclude that h_{i+1} is an isomorphism. ■

6.4 Conclusion

I would like to conclude with a comparison between the classical Fraïssé-Hintikka theorem and Theorem 6.19. First, let me remark some parallels between these results. Although the lack of prenexification puts a limit to a full recovery of Fraïssé-Hintikka theorem in QmbC, the non-compositionality of this system draws an analogy between the pairs of concepts quantifier-free formula/ $\neg\circ$-free formula, prenex normal form/ quasi-prenex normal form and quantifier rank/ $\neg\circ$-free quantifier rank. Moreover, even though, contrary to classical logic, in QmbC the agreement in sentences of $\neg\circ$-free quantifier rank $\leq k$ does not enable the construction of k-back and forth equivalence, at least in this system it is possible to construct k-back and forth equivalence in elementary extensions. In this sense, there is a structural similarity between classical Fraïssé-Hintikka theorem and Theorem 6.19.

Further, as in the classical case, it is easy to extend Theorem 6.19 to "full" QmbC languages, that is, QmbC languages with identity, functions and constants (in fact, the basic model theory for QmbC presented in section 6.2 holds for "full" languages). As I have shown in chapter 3, when considering "full" languages, Fraïssé-Hintikka theorem considers in particular the unnested formulas of the language. Now, this can be done as well in QmbC since this system also enables the translation of any formula into an unnested variant. Consequently, this logic has all the tools required for generalizing Theorem 6.19 for richer languages.

Despite these similarities, Theorem 6.19 states a result that diverges in many aspects from its classical counterpart as well. As I showed in chapter 3, the Fraïssé-Hintikka theorem implies the existence of Hintikka normal forms. On the other hand, apparently Theorem 6.19 does not imply an analogous result. Informally speaking, the game-normal formulas of quantifier rank k composing a Hintikka normal form characterize k-back and forth equivalence by determining, for any sequence of elements $\bar{a}$ of length k of a given model, in which ways $\bar{a}$ can behave with respect to the *finite* set of atomic formulas of a finite language.

So, for example, for a finite monadic language $\mathcal{L}$, the set of atomic formulas of $\mathcal{L}$ with x as free-variable is $\Phi = \{P(x) : P \text{ is a predicate of } \mathcal{L}\}$ and any game-normal formula θ of $\mathcal{L}$ of quantifier rank 1 is $\exists x[\bigwedge \Psi \wedge \bigwedge \neg(\Phi - \Psi)]$, for some $\Psi \subseteq \Phi$. In turn, in QmbC, given the non-compositionality of the system, an analogous notion of game-normal formula should also explicitly determine the satisfiability of formulas of the form $\neg\psi$ or $\circ\psi$. However, this enlarged set of formulas is infinite, a situation that is incompatible with the finitary character of the formulas of the language.

This divergence between classical Fraïssé-Hintikka theorem and Theorem 6.19 reflects a huge difference between classical logic and QmbC. First, based on game-normal formulas it is possible to define a *monotonic basis* for first order logic, i.e., it is possible to build chains $(\theta_k : k < \omega)$ of game-normal sentences such that the quantifier rank of θ_k is k, for any $k < q < \omega$, $\theta_q \rightarrow \theta_k$ is a validity of classical logic and $\bigcup(\theta_k : k < \omega)$ is a complete first order theory (RANTALA, 1987, pp. 60–62). Thus, the lack of game-normal formulas for QmbC means that there are not such monotonic bases in this system. Secondly, based on the syntactic properties of game-normal forms it is possible to describe a deductive system for classical logic. Now, the lack of game-normal formulas for QmbC means also that there is no such deductive system for this logic. Therefore, it is not possible to characterize the semi-decidability of QmbC based on Theorem 6.19, contrary to what can be done in classical logic.

Of course, the lack of a strong version of Fraïssé-Hintikka theorem for QmbC means that QmbC's logical framework does not provide enough resources for the formalization of TSI. So, the main result obtained in this chapter on the internalization of semantic information by LFIs is negative: in minimal systems of LFIs such as QmbC it is not possible to formalize TSI.

However, QmbC is just a minimal system of LFI. So, perhaps it is reasonable to expect that stronger LFIs validate stronger versions of the Fraïssé-Hintikka theorem. Thus, one could raise the following question: Are there LFI-extensions of QmbC that validate stronger versions of the Fraïssé-Hintikka theorem? This is a difficult issue for two reasons. First, given that the limitation of Theorem 6.19 is associated with the non-compositionality of QmbC, a property that stronger LFIs inherit, maybe a theorem as strong as classical Fraïssé-Hintikka theorem itself cannot be obtained in LFI-extensions. On the other hand, possibly QmbC's procedure of quasi-prenexification can be improved in stronger LFIs (for instance, the system *LFI1* seems to provide an improvement of that procedure). In this sense, perhaps it is reasonable to expect that LFI-extensions provide upgrades of Theorem 6.19, even if a result as strong as classical Fraïssé-Hintikka theorem cannot be obtained.

Chapter 7

Conclusion

I would like to finish this thesis with some *aporetic* remarks. These remarks do not undermine the results presented in this thesis. Rather, they advance a critical analysis of how this thesis changes the landscape of the debate on semantic information. This analysis will offer, then, a first evaluation of what I have obtained so far and of the work that still needs to be done on the subject.

In this thesis I have proposed to examine the following problem: how much semantic information does a sentence carry? In particular, I have tried to provide an answer to this problem that is not committed to SoD, the idea that logical truths are non-informative. Traditionally, in order to block SoD, philosophers have rejected TSI as an erroneous theory of semantic information. Now, my present analysis of this topic is motivated by the claim that TSI in fact provides a good measure of semantic information and SoD can be blocked by a better account of the nature of this concept. In this sense, I claimed here that ordinary speakers have a partial knowledge of truth-conditions associated with their semantic competence and this epistemic situation modulates the semantic information that they grasp from contentful sentences. Further, I argued that the epistemic situation of ordinary speakers is formally captured by urn semantics, a semantic system that provides a formalization of TSI without SoD.

The search for a theory of semantic information without SoD is included in a wider discussion on the ampliative character of logical knowledge: given that we can learn something that we did not know before by discovering that a certain sentence expresses a logical truth, SoD cannot be right. In this sense, an adequate analysis of semantic information provides also a good analysis of the ampliative character of logical knowledge.

The problem of the ampliative character of logical knowledge is best known in the literature as the problem of *logical omniscience*, a classical consequence of standard systems of *epistemic logic*. Epistemic logic is the branch of logic that studies the formal aspects of our use of expressions of the kind "S knows that ..." in discourse and reasoning. The standard approach to epistemic logic conceives "S knows that ..." as a modal operator K_s similar to necessity (FAGIN et al., 2004, pp. 18–ff.). Further, note that standard systems of epistemic logic are *normal* modal systems, that is, these systems have K as an axiom ($K_s(\phi \rightarrow \psi) \rightarrow (K_s\phi \rightarrow K_s\psi)$) and accept necessitation ($\vdash \phi \setminus \vdash K_s\phi$) as a logical rule. Logical omniscience is the thesis that, generally speaking, rational agents know all logical facts, that is, know all logical truths and all the logically valid patterns of inference. More formally stated: if S knows ϕ and ψ is a logical consequence of ϕ, then S knows ψ (HENDRICKS; SYMONS, 2015, sec. 1).

Now, logical omniscience is implausible: real rational agents ignore an infinite amount of logical facts (e.g., although the reader may know Peano's axioms, he certainly does not know all arithmetic theorems). Of course, there is certainly a minimum amount of logical facts that are known by any rational agent. However, what is the criterion that distinguishes obvious from non-obvious logical facts? Note that this question is the epistemological counterpart of the question of what logical truths (and what not) have in fact null information.[1]

Fagin et al. (2004, ch. 9) and Halpern and Pucella (2011) classify at least four general strategies of solution to the problem of logical omniscience: i) first, it is possible to block logical omniscience by rejecting the modal definition of K_s as a necessity operator; ii) secondly, we can solve the problem by revising the underlying semantic notion of truth; iii) thirdly, we can solve the problem by adding some extra clauses (e.g., awareness) to the modal definition of K_s; iv) finally, we can enrich the considered modal frames with epistemically possible worlds.

Perhaps there is no univocal answer to the problem of logical omniscience,[2] but strategies of type iv) provide nice solutions since they preserve the standard modal definition of K_s and offer a simple explanation of the falsity of logical omniscience: rational agents fail to know some logical facts because they usually assume that some logically impossible worlds are in fact possible. Considering in particular the correlative problem of semantic information, in this thesis I subscribed a solution of type iv). However, this kind of solution faces a big challenge: how can we determine which logically impossible worlds are epistemically possible?[3]

Type iv) solutions based on urn semantic were originally suggested by Rantala (1979) and Hintikka (1979).[4] This thesis adds some further reasons for claiming that urn semantics provides an adequate way of resolution of the problems of semantic information and logical omniscience. Now, this work represents just a first (although quite relevant) step in this research agenda. In this sense, it is important to recognize that, as a framework of solution of those problems, urn semantics suffers both of a *completeness* and a *soundness* flaw.

First, urn semantics is an unsound framework of resolution of the problems of semantic information and logical omniscience because there are some non-obvious logical facts (i.e., logical facts that can be ignored by some rational agents) that are considered obvious by urn

[1] In chapter 2, when examining ordinary speakers' partial knowledge of truth-conditions, I considered also a dual version of the logical omniscience problem: whilst example 2.2 presents a case in which a reasoner does not know that some sentence is a logical consequence of another one, example 2.1 presents a (dual) case in which a reasoner does not know that some sentences are compatible. This dual version of the problem of logical omniscience is in general unrecognized in the literature of the area.

[2] This is exactly the diagnosis by Halpern and Pucella (2011).

[3] Note that this question is independent of the metaphysical problem about the nature of impossible worlds.

[4] Later, Rantala (1982) radicalized these ideas by suggesting that every impossible world is epistemically possible.

semantics. In particular, urn semantics recognizes as obvious logical facts all the validities of classical propositional logic. With respect to semantic information, this unsoundness means that there are really informative logical truths (in particular, some validities of propositional logic) which are considered uninformative by TSI based on urn semantics. Therefore, even though there are good reasons for thinking that urn semantics is an interesting first step in the resolution of the problems of semantic information and logical omniscience, this framework must be *generalized* in order to obtain a more correct solution of these problems.

Moreover, urn semantics is an incomplete framework of resolution of the problems of semantic information and logical omniscience because it says that some obvious logical facts are non-obvious. In this sense, for instance, urn semantics says that the validity $\forall x \exists y (x = y)$ is non-obvious. However, it is pretty obvious that this sentence expresses a validity! With respect to semantic information, this completeness flaw means that there are some non-informative logical truths (in particular, some validities of first order logic) which are seen as really informative by TSI based on urn semantics. Hence, despite the interesting solution of the problems of logical omniscience and semantic information suggested by urn semantics, this semantic framework must be *restricted* in order to obtain a more complete solution of these problems.

So, on one hand, given the above mentioned incompleteness of urn semantics, this semantic system needs to be restricted, but, on the other hand, given the above mentioned unsoundness of urn semantics, this system needs to be generalized. Can we do both things at the same time? This is the problem that we need to fix if we are convinced that urn semantics provides a nice type iv) solution of the problems of logical omniscience and semantic information.

The recent literature on type iv) solutions abandon the use of urn semantics in the definition of epistemically possible worlds. Rather, these proposals explore a *bounded rationality account* of epistemic possibility. Important examples of this approach include D'Agostino (2010), Artemov and Kuznets (2009) and Jago (2006, 2009, 2013). In what follows I consider Jago's work in particular. The technical framework of his ideas varies in important ways between his works. For convenience, in what follows I focus on Jago (2013), whose ideas were already presented in subsection 4.3.2 of this thesis.

In subsection 4.3.2 I showed that, in Jago's account, an impossible world w is epistemically possible if and only if its characteristic sequent $w^+ \vdash w^-$ has great proof complexity. The idea is that, if the smallest proof of $w^+ \vdash w^-$ is big enough, then it is hard to prove the validity of this sequent and, consequently, it is reasonable to suppose that a rational agent could not know that it is valid. So, Jago's proposal explores the general insight that epistemic possibility must be defined with respect to hard provability, that is, a logical impossibility is epistemically possible if and only if the proof of its impossibility is so difficult that it usually exhausts rational agents' computational capacities.

Jago's account suffers from some drawbacks. First, Bjerring (2013) proved that if we assume that worlds have maximal theories, then, for every impossible world w, $w^+ \vdash w^-$ has minimum proof complexity, a quite undesirable result. Now, although Bjerring's result imposes an important challenge to the bounded rationality account of epistemic possibility, it also immediately suggests the consideration of partial worlds (i.e., worlds with non-maximal theories) as a way of solution. Epistemically possible partial worlds could then be defined as verifying only hardly provable logical facts (see Jago (2014) for a proposed revision of his previews work using partial worlds).

Secondly, one could ask what is the demarcation criterion between easily and hardly provable logical facts. In other words: how big must the smallest proof of $w^+ \vdash w^-$ be in order for this sequent to be considered hardly provable? As I showed before, in Jago's opinion, the notion of hard provability is a vague concept. Consequently, Jago understands that the concept of epistemic possibility is vague. Against this claim, it is enough to consider the objections that I presented in subsection 4.3.2.

Finally, I also mentioned already that the definite description "the smallest proof of $w^+ \vdash w^-$" is not always computable. So, Jago's concept of epistemic possibility is not computable as well. This is arguably a problem since it implies that we cannot effectively say which logical impossibilities are excluded by ordinary reasoners' epistemic situation and which are not. Note on the other hand that a characterization of epistemic possibility based on urn semantics such as the one proposed in this thesis does not suffer from any of these problems.

Now, it seems to me that the appeal to concepts such as great proof-complexity, hard provability etc. in order to explain epistemic possibility is a wrong move in this debate: the use of these concepts makes the corresponding solution to the problems of semantic information and logical omniscience depend on the particular cognitive capabilities of groups of individuals. Hence, this strategy of solution faces the risk of psychologism. On the other hand, the notion of obvious logical fact seems to have a *normative* status that is generally unrecognized in such studies: obvious logical facts characterize the minimum standards of logical knowledge that *must* be met by individuals in order for them to have the status of rational agents. In this sense, an obvious logical fact ϕ is not just knowledge shared by all rational agents, but also any rational agent, insofar as she is supposedly interacting with other rational agents, has the right to ascribe to any other rational agent the knowledge of ϕ.

In this sense, this thesis explored an alternative account of epistemic possibility – what could be called a *linguistic account* of epistemic possibility. Obvious logical facts characterize the partial knowledge of truth-conditions associated with the semantic competence of ordinary speakers. This account is not psychologistic: in fact, it is possible for individuals to ignore some obvious logical facts. However, an individual who ignores some obvious logical fact is not a *competent* user of the language (a normative status).

So, the work done here advances a first step in the direction of a linguistic account of epistemic possibility, an account that suggests an interesting type iv) solution to the problems of logical omniscience and semantic information without the difficulties associated with rival proposals. There is still a lot of work to do in the continuation of this research as I showed above. The question of the epistemology of logic is surely not an easy problem! However, I hope that the work developed here offers at least a small contribution in the examination of this difficult issue.

Bibliography

ADRIAANS, P. Information. In: ZALTA, E. N. (Ed.). *The Stanford Encyclopedia of Philosophy*. Stanford: Metaphysics Research Lab, Stanford University, 2013.

ARTEMOV, S.; KUZNETS, R. Logical omniscience as a computational complexity problem. In: HEIFETZ, A. (Ed.). *Proceedings of the 12th Conference on Theoretical Aspects of Rationality and Knowledge*. New York: ACM, 2009. p. 14–23.

BAR-HILLEL, Y.; CARNAP, R. *An Outline of a Theory of Semantic Information*. Cambridge: Research Laboratory of Electronics, MIT, 1952.

BAR-HILLEL, Y.; CARNAP, R. Semantic information. *The British Journal for the Philosophy of Science*, v. 4, n. 14, p. 147–157, 1953.

BARRIO, E. Models & proofs: LFIs without a canonical interpretation. *Manuscript submitted for publication*, 2017.

BENACERRAF, P. What numbers could not be. *The Philosophical Review*, v. 74, n. 1, p. 47–73, 1965.

BJERRING, J. C. Impossible worlds and logical omniscience: an impossibility result. *Synthese*, v. 190, n. 13, p. 2505–2524, 2013.

BOOLOS, G. S.; BURGESS, J. P.; JEFFREY, R. C. *Computability and Logic*. Cambridge: Cambridge university press, 2002.

BREMER, M. E. Do logical truths carry information? *Minds and Machines*, v. 13, n. 4, p. 567–575, 2003.

BURGE, T. Individualism and the mental. *Midwest Studies in Philosophy*, v. 4, n. 1, p. 73–121, 1979.

CARNIELLI, W. et al. On the way to a wider model theory: Completeness theorems for first-order logics of formal inconsistency. *The Review of Symbolic Logic*, v. 7, n. 3, p. 548–578, 2014.

CARNIELLI, W.; CONIGLIO, M. E.; MARCOS, J. Logics of formal inconsistency. In: *Handbook of Philosophical Logic*. Netherlands: Springer, 2007. p. 1–93.

CHALMERS, D. J. Epistemic two-dimensional semantics. *Philosophical Studies*, v. 118, n. 1, p. 153–226, 2004.

CRESSWELL, M. J. Urn models: A classical exposition. *Studia Logica*, v. 41, n. 2-3, p. 109–130, 1982.

CURD, M.; COVER, J. A. Science and pseudoscience: Introduction. In: CURD, M.; COVER, J. (Ed.). *Philosophy of Science: The Central Issues*. New York: Norton, 1998. p. 1–2.

D'AGOSTINO, M. Tractable depth-bounded logics and the problem of logical omniscience. In: HOSNI, H.; MONTAGNA, F. (Ed.). *Probability, Uncertainty and Rationality*. Netherlands: Springer, 2010. p. 245–275.

D'AGOSTINO, M.; FLORIDI, L. The enduring scandal of deduction. *Synthese*, v. 167, n. 2, p. 271–315, 2009.

EBBINGHAUS, H.-D. Chapter II: Extended logics: The general framework. In: *Model-theoretic Logics*. Netherlands: Springer, 1985. p. 25–76.

FAGIN, R. et al. *Reasoning about Knowledge*. Cambridge: MIT Press, 2004.

FETZER, J. H. Information: Does it have to be true? *Minds and Machines*, v. 14, n. 2, p. 223–229, 2004.

FINE, K. *Reasoning with Arbitrary Objects*. New Jersey: Blackwell, 1985.

FLORIDI, L. Outline of a theory of strongly semantic information. *Minds and Machines*, v. 14, n. 2, p. 197–221, 2004.

FLORIDI, L. Is semantic information meaningful data? *Philosophy and Phenomenological Research*, v. 70, n. 2, p. 351–370, 2005.

FLORIDI, L. Semantic conceptions of information. In: ZALTA, E. N. (Ed.). *The Stanford Encyclopedia of Philosophy*. Stanford: Metaphysics Research Lab, Stanford University, 2016.

FODOR, J. A. *Psychosemantics: The problem of Meaning in the Philosophy of Mind.* Cambridge: The MIT Press, 1987.

FRENCH, R. A sequent calculus for urn logic. *Journal of Logic, Language and Information*, v. 24, n. 2, p. 131–147, 2015.

GRICE, H. P.; STRAWSON, P. F. In defense of a dogma. *The Philosophical Review*, v. 65, n. 2, p. 141–158, 1956.

HALPERN, J. Y.; PUCELLA, R. Dealing with logical omniscience: expressiveness and pragmatics. *Artificial Intelligence*, v. 175, n. 1, p. 220–235, 2011.

HANSSON, S. O. Formalization in philosophy. *Bulletin of Symbolic Logic*, v. 6, n. 2, p. 162–175, 2000.

HAWTHORNE, J. A priority and externalism. *Internalism and Externalism in Semantics and Epistemology*, p. 201–218, 2007.

HENDRICKS, V.; SYMONS, J. Epistemic logic. In: ZALTA, E. N. (Ed.). *The Stanford Encyclopedia of Philosophy*. Stanford: Metaphysics Research Lab, Stanford University, 2015.

HINTIKKA, J. Are logical truths analytic? *The Philosophical Review*, p. 178–203, 1965.

HINTIKKA, J. Distributive normal forms in first-order logic. *Studies in Logic and the Foundations of Mathematics*, v. 40, p. 48–91, 1965.

HINTIKKA, J. Information, deduction, and the a priori. *Nous*, p. 135–152, 1970.

HINTIKKA, J. Surface information and depth information. In: HINTIKKA, J.; SUPPES, P. (Ed.). *Information and Inference*. Netherlands: Springer, 1970. p. 263–297.

HINTIKKA, J. Impossible possible worlds vindicated. In: SAARINEN, E. (Ed.). *Game-Theoretical Semantics*. Netherlands: Springer, 1979. p. 367–379.

HODGES, W. *A shorter model theory*. Cambridge: Cambridge university press, 1997.

HÁJEK, A. Interpretations of probability. In: ZALTA, E. N. (Ed.). *The Stanford Encyclopedia of Philosophy*. Stanford: Metaphysics Research Lab, Stanford University, 2012.

JACKSON, F. Reference and description revisited. *Nous*, v. 32, n. S12, p. 201–218, 1998.

JACKSON, F. Why we need a-intensions. *Philosophical Studies*, v. 118, n. 1, p. 257–277, 2004.

JAGO, M. Imagine the possibilities: information without overload. *Logique et Analyse*, p. 345–370, 2006.

JAGO, M. Hintikka and Cresswell on logical omniscience. *Logic and Logical Philosophy*, v. 15, n. 4, p. 325–354, 2007.

JAGO, M. Logical information and epistemic space. *Synthese*, v. 167, n. 2, p. 327–341, 2009.

JAGO, M. The content of deduction. *Journal of Philosophical Logic*, p. 1–18, 2013.

JAGO, M. The problem of rational knowledge. *Erkenntnis*, v. 79, n. 6, p. 1151–1168, 2014.

JUHL, C.; LOOMIS, E. *Analyticity*. New York: Routledge, 2009.

KAPLAN, D. Demonstratives. In: ALMOG, J.; PERRY, J.; WETTSTEIN, H. (Ed.). *Themes from Kaplan*. London: Oxford University Press, 1989. p. 481–564.

KOLMOGOROV, A. N. *Foundations of the Theory of Probability*. New York: Chelsea Pub. Co., 1950.

KRIPKE, S. A. A puzzle about belief. In: MARGALIT, A. (Ed.). *Meaning and Use*. Netherlands: Springer, 1979. p. 239–283.

LAU, J.; DEUTSCH, M. Externalism about mental content. In: ZALTA, E. N. (Ed.). *The Stanford Encyclopedia of Philosophy*. Stanford: Metaphysics Research Lab, Stanford University, 2016.

LOAR, B. Social content and psychological content. In: PESSIN, A.; GOLDBERG, S. (Ed.). *The Twin Earth Chronicles: Twenty Years of Reflection on Hilary Putnam's "The Meaning of 'meaning' "*. New York: ME Sharpe, 1996. p. 180.

MARKER, D. *Model theory: An Introduction*. Netherlands: Springer, 2006. v. 217.

MCGINN, M. *Elucidating the Tractatus: Wittgenstein's Early Philosophy of Logic and Language*. Oxford: Oxford University Press, 2006.

MOYAL-SHARROCK, D. The good sense of nonsense: a reading of Wittgenstein's Tractatus as nonself-repudiating. *Philosophy*, v. 82, n. 01, p. 147–177, 2007.

OLIN, P. Urn models and categoricity. *Journal of Philosophical Logic*, v. 7, n. 1, p. 331–345, 1978.

PUTNAM, H. Meaning and reference. *The Journal of Philosophy*, v. 70, n. 19, p. 699–711, 1973.

RANTALA, V. Urn models: a new kind of non-standard model for first-order logic. In: SAARINEN, E. (Ed.). *Game-Theoretical Semantics*. [S.l.]: Springer, 1979. p. 347–366.

RANTALA, V. Impossible worlds semantics and logical omniscience. *Acta Philosophica Fennica*, v. 35, p. 106–115, 1982.

RANTALA, V. Constituents. In: BOGDAN, R. (Ed.). *Jaakko Hintikka*. Netherlands: Springer, 1987. p. 43–76.

RESNIK, M. D. Mathematics as a science of patterns: ontology and reference. *Nous*, p. 529–550, 1981.

SCHROETER, L.; BIGELOW, J. Jackson's classical model of meaning. In: RAVENSCROFT, I. (Ed.). *Minds, Ethics, and Conditionals: Themes from the Philosophy of Frank Jackson*. Oxford: Oxford University Press, 2009.

SEQUOIAH-GRAYSON, S. The scandal of deduction. *Journal of Philosophical Logic*, v. 37, n. 1, p. 67–94, 2008.

SHANNON, C. E.; WEAVER, W. *The Mathematical Theory of Communication*. Urbana: Univ. Illinois Press, 1949.

SHOENFIELD, J. R. *Mathematical Logic*. Massachusetts: ASL, 1967.

SLATER, B. H. Paraconsistent logics? *Journal of Philosophical logic*, v. 24, n. 4, p. 451–454, 1995.

STALNAKER, R. Assertion. *Syntax and Semantics*, v. 9, p. 315–332, 1978.

STALNAKER, R. Narrow content. In: ANDERSON, C. A.; OWENS, J. (Ed.). *Propositional Attitudes: The Role of Content in Logic, Language, and Mind*. Stanford: CSLI, 1990.

SUPPES, P. *Axiomatic Set Theory*. Toronto: D. Van Nostrand Co., 1960.

VÄÄNÄNEN, J. *Models and Games*. Cambridge: Cambridge University Press, 2011. v. 132.

VERBRUGGE, R. L. Provability logic. In: ZALTA, E. N. (Ed.). *The Stanford Encyclopedia of Philosophy*. Stanford: Metaphysics Research Lab, Stanford University, 2017.

WITTGENSTEIN, L. *Tractatus Logico-Philosophicus. The German Text of Logisch-Philosophische Abhandlung, with a New Translation by D.F. Pears & B.F. Mcguinness, and with the Introduction By Bertrand Russell*. London: Routledge, 1963.