



Universidade Estadual de Campinas
Faculdade de Engenharia Elétrica e de Computação

Vinícius Lima Silva

**Norms and Stability of Stochastic Systems for
Which Control and State Variation Increase Uncer-
tainties**

**Normas e Estabilidade para Modelos Estocásticos
cuja Variação do Controle e do Estado Aumentam a
Incerteza**

Campinas
2018



UNIVERSIDADE ESTADUAL DE CAMPINAS
Faculdade de Engenharia Elétrica e de Computação

Vinícius Lima Silva

Norms and Stability of Stochastic Systems for which Control and State Variation Increase Uncertainties

Normas e Estabilidade para Modelos Estocásticos cuja Variação do Controle e do Estado Aumentam a Incerteza

Dissertação apresentada à Faculdade de Engenharia Elétrica e de Computação da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Mestre em Engenharia Elétrica, na Área de Automação.

Supervisor: Prof. Dr. João Bosco Ribeiro do Val

Este exemplar corresponde à versão final da tese defendida pelo aluno Vinícius Lima Silva, e orientada pelo Prof. Dr. João Bosco Ribeiro do Val

Campinas

2018

Agência(s) de fomento e nº(s) de processo(s): FAPESP, 2016/02208-6; FAPESP, 2017/10340-4

ORCID: <https://orcid.org/0000-0003-1859-9923>

Ficha catalográfica
Universidade Estadual de Campinas
Biblioteca da Área de Engenharia e Arquitetura
Luciana Pietrosanto Milla - CRB 8/8129

Si38n Silva, Vinícius Lima, 1991-
Norms and stability of stochastic systems for which control and state variation increase uncertainties / Vinícius Lima Silva. – Campinas, SP : [s.n.], 2018.

Orientador: João Bosco Ribeiro do Val.
Dissertação (mestrado) – Universidade Estadual de Campinas, Faculdade de Engenharia Elétrica e de Computação.

1. Teoria de controle estocástico. 2. Difusão - Modelos matemáticos. 3. Programação dinâmica. I. Val, João Bosco Ribeiro do, 1955-. II. Universidade Estadual de Campinas. Faculdade de Engenharia Elétrica e de Computação. III. Título.

Informações para Biblioteca Digital

Título em outro idioma: Normas e estabilidade para modelos estocásticos cuja variação do controle e do estado aumentam a incerteza

Palavras-chave em inglês:

Stochastic control theory

Diffusion processes - Mathematics

Dynamic programming

Área de concentração: Automação

Titulação: Mestre em Engenharia Elétrica

Banca examinadora:

João Bosco Ribeiro do Val [Orientador]

Oswaldo Luiz do Valle Costa

José Cláudio Geromel

Data de defesa: 25-06-2018

Programa de Pós-Graduação: Engenharia Elétrica

COMISSÃO JULGADORA — DISSERTAÇÃO DE MESTRADO

Candidato: Vinícius Lima Silva (RA 104313)

Data da defesa: 25 de junho de 2018

Título da tese: “Norms and Stability of Stochastic Systems for which Control and State Variation Increase Uncertainties” / “Normas e Estabilidade para Modelos Estocásticos cuja Variação do Controle e do Estado Aumentam a Incerteza”

Prof. Dr. João Bosco Ribeiro do Val (Presidente, FEEC / UNICAMP)

Prof. Dr. Oswaldo Luiz do Valle Costa (EP / USP)

Prof. Dr. José Cláudio Geromel (FEEC / UNICAMP)

A ata de defesa, com as respectivas assinaturas dos membros da Comissão Julgadora, encontra-se no processo de vida acadêmica do aluno.

To my family.

Acknowledgements

First and foremost I would like to thank my advisor, Prof. João Bosco, for the patience, enthusiasm and guidance during both the master's program and the previous experiences as an undergraduate research assistant. He has not only unwaveringly supported my research and guided me into the right direction when needed, but also gave me freedom to pursue various research projects. His support and my time as an undergraduate research student make up one of the main reasons why I decided to pursue an academic career in the first place.

I would also like to thank the master's thesis committee members, Prof. Geromel and Prof. Costa, for the careful reading of this rather lengthy work. Another special thanks to Prof. Geromel for his assistance with the graduate school application process.

A sincere thanks to Prof. Tamer Başar and the Coordinated Science Lab of the University of Illinois at Urbana Champaign for hosting me as a visiting student and accommodating me in his research group; and to Prof. Serdar Yüksel and the Department of Mathematics and Statistics of Queen's University at Kingston, Canada, for the short-term visit and insightful discussions amid the Canadian winter.

A special thanks to my family and friends, in particular my parents, José Nilton (in Memoriam) and Valdelice, and my siblings, Karoliny and Matheus, for the constant support through all these years.

I would also like to thank my colleagues at the Department of Systems and Energy, for the support and fruitful discussions; as well as the School of Electrical and Computer Engineering and the University of Campinas, where I spent a good many years as an undergraduate and master's student.

A final thanks to São Paulo Research Foundation (FAPESP), for the financial support for the master's research in Brazil (Fapesp grant no. 2016/02208-6) and research abroad internship (Fapesp grant no. 2017/10340-4); to the Brazilian government funding agencies *Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq)* and *Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES)*.

*“At last I can see why he can afford running around so freely. It’s because he has a place
he can always return to.”*
(Wong-Kar Wai, Happy Together (1997))

Abstract

This work discusses a new approach to the control of uncertain systems. Uncertain systems and their representation is a recurrent theme in control theory: approximate mathematical models, unmodeled dynamics and external disturbances are all sources of uncertainties in automated systems, and the topic has been extensively studied in the control literature, particularly within the stochastic and robust control research areas. Within the stochastic framework, a recent approach, named CVIU — *control variation increases uncertainty*, for short —, was recently proposed. The approach differs from previous models for assuming that a control action might actually increase the uncertainty about an unknown system, a notion represented by the use of stochastic noise depending on the absolute value of the control input. Moreover, the solution of the corresponding stochastic optimal control problem shows the existence of a region around the equilibrium point in which the optimal action is to keep the equilibrium control action unchanged. The CVIU control problem was previously solved by adopting a discounted quadratic cost formulation, and in this work we extend this previous result and study the corresponding long run average control problem. We also discuss possible relations between the CVIU approach and models from robust control theory, and present some potential applications of the theory presented here.

Keywords: Stochastic Control; Uncertainties; Robust Control.

Resumo

Essa dissertação de mestrado gira em torno da discussão sobre controle de sistemas incertos. Modelos matemáticos utilizados como base para o design de controladores automáticos são naturalmente uma representação aproximada do sistema real, o que, em conjunto com perturbações externas e dinâmica não modelada, gera incertezas a respeito dos sistemas estudados. Na literatura de controle, este tema vêm sendo discutido frequentemente, em particular nas sub-áreas de controle estocástico e controle robusto. Dentre as técnicas desenvolvidas dentro da teoria de controle estocástico, uma proposta recente se diferencia das demais por basear-se na idéia de que variações abruptas na política de controle possam acarretar em maiores incertezas a respeito do sistema. Matematicamente, essa noção é representada pelo uso de um ruído estocástico dependente do módulo da ação de controle, e a técnica foi apelidada de VCAI — acrônimo para *variação do controle aumenta a incerteza*. A definição da política de controle ótima correspondente, obtida por meio do método de programação dinâmica, mostra a existência de uma região ao redor do ponto de equilíbrio para a qual a política ótima é manter a ação de controle do equilíbrio inalterada, um resultado que parece particular à abordagem VCAI, mas que pode ser relacionado a políticas de gerenciamento cautelosas em áreas como economia e biologia. O problema de controle ótimo VCAI foi anteriormente resolvido ao adotar-se um critério de custo quadrático descontado e um horizonte de otimização infinito, e nessa dissertação nós utilizamos essa solução para atacar o problema de custo médio a longo prazo. Dada certa semelhança entre a estrutura do ruído estocástico na abordagem VCAI e modelos utilizados na teoria de controle robusto, discutimos ainda possíveis relações entre a abordagem proposta e controladores robustos. Discutimos ainda algumas possíveis aplicações do modelo proposto.

Palavras-chaves: Controle Estocástico; Incertezas; Controle Robusto.

List of Figures

Figure 1 – Open-loop and closed-loop control.	16
Figure 2 – $M - \Delta$ uncertainty model.	18
Figure 3 – An example of an optimal CVIU control policy.	22
Figure 4 – Simulation Path: CVIU control policy.	23
Figure 5 – CVIU optimal control policy: numerical solution.	51
Figure 6 – Correspondence between the representation of uncertainties in the CVIU and polytopic cases. Figure previously presented at (SILVA <i>et al.</i> , 2017).	87
Figure 7 – Relative operation cost as a function of error offsets in the linear pa- rameters. Figure previously presented at (SILVA <i>et al.</i> , 2017).	88
Figure 8 – CVIU and \mathcal{H}_2 control policies.	90
Figure 9 – CVIU and \mathcal{H}_2 control policies: sample path	92
Figure 10 – CVIU and \mathcal{H}_2 control policies: relative cost for different values of the nominal parameters A and B	92
Figure 11 – CVIU and \mathcal{H}_2 control policies: relative cost for different values of σ	93
Figure 12 – A 50% mismatch between the estimated parameter A and the actual system local approximation A^0 . Figure updated from (SILVA <i>et al.</i> , 2016).	101
Figure 13 – Relative gains when $\sigma = 1$. Figure updated from (SILVA <i>et al.</i> , 2016).	101
Figure 14 – Relative gains when $\sigma = 2$. Figure updated from (SILVA <i>et al.</i> , 2016).	102

Frequently Used Notation

\mathbb{R}^n	The n -dimensional Euclidean space.
I	Identity matrix.
A^\top	Transpose of matrix A .
$\text{diag}(v)$	Matrix formed by v as the main diagonal and zero elsewhere.
$\text{Diag}(A)$	Matrix formed by main diagonal of A with zeros elsewhere.
Ω	A given set.
ω	An element of set Ω .
\emptyset	Empty set.
\mathbb{F}	A subset of Ω .
\mathbb{F}^C	Complement of set \mathbb{F} .
\mathcal{F}	A σ -algebra of Ω .
\mathcal{B}	A Borel σ -algebra of Ω .
$P(\cdot)$	Probability measure.
$W(t)$	Standard Brownian motion or Wiener process.
$\mathbb{E}[\cdot]$	Expected value.
$\mathbb{E}[\cdot/\cdot]$	Conditional expected value.
\mathcal{A}	Infinitesimal generator of a diffusion process.
$C[0, \infty)^d$	Subspace of \mathbb{R}^d consisting of continuous functions.
$C^k[0, \infty)^d$	Subspace of \mathbb{R}^d consisting of continuous functions with continuous derivatives up to order k .
$C^{1,2}([0, T] \times \mathbb{R}^d)$	Class of continuous functions differentiable in the first argument and twice differentiable in the second.
$D^{1,-}v(x)$	First order sub-differential of function $v(x)$.
$D^{1,+}v(x)$	First order super-differential of function $v(x)$.

$D^{1,2,-}v(x)$	Second order sub-differential of function $v(x)$.
$D^{1,2,+}v(x)$	Second order super-differential of function $v(x)$.
S^l	Set of matrices A such that $\{A \in \mathbb{R}^l : A^\tau = A\}$.
S^{l+}	Set of matrices A such that $\{A \in S^l : A \geq 0\}$.
S^{l+}	Set of matrices A such that $\{A \in S^l : A > 0\}$.

Contents

1	Introduction	15
1.1	Control of uncertain systems	17
1.1.1	Robust Control	18
1.1.2	Stochastic Control	19
1.1.3	Data-Driven Control	21
1.2	The CVIU approach: a qualitative overview	21
1.3	Our Contribution	23
1.4	Outline	25
2	Preliminary concepts	26
2.1	Stochastic optimal control	26
2.1.1	Stochastic Processes, Brownian Motion and Stochastic Calculus . .	26
2.1.2	Stochastic Differential Equations	30
2.1.2.1	Simulation: the Euler-Maruyama Method	33
2.1.3	Diffusion Processes	33
2.1.4	Continuous-time Stochastic Optimal Control	35
2.1.4.1	Controlled Diffusion Processes	35
2.1.4.2	Cost structures	36
2.1.4.3	Problem formulation	36
2.1.4.4	Dynamic Programming	38
2.1.4.5	Sub- and superdifferentials and convex functions	40
2.1.4.6	Viscosity solutions	41
2.1.5	Extended generator	42
2.2	The CVIU control problem	44
2.2.1	Problem formulation	45
2.2.2	The inaction region	47
2.2.3	Infinite horizon and expected discounted cost	49
2.2.3.1	Design of a CVIU control policy	50
3	Norms and long run average formulation of the CVIU control problem . . .	52
3.1	What happens when σ_x is not zero?	52
3.2	Norm equivalence	56
3.2.1	Discounted cost and \mathcal{H}_2 norm	57
3.3	Long run average cost	60
3.3.1	Energy measurements and stability	62
3.3.1.1	α -observability	63
3.3.2	Stochastic Stability: the discounted case	68
3.3.3	The average case	69

3.3.4	The controlled case	71
3.3.4.1	Solution inside the inaction region	72
3.3.4.2	Asymptotic solution	74
4	Robust control of stochastic systems	77
4.1	Introduction	77
4.1.1	Related literature	78
4.1.2	Representation of uncertainties	80
4.2	Robust \mathcal{H}_2 and \mathcal{H}_∞ control of linear stochastic systems	81
4.3	The CVIU control problem and the deterministic \mathcal{H}_2 synthesis	86
4.4	Long run average cost and robust control	89
5	Applications	95
5.1	Introduction	95
5.2	Growth models in Biology	95
5.3	Uncertainties in fisheries management	96
5.4	Optimal harvesting problem	98
5.5	Numerical experiments	100
6	Conclusion	104
References		107
ANNEX A Robust \mathcal{H}_2 and \mathcal{H}_∞ control		116
A.1	\mathcal{H}_2 and \mathcal{H}_∞ spaces	116
A.1.1	\mathcal{H}_2 and \mathcal{L}_2 norms	116
A.1.2	\mathcal{H}_∞ and \mathcal{L}_∞ norms	117
A.2	The \mathcal{H}_2 optimal control problem	118
A.3	The \mathcal{H}_∞ optimal control problem	119

1 Introduction

As recent reports from the *IEEE Control Systems Society* (SAMAD; ANNASWAMY, 2014; LAMNABHI-LAGARRIGUE *et al.*, 2017) point out, control is *ubiquitous*, that is, it is omnipresent in our everyday lives — from airplanes to communication networks, from automobiles to power systems, from finance to synthetic biology, control theory and its methods have been successfully applied in different fields. There are examples of use of control as a technology as early as ancient times (ÅSTRÖM; KUMAR, 2014), while in the modern era the first use of automatic control is usually credited to James Watt’s centrifugal governor for steam engines (ÅSTRÖM; KUMAR, 2014; OGATA, 2001). Under a gross generalization, we can say control theory and its methods can be applied to regulate the operation of any system with some particular feature changing in time — take, for example, the altitude of an airplane, or the speed of a car. Under the same generalization, we can see the general control problem as a process to obtain a set of rules or policies to automatize the behavior of a dynamic system while guaranteeing its reliable operation. The control design process starts with the definition of a mathematical model which can adequately describe the dynamic behavior of the system we wish to control. The mathematical models used to describe the dynamics of the system to be controlled, however, are relatively simple, mathematically convenient approximations to the real system. The use of these models favor the design of realizable controllers, but one should take into consideration that uncertainties arising from unmodeled dynamics, estimation errors and variation of the system parameters, among others, are commonplace. The study of systems with uncertain dynamics is therefore a recurring theme in the control systems literature, and a multitude of techniques to tackle the issue of uncertainties and disturbances in dynamic systems have been developed.

One of the first concepts used to address this issue was the notion of feedback. Loosely speaking, feedback consists on measuring the output of the system, comparing it with a reference value, and using the difference as an input to the controller. This is not only an engineering concept, however (ÅSTRÖM; KUMAR, 2014). As Åström and Murray point out, feedback systems can also be found in biology, and a prime example is the regulation of sugar levels in the bloodstream (ÅSTRÖM; MURRAY, 2008). Once glucose levels rise, the pancreas releases insulin, a hormone which forces the liver to store excess glucose and therefore reduce the level of sugar in the bloodstream. On the other hand, when glucose levels are low, pancreas cells then release glucagon, a hormone which makes liver cells convert glycogen to glucose, thus increasing glucose levels. For general systems, this constant assessment of the system output and comparison with a reference value could then help to mitigate the effect of unforeseen disturbances or slight modeling

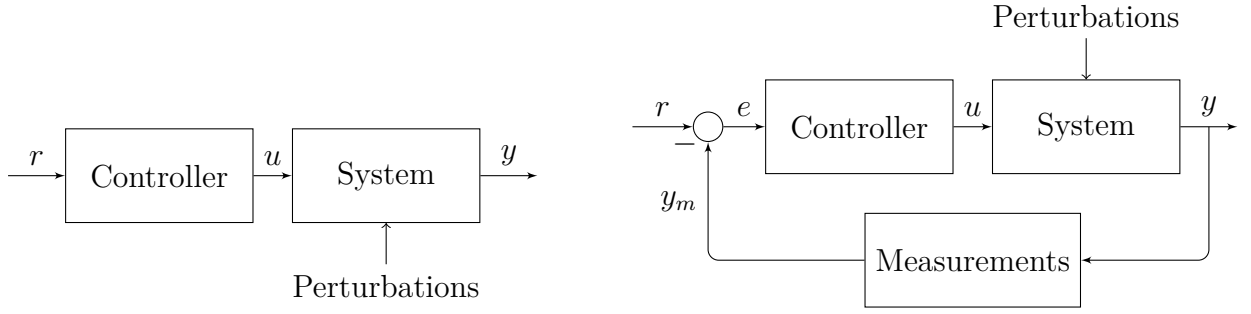


Figure 1 – Open-loop and closed-loop control.

errors: once the controller is aware the controlled output deviates from the reference input, the control action can be tuned in order to bring the controlled output back to the desired value. Control systems with feedback are called closed-loop, while their *feedback-less* counterpart are called open-loop. In Figure 1 we present simple examples of open-loop and closed-loop control.

Feedback alone cannot guarantee that the controlled system will always act according to the prescribed specifications, however, specially in face of uncertainties. That's actually when concepts such as robust control, stochastic control and, more recently, data-driven control come into mind. Each of these approaches, on its own way, aims to deal with uncertainties in dynamical systems. Broadly speaking, *robustness* refers to the control of uncertain plants subject to unknown disturbances (CHANDRASEHKARAN, 1996). Robust control then aims to provide a framework for the design of controllers which maintain stability and satisfactory performance against a set of unknown but bounded disturbances. A survey from Peterson and Tempo (PETERSEN; TEMPO, 2014) and classical books such as (ZHOU; DOYLE, 1998) give an overview of the main results in the area. The topic of uncertain systems has also been studied within the theory of stochastic control. In the stochastic case the disturbances affecting the system are modeled as stochastic processes, and a measure on the uncertainties is inbuilt in the models through the use of probability distributions. Here we point out to the survey paper from Kushner for an overview of the area (KUSHNER, 2014). More recently, approaches known as *data-driven* or *model-free* control put forward the use of statistical methods to design control algorithms based solely on data measurements, and mark a departure from modern control theory, where the design process starts with the definition of a sufficiently accurate model. A recent survey paper outlines the main results within this research direction (HOU; WANG, 2013).

This master's thesis revolves around the general topic of control of uncertain systems. Our objective is to analyze how the recently proposed CVIU approach relates, and compares to, classical techniques of the control systems literature. On the one hand, the CVIU approach differs from previous works in which it considers that any control action taken over a system with unknown dynamics might actually *increase* the uncertainties

about the real system operation. Moreover, the solution of the corresponding optimal stochastic control problem shows there exists a region around the equilibrium point where the optimal control policy is to keep the equilibrium control action unchanged. On the other hand, the mathematical representation of the CVIU model does resemble some works on robust control of stochastic systems, and a discussion on these similarities can yield a thorough description of the CVIU approach and its potential contributions. We investigate these questions in more detail in the next chapters. Before that, we briefly discuss methods currently applied to control uncertain systems, and give a simple overview of the CVIU approach.

1.1 Control of uncertain systems

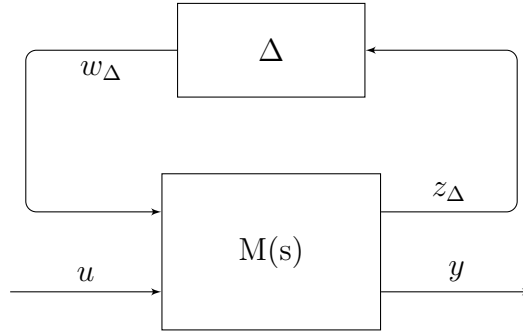
Control theory deals with the study of the behavior of dynamic systems and the development of mathematical methods to assure their reliable operation. Over the last decades, focus has been on the design of control algorithms under the assumption that mathematical models of the system were available, i.e., the first step in the design of a controller usually involves the choice of a sufficiently accurate model. When working with the time domain, the mathematical description of the system dynamics is usually done via differential equations in the continuous-time setting, or difference equations in the discrete-time case. Our work deals primarily with continuous-time systems, and we focus on them from now on. Under a general formulation, a linear, time-invariant system can be described by the system of equations

$$\begin{aligned} dx(t) &= (Ax(t) + Bu(t))dt, \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, and $y(t) \in \mathbb{R}^p$ the measured output. The system can also be modeled in terms of a transfer function matrix $G(s)$ which maps each possible control input value to the corresponding output,

$$\begin{aligned} Y(s) &= G(s)U(s), \\ G(s) &= C(sI_n - A)^{-1}B + D. \end{aligned} \tag{1.2}$$

Here, $Y(s)$ and $U(s)$ are the Laplace transforms of the measured output and control input, respectively, and I is the identity matrix. A mathematical model, however, cannot capture all the details regarding the operation of the system, and there naturally arise differences between the mathematical representation and the actual system. Sensor and actuator noise, as well as external disturbances, are further sources of uncertainties in control systems. It is possible, though, to include the description of uncertainties into the mathematical model of the system, and then take these uncertainties into account when designing a control policy. Different models to represent uncertainties were proposed in

Figure 2 – $M - \Delta$ uncertainty model.

the literature. A rather straightforward one coincides with the parametric uncertainty case, where the main idea is to assume that the matrices of the system model 1.1 or the transfer function are not exactly known but depend on a set q of uncertain parameters. In this case we get the state space representation,

$$\begin{aligned} dx(t) &= (A(q)x(t) + B(q)u(t))dt, \\ y(t) &= C(q)x(t) + D(q)u(t), \end{aligned} \quad (1.3)$$

and, correspondingly for the transfer function case, (PETERSEN; TEMPO, 2014),

$$Y(s) = G(s, q)U(s). \quad (1.4)$$

This model is used, for example, when some components are described inaccurately, and this innacuracy can be mathematically represented by the variation of parameters over a certain range (GU *et al.*, 2013). It can also be used to represent parametric uncertainties as bounds on the value of the unknown parameters, and cast the control problem in terms of a set of linear matrix inequalities (LMIs). Another model frequently used to represent uncertainties is the so-called $M - \Delta$ model (ZHOU; DOYLE, 1998; PETERSEN; TEMPO, 2014). The model is shown in Figure 2, where $M(s)$ consists of the known part of the system, i.e. the interconnections between plant and controller transfer functions, and Δ represents uncertainties. In this model, the uncertainty block Δ is usually assumed to be an unknown matrix or matrix transfer function which satisfies an appropriate matrix norm such as $\|\Delta\| \leq 1$. The $M - \Delta$ model is specially used to represent unstructured uncertainties, i.e., uncertainties about which there is not much information available — compare it to the parametric case, for example, when we assume the uncertainties are specifically related to unknown parameter values.

1.1.1 Robust Control

Under a general formulation, robust control concerns the use of mathematical methods to guarantee the reliable operation of a controlled system in face of possible disturbances affecting its operation or uncertainties regarding the exact model of the

system. The theory of robust control started to appear in the control systems literature during the 1970s. Before that, the focus of the control community was on optimality rather than robustness, and design methods usually assumed that a sufficiently accurate model was available. Failures in the use of multivariable controllers based on the classical, output-feedback linear quadratic Gaussian (LQG) approach in the early 1970s, however, motivated researchers to explore the importance of designing controllers tolerant to uncertainties (SAFONOV, 2012). Robust control theory therefore differs from previous approaches in the control systems literature in the sense that tolerance to uncertainties plays a central role in the design process. Robust control is now a vast field, and many methods to control uncertain systems were proposed within the area. Safonov's conference paper (SAFONOV, 2012), Petersen and Tempo survey paper (PETERSEN; TEMPO, 2014), and Zhou and Doyle book (ZHOU; DOYLE, 1998) give an overview of the main results regarding design methods and representation of uncertainties, as well as recent trends within the field.

Two of the most known robust control approaches credit their names to the corresponding optimization criteria. In the robust \mathcal{H}_2 case, the objective is to design a controller which stabilizes the system and minimizes the \mathcal{H}_2 norm of the closed-loop (uncertain) system. Likewise in the robust \mathcal{H}_∞ case, we look for stabilizing controllers which minimize the \mathcal{H}_∞ norm of the closed-loop system. Loosely speaking, the deterministic \mathcal{H}_2 norm measures the energy of the impulse response of the system, whereas its stochastic counterpart measures the expected power of the response to a white noise input. In the \mathcal{H}_∞ case, on the other hand, we measure the worst-case effect or maximal possible gain of an unknown (but bounded and square integrable) disturbance on the system output. The \mathcal{H}_2 control problem is related to the classical linear quadratic regulator problem and, on the \mathcal{H}_∞ side, the foundation for robust \mathcal{H}_∞ control is usually credited to Zames' seminal paper (ZAMES, 1981), followed by important contributions from other researchers during the 1980s.

1.1.2 Stochastic Control

Stochastic control can be seen as the branch of control theory which deals with systems where uncertainties are described as random processes. As a counterpart to the deterministic robust case, in stochastic control we assume the evolution of the system is subject to some noise which satisfies some known probability distribution. Under a general formulation, continuous time models used in stochastic control theory can be described by Itô's stochastic differential equation (OKSENDAL, 2007),

$$dx(t) = G(t, x(t), u(t))dt + \sigma(t, \cdot)dW(t). \quad (1.5)$$

As before, $x(t)$ represents the system state and $u(t)$ the control input. The first part of the equation, $G(t, x(t), u(t))$, is known as the *drift* vector and represents the evolution of

the system with no influence from the random noise. The second part, $\sigma(t, \cdot)$ corresponds to the stochastic term and as is known as the *diffusion* matrix. $W(t)$ is a standard Brownian motion, a stochastic process commonly used to represent uncertainties in stochastic systems, and corresponds, roughly speaking, to the integral of Gaussian white noise. The process is nowhere differentiable, so we write the differential equation in terms of the differential $dW(t)$ (KUO, 2006) instead of the mathematically inaccurate $\dot{W}(t)$. In the linear case, we can retrieve the original model for a continuous-time, linear system (1.1) to write

$$\begin{aligned} dx(t) &= (Ax(t) + Bu(t))dt + \sigma(t)dW(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{1.6}$$

Here, the stochastic term $\sigma(t)dW(t)$ aggregates the uncertainties regarding the system model and possible external disturbances.

Within the stochastic control framework, another widely used class of stochastic models corresponds to the so-called Markov Jump Linear Systems (MJLS). MJLS are used to describe dynamic systems in which abrupt changes due to environmental disturbances, sensor failures and switching, among others, are common (COSTA *et al.*, 2013). We can then imagine that, when one of these changes occurs, the system we are modeling switches among a set of linear models, each one describing a possible mode of operation. When the transition or jump mechanism between these models can be described by a Markov chain, we then get a Markov jump linear system. The mathematical model has a slightly different structure in this case,

$$\begin{aligned} dx(t) &= (A_{\theta(t)}(t)x(t) + B_{\theta(t)}(t)u(t))dt, \\ y(t) &= C_{\theta(t)}(t)x(t) + D_{\theta(t)}(t)u(t), \end{aligned} \tag{1.7}$$

with $\theta(t)$ representing the Markov chain which describes the jump process.

Given a model describing the dynamic behavior of a stochastic system, the stochastic control problem consists in minimizing a given optimality criterion subject to the evolution of the system, and can be stated under a general formulation as

$$\begin{aligned} \min J(t, x, u) \\ \text{s.t. } dx(t) &= G(t, x(t), u(t))dt + \sigma(t, \cdot)dW(t), \end{aligned} \tag{1.8}$$

where $J(t, x, u)$ is the cost functional used as optimization criterion. The most commonly used cost functionals are the discounted cost, where we introduce a decaying exponential factor and prioritize the cost evaluated at time instants closest to the initial time; long run average, where we evaluate the time-averaged cost over a finite or infinite horizon; and cost up to an exit time (BORKAR, 2005), where the cost functional is evaluated until the system state leaves a certain region. On this thesis we focus on the first class of continuous-time models described by stochastic differential equations, but similar problems can also be defined for the case of linear stochastic systems with Markov jumps or for discrete-time

systems. The solution of most stochastic control problems is based on Bellman's Dynamic Programming Principle, as we describe in chapter 2, where we present some results from the theory of stochastic control in more details.

1.1.3 Data-Driven Control

Modern control techniques usually assume that there is a mathematical model which provides at least a basic description of the system. Methods from optimal control theory, for example, start with the assumption that the mathematical representation of the system is sufficiently accurate, and aim to optimize a given criterion. Even though the focus of the control systems community has shifted with the introduction of robust control techniques in the 1980s, modern control theory is still largely based on the use of these mathematical models.

Recent approaches known as data-driven or model-free control, however, mark a departure from this framework, and from the aforementioned approaches. Here the idea is to rely only on measured data collected from the controlled system, and not on any explicit knowledge about the mathematical representation of the controlled process, to design adequate control policies (HOU; WANG, 2013). Advances in statistical learning theory and computing power, on the one hand, and difficulties in using physical and mathematical principles to model ever so complex industrial systems, on the other, justify the recent interest in this class of models.

1.2 The CVIU approach: a qualitative overview

The CVIU approach, short for *Control Variation Increases Uncertainties*, was proposed as an alternative to existing models aimed at controlling uncertain systems. It is based on the idea that, for systems with highly uncertain or unknown dynamics, a forceful control action may actually *increase* the uncertainties affecting the system. Mathematically speaking, the notion that any control action would increase the noise in the system, and that the amount of noise is proportional to the intensity of the control action, is represented by the use of the absolute value function. The CVIU approach first dealt with the discrete-time case, as presented in (PIN *et al.*, 2009). The initial idea was modeled by the addition of a stochastic noise term dependent on the absolute value of the control input to the stochastic differential equation (SDE) describing the system's dynamics. To solve the optimal control problem, a discounted cost functional was adopted. After this initial approach, the continuous-time case was developed and expanded with the addition of state-dependent noise (VAL; SOUTO, 2017). In that paper, the authors also adopted a discounted cost functional.

The solution of the optimal control problem obtained when we take the CVIU

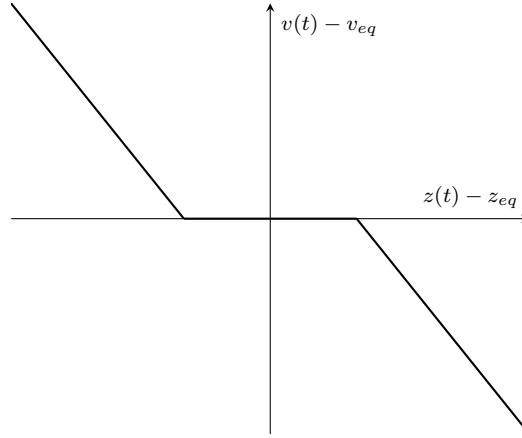


Figure 3 – An example of an optimal CVIU control policy.

approach to describe the dynamics of the controlled system shows the existence of the so-called *inaction region* around the equilibrium point, where the optimal control policy is to keep the equilibrium control action unchanged. We illustrate this result and the general structure of a state-feedback CVIU control policy for the unidimensional case in Figure 3. The figure shows the control policy obtained by the CVIU approach for a scalar system with uncertain parameters. The plot shows the difference between the control policy $v(t)$ and the equilibrium value v_{eq} for variations of the system state $z(t)$ around the equilibrium z_{eq} . The CVIU control policy exhibits a plateau around the equilibrium point in the origin, which indicates the optimal control action in that region equals the equilibrium policy v_{eq} . This plateau is what we call the *inaction region*, and this saturation-like behavior of the CVIU approach — the control policy is updated only if the state of the system leaves the boundaries of the inaction region — is a feature not shared by the previously mentioned approaches.

The existence of the inaction region in the CVIU model can be related to the adoption of cautionary control policies in other fields, such as economy and biology. In (STOKEY, 2008), for example, the author works with economic models for which the fixed cost of an action leads to the existence of an interval, also named the *inaction region*, where no control is performed; examples of such systems would be price adjustments, investment behavior and job creation. In (LOEHLE, 2006) the author suggests that, due to the uncertainties affecting the management of fisheries, aiming for a broader harvest target rather than a single peak optimum increases robustness, which can be related to a more general idea of the inaction region as a target interval where the control action remains constant.

An example from renewable resources management illustrates the dynamic behavior of a controller designed according to the CVIU approach. Two models commonly used in biology to describe the dynamics of renewable natural resources, such as fisheries, are the logistic and Gompertz equations (MURRAY, 2002). These equations, however,

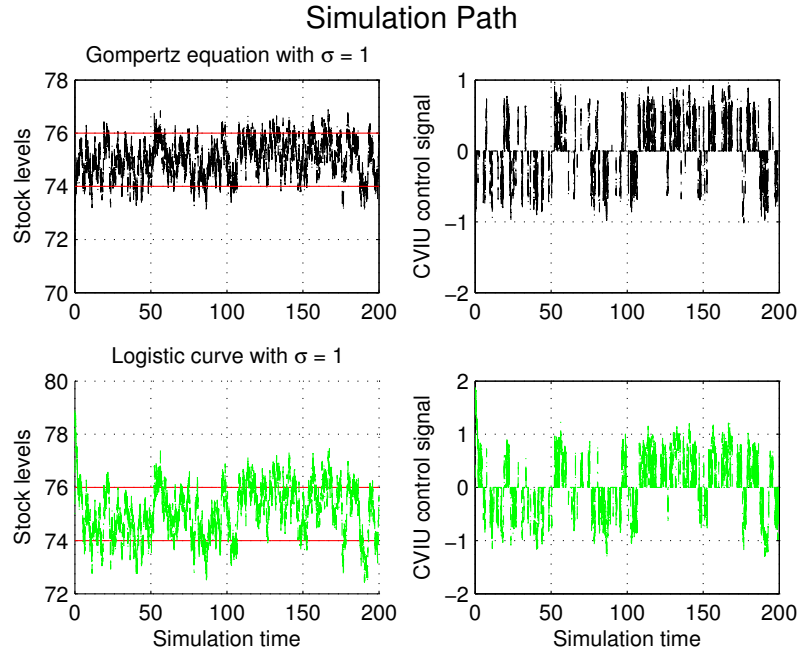


Figure 4 – Simulation Path: CVIU control policy.

yield an approximate description of the dynamics of these resources, and uncertainties are common, specially in fisheries management, where estimation and implementation errors, among others, occur frequently (SETHI *et al.*, 2005; ROUGHGARDEN; SMITH, 1996). Under these observations we designed a CVIU controller for the management of harvest in systems described by the mentioned models. A simulation for this case is shown in Figure 4, where we plot the path of the system state and the variation of the control signal for a system subject to additive white noise with diffusion coefficient $\sigma = 1$. The distinct feature of the CVIU approach is that the control action does not change while the system state remains within the boundaries of the inaction region. Referring once again to (LOEHLE, 2006), we can relate the inaction region around the equilibrium point to the *broader management target* mentioned by the author and speculate whether the CVIU approach can meet the requirements for robust, cautionary management practices in ecology.

1.3 Our Contribution

This thesis builds upon work done by Prof. João Bosco Ribeiro do Val's research group at the University of Campinas during the last few years. The initial proposal of the CVIU approach was done by Prof. do Val, Prof. Thomas Vallée (University of Nantes) and Dr. André Calmon for discrete-time systems (PIN *et al.*, 2009) adopting a discounted quadratic cost criteria and control-dependent noise, while the complete continuous-time model and solution with a discounted quadratic cost were summarized by Prof. do Val

and Dr. Rafael Souto in (VAL; SOUTO, 2017). Our work builds upon these previous results, and in this thesis we gather contributions to the following topics:

1. Expected discounted cost problem

The CVIU approach was presented and the corresponding control problem was solved on the paper (VAL; SOUTO, 2017) by considering one of the diffusion coefficients equal to zero. Here we review this assumption, and present a general solution for the discounted quadratic cost formulation. Moreover, we study the stability of the stochastic system by introducing a notion of observability for stochastic systems with multiplicative noise.

2. Long run average cost

The CVIU control problem was previously solved by adopting an expected discounted cost formulation, that is, by using a cost functional with a decaying exponential. On the one hand, the discounted cost functional stresses the first time instants, and is a common optimization criterion in economic applications, as the inflation rate makes a certain amount of money at a future instant worth less than the same amount at the initial time. Here we use a different cost functional and minimize an average quadratic measure of the system output over an infinite optimization horizon. We approach the problem in Chapter 3.

3. An alternative interpretation of the perturbation structure of the CVIU model.

The initial idea behind the CVIU approach (PIN *et al.*, 2009; VAL; SOUTO, 2017) was to propose a model to represent uncertain systems in which any action performed by the controller *necessarily* increases the noise affecting the system operation. This idea was mathematically represented by the introduction of stochastic noise terms depending on the absolute values of the control input and system state, and is the base of the work presented on the papers mentioned. State- and control-dependent noise structures, however, have been used in previous works and can be considered a representation of parametric uncertainties. This observation motivates the discussion on the relation of the CVIU approach with methods from the theory of robust control, and with previous works on robust control of systems with stochastic multiplicative noise. Furthermore, this characterization of possible relations between the CVIU approach and previous works on the literature of control of uncertain systems may enhance the description of the proposed control strategy and help to place it as an alternative to classical methods. This topic started with the paper presented at the *56th IEEE Conference on Decision and Control*, and it is treated in more details in chapter 4.

4. Applications

As discussed in the introduction, a unique feature of the CVIU approach is the saturation-like behavior exhibited by the optimal control policy around the equilibrium point. This feature, in turn, can be related to systems exhibiting this kind of inaction gap in economy and biology. Our work in this area has focused on the application of the CVIU approach to treat the management of fisheries in uncertain environments, as detailed on the papers presented at the *XXI Brazilian Congress on Automatic Control* and at the *6th IFAC Conference on Foundations of Systems Biology in Engineering*, and in Chapter 5.

Results from the master's research project have been previously presented in some conference papers,

- Vinícius L. Silva, João B.R. do Val and Rafael F. Souto. A Stochastic Approach for Robustness: a \mathcal{H}_2 -norm comparison. In *IEEE Conference on Decision and Control*, Melbourne, Australia, 2017.
- Vinícius L. Silva and João B.R. do Val. Stochastic Control with Poorly Known Biological Growth Models. In *6th IFAC Conference on Foundations of Systems Biology in Engineering*, Magdeburg, Germany, 2016. Extended abstract.
- Vinícius L. Silva, João B.R. do Val and Rafael F. Souto. Harvesting with stochastic control: when parameters are badly known. In *XXI Brazilian Congress on Automatic Control*, Vitória, Brazil, 2016.

Furthermore, the authors are preparing a journal paper detailing the long run average solution of the CVIU approach and its relations with robust control of stochastic systems.

1.4 Outline

In chapter 2 we recall results from the literature which we consider important for the problems studied in this dissertation. Special attention is given to the formulation of stochastic control problems. The second part of the chapter is used to present the CVIU approach. Within the chapter we indicate how the approach was developed, its unique features and previous results. The long run average control problem is treated in chapter 3. The following chapter, chapter 4, bridges the CVIU and robust control approaches. Here we discuss similarities between the two approaches, and show how the performance of the mentioned control policies can be compared. An overview of possible applications of the CVIU approach is given in chapter 5. We end this dissertation with some concluding remarks in chapter 6.

2 Preliminary concepts

In this chapter we recall results from the literature of control theory which we consider important for the development of the work presented in the next chapters. The first part of the chapter recalls concepts from the theory of stochastic control, while the second part gives an overview of the CVIU approach and its current developments.

2.1 Stochastic optimal control

2.1.1 Stochastic Processes, Brownian Motion and Stochastic Calculus

Let (Ω, \mathcal{F}, P) be a probability space, i.e., a triple which consists of the components Ω — the sample space, a set consisting of points ω which represent all possible outcomes of a random experiment —, the event space \mathcal{F} , representing the set of possible events (subsets of the sample space Ω), and the probability measure P , a function which maps events to probabilities (BILLINGSLEY, 1995). We assume the set \mathcal{F} is a σ -algebra (or σ -field) of Ω , that is, a nonempty collection of subsets of Ω satisfying

1. $\emptyset, \Omega \in \mathcal{F}$;
2. $A \in \mathcal{F}$ implies $A^C \in \mathcal{F}$;
3. $A_1, A_2, \dots, A_n \in \mathcal{F}$ implies $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$.

The second and third conditions imply that a σ -algebra is also closed under countable intersections. An important and widely used example of σ -algebra is the Borel σ -algebra, \mathcal{B} . The Borel σ -algebra \mathcal{B} defined on the set $\Omega = \mathbb{R}$ is the smallest σ -algebra on \mathbb{R} containing all the intervals (KLEBANER, 2012). We also assume that P is a probability measure as usually defined, i.e., a set function on \mathcal{F} which satisfies the conditions:

1. $P(\emptyset) = 0, P(\Omega) = 1$;
2. If $A \in \mathcal{F}$, then $0 \leq P(A) \leq 1$ and $P(A^C) = 1 - P(A)$;
3. If $A_1, A_2, \dots, A_n \in \mathcal{F}$ are mutually exclusive, then $P(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$.

In this probability space, we define a random variable on (Ω, \mathcal{F}) as a measurable function from (Ω, \mathcal{F}) to $(\mathbb{R}, \mathcal{B})$. By measurable we mean that the set $\{\omega : X(\omega) \in B\}$ belongs to \mathcal{F} for any Borel set $B \in \mathcal{B}$ (KLEBANER, 2012).

A collection of random variables ordered according to a set of numbers representing time is called a stochastic process. Formally, a stochastic process $X(t, \omega)$ corresponds to a measurable function defined on the space $[0, \infty] \times \Omega$ such that $X(\cdot, t)$ is a random variable and $X(\cdot, \omega)$ a measurable function (KUO, 2006). For ω fixed, $X(t)$ is a function of time t , known as a realization or sample path of the stochastic process $X(\omega, t)$. In a somewhat similar fashion to the case of random variables, we can also define the σ -algebra generated by a stochastic process, $\mathcal{F}_t = \sigma(X_u, u \leq t)$, as the smallest σ -algebra that contains all sets $\{a \leq X_u \leq b\}$ for $0 \leq u \leq t$ and $a, b \in \mathbb{R}$ (KLEBANER, 2012). It can be seen as the information available about the stochastic process X up to time t . The set $\mathbb{F} = \{\mathcal{F}_t\}$ of increasing σ -algebras on the space (Ω, \mathcal{F}) is called a filtration. Moreover, a filtration is called right-continuous if $\mathcal{F}_{t+} = \mathcal{F}_t$, where

$$\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s.$$

We call a probability space equipped with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$. Furthermore, we say a filtered probability space satisfies the usual conditions if it is right-continuous and complete, i.e., if the corresponding filtration is right-continuous and any set which is a subset of a set of zero probability is measurable with relation to the initial σ -algebra \mathcal{F}_0 (alternatively, we can say \mathcal{F}_0 contains all P -null sets).

As we mentioned in the introduction, Brownian motion is frequently used to portray uncertainties in continuous-time stochastic control problems. Formally speaking, a Brownian motion or Wiener Process is a stochastic process $\{W(t, \omega), t \geq 0\}$ on the probability space (Ω, \mathcal{F}, P) satisfying the following properties (KLEBANER, 2012; KUO, 2006) :

1. $P(w; W(0, \omega) = 0) = 1$.
2. (Independence of increments) $W(t)$ has independent increments, that is, for any $0 \leq t_1 < t_2 < \dots < t_n$, the random variables

$$W(t_1), W(t_2) - W(t_1), \dots, W(t_n) - W(t_{n-1}),$$

are independent.

3. (Normal increments) For any $0 \leq s < t$, the random variable $W(t) - W(s)$ has normal distribution with mean 0 and variance $t - s$. Thus, for any $a < b$,

$$P\{a \leq W(t) - W(s) \leq b\} = \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{-\frac{x^2}{2(t-s)}} dx.$$

4. (Continuity of paths) Almost all sample paths of $W(t, \omega)$ are continuous functions of t .

We can extend the above definition of Brownian motion for the multidimensional case as follows.

Definition 2.1. *A n -dimensional Brownian motion is defined as the random vector*

$$\mathbf{W}(\mathbf{t}) = [W^1(t), W^2(t), \dots, W^n(t)],$$

where all components $W^i(t)$ are independent one-dimensional Brownian motions.

It is interesting to note that almost every Brownian motion sample path $W(t)$ is nowhere differentiable, and thus we cannot formally write $\dot{W}(t)$ for its differential. Instead, we first define the stochastic integral, and a theorem analogous to the deterministic fundamental theorem of calculus gives meaning to the corresponding differential (KARATZAS; SHREVE, 1991). Since $W(t)$ has infinite variation over any interval, however, we cannot define the Itô integral (stochastic integral with relation to Brownian motion) pathwise, as usual in measure theory. The construction follows from the continuity and finite quadratic variation of Brownian motion paths — the quadratic variation of $W(t)$ over the interval $[0, t]$ is equal to t , for any t —, and details can be found in books on stochastic calculus. To define the Itô integral

$$\int_0^T X(t) dW(t), \quad (2.1)$$

we need to assume that the integrand $X(t)$ is square-integrable and *adapted* to the filtration $\{\mathcal{F}_t\}$, that is, $X(t)$ is \mathcal{F}_t measurable for all t . Another usual assumption is that $X(t)$ has *regular* sample paths, that is, the sample paths of the stochastic process $X(t)$ have only jump discontinuities. The Itô integral was the first stochastic integral to be defined, and it was introduced by K. Itô in the paper (ITÔ, 1944).

Theorem 2.1 ((KLEBANER, 2012), Theorem 4.3). *Let $X(t)$ be a regular adapted process such that with probability one $\int_0^T X^2(t) dt < \infty$. Then the Itô integral $\int_0^T X(t) dW(t)$ is defined and has the following properties:*

1. *Linearity: If Itô integrals of $X(t)$ and $Y(t)$ are defined and α and β are some constants, then*

$$\int_0^T (\alpha X(t) + \beta Y(t)) dW(t) = \alpha \int_0^T X(t) dW(t) + \beta \int_0^T Y(t) dW(t).$$

- 2.

$$\int_0^T X(t) I_{(a,b]}(t) dW(t) = \int_a^b X(t) dW(t).$$

The following two properties hold when the process satisfies an additional assumption

$$\int_0^T \mathbb{E}(X^2(t)) dW(t) < \infty. \quad (2.2)$$

3. *Zero mean property.* If condition (2.2) holds then

$$\mathbb{E} \left(\int_0^T X(t) dW(t) \right) = 0. \quad (2.3)$$

4. *Isometry property.* If condition (2.2) holds then

$$\mathbb{E} \left(\int_0^T X(t) dW(t) \right)^2 = \int_0^T \mathbb{E}(X^2(t)) dt. \quad (2.4)$$

A consequence of the isometry property follows.

Theorem 2.2 ((KLEBANER, 2012), Theorem 4.5). *Let $X(t)$ and $Y(t)$ be regular adapted processes, such that $\mathbb{E} \int_0^T X(t)^2 dt < \infty$ and $\mathbb{E} \int_0^T Y(t)^2 dt < \infty$. Then*

$$\mathbb{E} \left(\int_0^T X(t) dW(t) \int_0^T Y(t) dW(t) \right) = \int_0^T \mathbb{E}(X(t)Y(t)) dt. \quad (2.5)$$

The previous results concerning Brownian motion calculus can be generalized to the case of stochastic integrals defined with relation to continuous semimartingales. For details, see Kuo's (KUO, 2006), and Karatzas and Shreve (KARATZAS; SHREVE, 1991) books. One of the most important results in stochastic calculus is Itô's formula, which can be seen as a stochastic counterpart to the deterministic chain rule.

Theorem 2.3 ((KLEBANER, 2012), Theorem 4.13). *If $W(t)$ is a Brownian motion on $[0, T]$ and $f(x)$ is a twice continuously differentiable function on \mathbb{R} , then for any $t \leq T$*

$$f(W(t)) = f(0) + \int_0^t f'(W(s)) dW(s) + \frac{1}{2} \int_0^t f''(W(s)) ds. \quad (2.6)$$

Itô's formula can be generalized for the class of *Itô processes*. An Itô process is a stochastic process in the form

$$X(t) = X(0) + \int_0^t \mu(s) ds + \int_0^t \sigma(s) dW(s), \quad 0 \leq t \leq T, \quad (2.7)$$

where $X(0)$ is measurable with relation to \mathcal{F}_0 , and the processes $\mu(t)$ and $\sigma(t)$ are adapted to the filtration \mathcal{F}_t . We also require the processes $\mu(t)$ and $\sigma(t)$ to be absolutely integrable, that is, we assume the integrals

$$\int_0^T |\mu(t)| dt, \quad \int_0^T |\sigma(t)| dt$$

are finite. For the Itô process $X(t)$ we define the stochastic differential

$$dX(t) = \mu(t) dt + \sigma(t) dW(t).$$

Itô's formula, in this case, is given by the following theorem.

Theorem 2.4 ((KLEBANER, 2012), Theorem 4.16). *Let $X(t)$ have a stochastic differential for $0 \leq t \leq T$*

$$dX(t) = \mu(t)dt + \sigma(t)dW(t). \quad (2.8)$$

If $f(x)$ is twice continuously differentiable (C^2 function), then the stochastic differential of the process $Y(t) = f(X(t))$ exists and is given by

$$\begin{aligned} df(X(t)) &= f'(X(t))dX(t) + \frac{1}{2}f''(X(t))d[X, X](t) \\ &= f'(X(t))dX(t) + \frac{1}{2}f''(X(t))\sigma^2(t)dt \\ &= \left(f'(X(t))\mu(t) + \frac{1}{2}f''(X(t))\sigma^2(t) \right) dt + f'(X(t))\sigma(t)dW(t). \end{aligned} \quad (2.9)$$

In the above equation, $[X, X]$ stands for the quadratic variation of the process $X(t)$. Itô's formula can also be written in an integral form, cf. (KARATZAS; SHREVE, 1991; ITÔ, 1944),

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))dX(s) + \frac{1}{2} \int_0^t f''(X(s))ds.$$

In the multidimensional case, let $\hat{W}(t)$ a d -dimensional Brownian motion and $\hat{X}(t)$ a n -dimensional stochastic process with components

$$dX_i(t) = \mu_i(t) + \sum_{j=1}^d \sigma_{ij}(t)dW_j(t), \quad i = 1, \dots, n,$$

and $\sigma(t)$ a $n \times d$ matrix valued function (KLEBANER, 2012). $\hat{X}(t)$ is an Itô process and in vector form we get

$$d\hat{X}(t) = \hat{\mu}(t)dt + \hat{\sigma}(t)d\hat{W}(t). \quad (2.10)$$

Since the covariation of independent Brownian motions is identically zero, we can write the multidimensional version of Itô's formula as (KLEBANER, 2012),

$$\begin{aligned} df(X_1(t), X_2(t), \dots, X_n(t)) \\ &= \sum_{i=1}^n \frac{\partial}{\partial x_i} f(X_1(t), X_2(t), \dots, X_n(t))dX_i(t) \\ &\quad + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} f(X_1(t), X_2(t), \dots, X_n(t))d[X_i, X_j](t). \end{aligned}$$

2.1.2 Stochastic Differential Equations

Stochastic Differential Equations (SDEs) arise when white noise is introduced into ordinary differential equations describing the evolution of a dynamic system. Although a Brownian motion is nowhere differentiable, we can see the white noise process $\xi(t)$, in an

informal manner, as the derivative of Brownian motion with respect to time. Furthermore, if the intensity of the noise at point x and time t is given by $\sigma(x, t)$, then we can say, with a slight abuse of notation, that (KLEBANER, 2012):

$$\int_0^T \sigma(X(t), t) \xi(t) dt = \int_0^T \sigma(X(t), t) dW(t). \quad (2.11)$$

This allows us to make connections between systems under white-noise perturbations and the formal definition of stochastic differential equations. Formally, we say that a stochastic differential equation driven by Brownian motion is an equation of the form

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dW(t), \quad (2.12)$$

where we are given functions $\mu(x, t)$ and $\sigma(x, t)$ and wish to find the unknown process $X(t)$. In the above equation, the function representing the deterministic evolution of the system, $\mu(x, t)$ is called the *drift* coefficient or matrix; and the function $\sigma(x, t)$ representing the intensity of the noise is called the *diffusion* coefficient or vector. The drift vector can be seen as a local measure of the mean velocity of the random motion modeled by the stochastic process X , whereas the diffusion matrix estimates the variance of the displacement $X_t - x$ for t small (KARATZAS; SHREVE, 1991; KLEBANER, 2012). Note that, if we set $\sigma(\cdot) = 0$ we retrieve an ordinary differential equation.

In the following we recall two notions for solutions of stochastic differential equations.

Definition 2.2 ((KLEBANER, 2012), Definition 5.1). *A process $X(t)$ is called a strong solution of the SDE (2.12) if for all $t > 0$ the integrals $\int_0^t \mu(X(s), s)ds$ and $\int_0^t \sigma(X(s), s)dW(s)$ exist, with the second being an Itô integral, and*

$$X(t) = X(0) + \int_0^t \mu(X(s), s)ds + \int_0^t \sigma(X(s), s)dW(s). \quad (2.13)$$

Definition 2.3 ((KLEBANER, 2012), Definition 5.8). *If there exist a probability space with a filtration, a Brownian motion $\hat{W}(t)$ and a process $\hat{X}(t)$ adapted to that filtration, such that: $\hat{X}(0)$ has the given distribution, for all t the integrals below are defined, and $\hat{X}(t)$ satisfies*

$$\hat{X}(t) = \hat{X}(0) + \int_0^t \mu(\hat{X}(u), u)du + \int_0^t \sigma(\hat{X}(u), u)d\hat{W}(u), \quad (2.14)$$

then $\hat{X}(t)$ is called a weak solution to the SDE

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dW(t). \quad (2.15)$$

Definition 2.4 ((KLEBANER, 2012), Definition 5.9). *A weak solution is called unique if whenever $X(t)$ and $\tilde{X}(t)$ are two solutions (perhaps on different probability spaces) such that the distributions of $X(0)$ and $\tilde{X}(0)$ are the same, then all finite-dimensional distributions of $X(t)$ and $\tilde{X}(t)$ are the same.*

The difference between the notions of *strong* and *weak* solutions relies on the fact that, in the case of weak solutions, the probability space, filtration and Brownian motion driving the SDE are part of the solution (KARATZAS; SHREVE, 1991). Every strong solution is also a weak solution. Furthermore, as pointed out in (KLEBANER, 2012), weak solutions give meaning to stochastic differential equations for which there is no strong solution. The conditions for existence of weak solutions are also less strict than the corresponding conditions for strong solutions, as we can conclude from a comparison of the following theorems.

Theorem 2.5 ((KLEBANER, 2012), Theorem 5.4). *If the following conditions are satisfied*

1. *Coefficients are locally Lipschitz in x uniformly in t , that is, for every T and N such that for all $|x|, |y| \leq N$ and all $0 \leq t \leq T$*

$$|\mu(x, t) - \mu(y, t)| + |\sigma(x, t) - \sigma(y, t)| < K|x - y|, \quad (2.16)$$

2. *Coefficients satisfy the linear growth condition*

$$|\mu(x, t)| + |\sigma(x, t)| \leq K(1 + |x|), \quad (2.17)$$

3. *$X(0)$ is independent of $(W(t), 0 \leq t \leq T)$, and $\mathbb{E}X^2(0) < \infty$.*

Then there exists a unique strong solution $X(t)$ of the SDE (2.12). $X(t)$ has continuous paths, moreover

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} X^2(t) \right) < C(1 + \mathbb{E}(X^2(0))), \quad (2.18)$$

where constant C depends only on K and T .

Theorem 2.6 ((KLEBANER, 2012), Theorem 5.10). *If for each $t > 0$, functions $\mu(x, t)$ and $\sigma(x, t)$ are bounded and continuous then the SDE (2.12) has at least one weak solution starting at time s at point x , for all s , and x . If in addition their partial derivatives with respect to x up to order two are also bounded and continuous, then the SDE (2.12) has a unique weak solution starting at time s at point x . Moreover this solution has the strong Markov property.*

Although stochastic differential equations cannot usually be solved explicitly, multidimensional linear SDEs obtained when we take $\mu(t) = A(t)X(t) + a(t)$ and $\sigma(t) = \sum_{i=1}^m (B_i(t)X(t) + b_i(t))$, in such a way that

$$dX(t) = (A(t)X(t) + a(t))dt + \sum_{i=1}^m (B_i(t)X(t) + b_i(t))dW^i(t),$$

do have an explicit solution. For details, check (ARNOLD, 1974), chapter 8.

2.1.2.1 Simulation: the Euler-Maruyama Method

Throughout this work we use the so-called Euler-Maruyama method to simulate the random behavior of stochastic differential equations. This is a variation of the classical Euler method to solve ordinary differential equations, and relies on a discretized version of Brownian motion to compute the random paths.

Suppose we wish to simulate paths of a Brownian motion $W(t)$ depending on the time variable $t \in [0, T]$. We set $\delta t = T/N$, where N is an integer number, and define $W_j = W(t_j) = W(j\delta t)$. Since $W_0 = 0$ with probability one, we can calculate for the following values of t (HIGHAM, 2001)

$$W_j = W_{j-1} + dW_j, \quad j = 1, 2, \dots, N.$$

In the above equation, each dW_j is an independent random variable distributed according to $\sqrt{\delta t}N(0, 1)$.

Given the method to simulate paths of a Brownian motion outlined above, suppose now we wish to calculate numerically the stochastic differential equation

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dW(t), \quad X(0) = X_0, \quad 0 \leq t \leq T.$$

We first discretize the interval $[0, T]$. For that, we let $\Delta t = T/L$ and $\tau_j = j\Delta t$, with L a positive integer, and denote the approximation to $X(\tau_j)$ by X_j . The Euler-Maruyama method is then given by

$$X_j = X_{j-1} + \mu(X_{j-1})\Delta t + \sigma(X_{j-1})(W(\tau_j) - W(\tau_{j-1})), \quad j = 1, 2, \dots, N. \quad (2.19)$$

Further details and implementations using MATLAB[®] can be found in the paper (HIGHAM, 2001).

2.1.3 Diffusion Processes

Diffusion processes are Markov processes — stochastic processes for which the Markov property holds — with continuous sample paths. The Markov property states that the future value of the process only depends on its current value. If we denote the σ -algebra generated by the stochastic process up to time t by \mathcal{F}_t , then, for any $0 \leq s < t$, and $B \in \mathcal{F}_t$, the Markov property reads (KLEBANER, 2012),

$$P(X(t) \in B | \mathcal{F}_s) = P(X(t) \in B | X(s)) \text{ a.s.} \quad (2.20)$$

Formally, we can define a diffusion process as follows.

Definition 2.5 ((ARNOLD, 1974), Definition 2.5.1). *A Markov process $X(t)$, for $t_0 \leq t \leq T$, with values in \mathbb{R}^n and almost certainly continuous sample path functions is called*

a diffusion process if its transition probability $P(s, x, t, B)$ ¹ satisfies the following three conditions for every $s \in [t_0, T)$, $x \in \mathbb{R}^n$, and $\epsilon > 0$:

1. (Continuity)

$$\lim_{t \downarrow s} \frac{1}{t - s} \int_{|y-x| > \epsilon} P(s, x, t, dy) = 0; \quad (2.21)$$

2. (Drift coefficient) there exists an \mathbb{R}^n -valued function $\mu(s, x)$ such that

$$\lim_{t \downarrow s} \frac{1}{t - s} \int_{|y-x| \leq \epsilon} (y - x) P(s, x, t, dy) = \mu(s, x); \quad (2.22)$$

3. (Diffusion coefficient) there exists a $n \times n$ matrix valued function $\sigma(s, x)$ such that

$$\lim_{t \downarrow s} \frac{1}{t - s} \int_{|y-x| \leq \epsilon} (y - x)(y - x)' P(s, x, t, dy) = \sigma(s, x). \quad (2.23)$$

The functions μ and σ are called the coefficients of the diffusion process. In particular, μ is called the drift vector and σ is called the diffusion matrix. $\sigma(s, x)$ is symmetric and nonnegative-definite.

The first condition means that large variations of $X(t)$ over a short interval are unlikely, implying continuity of the diffusion process (ARNOLD, 1974). Recalling that solutions to SDEs satisfy the Markov property and have continuous sample paths, we can also see diffusion processes as solutions to stochastic differential equations (KLEBANER, 2012; KARATZAS; SHREVE, 1991).

We now associate to a solution $X(t)$ of the stochastic differential equation

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dW(t), \quad t \geq 0,$$

a second order differential operator L_t ,

$$L_t f(x, t) = (L_t f)(x, t) = \frac{1}{2} \sigma^2(x, t) \frac{\partial^2 f}{\partial x^2}(x, t) + \mu(x, t) \frac{\partial f}{\partial x}(x, t). \quad (2.24)$$

The operator L_t is called the generator of $X(t)$ (KLEBANER, 2012), and it allows us to write results from stochastic calculus in a compact manner. In particular, we outline here a result known as Dynkyn's Formula, useful when we wish to calculate the expectation of a function of a diffusion process.

Theorem 2.7 ((KLEBANER, 2012), Corollary 6.5; (OKSENDAL, 2007), Theorem 7.4.1). *Let $X(t)$ a solution of the SDE (2.12) with coefficients satisfying the existence and uniqueness conditions, and $f(x, t)$ a twice continuously differentiable in x and once in t function ($C^{2,1}$) with bounded first derivative in x . Then, for any $t, 0 \leq t \leq T$,*

$$\mathbb{E}f(X(t), t) = f(X(0), 0) + \mathbb{E} \int_0^t \left(L_u f + \frac{\partial f}{\partial t} \right) (X(u), u) du. \quad (2.25)$$

The result is also true if t is replaced by a bounded stopping time $\tau, 0 \leq \tau \leq T$.

¹ According to the notation on Arnold's book, $P(s, x, t, B)$ is the probability of transition from point x at time s into set B at time t .

For the case of multidimensional stochastic differential equations, let $\hat{X}(t)$ an n -dimensional diffusion process,

$$d\hat{X}(t) = \hat{\mu}(\hat{X}(t), t) + \hat{\sigma}(\hat{X}(t), t), \quad (2.26)$$

with $\hat{\sigma}$ an $n \times d$ matrix valued function. The generator of $\hat{X}(t)$ takes the form

$$(Lf)(\hat{X}(t)) = \sum_{i=1}^n \mu_i(t) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (\hat{\sigma} \hat{\sigma}^\top)_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(x), \quad (2.27)$$

and an analogous version of Dynkin's formula holds for the multidimensional case.

2.1.4 Continuous-time Stochastic Optimal Control

A stochastic optimal control problem consists in finding a control policy which optimizes a given performance criterion. A general structure of the control problem makes use of the following components (PHAM, 2009):

- **State of the system:** We represent by $x(t)$ the state of the dynamic system at time t . The system dynamics is described by a stochastic differential equation defined on a probability space (Ω, \mathcal{F}, P) .
- **Control:** The dynamics of the system can be influenced by a control policy, modeled as a stochastic process $u(t)$. We also need to define a set of control satisfying some constraints, and denote this set of *admissible* controls by \mathcal{U} .
- **Performance or cost criterion:** The objective of the problem is to optimize (over the set of admissible controls) a cost functional $J(t, x, u)$, and to find the control policy which achieves the optimal value of the cost functional.

2.1.4.1 Controlled Diffusion Processes

From now on we consider a model in which the state of the system is given by the following *controlled* stochastic differential equation (YONG; ZHOU, 1999; PHAM, 2009):

$$\begin{cases} dx(t) = \mu(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dW(t), \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \quad (2.28)$$

where $\mu(t, x(t), u(t)) : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$, and $\sigma(t, x(t), u(t)) : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^{n \times m}$. $W(t)$ is a m -dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ satisfying the usual conditions, and U is a given separable metric space.

We denote by $u(\cdot)$ the function representing the control action or policy of the decision-makers. Here we assume that, at any time, the controller or decision maker is aware of the information available about the system up to that moment, but cannot

anticipate the future behavior of the system. Mathematically, we say the control function $u(\cdot)$ is nonanticipative, or adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$, and define the set of feasible controls as (YONG; ZHOU, 1999):

$$\mathcal{U}[0, T] := \{u : [0, T] \times \Omega \rightarrow U \mid u(\cdot) \text{ is } \{\mathcal{F}_t\}_{t \geq 0} \text{ - adapted}\}. \quad (2.29)$$

If the control function is both feasible and square-integrable, we call it an *admissible* control.

2.1.4.2 Cost structures

Different cost structures are used as optimization criteria in stochastic optimal control problems. A first distinction is made between finite, indefinite and infinite time horizons. In the first case, the cost functional

$$\mathbb{E} \left[\int_0^T f(s, x, u) ds + g(x_T) \right] \quad (2.30)$$

is evaluated up to the terminal time $T < \infty$. Here $f(\cdot)$ is the running cost, and $g(\cdot)$ the terminal cost. Problems with indefinite time horizon, also known as *cost up to exit time*, use the cost functional

$$\mathbb{E} \left[\int_0^{\tau_u} f(s, x, u) ds + g(x_{\tau_u}) \right], \quad (2.31)$$

where the stopping time τ_u is the first exit time of $x(t)$ from the open set $D \subset \mathbb{R}^n$ with boundary ∂D . In the infinite horizon case, it is necessary to guarantee that the cost functional will not explode. We can avoid the explosion of the cost functional by introducing an exponential discount factor, $e^{-\alpha t}$, or averaging it over time. This leads to the infinite horizon discounted cost criterion,

$$\mathbb{E} \left[\int_0^\infty e^{-\alpha s} f(s, x, u) ds \right], \quad (2.32)$$

and the long run average cost functional (BORKAR, 2005),

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T f(s, x, u) ds \right]. \quad (2.33)$$

There exist yet other cost criteria in stochastic control theory, such as risk sensitive control, impulse control and optimal switching (BORKAR, 2005).

2.1.4.3 Problem formulation

Similar to the notions of *weak* and *strong* solutions of SDEs, we can define a strong and a weak formulation of stochastic control problems (YONG; ZHOU, 1999). In the following we consider a controlled diffusion process according to equation (2.28).

Moreover, $L^p_{\mathcal{F}}(0, T; \mathbb{R}^n)$ stands for the set of all $\{\mathcal{F}\}_{t \geq 0}$ -adapted \mathbb{R}^n -valued processes $X(\cdot)$ such that

$$\mathbb{E} \int_0^T |X(t)|^p dt < \infty,$$

and $L^p_{\mathcal{G}_T}(\Omega; \mathbb{R}^n)$ is the set of \mathbb{R}^n -valued \mathcal{G} -measurable random variables X such that $\mathbb{E}|X|^p < \infty$ ($p \in [1, \infty)$).

Strong formulation

Definition 2.6 ((YONG; ZHOU, 1999), Definition 4.1, p. 63). *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be given satisfying the usual conditions and let $W(t)$ be a given m -dimensional standard $\{\mathcal{F}_t\}_{t \geq 0}$ -Brownian motion. A control $u(\cdot)$ is called an s-admissible control, and $(x(\cdot), u(\cdot))$ an s-admissible pair, if*

1. $u(\cdot) \in \mathcal{U}[0, T]$;
2. $x(\cdot)$ is the unique solution of equation (2.28);
3. some prescribed state constraints are satisfied²;
4. $f(\cdot, x(\cdot), u(\cdot)) \in L^1_{\mathcal{F}}(0, T; \mathbb{R})$ and $g(x(T)) \in L^1_{\mathcal{F}_T}(\Omega; \mathbb{R})$.

Given the above definition, the strong formulation of the stochastic optimal control problem can be stated as to minimize the cost functional

$$J(t, x, u) = \mathbb{E} \left[\int_0^T f(t, x(t), u(t)) dt + g(x(T)) \right]$$

over the set of admissible controls $\mathcal{U}[0, T]$. Our goal is to find a $\bar{u}(\cdot) \in \mathcal{U}[0, T]$ which achieves the minimum of the cost functional $J(\cdot)$,

$$J(\cdot, \bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[0, T]} J(\cdot, u(\cdot)). \quad (2.34)$$

Weak formulation

Definition 2.7 ((YONG; ZHOU, 1999), Definition 4.2, p.64). *A 6-tuple $\pi = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P, W(\cdot), u(\cdot))$ is called a w-admissible control, and $(x(\cdot), u(\cdot))$ a w-admissible pair, if*

1. $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ is a filtered probability space satisfying the usual conditions;
2. $W(\cdot)$ is an m -dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$;

² Yong and Zhou's model includes possible state constraints, but we do not work with this concept in this monograph.

3. $u(\cdot)$ is an $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted process on (Ω, \mathcal{F}, P) taking values in U ;
4. $x(\cdot)$ is the unique solution of equation (2.28) on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ under $u(\cdot)$;
5. some prescribed state constraints are satisfied;
6. $f(\cdot, x(\cdot), u(\cdot)) \in L^1_{\mathcal{F}}(0, T; \mathbb{R})$ and $g(x(T)) \in L^1_{\mathcal{F}_T}(\Omega; \mathbb{R})$. Here, the spaces $L^1_{\mathcal{F}}(0, T; \mathbb{R})$ and $L^1_{\mathcal{F}_T}$ are defined on the given filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ associated with the 6-tuple π .

As in the previous case, we aim to find a control policy $\bar{\pi} \in \mathcal{U}[0, T]$ which achieves the minimum of the cost functional $J(\cdot)$,

$$J(\cdot, \bar{\pi}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[0, T]} J(\cdot, u(\cdot)). \quad (2.35)$$

Yong and Zhou point out in (YONG; ZHOU, 1999) that the weak formulation of a stochastic control problem serves as an auxiliary mathematical model to solve problems originally cast under the strong formulation, which is the formulation arising from practical applications. In stochastic control problems, we evaluate the expected value of a cost functional which depends on the distribution of the stochastic processes involved in the problem. In this sense, as long as the probability distribution of the solutions of equation (2.12) in different probability spaces is the same, we have some freedom to choose a more suitable probability space to work with. As the authors mention, that is the case, for example, of the dynamic programming approach to stochastic control problems.

2.1.4.4 Dynamic Programming

Dynamic programming is a mathematical technique used to solve optimization problems, including optimal control problems. It was developed by R. Bellman in the 1950s, and has been successfully applied to stochastic control problems as well. The method relies on breaking down the optimization horizon into smaller sub-problems, and finding the optimal solution for each sub-problem recursively. Loosely speaking, when we let the size of these partitions of the optimization horizon go to zero, we get the so-called Hamilton-Jacobi-Bellman (HJB) equation. The HJB or dynamic programming equation is, in this sense, the infinitesimal version of the dynamic programming principle (PHAM, 2009). It is a nonlinear, first-order (in the deterministic case) or second-order (in the stochastic case) partial differential equation (PDE). To present the formulation of the stochastic HJB equation, we first introduce the concept of value function (YONG; ZHOU, 1999), which can be seen as the optimal value of the cost functional $J(\cdot)$,

$$\begin{cases} V(x, t) &= \inf_{u(\cdot) \in \mathcal{U}[0, T]} J(t, x; u(\cdot)), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n, \\ V(x, T) &= g(x), \quad \forall x \in \mathbb{R}^n. \end{cases} \quad (2.36)$$

Now, let $T > 0$ be given and let U be a metric space. Recall that, for $s \in [0, T]$ fixed, the control process $u : [s, T] \times \Omega \rightarrow U$ is adapted to the filtration $\{\mathcal{F}_t^s\}_{t \geq s}$. In the following, a *Polish space* is a separable, completely metrizable topological space (SRIVASTAVA, 1998) — a space is called completely metrizable, in turn, if its topology is induced by a complete metric, that is, there exists a metric d which makes (U, d) a complete metric space. The standard example of Polish spaces is the set of real numbers \mathbb{R} . Moreover, there are some properties of Polish spaces, such as the fact that any closed subspace of a Polish space is also Polish, that makes them convenient spaces in which to define probability measures. Before presenting the infinitesimal version of the dynamic programming principle in a stochastic setting, we need to consider the following set of assumptions.

1. (U, d) is a Polish space and $T > 0$.
2. The maps $\mu : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^{n+m}$, $f : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ are uniformly continuous, and there exists a constant $L > 0$ such that for $\phi(t, x, u) = \mu(t, x, u)$, $\sigma(t, x, u)$, $f(t, x, u)$, $g(x)$,

$$\begin{cases} |\phi(t, x, u) - \phi(t, \hat{x}, u)| \leq L|x - \hat{x}|, \forall t \in [0, T], x, \hat{x} \in \mathbb{R}^n, u \in U, \\ |\phi(t, 0, u)| \leq L, \forall (t, u) \in [0, T] \times U. \end{cases} \quad (2.37)$$

Under the above assumptions, we have the following result.

Proposition 2.1 ((YONG; ZHOU, 1999), Proposition 3.5). *Suppose the above assumptions hold and the value function $V \in C^{1,2}([0, T] \times \mathbb{R}^n)$. Then V is a solution of the following terminal value problem of a (possibly degenerate) second-order partial differential equation:*

$$\begin{cases} -v_t - \inf_{u \in U} H(t, x, u, v_x, v_{xx}) = 0, (t, x) \in [0, T] \times \mathbb{R}^n, \\ v|_{t=T} = g(x), x \in \mathbb{R}^n, \end{cases} \quad (2.38)$$

where

$$\begin{aligned} H(t, x, u, p, P) &= \frac{1}{2} \text{tr}(P\sigma(t, x, u)\sigma(t, x, u)^\top) + \langle p, \mu(t, x, u) \rangle - f(t, x, u), \\ \forall (t, x, u, p, P) &\in [0, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathcal{S}^n. \end{aligned} \quad (2.39)$$

In the above equation, v_t , v_x and v_{xx} represent the partial derivatives of the candidate function v . $C^{1,2}([0, T] \times \mathbb{R}^n)$ is the set of all continuous functions $v : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that the partial derivatives v_t , v_x and v_{xx} are all continuous in (t, x) . $H(t, x, u, p, P)$ is the *Hamiltonian* function associated to the stochastic control problem.

The dynamic programming approach works under the assumption that there exist smooth enough solutions to the HJB equation, that is, that the value function has enough continuous derivatives. Therefore, if the value function belongs to $C^{1,2}([0, T])$, and further conditions on the coefficients of the stochastic differential equation are satisfied, we can use the above form of the HJB equation to solve the stochastic optimal control problem. This is not always the case, however, and this drawback has led to the introduction of the notion of viscosity solutions of the HJB equation (PHAM, 2009; YONG; ZHOU, 1999). The notion of viscosity solutions allows us to study the HJB equation for a more general class of functions not necessarily smooth but locally bound. It basically consists in writing a HJB inequality, and checking whether the inequality holds in both senses. Due to the possible nondifferentiability of the value function in this case, we need to consider the notions of sub- and superdifferentials, which will in turn allow us to write the conditions for the existence of viscosity solutions of the HJB equation in subsection 2.1.4.6.

2.1.4.5 Sub- and superdifferentials and convex functions

The notions of subdifferentials and superdifferentials in this thesis are as follows.

Definition 2.8. *First order sub-superdifferentials:*

$$D^{1,-}v(x) := \{\varphi_x(x) : \varphi : X \rightarrow \mathbb{R} \text{ is } C^1(X) \text{ and } v - \varphi \text{ has a local minimum at } x\}$$

$$D^{1,+}v(x) := \{\varphi_x(x) : \varphi : X \rightarrow \mathbb{R} \text{ is } C^1(X) \text{ and } v - \varphi \text{ has a local maximum at } x\}$$

Second order sub-superdifferentials:

$$D^{1,2-}v(x) := \{(\varphi_x(x), \varphi_{xx}(x)) : \varphi : X \rightarrow \mathbb{R} \text{ is } C^2(X) \text{ and } v - \varphi \text{ has a local minimum at } x\}$$

$$D^{1,2+}v(x) := \{(\varphi_x(x), \varphi_{xx}(x)) : \varphi : X \rightarrow \mathbb{R} \text{ is } C^2(X) \text{ and } v - \varphi \text{ has a local maximum at } x\}$$

Second order one-sided parabolic subdifferential of v at $(t, x) \in [0, T] \times \mathbb{R}^n$ denoted by $D_{t^+,x}^{1,2-}v(x)$, is the set such that $(q, p, P) \in D_{t^+,x}^{1,2-}v(t, x)$ if for each $y \in \mathbb{R}^n, s \geq t$,

$$v(s, y) \geq v(t, x) + q(s - t) + \langle p, y - x \rangle + \frac{1}{2}(y - x)^\top P(y - x) + o(s - t + \|y - x\|^2)$$

The second order one-sided parabolic superdifferential of v at (t, x) , $D_{t^+,x}^{1,2+}v(x)$ is defined by reversing the inequality above.

To get a clearer picture on the sets of sub-superdifferentials, their relations with other notions and in particular, taking into account that we will deal with a convex value function, we list the main properties used here. Suppose that $v : X \rightarrow \mathbb{R}^n$ is convex at x . Then,

- (i) $D_x^{1,-}v(x) = \partial v(x) = \partial_c v(x)$, where $\partial_c v(x)$ denotes the subgradient of v at x (ROCK-AFELLAR, 1970), and for a locally Lipschitzian function, $\partial v(x)$ stands for the

Clarke's generalized gradient (CLARKE, 1990) ((YONG; ZHOU, 1999, prop 2.6 p.172)).

- (ii) $D_x^{1,2-}v(x)$ is nonempty for almost all $x \in \mathbb{R}^n$. Alexandrov theorem for convex functions guarantees the existence of $(p, P) \in \mathbb{R}^n \times S^{n+}$ such that the expansion

$$v(y) = v(x) + \langle p, y - x \rangle + \frac{1}{2}(y - x)^\top P(y - x) + o(\|y - x\|^2) \quad (2.40)$$

holds for almost all $x \in \mathbb{R}^n$, cf. (NICULESCU; PERSSON, 2006, Th 3.11.2 p.154). Applying the definition of $D_x^{1,2-}v(x)$ in (YONG; ZHOU, 1999, p.191),

$$D_x^{1,2-}v(x) = \left\{ (p, P) \in \mathbb{R}^n \times S^n \left| \lim_{y \rightarrow x} \frac{1}{\|y - x\|^2} \left[v(y) - v(x) - \langle p, y - x \rangle - \frac{1}{2}(y - x)^\top P(y - x) \right] \geq 0 \right. \right\}, \quad (2.41)$$

yields the fact that $D_x^{1,2-}v(x) \neq \emptyset$ and there exists $(p, P) \in D_x^{1,2-}v(x)$ with $P \in S^{n+}$. The points x in (2.40) are called Alexandrov points.

- (iii) Either $D_x^{1,2+}v(x) = \emptyset$ or $D_x^{1,2+}v(x) \cap D_x^{1,2-}v(x)$ is a singleton $\{(p, P)\}$. In the second case, x is a point for which v is twice differentiable and hence, $(q, P) = (v_x(x), v_{xx}(x))$.

The characterization for the second order one-sided parabolic sub-superdifferential, $D_{t+,x}^{1,2-}v$ and $D_{t+,x}^{1,2+}v$, inherits the above properties of $D_x^{1,2-}v$ and $D_x^{1,2+}v$, respectively, provided that an upper bound on the time variation $v(t + h, x) - v(t, x)$ holds, cf. (GOZZI *et al.*, 2010).

2.1.4.6 Viscosity solutions

When the value function of the general stochastic control problem

$$\begin{aligned} \min J(x(\cdot), u(\cdot)) &= \mathbb{E} \left[\int_0^T f(t, x(t), u(t)) dt + g(x(T)) \right] \\ \text{s.t. } dx(t) &= \mu(t, x(t), u(t)) dt + \sigma(t, x(t), u(t)) dW(t), \quad x(0) = x_0, \end{aligned} \quad (2.42)$$

is not necessarily smooth, we cannot use the HJB (2.38) directly. Instead, we need to introduce the notion of viscosity solution in order to characterize the value function as the unique *viscosity* solution of the HJB equation (YONG; ZHOU, 1999).

Definition 2.9 ((YONG; ZHOU, 1999), Definition 5.1, p. 190). *A function $v \in C([0, T] \times \mathbb{R}^n)$ is called a viscosity subsolution of (2.38) if*

$$v(T, x) \leq g(x), \quad \forall x \in \mathbb{R}^n, \quad (2.43)$$

and for any $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^n)$, whenever $v - \varphi$ attains a local maximum at $(t, x) \in [0, T] \times \mathbb{R}^n$, we have

$$-\varphi_t(t, x) - \inf_{u \in \mathbb{U}} H(t, x, u, \varphi_x(t, x), \varphi_{xx}(t, x)) \leq 0. \quad (2.44)$$

A function $v \in C([0, T] \times \mathbb{R}^n)$ is called a viscosity supersolution of (2.38) if in (2.43) — (2.44) the inequalities “ \leq ” are changed to “ \geq ” and “local maximum” is changed to “local minimum”. Further, if $v \in C([0, T] \times \mathbb{R}^n)$ is both a viscosity subsolution and viscosity supersolution of (2.38), then it is called a viscosity solution of (2.38).

A first result follows.

Theorem 2.8 ((YONG; ZHOU, 1999), Theorem 5.2, p. 190). *Let (2.37) hold. Then the value function V is a viscosity solution of (2.38).*

Uniqueness of viscosity solutions is established by the following theorems.

Theorem 2.9 ((YONG; ZHOU, 1999), Theorem 6.1, p. 198). *Let (2.37) hold. Then the HJB equation (2.38) admits at most one viscosity solution $v(\cdot, \cdot)$ in the class of functions satisfying*

$$|V(s, y)| \leq K(1 + |y|), \forall (s, y) \in [0, T] \times \mathbb{R}^n, \quad (2.45a)$$

$$|V(s, y) - V(\hat{s}, \hat{y})| \leq K \left\{ |y - \hat{y}| + (1 + |y| \vee |\hat{y}|) |s - \hat{s}|^{1/2} \right\}, \forall s, \hat{s} \in [0, T], y, \hat{y} \in \mathbb{R}^n. \quad (2.45b)$$

Theorem 2.10 ((YONG; ZHOU, 1999), Theorem 6.2, p. 198). *Let (2.37) hold. Then the value function $V(\cdot, \cdot) \in C([0, T] \times \mathbb{R}^n)$ of the stochastic control problem is the only function that satisfies (2.45) and the following: For all $(t, x) \in [0, T] \times \mathbb{R}^n$,*

$$\begin{cases} -q - \inf_{u \in \mathbb{U}} H(t, x, u, p, P) \leq 0, \forall (q, p, P) \in D_{t+, x}^{1,2+} V(t, x), \\ -q - \inf_{u \in \mathbb{U}} H(t, x, u, p, P) \geq 0, \forall (q, p, P) \in D_{t+, x}^{1,2-} V(t, x), \\ V(T, x) = g(x). \end{cases} \quad (2.46)$$

2.1.5 Extended generator

The notions of stability which we investigate within the CVIU framework start with the concept of extended generator introduced in this section. For that, let us first define some conditions.

General conditions for a Ito diffusion process

The final time $T > 0$ is fixed, (U, d) is a Polish space, and the diffusion data functions μ, σ , and costs f, g are uniformly continuous and Lipschitz on the corresponding domains $[0, T] \times O \times U$ and $O \subseteq \mathbb{R}^n$, and $\text{abs}(b, \sigma, f, g)|_{x=0} \leq K$ (conditions 1 — 2 in page 42 (YONG; ZHOU, 1999).)

The weak formulation for the control problem

Read as in (YONG; ZHOU, 1999, chap.5, sec.4), for which, $U_{ad}^w[s, T]$, for $s \in [0, T]$ is the set of weakly admissible controls, the collection of all 5-tuples $(\Omega, \mathcal{F}, \mathbb{P}, W, u(\cdot))$. Denote $\mathcal{F}_t^s, s \leq t \leq T$ the augmented filtration generated by W with $W(s) = 0$ a.s.

Polynomial growth for viscosity subsolutions

(GOZZI *et al.*, 2005, lem 3.3) For $v \in C([0, T]) \times \mathbb{R}^n$) a viscosity subsolution of (2.43) in Definition 2.9, suppose that it satisfies,

$$\begin{aligned} |v(t, x)| &\leq C_1(1 + \|x\|^k) \quad \text{for some } k \geq 1, (t, x) \in (0, T) \times \mathbb{R}^n, \\ v(t+h, x) - v(t, x) &\leq C_2(1 + \|x\|^k), \quad k \geq 0, 0 < t < t+h \leq T, x \in \mathbb{R}^n, \\ v(t, \cdot) - C_3\|\cdot\|^2 &\text{ is concave on } \mathbb{R}^n \text{ for some } C_3 > 0. \end{aligned} \quad (2.47)$$

Theorem 2.11 ((VAL; SOUTO, 2017), Theorem 2). *For the stochastic control problem (2.42), suppose that the conditions above hold. Let $U \in C([0, T]) \times \mathbb{R}^n$ be a viscosity subsolution of (2.43). Fix $(s, y) \in [0, T] \times \mathbb{R}^n$ and let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be an admissible pair for problem $C_{s,y}$, such that there exists a triple*

$$(\bar{q}, \bar{p}, \bar{P}) \in L_{\mathcal{F}_t}^2(s, T; \mathbb{R}) \times L_{\mathcal{F}_t}^2(s, T; \mathbb{R}^n) \times L_{\mathcal{F}_t}^2(s, T; S^n) \quad (2.48)$$

satisfying

$$(\bar{q}(t), \bar{p}(t), \bar{P}(t)) \in D_{t^+, x}^{1,2,+} U(t, \bar{x}(t)), \quad \text{a.e. } t \in [s, T], \mathbf{P}\text{-a.s.} \quad (2.49)$$

and

$$\bar{q}(t) \leq -H(t, \bar{x}(t), \bar{u}(t), \bar{p}(t), \bar{P}(t)) = -\min_{u \in U} H(t, \bar{x}(t), u, \bar{p}(t), \bar{P}(t)) \quad (2.50)$$

Then $(\bar{x}(\cdot), \bar{u}(\cdot))$ is optimal and there exists measurable functions $t \rightarrow D_t J^(t, \bar{x}(t))$, $t \rightarrow \tilde{\mathcal{A}}^* J^*(t, \bar{x}(t))$ for the value function J^* such that*

$$\begin{aligned} E[J^*(t, \bar{x}(t))] - J^*(s, y) \\ = E\left[\int_s^t \left(D_t J^*(r, \bar{x}(r)) + \tilde{\mathcal{A}}^* J^*(r, \bar{x}(r))\right) dr\right], \quad s \leq t \leq T, \end{aligned} \quad (2.51)$$

and

$$D_t J^*(t, \bar{x}(t)) + \tilde{\mathcal{A}}^* J^*(t, \bar{x}(t)) \leq -f(\bar{x}(t), \bar{u}(t)), \quad \text{for a.a.t, } P\text{-a.s.} \quad (2.52)$$

$\tilde{\mathcal{A}}^*$ is the extended generator associated to the optimal control problem, and the cost function J^* belongs to the domain $D(\tilde{\mathcal{A}}^*)$, of such a generator.

The proof relies on evaluations developed for the verification theorem of (GOZZI *et al.*, 2005) and (GOZZI *et al.*, 2010), cf. (VAL; SOUTO, 2017).

2.2 The CVIU control problem

We now describe with more details the CVIU approach. As briefly discussed in the introduction, the CVIU approach was developed as an alternative to the control of stochastic systems with poorly known dynamics (VAL; SOUTO, 2017; PIN *et al.*, 2009), and the main idea behind the approach was to use stochastic terms dependent on the absolute values of the control input and system state to represent uncertainties about the system dynamics. The corresponding stochastic optimal control problem is presented with details on the paper (VAL; SOUTO, 2017), where the authors also present a solution of the problem when a discounted, quadratic cost formulation is adopted.

Here we analyze some topics of interest related to the CVIU approach. The first of them relates to this use of multiplicative — or state- and control-dependent — noise to represent uncertainties. Although the CVIU approach is unique in the sense that it seems to be the first model to use the absolute value to represent stochastic uncertainties, models in which disturbance depends on the state variable or control input of the system have been developed since the 1960s. Furthermore, this representation shares some similarities with models used in the theory of robust control, a topic which we will discuss with details in chapter 4. The second topic is the presence of the inaction region around the equilibrium point. In the CVIU approach, this results from the dependence of the stochastic noise on the absolute value of the system state and control input. In the following we first recall the formulation of the CVIU control problem. After that we review the assumptions that lead to the existence of the inaction region in the CVIU approach. We also revisit the solution of the CVIU control problem when the cost functional assumes a discounted form and the optimization horizon is infinite.

For now, recall that, in comparison with its deterministic counterparts, models used in stochastic control theory represent the disturbances affecting the operation of the systems as stochastic processes. Under a discrete-time framework, this leads to the use of difference equations with white noise disturbances, while in continuous-time settings the dynamics of the system can be described in terms of a stochastic differential equation of

the form (CHEN *et al.*, 1995; PHAM, 2009)

$$dx(t) = \mu(t, x_t, u_t)dt + \sigma(t, x_t, u_t)dW_t, \quad (2.53)$$

where $W(t)$ is a standard Brownian Motion of appropriate dimensions, $x_t \in \mathbb{R}^n$ represents the evolution of the system state, and $u \in U \subset \mathbb{R}^k$ is the control input, assumed to belong to the Borel set U , known as the set of admissible controls. The possibly multidimensional functions $\mu(t, x, u) : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{U} \rightarrow \mathbb{R}^n$ and $\sigma(t, x, u) : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{U} \rightarrow \mathbb{R}^{n \times m}$ are known as the drift and diffusion matrices, respectively.

The functions $\mu(t, x_t, u_t)$ and $\sigma(t, x_t, u_t)$ are assumed to be measurable and satisfy a uniform Lipschitz condition in the set of admissible controls \mathbb{U} (PHAM, 2009), that is, there exists a constant $L \geq 0$ such that condition (2.37) is satisfied. The set of admissible controls \mathbb{U} comprises Markov functions $t \rightarrow u(t, w) = u(t, x(t))$ and $u(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^r)$. Moreover, we consider, associated to the controlled system (2.53), the following cost functional,

$$J(t, x(t), u(t)) = \mathbb{E} \left[\int_0^T f(t, x(t), u(t))dt + g(x(T)) \right]. \quad (2.54)$$

2.2.1 Problem formulation

The class of controlled diffusion processes $z(t) \in \mathbb{R}^n$ that we study under the CVIU approach can be described by the stochastic differential equation

$$dz(t) = \mu(t, z(t), v(t))dt + \sigma(t)dW(t), \quad (2.55)$$

where $W(t)$ is a m -dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$; the control policy $v = \{v(t)\}_{t \geq 0}$ and the diffusion matrix $\sigma = \{\sigma(t)\}_{t \geq 0}$, $\sigma(\cdot) \in \mathbb{R}^{n \times m}$ are $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted stochastic process with $v(\cdot)$ taking values on the compact set $\mathbb{U} \subset \mathbb{R}^r$, and $\mu : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{U} \rightarrow \mathbb{R}^n$ represents the process drift.

Now, suppose the system governed by equation (2.55) operates around an equilibrium point z_{eq} and we wish to design a controller which would keep the system operating in that region. A natural first step in this case is to find a linear model valid in the vicinity of the equilibrium point, and base our design process on this linear, locally valid model. For that, take $x(t) = z(t) - z_{eq}$ and $u(t) = v(t) - v_{eq}$, and assume a rough description of the linearized model is given by

$$dx(t) = (Ax(t) + Bu(t))dt + \sigma(t)dW(t).$$

The corresponding model in the CVIU case is obtained by adding state- and control-dependent noise into the above equation (VAL; SOUTO, 2017). The complete CVIU representation for a locally valid linear model for the process (2.55) then takes the form

$$dx(t) = (Ax(t) + Bu(t))dt + \hat{\sigma}(x(t), u(t))d\hat{W}(t), \quad (2.56)$$

where $x(\cdot) \in \mathbb{R}^n$ represents the system state, the control input $u(\cdot)$ is adapted to the filtration $\{\mathcal{F}_t\}$ and takes values on the set $\mathbb{U} \in \mathbb{R}^r$, and

$$\begin{aligned}\hat{\sigma}(x(t), u(t)) &= [\sigma(t) \quad \sigma_x(t) + \bar{\sigma}_x(t) \operatorname{diag}(|x(t)|) \quad \sigma_u(t) + \bar{\sigma}_u(t) \operatorname{diag}(|u(t)|)], \\ \hat{W}(t) &= [W(t)^T \quad W^x(t)^T \quad W^u(t)^T].\end{aligned}\tag{2.57}$$

In a similar manner, the complete CVIU representation can also be written in the form

$$dx = (Ax(t) + Bu(t))dt + \sigma dW(t) + (\sigma_x + \bar{\sigma}_x |x(t)|)dW^x(t) + (\sigma_u + \bar{\sigma}_u |u(t)|)dW^u(t).$$

Here, $|v|$ is the element wise absolute value of vector v , and $\operatorname{diag} : \mathbb{R}^l \rightarrow \mathbb{R}^{l \times l}$ for $v \in \mathbb{R}^l$ is a linear operator such that $\operatorname{diag}(v)$ is the matrix formed by v as the main diagonal and zero elsewhere. An analogous operator for matrices $A \in \mathbb{R}^{l \times l}$ is given by $\operatorname{Diag} : \mathbb{R}^{l \times l} \rightarrow \mathbb{R}^{l \times l}$, where $\operatorname{Diag}(A)$ stands for the diagonal matrix obtained from the main diagonal of A with zeros elsewhere (VAL; SOUTO, 2017). Furthermore, $W(t), W^x(t), W^u(t)$ are standard, independent BM of appropriate dimensions, and the diffusion matrices $\sigma(t), \sigma_x(t), \bar{\sigma}_x(t), \sigma_u(t), \bar{\sigma}_u(t)$ are $\{\mathcal{F}\}_{t \geq 0}$ -adapted matrix functions. The CVIU optimal control problem for the continuous time case can then be stated as

$$\begin{aligned}\min J(s, x, u(\cdot)) \\ \text{s.a. } dx(t) &= (Ax(t) + Bu(t))dt + \hat{\sigma}(x(t), u(t))d\hat{W}(t),\end{aligned}\tag{2.58}$$

where $J(\cdot)$ is the cost function associated to the problem, given by the expected value of

$$J(s, x, u(\cdot)) := \mathbb{E}^x \left[\int_s^T f(t, x(t), u(t))dt + g(x(T)) \right].\tag{2.59}$$

In the above equation $\mathbb{E}^x[\cdot]$ corresponds to the expected value from starting the process (2.56) at state x . Furthermore, we assume that the cost functions $f : [0, T] \times \mathbb{R}^n \times \mathbb{U} \rightarrow \mathbb{R}_+$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}_+$, as well as the data for the problem, $A, B, \sigma, \sigma_x, \bar{\sigma}_x, \sigma_u, \bar{\sigma}_u$ are uniformly continuous functions. We also assume that functions $g(\cdot)$ and $f(t, \cdot, u)$ for each (t, u) are (strictly) convex functions and $f(t, x, \cdot)$ is convex and continuously differentiable for each (t, x) (VAL; SOUTO, 2017). Moreover, in the following we consider that the following holds in the semidefinite positive sense,

$$\begin{aligned}\sigma_x(t)\bar{\sigma}_x(t)^\top + \bar{\sigma}_x(t)\sigma_x(t)^\top &\geq 0 \\ \sigma_u(t)\bar{\sigma}_u(t)^\top + \bar{\sigma}_u(t)\sigma_u(t)^\top &\geq 0, \quad t \in [0, T].\end{aligned}\tag{2.60}$$

A key result for the CVIU approach is the convexity of the value function $J^*(s, x) := \inf_{u \in \mathbb{U}} J(s, x, u)$, stated in the following theorem.

Theorem 2.12 ((VAL; SOUTO, 2017), Theorem 1). *Under the choice in (2.60) and the assumption that $f(t, \cdot, \cdot)$, $t \in [0, T]$, and $g(\cdot)$ are (strictly) convex functions, the value $J^*(\cdot, \cdot)$ of the CVIU control problem is continuous and $J^*(t, \cdot)$ (strictly) convex for each $t \in [0, T]$.*

As we discussed in last section, the dynamic programming approach states that if the optimal value of the cost function $J^*(s, x) \in C^{1,2}$, then it is the unique solution of the Hamilton Jacobi Bellman equation

$$-\frac{\partial \varphi}{\partial t} - \inf_{u \in U} H(t, x, u, \varphi_x, \varphi_{xx}) = 0, \quad \varphi(T, x) = g(x), \quad (2.61)$$

where $H(t, x, u, \varphi_x, \varphi_{xx})$ is the Hamiltonian function associated to the optimization problem, φ_x the gradient and φ_{xx} the Hessian matrix of φ ,

$$H(t, x, u, p, P) := \frac{1}{2} \text{tr}(P \hat{\sigma}(t, x, u) \hat{\sigma}(t, x, u)^\top) + \langle p, (A(t)x + B(t)u) \rangle + f(t, x, u). \quad (2.62)$$

Since we cannot guarantee that the value function in the CVIU case is sufficiently smooth, we need to consider the notions of sub- and superdifferentials and viscosity solutions previously introduced.

2.2.2 The inaction region

Following (VAL; SOUTO, 2017), note that the Hamiltonian function (2.62) can be recast in the form

$$\begin{aligned} H(t, x, u, p, P) = & f(t, x, u) + \langle p, (A(t)x + B(t)u) \rangle \\ & + \frac{1}{2} (\Gamma_0(t, P) + \Gamma_1(t, x, u, P) + \Gamma_2(t, x, u, P)), \end{aligned} \quad (2.63)$$

where

$$\Gamma_0(t, P) = \frac{1}{2} \text{tr}(P(\sigma(t)\sigma(t)^\top + \sigma_x(t)\sigma_x(t)^\top + \sigma_u(t)\sigma_u(t)^\top)), \quad (2.64a)$$

$$\begin{aligned} \Gamma_1(t, x, u, P) = & \text{tr}((\bar{\sigma}_x(t)^\top P \sigma_x(t) + \sigma_x(t)^\top P \bar{\sigma}_x(t)) \text{diag}(|x|)) \\ & + \text{tr}((\bar{\sigma}_u(t)^\top P \sigma_u(t) + \sigma_u(t)^\top P \bar{\sigma}_u(t)) \text{diag}(|u|)), \end{aligned} \quad (2.64b)$$

$$\Gamma_2(t, x, u, P) = \text{tr}(\bar{\sigma}_x(t)^\top P \bar{\sigma}_x(t) \text{diag}(|x|)^2) + \text{tr}(\bar{\sigma}_u(t)^\top P \bar{\sigma}_u(t) \text{diag}(|u|)^2). \quad (2.64c)$$

Note that due to the presence of the absolute value function, the Hamiltonian function is nonlinear and nondifferentiable at the origin with relation to u at the origin of \mathbb{R}^r . To minimize the Hamiltonian function with relation to the control input u , we consider again the notion of subdifferentials. For the terms which depend on the absolute value of the state or control, the subdifferentials for $A \in \mathbb{R}^{l \times l}$ and $v \in \mathbb{R}^l$ take the form

$$\begin{aligned} D_v^{1,-} \text{tr}(A \text{diag}(|v|)^2) &= 2 \text{Diag}(A)v, \\ D_v^{1,-} \text{tr}(A \text{diag}(|v|)) &= \text{Diag}(A)\mathcal{S}(v), \end{aligned}$$

where $\mathcal{S}(v) := [\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_l]^\top$ is a vector of sets of form

$$\mathcal{S}_i = \begin{cases} +1, & \text{if } v_i > 0, \\ -1, & \text{if } v_i < 0, \\ [-1, +1] & \text{if } v_i = 0. \end{cases}$$

The first order condition in the HJB equation then takes the form (VAL; SOUTO, 2017)

$$\begin{aligned} D_u^{1,-} H(t, x, u, p, P)|_{u=u^*} &= \frac{\partial_u f(t, x, u)}{\partial u}|_{u=u^*} + B(t)^\top p \\ &+ \frac{1}{2} \text{Diag}(\bar{\sigma}_u(t)^\top P \sigma_u(t) + \sigma_u(t)^\top P \bar{\sigma}_u(t)) \mathcal{S}(u)|_{u=u^*} \\ &+ \text{Diag}(\bar{\sigma}_u(t)^\top P \bar{\sigma}_u(t)) u|_{u=u^*} = 0, \end{aligned} \quad (2.65)$$

where $D_u^{1,-}$ stands for the subdifferential with relation to u and $\mathcal{S}(u)$ indicates the sign of the components of u . To determine the optimal control policy u^* , we first need to determine the vector of signs $\mathcal{S}(u)$. This can be achieved by dividing the state space \mathbb{R}^n into distinct regions for which the sign of all components of the optimal control vector is known.

Lemma 2.1 ((VAL; SOUTO, 2017), Lemma 1). *Consider that $f(t, \cdot, \cdot), t \in [0, T]$ and $g(\cdot)$ are convex and $f(t, x, \cdot)$ is continuously differentiable for each t and x . Then the optimal control $u^* = [u_1^* \dots u_i^* \dots u_r^*]^\top$ for the problem in (2.58) satisfies the following sign conditions,*

$$\begin{cases} u_i^* > 0, & \text{if } x \in \mathcal{R}_i^+(t), \text{ with } \mathcal{R}_i^+(t) := \left\{ x \in \mathbb{R}^n : \lim_{u_i \downarrow 0} \frac{\partial H(t, x, u, p, P)}{\partial u_i} < 0 \right\}, \\ u_i^* < 0, & \text{if } x \in \mathcal{R}_i^-(t), \text{ with } \mathcal{R}_i^-(t) := \left\{ x \in \mathbb{R}^n : \lim_{u_i \uparrow 0} \frac{\partial H(t, x, u, p, P)}{\partial u_i} > 0 \right\}, \\ u_i^* = 0, & \text{if } x \in \mathcal{R}_i^0(t), \text{ with } \mathcal{R}_i^0(t) := \left\{ x \in \mathbb{R}^n : (\mathcal{R}_i^+(t) \cup \mathcal{R}_i^-(t))^C \right\}, \end{cases} \quad (2.66)$$

$i = 1, \dots, r$ for some p and P such that $(q, p, P) \in D_{t+,x}^{1,2,+} J^*(t, x)$, a.e. $t \in [0, T]$ P -a.s.

The region $\mathcal{R}_i^0(t)$, where the above lemma indicates the optimal policy (according to the CVIU model) is to keep the equilibrium control action for the component u_i unchanged, is called the *inaction* region for the i -th input. Conditions for the existence of the inaction region for the i -th control input as a nonempty hypervolume are given by the following lemma, valid if the value function is convex, cf. Theorem 2.12.

Lemma 2.2 ((VAL; SOUTO, 2017), Lemma 2). *Consider the previous assumptions on the CVIU model and that $f(t, x, \cdot)$ is a continuously differentiable function for each t and x . Let $P \in (q, p, P) \in D_{t+,x}^{1,2,+} J^*(t, x)$, for $(t, x) \in [0, T] \times \mathbb{R}^n$. If*

$$\delta_i(t) := (\bar{\sigma}_u(t)^\top P \sigma_u(t) + \sigma_u(t)^\top P \bar{\sigma}_u(t))_{ii} > 0, \quad (2.67)$$

for some $i = 1, \dots, r$, the region $\mathcal{R}_i^0(t) \subset \mathbb{R}^n$, $i = 1, \dots, r$ has nonempty hypervolume. Moreover,

$$\mathcal{R}_i^0(t) = \left\{ x \in \mathbb{R}^n : \left| \frac{\partial f(t, x, u)}{\partial u_i} \Big|_{u=u^*, u_i^*=0} + \langle B_i(t), p \rangle \right| \leq \delta_i(t) \right\}. \quad (2.68)$$

Note that the existence of the inaction region in the CVIU approach is associated with the use of the absolute value function to represent the intensity of the noise.

2.2.3 Infinite horizon and expected discounted cost

When we adopt a discounted cost functional evaluated along an infinite horizon,

$$\mathbb{E} \left[\int_0^\infty e^{-\alpha s} f(s, x, u) ds \right],$$

where $f(s, x, u)$ is a quadratic function with $f(s, x, u) = x(s)^\top Q x(s) + u(s)^\top R u(s)$, and assume the coefficient σ_x equals zero, we get a solution procedure as detailed in (VAL; SOUTO, 2017). Since we are working with an infinite horizon frame, we consider the coefficients do not depend explicitly on time, that is, we assume the matrices A and B , as well as σ , σ_x , $\bar{\sigma}_x$, σ_u and $\bar{\sigma}_u$ are time-invariant. The procedure consists basically in dividing the state space into distinct regions and calculating the optimal control policy for each one of these regions. Inside the overall inaction region, $\mathcal{R}_0 := \{x \in \mathbb{R}^n : x \in \cap_{i=r} \mathcal{R}_i^0\}$, we know the optimal control action for the locally valid linear model, u^* , equals zero, and by plugging this result into the HJB equation (2.61), we get a modified version of the Lyapunov equation,

$$A^\top X + XA - \alpha X + Q + \text{Diag}(\bar{\sigma}_x^\top X \bar{\sigma}_x) = 0. \quad (2.69)$$

The value function for the inaction region is then given in terms of the solution of the modified Lyapunov equation (VAL; SOUTO, 2017),

$$V^*(x) = x^\top X x + \frac{1}{\alpha} \text{tr}(X(\sigma\sigma^\top + \sigma_u\sigma_u^\top)), \quad (2.70)$$

whereas the limits of the inaction region result from Lemma 2.2,

$$\mathcal{R}_i^0 := \{x \in \mathbb{R}^n : -\delta_i(X) \leq B_i^\top X x \leq \delta_i(X)\}, \quad (2.71)$$

with $\delta_i(X) := (\bar{\sigma}_u^\top X \sigma_u + \sigma_u^\top X \bar{\sigma}_u)$.

We can also find a steady state solution for the HJB equation for regions with homoeogenous control signs, and for that we assume that x is sufficiently large. In this case, we need to consider that x is sufficiently distant from the origin so that the control sign remains the same for a nonempty region of the state space. The solution in this asymptotic case is then given in terms of the modified Riccati equation (VAL; SOUTO, 2017),

$$\begin{aligned} A^\top X + XA - \alpha X - XB(R')^{-1}B^\top X + Q' &= 0, \\ Q' &:= Q'(X) = Q + \text{Diag}(\bar{\sigma}_x^\top X \bar{\sigma}_x), \\ R' &:= R'(X) = R + \text{Diag}(\bar{\sigma}_u^\top X \bar{\sigma}_u). \end{aligned} \quad (2.72)$$

. The value function, in this case, tends asymptotically to

$$V_a^{\bar{s}} := x^\top X x + \langle v(\bar{s}), x \rangle + l(\bar{s}), \quad (2.73)$$

where \bar{s} is the vector of signs of the control policy u^* , with $\bar{s} = [s_1, s_2, \dots, s_r]^\top$ and $s_i \in [-1, +1]$. Moreover, v and l satisfy

$$v^\top (A - \alpha I - B(R')^{-1} B^\top X) = \bar{s}^\top \Delta_u (R')^{-1} B^\top X \quad (2.74a)$$

$$l(\bar{s}) = \frac{1}{\alpha} \left(\text{tr}(X(\sigma\sigma^\top + \sigma_u\sigma_u^\top)) - \frac{1}{4}(B^\top v + \Delta_u \bar{s}^\top (R')^{-1})(\bullet) \right) \quad (2.74b)$$

$$\Delta_u = \Delta_u(X) := \text{Diag}(\bar{\sigma}_u^\top X \sigma_u + \sigma_u^\top X \bar{\sigma}_u). \quad (2.74c)$$

The optimal control policy in this case tends asymptotically to

$$u^*(x) = -K_\infty x + M_\infty, \quad (2.75)$$

with

$$K_\infty = (R'(X))^{-1} B^\top X$$

$$M_\infty = \frac{1}{2} (R'(X))^{-1} (B^\top v + \Delta_u(X) \bar{s}).$$

2.2.3.1 Design of a CVIU control policy

With the solution of the CVIU control problem for the inaction region and asymptotic regions of the state space with homogeneous control signs, we can design an optimal CVIU control policy as follows.

1. Solve the generalized Lyapunov equation (2.70) using a relaxation procedure.
2. Calculate the limits of the inaction region according to equation (2.71).
3. Choose a x_∞ large enough for regions with homogeneous control signs.
4. Solve the generalized Riccati equation (2.72) using a relaxation procedure.
5. Calculate the asymptotic feedback policy (2.75).
6. Find a linear approximation for the control policy between the limit of the inaction region and x_∞ .

We illustrate this procedure, for the scalar case, in Figure 5, where the dashed line represents the numerical solution between the inaction region and the asymptotic region where the control signs can be considered homogeneous.

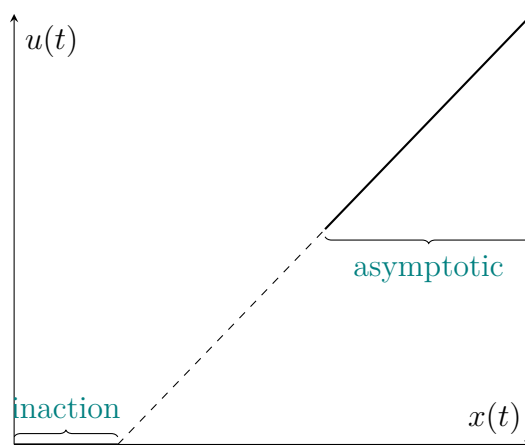


Figure 5 – CVIU optimal control policy: numerical solution.

3 Norms and long run average formulation of the CVIU control problem

In this chapter we collect some new results within the CVIU approach. In the first section we revisit the CVIU stochastic optimal control problem with discounted cost and present a slight generalization of the previous results by dropping the requirement that the diffusion coefficient σ_x equals zero. The second part of the chapter is concerned with the long run average formulation of the CVIU control problem.

3.1 What happens when σ_x is not zero?

The solution for the discounted quadratic cost solution of the CVIU control problem presented previously in (VAL; SOUTO, 2017) is based on the assumption that the coefficient σ_x equals zero. This is a convenient assumption, since it simplifies the corresponding Hamilton-Jacobi-Bellman equation, but it can be dropped in order to get a slightly more general solution of the control problem.

Recall then the definition of the vector of sets $\mathcal{S}(v) := [\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n]$, with

$$\mathcal{S}_i = \begin{cases} +1, & \text{if } v_i > 0, \\ -1, & \text{if } v_i < 0, \\ [-1, +1] & \text{if } v_i = 0. \end{cases}$$

Assume once more that we are considering the coefficients A , B , σ , σ_x , $\bar{\sigma}_x$, σ_u and $\bar{\sigma}_u$ to be time-invariant, since our optimization horizon is infinite. As before, we adopt a discounted cost functional, but write it in terms of the norm of the measured output $y(t)$. The control problem can be stated as

$$\begin{aligned} \min J(x, u, t) &:= \mathbb{E} \left[\int_0^\infty e^{-\alpha t} \|y(t)\|^2 dt \right], \\ \text{s.t.} \quad &\begin{cases} dx(t) = (Ax(t) + Bu(t))dt + \hat{\sigma}(x(t), u(t))d\hat{W}(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \end{aligned} \quad (3.1)$$

with $\|y(t)\| = [y(t)^\top y(t)]^{\frac{1}{2}}$. Note that we can retrieve the previous formulation of the control problem if we set $Q = C^\top C$, $R = D^\top D$ and $C^\top D = 0$.

The first-order condition for the HJB equation (2.65), as well as the boundary of the inaction region, remain the same, but the HJB equation for the discounted case now

takes the form

$$\begin{aligned} \alpha\varphi(x) - \inf_{u \in U} \left\{ x^\top \left(C^\top C + \frac{1}{2} \text{Diag}(\bar{\sigma}_x^\top P \bar{\sigma}_x) \right) x + u^\top \left(D^\top D + \frac{1}{2} \text{Diag}(\bar{\sigma}_u^\top P \bar{\sigma}_u) \right) u \right. \\ + 2x^\top C^\top D u + \langle p, (Ax + Bu) \rangle + \frac{1}{2} \text{tr}((\bar{\sigma}_x^\top P \sigma_x + \sigma_x^\top P \bar{\sigma}_x) \text{diag}(|x|)) \\ + \frac{1}{2} \text{tr}((\bar{\sigma}_u^\top P \sigma_u + \sigma_u^\top P \bar{\sigma}_u) \text{diag}(|u|)) \\ \left. + \frac{1}{2} \text{tr}(P(\sigma\sigma^\top + \sigma_x\sigma_x^\top + \sigma_u\sigma_u^\top)) \right\} = 0, \quad (3.2) \end{aligned}$$

with $p = \varphi_x$ and $P = \varphi_{xx}$ if φ is twice differentiable. Otherwise, the HJB equation (3.2) holds in the viscosity sense: for each $(p, P) \in D_x^{1,2+} \varphi(x)$ with the inequality sign ≤ 0 , which shows that φ is a viscosity subsolution; and for each $(p, P) \in D_x^{1,2-} \varphi(x)$ with the inequality sign ≥ 0 , which shows that φ is a viscosity supersolution. The solution follows along the general lines of the results of last chapter.

Lemma 3.1. *Consider the stochastic control problem (3.1). If there exists $X \in S_0^{n+}$ solution of the modified Lyapunov matrix equation*

$$A^\top X + XA - \alpha X + C^\top C + \text{Diag}(\bar{\sigma}_x^\top X \bar{\sigma}_x) = 0, \quad (3.3)$$

then the value function for each $x \in \mathcal{R}^0$ is

$$V^*(x) = x^\top X x + v^\top x + l, \quad (3.4)$$

with v and l satisfying

$$v = (A - \alpha I)^{-\top} \text{Diag}(\bar{\sigma}_x^\top X \bar{\sigma}_x) \mathcal{S}(x), \quad (3.5a)$$

$$l = \frac{1}{\alpha} \text{tr}(X(\sigma\sigma^\top + \sigma_x\sigma_x^\top + \sigma_u\sigma_u^\top)). \quad (3.5b)$$

Moreover, the inaction region \mathcal{R}^0 is defined by parallel hyperplanes

$$\mathcal{R}_i^0 = \{x \in \mathbb{R}^n : -\delta_i(X) \leq B_i^\top X x \leq +\delta_i(X)\}, \quad (3.6)$$

with $\delta_i(X) := (\bar{\sigma}_u^\top X \sigma_u + \sigma_u^\top X \bar{\sigma}_u)$.

Proof: In the following we set

$$Q' = Q'(X) = C^\top C + \text{Diag}(\bar{\sigma}_x^\top X \bar{\sigma}_x), \quad (3.7a)$$

$$\Delta_x(X) := \text{Diag}(\bar{\sigma}_x^\top X \sigma_x + \sigma_x^\top X \bar{\sigma}_x). \quad (3.7b)$$

Consider a quadratic candidate function $\varphi^c = x^\top X x + v^\top x + l$, such that $\varphi_x^c = 2Xx + v$ and $\varphi_{xx}^c = 2X$. Within the inaction region, $u^* = 0$, and therefore the HJB equation in this case takes the form

$$\begin{aligned} x^\top \left(C^\top C + \frac{1}{2} \text{Diag}(\bar{\sigma}_x^\top 2X \bar{\sigma}_x) \right) x + \langle xX, Ax \rangle + \langle Ax, Xx \rangle + \langle v, Ax \rangle \\ + \frac{1}{2} \Gamma_0(\cdot) + \frac{1}{2} \text{tr}((\bar{\sigma}_x^\top 2X \sigma_x + \sigma_x^\top 2X \bar{\sigma}_x) \text{diag}(|x(\cdot)|)) - \alpha\varphi(x) = 0, \end{aligned}$$

$$\begin{aligned}
& x^\top (C^\top C + \text{Diag}(\bar{\sigma}_x^\top X \bar{\sigma}_x)) x + 2\langle xX, Ax \rangle + \langle v, Ax \rangle + \frac{1}{2}\Gamma_0(\cdot) \\
& + \text{tr}((\bar{\sigma}_x^\top X \sigma_x + \sigma_x^\top X \bar{\sigma}_x) \text{diag}(|x(\cdot)|)) - \alpha(x^\top Xx + v^\top x + l) = 0, \\
& x^\top (A^\top X + XA - \alpha X + C^\top CQ + \text{Diag}(\bar{\sigma}_x^\top X \bar{\sigma}_x)) x + \langle v, Ax \rangle - \alpha v^\top x + \frac{1}{2}\Gamma_0(\cdot) \\
& + \mathcal{S}(x)^\top \text{Diag}((\bar{\sigma}_x^\top X \sigma_x + \sigma_x^\top X \bar{\sigma}_x)) x - \alpha l = 0. \quad (3.8)
\end{aligned}$$

From the last equation, we get that the modified Lyapunov equation within the inaction region is given by

$$\begin{aligned}
& A^\top X + XA - \alpha X + C^\top C + \text{Diag}(\bar{\sigma}_x^\top X \bar{\sigma}_x) = 0, \\
& \left(A - \frac{\alpha}{2}\right)^\top X + X \left(A - \frac{\alpha}{2}\right) + Q'(X) = 0.
\end{aligned}$$

The value function in this region corresponds to

$$V^*(x) = x^\top Xx + v^\top x + l, \quad (3.9)$$

with X the solution of the modified Lyapunov equation and v, l satisfying

$$\begin{aligned}
v &= (A - \alpha I)^{-\top} \text{Diag}(\bar{\sigma}_x^\top X \bar{\sigma}_x) \mathcal{S}(x), \\
l &= \frac{1}{\alpha} \text{tr}(X(\sigma \sigma^\top + \sigma_x \sigma_x^\top + \sigma_u \sigma_u^\top)).
\end{aligned}$$

□

We now consider the asymptotic solution of the CVIU control problem for regions in which the sign of the control input is homogeneous, in the sense that the control sign is either $u_i^*(\cdot) > 0$ or $u_i^*(\cdot) < 0$. For this analysis, we consider that the state x is distant enough from the origin in order for the corresponding control sign to remain constant, that is, if $\inf \|x - y\|$ is sufficiently large for $x \in \mathbb{R}^n, y \in \mathcal{R}_i^0$, then the control sign remains constant in a neighborhood of x . We follow the notation from (VAL; SOUTO, 2017) and, in the following, consider the situation in which $u_i^* \neq 0, \forall i$. Let then the sets, for $\ell \geq 0$,

$$\mathcal{T}_i^+(\ell) = \mathcal{T}_i^{\{s_i=+1\}}(\ell) := \{x \in \mathcal{R}_i^+ : \inf \|x - y\| \geq \ell, \forall y \in \cup_{j=1}^r \mathcal{R}_j^0\}, \quad i = 1, \dots, r, \quad (3.11)$$

$$\mathcal{T}_i^-(\ell) = \mathcal{T}_i^{\{s_i=-1\}}(\ell) := \{x \in \mathcal{R}_i^- : \inf \|x - y\| \geq \ell, \forall y \in \cup_{j=1}^r \mathcal{R}_j^0\}, \quad i = 1, \dots, r, \quad (3.12)$$

$$\mathcal{T}^{\bar{s}}(\ell) := \cap_{i=1}^r \mathcal{T}_i^{s_i}(\ell), \quad (3.13)$$

where $\bar{s} = [s_1 \ s_2 \ \dots \ s_r]^\top$ and each s_i takes the possible values $+1$ or -1 for \mathcal{T}_i^+ and \mathcal{T}_i^- , respectively. \mathcal{S} is the set of all possible values for \bar{s} , that is, $\mathcal{S} = \{\bar{s} = [s_1 \ s_2 \ \dots \ s_r]^\top : s_i = \pm 1, \forall i\}$. According to (VAL; SOUTO, 2017), $\mathcal{T}^{\bar{s}}(\ell)$ is symmetric to the origin to region $\mathcal{T}^{-\bar{s}}(\ell)$. Let $\mathcal{T}^{\bar{s}}(0)$ denote the entire region with controls with homogeneous signs, which are given by some \bar{s} . Each combination of signs \bar{s} in \mathcal{S} corresponds to a nonempty region $\mathcal{T}^{\bar{s}}(\ell) \subset \mathbb{R}^n$ (VAL; SOUTO, 2017), and for each possible region $\mathcal{T}^{\bar{s}}(\ell)$, we have the following result. Here, we consider L large enough.

Lemma 3.2. Consider the stochastic control problem (3.1). If there exists $X \in S_0^{n+}$ solution of the modified Riccati equation

$$\left(A - \frac{\alpha}{2}I\right)^\top X + X \left(A - \frac{\alpha}{2}I\right) - (C^\top D + XB)(R')^{-1}(B^\top X + D^\top C) + Q' = 0, \quad (3.14)$$

where $Q'(X) := Q + \text{Diag}(\bar{\sigma}_x^\top X \bar{\sigma}_x)$ and $R'(X) := R + \text{Diag}(\bar{\sigma}_u^\top X \bar{\sigma}_u)$, then the asymptotic solution of the value function in this case converges, in each region $\mathcal{T}^{\bar{s}}(L)$, to

$$V_a^{\bar{s}} := x^\top X x + \langle v(\bar{s}), x \rangle + l(\bar{s}), \quad (3.15)$$

with

$$v^\top \left(A - \alpha I - B(R')^{-1}F\right) = \mathcal{S}(u)^\top \Delta_u(X)(R')^{-1}F - \mathcal{S}(x)^\top \Delta_x(X), \quad (3.16a)$$

$$l = \frac{1}{\alpha} \left(\text{tr} \left(X(\sigma\sigma^\top + \sigma_x\sigma_x^\top + \sigma_u\sigma_u^\top) \right) - \frac{1}{4}(B^\top v + \Delta_u(X)\mathcal{S}(u))^\top (R')^{-1}(\bullet) \right), \quad (3.16b)$$

$$\Delta_x = \Delta_x(X) := \text{Diag}(\bar{\sigma}_x^\top X \sigma_x + \sigma_x^\top X \bar{\sigma}_x), \quad (3.16c)$$

$$\Delta_u = \Delta_u(X) := \text{Diag}(\bar{\sigma}_u^\top X \sigma_u + \sigma_u^\top X \bar{\sigma}_u), \quad (3.16d)$$

and X the solution of the modified Riccati equation. Moreover, the optimal control policy u^* tends asymptotically to

$$u^*(x) = -K_\infty x + M_\infty, \quad (3.17)$$

with

$$K_\infty = R'(X)^{-1}(B^\top X + D^\top C),$$

$$M_\infty = -\frac{1}{2}R'(X)^{-1}(B^\top v + \Delta_u(X)\mathcal{S}(u)).$$

Proof: Consider a quadratic test function $V(x) = x^\top X x + v^\top x + l$ with X a symmetric matrix and, once again, take

$$Q'(X) := Q + \text{Diag}(\bar{\sigma}_x^\top X \bar{\sigma}_x), \quad R'(X) := R + \text{Diag}(\bar{\sigma}_u^\top X \bar{\sigma}_u),$$

to write the HJB equation in a more compact form,

$$\begin{aligned} & -\alpha\varphi(x) + \min_{u \in U} \{x^\top Q'(X)x + u^\top R'(X)u + 2x^\top C^\top Du \\ & \quad + \langle Ax + Bu, Xx \rangle + \langle Xx, Ax + Bu \rangle + \langle v, Ax + Bu \rangle \\ & \quad + \langle \mathcal{S}(u), \Delta_u(X)u \rangle + \langle \mathcal{S}(x), \Delta_x(X)x \rangle + \frac{1}{2}\Gamma_0(\cdot)\} = 0. \end{aligned} \quad (3.18)$$

Now isolating the terms dependent on u and expanding the dot products, we get

$$\begin{aligned} & -\alpha(x^\top X x + v^\top x + l) + x^\top (A^\top X + XA + Q'(X))x + v^\top Ax + \mathcal{S}^\top(x)\Delta_x(x)x + \frac{1}{2}\Gamma_0(\cdot) \\ & \quad + \min_{u \in U} \{u^\top R'(X)u + 2x^\top C^\top Du + u^\top B^\top Xx + x^\top XBu + v^\top Bu + \mathcal{S}^\top(u)\Delta_u(X)u\} = 0. \end{aligned} \quad (3.19)$$

Now, define $F := B^\top X + D^\top C$ and $g := B^\top v + \Delta_u(X)\mathcal{S}(u)$. Furthermore, notice that the expression within the braces has a quadratic form, and we can use a matrix completion of squares technique to change it from the standard quadratic form to a more convenient expression. In our case this leads to

$$\left(u + R'(X)^{-1} \left(Fx + \frac{1}{2}g\right)\right)^\top R'(X)(\bullet) - \frac{1}{2} \left(Fx + \frac{1}{2}g\right)^\top R'(X)^{-1}(\bullet), \quad (3.20)$$

and, therefore,

$$\begin{aligned} u^* &= -R'(X)^{-1} \left(Fx + \frac{1}{2}g\right) \\ &= -R'(X)^{-1} \left((B^\top X + D^\top C)x + \frac{1}{2}(B^\top v + \Delta_u(X)\mathcal{S}(u))\right). \end{aligned} \quad (3.21)$$

With this choice of u^* , the HJB equation (3.19) can now be written as

$$\begin{aligned} x^\top &\left(\left(A - \frac{\alpha}{2}I\right)^\top X + X \left(A - \frac{\alpha}{2}I\right) - F^\top (R')^{-1} F + Q'(X)\right)x \\ &+ \langle v, (A - \alpha I)x \rangle + \langle \mathcal{S}(x), \Delta_x(X)x \rangle - \langle g, R'(X)^{-1} Fx \rangle \\ &- \alpha l + \frac{1}{2}\Gamma_0(\cdot) - \frac{1}{4}g^\top R'(X)^{-1} g = 0. \end{aligned} \quad (3.22)$$

From the last expression, the modified Riccati equation is given by

$$\left(A - \frac{\alpha}{2}I\right)^\top X + X \left(A - \frac{\alpha}{2}I\right) - (C^\top D + XB)(R')^{-1}(B^\top X + D^\top C) + Q' = 0,$$

and v and l should satisfy

$$\begin{aligned} v^\top &\left(A - \alpha I - B(R')^{-1}F\right) = \mathcal{S}(u)^\top \Delta_u(X)(R')^{-1}F - \mathcal{S}(x)^\top \Delta_x(X), \\ l &= \frac{1}{\alpha} \left(\text{tr}(X(\sigma\sigma^\top + \sigma_x\sigma_x^\top + \sigma_u\sigma_u^\top)) - \frac{1}{4}(B^\top v + \Delta_u(X)\mathcal{S}(u))^\top (R')^{-1}(\bullet)\right). \end{aligned}$$

□

The design of an optimal CVIU control policy here follows the procedure described in section 2.2.3.1.

3.2 Norm equivalence

A natural question we wished to address, when we first started to think about the relations between the CVIU and robust control policies, is how to relate the different optimization criteria used in these problems. Since we initially worked with the discounted quadratic solution of the CVIU control problem, our first idea was to derive a mathematical relation between the discounted cost functional adopted in the CVIU setting,

$$J(x, u(\cdot)) = E \left[\int_0^\infty e^{-\alpha t} \|y(t)\|^2 dt \right],$$

and the (stochastic) \mathcal{H}_2 norm of a linear, time-invariant stochastic system. That is the point we discuss in this section. Results have been previously presented in the conference paper (SILVA *et al.*, 2017).

3.2.1 Discounted cost and \mathcal{H}_2 norm

For the CVIU synthesis, the performance of the system is measured through a expected discounted cost associated to the CVIU model (2.57), and a relation between this functional and the H_2 norm of a LTI stochastic system allows us to compare the cost of operation of the mentioned controllers. Let then \mathcal{G} a LTI stochastic system,

$$\mathcal{G} : \begin{cases} dx(t) = Ax(t)dt + \sigma dW(t), \\ y(t) = Cx(t), \quad x(0) = 0, \end{cases} \quad (3.23)$$

with $x(\cdot) \in \mathbb{R}^n$, $y(\cdot) \in \mathbb{R}^m$, $W(\cdot)$ a r -dimensional BM, and A a stable matrix. The (stochastic) \mathcal{H}_2 norm of the above system is given by (DRAGAN *et al.*, 2006, sec. 7.1),

$$\left(\lim_{t \rightarrow \infty} \mathbb{E} \|y(t)\|^2 \right)^{\frac{1}{2}} = \lim_{T \rightarrow \infty} \left(\frac{1}{T} \left[\int_0^T \|y(t)\|^2 dt \right] \right)^{\frac{1}{2}}, \quad (3.24)$$

where $\mathbb{E}[\cdot]$ stands for the expected value when the stochastic process (4.7) starts at zero.

Here we wish to establish a relation between the \mathcal{H}_2 -norm of a LTI stochastic system with the cost functional

$$\lim_{T \rightarrow \infty} \mathbb{E} \left[\int_0^T e^{-\alpha t} \|y(t)\|^2 dt \right], \quad (3.25)$$

for some $\alpha > 0$. Note that, if we were working with deterministic systems, we could simply write

$$\begin{aligned} \int_0^T \|y(t)\|^2 dt &= \int_0^T \text{tr} \{ Cx(t)x(t)^\top C^\top \} dt = \\ &= \text{tr} \left\{ C \int_0^T e^{A(T-t)} B B^\top e^{A(T-t)^\top} dt C^\top \right\} = \\ &= \text{tr} \left\{ C \int_0^T e^{-\alpha s} e^{\bar{A}s} B B^\top e^{\bar{A}s^\top} ds C^\top \right\} = \int_0^T e^{-\alpha t} \|\bar{y}(t)\|^2 dt, \end{aligned} \quad (3.26)$$

where $\bar{A} := A + \frac{\alpha}{2}I$ and I is the identity matrix, and $\bar{y}(\cdot)$ is the output of the system (C, \bar{A}, B) . We wish to make this correspondence precise in the case of the stochastic \mathcal{H}_2 -norm. For that, consider now the modified system $\tilde{\mathcal{G}}$ and its \mathcal{H}_2 -norm,

$$\tilde{\mathcal{G}} : \begin{cases} d\tilde{x}(t) = \tilde{A}\tilde{x}(t)dt + \sigma dW(t), \\ \tilde{y}(t) = C\tilde{x}(t), \quad \tilde{x}(0) = 0, \end{cases} \quad (3.27)$$

with $\tilde{A} := A - \frac{\alpha}{2}I$ and I the identity matrix. The desired connection is established by the following lemma.

Lemma 3.3 ((SILVA *et al.*, 2017), Lemma 1). *For system (4.7) the discounted cost functional can be expressed as*

$$\mathbb{E} \left[\int_0^\infty e^{-\alpha t} \|y(t)\|^2 dt \right] = \frac{1}{\alpha} \|\tilde{\mathcal{G}}\|_2^2, \quad (3.28)$$

for $\alpha > 0$, where $\|\tilde{\mathcal{G}}\|_2$ is the \mathcal{H}_2 -norm of the modified system $\tilde{\mathcal{G}}$. Moreover, $\|\tilde{\mathcal{G}}\|_2^2 = \text{tr}\{CPC^\top\}$ where P is the unique symmetric positive semidefinite matrix that is solution of

$$\tilde{A}P + P\tilde{A}^\top + \sigma\sigma^\top = 0. \quad (3.29)$$

Proof: For some $t_0 \leq t$ with $x(t_0) = 0$, and using the shorter representation $M(\bullet)^\top$ for MM^\top , we first evaluate

$$\begin{aligned} \mathbb{E}[\|y(t)\|^2] &= \mathbb{E}[(Cx(t))^\top Cx(t)] = \mathbb{E}[\text{tr}\{Cx(t)(\bullet)^\top\}] = \\ &\quad \text{tr}\left\{\mathbb{E}\left[C \int_{t_0}^t e^{A(t-s)} \sigma dW(s) \int_{t_0}^t dW^\top(s) \sigma^\top e^{A^\top(t-s)} C^\top\right]\right\}, \end{aligned} \quad (3.30)$$

where we apply the integral version of the SDE in (4.7). We now recall a result from Itô's calculus.

Corollary 3.1 ((KUO, 2006), Corollary 4.3.6). *For any $f, g \in L_{ad}^2([a, b] \times \Omega)$, the following equality holds:*

$$E\left(\int_a^b f(t)dW(t) \int_a^b g(t)dW(t)\right) = \int_a^b E(f(t)g(t))dt$$

On equation (3.30) we apply the above result by taking $f(\cdot) = e^{A(t-s)}\sigma$ and $g(\cdot) = f(\cdot)^\top$. In our case $f(\cdot)$ and $g(\cdot)$ are deterministic, and therefore (3.31) holds. After a change of variable, we can then write (3.30) in the form

$$\text{tr}\left\{C\left(\int_{t_0}^t e^{A(t-v)}\sigma\sigma^\top e^{A^\top(t-v)}dv\right)C^\top\right\} = \text{tr}\left\{C\left(\int_0^{t-t_0} e^{A\eta}\sigma\sigma^\top e^{A^\top\eta}d\eta\right)C^\top\right\}. \quad (3.31)$$

Now, recall that the integral over the real line can be seen as a Lebesgue integral on a σ -finite measure space, and the same holds for the expectation operator. Furthermore the integrand $e^{-\alpha t}\|\cdot\|^2$ is nonnegative, thus we can apply Tonelli's theorem (BARTLE, 1966) and change the order of the integral and expectation signs. Thus, with $t_0 = 0$, and after changing the order of the expectation with the integral (3.31) yields

$$\mathbb{E}\left[\int_0^T e^{-\alpha t}\|y(t)\|^2 dt\right] = \mathbb{E}\left[\int_0^T e^{-\alpha t}(Cx(t))^\top(Cx(t))dt\right] = \text{tr}\{CX(T)C^\top\}, \quad (3.32)$$

where

$$X(T) := \int_0^T \int_0^t e^{-\alpha t} e^{A\eta} \sigma \sigma^\top e^{A^\top \eta} d\eta dt.$$

Now, by a change of integration order, one gets

$$X(T) = \frac{1}{\alpha} \int_0^T e^{-\alpha \eta} e^{A\eta} \sigma \sigma^\top e^{A^\top \eta} d\eta - \frac{1}{\alpha} e^{-\alpha T} \int_0^T e^{A\eta} \sigma \sigma^\top e^{A^\top \eta} d\eta. \quad (3.33)$$

Since we assumed matrix A in the original system \mathcal{G} is stable, it follows that

$$\lim_{T \rightarrow \infty} \left\| \int_0^T e^{A\eta} \sigma \sigma^\top e^{A^\top \eta} d\eta \right\| < k_0 < \infty,$$

and the second term in (3.33) vanishes as $T \rightarrow \infty$.

Therefore,

$$\lim_{T \rightarrow \infty} X(T) = \lim_{T \rightarrow \infty} \frac{1}{\alpha} \int_0^T e^{\tilde{A}\eta} \sigma \sigma^\top e^{\tilde{A}^\top \eta} d\eta, \quad (3.34)$$

From (3.32) and (3.34) one has that

$$\mathbb{E} \left[\int_0^\infty e^{-\alpha t} \|y(t)\|^2 dt \right] = \frac{1}{\alpha} \text{tr} \left\{ C \int_0^\infty e^{\tilde{A}\eta} \sigma \sigma^\top e^{\tilde{A}^\top \eta} d\eta C^\top \right\}. \quad (3.35)$$

Now recall system $\tilde{\mathcal{G}}$ (3.27). Its \mathcal{H}_2 -norm, according to equation (4.8), can be calculated as

$$\begin{aligned} \|\tilde{\mathcal{G}}\|_2^2 &= \lim_{t \rightarrow \infty} \mathbb{E} \|\tilde{y}(t)\|^2 = \lim_{t \rightarrow \infty} \mathbb{E} [\text{tr} \{ (C\tilde{x}(t))(\bullet)^\top \}] = \\ &= \lim_{t \rightarrow \infty} \text{tr} \{ C \mathbb{E} [(\tilde{x}(t))(\bullet)^\top] C^\top \} = \lim_{t \rightarrow \infty} \text{tr} \left\{ C \mathbb{E} \left[\left(\int_{t_0}^t e^{\tilde{A}(t-s)} \sigma dW(s) \right) (\bullet)^\top \right] C^\top \right\} = \\ &= \lim_{t \rightarrow \infty} \text{tr} \left\{ C \int_{t_0}^t (e^{\tilde{A}(t-s)} \sigma) (\bullet)^\top ds C^\top \right\} = \lim_{t \rightarrow \infty} \text{tr} \left\{ C \int_0^{t-t_0} e^{\tilde{A}\eta} \sigma \sigma^\top e^{\tilde{A}^\top \eta} d\eta C^\top \right\}, \end{aligned} \quad (3.36)$$

where we applied results from Itô's calculus (KUO, 2006) and a change of variable. Furthermore, with $t_0 = 0$, we get

$$\|\tilde{\mathcal{G}}\|_2^2 = \text{tr} \left\{ C \int_0^\infty e^{\tilde{A}\eta} \sigma \sigma^\top e^{\tilde{A}^\top \eta} d\eta C^\top \right\}. \quad (3.37)$$

Now, if we compare equations (3.35) and (3.37), we get the proposed connection between the discounted cost functional associated to system \mathcal{G} and the \mathcal{H}_2 norm of the (modified) system $\tilde{\mathcal{G}}$. For the second part of Lemma 3.3, we refer once again to the book (DRAGAN *et al.*, 2006). According to Theorem 7.1.4. on page 293, for a stable LTI stochastic system, the (stochastic) \mathcal{H}_2 norm can be characterized in terms of the controllability and observability Gramians. A simplified version of the theorem for the case with pure additive noise and no jumps, in connection with the discussion on the stochastic version of \mathcal{H}_2 norm in (COLANERI *et al.*, 1997) yields the desired result and concludes the proof. \square

Remark 3.1. Lemma 3.3 allows one to evaluate the H_2 -norm $\|\tilde{\mathcal{G}}\|_2$ of system $\tilde{\mathcal{G}}$ by means of the discounted functional (3.25) associated with system \mathcal{G} . Assuming $\tilde{\mathcal{G}}$ is such that \tilde{A} is asymptotically stable, note that there always exists $\alpha > 0$ sufficiently small in such a way that $A = \tilde{A} + \frac{\alpha}{2}I$ sets \mathcal{G} stable, and the identity in the lemma holds.

Lemma 3.3 states a relation between the discounted cost formulation associated to system \mathcal{G} and the (stochastic) \mathcal{H}_2 norm of $\tilde{\mathcal{G}}$, whereas equation (3.24) gives a characterization of the stochastic \mathcal{H}_2 norm of a LTI stochastic system in terms of an average running cost formulation. Connecting these two results, one gets

$$\begin{aligned} \mathbb{E} \left[\int_0^\infty e^{-\alpha t} \|y(t)\|^2 dt \right] &= \frac{1}{\alpha} \|\tilde{\mathcal{G}}\|_2^2 = \frac{1}{\alpha} \lim_{T \rightarrow \infty} \left(\frac{1}{T} \left[\int_0^T \|\tilde{y}(t)\|^2 dt \right] \right), \\ \alpha \mathbb{E} \left[\int_0^\infty e^{-\alpha t} \|y(t)\|^2 dt \right] &= \lim_{T \rightarrow \infty} \left(\frac{1}{T} \left[\int_0^T \|\tilde{y}(t)\|^2 dt \right] \right). \end{aligned} \quad (3.38)$$

Note the last equation resembles the so-called *vanishing discount* approach to ergodic control of diffusion processes. The approach is based on the relation between the convergence of the Cesaro and Abel means of a sequence of real numbers and consists on treating the ergodic control problem as a limiting case of the discounted cost formulation as the discount rate tends to zero (ARAPOSTATHIS *et al.*, 2011). In the previous equation, taking the limit as $\alpha \rightarrow 0$ and with a slight abuse of notation, we get

$$\lim_{\alpha \rightarrow 0} \alpha \mathbb{E} \left[\int_0^\infty e^{-\alpha t} \|y(t)\|^2 dt \right] = \lim_{\alpha \rightarrow 0} \lim_{T \rightarrow \infty} \left(\frac{1}{T} \left[\int_0^T \|\tilde{y}(t)\|^2 dt \right] \right). \quad (3.39)$$

This observation instigated our discussion on a possible solution for the long run average formulation of the CVIU approach, but an immediate drawback is that we did not consider controlled systems in our previous analysis. We then discuss the long run average formulation of the CVIU problem in the following section.

3.3 Long run average cost

The long run average cost functional can be formulated in the *mean average*,

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mathbb{E}[f(x(t), u(t))] dt, \quad (3.40)$$

or in the *pathwise average* version,

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(x(t), u(t)) dt, \quad (3.41)$$

as detailed in (ARAPOSTATHIS *et al.*, 2011; YONG; ZHOU, 1999). Unlike the discounted cost case, where short-term optimization is prioritized, long run average optimization criteria are used when one is interested in a steady state behavior of the system (ARAPOSTATHIS *et al.*, 2011). Under the above cost structure we can formulate the CVIU long run average control problem as

$$\begin{aligned} \min J(x, u, t) &:= \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mathbb{E}[x(t)^\top Q x(t) + u(t)^\top R u(t)] dt, \\ \text{s.t. } dx(t) &= (Ax(t) + Bu(t))dt + \hat{\sigma}(t, x(t), u(t))d\hat{W}(t). \end{aligned} \quad (3.42)$$

There exist two main approaches to solve the stochastic long run average problem. The so-called vanishing discount approach treats the long run average problem as a limiting case of the discounted cost problem as the discount factor α tends to zero (ROBIN, 1983; PRIETO-RUMEAU; HERNÁNDEZ-LERMA, 2010), whereas in the convex analytic approach (BORKAR, 2002; BORKAR, 1988) one makes use of the ergodic property to formulate the average running cost problem as an infinite dimensional convex programming problem. The classical vanishing discount approach starts from a sequence of

solutions of the discounted cost problem, with the value function for each control problem along a monotone sequence $\alpha \downarrow 0$ given by

$$V_\alpha = \inf_{u \in \mathbb{U}} J_\alpha,$$

$$J_\alpha = \lim_{T \rightarrow \infty} \int_0^T \mathbb{E}[e^{-\alpha t} (x(t)^\top Q x(t) + u(t)^\top R u(t))] dt.$$

If we simply took the above functional when the discount rate equals zero, we would get $J_\alpha^*(\cdot) \rightarrow \infty$, since the integrand would not necessarily converge. To overcome this drawback we usually fix a point and evaluate the difference of the cost functionals with relation to this point (ARAPOSTATHIS *et al.*, 2011). If we chose 0, for example, we would get the family of functions

$$\bar{V}_\alpha(x) = V_\alpha(x) - V_\alpha(0),$$

which would allow us to rewrite the HJB equation for the discounted case,

$$-\alpha V_\alpha(x) + \min_{u \in U} \{H(t, x, u, V_\alpha^x, V_\alpha^{xx})\} = 0,$$

as

$$\begin{aligned} -\alpha(V_\alpha(0^+) + V_\alpha(x) - V_\alpha(0^+)) + \min_{u \in U} \{H(t, x, u, V_\alpha^x, V_\alpha^{xx})\} &= 0, \\ -\alpha V_\alpha(0^+) - \alpha \bar{V}_\alpha(x) + \min_{u \in U} \{H(t, x, u, \bar{V}_\alpha^x, \bar{V}_\alpha^{xx})\} &= 0. \end{aligned} \quad (3.43)$$

Provided we can show that the sequence of functions $\bar{V}_\alpha(x)$ is equicontinuous and bounded, we can use the Arzela-Ascoli theorem to show that this sequence converges as $\alpha \downarrow 0$. Moreover, if we are also able to prove that the limit $V_\alpha(0^+) = \lim_{x \rightarrow 0^+} V_\alpha(x)$ exists with $\alpha V_\alpha(0^+)$ bounded (MORIMOTO, 2010), we get the HJB equation for the long run average problem,

$$-\eta + \min_{u \in U} \{H(t, x, u, V^x, V^{xx})\} = 0,$$

as a limiting case of the family of solutions of the discounted cost problem, with $\eta = \lim_{\alpha \downarrow 0} \alpha V_\alpha(0^+)$ and $V(x) = \lim \bar{V}_\alpha(x)$.

From the previous results for the discounted cost problem, we have that the value function of the discounted cost formulation of the CVIU problem is continuous in a nonempty subset around the origin and convex, and therefore the limit $\lim_{x \rightarrow 0^+} V_\alpha(x)$ exists for any $\alpha > 0$. Furthermore, the value function near the origin corresponds to a quadratic function $V_\alpha(x) = x^\top X x + \langle v, x \rangle + \frac{1}{\alpha} \text{tr}(X(\sigma \sigma^\top + \sigma_x \sigma_x^\top + \sigma_u \sigma_u^\top))$, which shows that $\alpha V_\alpha(0^+)$ is bounded for any $\alpha > 0$. The sequence $\alpha V_\alpha(0^+)$ therefore converges and $\eta = \lim_{\alpha \downarrow 0} \alpha V_\alpha(x) = \text{tr}(X(\sigma \sigma^\top + \sigma_x \sigma_x^\top + \sigma_u \sigma_u^\top))$ with X the solution of the modified Lyapunov equation (2.70). On the other hand, a quadratic function is continuous but not uniformly continuous over the real line, and therefore we cannot show that the sequence of functions \bar{V}_α is equicontinuous nor use the vanishing discount approach directly.

We follow then a slightly more conservative analysis, and we first deal with autonomous stochastic systems in order to pave the way for the stochastic control problem.

In a general framework, the noncontrolled diffusion process that we treat in the following is defined in a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t \geq 0}, P)$, and written as

$$\Phi : \begin{cases} dx(t) = Ax(t)dt + \sigma(x(t))dW(t), & x(0) = x_0, \\ y(t) = Cx(t), \end{cases} \quad (3.44)$$

where $A \in \mathbb{R}^n$, $C \in \mathbb{R}^{p \times n}$, and $D \in \mathbb{R}^{p \times r}$. The diffusion coefficient $x \rightarrow \sigma(x) \in \mathbb{R}^{n \times r}$, and W , W^y are multidimensional $\{\mathcal{F}\}_{t \geq 0}$ -adapted Brownian motions of dimension n and r , respectively. As usual, $x(t)$, $y(t)$ and $u(t)$ stand for the state, output path, and control input, respectively. In the CVIU case, we consider here $\sigma_x = 0$, and therefore $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ is such that $\sigma(x) = \bar{\sigma}_x \text{diag}(|x|)$ for $\bar{\sigma}_x \in \mathbb{R}^{n \times n}$. We study these systems in the spirit of perturbed linear systems, or semilinear stochastic differential equations — see Khasminskii's (KHASMINSKII, 2012) and Mao's (MAO, 1991) books. We first explore some structural properties.

3.3.1 Energy measurements and stability

For any $\{\mathcal{F}_t\}$ -adapted process $t \rightarrow z(t)$ defined on $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t \geq 0}, P)$, let us consider the following $L_2(\Omega, \mathcal{F}, P)$ mean energy measurements,

$$\mathcal{E}_2^\alpha(z(\cdot)) = E_{x_0} \left[\int_0^\infty e^{-\alpha t} |z(t)|^2 dt \right], \quad \alpha > 0, \quad (3.45)$$

$$\mathcal{E}_2(z(\cdot)) = \limsup_{T \rightarrow \infty} \frac{1}{T} E_{x_0} \left[\int_0^T |z(t)|^2 dt \right], \quad (3.46)$$

where the first measurement is related to a discounted cost functional, and the second one with the average power of process $z(\cdot)$. The above notions provide corresponding notions of stability.

Definition 3.1. *System Φ is α -stochastic stable if $\mathcal{E}_2^\alpha(x(\cdot)) < \infty$ for any $x(0) = x_0 \in \mathbb{R}^n$.*

Stochastic stability is usually seen in the literature in the sense of asymptotic stability; and the construction of this notion of stability assumes that the noise disappears when the system is in the 0-equilibrium point, that is, one assumes that, for the drift and diffusion matrices, $f(0) = 0$ and $\sigma(0) = 0$, respectively. For more details check, for instance, example 2.6 on page 116 of (MAO, 2007) or section 6.7 of (KHASMINSKII, 2012) for the multiplicative noise case. Here, if the system Φ is α -stochastic stable, we have from definition 3.1 that the above integral is finite and therefore the process $t \rightarrow e^{-\frac{\alpha}{2}t}x(t)$ has to converge to zero in the p -mean sense, with $0 \leq p \leq 2$. This in turn implies that $t \rightarrow e^{-\frac{\alpha}{2}t}x(t)$ is stable in such a p -mean sense, cf (LUKACS, 1975; BILLINGSLEY, 1995). Moreover, since Φ is in the class of perturbed linear systems, or semilinear stochastic differential equations (MAO, 1991), cap.4, pp 109; (KHASMINSKII, 2012), cap. 6.7., it then holds that Φ is almost sure exponentially stable. The second notion is related to finiteness of the output power, and we can apply the following definition.

Definition 3.2. System Φ is stable in the stochastic sense if $\mathcal{E}_2(x(\cdot)) < \infty$ for any $x_0 \in \mathbb{R}^n$.

The above notion is connected with recurrence and finiteness of mean recurrence time to a compact set and the existence of stationary distributions and ergodic behavior, cf. chapters 3 (recurrence) and 4 (ergodicity) of (KHASMINSKII, 2012). Note that the above notions can be found in the literature, and, since the system Φ is linear, then α -stability is equivalent to the stability to the origin of the process

$$dz(t) = (A - \frac{\alpha}{2}I)z(t) + e^{-\frac{\alpha}{2}t}\sigma(z(t))dW(t),$$

using a simple correspondence $z(t) := e^{-\frac{\alpha}{2}t}x(t)$.

An important issue in control theory is how the performance of a system, assessed in terms of measurements of the output energy, correlates with the internal stability of the system. In other words, can we find conditions under which finiteness of the measured energy implies stability of the system? In order to give an affirmative answer to this question, we wish to connect here the finiteness of measurements such as $\mathcal{E}_2^\alpha(y(\cdot))$ or $\mathcal{E}_2(y(\cdot))$ with the corresponding notions of α -stochastic stability or recurrence/ergodicity of system Φ , respectively. In the following, we denote by $\mathcal{E}_2^\alpha(z(\cdot))$ or $\mathcal{E}_2(z(\cdot))$ the evaluations

$$\mathcal{E}_2^\alpha(z(\cdot)) = E_{x_0} \left[\int_0^\infty e^{-\alpha t} \|z(x(\cdot))\|_F^2 dt \right], \quad \alpha > 0,$$

and

$$\mathcal{E}_2(z(\cdot)) = \limsup_{T \rightarrow \infty} \frac{1}{T} E_{x_0} \left[\int_0^\infty \|z(x(\cdot))\|_F^2 dt \right],$$

for the matrix process $t \rightarrow z(x(t))$. Here, $\|z(x(\cdot))\|_F = \text{tr}\{z(x(\cdot))z(x(\cdot))^\top\}^{1/2}$, the Frobenius norm of $z(x(\cdot))$.

3.3.1.1 α -observability

In this section we show that the usual notion of observability for linear, time-invariant systems can connect finiteness of the output energy with stability of the system. In control theory, a weaker notion establishing a similar relation is the deterministic concept of detectability. In this case it would be necessary, however, to require $\mathcal{E}_2^\alpha(\sigma(\cdot))$ to be bounded, an assumption that would exclude the class of semilinear stochastic differential equations introduced by the CVIU model, since the diffusion coefficient σ is an unbounded function of the state in the Markovian diffusion setting. We introduce, therefore, the following notion.

Definition 3.3 (α -observability). We say that (C, A) is α -observable when there exist $t_d \geq 0$ and $\gamma > 0$, such that $\mathcal{E}^{\alpha, t_d}(y(\cdot)) \geq \gamma \|x\|^2$ for each initial condition $x(0) = x \in \mathbb{R}^n$.

Let us consider now the linear operators $\mathcal{Z} : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$, and $\varpi : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^n$,

$$\mathcal{Z}(U) = \text{Diag}(\bar{\sigma}_x^\top U \bar{\sigma}_x), \quad (3.47)$$

$$\varpi(U) = \text{tr}\{U(\sigma\sigma^\top + \sigma_x\sigma_x^\top)\}, \quad (3.48)$$

Furthermore, we define $\mathcal{L}^\alpha : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$ as,

$$\mathcal{L}^\alpha(U) = -\alpha U + A'U + UA + \mathcal{Z}(U), \quad (3.49)$$

and let $L(t)$ and $\ell(t)$, $t \geq 0$ be defined by the following set of matrix differential equations, respectively,

$$\dot{L}(t) := \mathcal{L}^\alpha(L(t)) + C^\top C, \quad L(0) = 0, \quad t \geq 0, \quad (3.50a)$$

$$\dot{\ell}(t) := -\alpha \ell(t) + \varpi(L(t)), \quad \ell(0) = 0, \quad t \geq 0. \quad (3.50b)$$

for $i = 1, \dots, \iota$. Note that each operator \mathcal{L}^α is also linear, and $L(t), \ell(t)$ defined by (3.50) are unique. The following allows us to write the discounted functional in terms of the operators defined above.

Lemma 3.4. *Suppose that for each $U \in \mathbb{S}_0^{n \times n}$ and for each $x \in \mathbb{R}$,*

$$\text{tr}\{\text{Diag}(U\sigma(x)\sigma(x)^\top)\} = x^\top \mathcal{Z}(U)x + \varpi(U) \quad (3.51)$$

holds for operators $\mathcal{Z}(\cdot)$ and $\varpi(\cdot)$. Then for $x_0 = x$ and some $0 < \tau < T$,

$$\mathcal{E}^{\alpha, \tau}(y(\cdot)) := E_x \left[\int_0^\tau e^{-\alpha s} |y(s)|^2 ds \right] = x^\top L(\tau)x + \ell(\tau). \quad (3.52)$$

Proof: Let us assume that $P : [0, \infty) \rightarrow \mathbb{R}^{n \times n}$ and $s : [0, \infty) \rightarrow \mathbb{R}$ satisfy the following differential equations,

$$\begin{aligned} \dot{P}(t) + \mathcal{L}^\alpha(P(t)) + C^\top C &= 0, \\ \dot{p}(t) - \alpha p(t) + \varpi(P(t)) &= 0, \end{aligned} \quad (3.53)$$

for some $P(0) \in \mathcal{S}_0^{n \times n}$ and scalar $p(0)$, and consider the function $V(t, x(t)) := e^{-\alpha t} (x(t)^\top P(t)x(t) + p(t))$. We apply Itô's formula in connection with the stochastic differential equation

$$dx(t) = Ax(t)dt + \sigma(x(t))dW(t), \quad t \geq 0,$$

to obtain

$$\begin{aligned} dV(t, x(t)) &= -\alpha V(t, x(t))dt \\ &+ e^{-\alpha t} (x(t)^\top \dot{P}(t)x(t) + \dot{p}(t))dt + e^{-\alpha t} \langle P(t)x(t), Ax(t)dt + \sigma(x(t))dW(t) \rangle \\ &+ e^{-\alpha t} \langle Ax(t)dt + \sigma(x(t))dW(t), P(t)x(t) \rangle + e^{-\alpha t} \text{tr}\{\text{Diag}(P(t)\sigma(x(t))\sigma(x(t))^\top)\}dt, \end{aligned} \quad (3.54)$$

and, therefore,

$$\begin{aligned}
dV(t, x(t)) &= e^{-\alpha t} \left(x(t)^\top (\dot{P}(t) - \alpha P(t)) x(t) + \dot{p} - \alpha p \right) dt \\
&\quad + e^{-\alpha t} \left(\langle P(t)x(t), Ax(t) \rangle + e^{-\alpha t} \langle Ax(t), P(t)x(t) \rangle \right) dt \\
&\quad + e^{-\alpha t} \left(x(t)^\top \mathcal{Z}(P(t))x(t) + \varpi(P(t)) \right) dt + m(t) = \\
&= e^{-\alpha t} \left(x(t)^\top (\dot{P}(t) + \mathcal{L}^\alpha(P(t)))x(t) + \dot{p}(t) - \alpha p(t) + \varpi(P(t)) \right) dt + m(t) = -e^{-\alpha t} |y(t)|^2,
\end{aligned} \tag{3.55}$$

with

$$m(t) = e^{-\alpha t} \left(\langle \sigma(x(t)) dW(t), P(t)x(t) \rangle + \langle P(t)x(t), \sigma(x(t)) dW(t) \rangle \right), \quad t \leq \tau,$$

a martingale; Then (3.53)–(3.55) yield

$$E \left[\int_0^\tau e^{-\alpha s} |y(s)|^2 ds \right] = V(0, x) - E_x[V(\tau, x_\tau)]. \tag{3.56}$$

Since the solutions of equations (3.53) are unique, we can set $P(\tau) = L(0) = 0, r(\tau) = v(0) = 0$ and $p(\tau) = \ell(0) = 0$, and introduce a time reversed version into equations (3.53) to obtain the differential equations in (3.50). \square

Note that lemma 3.4 implies that $x^\top L(t)x + \ell(t) \geq 0, \forall t \geq 0$, and from the equations in (3.50) it implies that the operators \mathcal{Z} and ϖ are signal preserving in the positive semidefinite sense. In fact, we will require in the sequel that $x^\top L(t)x + \ell(t) > 0, \forall t \geq 0$ and $\forall x \in \mathbb{R}^n$. Moreover, since \mathcal{L}^α is a linear operator, the solution of (3.50a) can be written as

$$L(t) = \int_0^t e^{\mathcal{L}^\alpha(t-\tau)} (C^\top C) d\tau,$$

and

$$\left. \frac{d^k L(t)}{dt^k} \right|_{t=0} = (\mathcal{L}^\alpha)^k (C^\top C) := O(k). \tag{3.57}$$

In connection, let us then introduce the set of observability matrices $\mathcal{O} \in \mathbb{R}^{(n^2) \times n}$ of system Φ , given by

$$\mathcal{O} := [O(0) O(1) \cdots O(n^2 - 1)]^\top, \tag{3.58}$$

where each matrix $O(\cdot) \in \mathbb{R}^{n \times n}$ is defined as in (3.57). The set of observability matrices \mathcal{O} are analogous to the observability matrices of linear deterministic systems and it appears mutatis mutandis in the context studied in (COSTA; VAL, 2002; DRAGAN *et al.*, 2006; ZABCZYK, 2008).

Lemma 3.5. \mathcal{O} is of full rank if and only if $L(t) \in \mathbb{S}_{0+}^{n \times n}$ for all $t \geq 0$.

Proof: Let us deny that $L(t) \in \mathbb{S}_{0+}^{n \times n}$ for $t \geq 0$, and we assume that there exists τ and x , such that $x^\top L(\tau)x = 0$. Note that

$$\begin{aligned} L(t) &= \int_0^t e^{\mathcal{L}^\alpha(t-\tau)}(C^\top C)d\tau = \int_0^t \sum_{m=1}^{n^2} \alpha_m(\tau)(\mathcal{L}^\alpha)^{m-1}(C^\top C)d\tau \\ &= \sum_{m=1}^{n^2} (\mathcal{L}^\alpha)^{m-1}(C^\top C) \int_0^t \alpha_m(\tau)d\tau = \sum_{m=1}^{n^2} \hat{\alpha}_m(t)(\mathcal{L}^\alpha)^{m-1}(C^\top C), \end{aligned} \quad (3.59)$$

where α_m and $\hat{\alpha}_m$ are scalar functions. Then, if for some $\tau \geq 0$, $x^\top L(\tau)x = 0$, it holds that $x^\top (\mathcal{L}^\alpha)^{m-1}(C^\top C)x = 0$ for $m = 1, \dots, n^2$, or, equivalently, $x \in \mathcal{N}(\mathcal{O})$, which implies that $\mathcal{N}(\mathcal{O}) \neq \emptyset$. Conversely, if $L(t) \in \mathbb{S}_{0+}^{n \times n}$ for all $t \geq 0$, equation (3.59) implies that $x^\top (\mathcal{L}^\alpha)^{m-1}(C^\top C)x > 0$ for some $m = 1, \dots, n^2$ and $x \in \mathbb{R}^n$, which is equivalent to state that $\mathcal{N}(\mathcal{O}) = \emptyset$. \square

Theorem 3.1. *Consider system Φ . (C, A) is α -observable for some $\alpha \geq 0$ if and only if \mathcal{O} has full rank. Moreover,*

$$\mathcal{E}^\alpha(x(\cdot)) \leq \varrho \mathcal{E}^\alpha(y(\cdot)),$$

holds for some $\varrho > 0$.

Proof: If \mathcal{O} has full rank, we have from Lemma 3.5 that $L(t) \in \mathbb{S}_{0+}^{n \times n}$. Together with Lemma 3.4, we get,

$$\mathcal{E}^{\alpha, t_d}(y(\cdot)) \geq x^\top L(t_d)x + \ell(t_d) \geq \gamma|x - x_d|^2 \quad \forall x(0) = x \in \mathbb{R}^n,$$

with $x_d = -\frac{1}{2}L(t_d)^{-1}v(t_d)$. According to Definition 3.3, system Φ is then α -observable. Conversely, we apply Lemma 3.5 again to write

$$x^\top L(t_d)x + \ell(t_d) \geq \mathcal{E}^{\alpha, t_d}(y(\cdot)) \geq \gamma|x - x_d|^2 \quad \forall x(0) = x \in \mathbb{R}^n,$$

for some x_d , which implies that $L(t_d)$ is of full rank. Lemma 3.5 completes the necessity part of the first statement. To prove the second part, we need the following lemma.

Lemma 3.6. *Consider system Φ α -observable. Then, for scalars $\delta, \varrho > 0$,*

$$\mathcal{E}^{\alpha, T}(y(\cdot)) \geq \varrho E_x \left[\int_0^{T-\delta} e^{-\alpha s} (|x(s)|^2) ds \right],$$

holds for T sufficiently large and each $\alpha > 0$.

Proof: First consider from Lemma 3.4 that, for any $t > 0$,

$$\mathcal{E}^{\alpha, t}(y(\cdot)) = E_x \left[\int_0^t e^{-\alpha s} |y(s)|^2 ds \right] = x^\top L(t)x + \ell(t).$$

Set now $i^+ = \max\{i : it_d \leq T\}$ and write for $0 \leq \tau \leq t_d$,

$$\begin{aligned}
E_x \left[\int_0^T e^{-\alpha s} |y(s)|^2 ds \right] &\geq E_x \left[\sum_{i=0}^{i^+-2} \int_{\tau+it_d}^{\tau+(i+1)t_d} e^{-\alpha s} |y(s)|^2 ds \right] \\
&= E_x \left[\sum_{i=0}^{i^+-2} E \left[\int_{\tau+it_d}^{\tau+(i+1)t_d} e^{-\alpha s} |y(s)|^2 ds \middle| \mathcal{F}_{\tau+it_d} \right] \right] \\
&\geq \gamma E_x \left[\sum_{i=0}^{i^+-2} e^{-\alpha(\tau+it_d)} |L(t_d)^{\frac{1}{2}}(x_{\tau+it_d})|^2 \right], \quad (3.60)
\end{aligned}$$

where, for the last inequality, we apply the identity in (3.52) and the fact that $\ell(t_d) \geq 0$ and thus,

$$|L(t_d)^{\frac{1}{2}}x|^2 + \ell(t_d) \geq \gamma|x|^2.$$

By integrating both sides on the interval $[0, t_d)$ the above result yields,

$$\begin{aligned}
t_d E_x \left[\int_0^T e^{-\alpha s} |y(s)|^2 ds \right] &\geq \gamma E_x \left[\sum_{i=0}^{i^+-2} \int_0^{t_d} e^{-\alpha(\tau+it_d)} |(x_{\tau+it_d})|^2 d\tau \right] \\
&= \gamma E_x \left[\int_0^{(i^+-1)t_d} e^{-\alpha s} |(x(s))|^2 ds \right] \\
&\geq \gamma E_x \left[\int_0^{T-2t_d} e^{-\alpha s} (|x(s)|^2) ds \right]. \quad (3.61)
\end{aligned}$$

Set $\varrho = \gamma/t_d$, $\delta = 2t_d$ to complete the proof. \square

For $\alpha > 0$, the proof of Theorem 3.1 is completed with Lemma 3.6, by simply setting $T \rightarrow \infty$ in (3.61). When $\alpha = 0$, the evaluations in (3.61) in Lemma 3.6 can be repeated to get

$$t_d E_x \left[\int_0^T |y(s)|^2 ds \right] \geq \gamma E_x \left[\int_0^{T-2t_d} |x(s)|^2 ds \right],$$

and thus,

$$\limsup_{T \rightarrow \infty} \frac{1}{T} E_x \left[\int_0^T |y(s)|^2 ds \right] \geq \varrho \limsup_{T \rightarrow \infty} \frac{1}{T} E_x \left[\int_0^T |x(s)|^2 ds \right].$$

Therefore, if $\mathcal{E}^0(y(\cdot)) < \infty$, then $\mathcal{E}^0(x(\cdot)) < \infty$. \square

Note that if $\mathcal{E}^\alpha(y(\cdot)) < \infty$, for $\alpha > 0$, Φ is α -stochastically stable, that is, $\mathcal{E}^\alpha(x(\cdot)) < \infty$.

Corollary 3.2. *The following statements are equivalent. For any $\alpha \geq 0$,*

- (i) *System ϕ is α -observable.*
- (ii) *$\mathcal{O} = [(\mathcal{L}^\alpha)^k (C^\top C), k = 0, \dots, n^2 - 1]^\top$ has full rank.*
- (iii) *If $\mathcal{E}^\alpha(y(\cdot)) < \infty$, then $\mathcal{E}^\alpha(x(\cdot)) < \infty$ for any $x_0 = x \in \mathbb{R}^n$.*

Hence, α -observability connects finiteness of energy measurement of type $\mathcal{E}^\alpha(y(\cdot))$ for some α with α -stochastic stability of system Φ , and in the sequel, we suppose that system Φ is α -observable for some $\alpha \geq 0$.

3.3.2 Stochastic Stability: the discounted case

In the following we use the notions defined above to study the stability of stochastic systems. Consider then the diffusion process,

$$dx(t) = Ax(t)dt + \sigma(x(t))dW(t), \quad x_0 = x,$$

with $\sigma(\cdot)$ a Lipschitz function. Consider the output signal with $y(t) = Cx(t)$, $t \geq 0$, and the energy measurement $\mathcal{E}^\alpha(y(\cdot))$, here for $\alpha > 0$. Assume that the pair (C, A) is α -observable. Recall now the linear operators $\mathcal{Z} : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$, $\varpi : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$, and recall also $\mathcal{L}^\alpha : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$,

$$\mathcal{L}^\alpha(U) = -\alpha U + A'U + UA + \mathcal{Z}(U). \quad (3.62)$$

Here we consider $L \in S_0^n$ the algebraic solution of (3.63a), if it exists, together with equation (3.63b) in $\ell \geq 0$,

$$\mathcal{L}^\alpha(U) + C^\top C = 0, \quad (3.63a)$$

$$\ell = \alpha^{-1} \varpi(U), \quad (3.63b)$$

respectively.

Proposition 3.1 (General). *Suppose that $L^\alpha \in S_0^n$ is the solution of (3.63a). If (C, A) is α -observable, then Φ is α -stochastically stable. Moreover,*

$$\mathcal{E}^\alpha(y(\cdot)) = x_0^\top L^\alpha x_0 + \ell^\alpha, \quad (3.64)$$

where $\ell^\alpha \in \mathbb{R}$ satisfies (3.63b).

Proof: For $W^\alpha(x) := x^\top L^\alpha x + \ell^\alpha$, Itô's formula yields,

$$\begin{aligned} & E \left[\int_0^T e^{-\alpha t} dW^\alpha(t, x(t)) \right] \\ &= E \left[\int_0^T e^{-\alpha t} \left(x(t)^\top \mathcal{L}^\alpha(L^\alpha) x(t) - \alpha \ell^\alpha + \varpi(L^\alpha) \right) dt \right] \\ &= -E \left[\int_0^T e^{-\alpha t} x_t^\top C^\top C x_t dt \right] \end{aligned} \quad (3.65)$$

Hence,

$$\mathcal{E}^{\alpha, T}(y(\cdot)) = E \left[\int_0^T e^{-\alpha t} \|y(t)\|^2 dt \right] = E[W^\alpha(x_0) - e^{-\alpha T} W^\alpha(x_T)]. \quad (3.66)$$

Now, (C, A) α -observability yields that $\mathcal{E}^\alpha(x(\cdot)) < \infty$ provided that $\mathcal{E}^\alpha(y(\cdot)) < \infty$, hence it is easy to verify that $E[e^{-\alpha T} W^\alpha(x_T)] \rightarrow 0$ as $T \rightarrow \infty$, which completes the result. \square

3.3.3 The average case

Lemma 3.7 (Sequence of vanishing Lyapunovs). *Suppose that (C, A) is 0-observable. Let $\alpha_k \downarrow 0$ be a monotone sequence and $L^{\alpha_k}, k \geq 0$ be the corresponding sequence of solutions of (3.63a). It is an increasing sequence in the positive semidefinite sense; moreover, if there exists L , the unique semidefinite positive solution of*

$$\mathcal{L}^0(U) + C^\top C = A^\top U + UA + \mathcal{Z}(U) + C^\top C = 0, \quad (3.67)$$

then $L^{\alpha_k} \uparrow L$.

Proof: Here we use the notation $U \geq V, (U \leq V)$, for $U, V \in S^{n+}$ when $U - V \in S^{n+}, (V - U \in S^{n+})$. Consider the operators $\mathcal{Z} : S^n \rightarrow S^n$ defined in (3.47) and $\mathcal{L}^\alpha : S^n \rightarrow S^n$ in (3.62). Note that $\mathcal{Z}(U) = \text{Diag}(\bar{\sigma}_x^\top U \bar{\sigma}_x)$, for some matrix $U \in \mathbb{R}^{n \times n}$, hence, the operator \mathcal{Z} is monotone, in the sense that, for $U, V \in S^{n+}$ with $U \geq V$, $\mathcal{Z}(U) \geq \mathcal{Z}(V)$.

Let us also consider the operator $\tilde{\mathcal{L}}^\alpha : S^n \times S^n \rightarrow S^n$ such that $\tilde{\mathcal{L}}^\alpha(U, V) = A^\top U + UA - \alpha U + \mathcal{Z}(V)$. Since (C, A) is 0-observable, the solutions of $\mathcal{L}^\alpha(U) + C^\top C = 0$ and $\tilde{\mathcal{L}}^\alpha(U, V) + C^\top C = 0$ for some $V \in S^{n+}$ are unique matrices in S^{n+} for each $\alpha \geq 0$. Now, denote by \tilde{L} and L the following solutions:

$$-C^\top C = \tilde{\mathcal{L}}^\alpha(\tilde{L}, V) = \mathcal{L}^\alpha(L),$$

and from the monotonicity of \mathcal{Z} it follows that if $L \geq V$, then

$$\begin{aligned} & \mathcal{L}^\alpha(L) - \tilde{\mathcal{L}}^\alpha(\tilde{L}, V) \\ &= \tilde{\mathcal{L}}^\alpha(L, L) - \tilde{\mathcal{L}}^\alpha(\tilde{L}, V) = \tilde{\mathcal{L}}^\alpha(L - \tilde{L}, L - V) = \\ &= (A - \frac{\alpha}{2}I)^\top (L - \tilde{L}) + (L - \tilde{L})(A - \frac{\alpha}{2}I) + \mathcal{Z}(L - V) = 0, \end{aligned}$$

which implies that $L \geq \tilde{L}$.

Analogously, consider the operator $\bar{\mathcal{L}}^\alpha : S^n \times S^n \rightarrow S^n$ such that $\bar{\mathcal{L}}^\alpha(U, V) = A^\top U + UA + \mathcal{Z}(U) - \alpha V$. Then, the solutions \bar{L} and L in

$$-C^\top C = \bar{\mathcal{L}}^\alpha(\bar{L}, V) = \mathcal{L}^\alpha(L),$$

are such that if $L \geq V$, we get that $\bar{\mathcal{L}}^\alpha(\bar{L}, V) - \mathcal{L}^\alpha(L) = \mathcal{L}^0(\bar{L} - L) + \alpha(L - V) = 0$, which in turn implies $\bar{L} \geq L$. Finally, consider the solutions

$$-C^\top C = \mathcal{L}^\alpha(L_1) = \mathcal{L}^\beta(L_2), \quad \alpha \geq \beta \geq 0.$$

First note that the solutions \bar{L}_1 and \bar{L}_2 of

$$-C^\top C = \bar{\mathcal{L}}^\alpha(\bar{L}_1, V) = \bar{\mathcal{L}}^\beta(\bar{L}_2, V),$$

are such that if $\bar{L}_1, \bar{L}_2 \geq V$, we get that $\bar{\mathcal{L}}^\beta(\bar{L}_2, V) - \bar{\mathcal{L}}^\alpha(\bar{L}_1, V) = \mathcal{L}^0(\bar{L}_2 - \bar{L}_1) + (\alpha - \beta)V = 0$, which implies that $\bar{L}_2 \geq \bar{L}_1$. Now, set $V = L_1$ to conclude that

$$-C^\top C = \mathcal{L}^\alpha(L_1) = \bar{\mathcal{L}}^\alpha(L_1, L_1) = \bar{\mathcal{L}}^\beta(\bar{L}_2, L_1) = \mathcal{L}^\beta(L_2),$$

with $L_2 \geq \bar{L}_2 \geq L_1$. Finally, set $L_0 = 0$ and create the increasing sequence

$$-C^\top C = \tilde{\mathcal{L}}^{\alpha_1}(\tilde{L}_1, 0) = \mathcal{L}^{\alpha_1}(L_1) = \bar{\mathcal{L}}^{\alpha_2}(\bar{L}_2, L_1) = \mathcal{L}^{\alpha_2}(L_2),$$

in the sense that

$$0 \leq \tilde{L}_1 \leq L_1 \leq \bar{L}_1 \leq L_2 \cdots,$$

in such a way that L_k , the solution of $\mathcal{L}^{\alpha_k}(L_k) + C^\top C = 0, L_k \uparrow L_\infty$ in the semidefinite positive sense and L_∞ satisfies (3.67). From uniqueness, $L_\infty = L \in S^{n+}$, and that completes the proof. \square

Theorem 3.2 (Average case). *Suppose that (3.67) has a solution $L \in S^{n+}$. Then,*

$$\lim_{T \rightarrow \infty} \frac{1}{T} E \left[\int_0^T \|y(t)\|^2 dt \right] = \varpi(L). \quad (3.68)$$

Proof: Consider (3.64) in Proposition 3.1, and let $\{\alpha_k\}_{k \geq 0}$ be any sequence such that $\alpha_k > 0$ and $\alpha_k \downarrow 0$. Then,

$$\begin{aligned} \alpha_k \mathcal{E}^{\alpha_k}(y(\cdot)) &= \alpha_k E \left[\int_0^\infty e^{-\alpha_k t} \|y(t)\|^2 dt \right] \\ &= \alpha_k (x_0^\top L^{\alpha_k} x_0 + \ell^{\alpha_k}) \\ &= \alpha_k (x_0^\top L^{\alpha_k} x_0) + \varpi(L^{\alpha_k}) \rightarrow \varpi(L), \text{ as } \alpha_k \rightarrow 0, \end{aligned} \quad (3.69)$$

where, according to Lemma 3.7, L^{α_k} converges, in the semidefinite positive sense, to the solution of (3.67). Thus, we have shown that

$$\varpi(L) = \limsup_{\alpha \rightarrow 0} \alpha E \left[\int_0^\infty e^{-\alpha t} \|y(t)\|^2 dt \right]. \quad (3.70)$$

On the other hand, consider $V^\alpha(x) = x^\top L^\alpha x$ with L^α the solution of (3.63a). Similar to (3.71), Itô's formula applied here yields,

$$\begin{aligned} E \left[\int_0^T e^{-\alpha t} dV^\alpha(t, x(t)) \right] &= E \left[\int_0^T e^{-\alpha t} \left(x(t)^\top \mathcal{L}^\alpha(L^\alpha) x(t) + \varpi(L^\alpha) \right) dt \right] \\ &= E \left[\int_0^T e^{-\alpha t} (-x_t^\top C^\top C x_t + \varpi(L^\alpha)) dt \right]. \end{aligned} \quad (3.71)$$

Hence,

$$E \left[\int_0^T e^{-\alpha t} \|y(t)\|^2 dt \right] = \int_0^T e^{-\alpha t} \varpi(L^\alpha) dt + E[V^\alpha(x_0) - e^{-\alpha T} V^\alpha(x_T)] \quad (3.72)$$

Now set $\alpha = \alpha_k \rightarrow 0$. From the Dominated Convergence theorem and the fact that ϖ does not depend explicitly on T , we get

$$\begin{aligned} E \left[\int_0^T \|y(t)\|^2 dt \right] &= \lim_{\alpha \downarrow 0} \frac{1}{\alpha} (1 - e^{-\alpha T}) \varpi(L^\alpha) + E[V^\alpha(x_0) - e^{-\alpha T} V^\alpha(x_T)] \\ &= T\varpi(L) + E[V^0(x_0) - V^0(x_T)], \end{aligned}$$

with $V^0(x) = x^\top Lx$, and L the solution of $\mathcal{L}^0(U) + C^\top C = 0$. Therefore,

$$\limsup_{T \rightarrow \infty} \frac{1}{T} E \left[\int_0^T \|y(t)\|^2 dt \right] = \varpi(L) + \limsup_{T \rightarrow \infty} \frac{1}{T} E[V^0(x_0) - V^0(x_T)]. \quad (3.73)$$

Now, $V(x_0)$ is finite, and therefore $\limsup_{T \rightarrow \infty} \frac{1}{T} V(x_0) = 0$. On the other hand, if the pair (C, A) is 0-observable and equation (3.67) has a solution $L \in S^{n+}$, then

$$\limsup_{T \rightarrow \infty} \frac{1}{T} V(x_T) = 0$$

as long as $\mathcal{E}^0(y(\cdot))$ is finite. Together, (3.70) and (3.73) show the convergence holds as long as $\mathcal{E}^0(y(\cdot))$ is finite, which implies that $\mathcal{E}^0(x(\cdot)) < \infty$, and in particular that

$$\frac{\mathbb{E}[|x(t)|^2]}{t} \rightarrow 0$$

as $t \rightarrow \infty$. □

3.3.4 The controlled case

Consider now the controlled diffusion process

$$\begin{aligned} dx(t) &= (Ax(t) + Bu(t))dt + \hat{\sigma}(x(t), u(t))d\hat{W}(t), \\ y(t) &= Cx(t), \end{aligned}$$

where, as usual,

$$\begin{aligned} \hat{\sigma}(x, u) &= [\sigma \quad \sigma_x + \bar{\sigma}_x \text{diag}(|x|) \quad \sigma_u + \bar{\sigma}_u \text{diag}(|u|)], \\ \hat{W}(t) &= [W(t)^\top \quad W^x(t)^\top \quad W^u(t)^\top]^\top. \end{aligned}$$

In the following, the above coefficients are assumed to be time-invariant. Moreover, we take $\sigma_x = 0$. Consider as well the corresponding long run average stochastic optimal control problem

$$\begin{aligned} \min J(x, u, t) &:= \limsup_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T x(t)^\top Qx(t) + u(t)^\top Ru(t) dt \right], \\ \text{s.t. } dx(t) &= (Ax(t) + Bu(t))dt + \hat{\sigma}(t, x(t), u(t))d\hat{W}(t), \end{aligned} \quad (3.74)$$

and a state feedback control policy $u(t) := Kx(t)$ such that the closed-loop matrix $A_{cl} = A - BK$ is stable. Under an optimality point of view, the corresponding function V^0

can be seen as a suboptimal, upper bound of the CVIU analysis. This shows that the, under the previous evaluation for the autonomous case and under the assumption that the corresponding Lyapunov equation for the closed-loop case has a unique solution, then the corresponding cost function for the long run average case does not explode. This in turn allows us to write the HJB equation for the long run average control problem as

$$-\eta + \inf_{u \in U} \{H(t, u, x, p, P)\} = 0, \quad (3.75)$$

where $H(\cdot)$ is the Hamiltonian function associated with the control problem (3.74), or, equivalently, as

$$\min_{u \in U} \mathcal{L}^u V(x) - \eta + f(x, u) = 0, \quad (3.76)$$

where $\mathcal{L}^u = \sum_i b^i(x, u) \frac{\partial g}{\partial x^i}(x) + \frac{1}{2} \sum_{i,j} \sigma^i(x) \sigma^j(x, u) \frac{\partial^2 g}{\partial x^i \partial x^j}(x)$ is the Lyapunov operator associated to the diffusion process, and $f(x, u)$ the running cost function. Let us then get the corresponding HJB equation for the CVIU control problem,

$$x^\top Q x + u^\top R u + \langle p, Ax + Bu \rangle + \frac{1}{2} (\Gamma_0(P) + \Gamma_1(x, u, P) + \Gamma_2(x, u, P)) - \eta = 0, \quad (3.77)$$

where, retrieving the notation for the controlled case and considering time-invariant coefficients, we have

$$\Gamma_0(P) = \text{tr} (P(\sigma \sigma^\top + \sigma_x \sigma_x^\top + \sigma_u \sigma_u^\top)), \quad (3.78a)$$

$$\begin{aligned} \Gamma_1(x, u, P) &= \text{tr} ((\bar{\sigma}_x^\top P \sigma_x + \sigma_x^\top P \bar{\sigma}_x) \text{diag}(|x(t)|)) + \text{tr} ((\bar{\sigma}_u^\top P \sigma_u + \sigma_u^\top P \bar{\sigma}_u) \text{diag}(|u(t)|)) \\ &= \mathcal{S}(x)^\top \text{Diag} (\bar{\sigma}_x^\top P \sigma_x + \sigma_x^\top P \bar{\sigma}_x) x + \mathcal{S}(u)^\top \text{Diag} (\bar{\sigma}_u^\top P \sigma_u + \sigma_u^\top P \bar{\sigma}_u) u, \\ &= \mathcal{S}(x)^\top \Delta_x(P) x + \mathcal{S}(u)^\top \Delta_u(P) u \end{aligned} \quad (3.78b)$$

$$\begin{aligned} \Gamma_2(x, u, P) &= \text{tr} (\bar{\sigma}_x^\top P \bar{\sigma}_x \text{diag}(|x(t)|)^2) + \text{tr} (\bar{\sigma}_u^\top P \bar{\sigma}_u \text{diag}(|u(t)|)^2) \\ &= x^\top \text{Diag} (\bar{\sigma}_x^\top P \bar{\sigma}_x) x + u^\top \text{Diag} (\bar{\sigma}_u^\top P \bar{\sigma}_u) u. \end{aligned} \quad (3.78c)$$

Since $\sigma_x = 0$, the HJB equation can be rewritten as

$$\begin{aligned} x^\top \left(Q + \frac{1}{2} \text{Diag}(\bar{\sigma}_x^\top P \bar{\sigma}_x) \right) x + u^\top \left(R + \frac{1}{2} \text{Diag}(\bar{\sigma}_u^\top P \bar{\sigma}_u) \right) u \\ + \langle p, Ax + Bu \rangle + \frac{1}{2} \Gamma_0(P) + \frac{1}{2} \text{tr} ((\bar{\sigma}_u^\top P \sigma_u + \sigma_u^\top P \bar{\sigma}_u) \text{diag}(|u(\cdot)|)) - \eta = 0, \end{aligned} \quad (3.79)$$

In the following, we use the structure of the CVIU approach to solve the problem in steps: we first consider the inaction region, where we know the optimal value of the control function equals zero, and then consider the solution of the problem for asymptotic regions with homogeneous signs.

3.3.4.1 Solution inside the inaction region

Lemma 3.8. *Consider the stochastic control problem (3.74). If there exists $X \in S_0^{n+}$ solution of the modified Lyapunov matrix equation*

$$A^\top X + X A + Q + \text{Diag}(\bar{\sigma}_x^\top X \bar{\sigma}_x) = 0, \quad (3.80)$$

then the value function for each $x \in \mathcal{R}^0$ is

$$\eta = \text{tr} (X (\sigma \sigma^\top + \sigma_u \sigma_u^\top)). \quad (3.81)$$

Moreover, the inaction region \mathcal{R}^0 is defined by parallel hyperplanes

$$\mathcal{R}_i^0 = \{x \in \mathbb{R}^n : -\delta_i(X) \leq B_i^\top X x \leq +\delta_i(X)\}, \quad (3.82)$$

with $\delta_i(X) := (\bar{\sigma}_u^\top X \sigma_u + \sigma_u^\top X \bar{\sigma}_u)$.

Proof: In the long run average case, the HJB equation is given by

$$-\eta + \min_{u \in U} \{H(t, x, u, p, P)\} = 0, \quad (3.83)$$

which, in the CVIU case, can be written as

$$\begin{aligned} x^\top \left(Q + \frac{1}{2} \text{Diag}(\bar{\sigma}_x^\top P \bar{\sigma}_x) \right) x + u^\top \left(R + \frac{1}{2} \text{Diag}(\bar{\sigma}_u^\top P \bar{\sigma}_u) \right) u + \\ + \langle p, Ax + Bu \rangle + \frac{1}{2} \Gamma_0(\cdot) + \frac{1}{2} \text{tr} ((\bar{\sigma}_u^\top P \sigma_u + \sigma_u^\top P \bar{\sigma}_u) \text{diag}(|u(\cdot)|)) - \eta = 0. \end{aligned} \quad (3.84)$$

Since we know that inside the inaction region the optimal value of the control policy is given by $u^* = 0$, we get the corresponding equation,

$$x^\top \left(Q + \frac{1}{2} \text{Diag}(\bar{\sigma}_x^\top P \bar{\sigma}_x) \right) x + \langle p, Ax \rangle + \frac{1}{2} \Gamma_0(\cdot) - \eta = 0. \quad (3.85)$$

Now suppose that, since we are working with a quadratic cost structure, the auxiliary function $V(\cdot)$ inside the inaction region has a quadratic form

$$V(x) = x^\top X x + l,$$

with $V_x(x) = 2Xx$ and $V_{xx}(x) = 2X$. We plug in the above expressions into the HJB equation to obtain

$$\begin{aligned} x^\top \left(Q + \frac{1}{2} \text{Diag}(\bar{\sigma}_x^\top 2X \bar{\sigma}_x) \right) x + \langle xX, Ax \rangle + \langle Ax, Xx \rangle + \frac{1}{2} \Gamma_0(P) - \eta = 0, \\ x^\top (Q + \text{Diag}(\bar{\sigma}_x^\top X \bar{\sigma}_x)) x + \langle xX, Ax \rangle + \langle Ax, Xx \rangle + \frac{1}{2} \Gamma_0(P) - \eta = 0, \\ x^\top (A^\top X + XA + Q + \text{Diag}(\bar{\sigma}_x^\top X \bar{\sigma}_x)) x + \frac{1}{2} \Gamma_0(P) - \eta = 0. \end{aligned}$$

From the last expression we get the modified Lyapunov equation

$$A^\top X + XA + Q + \text{Diag}(\bar{\sigma}_x^\top X \bar{\sigma}_x) = 0,$$

and, if the above equation has a symmetric positive solution, then that solution is unique. Moreover, we get that

$$\eta = \frac{1}{2} \Gamma_0(P) = \frac{1}{2} \text{tr} (2X (\sigma \sigma^\top + \sigma_u \sigma_u^\top)) = \text{tr} (X (\sigma \sigma^\top + \sigma_u \sigma_u^\top))$$

is the optimal value of the value function inside the inaction region. Note that the Lyapunov equation has the same form as the limit of the previously defined sequence of Lyapunov equation as $\alpha_k \downarrow 0$ along a monotone sequence, as does the optimal value $J(u^*) = \Gamma_0(X) = \text{tr} (X (\sigma \sigma^\top + \sigma_u \sigma_u^\top))$. \square

3.3.4.2 Asymptotic solution

Lemma 3.9. *Consider the stochastic control problem (3.74). If there exists $X \in S_0^{n+}$ solution of the modified Riccati equation*

$$A^\top X + XA + Q'(X) - XBR'(X)^{-1}B^\top X = 0, \quad (3.86)$$

where $Q'(X) := Q + \text{Diag}(\bar{\sigma}_x^\top X \bar{\sigma}_x)$ and $R'(X) := R + \text{Diag}(\bar{\sigma}_u^\top X \bar{\sigma}_u)$, then the optimal control policy u^* tends asymptotically to

$$u^*(x) = -K_\infty x + M_\infty, \quad (3.87)$$

with

$$\begin{aligned} K_\infty &= R'(X)^{-1}B^\top X, \\ M_\infty &= -\frac{1}{2}R'(X)^{-1}(B^\top v + \Delta_u(X)\mathcal{S}_u). \end{aligned}$$

Proof: We now study the HJB equation for asymptotic regions with homogeneous signs, and consider a quadratic test function $V(x) = x^\top Xx + v^\top x + l$. Once again, we take

$$Q'(X) := Q + \text{Diag}(\bar{\sigma}_x^\top X \bar{\sigma}_x), \quad R'(X) := R + \text{Diag}(\bar{\sigma}_u^\top X \bar{\sigma}_u)$$

to write the HJB equation in a more compact form,

$$\begin{aligned} -\eta + \min_{u \in U} \{ & x^\top Q'(X)x + u^\top R'(X)u \\ & + \langle Ax + Bu, Xx \rangle + \langle Xx, Ax + Bu \rangle + \langle v, Ax + Bu \rangle \\ & + \langle \mathcal{S}(u), \Delta_u(X)u \rangle + \frac{1}{2}\Gamma_0(P) \} = 0. \end{aligned} \quad (3.88)$$

Now isolating the terms dependent on u and expanding the dot products, we get

$$\begin{aligned} -\eta + x^\top (A^\top X + XA + Q'(X))x + v^\top Ax + \frac{1}{2}\Gamma_0(P) \\ + \min_{u \in U} \{ u^\top R'(X)u + u^\top B^\top Xx + x^\top XBu + v^\top Bu + \mathcal{S}^\top(u)\Delta_u(X)u \} = 0. \end{aligned} \quad (3.89)$$

Notice that the expression within the braces has a quadratic form, and we can use a matrix completion of squares technique to change it from the standard quadratic form,

$$a + b^\top u + \frac{1}{2}x^\top Cx,$$

to a more convenient expression,

$$\frac{1}{2}(u - \zeta)^\top Z(u - \zeta) + \kappa.$$

In our case this leads to

$$\left(u + R'(X)^{-1} \left(B^\top Xx + \frac{1}{2}g \right) \right)^\top R'(X)(\bullet) - \frac{1}{2} \left(B^\top Xx + \frac{1}{2}g \right) R'(X)^{-1}(\bullet), \quad (3.90)$$

with $g = B^\top v + \Delta_u(X)\mathcal{S}_u$. From the last expression we get that the minimum is achieved for

$$\begin{aligned} u^\top &= -R'(X)^{-1} \left(B^\top Xx + \frac{1}{2}g \right) \\ &= -R'(X)^{-1} B^\top Xx - \frac{1}{2}R'(X)^{-1} (B^\top v + \Delta_u(X)\mathcal{S}_u). \end{aligned} \quad (3.91)$$

We now plug in this value of u back into the HJB equation to get

$$\begin{aligned} -\eta + x^\top (A^\top X + XA + Q'(X))x + v^\top Ax + \frac{1}{2}\Gamma_0 \\ - \left(B^\top Xx + \frac{1}{2}g \right) R'(X)^{-1}(\bullet) = 0, \\ -\eta + x^\top \left(A^\top X + XA + Q'(X) - XBR'(X)^{-1}B^\top X \right)x \\ + \left(v^\top A - g^\top R'(X)^{-1}B^\top X \right)x \\ + \frac{1}{2}\Gamma_0 - \frac{1}{4}g^\top R'(X)^{-1}g = 0. \end{aligned} \quad (3.92)$$

From the last expression we then get the modified Riccati equation,

$$A^\top X + XA + Q'(X) - XBR'(X)^{-1}B^\top X = 0,$$

and, furthermore,

$$v^\top (A - BR'(X)^{-1}B^\top X) = \mathcal{S}^\top(u)\Delta_u(X)R'(X)^{-1}B^\top X, \quad (3.93a)$$

$$\eta = \frac{1}{2}\Gamma_0 - \frac{1}{4}g^\top R'(X)^{-1}g = \frac{1}{2}\Gamma_0 - \frac{1}{4}(B^\top v + \Delta_u(X)\mathcal{S}(u))^\top R'(X)^{-1}(\bullet). \quad (3.93b)$$

□

Remark 3.2. We can also consider a formulation of the long run average problem in which the objective is to minimize the stochastic \mathcal{H}_2 norm of the measured output of the stochastic system. The control problem in this case can be written as

$$\begin{aligned} \min J(x, u, t) &:= \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \|y(t)\|^2 dt \right], \\ \text{s.t.} \quad \begin{cases} dx(t) = (Ax(t) + Bu(t))dt + \hat{\sigma}(x(t), u(t))d\hat{W}(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \end{aligned} \quad (3.94)$$

where $\|y(t)\| = \text{tr}\{y(t)^\top y(t)\}^{\frac{1}{2}}$. The solution procedure follows along the lines of section 3.1 and subsection 3.3.4, and we get the following results.

Lemma 3.10. *Consider the stochastic control problem (3.94). If there exists $X \in S_0^{n+}$ solution of the modified Lyapunov matrix equation*

$$A^\top X + XA + C^\top C + \text{Diag}(\bar{\sigma}_x^\top X \bar{\sigma}_x) = 0, \quad (3.95)$$

then the value function for each $x \in \mathcal{R}^0$ is

$$\eta = \text{tr} (X (\sigma \sigma^\top + \sigma_u \sigma_u^\top)). \quad (3.96)$$

Moreover, the inaction region \mathcal{R}^0 is defined by parallel hyperplanes

$$\mathcal{R}_i^0 = \{x \in \mathbb{R}^n : -\delta_i(X) \leq +\delta_i(X)\}, \quad (3.97)$$

with $\delta_i(X) := (\bar{\sigma}_u^\top X \sigma_u + \sigma_u^\top X \bar{\sigma}_u)$.

For asymptotic regions with homogeneous control signs, we have the following lemma.

Lemma 3.11. *Consider the stochastic control problem (3.94). If there exists $X \in S_0^{n+}$ solution of the modified Riccati equation*

$$A^\top X + XA - (C^\top D + XB) (R'(X))^{-1} (B^\top X + D^\top C) + Q'(X) = 0, \quad (3.98)$$

where $Q'(X) := Q + \text{Diag}(\bar{\sigma}_x^\top X \bar{\sigma}_x)$ and $R'(X) := R + \text{Diag}(\bar{\sigma}_u^\top X \bar{\sigma}_u)$, then the optimal control policy u^* tends asymptotically to

$$u^*(x) = -K_\infty x + M_\infty, \quad (3.99)$$

with

$$\begin{aligned} K_\infty &= R'(X)^{-1} (B^\top X + D^\top C), \\ M_\infty &= -\frac{1}{2} R'(X)^{-1} (B^\top v + \Delta_u(X) \mathcal{S}_u). \end{aligned}$$

The design of an optimal CVIU control policy here follows a procedure similar to the one described in section 2.2.3.1.

4 Robust control of stochastic systems

4.1 Introduction

The¹ basic idea behind the design of a robust controller is to guarantee stability and satisfactory performance against a set of possible disturbances affecting the operation of the system (ZHOU; DOYLE, 1998). As we mentioned in the introduction, one of the ways used to describe these disturbances is to consider the existence of parametric uncertainties in the model being studied. When working with deterministic models, this can be done by using a polytopic representation for the system matrices, for example, which allows us to cast the robust control problem as an optimization problem formulated via Linear Matrix Inequalities (LMIs) in a straightforward manner (BOYD *et al.*, 1994). In a stochastic framework, on the other hand, one can consider the multiplicative noise terms as stochastic perturbations of the nominal system matrices, which places them as a possible representation of parametric uncertainties and allows us to devise a robust approach to the control of stochastic systems. This interpretation can be found on previous works about the robust control of both stochastic diffusion processes and Markov jump linear systems (MJLS), see for example (HINRICHSSEN D. PRITCHARD, 1998; COSTA *et al.*, 1999; DRAGAN *et al.*, 2006). We present some papers where similar problems have been previously considered in section 4.1.1. The representation of uncertainties in the CVIU model, as indicated by equations (2.56) and (2.57), bears similarities with the models studied in these works, when one considers the state- and control- dependent noise terms as parametric uncertainties of the nominal system. This is the starting point for our study on the relation between the CVIU approach and the theory of robust control, and on how the performance of the CVIU control policy compares to that of robust \mathcal{H}_2 and \mathcal{H}_∞ controllers. Having that in mind we first discuss, in section 4.1.2, the representation of uncertainties in the CVIU and robust cases. Following the discussion on the representation of uncertainties, we give an overview of the main results on robust control of linear stochastic systems in section 4.2. A first challenge we face, when we wish to compare the performance of the aforementioned controllers in numerical terms, is how to bridge the gap between the different optimization criteria. We first dealt with this challenge by establishing a relation between the quadratic discounted cost functional and the \mathcal{H}_2 norm of an auxiliary linear stochastic system. That's the result previously presented at the *2017 IEEE Conference on Decision and Control*, and we discussed it in section 3.2. We conclude the chapter with a discussion on the more direct relation between the long run

¹ Some results from this chapter have been presented before on the paper presented at the 2017 IEEE Conference on Decision and Control (SILVA *et al.*, 2017).

average solution of the CVIU and robust \mathcal{H}_2 control problems in section 4.4.

4.1.1 Related literature

The simplest case of stochastic control problems corresponds to the pure additive white noise setting, when the diffusion matrix does not depend on the system state or control input. However, in applications such as mathematical finance, the dependence of the diffusion matrix on the system state or control input cannot be omitted (BORKAR, 2005), and works on control of systems with multiplicative noise have been developed since the 1960s.

Wonham, for example, studies in (WONHAM, 1967) the feedback control problem for a class of linear systems with additive and state-dependent white noise. He proves that, when the multiplicative noise is small enough, there exists a linear optimal control policy which minimizes a expected quadratic cost. McLane solves the state feedback control problem for linear stochastic systems with state- and control-dependent disturbances in (MCLANE, 1971). The paper adopts an integral quadratic optimality criterion, and gives examples of physical systems in which this class of disturbances is present. In the same year Hausmann published a paper (HAUSSMANN, 1971) on the steady state optimal linear regulator for systems with state- and control- dependent noise. He adopts a linear dependence on the state variable and control input for the multiplicative disturbances, and extends the results from Wonham by giving conditions under which an optimal control exists independently of the size of the control or state-dependent noise. In (HAUSSMANN, 1973) Hausmann published a paper on systems with control-dependent noise and additive disturbances, and gives necessary and sufficient conditions for the existence of stabilizing controllers without requiring the control-dependent noise to be small.

Due to the structure of the multiplicative noise that we discussed in the introduction, there is some overlap between the literature on stochastic systems with multiplicative noise and on robust control of linear stochastic systems. In this sense, previous works on robust control of linear stochastic systems tend to consider the state- and control-dependent stochastic noise terms as BM-driven perturbations of the nominal parameters, with a robustness measure given by the \mathcal{H}_2 or \mathcal{H}_∞ norm of the closed-loop system. In the \mathcal{H}_2 case, the problem consists in minimizing the energy of the measured output, and an appropriate choice of observation matrices C and D establishes a direct connection with the linear quadratic regulator problem. The \mathcal{H}_∞ problem, on the other hand, aims to minimize the induced gain of a \mathcal{L}_2 -bounded input on the output of the system, and in this case we need to consider both a control and a (bounded) disturbance inputs. The stochastic \mathcal{H}_∞ problem differs from the deterministic case in which it considers both white-noise perturbations and a bounded disturbance, against which the induced norm is evaluated.

An early reference on the topic is the paper from Willems and Willems (WILLEMS;

WILLEMS, 1976) on stabilizability of stochastic systems with state, control or both state and control-dependent noise. Here the authors investigate conditions under which this class of stochastic systems can be stabilized, in the mean square sense, via a feedback control policy. El Ghaoui discusses in (GHAOUI, 1995) the state feedback control of continuous-time stochastic systems with state-, control- and disturbance- dependent noise. The paper considers the problems of minimizing the expected energy of the system output and the induced \mathcal{L}_2 gain, which can be seen as \mathcal{H}_2 and \mathcal{H}_∞ -like control problems. The solutions are given in terms of Linear Matrix Inequalities (LMIs). Dragan et al published a series of papers on robust control approaches to linear stochastic systems. The results for continuous-time systems are summarized in their 2006 book (DRAGAN *et al.*, 2006), while the discrete-time counterpart is summarized in (DRĂGAN *et al.*, 2010). In the book the authors discuss, among other topics, \mathcal{H}_2 control and a version of the Bounded Real Lemma for linear systems subject to both Markov jumps and white noise perturbations. For the \mathcal{H}_2 case, the stochastic perturbation consists of both additive and multiplicative noise. However, the additive white noise is dropped when developing the stochastic BRL. The \mathcal{H}_2 solution and the stochastic BRL are given in terms of *stochastic generalized* Riccati algebraic equations. Results concerning \mathcal{H}_2 and \mathcal{H}_∞ control of linear stochastic systems with Markov jumps and additive white noise are summarized in the monographs from Costa, Fragoso and Marques on discrete-time MJLS (COSTA *et al.*, 2005) and Costa, Fragoso and Todorov on continuous-time MJLS (COSTA *et al.*, 2013). The topic was also studied in papers from Costa, do Val, Geromel, Farias and Gonçalves (COSTA *et al.*, 1999; VAL *et al.*, 2002; FARIAS *et al.*, 2000), among others

A definition of the stochastic \mathcal{H}_∞ norm and an \mathcal{H}_∞ -like theory for continuous-time stochastic systems are introduced in (HINRICHSSEN D. PRITCHARD, 1998). In that paper the authors consider stochastic systems with multiplicative noise but no Markov jumps. The authors present a stochastic version of the Bounded Real Lemma and give conditions for the existence of a controller which stabilizes the system and maintains the effect of disturbances on the system output under a given upper bound. The solution is given in terms of LMIs. Ugrinovskii studies a state-feedback \mathcal{H}_∞ problem with complete state measurement in (UGRINOVSKII, 1998). His model employs state-dependent noise, whereas Hinrichsen's uses state- and disturbance- dependent noise. The paper explores the relationship between \mathcal{H}_∞ control and differential games to solve the control problem. Sheng et al study in (SHENG *et al.*, 2015) a fuzzy approach for the \mathcal{H}_∞ control of systems with Markov jumps and stochastic noise dependent on the state, control and disturbance inputs. In that paper the authors deal with nonlinear stochastic systems, and they use fuzzy approach to rewrite the previous conditions — given in terms of Hamilton-Jacobi inequalities — as LMIs. Qi et al recently published two papers on the topic. In (QI *et al.*, 2017), the authors present conditions under which linear stochastic systems with multiplicative noise can be stabilized by output feedback, and in (SU *et al.*, 2017) the

authors study a type of \mathcal{H}_2 control problem for discrete-time stochastic systems.

4.1.2 Representation of uncertainties

Consider a dynamical system operating around a stable equilibrium point, and assume there is a locally valid linear model describing the behavior of the system in that region. As usual, we denote the state path of the system by $x(\cdot)$, the control input by $u(\cdot)$ and the observed output by $y(\cdot)$. This gives us a mathematical model of the system dynamics under a state-space representation,

$$\begin{aligned} dx(t) &= (Ax(t) + Bu(t))dt \\ y(t) &= Cx(t) + Du(t). \end{aligned} \quad (4.1)$$

We now assume that there is some degree of uncertainty about the value of the parameters of the local linear model, although we also assume that these uncertainties, despite unknown, belong to a certain set. In the single-input, single-output case, we could represent this knowledge about the local linear model by a differential equation of the type

$$dx(t) = ((A + \Delta_A)x(t) + (B + \Delta_B)u(t))dt, \quad (4.2)$$

where Δ_A and Δ_B represent parametric uncertainties. This is the approach taken, for example, when one chooses to use a polytopic representation for the uncertain parameters. Under a probabilistic viewpoint, such a model can be seen as an uniform distribution of the unknown parameter, i.e., the unknown parameter could assume any value within the bounded set with the same probability.

Now suppose we can give a sort of probabilistic interpretation to the parametric uncertainties mentioned above. Instead of implicitly assuming that the possible values of the uncertain parameter in question are uniformly distributed along the uncertain set or interval, as in the polytopic case, we attach a probability distribution to the set of uncertainties — it is not unreasonable to assume that, in a somewhat accurate model, the actual value of the parameter has a higher chance of being closer to the nominal value than to one of the extrema of the interval. Consider then the stochastic system described by

$$\begin{aligned} dx(t) &= (Ax(t) + Bu(t))dt + (\sigma_x x(t) + \sigma_u u(t))dW(t) + \sigma dW(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned} \quad (4.3)$$

In this stochastic setting, state- and control-dependent disturbance terms can be seen as stochastic perturbations of the parameters of the nominal system, A and B . We can also find a similar interpretation to the multiplicative noise in the CVIU approach. Note that equation (2.56) can be written as

$$\begin{aligned} dx(t) &= (Ax(t) + Bu(t))dt + \sigma(t)dW(t) \\ &\quad + (\sigma_x + \bar{\sigma}_x \text{diag}(|x(t)|))dW^x(t) + (\sigma_u + \bar{\sigma}_u \text{diag}(|u(t)|))dW^u(t), \end{aligned} \quad (4.4)$$

and the multiplicative noise terms above yield

$$Ax(t)dt + \bar{\sigma}_x \text{diag}(|x(t)|)dW^x(t) + Bu(t)dt + \bar{\sigma}_u \text{diag}(|u(t)|)dW^u(t) \equiv \\ (Ax(t)dt \pm \bar{\sigma}_x \text{diag}(x(t))dW^x(t)) + (Bu(t)dt \pm \bar{\sigma}_u \text{diag}(u(t))dW^u(t)), \quad (4.5)$$

equivalent in statistical terms due to symmetry of the Gaussian distribution around zero. We now expand the matrix notation for the terms related to the state vector to obtain

$$\begin{bmatrix} a_{11}dt \pm \bar{\sigma}_{x_{11}}dW_1^x(t) & \dots & a_{1n}dt \pm \bar{\sigma}_{x_{1n}}dW_n^x(t) \\ \vdots & \ddots & \vdots \\ a_{n1}dt \pm \bar{\sigma}_{x_{n1}}dW_1^x(t) & \dots & a_{nn}dt \pm \bar{\sigma}_{x_{nn}}dW_n^x(t) \end{bmatrix} x(t) + \begin{bmatrix} \sigma_{x_{11}} & \dots & \sigma_{x_{1n}} \\ \vdots & \ddots & \vdots \\ \sigma_{x_{n1}} & \dots & \sigma_{x_{nn}} \end{bmatrix} \begin{bmatrix} dW_1^x(t) \\ \vdots \\ dW_n^x(t) \end{bmatrix}. \quad (4.6)$$

Following a similar procedure for the control input, and recalling the standard BM is a zero-mean, symmetrically distributed process, we can thus see the state and control dependent noise terms in the above representation as an element-wise perturbation of the system matrices A and B . Moreover, the additive noise term serves as an extra representation term for the poorly known system.

4.2 Robust \mathcal{H}_2 and \mathcal{H}_∞ control of linear stochastic systems

The general robust control problem is linked to the attenuation of disturbances and the minimization of the effect of uncertainties on the operation of the controlled system. The control problem can be formulated as an optimization problem by use of the \mathcal{H}_2 and \mathcal{H}_∞ norms of the system as objective functions. This yields the so-called \mathcal{H}_2 and \mathcal{H}_∞ optimal control problems, and in the following we retrieve some results from the literature in order to describe how the \mathcal{H}_2 and \mathcal{H}_∞ control problems of linear stochastic systems can be formulated. For more details on the deterministic background, the reader can refer to annex A.

\mathcal{H}_2 control

Consider a LTI stochastic system subject to additive and multiplicative white noise perturbations. Assume $x(\cdot) \in \mathbb{R}^n$, $y(\cdot) \in \mathbb{R}^m$ and $W(\cdot)$, $W^x(\cdot)$ r - and n -dimensional BM, such that

$$\mathcal{G} : \begin{cases} dx(t) = A_0x(t)dt + Ax(t)dW^x(t) + \sigma dW(t), \\ y(t) = Cx(t), \quad x(0) = 0, \end{cases} \quad (4.7)$$

with A a stable matrix. We follow Morozan's definition of the stochastic \mathcal{H}_2 norm (DRAGAN *et al.*, 2006, ch. 7). The book develops the stochastic \mathcal{H}_2 norm for linear stochastic

systems subject to both Markov jumps and Brownian motion multiplicative and additive noise, but here we simplify the results for the pure white noise setting. This yields

$$\|\mathcal{G}\|_2 = \left[\lim_{t \rightarrow \infty} \mathbb{E}|y(t)|^2 \right]^{\frac{1}{2}}.$$

Under the assumption that the system is stable, the stochastic \mathcal{H}_2 norm can be given by

$$\|\mathcal{G}\|_2^2 = \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T |y(t)|^2 dt \right] \quad (4.8)$$

where $\mathbb{E}[\cdot]$ stands for the expected value from starting the process (4.7) at zero. For the corresponding control problem, consider the system described by

$$\begin{aligned} dx(t) &= (Ax(t) + Bu(t))dt + \sum_{k=1}^r (A_{d,k}x(t) + B_{d,k}u(t)) dW_k(t) + \sigma dW_0(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \quad (4.9)$$

adapted from (DRAGAN *et al.*, 2006) with a slight change of notation. Here, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $A, A_{d,k} \in \mathbb{R}^{n \times n}$ and $B, B_{d,k} \in \mathbb{R}^{n \times m}$. $W_k(t)$ are standard scalar Brownian motions and $W_0(t)$ a multidimensional one. As in the deterministic case, the stochastic \mathcal{H}_2 optimal control problem aims to minimize the (stochastic) \mathcal{H}_2 norm of the closed-loop system while guaranteeing stability.

For the case of perfect state measurements, we adapt results from (COSTA *et al.*, 2013) and (DRAGAN *et al.*, 2006) for the context of stochastic diffusions. For that, consider the controlled system G described by equation (4.9), and the output-feedback controller

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c u_c(t), \\ y_c(t) &= C_c x_c(t) + D_c u_c(t), \end{aligned} \quad (4.10)$$

with $x_c \in \mathbb{R}^{n_c}$, $u_c \in \mathbb{R}^n$ and $y_c \in \mathbb{R}^m$. The state-feedback case, $y_c(t) = D_c u_c(t)$, is included in the set of controllers described by the above equation. The resulting closed-loop system, once we couple system G and the above controller by taking $u_c(t) = x(t)$ and $u(t) = y_c(t)$, is given by (DRAGAN *et al.*, 2006)

$$\begin{aligned} dx_{cl}(t) &= A_{cl} x_{cl}(t) dt + \sum_{k=1}^r A_{dcl,k} x_{cl}(t) dW_k(t) + \sigma_{cl} dW_0(t), \\ y_{cl}(t) &= C_{cl} x_{cl}(t). \end{aligned} \quad (4.11)$$

In the above equation, we have

$$x_{cl} = \begin{bmatrix} x \\ x_c \end{bmatrix}; \quad (4.12a)$$

$$A_{cl} = \begin{bmatrix} A + BD_c & BC_c \\ B_c & A_c \end{bmatrix}; \quad (4.12b)$$

$$A_{dcl,k} = \begin{bmatrix} A_{d,k} + B_{d,k}D_c & B_{d,k}C_c \\ 0 & 0 \end{bmatrix}; \quad (4.12c)$$

$$\sigma_{cl} = \begin{bmatrix} \sigma \\ 0 \end{bmatrix}; \quad (4.12d)$$

$$C_{cl} = \begin{bmatrix} C + DD_c & DC_c \end{bmatrix}. \quad (4.12e)$$

We now adapt another result from (DRAGAN *et al.*, 2006) for the diffusion case with no jumps. First, consider the following version of a stochastic generalized Riccati algebraic equation

$$\begin{aligned} & A^\top X + XA + \sum_{k=1}^r A_{d,k}^\top X A_{d,k} \\ & - \left[XB + \sum_{k=1}^r A_{d,k}^\top X B_{d,k} + C^\top D \right] \\ & \quad \left[D^\top D + \sum_{k=1}^r B_{d,k}^\top X B_{d,k} \right]^{-1} \\ & \quad \left[B^\top X + \sum_{k=1}^r B_{d,k}^\top X A_{d,k} + D^\top C \right] + C^\top C = 0. \end{aligned} \quad (4.13)$$

The equation can be rewritten, in a more compact form, as

$$\mathcal{L}X - \mathcal{P}^\top(X)\mathcal{R}^{-1}(X)\mathcal{P}(X) + C^\top C = 0,$$

with \mathcal{L} the Lyapunov operator associated to the stochastic system, and \mathcal{P} given by (DRAGAN *et al.*, 2006)

$$\mathcal{P}(X) = B^\top X + \sum_{k=1}^r B_{d,k}^\top X A_{d,k} + D^\top C. \quad (4.14)$$

Moreover, let

$$\mathcal{R}(X) = D^\top D + \sum_{k=1}^r B_{d,k}^\top X B_{d,k}, \quad (4.15)$$

and consider the *generalized dissipation matrix* (DRAGAN *et al.*, 2006),

$$A^\Sigma = \begin{bmatrix} \mathcal{L}X + C^\top C & \mathcal{P}^\top(X) \\ \mathcal{P}(X) & \mathcal{R}(X) \end{bmatrix}.$$

We get the following result.

Theorem 4.1 ((DRAGAN *et al.*, 2006), Theorem 7.2.2). *Assume that the following conditions are fulfilled.*

1. *The system (4.9) is stabilizable.*
2. *It exists \hat{X} such that $A^\Sigma(\hat{X}) > 0$.*

Under these conditions we have

$$\min_{G_c \in \mathcal{K}_s(G)} \|G_{cl}\|_2 = \left[\text{tr} \left(\sigma^\top \tilde{X} \sigma \right) \right]^{\frac{1}{2}},$$

and the optimal control is

$$u(t) = \tilde{F}x(t), \quad (4.16)$$

where \tilde{X} is the stabilizing solution of the SGRAE (4.13) and \tilde{F} is the stabilizing gain

$$\tilde{F} = -\mathcal{R}^{-1}(\tilde{X})\mathcal{P}(\tilde{X}).$$

Remark 4.1. For the state-feedback case, a solution for the stochastic \mathcal{H}_2 control problem can also be found in (GHAOUI, 1995). El Ghaoui's model also considers disturbance-dependent stochastic white noise, and the solution is stated in terms of LMIs.

\mathcal{H}_∞ control

The formulation of the \mathcal{H}_∞ control problem in the stochastic case has some particular features. The performance index associated to the control problem is usually defined in terms of the \mathcal{H}_∞ norm of a perturbation operator, which maps the effect of mean-square stable, bounded stochastic disturbance inputs on the output of the system (HINRICHSSEN D. PRITCHARD, 1998; COSTA *et al.*, 2013; DRAGAN *et al.*, 2006). In these works the authors develop a stochastic version of the *Bounded Real Lemma* for a class of autonomous systems, in a similar manner to the deterministic case. According to the paper from Hinrichsen and Pritchard, let the linear stochastic system

$$\begin{aligned} dx(t) &= Ax(t)dt + A_0x(t)dw_1(t) + B_0v(t)dw_2(t) + Bv(t)dt, \\ z(t) &= Cx(t) + Dv(t), \end{aligned} \quad (4.17)$$

where

$$(A, A_0, B_0, B, C, D) \in \mathbb{K}^{n \times n} \times \mathbb{K}^{n \times n} \times \mathbb{K}^{n \times l} \times \mathbb{K}^{n \times l} \times \mathbb{K}^{q \times n} \times \mathbb{K}^{q \times l}, \quad \mathbb{K} = \mathbb{R} \text{ or } \mathbb{C}, \quad (4.18)$$

and $v(t)$ is an unknown, finite energy stochastic disturbance and $w_1, w_2(t)$ zero-mean, scalar Wiener processes. Define also the perturbation operator associated to system (4.17),

Definition 4.1 (Perturbation operator — (HINRICHSSEN D. PRITCHARD, 1998), Definition 2.3). *Suppose that (4.17) is externally stable. The operator*

$$\mathbb{L} : L_w^2(\mathbb{R}_+; L^2(\Omega, \mathbb{K}^l)) \Rightarrow L_w^2(\mathbb{R}_+; L_w^2(\Omega, \mathbb{K}^q))$$

is called the perturbation operator of (4.17). Its norm is defined as the minimal $\gamma \geq 0$ such that

$$\|\mathbb{L}\| = \sup_{v \in L_w^2(\mathbb{R}_+; L^2(\Omega, \mathbb{K}^l)), v \neq 0} \frac{\|Cx(\cdot, v, 0) + Dv(\cdot)\|_{L_w^2(\mathbb{R}_+; L_w^2(\Omega, \mathbb{K}^q))}}{\|v(\cdot)\|_{L_w^2(\mathbb{R}_+; L^2(\Omega, \mathbb{K}^l))}}. \quad (4.19)$$

$\|\mathbb{L}\|$ is a measure of the worst effect the stochastic disturbance $v(\cdot)$ may have on the to-be-controlled output $z(\cdot)$ of the system.

The definition of the perturbation operator is similar to that found in (COSTA *et al.*, 2013) for linear systems subject to Markov jumps and in (DRAGAN *et al.*, 2006) for stochastic systems subject to both Markov jumps and multiplicative white noise. We can associate a quadratic cost to the operator $\|\mathbb{L}\|$, in the same way as the parametrized cost from (BAŞAR; BERNHARD, 2008),

$$J_T^{\gamma^2}(x^0, v) = \int_0^T \mathbb{E}(\|\mathbb{L}v(t)\|^2 - \gamma^2\|v(t)\|^2)dt. \quad (4.20)$$

This functional slightly differs from the one adopted in (HINRICHSSEN D. PRITCHARD, 1998), given by

$$J_T^{\gamma^2}(x^0, v) = \int_0^T \mathbb{E}(\gamma^2\|v(t)\|^2 - \|Cx(t) + Dv(t)\|^2)dt,$$

although the general structure remains the same. The paper brings the following result.

Theorem 4.2 ((HINRICHSSEN D. PRITCHARD, 1998), Theorem 2.8). *For any set of data (4.18) and any positive real number γ , the following statements are equivalent:*

(i) *The system (4.17) is internally stable and $\|\mathbb{L}\| < \gamma$.*

(ii) *There exists $P \in \mathcal{H}_n(\mathbb{K})$ such that the inequality*

$$M(P) = \begin{bmatrix} PA + A^*P + q_{11}A_0^*P - C^*C & PB + q_{12}A_0^*PB_0 - C^*D \\ B^*P + q_{12}B_0^*A_0 - D^*C & \gamma^2 I_l + q_{22}B_0^*PB_0 - D^*D \end{bmatrix} > 0 \quad (4.21)$$

is satisfied.

Following the notation used in (HINRICHSSEN D. PRITCHARD, 1998, p. 1519), we have that the control problem is defined for the system

$$\begin{aligned} dx(t) &= Ax(t)dt + A_0x(t)dw_1(t) + B_0v(t)dw_2(t) + B_1v(t)dt + B_2u(t)dt, \\ z(t) &= C_1x(t) + D_{11}v(t) + D_{12}u(t), \\ y(t) &= C_2x(t) + D_{21}v(t), \end{aligned} \quad (4.22)$$

Conditions for the existence of a controller which stabilizes the system and guarantees the \mathcal{H}_∞ -norm of the controlled system has an upper limit γ , in a similar manner to the deterministic problem, are developed in these works.

4.3 The CVIU control problem and the deterministic \mathcal{H}_2 synthesis

We now discuss how we used our previous result 3.2, which establishes a link between the discounted cost functional and the \mathcal{H}_2 norm of a stochastic system, to compare the cost of operation of the CVIU and robust \mathcal{H}_2 controllers. Note that we deal with two types of control problems, and, in the CVIU case, we need to consider a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ satisfying the usual conditions (YONG; ZHOU, 1999, Ch 1, def 2.6)) on which a multidimensional standard Brownian motion of appropriate dimensions is defined. Admissible controls for a CVIU, denoted by $\mathcal{U}[0, \infty)$, are U -valued Markov functions $t \rightarrow u(t, \omega) = u(t, x(t))$ and $u(\cdot) \in L^2_{\mathcal{F}_t}(0, \infty; \mathbb{R}^r)$, with $L^p_{\mathcal{F}_t}(a, b; X) = \{\phi(\cdot) = \{\phi(t, \omega) : a \leq t \leq b\} \text{ such that } \phi(\cdot) \text{ is an } \{\mathcal{F}_t\}_{t \geq 0}\text{-adapted, } X\text{-valued measurable process on } [a, b], \text{ and } E[\int_a^b \|\phi(t, \omega)\|_X^p dt] < \infty\}$ (VAL; SOUTO, 2017; SILVA *et al.*, 2017).

For the state-feedback \mathcal{H}_2 robust control synthesis, on the other hand, we consider a controlled system

$$\tilde{\mathcal{G}}_c : \begin{cases} d\tilde{x}(t) = (\tilde{A} \pm \Delta\tilde{A})\tilde{x}(t) + (\tilde{B} \pm \Delta\tilde{B})\tilde{u}(t)dt + \sigma dW(t), \\ \tilde{y}(t) = C\tilde{x}(t) + D\tilde{u}(t), \quad \tilde{x}(0) = 0 \end{cases} \quad (4.23)$$

where the terms $\Delta\tilde{A}$ and $\Delta\tilde{B}$ represent uncertainties about the nominal matrices \tilde{A} and \tilde{B} . We assume the collection of pairs $(\tilde{A} \pm \Delta\tilde{A}, \tilde{B} \pm \Delta\tilde{B})$ are stabilizable and $(C, \tilde{A} \pm \Delta\tilde{A})$ detectable. We also assume that admissible controls are restricted to the class $t \rightarrow u(t, \omega) = u(t, x(t)) = Kx(t)$, with $K \in \mathbb{R}^{m \times n}$, that is, controls that are in static, linear feedback form. The problem is to find a controller which minimizes an upper bound of the \mathcal{H}_2 norm, $\|\tilde{\mathcal{G}}_c\|_2$, and keeps the system stable. Moreover, we state the problem in LMI form.

Now, recall once again that, in this case, the performance of the system for the CVIU synthesis is measured through the expected discounted cost,

$$J(x, u(\cdot)) = E^x \left[\int_s^\infty e^{-\alpha t} \|y(t)\|^2 dt \right], \quad (4.24a)$$

associated to the CVIU model,

$$\mathcal{G}_c : \begin{cases} dx(t) = Ax(t) + Bu(t)dt + \hat{\sigma}d\hat{W}(t), \\ y(t) = Cx(t) + Du(t), \quad x(0) = 0, \end{cases} \quad (4.24b)$$

for $t \in [0, \infty)$ and $\hat{\sigma}(x, u)$ as in (2.57). Assume that $J(x, u(\cdot)) < \infty$ for some $u(\cdot) \in \mathcal{U}[0, \infty)$, and the main objective is to find an admissible pair $u^*(\cdot) \in \mathcal{U}[0, \infty)$ and the

corresponding $x^*(\cdot)$ such that the minimum of $J(x, u(\cdot))$ is achieved over the class $\mathcal{U}[0, \infty)$. To set up a link between the CVIU and \mathcal{H}_2 problems, we follow the steps:

1. According to Lemma 3.3, consider $A = \tilde{A} + \frac{\alpha}{2}I$, $\tilde{B} = B$ for $\alpha > 0$ small enough.
2. Set $\bar{\sigma}_x = \frac{1}{\ell_x}\Delta\tilde{A}$ and $\bar{\sigma}_u = \frac{1}{\ell_u}\Delta\tilde{B}$. Here ℓ_x and ℓ_u are integer numbers that set a correspondence between the representation of uncertainties in the polytopic and in the CVIU cases. As previously discussed, when we use a polytopic representation for uncertainties, we are implicitly assuming the actual value of the parameter can be any value inside the polytopic set with equal chance, which resembles an uniform distribution. In the CVIU case, however, we get a Gaussian-like representation for the parametric uncertainties. Figure 6 illustrates this correspondence. Parameter matrices σ_x and σ_u

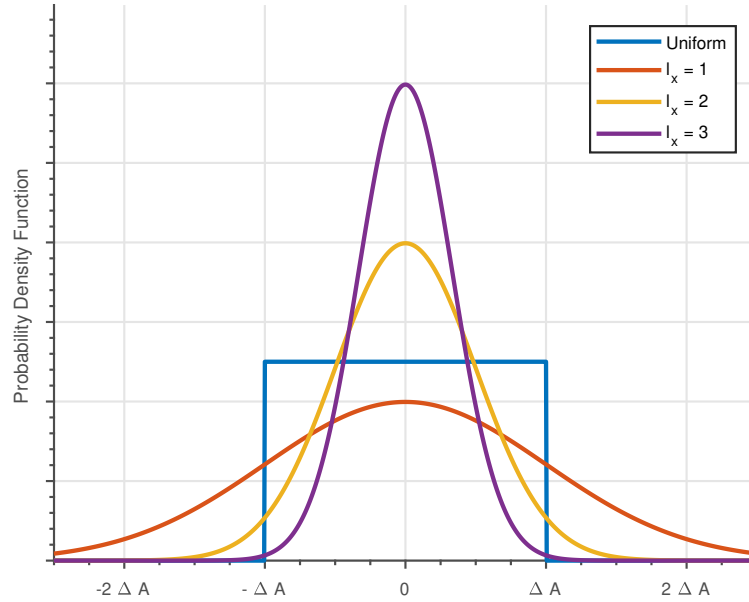


Figure 6 – Correspondence between the representation of uncertainties in the CVIU and polytopic cases. Figure previously presented at (SILVA *et al.*, 2017).

should satisfy the assumption in the CVIU model (≥ 0 stands on the positive semi-definite sense):

$$\begin{cases} \sigma_x \bar{\sigma}_x^\top + \bar{\sigma}_x \sigma_x^\top \geq 0, \\ \sigma_u \bar{\sigma}_u^\top + \bar{\sigma}_u \sigma_u^\top \geq 0. \end{cases} \quad (4.25)$$

In the situation indicated in 1., there exists $\alpha > 0$ small enough and $\gamma \neq 0$ in such a way that $A = \tilde{A} + \frac{\alpha}{2}I$, and $(A, \gamma\tilde{B})$ is a stabilizable pair. This is simple to check since (A, B) is a stabilizable pair if $\text{rank}[(\lambda_k I - A) : B] = n$ for each eigenvalue λ_k that lies outside the open left complex semiplane. Thus,

$$n = \text{rank}[(\tilde{\lambda}_k I - \tilde{A}) : \tilde{B}] = \text{rank}[(\tilde{\lambda}_k + \frac{\alpha}{2})I - A) : \gamma\tilde{B}],$$

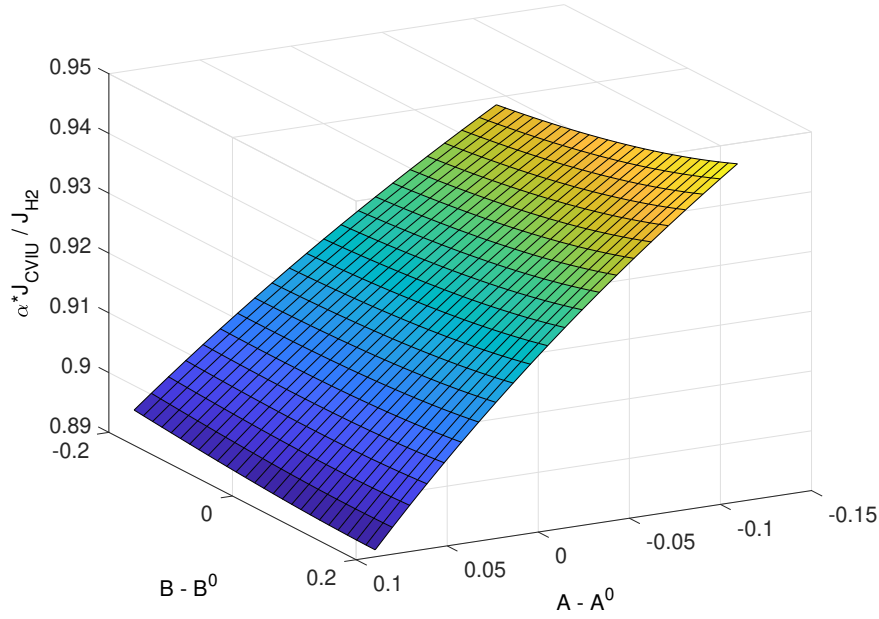


Figure 7 – Relative operation cost as a function of error offsets in the linear parameters. Figure previously presented at (SILVA *et al.*, 2017).

and if α is sufficiently small, all eigenvalues $\tilde{\lambda}_j + \alpha/2$ associated to a $\tilde{\lambda}_j$ lying inside the open left complex semi-plane, still lies in that region.

The above steps provide a strategy to compare the cost of operation between a robust controller designed by minimization of the \mathcal{H}_2 -norm and a controller designed according to the CVIU approach. To illustrate the procedure we refer once again to the scalar model mentioned in the introduction. We assume the linear system parameters, A and B , are uncertain but known to be in a bounded interval around the nominal values A^0 and B^0 . This allows us to create a polytopic representation for the uncertain parameters and solve the corresponding \mathcal{H}_2 synthesis problem in terms of LMI's using the packages SeDuMi (STURM, 1999) and Yalmip (LÖFBERG, 2004) in Matlab, while the SDE describing the system was computed according to the Euler-Maruyama method (HIGHAM, 2001). The cost of operation of the CVIU and \mathcal{H}_2 control policies was evaluated via Monte Carlo simulations with a time horizon of 100 seconds and 50 repetitions. The simulation results are shown in figure 7. The graph shows how the cost of operation of the CVIU control policy, evaluated with relation to the cost of the robust \mathcal{H}_2 controller, varies for different values of the errors $\Delta_A = A - A^0$ and $\Delta_B = B - B^0$. Within our simulation window the CVIU control policy offered a smaller cost of operation than the polytopic \mathcal{H}_2 controller.

4.4 Long run average cost and robust control

In the previous section we discussed the first approach we used to compare a CVIU controller with classical methods from robust control theory, which basically consisted in using an auxiliary system to bridge the discounted cost functional and norms of a LTI system. In this section, however, we aim to expand upon the previous results and follow a slightly different perspective. Instead of using an auxiliary system to relate the independently solved control problems, we now wish to use the solution of the long run average CVIU problem and the average running formulation of the stochastic \mathcal{H}_2 norm in order to establish a more direct relation between the CVIU approach and works on robust control of linear stochastic systems. For that, recall the CVIU model for a stochastic system, here rewritten as

$$dx(t) = (Ax(t) + Bu(t))dt + \bar{\sigma}_x \text{diag}(|x(t)|)dW^x(t) + \bar{\sigma}_u \text{diag}(|u(t)|)dW^u(t) + \begin{bmatrix} \sigma & \sigma_x & \sigma_u \end{bmatrix} \begin{bmatrix} dW(t) \\ dW^x(t) \\ dW^u(t) \end{bmatrix}, \quad (4.26)$$

in order to clear up the role of the state- and control-dependent stochastic noise in the CVIU approach. We also recall the model used by Morozan et al in (DRAGAN *et al.*, 2006),

$$dx(t) = (Ax(t) + Bu(t))dt + \sum_{k=1}^r (A_{d,k}x(t) + B_{d,k}u(t))dW_k(t) + \sigma dW_0(t), \quad (4.27)$$

$$y(t) = Cx(t) + Du(t)$$

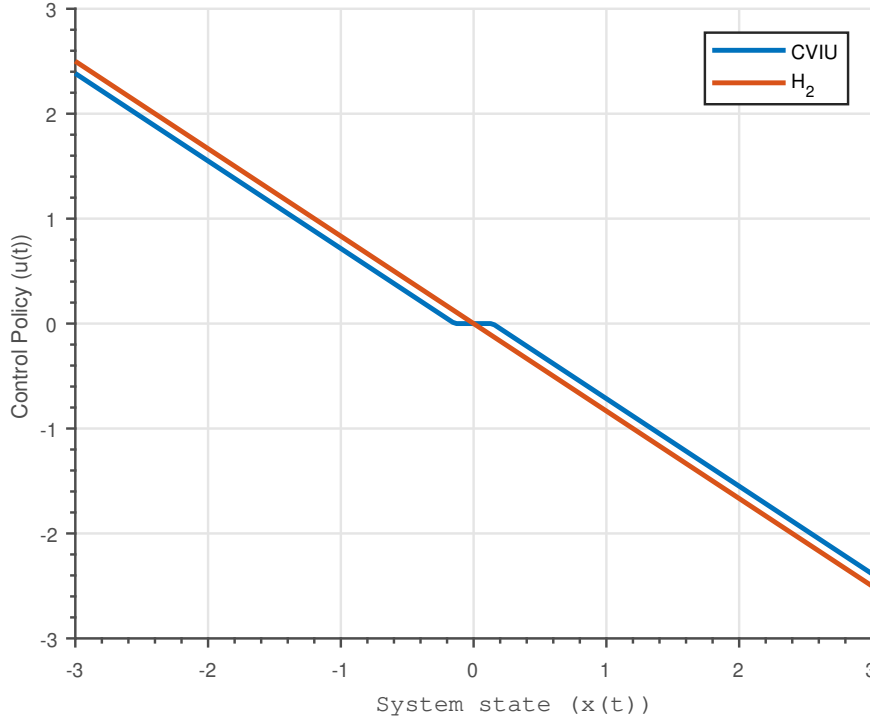
with a slight change of notation. Here, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $A, A_{d,k} \in \mathbb{R}^{n \times n}$ and $B, B_{d,k} \in \mathbb{R}^{n \times m}$. $W_k(t)$ are standard scalar Brownian motions and $W_0(t)$ a multidimensional one. In both models, we assume A and B represent the nominal system, and the diffusion coefficients are given by $\sigma_{x,u}$ in the CVIU model, and $A_{d,k}$, $B_{d,k}$, σ in the \mathcal{H}_2 model. In order to obtain a direct correspondence between the stochastic noise structure in the CVIU and stochastic \mathcal{H}_2 formulations, let us take $r = n$ and

$$A_{d,k} = \begin{bmatrix} 0 & \dots & a_{d,k_{1i}} & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & a_{d,k_{ni}} & \dots & 0 \end{bmatrix}.$$

Furthermore, recall that the \mathcal{H}_2 norm of a stochastic system is given by,

$$\|\mathcal{G}\|_2^2 = \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T |y(t)|^2 dt \right],$$

and we solved the corresponding problem for the CVIU approach (3.94) in the end of subsection 3.3.4.

Figure 8 – CVIU and \mathcal{H}_2 control policies.

Let us then devise a few numerical experiments to see how we can compare controllers designed according to the CVIU and stochastic \mathcal{H}_2 approaches. Consider then a linear, continuous-time stochastic system given by

$$\begin{aligned} dx(t) &= (Ax(t) + Bu(t))dt + \sigma dW(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \quad (4.28)$$

with $A = -5$, $B = 2$, $C = 1$, $D = 1.2$, $\sigma = 0.5$, $\sigma_x = 0.8$, $\bar{\sigma}_x = 0.95$, $\sigma_u = 0.15$ and $\bar{\sigma}_u = 0.95$. Here we consider the nature noise $W(t)$ a multidimensional Brownian-motion common to both the CVIU and \mathcal{H}_2 models. The design procedure follows along the previously outlined solution for the CVIU approach in chapter 3 and for the stochastic \mathcal{H}_2 problem in 4.2. In the CVIU case, the solution follows a relaxation procedure for the modified Lyapunov and Riccati equations for the inaction region and asymptotic behavior, respectively, and a numerical integration procedure is used to calculate the optimal solution between these two regions. Note that the modified Riccati equation of the stochastic \mathcal{H}_2 problem (4.13) has terms depending on the matrix variable X as in the CVIU case, and we also adopt a relaxation procedure to find the optimal solution.

In Figure 8 we plot the \mathcal{H}_2 and CVIU optimal control policies for system (4.28). Here $x(\cdot)$ and $u(\cdot)$ represent variations of the system state and control input with relation to the equilibrium values in the linearized model. The inaction region in the CVIU approach is indicated by the horizontal line around the equilibrium point. Given the similar

form of the Riccati equations in the CVIU,

$$A^\top X + XA - (C^\top D + XB) (D^\top D + \text{Diag}(\bar{\sigma}_u^\top X \bar{\sigma}_u))^{-1} (B^\top X + D^\top C) + C^\top C + \text{Diag}(\bar{\sigma}_x^\top X \bar{\sigma}_x) = 0,$$

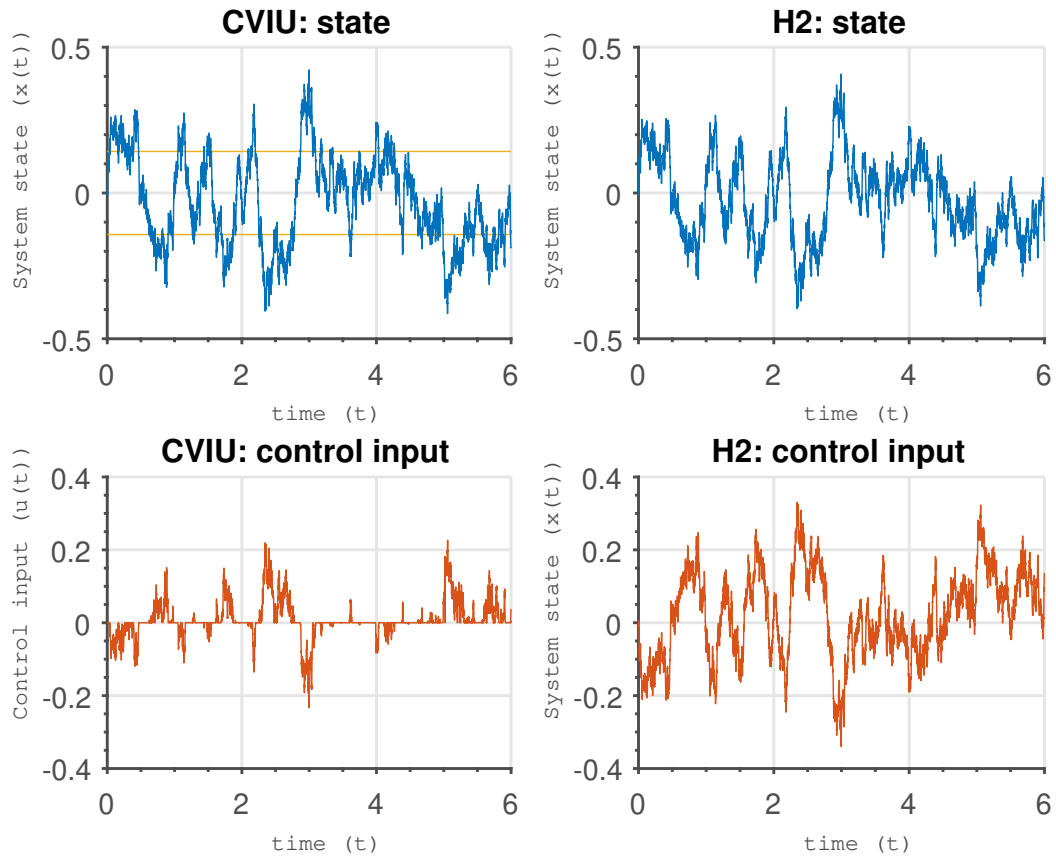
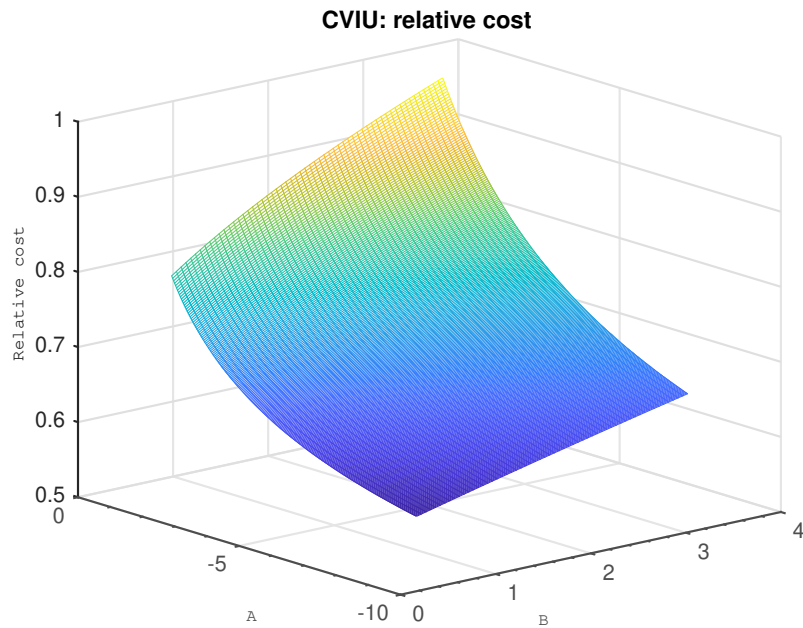
and stochastic \mathcal{H}_2 ,

$$\begin{aligned} A^\top X + XA + \sum_{k=1}^r A_{d,k}^\top X A_{d,k} \\ - \left[XB + \sum_{k=1}^r A_{d,k}^\top X B_{d,k} + C^\top D \right] \\ \left[D^\top D + \sum_{k=1}^r B_{d,k}^\top X B_{d,k} \right]^{-1} \\ [B^\top X + \sum_{k=1}^r B_{d,k}^\top X A_{d,k} + D^\top C] + C^\top C = 0, \end{aligned}$$

cases — recall that $R'(X) = R + \text{Diag}(\bar{\sigma}_u^\top X \bar{\sigma}_u) = D^\top D + \text{Diag}(\bar{\sigma}_u^\top X \bar{\sigma}_u)$, $Q'(X) = Q + \text{Diag}(\bar{\sigma}_x^\top X \bar{\sigma}_x) = C^\top C + \text{Diag}(\bar{\sigma}_x^\top X \bar{\sigma}_x)$, and we take $r = 1$ in the scalar case —, we see that the asymptotic solution of the CVIU control policy tends to the solution of the stochastic \mathcal{H}_2 problem.

We also illustrate the cautionary behavior of the CVIU solution in Figure 9. The plot shows the evolution of a controlled system in a given time horizon, and how the CVIU immediate control actions are updated within that simulation frame. In the CVIU case, the magenta lines indicate the boundaries of the inaction region. We bring a similar plot for the stochastic \mathcal{H}_2 case. As expected, the CVIU control policy is only updated when the system state leaves the inaction region, whereas the \mathcal{H}_2 control action tends to be updated regularly. We conclude the numerical experiments with an evaluation of the cost of operation of the mentioned controllers. As in case discussed in the previous section, we evaluate the cost of operation of the CVIU controller with relation to a stochastic \mathcal{H}_2 optimal control policy. We use the Euler-Maruyama method to simulate the dynamic evolution of the stochastic system, adopting a time horizon of 100 seconds and 50 simultaneous realizations of the stochastic differential equation. The results are shown in Figure 10. Here we plot the ratio $\frac{J_{\text{CVIU}}}{J_{\mathcal{H}_2}}$, with J_{CVIU} the computed average cost of the system controlled according to the CVIU approach and $J_{\mathcal{H}_2}$ the computed \mathcal{H}_2 norm of the system controlled via an optimal state-feedback \mathcal{H}_2 controller, as a function of possible variations on the actual values of the nominal parameters A and B . The idea behind this simulation is to evaluate how the controllers perform when the actual values of the system parameters are slightly different from those available in the design process.

Given the similar structure of the asymptotic CVIU and \mathcal{H}_2 solutions, we next evaluate how much of a role the inaction region plays in the relative cost between the two controllers. For that, let us simulate how the cost of the CVIU approach with relation to the \mathcal{H}_2 controller changes as a function of the noise intensity σ . Since the size of the

Figure 9 – CVIU and \mathcal{H}_2 control policies: sample pathFigure 10 – CVIU and \mathcal{H}_2 control policies: relative cost for different values of the nominal parameters A and B .

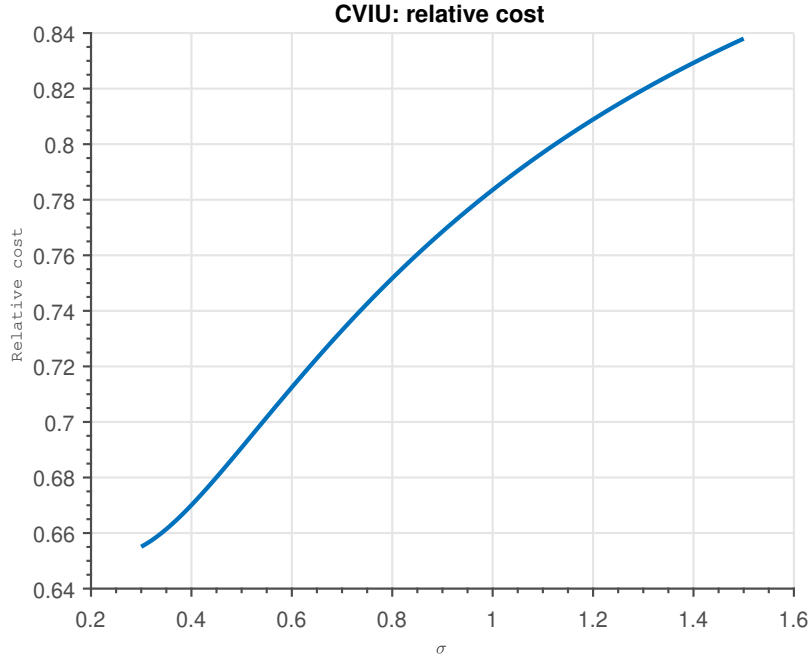


Figure 11 – CVIU and \mathcal{H}_2 control policies: relative cost for different values of σ .

inaction region does not depend on σ , we can expect that the system state will leave the inaction region more often, and the CVIU controller will therefore act more frequently, as the noise intensity increases. This in turn should lead to an increase in the cost of operation of the CVIU controller. We plot the simulation results in Figure 11. The plot shows the relative cost of the CVIU controller, $\frac{J_{\text{CVIU}}}{J_{\mathcal{H}_2}}$, as a function of the noise intensity σ . The simulation results seem to corroborate our hypothesis.

A natural extension to our discussion so far is the comparison between the CVIU approach and robust \mathcal{H}_∞ control of linear stochastic systems. When we compare the CVIU model with stochastic systems such as those treated in (HINRICHSSEN D. PRITCHARD, 1998) or (DRAGAN *et al.*, 2006), there is a natural correspondence between the multiplicative noise structure used in the aforementioned models, with results similar to those we discussed for the stochastic \mathcal{H}_2 case. If we were to follow a similar strategy to calculate the stochastic \mathcal{H}_∞ norm, however, we would need to consider an additional, norm-bounded and square-integrable disturbance, which would lead to a CVIU model in the form

$$\begin{aligned}
 dx(t) = & (Ax(t) + Bu(t))dt + \bar{\sigma}_x \text{diag}(|x(t)|)dW^x(t) + \bar{\sigma}_u \text{diag}(|u(t)|)dW^u(t) \\
 & + \begin{bmatrix} \sigma & \sigma_x & \sigma_u \end{bmatrix} \begin{bmatrix} dW(t) \\ dW^x(t) \\ dW^u(t) \end{bmatrix} + B_v v(t)dt,
 \end{aligned}$$

in order to define an induced norm of the corresponding input-output operator,

$$\|\mathbb{L}\| = \sup_{v \in L_w^2(\mathbb{R}_+; L^2(\Omega, \mathbb{K}^l)), v \neq 0} \frac{\|Cx(\cdot, v, 0) + Dv(\cdot)\|_{L_w^2(\mathbb{R}_+; L^2(\Omega, \mathbb{K}^q))}}{\|v(\cdot)\|_{L_w^2(\mathbb{R}_+; L^2(\Omega, \mathbb{K}^l))}}.$$

On the other hand, note that most existing literature on \mathcal{H}_∞ control of linear, continuous-time stochastic systems usually considers systems with only multiplicative noise, cf. for example (HINRICHSSEN D. PRITCHARD, 1998; UGRINOVSKII, 1998; ZHANG; CHEN, 2006; DRAGAN *et al.*, 2006; BERMAN; SHAKED, 2006; BERMAN; SHAKED, 2008; ZHANG *et al.*, 2014; SHENG *et al.*, 2015; DAMM *et al.*, 2017). Hinrichsen and Pritchard point out in (HINRICHSSEN D. PRITCHARD, 1998) that the use of additive white noise might pose additional, restrictive conditions on the the design of \mathcal{H}_∞ controllers. Another reason for the use of purely multiplicative noise seems to stem from the fact that it is easier to write the conditions for a stochastic version of the Bounded Real Lemma in this case. For that, consider once again a continuous-time stochastic system with state- and disturbance-dependent noise such as the one studied in (HINRICHSSEN D. PRITCHARD, 1998),

$$\begin{aligned} dx(t) &= Ax(t)dt + A_0x(t)dw_1(t) + B_0v(t)dw_2(t) + Bv(t)dt, \\ z(t) &= Cx(t) + Dv(t), \end{aligned}$$

which we can write in an integral form as

$$\begin{aligned} x(t) &= x(0) + \int_0^t (Ax(s) + Bu(s))ds + \int_0^t \begin{bmatrix} A_0x(s) & B_0v(s) \end{bmatrix} d \begin{bmatrix} w_1(s) \\ w_2(s) \end{bmatrix} \\ &= \int_0^t \varphi(s)ds + \int_0^t \Phi(s)d \begin{bmatrix} w_1(s) \\ w_2(s) \end{bmatrix}. \end{aligned}$$

From this integral form we can apply Itô's formula (Theorem 2.4) with a quadratic function $F(t, x(t)) = \langle x(t), P(t)x(t) \rangle$ to get (HINRICHSSEN D. PRITCHARD, 1998)

$$\begin{aligned} J_T^{\gamma^2}(x(0), v) + \mathbb{E}\langle x(T), P(T)x(T) \rangle - \langle x(0), P(0)x(0) \rangle \\ = \int_0^T \mathbb{E} \left[\langle x(t), \dot{P}(t)x(t) \rangle + \left\langle \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}, M(P(t)) \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} \right\rangle \right], \end{aligned}$$

from where theorem 4.2 and further analysis for the stochastic \mathcal{H}_∞ control problem in (HINRICHSSEN D. PRITCHARD, 1998) derive.

If we drop the additive white noise in the CVIU setting in order to follow the procedure detailed in the mentioned papers, however, we might violate the condition for non degeneracy of the inaction region outlined in lemma 2.2. An eventual loss of the inaction region, on the other hand, would leave the CVIU optimal control policy with a structure pretty much similar to that of previous works. This extension is therefore not immediate, and the problem demands a more careful analysis in order to decide the best strategy to tackle the issue of an \mathcal{H}_∞ -like approach for the CVIU model while keeping the fundamental structure which sets the CVIU approach apart from other models. That is a topic we plan to explore in the following months.

5 Applications

5.1 Introduction

In this chapter we investigate possible applications of the CVIU approach in renewable resources management. Results from this chapter have been discussed before in the conference papers presented at the XXI Brazilian Congress on Automatic Control (SILVA *et al.*, 2016) and at the 6th IFAC Conference on Foundations of Systems Biology in Engineering (SILVA; do Val, 2016). We begin the chapter with a description of growth models used to describe the dynamics of renewable resources in Biology, and a discussion on uncertainties in fisheries management. We then recall previous papers where control theory is employed to study renewable resources, and justify our belief that the CVIU approach is a good tool to address uncertainties in the field.

5.2 Growth models in Biology

The growth of a single species population, as observed in nature, often presents an exponential trait, which has led to the modeling of growth of biological populations as an exponential process. The rate at which the population grows, however, is not constant [(MURRAY, 2002)]: in part due to a limited availability of supplies in the environment, the growth dynamics of a single species population resembles a self-limiting process. As the population size increases and approaches the so called environmental carrying capacity, the growth rate slows down, which can be explained in terms of intra-species competition and dwindling environmental resources. Based on these observations, the mathematical representation of the growth of a single species population resembles a sigmoid curve, and the logistic, Richards and Gompertz functions are often used to approximate this self-limiting growth dynamics.

In order to capture the growth dynamics of a population with size controlled by a harvesting process, denote the total biomass amount — or the total size of the population — at a time instant by $Z(t)$ and suppose that population is harvested at a varying rate $h(t)$. Moreover, suppose that the estimation of the current stock size is affected by a random disturbance. In this case we get the following stochastic version of a differential logistic equation,

$$dZ(t) = \left(r \left(1 - \frac{Z(t)}{K} \right) Z(t) - h(t) \right) dt + \sigma dW(t) \quad (5.1)$$

in which $K \geq 0$ is the carrying capacity of the environment, usually determined by the available resources [(MURRAY, 2002)], and $r \geq 0$, the slope of the curve at $Z = 0$, is

called the intrinsic growth rate. In a similar fashion, we can write a stochastic version of the Gompertz equation as

$$dZ(t) = \left(\alpha \log \left(\frac{K}{Z(t)} \right) Z(t) - h(t) \right) dt + \sigma dW(t). \quad (5.2)$$

As we previously mentioned, both functions exhibit a dependence of the growth rate on the population size. The Gompertz equation, however, is not symmetric, while the logistic function is.

5.3 Uncertainties in fisheries management

According to reports from the Fisheries Department of the Food and Agriculture Organization of the United Nations (FAO, 2016) and nature conservation groups such as WWF, the fraction of world fisheries harvested at an unsustainable rate has increased in the last decades: in 2008, for example, 32.5% of commercial stocks were overfished, an amount which slightly decreased to 28.8% in 2011. In that same year about 61.3% of the stocks were fully fished, and only 9.9% of the stocks were considered underfished. Meanwhile the world per capita apparent fish consumption increased from an average of 9.9kg in the 1960s to 18.9kg in 2010 (SILVA *et al.*, 2016).

These numbers highlight the importance of studying optimal fisheries management techniques which take the conservation of fish stocks into account, as studied previously in (CLARK; KIRKWOOD, 1986), (ROUGHGARDEN; SMITH, 1996) and (SETHI *et al.*, 2005), to name a few samples. As a matter of fact, many governments around the world actively seek to maintain the fisheries harvest within a sustainable range, and this is usually done by establishing seasonal harvesting quotas. However, as previously pointed out in (ROUGHGARDEN; SMITH, 1996), uncertainties regarding the model and monitoring of fisheries dynamics may make the use of such harvest quotas ineffective. Take the case of Newfoundland's cod fishery collapse in Canada, for example [(WALTERS; MAGUIRE, 1996)]: despite the harvesting quotas set by Canada's fishery authorities being constantly obeyed by Newfoundland's fishers, the fishery still collapsed — and it wasn't until recently that the cod stocks in the area have started to rebound (ROSE; ROWE, 2015). In Brazil, according to a study published by the Ministry of Environment [(MMA, 2006)], 80% of the main fisheries were fully exploited, overfished or in recovery process in the early 2000s. The overfishing of Brazilian stocks may be aggravated by a loose inspection and monitoring of the harvest practices in the Brazilian coast and a lack of coordination between distinct government agencies responsible for the implementation of fishing policies. As reported in (AZEVEDO; PIERRI, 2014), for example, within the 2010 mullet fishing season, the Ministry of Fisheries and Aquaculture authorized 89 licences of fishing boats, despite IBAMA, the Brazilian Institute of Environment and Renewable Resources, having advocated for a maximum of 60 licences. Another example happened

in 2011, when the Ministry of Fisheries and Aquaculture issued new sardine fishing licences even though a previous decree by IBAMA forbade the admittance of new sardine fishing boats.

As stressed in previous works such as (ROUGHGARDEN; SMITH, 1996) and (SETHI *et al.*, 2005), environmental variability, inadequate estimation of stock sizes and management uncertainty such as those derived from a loose government coordination of the exploitation of renewable resources, as outlined above, are all possible causes of the collapse of the canadian cod and other fisheries around the world. These authors propose therefore harvest management techniques which aim to openly deal with such uncertainties. In this sense, (SETHI *et al.*, 2005) design a mathematical model comprising the three sources of uncertainties mentioned before, modeled as independent and uniformly distributed random variables, whereas (ROUGHGARDEN; SMITH, 1996) argue for a shift of the optimal harvesting point in order to operate around a more biologically stable stock level. Using the logistic function as a model for the growth of the total fish biomass, $Z(t)$, we get $dZ(t) = (r(1 - Z(t)/K)Z(t) - h(t))dt$, where K is the environment carrying capacity. The maximum sustainable yield (MSY), that is, the theoretical largest yield that can be harvested from a renewable resource such that the stock levels are still able to rebound, would then correspond to $Z(t) = K/2$. This, however, is an unstable equilibrium point and, as shown in (ROUGHGARDEN; SMITH, 1996), unexpected shocks to the environment or demand may cause the exploited fishery to collapse. Hence the paper proposes that fisheries should be harvested at a higher stock level, $Z(t) = 3K/4$, which provides lower immediate financial return but increases the stability of fish stocks in the long run.

Further models which take uncertainties into account can be found in the literature. One of the first papers to treat the issue of stochastic models in harvesting problems is (REED, 1978). In that paper the author considers a general, discrete-time Markov process multiplied by random variables with a distribution function on a finite interval around 1, and analyzes constant effort and constant catch policies. In (REED, 1979) the author revisits the previous stochastic model to analyze constant escapement harvest policies, where there is an attempt to maintain the stock size around some constant level (NATIONAL RESEARCH COUNCIL, 1998). The paper adopts a discounted revenue function. The same model is studied in (CLARK; KIRKWOOD, 1986), but the authors consider the case in which the stock assessment is uncertain. As usual, the objective of the problem is to maximize the expected, discounted revenue of future harvests. In a continuous-time setting, a stochastic version of the logistic model is proposed in (LUNGU; OKSENDAL, 1997), in which the intrinsic growth rate of the logistic equation is corrupted by white noise. The authors investigate the solution of the resulting stochastic differential equation, and the stochastic control problem makes use of a discounted cost functional. Another stochastic version of a logistic model is presented in (ALVAREZ;

SHEPP, 1998), in which the authors introduce a state-dependent stochastic noise. The problem adopts a discounted cost functional. A multivariable version of the harvest problem is treated in (TRAN; YIN, 2014). The interaction between different species is modeled after a Lotka-Volterra equation with a hidden Markov chain and the observation process is corrupted by white noise. The authors study feedback controls for permanence or extinction of the modeled species. In their subsequent paper (TRAN; YIN, 2015), the authors consider a competitive stochastic Lotka-Volterra model in which the competing species can be harvested. The multidimensional model considers Markovian switching and additive Gaussian white noise. Multispecies ecosystems modeled as stochastic systems with Markovian switching and observation white noise are also considered in (TRAN; YIN, 2016).

5.4 Optimal harvesting problem

Under this discussion on uncertainties in renewable biological systems, we can conclude that a sustainable fisheries management plan should include the analysis of uncertainties affecting the model chosen to portray the dynamics of the fish stock as well as the difficulties in implementing the desired actions. In this sense, we can see the problem of fisheries management with an uncertain model as the problem of managing a renewable resource with poorly known growth dynamics, subject to an also badly known consumption or harvest rate. Throughout this chapter, we assume that the growth dynamics of the exploited resource is roughly described by a logistic (5.1) or Gompertz (5.2) model with rather uncertain parameters, that is, the value of the carrying capacity K and the intrinsic growth rate r is not exactly known. Due to our earlier analysis on the use of the CVIU approach to deal with uncertain stochastic systems, and the existence of state- and control- dependent stochastic noise which could represent parametric and policy uncertainties, we discuss in the following how to design a management policy based on the CVIU approach to control the harvest rate of a fishery.

As we discussed in previous chapters, the design of a CVIU controller starts with a locally valid, linearized model of the system to be controlled. We then linearize the equations describing the logistic (5.1) and Gompertz (5.2) models around an equilibrium point (Z_e, h_e) . For that, let us take the variables of the local model as $x(t) = Z(t) - Z_e$ and $u(t) = h(t) - h_e$. In the logistic case this leads to

$$dX(t) = \left(r \left(1 - \frac{2Z_e}{K} \right) x(t) - u(t) \right) dt + \sigma dW(t), \quad (5.3)$$

whereas the local linear model of the Gompertz equation takes the form

$$dX(t) = \left(\alpha \left[\log \left(\frac{K}{Z_e} \right) - 1 \right] x(t) - u(t) \right) dt + \sigma dW(t). \quad (5.4)$$

There exist parametric uncertainties about the nominal model of the system, however, and in the CVIU approach we can represent those by introducing state-dependent and additive white noise into the linear model,

$$dX(t) = \left(r \left(1 - \frac{2Z_e}{K} \right) x(t) - u(t) \right) dt + \bar{\sigma}_x |x(t)| dW^x(t) + \bar{\sigma}_u |u(t)| dW^u(t) \\ + \sigma dW(t) + \sigma_x dW^x(t) + \sigma_u dW^u(t),$$

and in this sense the diffusion coefficient $\bar{\sigma}_x$ can be tuned to describe statistically the expected distribution of the parametric errors. In a similar fashion for the local representation of the Gompertz equation, we get

$$dX(t) = \left(\alpha \left[\log \left(\frac{K}{Z_e} \right) - 1 \right] x(t) - u(t) \right) dt + \bar{\sigma}_x |x(t)| dW^x(t) + \bar{\sigma}_u |u(t)| dW^u(t) \\ + \sigma dW(t) + \sigma_x dW^x(t) + \sigma_u dW^u(t).$$

The control-dependent stochastic noise, on the other hand, can be used to represent the inaccurate implementation of harvest policies or management uncertainty (SETHI *et al.*, 2005).

We evaluate the performance of the controller by adopting a quadratic discounted cost over an infinite time horizon. This is equivalent to assume that the operator wishes to minimize the control effort while striving to maintain the system around the equilibrium point. Moreover, this also means that future revenues are discounted to account for inflation rate, for example. In this case we can then formulate the corresponding optimal stochastic control problem as

$$\min \mathbb{E} \int_0^\infty e^{-\alpha t} (x^\top(t) Q x(t) + u^\top(t) R u(t)) dt, \quad (5.5)$$

$$\text{s.t. } dX(t) = F(x(t), u(t)) dt + \hat{\sigma}(t) d\hat{W}(t), \quad (5.6)$$

where $F(x(t), u(t))$ is replaced by (5.3) or (5.4) according to the chosen growth model.

With the corresponding CVIU models we can design a control strategy as presented in chapter 2. In the scalar case, the optimal CVIU cost function is symmetric with respect to the origin, and, starting from the inaction region, where the optimal control policy for the linear model means $u^* = 0$ and the value function has the form (2.70)

$$V^*(x) = x^\top X x + \frac{1}{\alpha} \text{tr} (X (\sigma \sigma^\top + \sigma_x \sigma_x^\top + \sigma_u \sigma_u^\top)),$$

we use a numerical procedure to approximate the value function from the boundary point of the inaction region, x_b , until a large enough x_∞ . From x_∞ onward, the asymptotic approximation for the value function (3.15),

$$V_a^{\bar{s}} := x^\top X x + \langle v(\bar{s}), x \rangle + l(\bar{s}),$$

can be considered valid. From the value function, we can calculate the optimal control policy for the entire state space, knowing that it assumes the equilibrium value h_e inside the inaction region and tends asymptotically to a state-feedback form.

5.5 Numerical experiments

We now discuss some numerical examples related to uncertainties such as those discussed in the introduction of this chapter to illustrate the use of a controller based on the CVIU approach in this setting. The first examples were previously presented at (SILVA *et al.*, 2016).

Our first analysis in that paper, and in the subsequent one (SILVA; do Val, 2016), was concerned with the relative performance of the CVIU approach when compared with other methods from the control systems literature. The performance of the proposed control strategy was then compared with that of an optimal LQG controller, designed according to the nominal system dynamics. The actual system, as described by (5.1), was simulated via Monte Carlo evaluations with 50 simultaneous repetitions and a time horizon of 200 seconds. The stochastic differential equation (5.1) was numerically computed by the Euler-Maruyama method (HIGHAM, 2001). The dynamics of the harvested population was described by a logistic (5.1) equation with nominal parameters $K = 100$ and $r = 0.3$. Moreover, we assumed the system to operate around an equilibrium point $x_e = 3\frac{K}{4}$, as proposed in (ROUGHGARDEN; SMITH, 1996).

A natural question, after a long discussion on the issue of uncertainties in fisheries management, is to analyze how the system behaves when the parameters on which the design of the controllers is based differ from the actual values of the system. That is, imagine that the fishery manager designs the harvest policy based on estimated parameters of the system, say $K = 90$ and $r = 0.35$, for example, but that the actual system operates according to the model with nominal parameters $K^0 = 100$ and $r^0 = 0.3$. Moreover, consider that the system dynamics is affected by a stochastic disturbance modeled as a standard, scalar Brownian motion with diffusion coefficient σ — a type of nature noise in the CVIU model. For that we, following a numerical experiment proposed in (SOUTO *et al.*, 2013) and (VAL; SOUTO, 2017), evaluate the performance of the controller as the mismatch $\delta = A^0 - A$ between the actual value of the parameter $A^0 = r^0 \left(1 - \frac{2Z_e}{K^0}\right)$, for the local linear model, and the estimated value $Ar \left(1 - \frac{2Z_e}{K}\right)$ increases. The relative gain of the CVIU control cost with relation to the LQG controller is shown in Figure 12. The plot shows the relative gain as the parameters $\bar{\sigma}_x$ and $\lambda_u = \frac{\sigma_u}{\bar{\sigma}_u}$, which can be seen as a sort of design variable in the CVIU case, vary. Here a maximum gain of 13.75% was observed.

We also simulated how fluctuations in the stock levels, modulated by the diffusion coefficient σ , impact on the controllers performance. For that we assume there is a 20% difference between the parameter A of the local linearized system and the estimate available during the design phase. Results are shown in figures 13 and 14. In both cases, $\sigma = 1$ or $\sigma = 2$, we observed a maximum gain of about 14% of the CVIU controller.

Following the arguments of (ROUGHGARDEN; SMITH, 1996) and (SETHI *et*

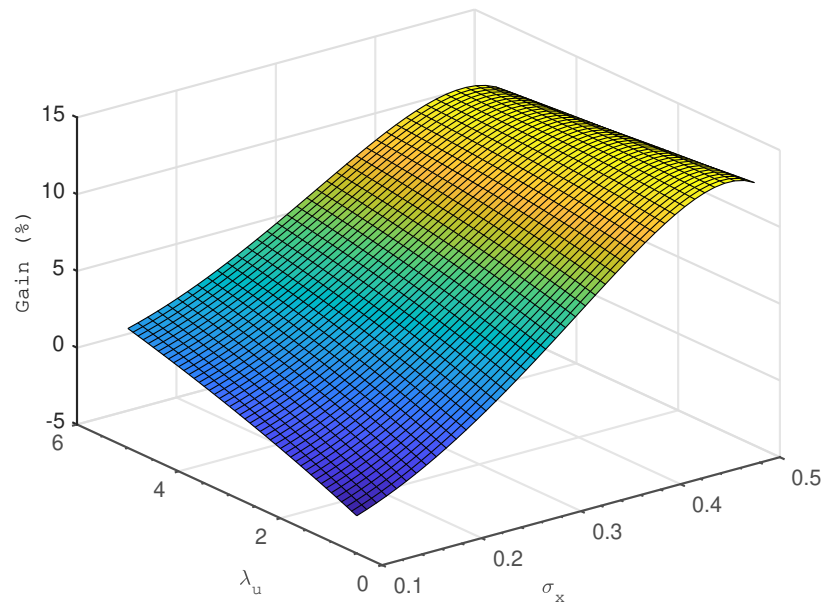


Figure 12 – A 50% mismatch between the estimated parameter A and the actual system local approximation A^0 . Figure updated from (SILVA *et al.*, 2016).

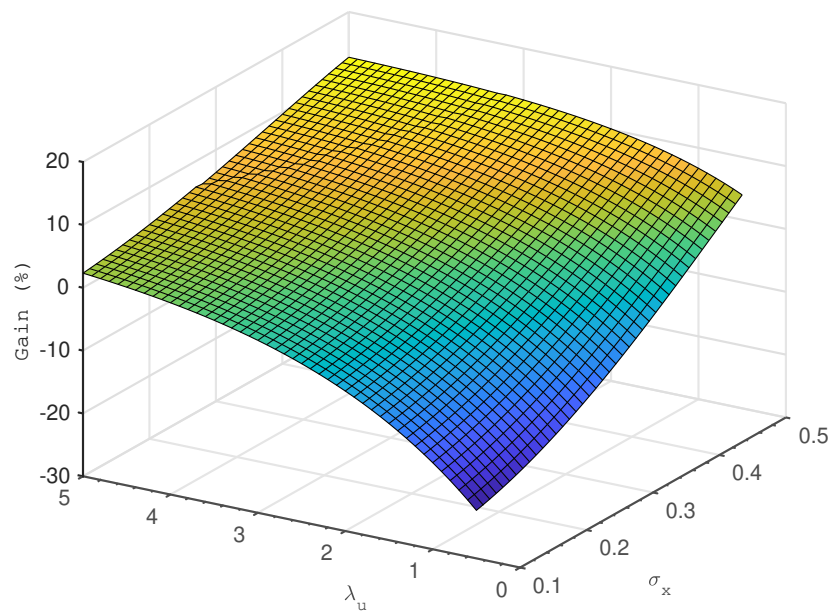


Figure 13 – Relative gains when $\sigma = 1$. Figure updated from (SILVA *et al.*, 2016).

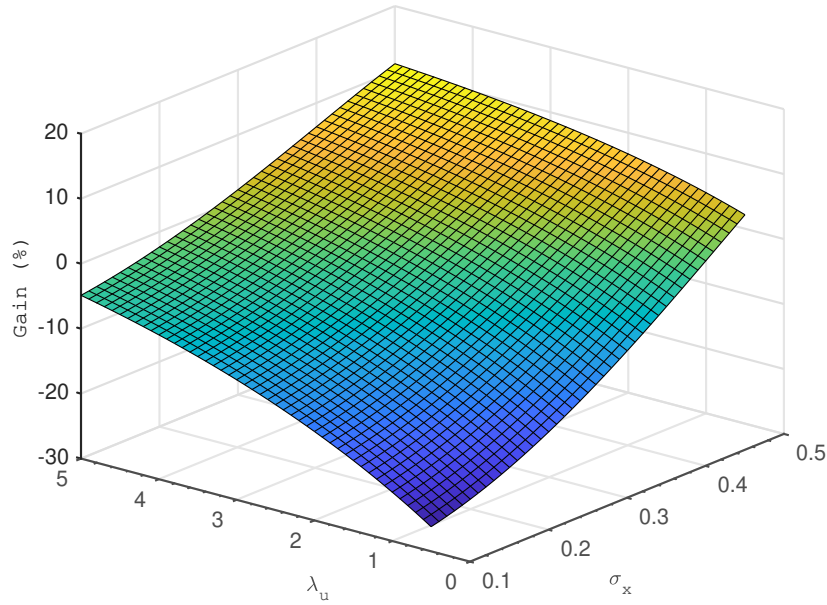


Figure 14 – Relative gains when $\sigma = 2$. Figure updated from (SILVA *et al.*, 2016).

al., 2005), we next evaluate how measurement errors affect the controlled harvest. The stock level assessment error was modeled as a random variable with uniform distribution $[1 - \delta, 1 + \delta]$ which is multiplied by the actual stock level. In Table 1 the relative gains of the CVIU approach when compared to the LQG controller are shown for $\delta = 0.1$ and $\delta = 0.2$.

Table 1 – Simulation results: CVIU gains (%) for stock level assessment errors

δ	$\bar{\sigma}_x$	min	mean	max
0.1	0.1	-18.27	-5.30	2.32
	0.2	-15.47	-3.0	4.20
	0.3	-10.91	0.72	6.85
	0.4	-4.35	5.76	10.80
	0.5	4.17	11.51	14.68
0.2	0.1	-17.95	-5.27	2.22
	0.2	-15.64	-3.01	4.21
	0.3	-10.53	0.81	6.69
	0.4	-4.47	5.74	10.55
	0.5	4.17	11.56	14.46

As presented in the works (SETHI *et al.*, 2005) and (ROUGHGARDEN; SMITH, 1996), another uncertainty source consists of harvest implementation errors. Moreover, examples such as those of the Canadian cod collapse and the disagreements between different government agencies [(AZEVEDO; PIERRI, 2014)] reinforce the perception that an evaluation of a possible mismatch between the attempted and the actual harvest might be interesting. Hence we simulated how the controlled system reacts to such disturbances.

Similarly to the previous simulation, the attempted control action is multiplied by a random variable with uniform distribution $[1 - \delta, 1 + \delta]$. Simulation results are shown in Table 2.

Table 2 – Simulation results: CVIU gains (%) for implementation errors

δ	$\bar{\sigma}_x$	min	mean	max
0.1	0.1	-18.22	-5.41	2.27
	0.2	-15.68	-3.17	3.84
	0.3	-11.08	0.57	6.72
	0.4	-4.39	5.59	10.36
	0.5	3.98	11.32	14.66
0.2	0.1	-18.30	-5.73	1.62
	0.2	-15.91	-3.47	3.63
	0.3	-10.95	0.29	6.17
	0.4	-4.99	5.25	10.29
	0.5	3.32	10.89	13.93

The numerical results presented in Tables 1 and 2 show the CVIU controller, when calibrated by the parameters $\bar{\sigma}_x$ and λ_u , performs better than the traditional LQG controller in the aforementioned scenarios, which were thought so as to mirror the uncertainties in the fisheries management literature.

6 Conclusion

This thesis brings results collected during the period the student spent as a master's student at the University of Campinas, under supervision from Prof. do Val. Broadly speaking, the underlying theme of the work presented here was to characterize the CVIU approach as an alternative method to control uncertain stochastic systems. Having that in mind, our effort focused on the solution of the control problem by adopting a different optimization criterion, on the one hand, and relations between the proposed approach and methods traditionally employed to design controllers tolerant to uncertainties, and on possible applications of the CVIU approach on the other.

The CVIU control problem: the discounted and long run average cost formulations

As we mentioned in the introduction, the CVIU approach first dealt with a discrete-time model and a discounted cost control problem, and the results are presented in (PIN *et al.*, 2009). Following that first conference paper, Prof. do Val and Dr. Souto started to work on a continuous-time version of the CVIU model, which culminated in the paper (VAL; SOUTO, 2017), where the continuous-time, discounted cost CVIU problem is discussed in details. Starting from the solution to the discounted cost problem, we began to analyze possible relations of the structure of uncertainty representation in the CVIU approach with models used in the analysis of robust control of stochastic systems. In order to compare the different performance criteria used in the CVIU and robust control problems, however, it was necessary to develop a mathematical tool which would allow us to express the discounted cost functional in terms of the \mathcal{H}_2 norm of an auxiliary stochastic system, as discussed in the conference paper (SILVA *et al.*, 2017) and in chapter 4. This tool, on the other hand, resembles a form of the so-called vanishing discount approach, a typical approach for the solution of long run average stochastic control problems in terms of a sequence of corresponding discounted cost problems, which are somewhat easier to solve. This was the starting point of our discussion on a possible solution of the CVIU control problem when a different optimization criteria is used. On the one hand, the basic structure of the CVIU approach should remain the same, since the existence of the inaction region does not depend on the particular cost structure used. On the other hand, by adopting a different optimization criteria, we are enriching the CVIU approach in the sense that we can now use different solutions of the corresponding CVIU control policy for different objectives: if we want to prioritize short-term optimization, then we can keep using the discounted cost solution; if our problem otherwise calls for a steady state solution, then we can now use the long run average case as presented in chapter

3. Furthermore, the CVIU long run average control problem can be directly related to models used in robust control of stochastic systems, as we discuss in the following.

The CVIU approach and robust control of stochastic systems

As we mentioned in the introduction, we first began investigating relations between the CVIU approach and robust stochastic control in our previous conference paper (SILVA *et al.*, 2017). We previously started from the discounted cost solution to the CVIU control problem, and connected the discounted and \mathcal{H}_2 cost functionals via an auxiliary system. Here, on the other hand, we study the relationship between the CVIU approach and \mathcal{H}_2 and \mathcal{H}_∞ robust control problems via the stochastic \mathcal{H}_2 norm. Recall that the stochastic \mathcal{H}_2 norm can be formulated in terms of an average running cost structure (DRAGAN *et al.*, 2006), and a simple relation between the quadratic cost matrices Q and R of the quadratic, long run average stochastic control problem, and the observation matrices C and D of a linear stochastic system can yield a direct relation between the robust and stochastic control problems. The solution of the CVIU long run average problem can then be (directly) related to the stochastic \mathcal{H}_2 control problem. Furthermore, the simulation results we presented at that conference were based on the comparison between the CVIU model and a polytopic approach to robust control. A comparison between the CVIU approach and robust methods specifically aimed at the control of stochastic systems, such as those presented in (DRAGAN *et al.*, 2006) and (GHAOUI, 1995), however, might be more adequate. This point was discussed with more details in chapter 4.

Applications

The motivation to use the CVIU approach to design adequate management policies for fisheries stems from the fact that uncertainties are a recurrent topic in the fisheries literature and that the CVIU approach is designed to deal with stochastic uncertain systems, on the one hand, and the presence of the inaction region and its relation with cautious management practices which may be more stable under a biological point of view, on the other. That is the point we make in our previous conference papers (SILVA *et al.*, 2016) and (SILVA; do Val, 2016), and which we discussed in chapter 5. Those papers, however, are largely based on a control theory point of view — we adopt a quadratic discounted cost formulation, for example, and do not evaluate the economic rate of return of the controlled fishery. The topic is explored in more details in Prof. do Val's paper with Prof. Guillotreau and Prof. Vallee (VAL *et al.*, 2018), where the authors also discuss the economic return of fisheries exploited according to a harvest policy based on the CVIU approach.

Discrete-time

Our work has dealt primarily with continuous-time stochastic systems, but Prof. do Val's group has also been discussing similar points for the discrete-time version of the CVIU approach, check for example the papers (PEDROSA *et al.*, 2017) and (PIN *et al.*, 2009).

References

- ALVAREZ, L. H. R.; SHEPP, L. A. Optimal harvesting of stochastically fluctuating populations. *Journal of Mathematical Biology*, v. 37, n. 2, p. 155–177, Aug 1998. ISSN 1432-1416. Available from Internet: <<https://doi.org/10.1007/s002850050124>>. Cited in page 98.
- ARAPOSTATHIS, A.; BORKAR, V. S.; GHOSH, M. K. *Ergodic Control of Diffusion Processes*. Cambridge: Cambridge University Press, 2011. (Encyclopedia of Mathematics and its Applications). Cited 2 times in pages 60 and 61.
- ARNOLD, L. *Stochastic Differential Equations: theory and applications*. [S.l.]: John Wiley, 1974. Cited 3 times in pages 32, 33, and 34.
- ÅSTRÖM, K. J.; KUMAR, P. Control: A perspective. *Automatica*, v. 50, n. 1, p. 3 – 43, 2014. Available from Internet: <<http://www.sciencedirect.com/science/article/pii/S0005109813005037>>. Cited in page 15.
- ÅSTRÖM, K. J.; MURRAY, R. M. *Feedback systems : an introduction for scientists and engineers*. Princeton, NJ: Princeton University Press, 2008. Available from Internet: <<http://resolver.caltech.edu/CaltechBOOK:2008.003>>. Cited in page 15.
- AZEVEDO, N.; PIERRI, N. A política pesqueira no brasil (2003-2011): a escolha pelo crescimento produtivo e o lugar da pesca artesanal. *Desenvolvimento e Meio Ambiente*, v. 32, p. 61–80, 2014. Available from Internet: <<https://revistas.ufpr.br/made/article/view/35547>>. Cited 2 times in pages 96 and 102.
- BARTLE, R. *The Elements of Integration*. [S.l.]: John Wiley & Sons Inc., 1966. Cited in page 58.
- BAŞAR, T.; BERNHARD, P. *H_∞ -Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach*. 2nd ed.. ed. [S.l.]: Birkhäuser, 2008. Cited 2 times in pages 85 and 120.
- BERMAN, N.; SHAKED, U. H_∞ -like control for nonlinear stochastic systems. *Systems & Control Letters*, v. 55, n. 3, p. 247 – 257, 2006. ISSN 0167-6911. Available from Internet: <<http://www.sciencedirect.com/science/article/pii/S0167691105001210>>. Cited in page 94.
- BERMAN, N.; SHAKED, U. H_∞ control for non-linear stochastic systems: the output-feedback case. *International Journal of Control*, Taylor & Francis, v. 81, n. 11, p. 1733–1746, 2008. Available from Internet: <<https://doi.org/10.1080/00207170701840136>>. Cited in page 94.
- BILLINGSLEY, P. *Probability and Measure*. 3rd ed.. ed. New Jersey, United States: John Wiley & Sons, 1995. (Wiley Series in Probability and Mathematical Statistics). Cited 2 times in pages 26 and 62.
- BORKAR, V. S. A convex analytic approach to markov decision processes. *Probability Theory and Related Fields*, v. 78, n. 4, p. 583–602, Aug 1988. ISSN 1432-2064. Available from Internet: <<https://doi.org/10.1007/BF00353877>>. Cited in page 60.

- BORKAR, V. S. Convex analytic methods in markov decision processes. In: _____. *Handbook of Markov Decision Processes: Methods and Applications*. Boston, MA: Springer US, 2002. p. 347–375. ISBN 978-1-4615-0805-2. Available from Internet: <https://doi.org/10.1007/978-1-4615-0805-2_11>. Cited in page 60.
- BORKAR, V. S. Controlled diffusion processes. *Probability Surveys*, The Institute of Mathematical Statistics and the Bernoulli Society, v. 2, p. 213–244, 2005. Available from Internet: <<https://projecteuclid.org/euclid.ps/1127136330>>. Cited 3 times in pages 20, 36, and 78.
- BOYD, S.; GHAOUI, L. E.; FERON, E.; BALAKRISHNAN, V. *Linear Matrix Inequalities in System and Control Theory*. Society for Industrial and Applied Mathematics, 1994. Available from Internet: <<https://web.stanford.edu/~boyd/lmibook/lmibook.pdf>>. Cited in page 77.
- CHANDRASEHKARAN, P. *Robust Control of Linear Dynamical Systems*. [S.l.]: Academic Press, 1996. Cited in page 16.
- CHEN, G.; CHEN, G.; HSU, S.-H. *Linear Stochastic Control Systems*. [S.l.]: CRC Press, 1995. Cited in page 45.
- CLARK, C. W.; KIRKWOOD, G. P. On uncertain renewable resource stocks: Optimal harvest policies and the value of stock surveys. *Journal of Environmental Economics and Management*, v. 13, n. 3, p. 235 – 244, 1986. ISSN 0095-0696. Available from Internet: <<http://www.sciencedirect.com/science/article/pii/0095069686900240>>. Cited 2 times in pages 96 and 97.
- CLARKE, F. *Optimization and Nonsmooth Analysis*. Society for Industrial and Applied Mathematics, 1990. Available from Internet: <<https://epubs.siam.org/doi/abs/10.1137/1.9781611971309>>. Cited in page 41.
- COLANERI, P.; GEROMEL, J.; LOCATELLI, A. *Control Theory and Design: an RH_2 and RH_∞ viewpoint*. [S.l.]: Academic Press, 1997. Cited in page 59.
- COSTA, E. F.; VAL, J. B. R. do. On the observability and detectability of continuous-time markov jump linear systems. *SIAM Journal on Control and Optimization*, v. 41, n. 4, p. 1295–1314, 2002. Available from Internet: <<https://doi.org/10.1137/S0363012901385460>>. Cited in page 65.
- COSTA, O.; VAL, J. do; GEROMEL, J. Continuous-time state-feedback H_2 -control of markovian jump linear systems via convex analysis. *Automatica*, v. 35, n. 2, p. 259 – 268, 1999. ISSN 0005-1098. Available from Internet: <<http://www.sciencedirect.com/science/article/pii/S0005109898001459>>. Cited 2 times in pages 77 and 79.
- COSTA, O. L.; FRAGOSO, M. D.; MARQUES, R. P. *Discrete-Time Markov Jump Linear Systems*. Springer, 2005. (Probability and its Applications). Available from Internet: <<https://link.springer.com/book/10.1007%2Fb138575>>. Cited in page 79.
- COSTA, O. L.; FRAGOSO, M. D.; TODOROV, M. G. *Continuous-Time Markov Jump Linear Systems*. Springer, 2013. (Probability and its Applications). Available from Internet: <<https://link.springer.com/book/10.1007%2F978-3-642-34100-7>>. Cited 5 times in pages 20, 79, 82, 84, and 85.

DAMM, T.; BENNER, P.; HAUTH, J. Computing the stochastic H^∞ -norm by a newton iteration. *IEEE Control Systems Letters*, v. 1, n. 1, p. 92–97, July 2017. ISSN 2475-1456. Available from Internet: <<https://doi.org/10.1109/LCSYS.2017.2707409>>. Cited in page 94.

DOYLE, J. C.; FRANCIS, B. A.; TANNENBAUM, A. R. *Feedback Control Theory*. Prentice Hall Professional Technical Reference, 1991. Available from Internet: <<http://www.control.utoronto.ca/people/profs/francis/dft.pdf>>. Cited in page 116.

DRAGAN, V.; MOROZAN, T.; STOICA, A.-M. *Mathematical Methods in Robust Control of Linear Stochastic Systems*. Springer-Verlag, 2006. Available from Internet: <<https://link.springer.com/book/10.1007%2F978-1-4614-8663-3>>. Cited 14 times in pages 57, 59, 65, 77, 79, 81, 82, 83, 84, 85, 89, 93, 94, and 105.

DRĂGAN, V.; MOROZAN, T.; STOICA, A.-M. *Mathematical Methods in Robust Control of Discrete-Time Linear Stochastic Systems*. New York, NY: Springer New York, 2010. ISBN 978-1-4419-0630-4. Available from Internet: <https://doi.org/10.1007/978-1-4419-0630-4_1>. Cited in page 79.

FAO. *The State of World Fisheries and Aquaculture 2014*. [S.l.], 2016. Available from Internet: <<http://www.fao.org/3/a-i5555e.pdf>>. Cited in page 96.

FARIAS, D. P. D.; GEROMEL, J. C.; VAL, J. B. R. D.; COSTA, O. L. V. Output feedback control of Markov jump linear systems in continuous-time. *IEEE Transactions on Automatic Control*, v. 45, n. 5, p. 944–949, May 2000. ISSN 0018-9286. Available from Internet: <<https://doi.org/10.1109/9.855557>>. Cited in page 79.

GHAOUI, L. E. State-feedback control of systems with multiplicative noise via linear matrix inequalities. *Systems & Control Letters*, v. 24, n. 3, p. 223 – 228, 1995. Available from Internet: <[https://doi.org/10.1016/0167-6911\(94\)00045-W](https://doi.org/10.1016/0167-6911(94)00045-W)>. Cited 3 times in pages 79, 84, and 105.

GOZZI, F.; SWIECH, A.; ZHOU, X. Y. A corrected proof of the stochastic verification theorem within the framework of viscosity solutions. *SIAM Journal on Control and Optimization*, v. 43, n. 6, p. 2009–2019, 2005. Available from Internet: <<https://doi.org/10.1137/S0363012903428184>>. Cited 2 times in pages 43 and 44.

GOZZI, F.; SWIECH, A.; ZHOU, X. Y. Erratum: “A Corrected Proof of the Stochastic Verification Theorem within the Framework of Viscosity Solutions”. *SIAM Journal on Control and Optimization*, v. 48, n. 6, p. 4177–4179, 2010. Available from Internet: <<https://doi.org/10.1137/090775567>>. Cited 2 times in pages 41 and 44.

GU, D.-W.; PETKOV, P. H.; KONSTANTINOV, M. M. *Robust Control Design with MATLAB®*. London: Springer London, 2013. ISBN 978-1-4471-4682-7. Available from Internet: <https://doi.org/10.1007/978-1-4471-4682-7_2>. Cited in page 18.

HAUSSMANN, U. G. Optimal stationary control with state control dependent noise. *SIAM Journal on Control*, v. 9, n. 2, p. 184–198, 1971. Available from Internet: <<https://doi.org/10.1137/0309016>>. Cited in page 78.

HAUSSMANN, U. G. Stability of Linear Systems with Control Dependent Noise. *SIAM Journal on Control*, v. 11, n. 2, p. 382–394, 1973. Available from Internet: <<https://doi.org/10.1137/0311030>>. Cited in page 78.

- HIGHAM, D. An algorithmic introduction to numerical simulation of stochastic differential equations. *SIAM Review*, v. 43, n. 3, p. 252–246, 2001. Available from Internet: <<https://doi.org/10.1137/S0036144500378302>>. Cited 3 times in pages 33, 88, and 100.
- HINRICHSSEN D. PRITCHARD, J. Stochastic H^∞ . *SIAM Journal on Control and Optimization*, v. 36, n. 5, p. 1504–1538, 1998. Available from Internet: <<https://doi.org/10.1137/S0363012996301336>>. Cited 6 times in pages 77, 79, 84, 85, 93, and 94.
- HOU, Z.-S.; WANG, Z. From model-based control to data-driven control: Survey, classification and perspective. *Information Sciences*, v. 235, p. 3 – 35, 2013. ISSN 0020-0255. Data-based Control, Decision, Scheduling and Fault Diagnostics. Available from Internet: <<http://www.sciencedirect.com/science/article/pii/S0020025512004781>>. Cited 2 times in pages 16 and 21.
- ITÔ, K. Stochastic integral. *Proc. Imp. Acad.*, The Japan Academy, v. 20, n. 8, p. 519–524, 1944. Available from Internet: <<https://doi.org/10.3792/pia/1195572786>>. Cited 2 times in pages 28 and 30.
- KARATZAS, I.; SHREVE, S. E. *Brownian Motion and Stochastic Calculus*. Springer-Verlag, 1991. Available from Internet: <<https://link.springer.com/book/10.1007%2F978-1-4612-0949-2>>. Cited 6 times in pages 28, 29, 30, 31, 32, and 34.
- KHASMINSKII, R. *Stochastic Stability of Differential Equations*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2012. ISBN 978-3-642-23280-0. Available from Internet: <https://doi.org/10.1007/978-3-642-23280-0_1>. Cited 2 times in pages 62 and 63.
- KLEBANER, F. C. *Introduction to Stochastic Calculus with Applications*. 3rd ed.. ed. London: Imperial College Press, 2012. Cited 9 times in pages 26, 27, 28, 29, 30, 31, 32, 33, and 34.
- KUO, H.-H. *Introduction to Stochastic Integration*. Springer-Verlag, 2006. Available from Internet: <<https://link.springer.com/book/10.1007%2F0-387-31057-6>>. Cited 5 times in pages 20, 27, 29, 58, and 59.
- KUSHNER, H. J. A partial history of the early development of continuous-time nonlinear stochastic systems theory. *Automatica*, v. 50, p. 303–334, 2014. Available from Internet: <<https://doi.org/10.1016/j.automatica.2013.10.013>>. Cited in page 16.
- LAMNABHI-LAGARRIGUE, F.; ANNASWAMY, A.; ENGELL, S.; ISAKSSON, A.; KHARGONEKAR, P.; MURRAY, R. M.; NIJMEIJER, H.; SAMAD, T.; TILBURY, D.; HOF, P. V. den. Systems & control for the future of humanity, research agenda: Current and future roles, impact and grand challenges. *Annual Reviews in Control*, v. 43, p. 1 – 64, 2017. ISSN 1367-5788. Available from Internet: <<http://www.sciencedirect.com/science/article/pii/S1367578817300573>>. Cited in page 15.
- LOEHLE, C. Control theory and the management of ecosystems. *Journal of Applied Ecology*, n. 43, p. 957–966, 2006. Available from Internet: <<https://doi.org/10.1111/j.1365-2664.2006.01208.x>>. Cited 2 times in pages 22 and 23.

LÖFBERG, J. YALMIP: A toolbox for modeling and optimization in MATLAB. In: *Proc. 2004 IEEE Int. Symp. on Comput. Aided Control Syst. Des.* Taipei, Taiwan: [s.n.], 2004. p. 284–289. Available from Internet: <<https://ieeexplore.ieee.org/document/1393890/>>. Cited in page 88.

LUKACS, E. Stochastic convergence (second edition). In: . 2nd ed.. ed. Academic Press, 1975, (Probability and Mathematical Statistics: A Series of Monographs and Textbooks). Available from Internet: <<https://www.sciencedirect.com/science/article/pii/B9780124598607500025>>. Cited in page 62.

LUNGU, E.; OKSENDAL, B. Optimal harvesting from a population in a stochastic crowded environment. *Mathematical Biosciences*, v. 145, n. 1, p. 47 – 75, 1997. ISSN 0025-5564. Available from Internet: <<http://www.sciencedirect.com/science/article/pii/S0025556497000291>>. Cited in page 97.

MAO, X. *Stability of Stochastic Differential Equations with Respect to Semimartingales*. New York: Longman Higher Education, 1991. Cited in page 62.

MAO, X. *Stochastic Differential Equations and Applications*. 2nd ed.. ed. Woodhead Publishing, 2007. ISBN 978-1-904275-34-3. Available from Internet: <<https://www.sciencedirect.com/science/article/pii/B9781904275343500013>>. Cited in page 62.

MCLANE, P. Optimal stochastic control of linear systems with state- and control-dependent disturbances. *IEEE Transactions on Automatic Control*, v. 16, n. 6, p. 793–798, Dec 1971. Available from Internet: <<https://ieeexplore.ieee.org/document/1099828/>>. Cited in page 78.

MMA. *Programa REVIZEE: Avaliação do Potencial Sustentável de Recursos Vivos na Zona Econômica Exclusiva - Relatório Executivo*. [S.l.], 2006. Available from Internet: <<http://www.mma.gov.br/biodiversidade/biodiversidade-aquatica/zona-costeira-e-marinha/programa-revizee>>. Cited in page 96.

MORIMOTO, H. *Stochastic control and mathematical modeling: applications in economics*. New York: Cambridge University Press, 2010. (Encyclopedia of Mathematics and its Applications). Cited in page 61.

MURRAY, J. *Mathematical Biology*. 3rd ed.. ed. [S.l.]: Springer-Verlag, 2002. I - An Introduction. Cited 2 times in pages 22 and 95.

NATIONAL RESEARCH COUNCIL. *Improving Fish Stock Assessments*. Washington, DC: The National Academies Press, 1998. ISBN 978-0-309-05725-7. Available from Internet: <<https://www.nap.edu/catalog/5951/improving-fish-stock-assessments>>. Cited in page 97.

NICULESCU, C. P.; PERSSON, L.-E. *Convex Functions and Their Applications*. Springer, 2006. Available from Internet: <<https://link.springer.com/book/10.1007/978-1-4020-387-3>>. Cited in page 41.

OGATA, K. *Modern Control Engineering*. 4th ed.. ed. Upper Saddle River, NJ, USA: Prentice Hall PTR, 2001. Cited in page 15.

OKSENDAL, B. *Stochastic Differential Equations: an introduction with applications*. 6th ed.. ed. [S.l.]: Springer-Verlag, 2007. Cited 2 times in pages 19 and 34.

PEDROSA, F. C.; NEREU, J. C.; VAL, J. B. R. do. Stochastic optimal control of systems for which control variation increases uncertainty: A contribution to the discrete time case. In: *2017 IEEE International Conference on Systems, Man, and Cybernetics (SMC)*. [s.n.], 2017. p. 2279–2284. Available from Internet: <<https://doi.org/10.1109/SMC.2017.8122960>>. Cited in page 106.

PETERSEN, I.; TEMPO, E. Robust control of uncertain systems: Classical results and recent developments. *Automatica*, v. 50, p. 1315 – 1335, 2014. Available from Internet: <<https://doi.org/10.1016/j.automatica.2014.02.042>>. Cited 3 times in pages 16, 18, and 19.

PHAM, H. *Continuous-time Stochastic Control and Optimization with Financial Applications*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2009. ISBN 978-3-540-89500-8. Available from Internet: <https://doi.org/10.1007/978-3-540-89500-8_1>. Cited 4 times in pages 35, 38, 40, and 45.

PIN, A. du; VALLÉE, T.; VAL, J. do. Control variation as a source of uncertainty: single input case. In: *American Control Conference*. [s.n.], 2009. Available from Internet: <<https://ieeexplore.ieee.org/document/5160556/>>. Cited 6 times in pages 21, 23, 24, 44, 104, and 106.

PRIETO-RUMEAU, T.; HERNÁNDEZ-LERMA, O. The vanishing discount approach to constrained continuous-time controlled markov chains. *Systems & Control Letters*, v. 59, n. 8, p. 504 – 509, 2010. ISSN 0167-6911. Available from Internet: <<http://www.sciencedirect.com/science/article/pii/S0167691110000769>>. Cited in page 60.

QI, T.; CHEN, J.; SU, W.; FU, M. Control Under Stochastic Multiplicative Uncertainties: Part I, Fundamental Conditions of Stabilizability. *IEEE Transactions on Automatic Control*, v. 62, n. 3, p. 1269–1284, March 2017. Available from Internet: <<https://ieeexplore.ieee.org/document/7501609/>>. Cited in page 79.

REED, W. J. The steady state of a stochastic harvesting model. *Mathematical Biosciences*, v. 41, n. 3, p. 273 – 307, 1978. ISSN 0025-5564. Available from Internet: <<http://www.sciencedirect.com/science/article/pii/002555647890041X>>. Cited in page 97.

REED, W. J. Optimal escapement levels in stochastic and deterministic harvesting models. *Journal of Environmental Economics and Management*, v. 6, n. 4, p. 350 – 363, 1979. ISSN 0095-0696. Available from Internet: <<http://www.sciencedirect.com/science/article/pii/0095069679900147>>. Cited in page 97.

ROBIN, M. Long-term average cost control problems for continuous time markov processes: A survey. *Acta Applicandae Mathematica*, v. 1, n. 3, p. 281–299, Sep 1983. ISSN 1572-9036. Available from Internet: <<https://doi.org/10.1007/BF00046603>>. Cited in page 60.

ROCKAFELLAR, R. T. *Convex Analysis*. [S.l.]: Princeton University Press, 1970. Cited in page 40.

- ROSE, G. A.; ROWE, S. Northern cod comeback. *Canadian Journal of Fisheries and Aquatic Sciences*, v. 72, n. 12, p. 1789–1798, 2015. Available from Internet: <<https://doi.org/10.1139/cjfas-2015-0346>>. Cited in page 96.
- ROUGHGARDEN, J.; SMITH, F. Why fisheries collapse and what to do about it. *Proceedings of the National Academy of Sciences*, v. 93, n. 10, p. 5078–5083, 1996. Available from Internet: <<https://www.ncbi.nlm.nih.gov/pmc/articles/PMC39409/>>. Cited 5 times in pages 23, 96, 97, 100, and 102.
- SAFONOV, M. G. Origins of robust control: Early history and future speculations. *IFAC Proceedings Volumes*, v. 45, n. 13, p. 1 – 8, 2012. ISSN 1474-6670. 7th IFAC Symposium on Robust Control Design. Available from Internet: <<http://www.sciencedirect.com/science/article/pii/S1474667015376540>>. Cited in page 19.
- SAMAD, T.; ANNASWAMY, A. *The Impact of Control Technology*. 2nd ed.. ed. [S.l.], 2014. Available from Internet: <<http://ieeecss.org/general/IoCT2-report>>. Cited in page 15.
- SETHI, G.; COSTELLO, C.; FISHER, A.; HANEMANN, M.; KARP, L. Fishery management under multiple uncertainty. *Journal of Environmental Economics and Management*, v. 50, n. 2, p. 300–318, 2005. Available from Internet: <<https://doi.org/10.1016/j.jeem.2004.11.005>>. Cited 5 times in pages 23, 96, 97, 99, and 102.
- SHENG, L.; GAO, M.; ZHANG, W.; CHEN, B.-S. Infinite horizon H_∞ control for nonlinear stochastic Markov jump systems with (x,u,v) -dependent noise via fuzzy approach. *Fuzzy Sets and Systems*, v. 273, p. 105 – 123, 2015. Available from Internet: <<https://doi.org/10.1016/j.fss.2014.10.015>>. Cited 2 times in pages 79 and 94.
- SILVA, V. L.; do Val, J. B. R. Stochastic control with poorly known biological growth models. In: *6th IFAC Conference on Foundations of Systems Biology in Engineering*. [S.l.: s.n.], 2016. Extended abstract. Cited 3 times in pages 95, 100, and 105.
- SILVA, V. L.; VAL, J. B. R. do; SOUTO, R. F. Harvesting with stochastic control: when parameters are badly known. In: *XXI Brazilian Congress on Automatic Control*. Vitória, Brazil: [s.n.], 2016. Available from Internet: <<https://ssl4799.websiteseuro.com/swge5/PROCEEDINGS/PDF/CBA2016-0947.pdf>>. Cited 7 times in pages , 95, 96, 100, 101, 102, and 105.
- SILVA, V. L.; VAL, J. B. R. do; SOUTO, R. F. A stochastic approach for robustness: A H_2 -norm comparison. In: *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*. [s.n.], 2017. p. 1094–1099. Available from Internet: <<https://doi.org/10.1109/CDC.2017.8263803>>. Cited 9 times in pages , 56, 57, 77, 86, 87, 88, 104, and 105.
- SOUTO, R.; VAL, J. do; OLIVEIRA, R. Controlling uncertain stochastic systems: Performance comparisons in a scalar system. In: *IEEE Conference on Decision and Control*. [s.n.], 2013. Available from Internet: <<https://ieeexplore.ieee.org/document/6760157/>>. Cited in page 100.
- SRIVASTAVA, S. M. *A Course on Borel Sets*. New York, NY: Springer New York, 1998. ISBN 978-0-387-22767-2. Available from Internet: <<https://rd.springer.com/book/10.1007%2Fb98956>>. Cited in page 39.

- STOKEY, N. L. *The Economics of Inaction: Stochastic Control Models with Fixed Costs*. Princeton University Press, 2008. Available from Internet: <<https://muse.jhu.edu/book/31105>>. Cited in page 22.
- STURM, J. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optim. Method Softw.*, v. 11, n. 1-4, p. 625–653, 1999. Available from Internet: <<https://doi.org/10.1080/10556789908805766>>. Cited in page 88.
- SU, W.; CHEN, J.; FU, M.; QI, T. Control Under Stochastic Multiplicative Uncertainties: Part II, Optimal Design for Performance. *IEEE Transactions on Automatic Control*, v. 62, n. 3, p. 1285–1300, March 2017. Available from Internet: <<https://ieeexplore.ieee.org/document/7501589/>>. Cited in page 79.
- TRAN, K.; YIN, G. Stochastic competitive lotka–volterra ecosystems under partial observation: Feedback controls for permanence and extinction. *Journal of the Franklin Institute*, v. 351, n. 8, p. 4039 – 4064, 2014. ISSN 0016-0032. Available from Internet: <<http://www.sciencedirect.com/science/article/pii/S0016003214001252>>. Cited in page 98.
- TRAN, K.; YIN, G. Optimal harvesting strategies for stochastic competitive lotka–volterra ecosystems. *Automatica*, v. 55, p. 236 – 246, 2015. ISSN 0005-1098. Available from Internet: <<http://www.sciencedirect.com/science/article/pii/S0005109815001284>>. Cited in page 98.
- TRAN, K.; YIN, G. Numerical methods for optimal harvesting strategies in random environments under partial observations. *Automatica*, v. 70, p. 74 – 85, 2016. ISSN 0005-1098. Available from Internet: <<http://www.sciencedirect.com/science/article/pii/S0005109816301054>>. Cited in page 98.
- UGRINOVSKII, V. A. Robust \mathcal{H}_∞ infinity control in the presence of stochastic uncertainty. *International Journal of Control*, Taylor & Francis, v. 71, n. 2, p. 219–237, 1998. Available from Internet: <<https://doi.org/10.1080/002071798221849>>. Cited 2 times in pages 79 and 94.
- VAL, J. B. do; GEROMEL, J. C.; GONÇALVES, A. P. The H_2 -control for jump linear systems: cluster observations of the Markov state. *Automatica*, v. 38, n. 2, p. 343 – 349, 2002. ISSN 0005-1098. Available from Internet: <<http://www.sciencedirect.com/science/article/pii/S0005109801002102>>. Cited in page 79.
- VAL, J. B. R. do; GUILLOTREAU, P.; VALLEE, T. Fishery management with poorly known dynamics. *Journal of Environmental Economics and Management*, 2018. Under review. Cited in page 105.
- VAL, J. B. R. do; SOUTO, R. F. Modeling and Control of Stochastic Systems With Poorly Known Dynamics. *IEEE Transactions on Automatic Control*, v. 62, n. 9, p. 4467–4482, Sept 2017. ISSN 0018-9286. Available from Internet: <<https://doi.org/10.1109/TAC.2017.2668359>>. Cited 14 times in pages 21, 24, 43, 44, 45, 46, 47, 48, 49, 52, 54, 86, 100, and 104.
- WALTERS, C.; MAGUIRE, J.-J. Lessons for stock assessment from the northern cod collapse. *Reviews in Fish Biology and Fisheries*, v. 6, n. 2, p. 125–137, 1996. Available from Internet: <<https://link.springer.com/article/10.1007/BF00182340>>. Cited in page 96.

- WILLEMS, J. L.; WILLEMS, J. C. Feedback stabilizability for stochastic systems with state and control dependent noise. *Automatica*, v. 12, n. 3, p. 277 – 283, 1976. Available from Internet: <[https://doi.org/10.1016/0005-1098\(76\)90029-7](https://doi.org/10.1016/0005-1098(76)90029-7)>. Cited in page 79.
- WONHAM, W. M. Optimal Stationary Control of a Linear System with State-Dependent Noise. *SIAM Journal on Control*, v. 5, n. 3, p. 486–500, 1967. Available from Internet: <<https://doi.org/10.1137/0305028>>. Cited in page 78.
- YONG, J.; ZHOU, X. Y. *Stochastic Controls - Hamiltonian Systems and HJB Equations*. New York: Springer, 1999. Available from Internet: <https://doi.org/10.1007/978-1-4612-1466-3_3>. Cited 11 times in pages 35, 36, 37, 38, 39, 40, 41, 42, 43, 60, and 86.
- ZABCZYK, J. *Mathematical Control Theory: An Introduction*. Boston, MA: Birkhäuser Boston, 2008. 1–9 p. ISBN 978-0-8176-4733-9. Available from Internet: <https://doi.org/10.1007/978-0-8176-4733-9_1>. Cited in page 65.
- ZAMES, G. Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses. *IEEE Transactions on Automatic Control*, v. 26, n. 2, p. 301–320, April 1981. ISSN 0018-9286. Available from Internet: <<https://doi.org/10.1109/TAC.1981.1102603>>. Cited 2 times in pages 19 and 119.
- ZHANG, W.; CHEN, B.-S. State Feedback H_∞ Control for a Class of Nonlinear Stochastic Systems. *SIAM Journal on Control and Optimization*, v. 44, n. 6, p. 1973–1991, 2006. Available from Internet: <<https://doi.org/10.1137/S0363012903423727>>. Cited in page 94.
- ZHANG, W.; CHEN, B. S.; TANG, H.; SHENG, L.; GAO, M. Some Remarks on General Nonlinear Stochastic H_∞ Control With State, Control, and Disturbance-Dependent Noise. *IEEE Transactions on Automatic Control*, v. 59, n. 1, p. 237–242, Jan 2014. ISSN 0018-9286. Available from Internet: <<https://doi.org/10.1109/TAC.2013.2270073>>. Cited in page 94.
- ZHOU, K.; DOYLE, J. *Essentials of Robust Control*. [S.l.]: Prentice Hall, 1998. Cited 9 times in pages 16, 18, 19, 77, 116, 117, 118, 119, and 120.
- ZHOU, K.; DOYLE, J. C.; GLOVER, K. *Robust and Optimal Control*. Upper Saddle River, NJ, USA: Prentice-Hall, Inc., 1996. ISBN 0-13-456567-3. Cited 2 times in pages 116 and 117.

ANNEX A – Robust \mathcal{H}_2 and \mathcal{H}_∞ control

The theory presented here is largely based on the books “Essentials of Robust Control” (ZHOU; DOYLE, 1998), “Robust and Optimal Control”(ZHOU *et al.*, 1996) and “Feedback Control Theory”(DOYLE *et al.*, 1991), and mentions on definitions and theorems from these books are indicated. We first bring the definition of \mathcal{H}_2 and \mathcal{H}_∞ spaces, and the corresponding norms.

Definition A.1 (*Definition 4.1* (ZHOU; DOYLE, 1998)). *Let V be a vector space over \mathbb{C} . An inner product¹ on V is a complex-valued function,*

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$$

such that for any $x, y, z \in V$ and $\alpha, \beta \in \mathbb{C}$

1. $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
3. $\langle x, x \rangle > 0$ if $x \neq 0$.

A vector space V with an inner product is called an inner product space.

A.1 \mathcal{H}_2 and \mathcal{H}_∞ spaces

The inner product as defined above induces the norm

$$\|x\| = \sqrt{\langle x, x \rangle},$$

and a complete² inner product space is called a *Hilbert space*. The \mathcal{H}_2 and \mathcal{H}_∞ spaces are Hilbert spaces with the corresponding norms defined in the following.

A.1.1 \mathcal{H}_2 and \mathcal{L}_2 norms

According to (ZHOU; DOYLE, 1998, Ch. 4), (ZHOU *et al.*, 1996, Ch. 4) we have

Definition ($\mathcal{L}_2(j\mathbb{R})$ Space). $\mathcal{L}_2(j\mathbb{R})$ is a Hilbert space consisting of complex matrix functions F for which the integral

$$\int_{-\infty}^{\infty} \text{tr}[F^*(j\omega)F(j\omega)d\omega] < \infty.$$

¹ Also dot or scalar product.

² A space X is said to be complete if every Cauchy sequence in X converges to a point in it.

For $F, G \in \mathcal{L}_2$ the inner product is defined as

$$\langle F, G \rangle := \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr}[F^*(j\omega)G(j\omega)]d\omega$$

and the corresponding norm given by

$$\|F\|_2 := \sqrt{\langle F, F \rangle}.$$

Definition (\mathcal{H}_2 Space). \mathcal{H}_2 is a subspace of $\mathcal{L}_2(j\mathbb{R})$ with matrix functions analytic³ in the open right half-plane and induced norm

$$\|F\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr}[F^*(j\omega)F(j\omega)]d\omega.$$

If the matrix (transfer) function F is stable, we can apply Parseval's theorem to get a representation of the \mathcal{H}_2 norm in the time-domain,

$$\|F\|_2^2 = \int_{-\infty}^{\infty} |G(t)|^2 dt.$$

Moreover, if we consider a state-space realization of F given by

$$\begin{cases} dx(t) = (Ax(t) + Bu(t))dt, \\ y(t) = Cx(t), \end{cases} \quad (\text{A.1})$$

where the matrix A is stable, then the \mathcal{H}_2 norm of the system can be calculated by

$$\|F\|_2^2 = \text{tr}(B^*L_oB) = \text{tr}(CL_cC^*), \quad (\text{A.2})$$

with L_o and L_c the observability and controllability Gramians associated to the system. In the above representation, the matrix D is suppressed so that the \mathcal{H}_2 norm is finite. The Gramians can be obtained by Lyapunov equations,

$$AP + PA^* + BB^* = 0, \quad A^*Q + QA + C^*C = 0. \quad (\text{A.3})$$

The \mathcal{L}_2 and \mathcal{H}_2 norms are related to the internal energy of a system.

A.1.2 \mathcal{H}_∞ and \mathcal{L}_∞ norms

Once again we refer to (ZHOU; DOYLE, 1998) and (ZHOU *et al.*, 1996) for the following definitions.

Definition (\mathcal{L}_∞ Space). \mathcal{L}_∞ is a Banach space⁴ of functions bounded on $j\mathbb{R}$. The corresponding norm is defined as

$$\|F\|_\infty := \text{ess sup}_{\omega \in \mathbb{R}} \bar{\sigma}[F(j\omega)].$$

³ A function is analytic in a region \mathcal{R} if it is differentiable at each point of \mathcal{R} .

⁴ Hilbert spaces are special cases of Banach spaces (for example L^p is a Banach space but not a Hilbert space for $p \neq 2$). A Banach space is a complete normed linear space, i.e., a normed linear space with the property that all Cauchy sequences are convergent. A Hilbert space is a complete inner product space, i.e., a Banach space with norm determined by an inner product.

Here, $\bar{\sigma}(A)$ represents the largest singular value of A .

Definition (\mathcal{H}_∞ Space). \mathcal{H}_∞ is a closed subspace of \mathcal{L}_∞ with functions analytic and bounded in the open right-half plane. The corresponding norm is given by

$$\|F\|_\infty := \sup_{\omega \in \mathbb{R}} \bar{\sigma}[F(j\omega)].$$

The \mathcal{H}_∞ norm can also be calculated in the state space formulation. Differently from the \mathcal{H}_2 case, there is not a closed solution, but the following lemma yields an upper bound for the \mathcal{H}_∞ norm, and one can find, by successive iterations — using the bisection algorithm for example —, the value corresponding to the \mathcal{H}_∞ norm (ZHOU; DOYLE, 1998).

Lemma A.1 (Lemma 4.5 (ZHOU; DOYLE, 1998)). *Let $\gamma > 0$ and*

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \in \mathcal{RL}_\infty \quad (\text{A.4})$$

Then $\|G\|_\infty < \gamma$ if and only if $\bar{\sigma}(D) < \gamma$ and the Hamiltonian matrix H has no eigenvalues on the imaginary axis where

$$H := \begin{bmatrix} A + BR^{-1}D^*C & BR^{-1}B^* \\ -C^*(I + DR^{-1}D^*)C & -(A + BR^{-1}D^*C)^* \end{bmatrix} \quad (\text{A.5})$$

*and $R = \gamma^2 I - D^*D$.*

In the above lemma, \mathcal{RL}_∞ is the rational subspace of \mathcal{L}_∞ ; it consists of all proper and real rational transfer matrices with no poles on the imaginary axis.

For a single-input, single-output system, the \mathcal{H}_∞ norm can be seen as the largest possible amplitude gain of the system response to sinusoidal inputs, and this observation can be extended to multiple-input, multiple-output systems (ZHOU; DOYLE, 1998). The \mathcal{H}_∞ norm of a system $G(s)$ can therefore be portrayed in a worst-case formulation,

$$\|G\|_\infty = \sup_{\substack{\omega \in L_2 \\ \|\omega\|_2 \neq 0}} \frac{\|G\omega\|_2}{\|\omega\|_2}, \quad (\text{A.6})$$

indicating the worst-possible effect of a bounded disturbance ω on the system output $G\omega$.

A.2 The \mathcal{H}_2 optimal control problem

Consider a system G described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1w(t) + B_2u(t) \\ z(t) &= C_1x(t) + D_{12}u(t) \\ y(t) &= C_2x(t) + D_{21}w(t), \end{aligned} \quad (\text{A.7})$$

where $x(t)$ stands for the system state, $y(t)$ is the measured output, $u(t)$ the control input, $w(t)$ a disturbance input, and $z(t)$ the controlled output. The state space representation of the system can also be given by

$$C = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right], \quad (\text{A.8})$$

where D_{11} and D_{22} are both 0, so that the \mathcal{H}_2 -norm of the closed-loop system is finite. We also make the following assumptions on the matrices of the above system

1. (A, B_2) is stabilizable and (C_2, A) is detectable;
2. $R_1 = D_{12}^* D_{12} > 0$ and $R_2 = D_{21}^* D_{21} > 0$;
3. $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω ;
4. $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω .

The \mathcal{H}_2 problem can then be stated as (ZHOU; DOYLE, 1998):

\mathcal{H}_2 Problem *The \mathcal{H}_2 control problem is to find a proper, real-rational controller K which stabilizes G internally and minimizes the \mathcal{H}_2 -norm of the transfer matrix T_{zw} from w to z .*

Under the previous assumptions, we get a closed-form solution for the general output-feedback \mathcal{H}_2 control problem, given by Theorem 13.7 on (ZHOU; DOYLE, 1998). In the state-feedback case, a simpler solution can be obtained. For that, consider the algebraic Riccati equation (ARE)

$$A^T X + X A - X B_2 (D_{12}^T D_{12})^{-1} B_2^T X + C_1^T C_1 = 0, \quad (\text{A.9})$$

and let X its unique symmetric positive solution. Then the minimizing state-feedback controller is given by

$$u(t) = K_{\text{opt}} x(t),$$

with $K_{\text{opt}} = -(D_{12}^T D_{12})^{-1} B_2^T X$.

A.3 The \mathcal{H}_∞ optimal control problem

The \mathcal{H}_∞ control problem traces its roots to the seminal paper from Zames (ZAMES, 1981), and was originally cast in the time-domain. Since our aim here is to relate robust

control methods to the stochastic models previously discussed, we focus on the state-space representation.

In the deterministic case the \mathcal{H}_∞ control problem corresponds to the search for admissible controllers $K(s)$ that minimize the \mathcal{H}_∞ -norm of the closed-loop system. Finding an optimal \mathcal{H}_∞ controller, however, is challenging and computationally expensive, and one usually goes for sub-optimal controllers which have norm close to that of the (theoretical) optimal controller yet are easier to obtain (ZHOU; DOYLE, 1998). In this sense an alternative formulation for the \mathcal{H}_∞ control problem can be stated as (ZHOU; DOYLE, 1998, Ch. 14)

Sub-optimal \mathcal{H}_∞ *Given $\gamma > 0$, find all admissible controllers $K(s)$, if there are any, such that $\|T_{zw}\|_\infty < \gamma$.*

In the above definition $\|T_{zw}\|_\infty$ corresponds to the \mathcal{H}_∞ -norm of the closed-loop system. We assume the controlled system has a state-space realization given by

$$\begin{aligned} dx(t) &= (Ax(t) + B_1w(t) + B_2u(t))dt \\ z(t) &= C_1x(t) + D_{12}u(t) \\ y(t) &= C_2x(t) + D_{21}w(t), \end{aligned} \tag{A.10}$$

and the pairs (A, B_1) , (A, B_2) , (C_1, A) , (C_2, A) are respectively controllable, stabilizable, observable and detectable. We also assume the following holds

$$\begin{aligned} D_{12}^* \begin{bmatrix} C_1 & D_{12} \end{bmatrix} &= \begin{bmatrix} 0 & I \end{bmatrix}, \\ \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^* &= \begin{bmatrix} 0 \\ I \end{bmatrix}. \end{aligned} \tag{A.11}$$

Under these conditions, Theorem 14.1 on (ZHOU; DOYLE, 1998) is valid, and one can solve the (suboptimal) deterministic \mathcal{H}_∞ control via an optimization procedure.

The \mathcal{H}_∞ control problem has yet an interpretation in terms of minimax games (BAŞAR; BERNHARD, 2008), with the controller seen as the minimizing player, and the disturbance as the maximizing player. The problem involves then minimization of the parametrized cost (BAŞAR; BERNHARD, 2008; ZHOU; DOYLE, 1998)

$$J_\gamma(u, w) := \|T_z w\|^2 - \gamma^2 \|w\|^2. \tag{A.12}$$