Atilio Gomes Luiz

# Graceful labellings and neighbour-distinguishing labellings of graphs 

Rotulações graciosas e rotulações semifortes em grafos

## Atilio Gomes Luiz

# Graceful labellings and neighbour-distinguishing labellings of graphs 

## Rotulações graciosas e rotulações semifortes em grafos

> Tese apresentada ao Instituto de Computação da Universidade Estadual de Campinas como parte dos requisitos para a obtenção do título de Doutor em Ciência da Computação.

> Dissertation presented to the Institute of Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Doctor in Computer Science.

Supervisor/Orientadora: Profa. Dra. Christiane Neme Campos

Este exemplar corresponde à versão final da Tese defendida por Atilio Gomes Luiz e orientada pela Profa. Dra. Christiane Neme Campos.

Ficha catalográfica
Universidade Estadual de Campinas
Biblioteca do Instituto de Matemática, Estatística e Computação Científica Ana Regina Machado - CRB 8/5467

```
Luiz, Atílio Gomes, 1987-
L968g Graceful labellings and neighbour-distinguishing labellings of graphs / Atílio Gomes Luiz. - Campinas, SP : [s.n.], 2018.
Orientador: Christiane Neme Campos.
Tese (doutorado) - Universidade Estadual de Campinas, Instituto de Computação.
1. Teoria dos grafos. 2. Teoria da computação. 3. Árvores (Teoria dos grafos). 4. Coloração de grafos. 5. Rotulação de grafos. I. Campos, Christiane Neme, 1972-. II. Universidade Estadual de Campinas. Instituto de Computação. III. Título.
```


## Informações para Biblioteca Digital

## Título em outro idioma: Rotulações graciosas e rotulações semifortes em grafos

 Palavras-chave em inglês:Graph theory
Theory of computing
Trees (Graph theory)
Graph coloring
Graph labelings
Área de concentração: Ciência da Computação
Titulação: Doutor em Ciência da Computação
Banca examinadora:
Christiane Neme Campos [Orientador]
Claudia Linhares Sales
Daniel Morgato Martin
Célia Picinin de Mello
João Meidanis
Data de defesa: 21-05-2018
Programa de Pós-Graduação: Ciência da Computação

## Atilio Gomes Luiz

## Graceful labellings and neighbour-distinguishing labellings of graphs

Rotulações graciosas e rotulações semifortes em grafos

## Banca Examinadora:

- Profa. Dra. Christiane Neme Campos

Instituto de Computação - UNICAMP

- Profa. Dra. Cláudia Linhares Sales

Departamento de Computação - UFC

- Prof. Dr. Daniel Morgato Martin

Centro de Matemática, Computação e Cognição - UFABC

- Profa. Dra. Célia Picinin de Mello

Instituto de Computação - UNICAMP

- Prof. Dr. João Meidanis

Instituto de Computação - UNICAMP

A ata da defesa com as respectivas assinaturas dos membros da banca encontra-se no processo de vida acadêmica do aluno.

To my parents, Luiz and Francisca.

One, remember to look up at the stars and not down at your feet. Two, never give up work. Work gives you meaning and purpose and life is empty without it. Three, if you are lucky enough to find love, remember it is there and don't throw it away.
(Stephen Hawking)

## Acknowledgements

First of all, I would like to thank my supervisor, Christiane, for her guidance, friendship and support throughout all these years. Thank you Chris for your help!
I would like to thank my girlfriend, Karina, for being such a loving partner, for helping me during many difficult moments and for making my life brighter.

* I would like to express my gratitude to my parents, Francisca and Luiz, and to my sisters, Gabi and Amanda, for supporting me in my crazy wish to continue studying and learning.
I would like to thank my friends Victor Menezes, Murilo Souza and Luiz Simione for their company, for our quarrels, and for all the interesting discussions we had during the last years.
( I am also very grateful to all my colleagues and friends at the Institute of Computing, at University of Campinas, for creating such a pleasant and friendly working environment. I would like to send special thanks to Judy Guevara, Luciano Chaves, Rodrigo Silva, Celso Weffort and to professors Lehilton Pedrosa and Rafael Schouery.
I would like to thank professors Cláudia Sales, Daniel Martin, Célia Picinin and João Meidanis for joining my thesis committee and for the valuable and constructive feedback they provided. I am also immensely grateful to professor Ricardo Dahab for having read an earliest version of this thesis and for the corrections and improvements he suggested.
I would like to thank my co-authors Bruce Richter, Diana Sasaki, Sheila de Almeida and Simone Dantas for working with me on the problems presented in this thesis. I am also grateful to professor Celina Miraglia and Rodrigo Zhou for drawing my attention to the graceful labellings of families of graphs other than trees.
姆 I would like to thank John Hofstetter for making me feel welcome in his home during my stay at Waterloo.
( I would like to acknowledge the financial support provided by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES process numbers 1364607 and 1406892) and São Paulo Research Foundation (FAPESP grants 2014/16861-8 and 2015/03372-1).
Finally, I am specially grateful to the one that made this whole universe possible.
Thank you very much to all of you!


## Resumo

Três problemas de rotulação em grafos são investigados nesta tese: a Conjetura das Árvores Graciosas, a Conjetura 1,2,3 e a Conjetura 1,2 .

Uma rotulação graciosa de um grafo simples $G=(V(G), E(G))$ é uma função injetora $f: V(G) \rightarrow\{0, \ldots,|E(G)|\}$ tal que $\{|f(u)-f(v)|: u v \in E(G)\}=\{1, \ldots,|E(G)|\}$. A Conjetura das Árvores Graciosas, proposta por Rosa e Kotzig em 1967, afirma que toda árvore possui uma rotulação graciosa. Um problema relacionado à Conjetura das Árvores Graciosas consiste em determinar se, para todo vértice $v$ de uma árvore $T$, existe uma rotulação graciosa de $T$ que atribui o rótulo 0 a $v$. Árvores com tal propriedade são denominadas 0 -rotativas. Nesta tese, apresentamos famílias infinitas de caterpillars 0rotativos. Nossos resultados reforçam a conjetura de que todo caterpillar com diâmetro pelo menos cinco é 0-rotativo.

Também investigamos uma rotulação graciosa mais restrita, chamada rotulação- $\alpha$. Uma rotulação graciosa $f$ de $G$ é uma rotulação- $\alpha$ se existir um inteiro $k, 0 \leq k \leq|E(G)|$, tal que, para toda aresta $u v \in E(G), f(u) \leq k<f(v)$ ou $f(v) \leq k<f(u)$. Nesta tese, apresentamos duas famílias de lobsters com grau máximo três que possuem rotulações$\alpha$. Nossos resultados contribuem para uma caracterização de todos os lobsters com grau máximo três que possuem rotulações- $\alpha$.

Na segunda parte desta tese, investigamos generalizações da Conjetura 1,2,3 e da Conjetura 1,2. Dado um grafo simples $G=(V(G), E(G))$ e $\mathcal{L} \subset \mathbb{R}$, dizemos que $\pi: E(G) \rightarrow \mathcal{L}$ é uma $\mathcal{L}$-rotulação de arestas de $G$ e dizemos que $\pi: V(G) \cup E(G) \rightarrow \mathcal{L}$ é uma $\mathcal{L}$-rotulação total de $G$. Para todo $v \in V(G)$, a cor de $v$ é definida como $C_{\pi}(v)=\sum_{u v \in E(G)} \pi(u v)$, se $\pi$ for uma $\mathcal{L}$-rotulação de arestas de $G$, ou $C_{\pi}(v)=\pi(v)+\sum_{u v \in E(G)} \pi(u v)$, se $\pi$ for uma $\mathcal{L}$ rotulação total de $G$. O par $\left(\pi, C_{\pi}\right)$ é uma $\mathcal{L}$-rotulação de arestas semiforte ( $\mathcal{L}$-rotulação total semiforte) se $\pi$ for uma rotulação de arestas (rotulação total) e $C_{\pi}(u) \neq C_{\pi}(v)$, para toda aresta $u v \in E(G)$. A Conjetura 1,2,3, proposta por Karónski et al. em 2004, afirma que todo grafo simples e conexo com pelo menos três vértices possui uma $\{1,2,3\}$ rotulação de arestas semiforte. A Conjetura 1,2, proposta por Przybyło e Woźniak em 2010, afirma que todo grafo simples possui uma $\{1,2\}$-rotulação total semiforte.

Sejam $a, b, c$ três reais distintos. Nesta tese, nós investigamos $\{a, b, c\}$-rotulações de arestas semifortes e $\{a, b\}$-rotulações totais semifortes para cinco famílias de grafos: as potências de caminho, as potências de ciclo, os grafos split, os grafos cobipartidos regulares e os grafos multipartidos completos. Provamos que essas famílias possuem tais rotulações para alguns valores reais $a, b, c$. Como corolário de nossos resultados, obtemos que a Conjetura $1,2,3$ e a Conjetura 1,2 são verdadeiras para essas famílias. Além disso, também mostramos que nossos resultados em rotulações de arestas semifortes implicam resultados similares para outro problema de rotulação de arestas relacionado.

## Abstract

This thesis addresses three labelling problems on graphs: the Graceful Tree Conjecture, the 1,2,3-Conjecture, and the 1,2-Conjecture.

A graceful labelling of a simple graph $G=(V(G), E(G))$ is an injective function $f: V(G) \rightarrow\{0, \ldots,|E(G)|\}$ such that $\{|f(u)-f(v)|: u v \in E(G)\}=\{1, \ldots,|E(G)|\}$. The Graceful Tree Conjecture, posed by Rosa and Kotzig in 1967, states that every tree has a graceful labelling. A problem connected with the Graceful Tree Conjecture consists of determining whether, for every vertex $v$ of a tree $T$, there exists a graceful labelling of $T$ that assigns label 0 to $v$. Trees with such a property are called 0 -rotatable. In this thesis, we present infinite families of 0 -rotatable caterpillars. Our results reinforce a conjecture that states that every caterpillar with diameter at least five is 0-rotatable.

We also investigate a stronger type of graceful labelling, called $\alpha$-labelling. A graceful labelling $f$ of $G$ is an $\alpha$-labelling if there exists an integer $k \in\{0, \ldots,|E(G)|\}$ such that, for each edge $u v \in E(G)$, either $f(u) \leq k<f(v)$ or $f(v) \leq k<f(u)$. In this thesis, we prove that the following families of lobsters have $\alpha$-labellings: lobsters with maximum degree three, without $Y$-legs and with at most one forbidden ending; and lobsters $T$ with a perfect matching $M$ such that the contracted tree $T / M$ has a balanced bipartition. These results point towards a characterization of all lobsters with maximum degree three that have $\alpha$-labellings.

In the second part of the thesis, we focus on generalizations of the 1,2,3-Conjecture and the 1,2 -Conjecture. Given a simple graph $G=(V(G), E(G))$ and $\mathcal{L} \subset \mathbb{R}$, we call $\pi: E(G) \rightarrow \mathcal{L}$ an $\mathcal{L}$-edge-labelling of $G$, and $\pi: V(G) \cup E(G) \rightarrow \mathcal{L}$ an $\mathcal{L}$-total-labelling of $G$. For each $v \in V(G)$, the colour of $v$ is defined as $C_{\pi}(v)=\sum_{u v \in E(G)} \pi(u v)$, if $\pi$ is an $\mathcal{L}$-edge-labelling, and $C_{\pi}(v)=\pi(v)+\sum_{u v \in E(G)} \pi(u v)$, if $\pi$ is an $\mathcal{L}$-total-labelling. The pair $\left(\pi, C_{\pi}\right)$ is a neighbour-distinguishing $\mathcal{L}$-edge-labelling (neighbour-distinguishing $\mathcal{L}$-total-labelling) if $\pi$ is an edge-labelling (total-labelling) and $C_{\pi}(u) \neq C_{\pi}(v)$, for every edge $u v \in E(G)$. The 1,2,3-Conjecture, posed by Karónski et al. in 2004, states that every connected simple graph with at least three vertices has a neighbour-distinguishing $\{1,2,3\}$-edge-labelling. The 1,2-Conjecture, posed by Przybyło and Woźniak in 2010, states that every simple graph has a neighbour-distinguishing $\{1,2\}$-total-labelling.

Let $a, b, c \in \mathbb{R}$ be distinct. In this thesis, we investigate neighbour-distinguishing $\{a, b, c\}$-edge-labellings and neighbour-distinguishing $\{a, b\}$-total labellings for five families of graphs: powers of paths, powers of cycles, split graphs, regular cobipartite graphs and complete multipartite graphs. We prove that these families have such labellings for some real values $a, b$, and $c$. As a corollary of our results, we obtain that the $1,2,3-$ Conjecture and the 1,2 -Conjecture are true for these families. Furthermore, we also show that our results on neighbour-distinguishing edge-labellings imply similar results on a closely related problem called detectable edge-labelling of graphs.

## List of Notation

$[p, q] \quad$ set of consecutive integers $\{p, \ldots, q\}$, with $p \leq q$, page 18
[q] set of consecutive integers $\{1, \ldots, q\}$, with $q \geq 1$, page 18
$\chi(G) \quad$ chromatic number of graph $G$, page 22
$\chi_{\Sigma}^{\prime}(G) \quad$ neighbour-distinguishing edge chromatic number of $G$, page 113
$\chi_{\Sigma}^{\prime \prime}(G) \quad$ neighbour-distinguishing total chromatic number of $G$, page 114
$\Delta(G) \quad$ maximum degree of graph $G$, page 16
$\delta(G) \quad$ minimum degree of graph $G$, page 16
$\bar{G} \quad$ complement of graph $G$, page 16
$A<B \quad \max \{A\}<\min \{B\}$, where $A, B$ are sets of numbers, page 18
$d_{G}(u, v) \quad$ distance between vertices $u, v \in V(G)$, page 19
$d_{G}(v) \quad$ degree of vertex $v$ in a graph $G$, page 16
$\operatorname{diam}(G) \quad$ diameter of graph $G$, page 19
$E_{G}[X, Y] \quad$ set of edges linking two sets of vertices, page 17
$G+e \quad$ addition of edge, page 20
$G+w \quad$ addition of vertex, page 20
$G+X \quad$ addition of elements of $X$ to $G$, page 20
$G \cong H \quad$ isomorphism between $G$ and $H$, page 16
$G-S \quad$ deletion of set of vertices $S$, page 20
$G-v \quad$ deletion of vertex, page 20
$G[X] \quad$ induced subgraph, page 17
$G \backslash e \quad$ deletion of edge, page 20
$G \backslash S \quad$ deletion of subset of edges $S$, page 20
$\operatorname{grac}(G) \quad$ gracefulness of $G$, page 47
$H \subseteq G \quad$ subgraph, page 17
$L_{E(G)}^{f} \quad$ set of edge labels, page 27
$L_{V(G)}^{f} \quad$ set of vertex labels, page 27

## Contents

1 Introduction ..... 14
1.1 Basic definitions ..... 14
1.2 Special families of graphs ..... 18
1.3 Operations on graphs ..... 20
1.4 Graph Decomposition ..... 21
1.5 Graph labellings ..... 22
2 The Graceful Tree Conjecture ..... 26
2.1 Early results and constructions ..... 31
2.1.1 Graceful labellings of families of graphs ..... 38
2.1.2 A necessary condition for graceful labellings ..... 44
2.2 Relaxed versions of graceful labellings ..... 46
2.2.1 Forbidden subgraphs ..... 52
2.3 The technique of transfers ..... 53
2.4 Trees with no $\alpha$-labelling ..... 60
2.5 Strongly-graceful labellings ..... 66
3 0-rotatable graceful caterpillars ..... 70
3.1 Results ..... 72
3.1.1 Caterpillars with diameter five ..... 77
3.1.2 Caterpillars with diameter six ..... 82
3.1.3 Caterpillars with diameter at least seven ..... 87
3.2 Concluding remarks ..... 89
$4 \alpha$-labellings of lobsters with $\Delta(G)=3$ ..... 90
4.1 A graphical representation of $\alpha$-labellings ..... 94
4.2 Lobsters with maximum degree three ..... 95
4.3 Trees with a perfect matching ..... 105
4.4 Concluding remarks ..... 108
5 Neighbour-distinguishing labellings ..... 110
5.1 Upper bounds on $\chi_{\Sigma}^{\prime}(G)$ and $\chi_{\Sigma}^{\prime \prime}(G)$ ..... 115
5.2 Known results for some families of graphs ..... 120
6 Neighbour-distinguishing labellings of families of graphs ..... 129
6.1 Powers of paths ..... 131
6.2 Powers of cycles ..... 141
6.3 Split graphs ..... 147
6.4 Regular cobipartite graphs ..... 152
6.5 Complete multipartite graphs ..... 154
6.6 The related problem of detectable edge-labellings ..... 156
7 Conclusions and future work ..... 160
Bibliography ..... 165
Index of definitions ..... 173

## Chapter 1

## Introduction

Graph labelling is an area of Graph Theory whose main concern is to determine the feasibility of assigning labels to the elements of a graph satisfying certain conditions. Usually, the labels are elements of a set that supports some kind of mathematical operation (for example, the set of nonnegative real numbers). Labelling a graph arbitrarily is a simple exercise since, with no effort, we can assign numbers to the vertices or edges of a graph and obtain a labelled graph. However, adding restrictions to the labelling may turn the labelling problem into a big challenge.

In this thesis, we present new results on three types of labellings of graphs, namely, graceful labellings, neighbour-distinguishing edge-labellings and neighbour-distinguishing total-labellings. These three labellings are formally defined, discussed and investigated in the next chapters of this thesis.

Sections 1.1 through 1.4 introduce basic definitions as well as some classic families of graphs and basic operations on graphs. Section 1.5 discusses labellings in a broader context and details the organization of the thesis.

### 1.1 Basic definitions

A graph $G$ is an ordered pair $(V(G), E(G))$ consisting of a finite nonempty set $V(G)$ and a finite set $E(G)$, disjoint from $V(G)$, together with an incidence function $\psi_{G}$ that associates each element of $E(G)$ with an unordered pair of (not necessarily distinct) elements of $V(G)$. The elements of $V(G)$ are called vertices and the elements of $E(G)$ are called edges. Moreover, set $V(G)$ is called the vertex set of $G$ and $E(G)$ the edge set of $G$. An element of a graph is a vertex or an edge of the graph. A graph that has only one vertex is called trivial. The order of a graph $G$ is the cardinality of its vertex set and the size of $G$ is the cardinality of its edge set.

If $e \in E(G)$ and $\psi_{G}(e)=\{u, v\}$ for $u, v \in V(G)$, we say that $u$ and $v$ are the endpoints of edge $e$ and that edge e links vertices $u$ and $v$. The endpoints of an edge are said to be incident with the edge, and vice versa. Two vertices that are incident with a common edge are adjacent, as are two edges incident with a common vertex. Two distinct adjacent vertices are neighbours. The set of neighbours of a vertex $v \in V(G)$ is denoted by $N_{G}(v)$. As an example, we present the graph $G=(V(G), E(G))$ with vertex
set $V(G)=\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$, edge set $E(G)=\left\{e_{1}, e_{2}, e_{3}, e_{4}, e_{5}, e_{6}, e_{7}, e_{8}\right\}$ and incidence function $\psi_{G}$ defined by

$$
\begin{aligned}
& \psi_{G}\left(e_{1}\right)=\left\{v_{1}\right\}, \quad \psi_{G}\left(e_{2}\right)=\left\{v_{1}, v_{2}\right\}, \quad \psi_{G}\left(e_{3}\right)=\left\{v_{1}, v_{4}\right\}, \quad \psi_{G}\left(e_{4}\right)=\left\{v_{1}, v_{3}\right\}, \\
& \psi_{G}\left(e_{5}\right)=\left\{v_{2}, v_{4}\right\}, \quad \psi_{G}\left(e_{6}\right)=\left\{v_{3}, v_{4}\right\}, \quad \psi_{G}\left(e_{7}\right)=\left\{v_{1}, v_{3}\right\}, \quad \psi_{G}\left(e_{8}\right)=\left\{v_{2}, v_{3}\right\} .
\end{aligned}
$$

Graphs are so named because they can be represented graphically in the plane. We draw a graph in the plane such that each vertex is represented by a small circle and each edge is a line linking its endpoints. It is common to refer to a graphical representation of a graph as the graph per se. As an example, Figure 1.1 presents the graph $G$ previously defined.


Figure 1.1: Graph $G$.

An edge with identical endpoints is called a loop. Two or more edges with the same pair of endpoints are said to be multiple edges. For instance, in the graph $G$ of Figure 1.1, edge $e_{1}$ is a loop and edges $e_{4}$ and $e_{7}$ are multiple edges. A graph that has no loops or multiple edges is a simple graph.

In a simple graph $G=(V(G), E(G))$, an edge $e \in E(G)$ such that $\psi_{G}(e)=\{u, v\}$ is completely determined by its endpoints since $e$ is the unique edge linking $u$ and $v$. Therefore, in simple graphs, we can ignore the formality of the incidence function and denote the edge $e$ by $\{u, v\}$. Moreover, for simplicity, edge $\{u, v\} \in E(G)$ is also denoted by $u v$ or $v u$. The graph $G=(V(G), E(G))$ with vertex set $V(G)=\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$ and edge set $E(G)=\left\{v_{1} v_{2}, v_{1} v_{3}, v_{1} v_{4}, v_{2} v_{3}, v_{2} v_{4}, v_{3} v_{4}\right\}$ is an example of this notation for simple graphs and is illustrated in Figure 1.2.


Figure 1.2: A simple graph $G$ with four vertices and six edges. Each edge is labelled with its endpoints. Names of edges are omitted in future drawings of simple graphs.

Let $G$ be a graph and $v \in V(G)$. The degree $d_{G}(v)$ of $v$ is the number of occurrences of $v$ as an endpoint of an edge. For example, in Figure 1.1, we have that $d_{G}\left(v_{1}\right)=6$ and $d_{G}\left(v_{3}\right)=4$. A vertex of degree 0 is an isolated vertex. A vertex that is adjacent to all other vertices of the graph is called a universal vertex. The maximum degree of graph $G$ is $\Delta(G)=\max \left\{d_{G}(v): v \in V(G)\right\}$ and the minimum degree of $G$ is $\delta(G)=$ $\min \left\{d_{G}(v): v \in V(G)\right\}$. For instance, in Figure 1.1, $\Delta(G)=6$ and $\delta(G)=3$. If $G$ is a graph with $n$ vertices $v_{1}, \ldots, v_{n}$ with degrees $d_{G}\left(v_{1}\right) \leq \cdots \leq d_{G}\left(v_{n}\right)$, then the $n$-tuple $\left(d_{G}\left(v_{1}\right), \ldots, d_{G}\left(v_{n}\right)\right)$ is called the degree sequence of $G$. For example, the degree sequence of graph $G$ in Figure 1.1 is $(3,3,4,6)$ and the degree sequence of the graph $G$ in Figure 1.2 is $(3,3,3,3)$. A graph is $k$-regular if all of its vertices have the same degree $k$; we may simply say $G$ is regular when the value of $k$ is not relevant for the discussion.

The complement $\bar{G}$ of a simple graph $G$ is the simple graph that has $V(\bar{G})=V(G)$, with two vertices of $\bar{G}$ being adjacent if and only if they are nonadjacent in $G$. Figure 1.3 shows a simple graph $G$ and its complement $\bar{G}$.


Figure 1.3: A simple graph $G$ and its complement $\bar{G}$.
A matching $M$ in a graph $G$ is a subset of pairwise nonadjacent edges of $E(G)$. We say that a vertex $v \in V(G)$ is saturated by $M$ if $v$ is an endpoint of some edge of $M$; otherwise, $v$ is unsaturated. A perfect matching is a matching that saturates all vertices of the graph. As an illustration, Figure 1.4 shows a graph $H$ with a matching and a graph $G$ with a perfect matching.


Figure 1.4: A graph $H$ with matching $M=\left\{v_{2} v_{6}, v_{4} v_{5}\right\}$ and a graph $G$ with perfect matching $M=\left\{v_{1} v_{4}, v_{2} v_{3}, v_{5} v_{6}\right\}$.

Set $S \subseteq V(G)$ is called an independent set of vertices if its elements are pairwise nonadjacent and is called a clique if its elements are pairwise adjacent. For instance, in Figure 1.4, $S=\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$ is an independent set of $H$ and $S=\left\{v_{1}, v_{4}, v_{5}\right\}$ is a clique of $G$.

Two graphs $G$ and $H$ are isomorphic, written $G \cong H$, if there are bijections $\theta: V(G) \rightarrow$ $V(H)$ and $\phi: E(G) \rightarrow E(H)$ such that $\psi_{G}(e)=u v$ if and only if $\psi_{H}(\phi(e))=\theta(u) \theta(v)$;
such a pair of mappings is called an isomorphism between $G$ and $H$. Figure 1.5 presents two isomorphic graphs.


G


H

Figure 1.5: Two isomorphic graphs $G$ and $H$. Isomorphism $(\theta, \phi)$, where $\theta\left(v_{1}\right)=e$, $\theta\left(v_{2}\right)=b, \theta\left(v_{3}\right)=c, \theta\left(v_{4}\right)=d, \theta\left(v_{5}\right)=a, \phi\left(e_{1}\right)=k, \phi\left(e_{2}\right)=\ell, \phi\left(e_{3}\right)=g, \phi\left(e_{4}\right)=m$, $\phi\left(e_{5}\right)=i, \phi\left(e_{6}\right)=j, \phi\left(e_{7}\right)=f, \phi\left(e_{8}\right)=h$, and $\phi\left(e_{9}\right)=n$.

Given two graphs $G$ and $H$, we say that $H$ is a subgraph of $G$, written $H \subseteq G$, if $V(H) \subseteq V(G), E(H) \subseteq E(G)$, and the incidence function $\psi_{H}$ is a restriction of $\psi_{G}$ to $E(H)$. In this case, we also say that $G$ contains $H$ or that $H$ is in $G$. For example, graph $G$ of Figure 1.4 contains graph $H$ in that same figure. A graph $G$ contains a copy of a graph $H$ if $G$ has a subgraph isomorphic to $H$. Let $X \subseteq V(G)$. We denote $G[X]$ the subgraph of $G$ such that $V(G[X])=X$ and $E(G[X])$ comprises all edges of $G$ that have both endpoints in $X$. We say that $G[X]$ is the subgraph of $G$ induced by $X$. Similarly, given $Y \subseteq E(G)$, we denote $G[Y]$ the subgraph of $G$ such that $E(G[Y])=Y$ and $V(G[Y])$ comprises all vertices of $G$ that are endpoints of edges in $Y$. We say that $G[Y]$ is the subgraph of $G$ induced by $Y$.

Two graphs $G$ and $H$ are disjoint if $V(G) \cap V(H)=\emptyset$, and they are edge-disjoint if $E(G) \cap E(H)=\emptyset$. The union of two graphs $G$ and $H$ is the graph $G \cup H$ with vertex set $V(G \cup H)=V(G) \cup V(H)$ and edge set $E(G \cup H)=E(G) \cup E(H)$. If $G$ and $H$ are disjoint, we refer to their union as disjoint union. We denote $k G$ the disjoint union of $k$ copies of graph $G$.

A graph $G$ is connected if, for every partition of $V(G)$ into two nonempty subsets $X$ and $Y$, there exists an edge with one endpoint in $X$ and the other endpoint in $Y$. A connected component of a graph $G$ is a maximal connected subgraph of $G$. Figure 1.6 exhibits a graph with three connected components.


Figure 1.6: A non-connected graph with three connected components.
Let $X, Y \subset V(G)$ be disjoint subsets. We denote $E_{G}[X, Y]$ the set of edges of $G$ with one endpoint in $X$ and the other endpoint in $Y$. When $Y=V(G) \backslash X$, set $E_{G}[X, Y]$ is called the edge cut of $G$ associated with $X$ and is denoted by $\partial(X)$. Note that $\partial(X)=$ $\partial(Y)$ and $\partial(V)=\emptyset$. As an illustration, Figure 1.7 shows a graph $G$ and one of its edge cuts.


Figure 1.7: A graph with edge cut $\partial(X)=\left\{v_{2} v_{5}, v_{3} v_{5}, v_{3} v_{6}, v_{3} v_{7}\right\}$ shown in bold edges.

Let $p, q \in \mathbb{Z}$ such that $p \leq q$. In this thesis, the set of consecutive integers $\{p, \ldots, q\}$ is denoted by $[p, q]$. Moreover, the specific set of consecutive integers $\{1, \ldots, q\}$, with $q \geq 1$, is denoted by $[q]$. For two sets $A, B$ of positive integers, we write $A<B$ if, for every $a \in A$ and $b \in B, a<b$.

### 1.2 Special families of graphs

Many problems in Graph Theory are hard to solve for arbitrary graphs. In order to face these difficulties, a common approach consists of splitting the original graph into smaller subgraphs for which the solutions are known and, from these solutions, try to obtain the solution for the original graph. In this section, we define some families of graphs commonly used in these approaches.

An empty graph is a graph whose edge set is empty. A complete graph is a simple graph in which any two vertices are adjacent. A complete graph with $n$ vertices is denoted by $K_{n}$, for $n \geq 1$. For instance, the graph in Figure 1.2 is the complete graph $K_{4}$.

A graph is bipartite if its vertex set can be partitioned into two subsets $X$ and $Y$ so that each of its edges has one endpoint in $X$ and the other endpoint in $Y$; such a partition $\{X, Y\}$ is called a bipartition of the graph, and $X$ and $Y$ are its parts. If $G$ is a bipartite simple graph with bipartition $\{X, Y\}$ and every vertex in $X$ is adjacent to every vertex in $Y$, then $G$ is a complete bipartite graph and is denoted by $K_{r, s}$, where $r=|X|$ and $s=|Y|$. A star is a complete bipartite graph $K_{r, s}$ with $r=1$ or $s=1$. As an illustration, $K_{3,3}$ is the first graph exhibited in Figure 1.8.

$K_{3,3}$

$K(3,2)$

Figure 1.8: The complete bipartite graph $K_{3,3}$ and the complete equipartite graph $K(3,2)$.

An r-partite graph (also called a multipartite graph) is one whose vertex set can be partitioned into $r$ subsets, or parts, in such a way that no edge has both endpoints in the same part. An $r$-partite graph is complete if any two vertices in different parts are adjacent. When all $r$ parts of a complete multipartite graph $G$ have the same number $n$
of vertices, $n \geq 1, G$ is also called a complete equipartite graph and is denoted by $K(r, n)$. Figure 1.8 exhibits graph $K(3,2)$.

A path $P_{n}$, with $n \geq 1$ vertices, is a simple graph whose vertices can be arranged in a linear sequence $\left(v_{0}, \ldots, v_{n-1}\right)$ such that two vertices are adjacent if and only if they are consecutive in the sequence. Similarly, a cycle $C_{n}$, with $n \geq 3$ vertices, is a simple graph whose vertices can be arranged in a cyclic sequence $\left(v_{0}, \ldots, v_{n-1}\right)$, such that two vertices are adjacent if and only if they are consecutive in the cyclic sequence. The length of a path or a cycle is its number of edges. Figure 1.9 shows a path with five vertices and a cycle with seven vertices.


Figure 1.9: A path and a cycle.
Given a graph $G$, the distance between two vertices $u, v \in V(G)$, written $d_{G}(u, v)$, is the number of edges in a shortest path connecting $u$ and $v$ in $G$. If $G$ has no such path, then $d_{G}(u, v)=\infty$. As an example, $d_{G}\left(v_{4}, v_{5}\right)=2$ in Figure 1.10. The diameter of a graph $G$ is the greatest distance between two of its vertices and is denoted by $\operatorname{diam}(G)$. Graph $G$ of Figure 1.10 has $\operatorname{diam}(G)=3$. The central vertex of a graph $G$ is a vertex $u \in V(G)$ such that $\max \left\{d_{G}(u, v): v \in V(G)\right\}$ is as small as possible. For example, the central vertices of graph $G$ in Figure 1.10 are vertices $v_{2}, v_{3}, v_{5}$ and $v_{7}$.


Figure 1.10: A simple graph $G$.
A simple graph $G$ is a forest if it is acyclic, i.e., it does not contain a cycle. Each connected component of a forest is called a tree. In a tree, every vertex with degree one is called a leaf. A spine of a tree $T$ is a path in $T$ that connects two vertices at maximum distance in $T$, that is, a path in $T$ with length equal to $\operatorname{diam}(T)$. A rooted tree is a tree $T$ with a distinguished vertex $x \in V(T)$, called the root of $T$. Given a rooted tree $T$ with root $x$, the level of a vertex $v \in V(T)$ is the number $d_{G}(x, v)$.

Let $T$ be a tree and $P$ be one of its spines. We say that $T$ is $k$-distant if all of its vertices are at distance at most $k$ from $P$. The $k$-distant trees with $k \in\{0,1,2\}$ have
specific names in the literature: 0-distant trees are paths, 1-distant trees are caterpillars, and 2-distant trees are lobsters. Figure 1.11 illustrates these definitions.


Figure 1.11: A path, a caterpillar and a lobster.

### 1.3 Operations on graphs

Given a graph $G$, we may be interested in obtaining a subgraph of $G$ from an operation that removes vertices and/or edges from $G$. The deletion of an element from graph $G$ may be suitable, for example, in the context of demonstrating a property of $G$ by means of mathematical induction. In this section, we define some operations on graphs that allow us to obtain new graphs from others. In all the operations defined below, the incidence function of the graphs under consideration are implicit and are properly updated.

Two natural ways of obtaining smaller graphs from $G$ is by removing vertices or edges from $G$. Formally, given an edge $e \in E(G)$, an edge deletion is the operation of removing $e$ from $E(G)$. The resulting graph is denoted by $G \backslash e$. Given $S \subseteq E(G)$, we denote $G \backslash S$ the subgraph of $G$ obtained by deletion of all edges of $S$ from $E(G)$. Given a vertex $v \in V(G)$, a vertex deletion is the operation of removing vertex $v$ from $V(G)$ and also removing all edges that have $v$ as an endpoint. The resulting graph is denoted by $G-v$. Given $S \subseteq V(G), G-S$ is the subgraph $G[V(G) \backslash S]$.

Let $G$ be a graph and $u, v$ two nonadjacent vertices of $G$. Graph $G+e$ is obtained from $G$ by adding an edge $e=u v$ to the set $E(G)$, that is, $V(G+e)=V(G)$ and $E(G+e)=E(G) \cup\{e\}$. This operation is called edge addition. Now, let $w \notin V(G)$. Graph $G+w$ is obtained from $G$ by adding $w$ to $V(G)$, that is, $V(G+w)=V(G) \cup\{w\}$ and $E(G+w)=E(G)$. This operation is called vertex addition. Let $X$ be a nonempty set such that $X \cap(V(G) \cup E(G))=\emptyset$. We denote $G+X$ the graph obtained from $G$ by successive additions of elements of $X$.

Given a graph $G$ and two nonadjacent vertices $x, y \in V(G)$, the identification of $x$ and $y$ is an operation performed in three steps: (i) add a new vertex $v_{x, y}$ to $G$; (ii) for every $e=\{u, v\} \in E(G)$, with $v \in\{x, y\}$, remove $e$ from $E(G)$ and add a new edge $e^{\prime}=\left\{u, v_{x, y}\right\}$; (iii) delete $x$ and $y$ from $G$. This operation is illustrated in Figure 1.12.


Figure 1.12: Identification of vertices $x$ and $y$ and of vertices $w$ and $z$.

Given a graph $G$, the contraction of an edge $e \in E(G)$ is the deletion of the edge followed by (if the endpoints of $e$ are different) the identification of its endpoints. The resulting graph is denoted by $G / e$. This operation is illustrated in Figure 1.13.


Figure 1.13: Contraction of an edge.

In a graph $G$, the subdivision of edge $u v \in E(G)$ is the deletion of $u v$ followed by the addition of a new vertex $w$ and edges $w u$ and $w v$. Any graph obtained from $G$ by a sequence of edge subdivisions is called a subdivision of $G$. This operation is illustrated in Figure 1.14.


Figure 1.14: Subdivision of edges $v_{1} v_{2}$ and $v_{1} v_{3}$ of a graph $G$.

### 1.4 Graph Decomposition

A decomposition of a graph $G$ is a set $\mathcal{D}=\left\{H_{0}, \ldots, H_{t-1}\right\}$ of nonempty edge-disjoint subgraphs of $G$ such that

$$
\bigcup_{i=0}^{t-1} E\left(H_{i}\right)=E(G) .
$$

Graph $G$ is called decomposable into subgraphs $H_{0}, \ldots, H_{t-1}$. If $H_{i}$ is isomorphic to a fixed graph $H$, for all $i \in[0, t-1]$, then we say that $\mathcal{D}$ is an $H$-decomposition of $G$ and that $G$ is $H$-decomposable. Figure 1.15 exhibits a $P_{4}$-decomposition of a graph $G$.

$G$

$H_{0}$

$H_{1}$

$\mathrm{H}_{2}$

Figure 1.15: A graph $G$ and a decomposition of $G$ into three copies of $P_{4}$.

In this section, we are interested in a special kind of decomposition called cyclic decomposition. We define this decomposition in the context of complete graphs. Let $K_{n}$ be the complete graph with vertex set $V\left(K_{n}\right)=\left\{v_{0}, \ldots, v_{n-1}\right\}$. We draw $K_{n}$ in the plane as a regular $n$-gon, whose vertices are, in order, $v_{0}, v_{1}, \ldots, v_{n-1}$. Figure 1.16(a) illustrates $K_{6}$. The reach of an edge $v_{i} v_{j} \in E\left(K_{n}\right)$ is $\min \{|i-j|, n-|i-j|\}$. For example, edge $v_{1} v_{5} \in E\left(K_{6}\right)$ has reach 2 and edge $v_{1} v_{2} \in E\left(K_{6}\right)$ has reach 1 . The rotation of an edge $v_{i} v_{j} \in E\left(K_{n}\right)$ is the mapping of $v_{i} v_{j}$ into $v_{(i+1) \bmod n} v_{(j+1) \bmod n}$. Note that this operation preserves the reach of the edge. The rotation of a subgraph $H \subseteq K_{n}$ is the operation by which we obtain a new subgraph $H^{\prime} \subseteq K_{n}, H^{\prime} \cong H$, by simultaneous rotation of all edges of $H$. A decomposition $\mathcal{D}$ of $K_{n}$ is called cyclic if, for any graph $H \in \mathcal{D}$, graph $H^{\prime}$, obtained by rotation of $H$, is also in $\mathcal{D}$. Figure 1.16 presents a cyclic decomposition of $K_{6}$.

(a) Complete graph $K_{6}$.


$H_{1}$

(b) A cyclic decomposition of $K_{6}$ into five subgraphs.

Figure 1.16: A cyclic decomposition $\mathcal{D}=\left\{H_{0}, \ldots, H_{4}\right\}$ of $K_{6}$. By rotating any subgraph in subsets $\left\{H_{0}, H_{1}\right\}$ and $\left\{H_{2}, H_{3}, H_{4}\right\}$, we obtain another subgraph in the same subset.

### 1.5 Graph labellings

Let $G$ be a graph and $\mathcal{L}$ be a nonempty set. In general, a labelling of $G$ is an assignment of elements (labels) of the set $\mathcal{L}$ to elements of $G$ satisfying certain conditions. In most graph labelling problems, the labels are numbers. However, the idea of assigning symbols other than numbers to the elements of a graph is almost as old as the study of graphs. For example, an old and very studied problem in graph theory consists of assigning colours to the vertices of a graph such that any two adjacent vertices receive distinct colours. Such assignment is called a proper-vertex-colouring of the graph and is formally defined as follows: a $k$-vertex-colouring of a graph $G$ is an assignment $f: V(G) \rightarrow \mathcal{L}$, where $\mathcal{L}$ is a set of $k$ colours. The set $\mathcal{L}$ of colours is usually $[1, k]$. We say that a vertex-colouring $f$ of $G$ is proper if no two adjacent vertices are assigned the same colour. The minimum $k$ for which a simple graph $G$ has a proper- $k$-vertex-colouring is called the chromatic number of $G$ and denoted by $\chi(G)$. Figure 1.17 shows two proper-vertex-colourings of a tree.

A $k$-vertex-colouring of a graph $G$ can also be viewed as a partition $\left\{V_{1}, \ldots, V_{k}\right\}$ of $V(G)$, where $V_{i}$ denotes the set of vertices assigned colour $i$. The sets $V_{i}$ are called the
colour classes of the colouring. A proper- $k$-vertex-colouring is then a $k$-vertex-colouring in which each colour class is an independent set. The idea of constrained partitioning is so fundamental that many variations and generalizations of vertex-colouring were proposed. For example, we may allow colour classes to induce subgraphs other than independent sets [74] or we may restrict the colours allowed on each vertex [41]. We can also ask questions involving numerical values when the colours are numbers. An example of the latter is given in the following.

Given a proper-vertex-colouring $f: V(G) \rightarrow \mathbb{N}_{>0}$ of a simple graph $G$, the colour sum of $f$ is the number $\sum_{v \in V(G)} f(v)$. The chromatic sum of a graph $G$, written $\sum(G)$, is the smallest colour sum among all proper-vertex-colourings of $G$ with elements of $\mathbb{N}_{>0}$. The strength of a graph $G$, written strength $(G)$, is the minimum number of colours necessary to obtain its chromatic sum. The chromatic sum problem was introduced by Kubicka [72], in 1989. The author proved that the problem of computing the chromatic sum for arbitrary graphs is $\mathcal{N} \mathcal{P}$-complete. She also observed that minimizing the colour sum may require using more than $\chi(G)$ colours. For instance, the proper-2-vertex-colouring of the tree in Figure 1.17 (a) has colour sum 12, while there exists a proper-3-vertex-colouring of the same tree with colour sum 11, as shown in Figure 1.17(b).


Figure 1.17: Two proper-vertex-colourings of the same tree.
Chromatic sums were investigated by Thomassen et al. [111], who proved that any connected graph $G$ has $\lceil\sqrt{8 e}\rceil \leq \sum(G) \leq\left\lfloor\frac{3}{2}(|E(G)|+1)\right\rfloor$. An interesting conjecture posed by Hajiabolhassan et al. [54] states that strength $(G) \leq\lceil(\chi(G)+\Delta(G)) / 2\rceil$.

In the last decades, new contexts have emerged where it is required to label the vertices or the edges of a given graph with numbers. More formally, given a graph $G$ and a set $\mathcal{L} \subset \mathbb{R}$, we want to construct a vertex-labelling $f: V(G) \rightarrow \mathcal{L}$ and an (induced) edgelabelling $g: E(G) \rightarrow \mathbb{R}$ such that the value of $g(u v)$ is a function of $f(u)$ and $f(v)$, for all $u v \in E(G)$, and $g$ respects some specified restrictions. Two possible ways of defining the induced function $g$ are: $g(u v)=f(u)+f(v)$ or $g(u v)=f(u) \cdot f(v)$, for all $u v \in E(G)$. Furthermore, an example of restriction imposed on $g$ could be that any two distinct edges of the graph are assigned distinct labels. As observed by Ringel and Hartsfield [97], labelling problems with these properties are different from vertex-colouring problems, since we shall use in their solution properties of numbers such as ordering and addition that are not properties of colours.

One of the oldest and most studied graph labellings that uses numbers as labels is the graceful labelling, defined as follows. Given a simple graph $G$ with $m$ edges, a graceful labelling of $G$ is an injective function $f: V(G) \rightarrow[0, m]$ such that

$$
\{|f(u)-f(v)|: u v \in E(G)\}=[1, m] .
$$

Note that the induced edge-labelling $g$ is implicitly defined: $g(u v)=|f(u)-f(v)|$ for all $u v \in E(G)$. Figure 1.18 shows $K_{4}$ with a graceful labelling. Graceful labellings are discussed in Chapter 2, Chapter 3 and Chapter 4 of this thesis.


Figure 1.18: Complete graph $K_{4}$ with a graceful labelling. The number in each edge $u v \in E\left(K_{4}\right)$ is its induced label $|f(u)-f(v)|$. All induced edge labels are distinct.

Note that a graceful labelling is a vertex-labelling that induces an edge-labelling of a graph. Different from graceful labellings, there are labelling problems which aim at primarily assigning an edge-labelling to a graph $G$ so as to induce a vertex-labelling of $G$ satisfying certain properties. An example is presented below.

Let $G=(V(G), E(G))$ be a simple graph and let $g: E(G) \rightarrow \mathcal{L}$ be an edge-labelling of $G$, where $\mathcal{L} \subset \mathbb{R}$. From $g$ we define $f: V(G) \rightarrow \mathbb{R}$ as $f(v)=\sum_{u v \in E(G)} g(u v)$ (that is, $v$ is assigned the sum of the labels of its incident edges). There are many conditions one can impose on $g$ and $f$ and then ask if they can simultaneously exist satisfying such conditions. For example, we can require $g$ to be any function from $E(G)$ to $\mathcal{L}$ but require $f$ to be a proper-vertex-colouring of $G$. In fact, this labelling is called a neighbour-distinguishing edge-labelling of $G$ and is investigated in Chapter 5 and Chapter 6 of this thesis. A graph with a neighbour-distinguishing edge-labelling is presented in Figure 1.19.


Figure 1.19: Graph $G$ with a neighbour-distinguishing edge-labelling $g: E(G) \rightarrow\{1,2\}$. The number inside each vertex $v \in V(G)$ corresponds to the induced label $f(v)=$ $\sum_{u v \in E(G)} g(u v)$. Note that $f$ is a proper-vertex-colouring of $G$.

There are labelling problems which aim at assigning labels to every element of a graph so as to obtain a second vertex-labelling of the graph. For example, let $G$ be a simple graph and let $g: V(G) \cup E(G) \rightarrow \mathcal{L}$ be a total-labelling of $G$, where $\mathcal{L} \subset \mathbb{R}$. From total-labelling $g$, define $f: V(G) \rightarrow \mathbb{R}$ such that, for each vertex $v \in V(G), f(v)=g(v)+\sum_{u v \in E(G)} g(u v)$.

Here, $g$ is allowed to be any function but $f$ is required to be a proper-vertex-colouring of $G$. This labelling is called a neighbour-distinguishing total-labelling of $G$ and is also investigated in Chapter 5 and Chapter 6 of this thesis. Figure 1.20 shows a graph with a neighbour-distinguishing total-labelling.


Figure 1.20: Graph $G$ with a neighbour-distinguishing total-labelling $g: V(G) \cup E(G) \rightarrow$ $\{1,2\}$. The number inside each vertex $v \in V(G)$ corresponds to the induced label $f(v)=$ $g(v)+\sum_{u v \in E(G)} g(u v)$. Note that $f$ is a proper-vertex-colouring of $G$.

Since the 1950's, many other labelling problems have been proposed and investigated. Some of the most studied graph labellings are graceful labellings [98], harmonious labellings [53], edge-magic total-labellings [101], antimagic labellings [97], and neighbourdistinguishing edge-labellings [65]. A survey on graph labellings containing recent results on the previous mentioned labellings has been continuously updated by Gallian [49]. Many other labelling problems are presented and investigated in the books of López and Muntaner-Batle [82], Marr and Wallis [83], Zhang [118] and also in the collection edited by Acharya, Arumugam and Rosa [4].

The remainder of the thesis is divided into six chapters. Chapter 2 is dedicated to graceful labellings of trees. In that chapter, we present the context in which graceful labellings were introduced and also survey the main results known in the literature related to the Graceful Tree Conjecture, which states that every tree has a graceful labelling. Chapters 3 and 4 contain our results on the Graceful Tree Conjecture. In Chapter 3, we present our results on graceful labellings of caterpillars and, in Chapter 4 we present our results on graceful labellings of lobsters with maximum degree three.

Chapter 5 describes the context in which neighbour-distinguishing edge-labellings and neighbour-distinguishing total-labellings of graphs were introduced. In that chapter, we briefly survey some known results on these labellings, including those that are used in our proofs. In Chapter 6, we present our results on neighbour-distinguishing edge-labellings and neighbour-distinguishing total-labellings of some families of graphs. That chapter is finalized with a discussion on the relation between neighbour-distinguishing edge-labellings and another kind of labelling called detectable edge-labelling. Chapter 7 summarises the contributions of this thesis and suggests some problems for future research.

## Chapter 2

## The Graceful Tree Conjecture

> When does a combinatorial problem become a disease? Certainly, the extreme ease of formulating the problem has something to do with it: most identified "diseases" are understandable to undergraduates or even to good high school students. They are highly contagious, and so they attract the attention of not only professional mathematicians but also of scores of layman mathematicians.

-Alexander Rosa [100]
In this chapter, we discuss graceful labellings of graphs. We start by giving a historical background of this problem, presenting classical results related to graceful labellings of trees and other families of graphs. We are particularly interested in graceful labellings of trees, specifically in the Graceful Tree Conjecture, posed by Anton Kotzig and popularized by Alexander Rosa's 1967 paper [98]. This conjecture remains open for about fifty years and has drawn much attention for its remarkably short and easily understandable statement and hardness to settle.

Let $G=(V(G), E(G))$ be a simple graph with $m$ edges. A graceful labelling of $G$ is a vertex-labelling $f: V(G) \rightarrow[0, m]$ such that
(i) $f$ is injective, and
(ii) $\{|f(u)-f(v)|: u v \in E(G)\}=[1, m]$.

If a graph $G$ admits a graceful labelling, we say that $G$ is graceful. Some examples of graceful graphs are illustrated in Figure 2.1.


Figure 2.1: Four gracefully labelled graphs. In each graph $G$, the edge $u v \in E(G)$ is labelled with its induced edge label $|f(u)-f(v)|$.

The origin of graceful labellings traces back to the Symposium on Graph Theory and Its Applications held in Smolenice in 1963, Czechoslovakia. At that symposium, Gerhard Ringel posed the following conjecture.

Conjecture 2.1 (Ringel's Conjecture [96]). If $T$ is an arbitrary tree with $m$ edges, then the complete graph $K_{2 m+1}$ can be decomposed into $2 m+1$ subgraphs isomorphic to $T$.

Conjecture 2.1 is known as Ringel's Conjecture and it remains open until the conclusion of this thesis. The introduction and development of graceful labellings and most of their variants were motivated by Ringel's Conjecture. In 1966, in the next symposium, Rosa [98] reported Kotzig's suggestion of adding a stronger constraint to Ringel's Conjecture as a way of approaching it.

Conjecture 2.2 (Ringel-Kotzig Conjecture). If $T$ is an arbitrary tree with $m$ edges, then the complete graph $K_{2 m+1}$ can be cyclically decomposed into $2 m+1$ subgraphs isomorphic to $T$.

Intuitively, such a cyclic decomposition of $K_{2 m+1}$ can be accomplished by: (a) choosing an arbitrary tree $T$ with $m$ edges; (b) identifying the edges of $T$ with a suitable set of edges of $K_{2 m+1}$; and then (c) "rotating" the subgraph $T \subset K_{2 m+1} 2 m$ times from its original position. After the final rotation, every edge of $K_{2 m+1}$ will have been covered exactly once by one of the $2 m+1$ positions of $T$. Figure 2.2 shows one such cyclic decomposition of $K_{5}$ into five copies of $P_{3}$.


Figure 2.2: A cyclic decomposition of $K_{5}$.
Rosa observed that steps (b) and (c) of the above procedure could be accomplished by assigning an appropriate labelling to the vertices of the tree $T$. As a way to attack the Ringel-Kotzig Conjecture, he introduced four types of labellings [98], defined below.

Let $G$ be a simple graph with $m$ edges and with an injective labelling $f: V(G) \rightarrow \mathbb{Z}_{\geq 0}$. Let $L_{V(G)}^{f}=\{f(v): v \in V(G)\}$ be the set that comprises the labels of the vertices of $G$ and let $L_{E(G)}^{f}=\{|f(u)-f(v)|: u v \in E(G)\}$ be the set that comprises the induced labels of the edges of $G$. Consider the following statements:
(a) $L_{V(G)}^{f} \subseteq\{0,1, \ldots, m\}$;
(b) $L_{V(G)}^{f} \subseteq\{0,1, \ldots, 2 m\}$;
(c) $L_{E(G)}^{f}=\{1,2, \ldots, m\}$;
(d) $L_{E(G)}^{f}=\left\{x_{1}, x_{2}, \ldots, x_{m}\right\}$, such that $x_{i}=i$ or $x_{i}=2 m+1-i$;
(e) there exists $k$, with $k \in\{0,1, \ldots, m\}$, such that, for each edge $u v \in E(G)$, either $f(u) \leq k<f(v)$ or $f(v) \leq k<f(u)$. The constant $k$ is called the separator of the labelling $f$.

Based on the previous five statements, Rosa [98] defined the following labellings:
(i) If $f$ satisfies (a), (c) and (e), then $f$ is called an $\alpha$-labelling.
(ii) If $f$ satisfies (a) and (c), then $f$ is called a $\beta$-labelling.
(iii) If $f$ satisfies (b) and (c), then $f$ is called a $\sigma$-labelling.
(iv) If $f$ satisfies (b) and (d), then $f$ is called a $\rho$-labelling.

Figure 2.3 illustrates a tree $T$ with each one of these labellings.

(a) $\alpha$-labelling of $T$ with separator $k=2$.

(b) $\beta$-labelling of $T$.

(c) $\sigma$-labelling of $T$.

(d) $\rho$-labelling of $T$.

Figure 2.3: Four different labellings of a tree $T$ with five edges.
Observe that, by the definition: an $\alpha$-labelling is also a $\beta$-, $\sigma$-, and $\rho$-labelling; a $\beta$ labelling is also a $\sigma$ - and $\rho$-labelling; and a $\sigma$-labelling is also a $\rho$-labelling. However, the converse is not true, that is, a $\rho$-labelling may not be a $\sigma$-labelling (see Figure 2.3(d)), a $\sigma$-labelling may not be a $\beta$-labelling (see Figure 2.3(c)), and a $\beta$-labelling may not be an $\alpha$-labelling (see Figure 2.3(b)).

Note that $\beta$-labellings and graceful labellings are the same. In fact, the name graceful was created by Solomon Golomb [52] in 1972, in a independent work, and it was later popularized by Martin Gardner [51]. Since then, the name graceful labelling is commonly used.

Rosa [98] observed that if a simple graph $G$ with $m$ edges cyclically decomposes the complete graph $K_{2 m+1}$, then $G$ naturally possesses a $\rho$-labelling. In fact, Rosa proved that possessing a $\rho$-labelling is a sufficient condition for $G$ to cyclically decompose $K_{2 m+1}$.

Theorem 2.3 (Rosa [98]). Let $G$ be a simple graph with $m$ edges. The complete graph $K_{2 m+1}$ has a cyclic decomposition into $2 m+1$ copies of $G$ if and only if $G$ has a $\rho$-labelling.

Proof. Let $G$ be a simple graph with $m$ edges and $K_{2 m+1}$ a complete graph with vertex set $V\left(K_{2 m+1}\right)=\left\{v_{0}, \ldots, v_{2 m}\right\}$. From the definition of reach ${ }^{1}$ of an edge of $K_{n}$, it follows that there are only edges with reaches $1, \ldots, m$ in $K_{2 m+1}$ and, for any fixed $i \in\{1, \ldots, m\}$, exactly $2 m+1$ edges of $K_{2 m+1}$ have reach $i$ (these are the edges $v_{0} v_{i}, v_{1} v_{i+1}, \ldots, v_{2 m} v_{i+2 m}$, where operations on indices are taken modulo $2 m+1$ ) obtained, for example, by rotating consecutively $2 m$ times any one of the edges with reach $i$. By rotating an edge of reach $i$, a new edge with the same reach is obtained.

First, suppose that $K_{2 m+1}$ has a cyclic decomposition into $2 m+1$ copies of $G$. Take an arbitrary subgraph $H \cong G$ of the $2 m+1$ subgraphs of this decomposition. Next, we

[^0]prove that the edges of $H$ have mutually different reaches in $K_{2 m+1}$. Suppose that $H$ contains two edges of reach $i, 1 \leq i \leq m$, for example, $v_{x} v_{x+i}$ and $v_{y} v_{y+i}$ with $x \neq y$ and $x<y$. By the definition of a cyclic decomposition, this decomposition contains the graph $H^{(y-x)}$, obtained from $H$ by rotating it $y-x$ times. But then this graph contains edge $v_{x+(y-x)} v_{x+i+(y-x)}=v_{y} v_{y+i}$, which is a contradiction to the definition of decomposition of a graph. Thus, all edges of $H$ have different reaches in $K_{2 m+1}$, which means that the injective vertex-labelling $f: V(H) \rightarrow[0,2 m]$ defined by $f\left(v_{i}\right)=i$, for every $v_{i} \in V(H)$, is a $\rho$-labelling of $H$. Finally, since $H \cong G$, we obtain that $G$ has a $\rho$-labelling, as asserted.

Now, suppose that $G$ has a $\rho$-labelling $f: V(G) \rightarrow[0,2 m]$. We obtain a subgraph $G_{0} \subset K_{2 m+1}, G_{0} \cong G$, as follows: the edge with endpoints $v_{i}$ and $v_{j}$, as well as the vertices $v_{i}$ and $v_{j}$, belong to $G_{0}$ if and only if $G$ contains an edge whose endpoints are labelled $i$ and $j$ under labelling $f$. By the definition of $\rho$-labelling, the set of induced edge labels under $f$ is $\left\{x_{1}, \ldots, x_{m}\right\}$, where $x_{\ell}=\ell$ or $x_{\ell}=2 m+1-\ell$, for $\ell \in\{1, \ldots, m\}$. Recall that the reach of edge $v_{i} v_{j} \in E\left(K_{2 m+1}\right)$ is $|i-j|$ if $|i-j| \leq m$, or $2 m+1-|i-j|$ if $|i-j|>m$. These facts imply that the edges of $G_{0}$ have mutually different reaches in $K_{2 m+1}$ and these reaches are $1,2, \ldots, m$.

For each $k \in\{1, \ldots, 2 m\}$, we obtain a new subgraph $G_{k} \subset K_{2 m+1}$ by rotating $G_{0}$ exactly $k$ times. Note that each of the subgraphs $G_{0}, G_{1}, \ldots, G_{2 m}$ has exactly one edge of reach $i$, for $i \in\{1, \ldots, m\}$. Moreover, by the construction of subgraphs $G_{0}, \ldots, G_{2 m}$, each one of the $2 m+1$ edges of reach $i, 1 \leq i \leq m$, belongs to exactly one of these subgraphs. Since $G_{0} \cong G$, the same holds for each $G_{i}$, with $1 \leq i \leq 2 m$. Therefore, $\left\{G_{0}, \ldots, G_{2 m}\right\}$ is a cyclic decomposition of $K_{2 m+1}$ into $2 m+1$ copies of $G$.

Figure 2.4 shows a cyclic decomposition of $K_{9}$ into nine subgraphs isomorphic to $C_{4}$. Since every $\alpha$-labelling is also a $\rho$-labelling, by Theorem 2.3 , we conclude that if a simple graph $G$ with $m$ edges has an $\alpha$-labelling, then $K_{2 m+1}$ has a cyclic decomposition into $2 m+1$ subgraphs isomorphic to $G$. In fact, Rosa proved a stronger result concerning $\alpha$-labellings, as follows.

Theorem 2.4 (Rosa [98]). If a graph $G$ with $m$ edges has an $\alpha$-labelling, then there exists a cyclic decomposition of $K_{2 p m+1}$ into subgraphs isomorphic to $G$, where $p$ is an arbitrary positive integer.

Proof. Let $G$ be a graph with $m$ edges and $p$ an arbitrary positive integer. Suppose that $G$ has an $\alpha$-labelling $f$. By the definition of $\alpha$-labellings, $L_{V(G)}^{f} \subseteq\{0, \ldots, m\}$. Adjust notation so that $V(G) \subseteq\left\{v_{0}, v_{1}, \ldots, v_{m}\right\}$ with $f\left(v_{i}\right)=i$. Also, by the definition of $\alpha$ labellings, there exists $k \in[0, m]$ such that, for each edge $v_{i} v_{j} \in E(G)$, either $i \leq k<j$ or $j \leq k<i$. Adjust notation so that, for each $v_{i} v_{j} \in E(G), i \leq k<j$.

Next, we construct $p$ edge-disjoint graphs $G^{1}, \ldots, G^{p}$, each isomorphic to $G$. Let $A=\left\{v_{i} \in V(G): i \leq k\right\}$ and $B=V(G) \backslash A$. For each $r \in\{1, \ldots, p\}$, define graph $G^{r}$ as follows:
(i) $V\left(G^{r}\right)=A \cup B^{r}$, where $B^{r}=\left\{v_{j+m(r-1)}: j \in L_{B}^{f}\right\}$;
(ii) $E\left(G^{r}\right)=\left\{v_{i} v_{j+m(r-1)}: v_{i} \in A, j \in L_{B}^{f}\right.$ and $\left.v_{i} v_{j} \in E(G)\right\}$.


Figure 2.4: Cycle $C_{4}$ as a subgraph of $K_{9}$. Note that the subgraph on the upper left corner is shown with a graceful labelling (which is also a $\rho$-labelling). The subgraphs obtained by rotating the first subgraph eight times form a cyclic decomposition of $K_{9}$.

By the definition, each $G^{r}$ is isomorphic to $G$. One can see this analysing sets $A$ and $B^{r}$ and observing the one-to-one relation between the edges of $G$ and the edges of $G^{r}$. Note that $G^{1}=G$. Also, since each edge in $G^{r}$ has one endpoint in $A$ and the other endpoint in $B^{r}$ and the sets $B^{1}, \ldots, B^{p}$ are pairwise disjoint, it follows that graphs $G^{1}, \ldots, G^{p}$ are edge-disjoint.

Now, define $G^{\prime}=\cup_{r=1}^{p} G^{r}$. Since $G^{1}, \ldots, G^{p}$ are edge-disjoint and each isomorphic to $G$, we have that $\left|E\left(G^{\prime}\right)\right|=p|E(G)|=p m$. By construction, $V\left(G^{\prime}\right) \subseteq\left\{v_{0}, \ldots, v_{p m}\right\}$. Define labelling $g: V\left(G^{\prime}\right) \rightarrow\{0, \ldots, p m\}$ as $g\left(v_{i}\right)=i$, for $v_{i} \in V\left(G^{\prime}\right)$.

In order to see that $g$ is an $\alpha$-labelling of $G^{\prime}$, first note that $g$ is injective, by construction of $G^{1}, \ldots, G^{p}$. Moreover, for each edge $v_{i} v_{j+m(r-1)} \in E\left(G^{r}\right)$, we have

$$
\begin{align*}
\left|g\left(v_{j+m(r-1)}\right)-g\left(v_{i}\right)\right| & =|(j+m(r-1))-i| \\
& =(j-i)+m(r-1) \\
& =\left(f\left(v_{j}\right)-f\left(v_{i}\right)\right)+m(r-1) . \tag{2.1}
\end{align*}
$$

By (2.1) and the fact that $f$ is an $\alpha$-labelling of $G$, we have that $L_{E\left(G^{r}\right)}^{g}=\{\ell+m(r-$ $1): 1 \leq \ell \leq m\}$, for $1 \leq r \leq p$. This implies that $L_{E\left(G^{\prime}\right)}^{g}=\cup_{r=1}^{p} L_{E\left(G^{r}\right)}^{g}=\{1, \ldots, p m\}$. Finally, by the definition of $G^{\prime}$, for each edge $v_{i} v_{j+m(r-1)} \in E\left(G^{\prime}\right)$, we have that $i \leq k$ and $j+m(r-1) \geq j+m(1-1)=j>k$. Hence, $g$ is an $\alpha$-labelling.

Since $g$ is an $\alpha$-labelling, it is also a $\rho$-labelling. Therefore, by Theorem 2.3, there exists a cyclic decomposition of $K_{2 p m+1}$ into $2 p m+1$ subgraphs isomorphic to $G^{\prime}$. Since $G^{\prime}$ is the edge-disjoint union of $p$ copies of $G$, the result follows.

Theorem 2.3 and Theorem 2.4 stress the importance of Rosa's labellings to the cyclic
decomposition of complete graphs. Theorem 2.3 implies that the Ringel-Kotzig Conjecture is true if and only if every tree has a $\rho$-labelling. On the other hand, by Theorem 2.4, proving that a graph $G$ with $m$ edges has an $\alpha$-labelling is stronger than simply showing that it has a $\rho$-labelling. For this reason, when considering families of graphs, many authors have put their efforts into finding the strongest labellings in Rosa's hierarchy, namely $\alpha$-labellings.

In fact, Rosa [98] was the first to follow this approach. When considering the family of trees, he proved that not all trees have an $\alpha$-labelling. For example, the smallest tree that does not have an $\alpha$-labelling is exhibited in Figure 2.5. (In Theorem 2.47 it is proved that all trees with diameter four that are not caterpillars do not have $\alpha$-labellings.)


Figure 2.5: The smallest tree that does not have an $\alpha$-labelling.
Subsequently, Rosa considered graceful labellings of trees and, as he did not find any trees without a graceful labelling and proved that many subfamilies of trees have graceful labellings, he posed the following conjecture.

Conjecture 2.5 (The Graceful Tree Conjecture [98]). All trees are graceful.
The Graceful Tree Conjecture is a very important and challenging open problem in Graph Theory, with hundreds of papers about it [49]. Although this conjecture has been extensively studied, little is known about its validity for arbitrary trees. However, it has been verified for several subfamilies of trees [24,58,61,98]. Currently, it is known that all trees with at most 35 vertices are graceful [46]. Recent results involving graceful labelling are maintained and annually updated by Gallian in his dynamic survey on graceful and related labellings [49].

### 2.1 Early results and constructions

In the literature of graceful labellings of trees, there are some constructions and results that have been extensively used since the very beginning of the study of these labellings. Most of these constructions trace back to the first articles of Kotzig and Rosa on graceful labellings $[69,98,99]$. We start by presenting some of these techniques that are used later.

A result that follows immediately from the definition of graceful labelling is that, if a graph $G$ has $|E(G)| \leq|V(G)|-2$, then $G$ is not graceful, since the number of available labels is not enough to label all the vertices of $G$. Another simple but important observation, this time concerning $\alpha$-labellings, is established in Lemma 2.6 below.

Lemma 2.6 (Rosa [98]). If a graph $G$ has an $\alpha$-labelling, then $G$ is a bipartite graph. Moreover, $G$ has a bipartition $\{A, B\}$ such that the labels of the vertices in $A$ are smaller than the labels of the vertices in $B$.

Proof. Let $G$ be a graph with an $\alpha$-labelling $f$ that has separator ${ }^{2} k$. Define a partition of $V(G)$ into sets $A$ and $B$ such that $A=\{v \in V(G): f(v) \leq k\}$ and $B=V(G) \backslash A$. For each edge $u v \in E(G)$, we have that either $f(u) \leq k<f(v)$ or $f(v) \leq k<f(u)$ since $f$ is an $\alpha$-labelling. This implies that, for each edge $u v \in E(G), u$ and $v$ belong to different parts. Therefore, $\{A, B\}$ is a bipartition of $G$ with the required properties.

From Lemma 2.6, we obtain the following corollary.
Corollary 2.7. If a tree $T$ has an $\alpha$-labelling $f$, then there exists a bipartition $\{A, B\}$ of $T$ such that $L_{A}^{f}=\{0, \ldots,|A|-1\}$ and $L_{B}^{f}=\{|A|, \ldots,|A|+|B|-1\}$.

Let $G$ be a simple graph with $m$ edges and a graceful labelling $f$. The complementary labelling of $f$ is the function $\bar{f}: V(G) \rightarrow[0, m]$ such that $\bar{f}(v)=m-f(v)$, for every $v \in V(G)$. The function $\bar{f}$ is also a graceful labelling of $G$ since:
(i) $\bar{f}$ is an injective labelling from $V(G)$ to $[0, m]$; and
(ii) for every edge $u v \in E(G),|\bar{f}(u)-\bar{f}(v)|=|m-f(u)-(m-f(v))|=|f(v)-f(u)|$.

Therefore, the induced edge labels remain unchanged in the complementary labelling of $f$. Moreover, vertex $v \in V(G)$ with $f(v)=0$ has label $m$ in the complementary labelling $\bar{f}$. Figure 2.6(a) shows a tree $T$ with an $\alpha$-labelling $f$ and Figure 2.6(b) shows its complementary labelling $\bar{f}$. The next lemma shows that if $f$ is an $\alpha$-labelling, then also is $\bar{f}$.

Lemma 2.8 (Rosa [99]). Let $G$ be a graph with $m$ edges. If $G$ has an $\alpha$-labelling $f$ with separator $k$, then its complementary labelling $\bar{f}$ is an $\alpha$-labelling with separator $m-k-1$.

Proof. Let $\bar{f}$ be the complementary labelling of $f$. Since $f$ is a graceful labelling, $\bar{f}$ is also graceful. Let $u v \in E(G)$. Adjust notation so that $f(u) \leq k<f(v)$. Then,

$$
\begin{array}{r}
f(u) \leq k<f(v), \\
-f(u) \geq-k>-f(v), \\
m-f(u) \geq m-k>m-f(v) .
\end{array}
$$

Since $\bar{f}(w)=m-f(w)$ for all $w \in V(G)$, we have that

$$
\begin{aligned}
\bar{f}(u) \geq m-k & >\bar{f}(v), \\
\bar{f}(u)>m-k-1 & \geq \bar{f}(v) .
\end{aligned}
$$

Therefore, $\bar{f}$ is an $\alpha$-labelling of $G$ with separator $m-k-1$.
Let $G$ be a graph with $m$ edges and an $\alpha$-labelling $f$ with separator $k$. The reverse labelling of $f$ is defined by:

$$
\hat{f}(v)= \begin{cases}k-f(v), & \text { if } f(v) \leq k ; \\ m+k+1-f(v), & \text { if } f(v)>k\end{cases}
$$

[^1]Figure 2.6(a) shows a tree $T$ with an $\alpha$-labelling $f$ and Figure 2.6(c) shows the reverse labelling of $f$. The next lemma shows that $\hat{f}$ is also an $\alpha$-labelling of $G$.

Lemma 2.9 (Rosa [99]). If a graph $G$ has an $\alpha$-labelling $f$ with separator $k$, then its reverse labelling $\hat{f}$ is also an $\alpha$-labelling with separator $k$.

Proof. Let $G$ and $f$ be as stated in the hypothesis and $m=|E(G)|$. Let $u v \in E(G)$. Adjust notation so that $0 \leq f(u) \leq k<f(v) \leq m$. By a similar reasoning presented in the proof of Lemma 2.8, we have that $\hat{f}(u) \leq k<\hat{f}(v)$. Moreover, $|\hat{f}(v)-\hat{f}(u)|=$ $|(m+k+1-f(v))-(k-f(u))|=|(m+1)-(f(v)-f(u))|$. Therefore, the induced edge labels under $\hat{f}$ are also $1,2, \ldots, m$ and $\hat{f}$ is an $\alpha$-labelling of $G$ with separator $k$.

Let $G$ be a bipartite graph and $v \in V(G)$. Note that, if $G$ has an $\alpha$-labelling $f$ with separator $k$ such that $f(v)=0$, then the reverse labelling $\hat{f}$ assigns label $k$ to $v$. Labelling $\hat{f}$ is called the "reverse" of $f$ because it reverses the order of the edge labels, that is, for $u v \in E(G)$, we have $|\hat{f}(v)-\hat{f}(u)|=(m+1)-|f(v)-f(u)|$.

(a) $\alpha$-labelling $f$.

(b) Complementary labelling $\bar{f}$.

(c) Reverse labelling $\hat{f}$.

Figure 2.6: An $\alpha$-labelling $f$ of a tree $T$ and its complementary and reverse labellings.
One of the great challenges in solving the Graceful Tree Conjecture is how to arbitrarily combine two gracefully labelled trees in order to obtain a larger graceful tree. No general method that allows us to do such an arbitrary gluing and subsequent adjustment of labellings is known. Nevertheless, there are some constructions that allow us to take two gracefully labelled trees and combine them in specific ways in order to obtain a larger graceful tree [59, 108]. In the next result, we present one of these constructions, due to Huang, Kotzig and Rosa [59].

Theorem 2.10 (Huang et al. [59]). Let $T_{1}$ and $T_{2}$ be disjoint trees with $v_{1} \in V\left(T_{1}\right)$ and $v_{2} \in V\left(T_{2}\right)$. Let $f_{1}$ be an $\alpha$-labelling of $T_{1}$ with $f_{1}\left(v_{1}\right)=0$, and $f_{2}$ be a graceful labelling ( $\alpha$-labelling) of $T_{2}$ with $f_{2}\left(v_{2}\right)=0$. Then, tree $T$, obtained by identifying $v_{1}$ and $v_{2}$, has a graceful labelling ( $\alpha$-labelling).

Proof. Let $T, T_{1}, T_{2}, f_{1}$, and $f_{2}$ be as stated in the hypothesis. Let $k$ be the separator of $\alpha$ labelling $f_{1}$. Define $m_{1}=\left|E\left(T_{1}\right)\right|$ and $m_{2}=\left|E\left(T_{2}\right)\right|$. By Lemma 2.9, the reverse labelling $\hat{f}_{1}$ is an $\alpha$-labelling of $T_{1}$ with separator $k$. Define labelling $f: V(T) \rightarrow\left[0, m_{1}+m_{2}\right]$ as follows:

$$
f(v)= \begin{cases}\hat{f}_{1}(v), & \text { if } v \in V\left(T_{1}\right) \text { and } \hat{f}_{1}(v) \leq k ; \\ \hat{f}_{1}(v)+m_{2}, & \text { if } v \in V\left(T_{1}\right) \text { and } \hat{f}_{1}(v)>k \\ k+f_{2}(v), & \text { if } v \in V\left(T_{2}\right)\end{cases}
$$

First, recall that $\hat{f}_{1}\left(v_{1}\right)=k$ and $f_{2}\left(v_{2}\right)=0$. Therefore, $f\left(v_{1}\right)=\hat{f}_{1}\left(v_{1}\right)=k=$ $k+0=k+f_{2}\left(v_{2}\right)=f\left(v_{2}\right)$ and labelling $f$ is well-defined. Next, we prove that $f$ is graceful. Observe that $f$ is injective, $L_{V\left(T_{1}\right)}^{f}=[0, k] \cup\left[k+1+m_{2}, m_{1}+m_{2}\right]$ and $L_{V\left(T_{2}\right) \backslash\left\{v_{2}\right\}}^{f}=\left[k+1, k+m_{2}\right]$. Therefore, $L_{V(T)}^{f}=\left[0, m_{1}+m_{2}\right]$. It remains to show that $L_{E(T)}^{f}=\left[1, m_{1}+m_{2}\right]$. Since every vertex $v \in V\left(T_{2}\right)$ has $f(v)=k+f_{2}(v)$, for every edge $u v \in E\left(T_{2}\right),|f(v)-f(u)|=\left|f_{2}(v)-f_{2}(u)\right|$. Therefore, $L_{E\left(T_{2}\right)}^{f}=L_{E\left(T_{2}\right)}^{f_{2}}=\left[1, m_{2}\right]$. Additionally, $L_{E\left(T_{1}\right)}^{f}=\left[m_{2}+1, m_{2}+m_{1}\right]$ since, for each edge $u v \in E\left(T_{1}\right),|f(v)-f(u)|=$ $\left|\hat{f}(v)+m_{2}-\hat{f}(u)\right|=m_{2}+|\hat{f}(v)-\hat{f}(u)|$. Therefore, $L_{E(T)}^{f}=\left[1, m_{1}+m_{2}\right]$ and $f$ is a graceful labelling of $T$.

Now, suppose that $f_{2}$ is an $\alpha$-labelling of $T_{2}$ with separator $k_{2}$, for $k_{2} \in\left[0, m_{2}\right]$. Since $f$ is graceful, it remains to show that $f$ has a separator $k+k_{2}$. By Lemma 2.9, $\hat{f}_{1}$ is an $\alpha$-labelling of $T_{1}$ with separator $k$. Thus, for each edge $u v \in E\left(T_{1}\right)$, we have that $\hat{f}_{1}(u) \leq k<\hat{f}_{1}(v)$. By this fact and by the definition of $f$, for every edge $u v \in E\left(T_{1}\right)$, $f(u)=\hat{f}_{1}(u) \leq k \leq k+k_{2}<(k+1)+k_{2} \leq \hat{f}_{1}(v)+m_{2}=f(v)$, as required. Since $f_{2}$ is an $\alpha$-labelling of $T_{2}$, for each edge $u v \in E\left(T_{2}\right), f_{2}(u) \leq k_{2}<f_{2}(v)$. By this fact and by the definition of $f$ we have that, for each edge $u v \in E\left(T_{2}\right), f(u)=k+f_{2}(u) \leq k+k_{2}<$ $k+f_{2}(v)=f(v)$. Therefore, for each edge $u v \in E(T), f(u) \leq k+k_{2}<f(v)$, and the result follows.

Note that we can easily grow a gracefully labelled tree $T$ by adding $k$ new leaves to the vertex with label 0 and expanding the graceful labelling by assigning labels from the set $[|E(T)|+1,|E(T)|+k]$ to these new leaves. From this observation, we obtain the following result.

Lemma 2.11. Let $T$ be a tree and $T^{\prime}$ obtained from $T$ by adding $k$ new leaves $u_{1}, \ldots, u_{k}$ to a vertex $v \in V(T)$. If there exists a graceful labelling $f$ of $T$ such that $f(v)=0$, then there exists a graceful labelling $f^{\prime}$ of $T^{\prime}$ such that

$$
f^{\prime}(u)= \begin{cases}i+|E(T)|, & \text { if } u=u_{i}, 1 \leq i \leq k ; \\ f(u), & \text { otherwise }\end{cases}
$$

The procedure presented in Lemma 2.11 also works if we add $k$ new leaves to the vertex with label $|E(T)|$ : in this case, we first apply the complementary labelling to $T$ and label the $k$ new leaves as before. On the other hand, if the $k$ new leaves are added to any other vertex of $T$ other than the vertices with labels 0 and $|E(T)|$, no general method is known to relabel $T$ so as to obtain a gracefully labelled tree.

In the context of the Ringel-Kotzig Conjecture, Kotzig [69] proved that if we add any number of leaves to arbitrary vertices of a gracefully labeled tree, the resulting tree $T$ has a $\sigma$-labelling and, therefore, $T$ cyclically decomposes the complete graph $K_{2|E(T)|+1}$, by Theorem 2.3.

Theorem 2.12 (Kotzig [69]). Let $T_{1}$ be an arbitrary tree and $T_{2}$ be the tree obtained from $T_{1}$ by adding an arbitrary nonnegative number of leaves to each vertex of $T_{1}$. If $T_{1}$ has a graceful labelling, then $T_{2}$ has a $\sigma$-labelling.

Proof. Let $T_{1}$ and $T_{2}$ be as stated in the hypothesis, $m_{1}=\left|E\left(T_{1}\right)\right|$ and $m_{2}=\left|E\left(T_{2}\right)\right|$. Suppose that $T_{1}$ has a graceful labelling $f: V\left(T_{1}\right) \rightarrow\left[0, m_{1}\right]$. Adjust notation so that $V\left(T_{1}\right)=\left\{v_{0}, \ldots, v_{m_{1}}\right\}$ and $f\left(v_{i}\right)=i$, for $0 \leq i \leq m_{1}$.

If $m_{1}=m_{2}$, the result follows. Then, suppose $m_{1}<m_{2}$. By the definition of $T_{2}$, set $S=V\left(T_{2}\right) \backslash V\left(T_{1}\right)$ contains only leaves from $T_{2}$. Let $s=|S|$. Denote by $u_{1}, \ldots, u_{s}$ the vertices of $S$ and let $v_{x(i)}$ be the neighbour of $u_{i}$ in $T_{2}$. Adjust notation so that, for $u_{i}$ and $u_{j}$ with $i<j, x(i) \leq x(j)$. Define the labels of vertices of $S$ as follows: $f\left(u_{i}\right)=m_{1}+i+x(i)$, for $1 \leq i \leq s$.

In order to see that $f$ is a $\sigma$-labelling of $T_{2}$, first note that, for every edge $u_{i} v_{x(i)} \in E\left(T_{2}\right)$, $\left|f\left(u_{i}\right)-f\left(v_{x(i)}\right)\right|=\left|\left(m_{1}+i+x(i)\right)-x(i)\right|=m_{1}+i$, for $1 \leq i \leq s$. This implies that $L_{E\left(T_{2}\right)}^{f}=L_{E\left(T_{1}\right)}^{f} \cup\left[m_{1}+1, m_{1}+s\right]=\left[1, m_{1}\right] \cup\left[m_{1}+1, m_{1}+s\right]=\left[1, m_{2}\right]$. It remains to show that $f$ is injective and that $L_{V\left(T_{2}\right)}^{f} \subseteq\left[0,2 m_{2}\right]$. By the hypothesis, $\left\{f(v): v \in V\left(T_{1}\right)\right\}=\left[0, m_{1}\right]$. Furthermore, since $f\left(u_{i}\right)=m_{1}+i+x(i)$, where $1 \leq i \leq s$ and $0 \leq x(i) \leq m_{1}$, we conclude that: (i) any two vertices $u_{i}$ and $u_{j}$ with $i<j$ have $f\left(u_{i}\right)<f\left(u_{j}\right)$; (ii) for every $i \in[1, s], m_{1}+1 \leq f\left(u_{i}\right) \leq 2 m_{1}+s<2\left(m_{1}+s\right)=2 m_{2}$. These facts imply that $f$ is injective, that $L_{V\left(T_{2}\right)}^{f} \subseteq\left[0,2 m_{2}\right]$, and the result follows.

Figure 2.7 illustrates the construction presented in the proof of Theorem 2.12.

(a) Tree $T_{1}$ with a graceful labelling.

(b) Tree $T_{2}$ with a $\sigma$-labelling.

Figure 2.7: A tree $T_{2}$ with a $\sigma$-labelling obtained from a tree $T_{1}$ with a graceful labelling.

From Theorem 2.12, we immediately obtain the following corollary.
Corollary 2.13 (Kotzig [69]). Let $T$ be a tree and $T^{\prime} \subset T$ be the connected subgraph obtained from $T$ by removing all of its leaves. If $T^{\prime}$ has a graceful labelling, then $T$ has a $\sigma$-labelling.

In Theorem 2.10 and Lemma 2.11, we saw that it is possible to expand a graceful tree by adding new leaves to the vertex with label 0 and also that, by identifying the 0 -labelled vertices of two gracefully labelled trees, we obtain a graceful tree. In fact, many techniques were proposed aiming at generating new families of graceful trees through some kind of product or identification of two trees on specific vertices [67,68,84,108,116]. The following theorem presents one of the earliest techniques, proposed by Stanton and Zarnke [108].

Theorem 2.14 (Stanton and Zarnke [108]). Let $S$ and $T$ be trees with $n_{S}$ and $n_{T}$ vertices, respectively, and let $\{A, B\}$ be a bipartition of $T$. Also, let $V(S)=\left\{u_{0}, \ldots, u_{n_{S}-1}\right\}$, $V(T)=\left\{v_{0}, \ldots, v_{n_{T}-1}\right\}$ and $v^{*}$ be a distinguished vertex of $T$. Let $T_{0}, T_{1}, \ldots, T_{n_{S}-1}$ be $n_{S}$ copies of tree $T$. Finally, let $S \Delta T$ be the tree obtained by identifying each vertex $u_{i}$ of $S$
with the vertex $v^{*}$ of the copy $T_{i}$ of $T, 0 \leq i \leq n_{S}-1$. If $S$ and $T$ have graceful labellings $f_{S}$ and $f_{T}$, respectively, then $S \Delta T$ has a graceful labelling $f$ defined as follows: for each $i \in\left[0, n_{S}-1\right]$, vertex $v_{j} \in V\left(T_{i}\right)$ receives label

$$
f\left(v_{j}\right)= \begin{cases}f_{S}\left(u_{i}\right) n_{T}+f_{T}\left(v_{j}\right), & \text { if } v_{j} \in A \\ \left(n_{S}-f_{S}\left(u_{i}\right)-1\right) n_{T}+f_{T}\left(v_{j}\right), & \text { if } v_{j} \in B\end{cases}
$$

The construction presented in the statement of Theorem 2.14 is called $\Delta$-construction and it is illustrated in Figure 2.8.

(a) Tree $S$ with a graceful labelling $f_{S}$, vertex set $V(S)=\left\{u_{0}, \ldots, u_{7}\right\}$ such that $f_{S}\left(u_{i}\right)=i$, for $0 \leq i \leq 7$.

(b) Tree $T$ with a graceful labelling $f_{T}$ and with $V(T)=$ $\left\{v_{0}, \ldots, v_{3}\right\}$ such that $f_{T}\left(v_{i}\right)=i$, for $0 \leq i \leq 3$. $T$ has bipartition $\{A, B\}$ such that $A=\left\{v_{0}\right\}$ and $B=V(T) \backslash A$. Moreover, $v^{*}=v_{0}$.

(c) Tree $S \Delta T$ with a graceful labelling $f$.

Figure 2.8: Trees $S, T$ and $S \Delta T$ with graceful labellings $f_{S}, f_{T}$ and $f$, respectively. Graceful labelling $f$ is constructed from graceful labellings $f_{S}$ and $f_{T}$ as defined in the statement of Theorem 2.14.

It is notorious the importance of label 0 in a graceful labelling of a tree $T$ : first, by Lemma 2.11, we know that we can grow a gracefully labelled tree $T$ by adding $k$ new leaves to its 0-labelled vertex; furthermore, by Theorem 2.10, it is possible to combine any tree with an $\alpha$-labelling and any tree with a graceful labelling, by identifying the vertices labelled 0 , such that the resulting tree is graceful. A natural question that arises from this observation is the following: given a tree $T$ and an arbitrary vertex $v \in V(T)$, does there exist a graceful labelling of $T$ that assigns label 0 to vertex $v$ ? This question was first studied by Rosa [98]. In fact, the author proved that the answer is affirmative for all paths.

Theorem 2.15 (Rosa [99]). Let $n$ be an arbitrary natural number and let $v$ be an arbitrary vertex of the path $P_{n}$. Then,
(i) there exists an $\alpha$-labelling of $P_{n}$ with $f(v)=0$ if and only if $v$ is not the central vertex of $P_{5}$;
(ii) if $v$ is the central vertex of $P_{5}$, then $P_{5}$ has a graceful labelling $f$ with $f(v)=0$.

In the context of the importance of label 0 for graceful labellings, we say that a tree $T$ is 0 -rotatable if, for every vertex $v \in V(T)$, there exists a graceful labelling $f$ of $T$ such that $f(v)=0$. By Theorem 2.15, we have that all paths are 0-rotatable. However, not all trees are 0-rotatable [40]. As an example, the smallest non-0-rotatable tree is shown in Figure 2.9.


Figure 2.9: This tree has no graceful labelling that assigns label 0 to the black vertex $v$.
In spite of the fact that not all trees are 0 -rotatable, this property was revisited and investigated by some authors [11,27,36]. For instance, after Rosa's work on 0-rotatability of paths, Chung and Hwang [36] investigated the 0-rotatability of trees generated through $\Delta$-construction and proved the following interesting result.

Theorem 2.16 (Chung and Hwang [36]). If $S$ and $T$ are 0 -rotatable trees, then the tree $S \Delta T$ is 0-rotatable.

Proof. Let $S$ and $T$ be two 0 -rotatable trees with $n_{S}$ and $n_{T}$ vertices, respectively. Also, let $f_{S}$ and $f_{T}$ be graceful labellings of $S$ and $T$, and let $w$ be an arbitrary vertex of $S \Delta T$. We show that there exists a graceful labelling $f$ of $S \Delta T$ that assigns label 0 to $w$. Suppose that $w$ is in the copy $T_{i}$ of $T$ attached at vertex $u_{i}$ of $S$, and suppose $w$ corresponds to vertex $v_{j}$ of $T$ (Note that $w$ and $u_{i}$ can be the same vertex in $S \Delta T$ ). Since $S$ and $T$ are 0-rotatable, we can choose $f_{S}$ and $f_{T}$ such that $f_{S}\left(u_{i}\right)=f_{T}\left(v_{j}\right)=0$, and we can choose the bipartition $\{A, B\}$ of $T$ mentioned in the statement of Theorem 2.14, such that $v_{j} \in A$. Thus, by Theorem 2.14, $S \Delta T$ has a graceful labelling $f$ such that $f\left(v_{j}\right)=f\left(u_{i}\right) n_{T}+f_{T}\left(v_{j}\right)=0$. Therefore, $f(w)=0$. Since $w$ is arbitrary, $S \Delta T$ is 0 -rotatable.

The following corollary is a consequence of Theorem 2.15 and Theorem 2.16.
Corollary 2.17 (Chung and Hwang [36]). Let $k$ be a positive integer and let $T$ be $a$ caterpillar such that, for every non-leaf vertex $v \in V(T)$, the number of leaves adjacent to $v$ is $k$. Then, $T$ is 0 -rotatable.

Proof. Let $T$ be a tree as stated in the hypothesis. Note that $T \cong P_{n} \Delta K_{1, t}$, where $n=$ $\operatorname{diam}(T)-1$ and the distinguished vertex of $K_{1, t}$ is its central vertex. By Theorem 2.15, path $P_{n}$ is 0 -rotatable. Moreover, every star $K_{1, t}$ is also 0-rotatable: a graceful labelling $f$ for $K_{1, t}$ is obtained by assigning label 0 to its central vertex and labels $1, \ldots, t-1$ to its leaves; by taking the complementary labelling $\bar{f}$ we obtain label 0 in a leaf. Therefore, $K_{1, t}$ is 0 -rotatable and, by Theorem 2.16, $T$ is also 0 -rotatable.

Chung and Hwang [36] have further shown that another subfamily of caterpillars, presented in Theorem 2.18, are 0-rotatable. Figure 2.10 illustrates the subtle difference between the two families of caterpillars presented in Theorem 2.17 and Theorem 2.18.

Theorem 2.18 (Chung and Hwang [36]). Let $T$ be a caterpillar whose non-leaf vertices all have the same degree. Then, $T$ is 0 -rotatable.


Figure 2.10: A caterpillar $T_{1}$ such that each one of its non-leaf vertices is adjacent to three leaves and a caterpillar $T_{2}$ whose non-leaf vertices all have the same degree.

### 2.1.1 Graceful labellings of families of graphs

Since Rosa's article [98], graceful labellings have been investigated mostly for families of trees. However, after Golomb's work [52], graceful labellings were further investigated for other classes of graphs.

In this section, we present selected results for some families of trees and other classic families of graphs. First, we consider the family of caterpillars and show that every caterpillar has an $\alpha$-labelling.

Theorem 2.19 (Rosa [98]). Let $T$ be a caterpillar with a spine $P$ and let $v \in V(P)$ be a leaf of $P$. Then, $T$ has an $\alpha$-labelling $f$ such that $f(v)=0$.

Proof. Let $T, P$ and $v$ be as stated in the hypothesis. We use induction on $|V(T)|$ to prove that $T$ has an $\alpha$-labelling $f$ such that $f(v)=0$. When $T$ is a trivial graph, $f$ is obtained by assigning label 0 to $v$ and we are done. Thus, consider $T$ with $k$ vertices, $k \geq 2$. Let $u$ be the neighbour of $v$. We consider two cases depending on the number of leaves adjacent to vertex $u$.

Case 1. $u$ is adjacent to more than one leaf.
Let $w$ be a leaf, neighbour of $u$, such that $w \neq v$. Note that path $P^{\prime}$, induced by $(V(P) \backslash\{v\}) \cup\{w\}$, is a spine of subgraph $T^{\prime}=T-v$ and $w$ is a leaf of $P^{\prime}$. Since $T^{\prime}$ is a caterpillar with $k-1$ vertices, by the induction hypothesis, $T^{\prime}$ has an $\alpha$-labelling $g$ with $g(w)=0$. Since $w$ has label 0 , its neighbour $u$ has label $g(u)=\left|E\left(T^{\prime}\right)\right|$. Thus, the complementary labelling $\bar{g}$ assigns label 0 to vertex $u$. By taking $T^{\prime}$ labelled with $\bar{g}$, we obtain an $\alpha$-labelling of $T$ by adding vertex $v$ and edge $u v$ to $T^{\prime}$ and assigning label $\left|E\left(T^{\prime}\right)\right|+1$ to vertex $v$. Finally, by applying the complementary labelling to $T$, we obtain the required $\alpha$-labelling of $T$ that assigns label 0 to $v$.

Case 2. $u$ is adjacent to exactly one leaf.
In this case, vertex $v$ is the unique leaf adjacent to $u$. Recall that all leaves of $T$ are at distance at most 1 from the spine $P$. Therefore, all leaves of $T$ are at distance at most 1 from the subpath $P^{\prime}=P-v$ and $P^{\prime}$ is a spine of subgraph $T^{\prime}=T-v$. Moreover, $u$
is a leaf of $P^{\prime}$. Since $T^{\prime}$ is a caterpillar with $k-1$ vertices, by the induction hypothesis, $T^{\prime}$ has an $\alpha$-labelling $g$ with $g(u)=0$. We extend $g$ to an $\alpha$-labelling of $T$ by adding vertex $v$ and edge $u v$ to $T^{\prime}$, and assigning label $\left|E\left(T^{\prime}\right)\right|+1$ to vertex $v$. By applying the complementary labelling to $T$, we obtain the required $\alpha$-labelling of $T$ that assigns label 0 to $v$, and the result follows.

Figure 2.11 illustrates a caterpillar with an $\alpha$-labelling.


Figure 2.11: A caterpillar with an $\alpha$-labelling with separator $k=6$. Note that the labels of black vertices are smaller than the labels of white vertices.

From Theorem 2.19, we obtain the following corollary.
Corollary 2.20 (Rosa [98]). Let $T$ be a caterpillar and $P$ be a spine of $T$. Also, let $v \in V(P)$ such that $v$ is a leaf of $P$ or is adjacent to a leaf of $P$. Then, $T$ has an $\alpha$-labelling that assigns label 0 to vertex $v$.

Proof. By Theorem 2.19 and by the choice of $v, T$ has an $\alpha$-labelling $f$ that assigns label 0 to $v$, if $v$ is a leaf, or assigns label 0 to a leaf adjacent to $v$, otherwise. In the first case, we are done. In the second, the complementary labelling of $f$ gives the desired labelling.

Another class for which graceful labellings were studied is that of complete graphs. In fact, Golomb [52] completely characterized graceful labellings of complete graphs, as shown in the next theorem.

Theorem 2.21 (Golomb [52]). The complete graph $K_{n}$ is graceful if and only if $1 \leq n \leq 4$.
Proof. Graceful labellings of $K_{n}$, for $1 \leq n \leq 4$, are illustrated in Figure 2.12. Next, we prove that, if $K_{n}$ is graceful, then $n \leq 4$. We proceed by contradiction.

Suppose that $K_{n}, n \geq 5$, has a graceful labelling $f$. Let $m=\left|E\left(K_{n}\right)\right|$. Then, $L_{V\left(K_{n}\right)}^{f} \subseteq\{0, \ldots, m\}$ and $L_{E\left(K_{n}\right)}^{f}=\{1, \ldots, m\}$. Note that 0 and $m$ are vertex labels of $K_{n}$ since this is the only way of generating edge label $m$. In order to obtain an edge with label $m-1$, either 1 or $m-1$ has to be a vertex label. For any graceful graph with $m$ edges, the replacement of every vertex label $i$ by $m-i$ does not change the edge labels (this is the complementary labelling). Hence, without loss of generality, we can suppose that label 1 appears in the labelling $f$ instead of label $m-1$. With vertex labels 0,1 , and $m$, the edge labels $m, m-1$, and 1 are generated.

In order to obtain $m-2$ as an edge label, 0 and $m-2$, or 1 and $m-1$, or 2 and $m$ have to occur as vertex labels. Since labels 0,1 , and $m$ are already used for vertices, there must be a vertex with label $m-2, m-1$, or 2 . However, vertex labels 2 and $m-1$ both generate a repeated edge label 1 with the existing vertex labels 1 and $m$, respectively.

Therefore, only $m-2$ can be used as the next vertex label, generating edge labels 2 , $m-2$, and $m-3$.

With vertex labels $0,1, m-2$, and $m$, we get edge labels $1,2, m-3, m-2, m-1$, and $m$. By a similar reasoning, we conclude that the only way to yield edge label $m-4$ is having a vertex with label 4.

With vertex labels $0,1,4, m-2$, and $m$, we have edge labels $1,2,3,4, m-6, m-4$, $m-3, m-2, m-1$, and $m$. This already proves that $K_{5}$ is not graceful since the edge labels 4 and $m-6$ are equal for $m=\left|E\left(K_{5}\right)\right|=10$. Thus, consider $n>5$. In this case, $m>10$ and so $4<m-6$ and there is no repetition of edge labels at this point. However, we still need to generate the edge label $m-5$. In fact, there is no way of obtaining this edge label without repeating an existent edge label. This is a contradiction to our assumption that $K_{n}$ is graceful for all $n>5$, completing our proof.


Figure 2.12: Graceful labellings of $K_{n}, 1 \leq n \leq 4$.

The next class of graphs we present is the class of complete bipartite graphs.
Theorem 2.22 (Rosa [98]). Every complete bipartite graph has an $\alpha$-labelling.
Proof. Let $p$ and $q$ be positive integers with $p \leq q$. Let $G=K_{p, q}$ be a complete bipartite graph and $\{X, Y\}$ be a bipartition of $G$ such that $X=\left\{x_{1}, \ldots, x_{p}\right\}$ and $Y=\left\{y_{1}, \ldots, y_{q}\right\}$. Note that $|E(G)|=p q$. Define a labelling $f: V(G) \rightarrow[0, p q]$ as follows: $f\left(x_{i}\right)=i-1$, for $i \in[1, p]$, and $f\left(y_{j}\right)=j p$, for $j \in[1, q]$.

In order to see that $f$ is an $\alpha$-labelling, first note that $f$ is an injective function that maps $V(G)$ into $[0, p q]$. Moreover, note that $f$ has separator $p-1$ since, for every edge $x_{i} y_{j} \in E(G), f\left(x_{i}\right) \leq p-1$ and $f\left(y_{i}\right)>p-1$. In order to conclude the proof, it remains to show that $L_{E(G)}^{f}=[1, p q]$. By the definition of $f,\left|f\left(y_{j}\right)-f\left(x_{i}\right)\right|=f\left(y_{j}\right)-f\left(x_{i}\right)=j p-i+1$. Thus, for $1 \leq j \leq q$, the labels of the edges incident with vertex $y_{j}$ are the labels in $L_{j}=\left\{\left|f\left(y_{j}\right)-f\left(x_{i}\right)\right|: 1 \leq i \leq p\right\}=\{j p-(p-1), j p-(p-2), \ldots, j p\}$. By the definition of $L_{j}$, we have that:
(i) for all $j \in[1, q-1]$, all labels in $L_{j}$ are smaller than all labels in $L_{j+1}$, which implies that $L_{1}<L_{2}<\cdots<L_{q}$;
(ii) for all $j \in[1, q],\left|L_{j}\right|=p$; and
(iii) $\min \left\{L_{1}\right\}=1$ and $\max \left\{L_{q}\right\}=p q$.

These facts imply that that $L_{E(G)}^{f}=[1, p q]$, as required. Therefore, $f$ is an $\alpha$-labelling. Figure 2.13 illustrates the $\alpha$-labelling $f$ of $K_{3,4}$ as defined in this proof.


Figure 2.13: Complete bipartite graph $K_{3,4}$ with an $\alpha$-labelling.

Note that complete bipartite graphs are a subclass of complete multipartite graphs. Beutner and Harborth [18] studied the graceful labelling of these graphs and proved that the complete multipartite graphs $K_{1, p, q}, K_{2, p, q}$, and $K_{1,1, p, q}$ are also graceful, for any positive integers $p$ and $q$. The graceful labellings of these families are illustrated in Figure 2.14. Furthermore, Beutner and Harborth [18] conjectured that these are the only complete multipartite graphs which are graceful and checked computationally that this conjecture is valid for all complete multipartite graphs up to 23 vertices.


Figure 2.14: Graceful labellings of $K_{p, q}, K_{1, p, q}, K_{2, p, q}$, and $K_{1,1, p, q}$.

A closed trail of a graph $G$ is a sequence $W=v_{0} e_{1} v_{1} \cdots v_{\ell-1} e_{\ell} v_{\ell}$, whose terms are alternately vertices and edges of $G$, such that: $e_{i}=\left\{v_{i-1} v_{i}\right\}$ for $1 \leq i \leq \ell ; v_{0}=v_{\ell}$; and all edges in the sequence are distinct.

A graph in which each vertex has even degree is called an even graph. Even graphs are extensively studied in the literature [48]. Rosa [98] and Golomb [52] proved that a necessary condition for an even graph $G$ to be graceful is that $|E(G)| \equiv 0,3(\bmod 4)$ (see Theorem 2.25). In the proof of this result, we use the following two lemmas.

Lemma 2.23 (Veblen [113]). A graph $G$ admits a decomposition into cycles if and only if $G$ is even.

Lemma 2.24 (Rosa [98], Golomb [52]). Suppose that nonnegative integers, not necessarily distinct, are assigned to the vertices of a graph $G$, and that each edge of $G$ is assigned a
number equal to the absolute difference of the labels of its endpoints. Then, the sum of the edge labels of any closed trail of $G$ is even.

Proof. Let $G$ be a graph labelled as in the hypothesis. Let the consecutive vertex labels of a closed trail $C$ of $G$ be $a_{0}, a_{1}, \ldots, a_{r-1}$, where $a_{r-1}=a_{0}$. Then, the consecutive edge labels are $\left|a_{0}-a_{1}\right|,\left|a_{1}-a_{2}\right|, \ldots,\left|a_{r-2}-a_{r-1}\right|$. Thus, the sum of the edge labels in $C$ is

$$
\sum_{i=0}^{r-2}\left|a_{i}-a_{i+1}\right|
$$

Note that this sum is even since each $a_{i}$ appears exactly twice either with the same signal or opposite signals.

Theorem 2.25 (Rosa [98], Golomb [52]). If an even graph $G$ is graceful, then

$$
|E(G)| \equiv 0,3 \quad(\bmod 4)
$$

Proof. Let $G$ be an even graph with a graceful labelling $f$ and let $m=|E(G)|$. Suppose that $m \equiv 1,2(\bmod 4)$. By the definition of $f$, the sum of all edge labels of $G$ is

$$
\sum_{u v \in E(G)}|f(u)-f(v)|=\sum_{i=1}^{m} i=\frac{m(m+1)}{2}
$$

Since $m \equiv 1,2(\bmod 4)$, the sum of all edge labels of $G$ is an odd integer. On the other hand, we know that $G$ is an even graph. Thus, by Lemma 2.23, $G$ has a decomposition $\mathcal{D}$ into cycles. Let $C \in \mathcal{D}$ be an arbitrary cycle. By Lemma 2.24, the sum $\sum_{u v \in E(C)}|f(u)-f(v)|$ is even. Thus, by adding all these sums over all cycles in decomposition $\mathcal{D}$, we obtain that the sum of all edge labels of $G$ is even, which is a contradiction. Therefore, $m \not \equiv 1,2(\bmod 4)$, and the result follows.

Kotzig $[70,71]$ proved that having $|E(G)| \equiv 0,3(\bmod 4)$ is not a sufficient condition for an even graph $G$ to be graceful. For instance, he showed that graphs $2 C_{3} \cup C_{5}$ and $3 C_{5}$ are not graceful despite having $\left|E\left(2 C_{3} \cup C_{5}\right)\right| \equiv\left|E\left(3 C_{5}\right)\right| \equiv 3(\bmod 4)$. Kotzig [69] also improved the necessary condition presented in Theorem 2.25 for the case when $G$ is both even and bipartite.

Theorem 2.26 (Kotzig [69]). If an even bipartite graph $G$ is graceful, then

$$
|E(G)| \equiv 0 \quad(\bmod 4)
$$

Proof. Let $G$ be an even bipartite graceful graph. Since $G$ is even, $G$ has a decomposition $\mathcal{D}$ into cycles. Since $G$ is bipartite, every cycle in $\mathcal{D}$ has even length. This implies that $G$ has an even number of edges. By this fact and by Theorem 2.25, we obtain that $|E(G)| \equiv 0(\bmod 4)$.

Since cycles are even graphs, by Theorem 2.25, if a cycle $C_{n}$ is graceful, then $n \equiv 0,3$ $(\bmod 4)$. In fact, Rosa [98] proved that this necessary condition is also sufficient.

Theorem 2.27 (Rosa [98]). Cycle $C_{n}$ has a graceful labelling if and only if $n \equiv 0,3$ $(\bmod 4)$. Moreover, $C_{n}$ has an $\alpha$-labelling if and only if $n \equiv 0(\bmod 4)$.

Proof. Let $C_{n}$ be a cycle with $n$ vertices such that $V\left(C_{n}\right)=\left\{v_{0}, \ldots, v_{n-1}\right\}$ and $E\left(C_{n}\right)=$ $\left\{v_{i} v_{(i+1)}(\bmod n): 0 \leq i \leq n-1\right\}$. Since cycles are even graphs, by Theorem 2.25, an even cycle $C_{n}$ is graceful only if $n \equiv 0(\bmod 4)$ and an odd cycle $C_{n}$ is graceful only if $n \equiv 3$ $(\bmod 4)$. Thus, in order to prove the result, it remains to show that, if $n \equiv 0(\bmod 4)$, then $C_{n}$ has an $\alpha$-labelling and that, if $n \equiv 3(\bmod 4)$, then $C_{n}$ has a graceful labelling.

First, consider $n \equiv 0(\bmod 4)$. Define labelling $f: V\left(C_{n}\right) \rightarrow[0, n]$ as follows:

$$
f\left(v_{i}\right)= \begin{cases}i / 2, & \text { if } i \text { is even; } \\ n+1-(i+1) / 2, & \text { if } i \text { is odd and } i \leq n / 2-1 \\ n-(i+1) / 2, & \text { if } i \text { is odd and } i>n / 2-1\end{cases}
$$

Figure 2.15 shows such labellings for $C_{4}$ and $C_{8}$. Next, we prove that $f$ is an $\alpha$ labelling. By the definition of $f$, vertex $v_{0}$ is assigned the smallest label (label 0 ) and vertex $v_{1}$ is assigned the greatest label (label $n$ ). It is not difficult to check that any two distinct vertices with both indices even or with both indices odd are assigned distinct labels. Furthermore, equations $i / 2=n+1-(j+1) / 2$ and $i / 2=n-(j+1) / 2$ give rise to a contradiction. These facts imply that any two vertices are assigned distinct labels from $[0, n]$. Therefore, $f$ is an injective function. Now, we show that $L_{E\left(C_{n}\right)}^{f}=[1, n]$. By the definition of $f$ :
(i) the set of edge labels of subpath $\left(v_{0}, v_{1}, \ldots, v_{n / 2}\right)$ of $C_{n}$ is

$$
L_{1}=\left\{\left|\frac{i}{2}-\left(n+1-\frac{(i+1)+1}{2}\right)\right|: 0 \leq i \leq \frac{n}{2}-1\right\}=\left\{n, n-1, \ldots, \frac{n}{2}+1\right\}
$$

(ii) the set of edge labels of subpath $\left(v_{n / 2}, \ldots, v_{n-1}\right)$ of $C_{n}$ is

$$
L_{2}=\left\{\left|\frac{i}{2}-\left(n-\frac{(i+1)+1}{2}\right)\right|: \frac{n}{2} \leq i \leq n-2\right\}=\left\{\frac{n}{2}-1, \ldots, 1\right\} ; \text { and }
$$

(iii) edge $v_{0} v_{n-1}$ has label $\frac{n}{2}$.

Therefore, $L_{E\left(C_{n}\right)}^{f}=L_{1} \cup\left\{\left|f\left(v_{0}\right)-f\left(v_{n-1}\right)\right|\right\} \cup L_{2}=[1, n]$, as required. In order to conclude this case, note that $f$ has separator $\frac{n-2}{2}$ since, for every $i \in\{1,3, \ldots, n-1\}$, we have that $f\left(v_{i-1}\right) \leq \frac{n-2}{2}<f\left(v_{i}\right)$ and $f\left(v_{i+1}\right) \leq \frac{n-2}{2}<f\left(v_{i}\right)$. Therefore, $f$ is an $\alpha$-labelling.

Now, consider $n \equiv 3(\bmod 4)$. Define a labelling $f: V\left(C_{n}\right) \rightarrow[0, n]$ as follows:

$$
f\left(v_{i}\right)= \begin{cases}n+1-(i+1) / 2, & \text { if } i \text { is odd; } \\ i / 2, & \text { if } i \text { is even and } i \leq(n-3) / 2 \\ (i+2) / 2, & \text { if } i \text { is even and } i>(n-3) / 2\end{cases}
$$

Figure 2.15 shows such labellings for $C_{3}$ and $C_{7}$. In the following, we prove that $f$ is a graceful labelling. As in the previous case, vertex $v_{0}$ is assigned label 0 and vertex $v_{1}$ is assigned label $n$. Also, any two distinct vertices with indices of the same parity are assigned distinct labels. Furthermore, equations $n+1-(i+1) / 2=j / 2$ and
$n+1-(i+1) / 2=(j+2) / 2$ give rise to a contradiction, implying that any two vertices with indices of different parities are assigned distinct labels. Since any two vertices are assigned distinct labels, we obtain that $f$ is injective. In order to conclude the proof, it remains to show that $L_{E\left(C_{n}\right)}^{f}=[1, n]$. By the definition of $f$ :
(i) the set of edge labels of subpath $\left(v_{0}, v_{1}, \ldots, v_{(n-1) / 2}\right)$ of $C_{n}$ is

$$
L_{1}=\left\{\left|\frac{i}{2}-\left(n+1-\frac{(i+1)+1}{2}\right)\right|: 0 \leq i \leq \frac{n-3}{2}\right\}=\left\{n, n-1, \ldots, \frac{n+3}{2}\right\}
$$

(ii) the set of edge labels of subpath $\left(v_{(n-1) / 2}, \ldots, v_{n-1}\right)$ of $C_{n}$ is

$$
L_{2}=\left\{\left|n+1-\left(\frac{i+1}{2}\right)-\left(\frac{(i+1)+2}{2}\right)\right|: \frac{n-1}{2} \leq i \leq n-2\right\}=\left\{\frac{n-1}{2}, \ldots, 1\right\} ; \text { and }
$$

(iii) edge $v_{0} v_{n-1}$ has label $\frac{n+1}{2}$.

Therefore, $L_{E\left(C_{n}\right)}^{f}=L_{1} \cup\left\{\left|f\left(v_{0}\right)-f\left(v_{n-1}\right)\right|\right\} \cup L_{2}=[1, n]$, and the result follows.


Figure 2.15: $\alpha$-labellings of $C_{4}$ and $C_{8}$, and graceful labellings of $C_{3}$ and $C_{7}$.
A complete characterization of graceful labellings for the family of 2-regular graphs is still not known. A first step towards such a characterization was conducted by Abrham and Kotzig [3], who proved that the 2-regular graph $C_{p} \cup C_{q}$ is graceful if and only if $p+q \equiv 0,3(\bmod 4)$. They also proved that $C_{p} \cup C_{q}$ has an $\alpha$-labelling if and only if $p, q$ are even and $p+q \equiv 0(\bmod 4)$. In 1997, Eshghi [44] proved that, with exception of $3 C_{4}$, every 2-regular bipartite graph $G$ with three components has an $\alpha$-labelling if and only if $|E(G)| \equiv 0(\bmod 4)$. In general, it is still unknown which 2-regular graphs with $|E(G)| \equiv 0,3(\bmod 4)$ and with more than three connected components are graceful. However, some subfamilies of these graphs have been proved to be graceful [1, 45, 70, 71].

### 2.1.2 A necessary condition for graceful labellings

In Theorem 2.25, it was presented a necessary condition for an even graph to be graceful. Such a condition is very useful and can be easily verified since it only requires to check whether the number of edges of a graph is congruent to 0 or 3 modulo 4 . In this section, it is presented a necessary condition for the existence of graceful labellings in arbitrary graphs.

Let $G$ be a graph with $m$ edges and let $S$ be a nonempty subset of $V(G)$. We say that an edge cut $\partial(S)$ with $|\partial(S)|=\left\lceil\frac{m}{2}\right\rceil$ is a half cut. An example of graph that has a half cut is the complete graph with four vertices: for any $v \in V\left(K_{4}\right)$, by taking $S=\{v\}$, we have
that $|\partial(S)|=d(v)=3=\left\lceil\left|E\left(K_{4}\right)\right| / 2\right\rceil$. In 1972, Golomb [52] showed that a necessary condition for a nontrivial graph $G$ to be graceful is to have a half cut.

Lemma 2.28 (Golomb [52]). If a nontrivial graph $G$ is graceful, then $G$ has a half cut.
Proof. Let $f$ be a graceful labelling of a nontrivial graph $G$ and let $m=|E(G)|$. Also, let $V_{\mathcal{O}}=\{v \in V(G): f(v)$ is odd $\}$ and $V_{\mathcal{E}}=V(G) \backslash V_{\mathcal{O}}$. Note that all edges with odd label have one endpoint in $V_{\mathcal{O}}$ and the other endpoint in $V_{\mathcal{E}}$, and these are the unique edges in the edge cut $\partial\left(V_{\mathcal{O}}\right)$. Therefore, $\left|\partial\left(V_{\mathcal{O}}\right)\right|$ is equal to the number of odd labels in the set $\{1, \ldots, m\}$, which is $\left\lceil\frac{m}{2}\right\rceil$.

Golomb [52] also observed that having a half cut is not a sufficient condition for a graph to be graceful. For example, Theorem 2.29 shows an infinite subfamily of complete graphs that have a half cut but, by Theorem 2.21, all complete graphs with more than five vertices are not graceful.
Theorem 2.29 (Golomb [52], Sucupira et al. [109]). Let $n \in \mathbb{Z}$ with $n>1$. The complete graph $K_{n}$ has a half cut if and only if $n=q^{2}$ or $n=q^{2}+2$, for $q \in \mathbb{Z}$.

Proof. Let $n \in \mathbb{Z}$ with $n>1$. The complete graph $K_{n}$ has $m=\frac{n(n-1)}{2}$ edges. Note that any edge cut $\partial(S)$ of $K_{n}$ has cardinality equal to $s(n-s)$, where $s=|S|$. Since a half cut in $K_{n}$ has cardinality $\left\lceil\frac{m}{2}\right\rceil=\left\lceil\frac{n(n-1)}{4}\right\rceil, K_{n}$ has a half cut if and only if there exists a choice of $s$ for which $s(n-s)=\left\lceil\frac{n(n-1)}{4}\right\rceil$. We analyse four cases depending on the values of $n$ modulo 4 .

First, if $n \equiv 0,1(\bmod 4)$, then $\frac{n(n-1)}{4}$ is an integer and the solutions of the equation $s(n-s)=\frac{n(n-1)}{4}$ are $s=\frac{n+\sqrt{n}}{2}$ and $s=\frac{n-\sqrt{n}}{2}$. Since $s$ is a positive integer, we conclude that $n$ is a perfect square, that is, $n=q^{2}$, for $q \in \mathbb{Z}$.

If $n \equiv 3(\bmod 4)$, then $n=4 k+3, k \in \mathbb{Z}_{\geq 0}$, and we have that $s(n-s)=\left\lceil\frac{n(n-1)}{4}\right\rceil=$ $\left\lceil\frac{(4 k+3)(4 k+2)}{4}\right\rceil=4 k^{2}+5 k+2$. Solving the equation $s(n-s)=4 k^{2}+5 k+2$, we obtain that $s=\frac{n+\sqrt{4 k+1}}{2}$ or $s=\frac{n-\sqrt{4 k+1}}{2}$. Since $s$ is an integer, $4 k+1$ is a perfect square. Moreover, since $4 k+1=n-2$, we have that $n-2$ is also a perfect square and, thus, $n=q^{2}+2$, for some $q \in \mathbb{Z}$.

Now, consider $n \equiv 2(\bmod 4)$. In this case, $n=4 k+2, k \in \mathbb{Z}_{\geq 0}$, and we have that $s(n-s)=\left\lceil\frac{n(n-1)}{4}\right\rceil=\left\lceil\frac{(4 k+2)(4 k+1)}{4}\right\rceil=4 k^{2}+3 k+1$. Solving the equation $s(n-s)=$ $4 k^{2}+3 k+1$, we obtain that $s=(2 k+1)+\sqrt{k}$ or $s=(2 k+1)-\sqrt{k}$. Since $s$ is an integer, $k$ is a perfect square, that is, $k=p^{2}$ and $n=4 p^{2}+2$, for some $p \in \mathbb{Z}$. Furthermore, note that $4 p^{2}+2=q^{2}+2$, for $q=2 p$. Hence, in this case, $n$ also has the form $q^{2}+2$, for $q \in \mathbb{Z}$ and $q$ even. Therefore, $K_{n}$ has a half cut if and only if $n=q^{2}$ or $n=q^{2}+2$, for $q \in \mathbb{Z}$.

In general, it may not be easy to check for the existence of a half cut in an arbitrary graph. In fact, Cairnie and Edwards [28] proved that it is $\mathcal{N} \mathcal{P}$-complete to decide whether an arbitrary simple graph $G$ with even number of edges has a half cut. In spite of this result, it may be possible to determine if some classes of graphs have a half cut, as it was determined for complete graphs in Theorem 2.29. In the context of the Graceful Tree Conjecture, Golomb [52] proved, in 1972, that all nontrivial trees have a half cut.

Theorem 2.30 (Golomb [52]). Let $T$ be a tree with at least two vertices. Then, there exists $S \subset V(T)$, with $|S|=\lfloor|V(T)| / 2\rfloor$, such that $\partial(S)$ is a half cut.

Proof. We prove the result by induction on the number of edges of $T$. If $T$ has one edge or two edges, it is not difficult to see that $\partial(S)$, where $S=\{v\}$ and $v \in V(T)$ is a leaf, is a half cut and $|S|=1=\lfloor|V(T)| / 2\rfloor$. Now, suppose that every tree with at most $k$ edges, $k \geq 2$, has a half cut satisfying all the conditions of the theorem.

Let $T$ be any tree with $|E(T)|=k+1$ and define $m_{T}=|E(T)|$. Let $P=v_{0}, \ldots, v_{r}$ be a spine of $T$ with leaves $v_{0}$ and $v_{r}$. We consider two cases depending on the degree of the neighbour of $v_{r}$, vertex $v_{r-1}$.

Case 1. $d_{T}\left(v_{r-1}\right)>2$.
By the maximality of the path $P$, vertex $v_{r-1}$ is adjacent only to vertices $v_{r-2}, v_{r}$ and to leaves $w_{1}, \ldots, w_{s}$, for $s \geq 1$. Let $T^{\prime}$ be the tree obtained from $T$ by removing the leaves $v_{r}$ and $w_{1}$. Define $m_{T^{\prime}}=\left|E\left(T^{\prime}\right)\right|$. By the induction hypothesis, there exists $S^{\prime} \subset V\left(T^{\prime}\right)$, with $\left|S^{\prime}\right|=\left\lfloor\left|V\left(T^{\prime}\right)\right| / 2\right\rfloor$, such that $\left|\partial\left(S^{\prime}\right)\right|=\left\lceil\frac{m_{T^{\prime}}}{2}\right\rceil$. Let $S \subset V(T)$ be defined by $S=S^{\prime} \cup\left\{v_{r}\right\}$.

Next, we prove that $|S|=\lfloor|V(T)| / 2\rfloor$ and that $|\partial(S)|=\left\lceil\frac{m_{T}}{2}\right\rceil$. First, note that $|S|=\left|S^{\prime}\right|+1=\left\lfloor\left|V\left(T^{\prime}\right)\right| / 2\right\rfloor+1=\lfloor|V(T)| / 2\rfloor$, where the last equality holds by the fact that $\left|V\left(T^{\prime}\right)\right|=|V(T)|-2$. Second, since $\partial(S) \backslash \partial\left(S^{\prime}\right)=\left\{w_{1} v_{r-1}\right\}$, we have that $|\partial(S)|=$ $\left|\partial\left(S^{\prime}\right)\right|+1$. By this fact and by the facts that $\left|\partial\left(S^{\prime}\right)\right|=\left\lceil\frac{m_{T^{\prime}}}{2}\right\rceil$ and $m_{T^{\prime}}=m_{T}-2$, we obtain that $|\partial(S)|=\left\lceil\frac{m_{T}}{2}\right\rceil$. Therefore, $\partial(S)$ is a half cut of $T$ such that $|S|=\lfloor|V(T)| / 2\rfloor$.

Case 2. $d_{T}\left(v_{r-1}\right)=2$.
In this case, let $T^{\prime}$ be the tree obtained from $T$ by removing vertices $v_{r}$ and $v_{r-1}$. Define $m_{T^{\prime}}=\left|E\left(T^{\prime}\right)\right|$. By the induction hypothesis, there exists $S^{\prime} \subset V\left(T^{\prime}\right)$, with $\left|S^{\prime}\right|=$ $\left\lfloor\left|V\left(T^{\prime}\right)\right| / 2\right\rfloor$, such that $\left|\partial\left(S^{\prime}\right)\right|=\left\lceil\frac{m_{T^{\prime}}}{2}\right\rceil$. Note that, either $v_{r-2} \in S^{\prime}$ or $v_{r-2} \in V\left(T^{\prime}\right) \backslash S^{\prime}$. Let $S \subset V(T)$ defined by

$$
S= \begin{cases}S^{\prime} \cup\left\{v_{r-1}\right\}, & \text { if } v_{r-2} \in S^{\prime} \\ S^{\prime} \cup\left\{v_{r}\right\}, & \text { if } v_{r-2} \in V\left(T^{\prime}\right) \backslash S^{\prime}\end{cases}
$$

By a similar reasoning to that used in the previous case, we conclude that $|S|=$ $\lfloor|V(T)| / 2\rfloor$ and that $|\partial(S)|=\left\lceil\frac{m_{T}}{2}\right\rceil$. Therefore, $\partial(S)$ is a half cut of $T$ such that $|S|=$ $\lfloor|V(T)| / 2\rfloor$.

### 2.2 Relaxed versions of graceful labellings

We have seen that it may not be possible to assign distinct labels from the set $[0,|E(G)|]$ to the vertices of an arbitrary graph $G$ so that the induced edge labelling is also injective (the induced edge labelling assigns to each edge of $G$ the absolute difference of the labels of its endpoints). However, as it was first observed by Golomb [52], such a labelling is always possible if the set of vertex labels is expanded. This relaxed labelling was named range-relaxed graceful labelling by Bussel [26].

Formally, a range-relaxed graceful labelling of a simple graph $G$ is an injective function $f: V(G) \rightarrow[0, k]$, with $k \geq|E(G)|$, such that the set $\{|f(u)-f(v)|: u v \in E(G)\}$ comprises exactly $|E(G)|$ distinct integers from the set $[1, k]$. Golomb defined the gracefulness of a graph $G$, denoted $\operatorname{grac}(G)$, as the smallest positive integer $k$ for which there exists a range-relaxed graceful labelling $f: V(G) \rightarrow[0, k]$ of $G$. Note that, if $\operatorname{grac}(G)=|E(G)|$, then $G$ is graceful.

Range-relaxed graceful labellings have been investigated for as long as graceful labellings. For example, the $\sigma$-labelling and the $\rho$-labelling are special types of range-relaxed graceful labellings and were introduced by Rosa [98] along with graceful labellings. The first investigations on range-relaxed graceful labellings were restricted to graphs that are not graceful [19,52], such as complete graphs with more than five vertices. The gracefulness of a graph $G$ can be thought as a measure of how close $G$ is of being graceful. The next result presents a simple upper bound for the parameter $\operatorname{grac}(G)$, for every simple graph $G$.

Proposition 2.31. If $G$ is a simple graph, then $\operatorname{grac}(G) \leq 2^{|V(G)|-1}-1$.
Proof. Let $G$ be a simple graph with vertex set $V(G)=\left\{v_{0}, \ldots, v_{n-1}\right\}$. Define an injective labelling $f: V(G) \rightarrow\left[0,2^{n-1}-1\right]$ as follows: $f\left(v_{i}\right)=2^{i}-1$, for $0 \leq i \leq n-1$. Suppose that there exist two edges $v_{i} v_{j}, v_{r} v_{s} \in E(G)$ with the same induced edge label. Without loss of generality, consider $f\left(v_{j}\right)>f\left(v_{i}\right)$ and $f\left(v_{s}\right)>f\left(v_{r}\right)$. This implies that $j>i$ and $s>r$, which in turn implies that $j=i+x$ and $s=r+y$, for $x, y \in \mathbb{N}$. Since $f\left(v_{j}\right)-f\left(v_{i}\right)=f\left(v_{s}\right)-f\left(v_{r}\right)$, we have

$$
\begin{align*}
f\left(v_{j}\right)-f\left(v_{i}\right) & =f\left(v_{s}\right)-f\left(v_{r}\right) \\
2^{j}-2^{i} & =2^{s}-2^{r} \\
2^{i+x}-2^{i} & =2^{r+y}-2^{r} \\
2^{i}\left(2^{x}-1\right) & =2^{r}\left(2^{y}-1\right) \tag{2.2}
\end{align*}
$$

Equation (2.2) implies that $r=i, y=x$ and, since $s=r+y$, we obtain $s=j$; which in turn implies that $f\left(v_{j}\right)=f\left(v_{s}\right)$ and $f\left(v_{i}\right)=f\left(v_{r}\right)$, a contradiction. Therefore, the induced edge labels are pairwise distinct, and the result follows.

A better upper bound for the gracefulness of trees was published in 2002 [26]. Next, we present this result.

Theorem 2.32 (Bussel [26]). If $T$ is a tree with $m$ edges, then $\operatorname{grac}(T) \leq 2 m-\operatorname{diam}(T)$.
Proof. Let $T$ be a tree with $m$ edges and $u_{0}$ be an arbitrary vertex of $T$. Consider $T$ as a tree rooted at $u_{0}$, drawn in the plane such that vertices at the same distance from $u_{0}$ are drawn on the same level, and edges are not allowed to cross each other. We assume that a longest path from $u_{0}$ to a leaf is the leftmost path in this representation of $T$. Let the length of this path be $\ell$, let the vertex on this path at level $i$ be denoted by $u_{i, 0}$, $i \in\{1, \ldots, \ell\}$, and $k_{i}$ be the number of vertices at level $i$. The following construction provides a vertex labelling for $T$ in the range [ $0,2 m-\ell]$ :
(i) Label $u_{0}$ temporarily with an arbitrary integer $\alpha$ and $u_{1,0}$ with $\alpha+1$. After labelling all vertices, we shift all labels by a constant so that the smallest value is 0 ;
(ii) For $i>1$, each vertex $u_{i, 0}$ on the leftmost path receives label

$$
f\left(u_{i, 0}\right)= \begin{cases}f\left(u_{i-2,0}\right)-k_{i-2}-k_{i-1}+1=\alpha-\sum_{j=0}^{i-1} k_{j}+\frac{i}{2}, & \text { if } i \text { is even; } \\ f\left(u_{i-2,0}\right)+k_{i-2}+k_{i-1}-1=\alpha+\sum_{j=0}^{i-1} k_{j}-\frac{i-1}{2}, & \text { if } i \text { is odd. }\end{cases}
$$

(iii) Denote $u_{i, j}$ the vertex at the $i$-th level that is located $j$ places to the right of $u_{i, 0}$. For $0 \leq j \leq k_{i}-1$, vertex $u_{i, j}$ receives label

$$
f\left(u_{i, j}\right)= \begin{cases}f\left(u_{i, 0}\right)-j, & \text { if } i \text { is even; } \\ f\left(u_{i, 0}\right)+j, & \text { if } i \text { is odd }\end{cases}
$$

Figure 2.16 exhibits a labelling $f$ of a tree $T$ obtained using this construction. By the construction, all vertex labels are distinct: (i) vertex labels with even level are smaller than or equal to $\alpha$; they monotonically decrease as we go from left to right and from top to bottom; (ii) on the other hand, vertex labels with odd level are greater than $\alpha$; they monotonically increase as we go from left to right and from top to bottom.

Next, we show that the induced edge labels increase as we go from left to right and from top to bottom in the tree.
(a) Consider two edges $u_{i, r} u_{i+1, p}$ and $u_{i, s} u_{i+1, q}$ between vertices with consecutive levels $i$ and $i+1$, where $u_{i, r}$ is to the left of $u_{i, s}$ and $i$ is even. By part (iii) of the construction, since edges cannot cross, we have that $f\left(u_{i, r}\right)>f\left(u_{i, s}\right)$ and $f\left(u_{i+1, p}\right)<f\left(u_{i+1, q}\right)$. Then, $\left|f\left(u_{i+1, p}\right)-f\left(u_{i, r}\right)\right|=f\left(u_{i+1, p}\right)-f\left(u_{i, r}\right)<f\left(u_{i+1, q}\right)-f\left(u_{i, s}\right)=\mid f\left(u_{i+1, q}\right)-$ $f\left(u_{i, s}\right) \mid$. The same follows for $i$ odd by an analogous proof.
(b) Now, consider $i$ even and let $u_{i, k_{i}-1}$ and $u_{i+1, k_{i+1}-1}$ be the rightmost vertices with levels $i$ and $i+1$ respectively. Note that $u_{i, k_{i}-1}$ and $u_{i+1, k_{i+1}-1}$ have, respectively, the minimum and the maximum labels on levels $i$ and $i+1$. Thus, by item (a), any edge $u_{i, r} u_{i+1, s}$ from level $i$ to $i+1$ has label at most $f\left(u_{i+1, k_{i+1}-1}\right)-f\left(u_{i, k_{i}-1}\right)$, that is, $f\left(u_{i+1, s}\right)-f\left(u_{i, r}\right) \leq f\left(u_{i+1, k_{i+1}-1}\right)-f\left(u_{i, k_{i}-1}\right)$. On the other hand, the leftmost difference $f\left(u_{i+1,0}\right)-f\left(u_{i+2,0}\right)$ is a lower bound on the labels of the edges between levels $i+1$ and $i+2$. So, it suffices to show that $f\left(u_{i+1, k_{i+1}-1}\right)-f\left(u_{i, k_{i}-1}\right)<$ $f\left(u_{i+1,0}\right)-f\left(u_{i+2,0}\right)$. By the definition of $f, f\left(u_{i+1, k_{i+1}-1}\right)-f\left(u_{i, k_{i}-1}\right)=f\left(u_{i+1,0}\right)+$ $k_{i+1}-1-\left(f\left(u_{i, 0}\right)-k_{i}+1\right)<f\left(u_{i+1,0}\right)-\left(f\left(u_{i, 0}\right)-k_{i}-k_{i+1}+1\right)=f\left(u_{i+1,0}\right)-f\left(u_{i+2,0}\right)$. Note that the last equality holds by part (ii) of the construction. Again, the same holds for $i$ odd by an analogous proof.

Last, we show that the range of the vertex labels is $[0,2 m-\ell]$. Let $f_{\text {MIN }}$ and $f_{M A X}$ be, respectively, the minimum and the maximum vertex labels generated by labelling $f$. Recall that $\ell$ is the length of the leftmost path of $T$. If $\ell$ is even, then the vertex with largest label is the rightmost vertex on level $\ell-1$ and the vertex with the smallest label
is the rightmost on level $\ell$. Hence, by construction of $f$,

$$
\begin{aligned}
& f_{M A X}=f\left(u_{\ell-1,0}\right)+k_{\ell-1}-1=\left(\alpha+\sum_{j=0}^{\ell-2} k_{j}-\frac{\ell-2}{2}\right)+k_{\ell-1}-1=\alpha+m+1-k_{\ell}-\frac{\ell}{2} \\
& f_{M I N}=f\left(u_{\ell, 0}\right)-k_{\ell}+1=\left(\alpha-\sum_{j=0}^{\ell-1} k_{j}+\frac{\ell}{2}\right)-k_{\ell}+1=\alpha-m+\frac{\ell}{2}
\end{aligned}
$$

From these we obtain the bound on the range:

$$
\begin{aligned}
f_{M A X}-f_{M I N} & =\left(\alpha+m+1-k_{\ell}-\frac{\ell}{2}\right)-\left(\alpha-m+\frac{\ell}{2}\right) \\
& =2 m-\ell-k_{\ell}+1 \\
& \leq 2 m-\ell
\end{aligned}
$$

By an analogous reasoning, the inequality $f_{M A X}-f_{M I N} \leq 2 m-\ell$ also holds when $\ell$ is odd. In order to conclude the proof, we remark that, if we choose the root $u_{0}$ as one of the endpoints of a longest path of $T$, we obtain a labelling $f$ in the range $[0,2 m-\operatorname{diam}(T)]$.


Figure 2.16: A rooted tree $T$ with a vertex labelling $f$ obtained from the construction presented in the proof of Theorem 2.32 by using $\alpha=3$ as starting value. Note that the edge labels induced by $f$ are pairwise distinct. Moreover, by adding the constant 8 to each vertex label, we obtain a new vertex labelling $f^{\prime}$ of $T$ in the range $[0,2|E(T)|-4]$.

Note that the bound given by Theorem 2.32 is tight for paths, since $\operatorname{grac}\left(P_{n}\right) \leq 2(n-$ 1) $-\operatorname{diam}\left(P_{n}\right)=n-1=\left|E\left(P_{n}\right)\right|$. In 2017, Barrientos and Krop [13] made improvements to the ideas of Bussel and were able to show better bounds for the parameter $\operatorname{grac}(T)$ of some trees. The authors showed that every lobster $T$ with $m$ edges and diameter $d$ has a range-relaxed graceful labelling with vertex labels no greater than $\frac{3}{2} m-\frac{1}{2} d$. Also in 2017, Sethuraman et al. [105] presented a better upper bound for the parameter $\operatorname{grac}(G)$ of trees.

The family of complete graphs is another family of graphs for which the parameter $\operatorname{grac}(G)$ has been investigated. In fact, until the conclusion of this thesis, the exact value of $\operatorname{grac}\left(K_{n}\right)$ is not known for $n \geq 24$ [106].

Range-relaxed graceful labellings of complete graphs are equivalent to another problem popularized by Golomb and that became known as Golomb ruler.

An $n$-mark Golomb ruler is a straight line containing $n$ distinct nonnegative integers $a_{1}, a_{2}, \ldots, a_{n}$ (with $0=a_{1}<a_{2}<\ldots<a_{n}$ ), called marks, such that the positive differences $\left|a_{i}-a_{j}\right|$, for all pairs $i, j$ with $i \neq j$, are distinct. The 0 mark and the last mark are the ends of the ruler. The length of the ruler is the largest mark. Figure 2.17 shows two Golomb rulers.


Figure 2.17: The first Golomb ruler has 4 marks and length 6 and the second Golomb ruler has 5 marks and length 11 .

A Golomb ruler that is of minimum length for a given number of marks is called an optimal Golomb ruler. A perfect Golomb ruler of $n$ marks is a Golomb ruler in which every integer from 1 up to the length of the ruler can be measured as the distance of exactly two marks. For example, the two Golomb rulers exhibited in Figure 2.17 are optimal but only the first is perfect (the Golomb ruler with five marks is not perfect because the distance 6 cannot be measured). The following result, originally proved by Simmons [107], is obtained as a consequence of Theorem 2.21.

Theorem 2.33 (Simmons [107]). A perfect Golomb ruler of $n$ marks exists if and only if $n \in\{1,2,3,4\}$.

Proof. A ruler whose only mark is 0 is trivially a perfect Golomb ruler. By inspection, one can verify that the rulers $R_{2}=\{0,1\}, R_{3}=\{0,1,3\}$ and $R_{4}=\{0,1,4,6\}$, where the elements of the sets label the ruler's marks, are perfect Golomb rulers on 2, 3 and 4 marks, respectively. Now, suppose that there exists a perfect Golomb ruler on $n$ marks, for $n \geq 5$. By the definition, every integer from 1 up to the length of the ruler can be measured as the distance of exactly two marks and all distances are distinct. Since there are $n$ marks, the total number of distances is $\binom{n}{2}$. Note that $\binom{n}{2}$ is the total number of edges of complete graph $K_{n}$. Then, we can use the labels on the $n$ marks to gracefully label the vertices of $K_{n}$, contradicting Theorem 2.21. Therefore, there exists no perfect Golomb ruler on $n \geq 5$ marks.

Determining $\operatorname{grac}\left(K_{n}\right)$ is equivalent to determining the length of an optimal Golomb ruler on $n$ marks [52]. By Theorem 2.33 and also by Theorem 2.21, we have that $\operatorname{grac}\left(K_{n}\right)>\left|E\left(K_{n}\right)\right|$ for all $n \geq 5$. Rosa [98] observed that every complete graph $K_{p^{t}+1}$, where $p$ is prime and $t \in \mathbb{N}_{>0}$, has a $\rho$-labelling. Therefore, for these complete graphs we have that $\operatorname{grac}\left(K_{p^{t}+1}\right) \leq 2\left|E\left(K_{p^{t}+1}\right)\right|$. However, as previously remarked, the exact value of $\operatorname{grac}\left(K_{n}\right)$ is not known for $n \geq 24$. Table 2.1 shows some known optimal Golomb rulers on up to 11 marks. For big values of $n$, some of the known optimal Golomb rulers with $n$ marks have been obtained using heavy computer calculations.

As pointed out by López and Muntaner-Batle [82], the study of Golomb rulers is still a very active area of research and very few optimal Golomb rulers are known. Most of the

| Order $n$ | Length | Marks |
| :--- | :--- | :--- |
| 1 | 0 | 0 |
| 2 | 1 | 01 |
| 3 | 3 | 013 |
| 4 | 6 | 0146 |
| 5 | 11 | 014911 |
| 6 | 17 | 014101217 |
| 7 | 25 | 01410182325 |
| 8 | 34 | 014915223234 |
| 9 | 44 | 015122527354144 |
| 10 | 55 | 01610232634415355 |
| 11 | 72 | 0141328334754647072 |

Table 2.1: Optimal Golomb rulers on $n$ marks, for $1 \leq n \leq 11$.
current work done on Golomb rulers involves the use of parallel computing to find and verify optimal Golomb rulers [38,95]. A list of optimal Golomb rulers on up to 23 marks can be found at James B. Shearer's page [106].

As discussed above, one of the forms of relaxing the graceful labelling is to expand the set of available vertex labels but still requiring the induced edge labels to be all distinct. However, other relaxations are possible like assigning distinct labels $0, \ldots,|E(G)|$ to the vertices a graph $G$ and allowing repeated induced edge labels. Such a labelling is called an edge-relaxed graceful labelling. Note that any graph $G$ has an edge-relaxed graceful labelling since any injective function $f: V(G) \rightarrow[0,|E(G)|]$ is an example of such a labelling. The difficult task is to find an edge-relaxed graceful labelling of $G$ with the minimum number of repeated induced edge labels. Given a graph $G$, the gracesize of $G$, denoted $g s(G)$, is the maximum number of distinct induced edge labels taken over all edge-relaxed graceful labellings of $G$. Note that, if $g s(G)=|E(G)|$, then $G$ is graceful.

The concept of gracesize was introduced by Heinrich and Hell [57]. The pioneer work on gracesize of trees is due to Rosa and Širáň [102], who proved that $g s(T) \geq 5 n / 7$, for every tree $T$ of order $n \geq 4$. Later, Bonnington and Širáň [20] improved this lower bound by proving that every tree $T$ of order $n \geq 12$ and with maximum degree three has $g s(T) \geq 5 n / 6$. Brankovic, Rosa and Širáň [22] improved this lower bound by proving that $g s(T) \geq\lfloor 6 n / 7\rfloor-1$ for every tree $T$ with $n$ vertices and with maximum degree three. Later, Brankovic et al. [21] proved that every tree $T$ of order $n$ with maximum degree three and a perfect matching has $g s(T) \geq\lfloor 14 n / 15\rfloor-1$.

A third way of relaxing the graceful labelling is to allow the function $f: V(G) \rightarrow$ $[0,|E(G)|]$ to be non-injective but still requiring that the induced edge labels be pairwise distinct. A labelling $f$ satisfying these two conditions is called a vertex-relaxed graceful labelling. Bussel [27] investigated vertex-relaxed graceful labellings of trees and proved that every tree $T$ with $n$ vertices has a vertex-relaxed graceful labelling such that the number of distinct vertex labels is greater than $n / 2$.

We have previously seen that all simple graphs have a range-relaxed graceful labelling and an edge-relaxed graceful labelling. However, not all graphs have a vertex-relaxed graceful labelling. Examples of graphs that do not have a vertex-relaxed graceful labelling
are the complete graphs with more than four vertices. This result is immediately obtained from Theorem 2.21 and the next proposition.

Proposition 2.34 (Bussel [27]). Let $G$ be a graph that has a universal vertex. If $G$ has a vertex-relaxed graceful labelling, then $G$ is graceful.

Proof. Note that the presence of a universal vertex $v$ in the graph $G$ precludes the use of repeated vertex labels since, in order to generate distinct edge labels at the edges incident with $v$, the labels of all the other vertices of $G$ must be distinct. This implies that any vertex-relaxed graceful labelling $f: V(G) \rightarrow[0,|E(G)|]$ of $G$ must be an injective function and, therefore, is a graceful labelling of $G$.

The ultimate goal of these relaxed versions of graceful labelling is to push the bounds so as to completely characterize graceful graphs. For trees, this implies to solve the Graceful Tree Conjecture. As we have seen, most of these relaxed labellings have been investigated for trees and complete graphs but remain completely open for many other families of graphs.

### 2.2.1 Forbidden subgraphs

It is natural in graph theory to think of substructures that forbid graphs in a given class from satisfying a certain property. Such substructures can be subgraphs or induced subgraphs and they are called forbidden subgraphs for the graph class. Although one might think of finding forbidden subgraphs for the class of graceful graphs, Acharya et al. [5] proved that the class of graceful graphs has no forbidden subgraphs.

Theorem 2.35 (Acharya et al. [5]). Every simple graph is an induced subgraph of a graceful graph.

Proof. Let $G$ be a simple graph. If $G$ is graceful, the result follows. Then, suppose $G$ is not graceful. Next, we construct a graceful graph $H$ that contains $G$ as an induced subgraph. Let $f: V(G) \rightarrow[0, k]$ be a range-relaxed graceful labelling of $G$ with $k=\operatorname{grac}(G)$. Since $k=\operatorname{grac}(G)$, there exist vertices $u, v \in V(G)$ such that $f(u)=0$ and $f(v)=k$.

Let $\left\{\ell_{1}, \ldots, \ell_{r}\right\}$ be the set of missing induced edge labels. Let $\left\{L_{V}, L_{\bar{V}}\right\}$ be a bipartition of $\left\{\ell_{1}, \ldots, \ell_{r}\right\}$ such that $\ell_{i} \in L_{V}$ if and only if there exists $w \in V(G)$ such that $f(w)=\ell_{i}$. For each $\ell_{i} \in L_{\bar{V}}$, add a vertex $w_{i}$ with label $\ell_{i}$ and add an edge connecting $w_{i}$ to $u$ so that $\left|f(u)-f\left(w_{i}\right)\right|=\ell_{i}$. For each $\ell_{i} \in L_{V}$, add a vertex $w_{i}$ with label $k+\ell_{i}$ and connect $w_{i}$ to $u$ and $v$ so that $\left|f(u)-f\left(w_{i}\right)\right|=k+\ell_{i}$ and $\left|f(v)-f\left(w_{i}\right)\right|=\ell_{i}$. At this point, each $\ell_{i}$ is an edge-label and new missing edge labels with values greater than $k$ may have been introduced. However, these new missing edge labels are not vertex-labels. Thus, for each new missing edge label $t$, add a new vertex $w_{t}$ with label $t$ and connect $w_{t}$ to $u$ so that $\left|f(u)-f\left(w_{t}\right)\right|=t$. By the construction, the resulting graph $H$ is graceful and contains $G$ as an induced subgraph.

The construction presented in the proof of Theorem 2.35 is illustrated in Figure 2.18.


Figure 2.18: Constructing a graceful graph $H$ that contains the forest $G=K_{1,3} \cup K_{1,3}$ as an induced subgraph. Note that $G$ does not have a graceful labelling since $|V(G)|>|E(G)|$.

### 2.3 The technique of transfers

In this section, we present a technique that allows us to transform a graceful tree $T$ into a new graceful tree $T^{\prime}$ with the same order and size as $T$. First, we need some definitions.

Let $T$ be a tree and $u v \in E(T)$ be an arbitrary edge. We define $T_{u, v}$ as the connected component of $T-u v$ that contains vertex $v$. We also say that $T_{u, v}$ is a component incident with vertex $u$. Let $w$ be a vertex distinct from $u$ and $v$, that does not belong to $V\left(T_{u, v}\right)$. We call transfer the operation of deleting edge $u v$ from $T$ and adding edge $v w$. After the transfer operation, we say that the component $T_{u, v}$ has been transferred or moved from vertex $u$ to vertex $w$. For any two distinct vertices $u$ and $w$ of a gracefully labelled tree $T$, the notation $u \rightarrow w$ means that we moved some components incident with vertex $u$ to vertex $w$. We say that a transfer $u \rightarrow w$ applied to a graceful tree is safe if the resulting tree is also graceful. Henceforth, given a graceful labelling $f$ of a tree $T$, we denote a vertex $u \in V(T)$ with $f(u)=k$ by $v_{k}$.

The following lemma establishes when a transfer performed on a graceful tree generates another graceful tree.

Lemma 2.36 (Wang et al. [114], Hrnčiar and Haviar [58]). Let $f$ be a graceful labelling of a tree $T$ and let $u, v \in V(T)$ be two distinct vertices. If $u$ is adjacent to (not necessarily distinct) vertices $u_{1}, u_{2} \in V(T)$, such that $v \notin V\left(T_{u u_{1}} \cup T_{u u_{2}}\right)$ and $f\left(u_{1}\right)+f\left(u_{2}\right)=$ $f(u)+f(v)$, then tree $T^{\prime}$, obtained from $T$ by moving components $T_{u u_{1}}$ and $T_{u u_{2}}$ from $u$ to $v$, is also graceful.

Proof. Let $T, T^{\prime}$ and $f$ be as stated in the hypothesis. Note that $V(T)=V\left(T^{\prime}\right)$ and all vertices in $T^{\prime}$ have the same labels under $f$ as they have in $T$ since the transfer operation does not modify vertex labels. Moreover, with exception of the edges $v u_{1}, v u_{2} \in E\left(T^{\prime}\right)$, all other edges of $E\left(T^{\prime}\right)$ are also edges of $E(T)$ and have the same induced edge labels under $f$ as they have in $T$. Therefore, in order to prove the result, it is sufficient to show that edges $u u_{1}, u u_{2} \in E(T)$ have the same induced labels as edges $v u_{1}, v u_{2} \in E\left(T^{\prime}\right)$. We consider two cases depending on whether $u_{1}$ and $u_{2}$ are the same vertex.

First, consider that $u_{1} \neq u_{2}$. By the hypothesis, we known that $f\left(u_{1}\right)+f\left(u_{2}\right)=$ $f(u)+f(v)$. This implies that $\left|f\left(u_{1}\right)-f(u)\right|=\left|f(u)+f(v)-f\left(u_{2}\right)-f(u)\right|=\left|f(v)-f\left(u_{2}\right)\right|$
and $\left|f\left(u_{2}\right)-f(u)\right|=\left|f(u)+f(v)-f\left(u_{1}\right)-f(u)\right|=\left|f(v)-f\left(u_{1}\right)\right|$, as required. This case is illustrated in Figure 2.19.

Now, consider that $u_{1}=u_{2}$. By the hypothesis, we know that $f\left(u_{1}\right)=\frac{f(u)+f(v)}{2}$. This implies that $\left|f\left(u_{1}\right)-f(u)\right|=\left|\frac{f(u)+f(v)}{2}-f(u)\right|=\left|\frac{f(v)-f(u)}{2}\right|$ and also that $\left|f\left(u_{1}\right)-f(v)\right|=$ $\left|\frac{f(u)+f(v)}{2}-f(v)\right|=\left|\frac{f(u)-f(v)}{2}\right|=\left|\frac{f(v)-f(u)}{2}\right|$. Therefore, $\left|f\left(u_{1}\right)-f(u)\right|=\left|f\left(u_{1}\right)-f(v)\right|$, and the result follows. This case is illustrated in Figure 2.20.

(a) Tree $T$.

(b) Tree $T^{\prime}$.

Figure 2.19: Trees $T$ and $T^{\prime}$, each with a graceful labelling. Tree $T^{\prime}$ is obtained from $T$ by transferring components $T_{v_{0}, v_{1}}$ and $T_{v_{0}, v_{7}}$ from vertex $v_{0}$ to vertex $v_{8}$. Since $f\left(v_{1}\right)+f\left(v_{7}\right)=$ $8=f\left(v_{0}\right)+f\left(v_{8}\right)$, by Lemma 2.36, we have that this transfer is safe and, therefore, $T^{\prime}$ is graceful.

(a) Tree $T$.

(b) Tree $T^{\prime}$.

Figure 2.20: Trees $T$ and $T^{\prime}$, each one with a graceful labelling. Tree $T^{\prime}$ is obtained from $T$ by moving the component $T_{v_{1}, v_{4}}$ from vertex $v_{1}$ to vertex $v_{7}$. Since $2 f\left(v_{4}\right)=8=$ $f\left(v_{1}\right)+f\left(v_{7}\right)$, by Lemma 2.36, we have that this transfer is safe and, therefore, $T^{\prime}$ is graceful.

In this thesis, we always apply safe transfers in order to move leaves $u_{1}, \ldots, u_{r}$ adjacent to a vertex $u$ to another vertex $v$ such that $v \notin\left\{u_{1}, \ldots, u_{r}\right\}$. Therefore, in order to simplify further definitions and statements of results related to the transfer technique, we always consider that we want to move leaves. Nevertheless, we draw the reader's attention to the fact that this restriction does not affect the general case in which the moved components have more than one vertex: in order to obtain a tree after the transfer of component $T_{u, u_{i}}$ from a vertex $u$ to a vertex $v$, we only have to guarantee that $v \notin V\left(T_{u, u_{i}}\right)$. With this simpler context in mind, Lemma 2.36 can be restated as follows.

Corollary 2.37 (Wang et al. [114], Hrnčiar and Haviar [58]). Let $f$ be a graceful labelling of a tree $T$ and let $u, v \in V(T)$ be two distinct vertices. If $u$ is adjacent to (not necessarily distinct) leaves $u_{1}, u_{2} \in V(T)$, such that $v \neq u_{1}, v \neq u_{2}$ and $f\left(u_{1}\right)+f\left(u_{2}\right)=f(u)+$
$f(v)$, then the tree $T^{\prime}$, obtained from $T$ by moving leaves $u_{1}$ and $u_{2}$ from $u$ to $v$, is also graceful.

Let $T$ be a gracefully labelled tree and $u$ and $v$ be two distinct vertices of $T$ such that $u$ is incident with leaves $u_{1}, \ldots, u_{r}$ and $v \notin\left\{u_{1}, \ldots, u_{r}\right\}$. It is often possible to safely transfer many leaves from $u$ to $v$ at the same time. Next, we describe two ways of doing this. We say that a transfer $u \rightarrow v$ is a transfer of the first type if $f(u)+f(v)=2 k+p$ and the labels of the transferred leaves comprise the set $[k, k+p]$. A transfer of the first type is also denoted by $u \xrightarrow{[k, k+p]} v$. On the other hand, if $u$ and $v$ are vertices such that $f(u)+f(v)=k+l+p$ and the labels of the transferred leaves comprise the set $[k, k+p] \cup[l, l+p]$, with $k+p<l$, then this transfer $u \rightarrow v$ is of the second type. A transfer of the second type is also denoted by $u \xrightarrow{[k, k+p],[l, l+p]} v$. Note that a transfer of the second type $u \xrightarrow{[k, k+p],[l, l+p]} v$ is equivalent to $u \xrightarrow{[k, l+p]} v$ followed by $v \xrightarrow{[k+p+1, l-1]} u$, which are two transfers of the first type. These two types of transfers are illustrated in Figure 2.21 and Lemma 2.38 shows that they are safe transfers.

(a) Graceful tree $T$.

(b) Graceful tree $T^{\prime}$.

(c) Graceful tree $T^{\prime \prime}$.

Figure 2.21: Three graceful trees $T, T^{\prime}$ and $T^{\prime \prime}$. Tree $T^{\prime}$ is obtained from $T$ by transfer $v_{11} \xrightarrow{[5,7]} v_{1}$ of the first type and tree $T^{\prime \prime}$ is obtained from $T$ by applying transfer $v_{11} \xrightarrow{[3,5],[7,9]}$ $v_{1}$ of the second type.

Lemma 2.38 (Mishra and Panigrahi [87]). Let $T$ be a tree with a graceful labelling $f$. Also, let $u$ and $v$ be two distinct vertices of $T$ such that $u$ is adjacent to a set of leaves $\mathcal{S}$ with labels $s, \ldots, s+p$, with $s<s+p$, satisfying the following properties:
(i) $\{s, \ldots, s+p\} \cap\{f(u), f(v)\}=\emptyset$; and
(ii) either $f(u)+f(v)=2 s+p+1$ or $f(u)+f(v)=2 s+p-1$.

Then, the following statements are true:
(a) let $n_{u}=2 r+1$ be an odd integer such that $1 \leq n_{u}<|\mathcal{S}|$. If $f(u)+f(v)=2 s+p+1$, then it is possible to make a safe transfer $u \xrightarrow{[s+r+1, s+p-r]} v$ of the first type, leaving $n_{u}$ leaves of $\mathcal{S}$ adjacent to vertex $u$. Moreover, if $f(u)+f(v)=2 s+p-1$, then it is possible to make a safe transfer $u \xrightarrow{[s+r, s+p-r-1]} v$ of the first type, leaving $n_{u}$ leaves of $\mathcal{S}$ adjacent to vertex $u$.
(b) let $n_{u}=2 r$ be an even integer such that $2 \leq n_{u}<|\mathcal{S}|$. If $|\mathcal{S}|$ is even and $f(u)+f(v)=$ $2 s+p+1$, then it is possible to make a safe transfer $u \xrightarrow{\left[s+r, \frac{2 s+p-1}{2}\right],\left[\frac{2 s+p+3}{2}, s+p+1-r\right]} v$ of the second type, leaving $n_{u}$ leaves of $\mathcal{S}$ adjacent to vertex $u$. Moreover, if $|\mathcal{S}|$ is even and $f(u)+f(v)=2 s+p-1$, then it is possible to make a safe transfer
$u \xrightarrow{\left[s+r-1, \frac{2 s+p-3}{2}\right] \cup\left[\frac{2 s+p+1}{2}, s+p-r\right]} v$ of the second type, leaving $n_{u}$ leaves of $\mathcal{S}$ adjacent to vertex $u$.

Proof. Let $T, f, u, v, \mathcal{S}$ be as stated in the hypothesis. We analyse two cases.
Case (a). Let $n_{u}=2 r+1$ be an odd integer with $1 \leq n_{u}<|\mathcal{S}|$. First, assume that $f(u)+f(v)=2 s+p+1$. In this case, we leave $2 r+1$ leaves adjacent to $u$, which consist of the leaf with label $s$ plus the pairs of leaves with labels $s+1+i$ and $s+p-i$, for $0 \leq i \leq r-1$, and we move the rest of the leaves of $\mathcal{S}$ to $v$. Note that the labels of the leaves that are moved to $v$ are the integers in the set $[s+r+1, s+p-r]$. Moreover, by Corollary 2.37, the pairs of leaves with labels $s+r+1+i$ and $s+p-r-i$, for $0 \leq i \leq\left\lceil\frac{p-2 r}{2}\right\rceil$, can be safely transferred to $v$ since $(s+r+1+i)+(s+p-r-i)=2 s+p+1=f(u)+f(v)$. Therefore, the transfer of the first type $u \xrightarrow{[s+r+1, s+p-r]} v$ is safe, as asserted.

Now, suppose that $f(u)+f(v)=2 s+p-1$. In this case, we move from $u$ to vertex $v$ all the leaves with labels in the set $[s+r, s+p-r-1]$ and leave the rest of the leaves of $\mathcal{S}$ adjacent to $u$. By a reasoning similar to the previous case, the transfer of the first type $u \xrightarrow{[s+r, s+p-r-1]} v$ is safe and exactly $2 r+1$ leaves of $\mathcal{S}$ are left adjacent to vertex $u$, as required.

Case (b). Let $n_{u}=2 r$ be an even integer, with $2 \leq n_{u}<|\mathcal{S}|$, and let $|\mathcal{S}|$ be even. This implies that $p$ is odd. First, suppose that $f(u)+f(v)=2 s+p+1$. In this case, we move from vertex $u$ to vertex $v$ the pairs of leaves with labels $s+r+i$ and $s+p+1-r-i$, for $0 \leq i \leq \frac{p-2 r-1}{2}$. Note that the leaves of $\mathcal{S}$ which are left adjacent to vertex $u$ have labels in the set $[s, s+r-1] \cup\left\{\frac{2 s+p+1}{2}\right\} \cup[s+p+2-r, s+p]$ and the cardinality of this set is exactly $2 r$. Moreover, by Corollary 2.37 , for each $i \in\left[0, \frac{p-2 r-1}{2}\right]$, the pairs of leaves with labels $s+r+i$ and $s+p+1-r-i$ can be safely transferred to $v$ since $(s+r+i)+(s+p+1-r-i)=2 s+p+1=f(u)+f(v)$. Therefore, the transfer of the second type $u \xrightarrow{\left[s+r, \frac{2 s+p-1}{2}\right],\left[\frac{2 s+p+3}{2}, s+p+1-r\right]} v$ is safe, as asserted. The proof for the case $f(u)+f(v)=2 s+p-1$ is similar.

Let $T$ be a tree with a graceful labelling $f$ and let $v_{1}, \ldots, v_{k}$ be $k$ distinct vertices of $T$ such that $v_{1}$ is adjacent to a set of leaves $\mathcal{S}=\left\{u_{1}, \ldots, u_{r}\right\}$, with $\mathcal{S} \cap\left\{v_{1}, \ldots, v_{k}\right\}=\emptyset$. Many authors $[12,58,87,110,114]$ have observed that, if the labels of the vertices $v_{1}, \ldots, v_{k}$ and $u_{1}, \ldots, u_{r}$ satisfy certain conditions, then it is possible to start at vertex $v_{1}$ and make a sequence of safe transfers $v_{1} \rightarrow v_{2} \rightarrow v_{3} \rightarrow \ldots \rightarrow v_{k}$ such that, at the end, each vertex $v_{i}$ of the sequence is adjacent to a positive number of leaves of the original set $\mathcal{S}$. Two conditions frequently imposed to vertices $v_{1}, \ldots, v_{k}$ are:
(i) for every $i, j \in[1, k]$, with $i$ odd and $j$ even, we have that $f\left(v_{i}\right)=f\left(v_{i+2}\right)+1$ and $f\left(v_{j}\right)=f\left(v_{j+2}\right)-1$, that is, labels $f\left(v_{1}\right), f\left(v_{2}\right), f\left(v_{3}\right), f\left(v_{4}\right), \ldots$ form a sequence $a, b, a-1, b+1, \ldots$, where $a, b \in[0,|E(T)|] ;$
(ii) for every $i, j \in[1, k]$, with $i$ odd and $j$ even, we have that $f\left(v_{i}\right)=f\left(v_{i+2}\right)-1$ and $f\left(v_{j}\right)=f\left(v_{j+2}\right)+1$, that is, labels $f\left(v_{1}\right), f\left(v_{2}\right), f\left(v_{3}\right), f\left(v_{4}\right), \ldots$ form a sequence $a, b, a+1, b-1, \ldots$, where $a, b \in[0,|E(T)|]$.

When vertices $v_{1}, \ldots, v_{k}$ satisfy any of these two conditions, we say that the labels of $v_{1}, \ldots, v_{k}$ form an alternating sequence of labels.

Lemma 2.39 and Lemma 2.40 show how to use vertices with an alternating sequence of labels along with some additional conditions in order to make a sequence of safe transfers of the second type in a gracefully labelled tree. The way Lemma 2.39 works is illustrated in Figure 2.22 and item (a) of Lemma 2.40 is illustrated in Figure 2.23.

Lemma 2.39 (Mishra and Panigrahi [87]). Let $T$ be a tree with a graceful labelling $f$ satisfying the following two properties:
(i) there exist distinct vertices in $T$ with labels $a-r_{1}, \ldots, a, b, \ldots, b+r_{2}$ such that $a<b$ and $r_{1}, r_{2} \in \mathbb{Z}_{\geq 0}$;
(ii) the vertex with label $a$ is adjacent to a set of leaves $\mathcal{S}$ with labels $s, \ldots, s+p$, such that:
(a) $p \geq 2$;
(b) $\{s, \ldots, s+p\} \cap\left\{a-r_{1}, \ldots, a, b, \ldots, b+r_{2}\right\}=\emptyset$; and
(c) either $a+b=2 s+p+1$ or $a+b=2 s+p-1$.

For each $x \in X=\left\{a-r_{1}, \ldots, a, b, \ldots, b+r_{2}\right\}$, let $v_{x}$ be the vertex of $T$ with label $x$ and let $n_{x}$ be an even positive integer. If $\sum_{x \in X} n_{x}=|\mathcal{S}|$, then it is possible to make the sequence of safe transfers of the second type $v_{a} \rightarrow v_{b} \rightarrow v_{a-1} \rightarrow v_{b+1} \rightarrow v_{a-2} \rightarrow v_{b+2} \rightarrow$ $\ldots \rightarrow v_{z}$, where $z=a-r_{1}$ or $z=b+r_{2}$, leaving, for each $v_{x}$ in the sequence, precisely $n_{x}$ (additional) leaves adjacent to $v_{x}$.

Lemma 2.40 (Mishra and Panigrahi [87]). Let $T$ be a tree with a graceful labelling $f$ satisfying the following two properties:
(i) there exist distinct vertices in $T$ with labels $a, \ldots, a+r_{1}, b-r_{2}, \ldots, b$ such that $a<b$, $a+r_{1}<b-r_{2}$, and $r_{1}, r_{2} \in \mathbb{Z}_{\geq 0}$;
(ii) the vertex with label $a$ is adjacent to $a$ set of leaves $\mathcal{S}$ with labels $s, \ldots, s+p$, such that:
(a) $p \geq 2$;
(b) $\{s, \ldots, s+p\} \cap\left\{a, \ldots, a+r_{1}, b-r_{2}, \ldots, b\right\}=\emptyset$; and
(c) either $a+b=2 s+p+1$ or $a+b=2 s+p-1$.

For each $x \in X=\left\{a, \ldots, a+r_{1}, b-r_{2}, \ldots, b\right\}$, let $v_{x}$ be the vertex of $T$ with label $x$ and let $n_{x}$ be a positive integer. Then, the following statements are true:
(a) if $n_{a}+n_{b}+n_{a+1}=|\mathcal{S}|$ with $n_{a}$ odd, $n_{b}$ and $n_{a+1}$ even, then it is possible to make a safe transfer of the first type $v_{a} \rightarrow v_{b}$, leaving $n_{a}$ (additional) vertices adjacent to $v_{a}$, followed by a safe transfer of the second type $v_{b} \rightarrow v_{a+1}$ leaving $n_{b}$ (additional) leaves adjacent to $v_{b}$.
(b) if $\sum_{x \in X} n_{x}=|\mathcal{S}|$ and each $n_{x}$ is even, then it is possible to make the sequence of safe transfers of the second type $v_{a} \rightarrow v_{b} \rightarrow v_{a+1} \rightarrow v_{b-1} \rightarrow v_{a+2} \rightarrow v_{b-2} \rightarrow \ldots \rightarrow v_{z}$, where $z=a+r_{1}$ or $z=b-r_{2}$, leaving, for each $v_{x}$ in the sequence, precisely $n_{x}$ (additional) leaves adjacent to $v_{x}$.

(a) Tree $T$ with a graceful labelling $f$. Note that vertex $v_{2}$ is adjacent to a set $\mathcal{S}$ of leaves with labels $3, \ldots, 14$ and $T$ has an alternating sequence of vertices $v_{2}, v_{16}, v_{1}, v_{17}, v_{0}$. In order to apply Lemma 2.39, we set $a=2, b=16, r_{1}=2, r_{2}=1, s=3, s+p=14$ and $z=a-r_{1}=0$. By inspection, it is possible to verify that these values satisfy the hypothesis of Lemma 2.39. Therefore, we can apply the lemma in order to perform a sequence of safe transfers of the second type $v_{2} \rightarrow v_{16} \rightarrow v_{1} \rightarrow$ $v_{17} \rightarrow v_{0}$ leaving a positive even number of leaves of $\mathcal{S}$ adjacent to each vertex of the sequence. The quantity of leaves we will leave adjacent are, respectively, $n_{2}=2, n_{16}=2, n_{1}=4, n_{17}=2, n_{0}=2$. Note that $n_{2}+n_{16}+n_{1}+n_{17}+n_{0}=12=|\mathcal{S}|$, as required. In order to make the first transfer of the sequence, we leave $v_{3}$ and $v_{9}$ adjacent to $v_{2}$ and move the remaining leaves of $\mathcal{S}$ to $v_{16}$. By Corollary 2.37, each pair of vertices with labels $4+i$ and $14-i$, for $0 \leq i \leq 4$, can be safely transferred to $v_{16}$ since $(4+i)+(14-i)=18=f\left(v_{2}\right)+f\left(v_{16}\right)$. The resulting tree is shown in Figure 2.22(b).

(b) In order to make the required transfer of the second type $v_{16} \rightarrow v_{1}$, we leave $v_{8}$ and $v_{14}$ adjacent to $v_{16}$ and move the remaining leaves from $v_{16}$ to $v_{1}$. By Corollary 2.37, each pair of vertices with labels $4+i$ and $13-i$, for $0 \leq i \leq 3$, can be safely transferred to $v_{1}$ since $(4+i)+(13-i)=$ $17=f\left(v_{16}\right)+f\left(v_{1}\right)$. The resulting tree is shown in Figure 2.22(c).

(c) In order to make the required transfer of the second type $v_{1} \rightarrow v_{17}$, we leave $v_{4}, v_{5}, v_{10}$ and $v_{13}$ adjacent to $v_{1}$ and move the remaining leaves from $v_{1}$ to $v_{17}$. By Corollary 2.37, each pair of vertices with labels $6+i$ and $12-i$, for $0 \leq i \leq 1$, can be safely transferred to $v_{17}$ since $(6+i)+(12-i)=$ $18=f\left(v_{1}\right)+f\left(v_{17}\right)$. The resulting tree is shown in Figure $2.22(\mathrm{~d})$.

(d) In order to make the required transfer of the second type $v_{17} \rightarrow v_{0}$, we leave $v_{7}$ and $v_{12}$ adjacent to $v_{17}$ and move the remaining leaves from $v_{17}$ to $v_{0}$. By Corollary 2.37, the vertices $v_{6}$ and $v_{11}$ can be safely transferred to $v_{0}$ since $6+11=17=f\left(v_{17}\right)+f\left(v_{0}\right)$. The resulting tree is shown in Figure 2.22(e).

(e) Resulting tree obtained from $T$ by a sequence of safe transfers of the second type.

Figure 2.22: Illustration of Lemma 2.39.

(a) Tree $T$ with a graceful labelling $f$. Note that vertex $v_{1}$ is adjacent to a set of leaves $\mathcal{S}=$ $\left\{v_{4}, \ldots, v_{16}\right\}$ and $T$ has an alternating sequence of vertices $v_{1}, v_{18}, v_{2}$. In order to apply item (a) of Lemma 2.40, we set $a=1, b=18, r_{1}=1, r_{2}=0, s=4, s+p=16, n_{a}=5, n_{b}=2$, and $n_{a+1}=6$. By inspection, it is possible to verify that these values satisfy the hypothesis of Lemma 2.40. Therefore, we can apply the item (a) of the lemma in order to make a transfer of the first type $v_{1} \rightarrow v_{18}$, leaving $n_{a}=5$ leaves of $\mathcal{S}$ adjacent to $v_{1}$, followed by a transfer of the second type $v_{18} \rightarrow v_{2}$, leaving $n_{b}=2$ leaves $\mathcal{S}$ adjacent to $v_{18}$. In order to make the first transfer, we leave $v_{4}, v_{5}, v_{14}, v_{15}, v_{16}$ adjacent to $v_{1}$ and move the remaining leaves of $\mathcal{S}$ to $v_{18}$. By Corollary 2.37, each pair of vertices with labels $6+i$ and $13-i$, for $0 \leq i \leq 3$, can be safely transferred to $v_{18}$ since $(6+i)+(13-i)=19=f\left(v_{1}\right)+f\left(v_{18}\right)$. The resulting tree is shown in Figure 2.23(b).

(b) In order to make the required transfer of the second type $v_{18} \rightarrow v_{1}$, we leave $v_{6}$ and $v_{10}$ adjacent to $v_{18}$ and move the remaining leaves from $v_{18}$ to $v_{2}$. By Corollary 2.37, each pair of vertices with labels $7+i$ and $13-i$, for $0 \leq i \leq 2$, can be safely transferred to $v_{1}$ since $(7+i)+(13-i)=20=f\left(v_{18}\right)+f\left(v_{2}\right)$. The resulting tree is shown in Figure 2.23(c).

(c) Resulting tree obtained from $T$ by a sequence of safe transfers $v_{1} \rightarrow v_{18} \rightarrow v_{2}$.

Figure 2.23: Illustration of item (a) of Lemma 2.40.

One of the first appearances of the technique of transfers is in the work of Wang et al. [114], in 1994, in which the authors use transfers to construct a class of graceful lobsters. Nevertheless, it was only seven years later that the technique of transfers became wellknown, due to the work of Hrnčiar and Haviar [58]. By using transfers, complementary labellings and other ideas they proved that all trees with diameter four or five are graceful. It is still unknown if all trees with diameter greater than five are graceful. The technique of transfers was also used by Mishra and Panigrahi [85-89] to construct many families of graceful lobsters. In 2007, Balbuena et al. [12] used transfers to prove that some families of rooted trees are graceful. In 2013, Superdock [110] used transfers and other ideas to show that some families of trees with diameter six are graceful.

In Chapter 3 of this thesis, we use the technique of transfers to prove that many infinite families of caterpillars are 0-rotatable. Our results in Chapter 3 reinforce a conjecture that says that all caterpillars with diameter at least five are 0-rotatable.

### 2.4 Trees with no $\alpha$-labelling

Many advancements towards a solution of the Graceful Tree Conjecture use $\alpha$-labellings as a main tool. For example, all lower bounds on the gracesize of trees presented in Section 2.2 were obtained using $\alpha$-labellings of trees. In fact, as can be seen in Chapter 4, $\alpha$-labellings possess special properties that allow us to combine two trees with $\alpha$-labellings in order to obtain a larger tree that also has an $\alpha$-labelling. A first example of these gluing properties was shown in the proof of Theorem 2.10.

As noted in page 31, not all trees have $\alpha$-labellings. For example, Rosa [98] proved that by subdividing each edge of a star $K_{1, q}$ exactly once, $q \geq 3$, we obtain a tree that does not have $\alpha$-labellings. Thus, since not all trees have $\alpha$-labellings and given the importance of $\alpha$-labellings in the developments of the Graceful Tree Conjecture, it seems relevant to exhibit classes of trees without $\alpha$-labellings. In this section, we present some known families of trees that do not have $\alpha$-labellings. We first introduce some definitions and auxiliary lemmas.

Let $G$ be a bipartite graph. A labelling $f: V(G) \rightarrow \mathbb{Z}$ is bipartite if there exists an integer $k$ such that, for every edge $u v \in E(G)$, either $f(u) \leq k<f(v)$ or $f(v) \leq k<f(u)$. Note that an $\alpha$-labelling is a bipartite graceful labelling. Figure 2.24 shows a tree with a bipartite labelling and with a bipartite graceful labelling ( $\alpha$-labelling).


Figure 2.24: A path $P_{6}$ with a bipartite labelling and with a bipartite graceful labelling. Each edge $u v \in E\left(P_{6}\right)$ is shown with its induced label $|f(u)-f(v)|$.

Let $T$ be a tree with an injective labelling $f: V(T) \rightarrow[0,|E(T)|]$. Define the edge parity of $T$ to be $\left(\sum_{i=1}^{|E(T)|} i\right) \bmod 2$. Note that $\left(\sum_{i=1}^{|E(T)|} i\right) \bmod 2=\frac{1}{2}(|V|-1)|V| \bmod 2$. Hence, the edge parity of $T$ is 0 if $|V| \equiv 0,1(\bmod 4)$ and 1 , if $|V| \equiv 2,3(\bmod 4)$. Moreover, if $f$ is a graceful labelling, then $\left(\sum_{i=1}^{|E(T)|} i\right) \bmod 2$ is the sum of all edge labels modulo 2. Define the vertex parity of $T$ under $f$ to be $\left(\sum_{v \in V(G)} d(v) f(v)\right) \bmod 2$ or, equivalently, to be the parity of the number of vertices of odd degree with odd label.

Brinkmann et al. [23] presented the following necessary condition for an injective bipartite labelling to be an $\alpha$-labelling.

Theorem 2.41 (Brinkmann et al. [23]). Let $f: V(T) \rightarrow[0,|E(T)|]$ be an injective bipartite labelling of a tree $T$. If $f$ is an $\alpha$-labelling, then the edge parity of $T$ and the vertex parity of $T$ under $f$ are equal.

Proof. Let $T$ be a tree with an $\alpha$-labelling $f$. Also, let $\left\{V_{1}, V_{2}\right\}$ be the bipartition of $T$ such that the labels of the vertices in $V_{1}$ are smaller than the labels of the vertices in $V_{2}$ (this bipartition is induced by $f$ ). By the definition, the edge parity of $T$ is $\left(\sum_{i=1}^{|E(T)|} i\right) \bmod 2$,
which is the sum modulo 2 of all edge labels of $T$. Therefore, the edge parity of $T$ is

$$
\begin{equation*}
\left(\sum_{\substack{v_{1} v_{2} \in E(T) \\ v_{1} \in V_{1}, v_{2} \in V_{2}}}\left(f\left(v_{2}\right)-f\left(v_{1}\right)\right)\right) \bmod 2=\left(\sum_{\substack{v_{1} v_{2} \in E(T) \\ v_{1} \in V_{1}, v_{2} \in V_{2}}}\left(f\left(v_{2}\right)+f\left(v_{1}\right)\right)\right) \bmod 2 \tag{2.3}
\end{equation*}
$$

Note that, for each $v \in V(T)$, the value $f(v)$ occurs exactly $d(v)$ times in the sum at the right side of equation (2.3). Therefore,

$$
\left(\sum_{\substack{v_{1} v_{2} \in E(T) \\ v_{1} \in V_{1}, v_{2} \in V_{2}}}\left(f\left(v_{2}\right)+f\left(v_{1}\right)\right)\right) \bmod 2=\left(\sum_{v_{1} \in V_{1}} d\left(v_{1}\right) f\left(v_{1}\right)+\sum_{v_{2} \in V_{2}} d\left(v_{2}\right) f\left(v_{2}\right)\right) \bmod 2
$$

which is, in fact, the vertex parity of $T$ under $f$. Therefore, the edge parity of $T$ and the vertex parity of $T$ under $f$ are equal.

As noted by Brinkmann et al. [23], while the edge parity is a property of the tree that does not depend on the labelling, the vertex parity in general does. However, Brinkmann et al. [23] proved that, for some families of trees, the vertex parity does not depend on the labelling but only on the tree. This result is presented as follows.

Lemma 2.42 (Brinkmann et al. [23]). Let $T$ be a tree with bipartition $\left\{V_{1}, V_{2}\right\}$ and let $f: V(T) \rightarrow[0,|E(T)|]$ be an injective bipartite labelling of $T$. Then, the following statements are true:
(i) if all vertices of $T$ have odd degree, then the vertex parity of $T$ under $f$ is the parity of the number of odd integers in the set $[0,|E(T)|]$;
(ii) if all vertices in $V_{1}$ have even degree and all vertices in $V_{2}$ have odd degree, then the vertex parity of $T$ under $f$ is equal to $\frac{\left|V_{2}\right|}{2} \bmod 2$.

Proof. Let $T$ be a tree with bipartition $\left\{V_{1}, V_{2}\right\}$. Also, let $f: V(T) \rightarrow[0,|E(T)|]$ be an injective bipartite labelling of $T$. First, suppose that all vertices of $T$ have odd degree. By the definition, the vertex parity of $T$ under $f$ is $\left(\sum_{v \in V(G)} d(v) f(v)\right) \bmod 2$, which is the parity of the number of vertices of odd degree with odd label. Since all vertices of $T$ have odd degree, this number is equal to parity of the number of odd integers in the set $[0,|E(T)|]$.

Now, suppose that all vertices in $V_{1}$ have even degree and all vertices in $V_{2}$ have odd degree. By the definition, the vertex parity of $T$ under $f$ is $\left(\sum_{v \in V(G)} d(v) f(v)\right) \bmod 2$. Since all vertices of odd degree belong to $V_{2}$, the vertex parity of $T$ is the parity of the number of vertices of $V_{2}$ with odd label. Since the number of vertices with odd degree in a graph is even, $\left|V_{2}\right|$ is even. Moreover, since $f$ is a bipartite labelling and $T$ is a tree, the labels assigned to the vertices of $V_{2}$ are either in the set $\left[0,\left|V_{2}\right|-1\right]$ or in the set $\left[\left|V_{1}\right|,\left|V_{1}\right|+\left|V_{2}\right|-1\right]$. This implies that the vertex parity of $T$ is either the parity of the number of odd integers in $\left[0,\left|V_{2}\right|-1\right]$ or in $\left[\left|V_{1}\right|,\left|V_{1}\right|+\left|V_{2}\right|-1\right]$. Since $\left|V_{2}\right|$ is even, these parities are the same and are equal to $\frac{\left|V_{2}\right|}{2} \bmod 2$.

In view of Lemma 2.42, when a tree $T$ has a bipartition $\left\{V_{1}, V_{2}\right\}$ so that the parity of the vertex degrees is the same for all vertices in the same part, the vertex parity of $T$ is the same for every bipartite labelling $f$ assigned to $T$ and depends only on $T$. In this case, we talk about the vertex parity of the tree.

We say that a tree $T$ has the parity property if: (a) the parity of the vertex degrees is the same for all vertices in the same part of the bipartition of $V(T)$; and (b) the vertex parity of $T$ and the edge parity of $T$ are different. The following result presents some properties of trees that have the parity property.

Lemma 2.43 (Brinkmann et al. [23]). If a tree $T$ has the parity property, then
(i) $T$ does not have an $\alpha$-labelling;
(ii) All vertices in one of the parts of $T$ have even degree;
(iii) $T$ has an odd number of vertices.

Proof. Let $T$ be a tree that has the parity property. Case (i). By Theorem 2.41, $T$ does not have an $\alpha$-labelling since the vertex parity of $T$ and the edge parity of $T$ are different.

Case (ii). Suppose $T$ has only vertices of odd degree. By the definition, the vertex parity of $T$ is $\left(\sum_{v \in V(T)} d(v) f(v)\right) \bmod 2=\left(\sum_{v \in V(T)} f(v)\right) \bmod 2=\left(\sum_{i=0}^{|V(T)|-1} i\right) \bmod 2=$ $\left(\sum_{i=1}^{|V(T)|-1} i\right) \bmod 2$. However, since $\left(\sum_{i=1}^{|V(T)|-1} i\right) \bmod 2$ is also the edge parity of $T$, we conclude that $T$ does not have the parity property, which is a contradiction. Therefore, $T$ contains at least one vertex $v$ of even degree and, since $T$ has the parity property, all vertices in the same part of $v$ have even degree.

Case (iii). By the previous case, one of the bipartition classes of $T$ contains only vertices of even degree. This implies that $|E(T)|$ is also even. Therefore, the number of vertices of $T,|E(T)|+1$, is an odd number.

Using the facts provided by Theorem 2.41, Lemma 2.42, and Lemma 2.43, it is not difficult to construct examples of trees that do not have $\alpha$-labellings. An example of such a tree is shown in Figure 2.25.

A family of trees that do not have $\alpha$-labellings is presented in Theorem 2.44.
Theorem 2.44 (Huang et al. [59], Brinkmann et al. [23]). Let $T$ be a tree with $4 k$ vertices, $k \geq 1$, all of odd degree, and let $T^{\prime}$ be the tree obtained from $T$ by subdividing each edge of $T$ exactly once. Then, $T^{\prime}$ has no $\alpha$-labelling.

Proof. Let $T$ and $T^{\prime}$ be as stated in the hypothesis and let $\left\{V_{1}, V_{2}\right\}$ be a bipartition of $T^{\prime}$ such that $V_{1}$ is the set of all vertices of degree two and $V_{2}$ is the set of all vertices of odd degree. By Lemma 2.43, in order to prove the result, it is sufficient to show that $T^{\prime}$ has the parity property. By the definition of $V_{2}$ and by the fact that the number of vertices of odd degree in a graph is even, we have that $\left|V_{2}\right|$ is even. Since $\left|V\left(T^{\prime}\right)\right|=8 k-1$, the edge parity of $T^{\prime}$ is 1 . On the other hand, the vertex parity of $T^{\prime}$ is the parity of the number of vertices of odd degree with odd label. By Lemma 2.42, the vertex parity of $T^{\prime}$ is $\frac{\left|V_{2}\right|}{2} \bmod 2=\frac{4 k}{2} \bmod 2=0$. Therefore, since the edge parity of $T$ and the vertex parity of $T$ are different, we conclude that $T$ has the parity property.


Figure 2.25: A tree $T$ that does not have $\alpha$-labellings. Note that $V(T)$ has a part formed by the black vertices, all of odd degree, and another formed by the white vertices, all of even degree. Since $|V(T)|=37$, the edge parity of $T$ is $\frac{1}{2}(37-1) 37 \bmod 2=0$. On the other hand, since all odd degree vertices belong to the same part and there are 22 of them, by Lemma 2.42, the vertex parity of $T$ is $\frac{22}{2} \bmod 2=1$. Therefore, $T$ has the parity property and, by Lemma 2.43, $T$ does not have an $\alpha$-labelling.

In Theorem 2.45 we present a new family of trees that do not have $\alpha$-labellings. This family is a generalization of the family presented in Theorem 2.44.

Theorem 2.45. Let $t$ be a positive odd integer and $T$ be a tree with $4 k$ vertices, $k \geq 1$, all with odd degree. Then, tree $T^{\prime}$, obtained from $T$ by performing the three operations described below, does not have $\alpha$-labellings:
(i) for each leaf $w \in V(T)$, add $t$ new vertices, linking them to $w$;
(ii) subdivide each edge $u v \in E(T)$ with $d_{T}(u)>1$ and $d_{T}(v)>1$, thereby creating a new vertex $x_{u v}$;
(iii) for each vertex $x_{u v}$ created in item (ii), add $t-1$ new vertices, linking them to $x_{u v}$.

Proof. Let $t, T$ and $T^{\prime}$ be as stated in the hypothesis. By Lemma 2.43, in order to prove the result, it is sufficient to show that $T^{\prime}$ has the parity property.

Let $\left\{V_{\mathcal{O}}, V_{\mathcal{E}}\right\}$ be a bipartition of $V\left(T^{\prime}\right)$ such that $V_{\mathcal{E}}=\left\{v \in V\left(T^{\prime}\right): d_{T^{\prime}}(v)\right.$ is even $\}$ and $V_{\mathcal{O}}=V\left(T^{\prime}\right) \backslash V_{\mathcal{E}}$. Set $V_{\mathcal{E}}$ comprises all vertices $x_{u v}$, created in item (ii), and all vertices $w \in V\left(T^{\prime}\right)$ corresponding to leaves of $T$. Note that the vertices of $V_{\mathcal{E}}$ are pairwise nonadjacent since: (i) vertex $w \in V_{\mathcal{E}}$ is adjacent to $t$ leaves and to a vertex $u \in V\left(T^{\prime}\right)$ that corresponds to a non-leaf vertex in $T$; and (ii) vertex $x_{u v} \in V_{\mathcal{E}}$ is adjacent to $t-1$ new leaves and to two vertices $u, v \in V\left(T^{\prime}\right)$ that correspond to non-leaf vertices in $T$. Moreover, note that every vertex of $V_{\mathcal{E}}$ has degree $t+1$, which is even.

By the definition, $V_{\mathcal{O}}=V\left(T^{\prime}\right) \backslash V_{\mathcal{E}}$, that is, $V_{\mathcal{O}}$ comprises all leaves of $T^{\prime}$ and all vertices $u, v$ that are neighbours of vertices $x_{u v}$. Leaves have degree one and, for each vertex $x_{u v} \in V\left(T^{\prime}\right)$, the degrees of its neighbours $u$ and $v$ in $T^{\prime}$ are the same as their corresponding in $T$, which are odd by the hypothesis. Hence, all vertices of $V_{\mathcal{O}}$ have odd degree. Furthermore, the vertices of $V_{\mathcal{O}}$ are pairwise nonadjacent since a leaf of $T^{\prime}$ is adjacent either to a vertex $x_{u v}$ or a vertex that corresponds to a leaf in $T$. Therefore, we conclude that $\left\{V_{\mathcal{O}}, V_{\mathcal{E}}\right\}$ is a bipartition of $T^{\prime}$ such that all vertices of $V_{\mathcal{E}}$ have even degree and all vertices of $V_{\mathcal{O}}$ have odd degree; it remains to show that the edge parity of $T^{\prime}$ is different from the vertex parity of $T^{\prime}$.

By the definition of $T^{\prime},\left|V\left(T^{\prime}\right)\right|=|V(T)|(t+1)-t$ (to see this, observe that $V(T) \subset$ $V\left(T^{\prime}\right)$ and that, for each edge of $T$, exactly $t$ new vertices are added so as to construct $\left.T^{\prime}\right)$. Since $|V(T)|=4 k$, we have that $\left|V\left(T^{\prime}\right)\right|=4 k(t+1)-t$. The edge parity of $T^{\prime}$ is given by $\left(\frac{1}{2}\left(\left|V\left(T^{\prime}\right)\right|-1\right)\left|V\left(T^{\prime}\right)\right|\right) \bmod 2$. Hence, considering the possible values of $t$ modulo 4, the edge parity of $T^{\prime}$ is:

$$
\left(\frac{(4 k(t+1)-t-1)(4 k(t+1)-t)}{2}\right) \bmod 2=\left\{\begin{array}{lll}
1, & \text { if } t \equiv 1 & (\bmod 4)  \tag{2.4}\\
0, & \text { if } t \equiv 3 & (\bmod 4)
\end{array}\right.
$$

By item (ii) of Lemma 2.42, the vertex parity of $T^{\prime}$ is equal to $\left(\left|V_{\mathcal{O}}\right| / 2\right) \bmod 2$. We claim that $\left|V_{\mathcal{O}}\right|=(|V(T)|-1) t+1$. In order to see this, let $|V(T)|=|I|+|L|$ such that $L$ is the set of leaves of $T$ and $I=V(T) \backslash L$. By the construction of $T^{\prime}$,

$$
\begin{aligned}
\left|V_{\mathcal{O}}\right| & =(|I|-1)(t-1)+|I|+|L| t \\
& =(|I|+|L|) t-t+1 \\
& =(|V(T)|-1) t+1 .
\end{aligned}
$$

Thus, $\left|V_{\mathcal{O}}\right|=(4 k-1) t+1$ and, again considering the possible values of $t$ modulo 4 , we conclude that the vertex parity of $T^{\prime}$ is:

$$
\left(\frac{(4 k-1) t+1}{2}\right) \bmod 2=\left\{\begin{array}{lll}
0, & \text { if } t \equiv 1 & (\bmod 4) ;  \tag{2.6}\\
1, & \text { if } t \equiv 3 & (\bmod 4) .
\end{array}\right.
$$

Therefore, we conclude that the edge parity and the vertex parity of $T^{\prime}$ are different, completing the proof.

In the same paper where Huang, Kotzig and Rosa [59] proved Theorem 2.44, they also proved that all lobsters with diameter four not isomorphic to caterpillars do not have $\alpha$-labellings. This result is presented in Theorem 2.47 and its proof uses the following lemma.

Lemma 2.46 (Huang et al. [59]). Let $T$ be a caterpillar with diameter four and let $w \in V(T)$ be the central vertex of $T$. Then, there is no $\alpha$-labelling $f$ of $T$ such that $f(w)=0$.

Proof. Let $T$ be a caterpillar with diameter four and $m$ edges. Let $w$ be the central vertex of $T$ and $u_{1}, \ldots, u_{r}$ be the leaves adjacent to $w$. Let $v_{1}, v_{2}$ be the non-leaf vertices adjacent to $w$ such that $v_{1,1}, v_{1,2} \ldots, v_{1, q_{1}}$ are the leaves adjacent to $v_{1}$ and $v_{2,1}, v_{2,2} \ldots, v_{2, q_{2}}$ are the leaves adjacent to $v_{2}$. See Figure 2.26 for a sketch of $T$.

Suppose there exists an $\alpha$-labelling $f$ of $T$ with separator $k \in[0, m]$ such that $f(w)=0$. Then, by the definition of $f$, vertices $v_{1}, v_{2}, u_{1}, \ldots, u_{r}$ have labels greater than $k$ and vertices $v_{1,1}, \ldots, v_{1, q_{1}}, v_{2,1}, \ldots, v_{2, q_{2}}$ have labels smaller than or equal to $k$. This implies that the labels of the edges incident with $w$ are exactly $k+1, k+2, \ldots, m$. Without loss of generality, let $f\left(v_{2,1}\right)=1$. Then, $f\left(v_{2}\right)=k+1$ because all edge labels greater than $k$ already appear at the edges incident with $w$. Let $j$ be the smallest label not appearing at the vertices $v_{2,1}, v_{2,2}, \ldots, v_{2, q_{2}}$. Note that $j \geq 2$. This implies that $\{1, \ldots, j-1\} \subseteq$ $L_{N_{T}\left(v_{2}\right) \backslash\{w\}}^{f}$ and all the labels $k, \ldots, k-j+2$ appear at edges incident with $v_{2}$. By the
definition of $j$, there exists $v_{1, i}$ such that $f\left(v_{1, i}\right)=j$. Moreover, $f\left(v_{1}\right)-j \leq k-j+1$ since all edge labels greater than $k-j+1$ appear at edges incident with $v_{2}$ or $w$. However, this implies that $f\left(v_{1}\right) \leq k+1$, which is impossible to satisfy since all labels smaller than or equal to $k+1$ are already assigned to other vertices.


Figure 2.26: Sketch of a caterpillar with diameter four.

Theorem 2.47 (Huang et al. [59]). Let $T$ be a lobster with diameter four such that $T$ is not a caterpillar. Then, $T$ does not have an $\alpha$-labelling.

Proof. Suppose that there exist lobsters with diameter four, not isomorphic to caterpillars, and with $\alpha$-labellings. Let $T$ be such a lobster with the minimum number $m$ of edges. We refer to the vertices of $T$ as follows: $T$ has a central vertex $w$, adjacent to leaves $u_{1}, \ldots, u_{r}$, and to non-leaf vertices $v_{1}, v_{2}, \ldots, v_{p}$. Moreover, for each $i \in\{1, \ldots, p\}$, vertex $v_{i}$ is adjacent to leaves $v_{i, 1}, v_{i, 2} \ldots, v_{i, q_{i}}$ (see Figure 2.26 for an example in which $p=2$ ).

Let $f: V(T) \rightarrow[0, m]$ be an $\alpha$-labelling of $T$ with separator $k, k \in[0, m]$. Let $e$ be the edge of $T$ with label $f(e)=m$. We consider three cases depending on the endpoints of $e: e=w u_{i} ; e=w v_{i}$; or $e=v_{i} v_{i, j}$.

Case 1. $e$ is an edge $w u_{i}$.
Suppose that $f(w)=0$ and $f\left(u_{i}\right)=m$ (otherwise, take the complementary labelling). Let $T^{\prime}=T-u_{i}$. Note that $f$ restricted to $T^{\prime}$ is an $\alpha$-labelling of $T^{\prime}$ and $\left|E\left(T^{\prime}\right)\right|<|E(T)|$, contradicting the choice of $T$.

Case 2. $e$ is an edge $v_{i} v_{i, j}$.
In this case, $e=v_{i} v_{i, j}$, where $v_{i, j}$ is a leaf and $i \in[1, p]$. Without loss of generality, we can assume that $f\left(v_{i, j}\right)=m$ and $f\left(v_{i}\right)=0$. Let $T^{\prime}$ be the tree obtained from $T$ by removing $v_{i, j}$. Note that the restriction of $f$ to $T^{\prime}$ is an $\alpha$-labelling. By this fact and by the minimality of $T$, we obtain that $T^{\prime}$ is a caterpillar with diameter four and so, $v_{i}$ is a vertex of degree two in $T$, adjacent to $v_{i, j}$ and $w$. Note that this implies $f(w)=m-1$. Hence $v_{i}$ is a leaf in $T^{\prime}$ and $T^{\prime}$ admits an $\alpha$-labelling $f^{\prime}$ in which $v_{i}$ has label $m-1$ and $w$ is labelled 0 , by complementary labelling. Thus, $f^{\prime}$ is an $\alpha$-labelling of the caterpillar $T^{\prime}$ that assigns label 0 to $w$, contradicting Lemma 2.46.

Case 3. $e$ is an edge $w v_{i}$.
Without loss of generality, consider $e=w v_{1}, f\left(v_{1}\right)=0$ and $f(w)=m$. Then, the neighbours of $w$ have labels $0,1, \ldots, k$ and the edge labels $m, m-1, \ldots, m-k$ appear at the edges incident with vertex $w$ (note that $k \geq 2$ ). Since edge label $m-1$ appears at an edge incident with $w$, vertex label $m-1$ cannot be assigned to any neighbour of $v_{1}$.

Thus, without loss of generality, assume that $f\left(v_{2,1}\right)=m-1$. This implies that $f\left(v_{2}\right)=k$ since any other value smaller than $k$ would generate a repeated edge label on the edge $v_{2} v_{2,1}$. Let $t$ be the largest label not appearing at the vertices of $\left\{v_{2,1}, v_{2,2}, \ldots, v_{2, q_{2}}\right\}$. This implies that labels $t+1, \ldots, m-1$ all appear in some vertex of $\left\{v_{2,1}, v_{2,2}, \ldots, v_{2, q_{2}}\right\}$ and $m-k-1, \ldots, t+1-k$ appear at edges incident with $v_{2}$. Moreover, vertex label $t$ appears at some leaf $v_{i, j}$ adjacent to a vertex $v_{i}, i \neq 2$, with label at most $k-1$. However, this is a contradiction since the label of the edge $v_{i} v_{i, j}$ is greater than or equal to $t+1-k$ and, therefore, is a repeated edge label.

As shown in Theorem 2.4, if a bipartite graph with $m$ edges has an $\alpha$-labelling, then it cyclically decomposes the complete graph $K_{2 p m+1}$, for $p$ an arbitrary positive integer. Therefore, for graph decompositions of the complete graph $K_{2 p m+1}$, the most desirable labelling (among $\alpha, \beta, \sigma$ and $\rho$ ) of a bipartite graph would be an $\alpha$-labelling.

Some families of trees are known to possess $\alpha$-labellings [49]. For example, by Theorem 2.19, all caterpillars have $\alpha$-labellings. A next step would be to consider the class of lobsters. By Theorem 2.47, we know that not all lobsters have $\alpha$-labellings. In fact, it is still unknown if all lobsters have a graceful labelling and the lobsters that do not have $\alpha$-labellings are also not fully characterized. In fact, in one of our results, presented in Chapter 4, we prove that some families of lobsters with maximum degree three have $\alpha$-labellings.

Despite the fact that the Graceful Tree Conjecture is still open for lobsters, it is wellknown that all lobsters have $\sigma$-labellings [49] (this result is shown in Corollary 4.1), which implies that the Ringel-Kotzig Conjecture is true for lobsters.

### 2.5 Strongly-graceful labellings

In this section, we discuss a special type of graceful labelling, introduced by Broersma and Hoede $[24]$ and that is defined only for trees that have a perfect matching.

Let $T$ be a tree with a perfect matching $M$. A labelling $f$ of $T$ is strongly-graceful if $f$ is a graceful labelling and, additionally, $f(u)+f(v)=|E(T)|$ for every edge $u v \in M$. Figure 2.27 exhibits a tree $T$ with a strongly-graceful labelling.


Figure 2.27: A tree with a perfect matching and a strongly-graceful labelling.

Let $f$ be a strongly-graceful labelling of a tree $T$ with a perfect matching $M$. There are some properties that arise directly from the definition. For instance, for every edge $u v \in$ $M$, the induced label of $u v$ is $|f(u)-f(v)|=|f(u)-(|E(T)|-f(u))|=|2 f(u)-|E(T)||$, which is an odd number since $|E(T)|$ is odd. Thus, the parities of the endpoints of
each edge in $M$ are different. Moreover, since $f$ is a graceful labelling, the labels of the edges in $E(T) \backslash M$ are the even numbers in set $\{1, \ldots,|E(T)|\}$. Therefore, the endpoints of each edge in $E(T) \backslash M$ have the same parity. These observations are summarized in Proposition 2.48 .

Proposition 2.48. Let $f$ be a strongly-graceful labelling of a tree $T$ with a perfect matching $M$. Then, $f(u) \not \equiv f(v)(\bmod 2)$, if $u v \in M$ and $f(u) \equiv f(v)(\bmod 2)$, otherwise. Thus, $L_{M}^{f}=\{2 i+1: 0 \leq i \leq\lfloor|E(T)| / 2\rfloor\}$ and $L_{E(T) \backslash M}^{f}=\{2 i: 1 \leq i \leq\lfloor|E(T)| / 2\rfloor\}$.

Given a tree $T$ with perfect matching $M$, the contree of $T$ is the tree $T^{\prime}$ obtained from $T$ by contracting all the edges of $M$. In their article, Broersma and Hoede [24] also proved the following result.

Theorem 2.49 (Broersma and Hoede [24]). If the contree of a tree $T$ with a perfect matching has a graceful labelling, then $T$ has a strongly-graceful labelling.

In order to prove Theorem 2.49, Broersma and Hoede [24] defined a construction (described below) that is used in some of our proofs in Chapter 4. This construction allows us to obtain a strongly-graceful labelling of a tree $T$ with a perfect matching from any graceful labelling of its contree.

Broersma-Hoede's construction. Let $T$ be a tree with a perfect matching $M$. By Proposition 2.48, in a strongly-graceful labelling $f$ of $T$, we have $f(u) \not \equiv f(v)(\bmod 2)$, if $u v \in M$, and $f(u) \equiv f(v)(\bmod 2)$, otherwise. Note that, once the parity of the label of one vertex of $T$ is known, the parities of the labels of the other vertices are uniquely determined since $T$ is a tree. Thus, the first step of the construction is to choose the parity of the label of an arbitrary vertex $v$ of $T$ and, then, obtain the parity of the labels of the remaining vertices. This step is illustrated in Figure 2.28(a). Next, consider a graceful labelling $f^{\prime}$ of the contree $T^{\prime}$ of $T$. Modify $f^{\prime}$ so that, for each vertex $x_{u v} \in V\left(T^{\prime}\right)$, $x_{u v}$ is assigned label $2 f^{\prime}\left(x_{u v}\right)$, as illustrated in Figure 2.28(b) and Figure 2.28(c). Let $u v \in M$. Considering that $v$ has even parity and $u$ has odd parity, assign label $2 f^{\prime}\left(x_{u v}\right)$ to $v$ and label $|E(T)|-2 f^{\prime}\left(x_{u v}\right)$ to $u$. Broersma and Hoede proved that this assignment is a strongly-graceful labelling of $T$.

Let $T$ be a tree with a perfect matching $M$ and let $u v \in M, u, v \in V(T)$. Let $T^{\prime}$ be the contree of $T$ and $x_{u v} \in V\left(T^{\prime}\right)$ be the vertex of $T^{\prime}$ that corresponds to edge $u v$. One of the first steps of Broersma-Hoede's construction is to choose the parity of the labels of the vertices of $T$. By Proposition 2.48, there are only two possibilities for the labels of the endpoints of the edge $u v$ : either $u$ is even and $v$ is odd or $u$ is odd and $v$ is even. Moreover, as we discussed, each one of these two possibilities completely determines the parities of the labels of the other vertices of $T$. This implies that a given graceful labelling of the contree of $T$ can, in fact, generate two strongly-graceful labellings $f$ and $f^{\prime}$ of the tree $T$. Furthermore, it is not difficult to see that $f$ and $f^{\prime}$ are complementary labellings since, for each edge of $M$, its endpoints have labels $x$ and $|E(T)|-x$, with different parities. These observations lead us to the following lemma.

Lemma 2.50 (Broersma and Hoede [24]). Let $T$ be a tree with a perfect matching $M$ and $u v \in M, u, v \in V(T)$. Let $T^{\prime}$ be the contree of $T$ and let $x \in V\left(T^{\prime}\right)$ be the vertex

(a) A tree $T$ with a perfect matching. Each vertex of $T$ is labelled with letters $O$ or $E$, where letter $O$ means odd parity and letter $E$ means even parity.

(c) A new labelling of $T^{\prime}$ obtained by assigning label $2 f^{\prime}(v)$ to each vertex $v$.

(d) Strongly-graceful labelling $f$ of $T$. The even label $f(v)$ is taken from the previous labelling of $T^{\prime}$ and the odd label is $|E(T)|-f(v)$.

Figure 2.28: Construction of a strongly-graceful labelling $f$ for a tree $T$ with a perfect matching. Note that, in this case, $f$ is also an $\alpha$-labelling.
corresponding to edge uv. If $T^{\prime}$ has a graceful labelling $f^{\prime}$, with $f^{\prime}(x)=0$, then $T$ has two strongly graceful labellings $f_{1}$ and $f_{2}$, such that: (i) $f_{1}(u)=0$ and $f_{1}(v)=|E(T)|$; (ii) $f_{2}(u)=|E(T)|$ and $f_{2}(v)=0$.

Broersma and Hoede [24] also showed the following equivalence between graceful trees and strongly-graceful trees.

Theorem 2.51 (Broersma and Hoede [24]). Every tree has a graceful labelling if and only if every tree with a perfect matching has a strongly-graceful labelling.

Proof. First, suppose that every tree has a graceful labelling and let $T$ be a tree with a perfect matching. Since the contree of $T$ is also a tree, it has a graceful labelling and, by Theorem 2.49, $T$ has a strongly-graceful labelling. Now, suppose that every tree with a perfect matching has a strongly-graceful labelling and let $T$ be an arbitrary tree with vertex set $V(T)=\left\{v_{1}, \ldots, v_{n}\right\}$. We construct a new tree $T^{\prime}$ with a perfect matching, such that $T \subset T^{\prime}$, as follows: for each vertex $v_{i} \in V(T), 1 \leq i \leq n$, add a new vertex $u_{i}$ and an edge $v_{i} u_{i}$ linking $v_{i}$ to $u_{i}$. It is not difficult to see that the set $M=\left\{u_{i} v_{i}: 1 \leq i \leq n\right\}$ is a perfect matching of $T^{\prime}$. By the hypothesis, $T^{\prime}$ has a stronglygraceful labelling $f: V\left(T^{\prime}\right) \rightarrow[0,2 n-1]$. By Proposition 2.48, the edges in $E\left(T^{\prime}\right) \backslash M$ have even labels $2,4, \ldots, 2 n-2$ under $f$ and the edges in $M$ have odd labels $1,3, \ldots, 2 n-1$. Without loss of generality, we may assume that all $f\left(v_{i}\right)$ are even and all $f\left(u_{i}\right)$ are odd (otherwise, we use the complementary labelling of $f$ ). Thus, define $g\left(v_{i}\right)=\frac{1}{2} f\left(v_{i}\right)$. Then, $\left\{g\left(v_{i}\right): 1 \leq i \leq n\right\}=[0, n-1]$ and the edges of $T$ have labels $1,2, \ldots, n-1$. Therefore, $g: V(T) \rightarrow[0, n-1]$ is a graceful labelling of $T$.

Theorem 2.51 implies that the Graceful Tree Conjecture is equivalent to Conjecture 2.52 .

Conjecture 2.52 (Broersma and Hoede [24]). Every tree with a perfect matching has a strongly-graceful labelling.

In their seminal article, Broersma and Hoede [24] also showed some families of trees with perfect matching that have strongly-graceful labellings. Posteriorly, Yao et al. [117] proved that all trees with perfect matching and diameter at most five have stronglygraceful labellings. Wang et al. [116] improved this result by showing that all trees with perfect matching and diameter at most seven have strongly-graceful labellings. Stronglygraceful labellings were also independently discovered by Haviar and Ivaška [56], who proved similar results as those by Broersma and Hoede [24], Yao et al. [117] and Wang et al. [116]. Despite all these findings, Conjecture 2.52 remains open.

## Chapter 3

## 0-rotatable graceful caterpillars

As soon as one starts assigning labels to the vertices of a graph so as to obtain a graceful labelling, one of the first questions that arises is "which label goes where?" A natural starting point is to assign labels 0 and $m$ to the endpoints of a certain edge since every gracefully labelled graph has an edge with induced label $m$. If $G$ is a tree, then its vertices have all labels in the set $\{0, \ldots, m\}$. However, for an arbitrary connected graph $G$ with $|V(G)|<|E(G)|+1$, we know that labels 0 and $m$ must be present but we cannot say the same for any other labels. Some work in this direction was initiated by Abrham and Kotzig [2], who proved that in any $\alpha$-labelling of a 2 -regular graph $G$ with $4 k$ vertices, exactly one of integers $k$ and $3 k$ is not assigned to any vertex of $G$.

We now illustrate possible ways of approaching the construction of graceful labellings using properties derived earlier, in Chapter 2. Let $G$ be a graph with $m$ edges and with an injective vertex-labelling $f$. Given $v \in V(G)$ :
(i) if $f$ is graceful and $f(v)=0$, then $v$ has a neighbour $u$ such that $f(u)=m$ since this is the only way edge label $m$ can be generated;
(ii) if $f$ is graceful and $f(v)=0$, then $G$ also has a graceful labelling $\bar{f}$ such that $\bar{f}(v)=m$ (complementary labelling);
(iii) if $G$ is a tree, $f$ is an $\alpha$-labelling with $f(v)=0$, and $s$ is the size of the part containing $v$, then there are $\alpha$-labellings $f^{\prime}$ and $f^{\prime \prime}$ of $G$ such that $f^{\prime}(v)=s-1$ and $f^{\prime \prime}(v)=m-s-1$ (reverse labelling);
(iv) if $v$ has a neighbour $u$ with degree 1 and graph $G^{\prime}=G-u$ has a graceful labelling $f^{\prime}$ such that $f^{\prime}(v)=0$, then $G$ has a graceful labelling that assigns label 0 to $v$ and label $m$ to $u$ (by Lemma 2.11).
These observations and the construction presented in Theorem 2.10 stress the importance of knowing how to construct graceful labellings (or $\alpha$-labellings) of a tree with the label 0 appearing in a selected vertex. As defined in Section 2.1, trees $T$ for which there exists a graceful labelling of $T$ that assigns label 0 to $v$, for every vertex $v \in V(T)$, are called 0-rotatable. Figure 3.1 shows a 0 -rotatable tree.

The first work published on 0-rotatable trees is due to Rosa [99], who proved that all paths are 0-rotatable (see Theorem 2.15). In fact, for paths we have a stronger result: in most cases we can find a graceful labelling such that a chosen vertex receives a chosen label, as the following two theorems show.


Figure 3.1: Six graceful labellings of a 0-rotatable tree $T$.

Theorem 3.1 (Flandrin et al. [47]). Let $n \in \mathbb{Z}$ with $n \geq 9$. For every vertex $v$ of path $P_{n}$ and for every integer $k$, with $k \in[0, n-1]$, there exists a graceful labelling $f$ of $P_{n}$ such that $f(v)=k$.

Theorem 3.2 (Cattell [29]). Let $n \in \mathbb{Z}_{\geq 1}$. Given a path $P_{n}$ and any vertex $v \in P_{n}$, there exists a graceful labelling of $P_{n}$ in which $v$ has label $i$, for any $i \in\{0, \ldots, n-1\}$, whenever at least one of the following conditions is met:
(i) $n$ is even;
(ii) $n \equiv 5$ or $9(\bmod 12)$;
(iii) $v$ is in the larger of the two partite sets of vertices;
(iv) $i \neq \frac{n-1}{2}$.

As discussed in Section 2.1, it is well-known that not all trees are 0-rotatable. For example, the smallest non-0-rotatable tree is illustrated in Figure 3.2.


Figure 3.2: A tree $T$ that is not 0 -rotatable. There is no graceful labelling of $T$ with label 0 assigned to the black vertex $v$.

In 2004, Bussel [27] showed that all trees with diameter at most three are 0-rotatable. Additionally, the author showed that there exist non-0-rotatable trees with diameter four. In fact, he completely determined the non-0-rotatable trees of diameter four, starting with the following result:

Theorem 3.3 (Bussel [27]). Let $T$ be a tree of diameter four such that its central vertex $v$ has degree two. Let $v_{1}, v_{2}$ be the vertices adjacent to $v$ and $m_{1}, m_{2}$ be the number of leaves adjacent to $v_{1}, v_{2}$, respectively. Assume $m_{1} \geq m_{2}$. Tree $T$ has a graceful labelling $f$ with $f(v)=0$ if and only if there exist integers $x$ and $r$ such that $m_{1}=\left(m_{2}+2-x\right)(r-1)-x$, with:
(i) $x$, r not both odd;
(ii) $2 \leq r \leq|E(T)| / 2$; and
(iii) $0 \leq x \leq \min \left\{r-1, m_{2}\right\}$.

In order to characterize the non-0-rotatable trees of diameter four, Bussel [27] defined $\mathcal{D}$ as a class of diameter-four trees whose central vertex has degree two and that do not satisfy the conditions of Theorem 3.3. Figure 3.3 shows the three smallest trees with diameter four that belong to class $\mathcal{D}$.


Figure 3.3: The three smallest diameter-four trees that are not 0-rotatable.

Additionally, he defined $\mathcal{D}^{\prime}$ as the class of trees built by identifying a leaf of an arbitrary path $P_{n}, n \geq 1$, with the central vertex of a tree in $\mathcal{D}$. Bussel [27] proved that, given a tree $T$ with diameter four, $T$ is 0-rotatable if and only if $T \notin \mathcal{D}^{\prime}$. Moreover, he showed that all trees with at most 14 vertices and that are not 0 -rotatable belong to class $\mathcal{D}^{\prime}$. Thus, based on these results, the author posed the following conjecture.

Conjecture 3.4 (Bussel [27]). The class $\mathcal{D}^{\prime}$ contains all non-0-rotatable trees.
From the time it was first studied, 0-rotatability of trees has been considered a possible way to approach the Graceful Tree Conjecture and also a challenging problem by itself. Even for arbitrary caterpillars the result is not known. In fact, note that, if Conjecture 3.4 is true, then it implies that every caterpillar with diameter at least five is 0 -rotatable.

Conjecture 3.5. Every caterpillar with diameter at least five is 0 -rotatable.
In this chapter we prove that the following families of caterpillars are 0-rotatable:
(i) caterpillars with a perfect matching;
(ii) caterpillars obtained by identifying a central vertex of a path $P_{n}$ with a vertex of $K_{2}$;
(iii) caterpillars obtained by linking one leaf of the star $K_{1, s-1}$ to a leaf of a path $P_{n}$, $n \geq 3$ and $s \geq\left\lceil\frac{n}{2}\right\rceil ;$
(iv) caterpillars with diameter five or six; and
(v) caterpillars $T$ with $\operatorname{diam}(T) \geq 7$ such that, for every non-leaf vertex $v \in V(T)$, the number of leaves adjacent to $v$ is even and is at least $2+2((\operatorname{diam}(T)-1) \bmod 2)$.

These results reinforce Conjecture 3.5. In particular, the last three families show that, for each integer $d \geq 5$, there exist 0 -rotatable caterpillars with diameter $d$ and arbitrary number of vertices.

### 3.1 Results

In this section, we prove our main results. We start by showing an interesting result which is useful for proving that certain families of trees with a perfect matching are 0-rotatable.

Theorem 3.6. Let $T$ be a tree with a perfect matching. If the contree of $T$ is 0 -rotatable, then $T$ is 0 -rotatable.

Proof. Let $T$ be a tree with perfect matching $M$ and $u v \in M$. Let $T^{\prime}$ be the contree of $T$ and $x \in V\left(T^{\prime}\right)$ be the vertex corresponding to edge $u v$. Suppose $T^{\prime}$ is 0 -rotatable. Hence, $T^{\prime}$ has a graceful labelling $f^{\prime}$ such that $f^{\prime}(x)=0$. Thus, by Lemma 2.50, $T$ has two strongly graceful labellings $f_{1}$ and $f_{2}$ such that: $f_{1}(u)=0$ and $f_{1}(v)=|E(T)|$; $f_{2}(u)=|E(T)|$ and $f_{2}(v)=0$. Therefore, there exist strongly graceful labellings of $T$ which assign label 0 to vertices $u$ and $v$. Since $u v$ is an arbitrary edge of $M$, we conclude that $T$ is 0 -rotatable.

Theorem 3.7. Every caterpillar with a perfect matching is 0-rotatable.
Proof. The result follows from Theorem 3.6 and the fact that the contree of a caterpillar with a perfect matching is a path, which is 0 -rotatable by Theorem 2.15.

Theorem 3.10 and Theorem 3.12 prove that two families of caterpillars are 0-rotatable. Before presenting these results, it is necessary to establish some auxiliary lemmas.

Lemma 3.8. Let $n, p, q$ be positive integers and let $X=[0, n-1], Y=[n, n+p-1]$, $Z=[n+p, n+p+q-1]$. If $p \geq n$ and $p \geq q$, then, for every $\ell \in X \cup Z$, there exists $t \in Y$ such that $|\ell-t|=|Y|$.

Proof. The result follows by letting $t=\ell+|Y|$ when $\ell \in X$, and letting $t=\ell-|Y|$ when $\ell \in Z$.

Lemma 3.9. Let $T$ be either a path $P_{n}$, with $n \geq 1$, or a star $K_{1, n-1}$, with $n \geq 2$. Let $v \in V(T)$ be a leaf of $T, t$ be a positive integer and $S=\{t, t+1, \ldots, t+n-1\}$. Then, for each $i \in S$, there exists an injective labelling $f: V(T) \rightarrow S$ such that $f(v)=i$ and $L_{E(T)}^{f}=\{1, \ldots, n-1\}$.

Proof. Let $T, S$ and $v$ be as stated in the hypothesis. First, suppose $T$ is a path $P_{n}$, with $n \geq 1$. By items (i) and (iii) of Theorem 3.2, for each $j \in\{0,1, \ldots, n-1\}, P_{n}$ has a graceful labelling $g: V\left(P_{n}\right) \rightarrow\{0,1, \ldots, n-1\}$ such that $g(v)=j$. In order to obtain the required labelling $f$, it is sufficient to define $f(v)=g(v)+t$, for all $v \in V\left(P_{n}\right)$. Note that the vertex labels of $P_{n}$ under $f$ are $t, t+1, \ldots, t+n-1$ and, for every edge $u v \in E\left(P_{n}\right)$, $|f(u)-f(v)|=|(t+g(u))-(t+g(v))|=|g(u)-g(v)|$.

Now, suppose $T$ is a star $K_{1, n-1}$ with central vertex $x \in V(T)$. Let $i \in S$. We construct two labellings $f$ and $f^{\prime}$ depending on the value of $i$. If $i \neq t$, define $f: V(T) \rightarrow S$ such that: $f(v)=i, f(x)=t$, and the remaining leaves are assigned distinct labels in $S \backslash\{t, i\}$. If $i=t$, define $f^{\prime}: V(T) \rightarrow S$ such that: $f^{\prime}(v)=t, f^{\prime}(x)=t+n-1$, and the remaining leaves are assigned distinct labels in $S \backslash\{t, t+n-1\}$. Note that $f$ and $f^{\prime}$ are injective labellings from $V(T)$ to $S$ and that $L_{E(T)}^{f}=L_{E(T)}^{f^{\prime}}=\{1,2, \ldots, n-1\}$. Therefore, the result follows.

Theorem 3.10. Every caterpillar obtained by identifying a vertex of $K_{2}$ with a central vertex of $P_{n}$ is 0 -rotatable.

Proof. Let $P_{n}=\left(v_{0}, \ldots, v_{n-1}\right)$ be a path, with $n \geq 1$. Let $T$ be the caterpillar obtained by identifying a vertex of $K_{2}$ with the central vertex $v_{\lfloor(n-1) / 2\rfloor}$ of $P_{n}$. Let $v_{n}$ be the leaf of $T$ adjacent to vertex $v_{\lfloor(n-1) / 2\rfloor}$.

If $\operatorname{diam}(T) \in\{1,2,3,4\}$, the result follows from Corollary 2.20 and Theorem 3.7. Now, consider $\operatorname{diam}(T) \in\{5,6,7\}$. By Corollary 2.20 , for $v \in\left\{v_{0}, v_{1}, v_{n-2}, v_{n-1}\right\}$, there exists a graceful labelling $f$ of $T$ such that $f(v)=0$. Moreover, Figure 3.4 exhibits two distinct graceful labellings $f_{5}^{1}, f_{5}^{2}$ of $T$ with $\operatorname{diam}(T)=5$, such that $f_{5}^{1}\left(v_{2}\right)=0$ and $f_{5}^{2}\left(v_{3}\right)=0$. The complementary labelling of $f_{5}^{1}$ assigns label 0 to $v_{6}$. Figure 3.5 exhibits three distinct graceful labellings $f_{6}^{1}, f_{6}^{2}, f_{6}^{3}$ of $T$ with $\operatorname{diam}(T)=6$, such that $f_{6}^{1}\left(v_{2}\right)=0, f_{6}^{2}\left(v_{3}\right)=0$, and $f_{6}^{3}\left(v_{4}\right)=0$. The complementary labelling of $f_{6}^{2}$ assigns label 0 to $v_{7}$. Finally, Figure 3.6 exhibits three distinct graceful labellings $f_{7}^{1}, f_{7}^{2}, f_{7}^{3}$ of $T$ with $\operatorname{diam}(T)=7$, such that $f_{7}^{1}\left(v_{2}\right)=0, f_{7}^{2}\left(v_{3}\right)=0, f_{7}^{3}\left(v_{4}\right)=0$. The complementary labelling of $f_{7}^{3}$ assigns label 0 to $v_{5}$, the complementary labelling of $f_{7}^{2}$ assigns label 0 to $v_{8}$, and the result follows.


Figure 3.4: Two graceful labellings of a caterpillar $T$ with $\operatorname{diam}(T)=5$.

(a) Graceful labelling $f_{6}^{1}$.

(b) Graceful labelling $f_{6}^{2}$.

(c) Graceful labelling $f_{6}^{3}$.

Figure 3.5: Three graceful labellings of a caterpillar $T$ with $\operatorname{diam}(T)=6$.


Figure 3.6: Three graceful labellings of a caterpillar $T$ with $\operatorname{diam}(T)=7$.

Now, we consider the remaining case in which $\operatorname{diam}(T) \geq 8$. Let $P \subset T$ be the subgraph induced by vertex set $\left\{v_{0}, v_{1}, \ldots, v_{\lfloor(n-1) / 2\rfloor}, v_{n}\right\}$ and let $Q \subset T$ be the subgraph induced by vertex set $V(T) \backslash V(P)$. Let $n_{P}$ and $n_{Q}$ denote the order of $P$ and $Q$, respectively, and let $m_{P}$ and $m_{Q}$ denote the sizes of $P$ and $Q$, respectively. Note that both $P$ and $Q$ are paths. Moreover, since $\operatorname{diam}(T) \geq 8, \operatorname{diam}(P) \geq 5$.

First, we prove that, for $v \in V(P)$, there exists a graceful labelling $f$ of $T$ such that $f(v)=0$. By Theorem 2.15, $P$ has an $\alpha$-labelling $g: V(P) \rightarrow\left\{0,1, \ldots, m_{P}\right\}$ such that $g(v)=0$. By Corollary 2.7, there exists a bipartition $\{A, B\}$ of $P$ such that $L_{A}^{g}=$
$\{0,1, \ldots,|A|-1\}$ and $L_{B}^{g}=\{|A|, \ldots,|A|+|B|-1\}$. Using this bipartition, we modify $g$ in order to obtain another labelling $f_{P}$ of $P$ as follows:

$$
f_{P}(u)= \begin{cases}g(u), & \text { if } u \in A ; \\ g(u)+n_{Q}, & \text { if } u \in B\end{cases}
$$

Therefore, we obtain $f_{P}: V(P) \rightarrow A \cup B^{\prime}$ such that $A=\{0,1, \ldots,|A|-1\}$ and $B^{\prime}=\left\{|A|+n_{Q},|A|+1+n_{Q}, \ldots,|A|+|B|-1+n_{Q}\right\}$. Since each label in $B$ was increased by $n_{Q}, L_{E(P)}^{f_{P}}=\left\{1+n_{Q}, 2+n_{Q}, \ldots, m_{P}+n_{Q}\right\}$.

Note that the vertex labels $|A|,|A|+1, \ldots,|A|+n_{Q}-1$ are missing in $f_{P}$, as well as the induced edge labels $1,2, \ldots, n_{Q}$. Let $C=\left\{|A|,|A|+1, \ldots,|A|+n_{Q}-1\right\}$ and let $\ell=f_{P}\left(v_{\lfloor(n-1) / 2\rfloor}\right)$. Next, we show that there exists an integer $t \in C$, such that $|\ell-t|=|C|=n_{Q}$.

By the definition of $P$, we have that $|A|+\left|B^{\prime}\right|=n_{P}=\left\lceil\frac{n}{2}\right\rceil+1$. Moreover, since $P$ is a path, one of the following holds: (i) $|A|=\left|B^{\prime}\right|=\left(\left\lceil\frac{n}{2}\right\rceil+1\right) / 2$; (ii) $|A|=\left\lfloor\left(\left\lceil\frac{n}{2}\right\rceil+1\right) / 2\right\rfloor$ and $\left|B^{\prime}\right|=|A|+1$; or (iii) $\left|B^{\prime}\right|=\left\lfloor\left(\left\lceil\frac{n}{2}\right\rceil+1\right) / 2\right\rfloor$ and $|A|=\left|B^{\prime}\right|+1$. Since $|C|=n_{Q}=\left\lfloor\frac{n}{2}\right\rfloor$ and $\left\lfloor\frac{n}{2}\right\rfloor>\left\lceil\left(\left\lceil\frac{n}{2}\right\rceil+1\right) / 2\right\rceil$ for $n \geq 9$, we obtain that $|C|>|A|$ and $|C|>\left|B^{\prime}\right|$. Thus, considering $X=A, Y=C, Z=B^{\prime}$, and $\ell$ as previously chosen, by Lemma 3.8, there exists $t \in Y$, such that $|\ell-t|=|Y|=|C|$, as required.

By Lemma 3.9, there exists an injective labelling $f_{Q}: V(Q) \rightarrow C$ such that: (i) $f_{Q}\left(v_{\lfloor(n-1) / 2\rfloor+1}\right)=t$; and (ii) $L_{E(Q)}^{f_{Q}}=\left\{1, \ldots, n_{Q}-1\right\}$. Define a labelling $f: V(T) \rightarrow$ $\{0,1, \ldots,|E(T)|\}$ such that:

$$
f(u)= \begin{cases}f_{P}(u), & \text { if } u \in P \\ f_{Q}(u), & \text { if } u \in Q\end{cases}
$$

Labelling $f$ is a graceful labelling of $T$ since: (i) $f$ is an injective function from $V(T)$ to $\left\{0,1, \ldots, m_{P}+m_{Q}+1=|E(T)|\right\}$; (ii) the induced edge labels of $Q$ are $1,2, \ldots, n_{Q}-1$; (iii) the induced edge labels of $P$ are $n_{Q}+1, n_{Q}+2, \ldots,|E(T)|$; and (iv) $\mid f\left(v_{\lfloor(n-1) / 2\rfloor}\right)-$ $f\left(v_{\lfloor(n-1) / 2\rfloor+1}\right) \mid=n_{Q}$.

In order to conclude the proof, we have to show that, for each vertex $v \in V(Q)$, there exists a graceful labelling $f$ of $T$ such that $f(v)=0$. It can be done by the previous reasoning, considering $V(P)=\left\{v_{\lfloor(n-1) / 2\rfloor}, \ldots, v_{n-1}, v_{n}\right\}$ and $V(Q)=V(T) \backslash V(P)$.

Theorem 3.12 proves that every caterpillar obtained by linking one leaf of star $K_{1, s-1}$ to a leaf of path $P_{n}$, with $n \geq 3$ and $s \geq\left\lceil\frac{n}{2}\right\rceil$, is 0 -rotatable. In our proof we use a specific labelling of a caterpillar which is presented in the next lemma.

Lemma 3.11. Let $T$ be the caterpillar obtained by linking one leaf of star $K_{1, s-1}, s \geq 3$, to a leaf of path $P_{5}$. If $v$ is the central vertex of $P_{5}$, then there exists a graceful labelling $f$ of $T$ such that $f(v)=0$.

Proof. Let $P_{5}=\left(v_{0}, \ldots, v_{4}\right)$ and $V\left(K_{1, s-1}\right)=\left\{x_{0}, \ldots, x_{s-1}\right\}$, with $x_{s-1}$ its central vertex. Let $T$ be the caterpillar obtained by linking $x_{0}$ to $v_{0}$. In the following, we construct a graceful labelling $f$ of $T$ such that $f\left(v_{2}\right)=0$.

Let $m=|E(T)|$. Define labelling $f$ as follows: (i) vertices $x_{s-1}, x_{0}, v_{0}, v_{1}, v_{2}, v_{3}, v_{4}$ are assigned labels $2,4,3, m, 0, m-1,1$, respectively; and (ii) the leaves adjacent to $x_{s-1}$ are assigned labels $5,6, \ldots, m-2$. A scheme of the labelling $f$ is exhibited in Figure 3.7.


Figure 3.7: Scheme of a graceful labelling of a tree $T$.

By the definition, $f$ is an injective map from $V(T)$ to $\{0, \ldots, m\}$. By inspection, $L_{E\left(P_{5}\right)}^{f}=\{m-3, m-2, m-1, m\}, L_{E\left(K_{1, s-1}\right)}^{f}=\{2, \ldots, m-4\}$, and $\left|f\left(x_{0}\right)-f\left(v_{0}\right)\right|=1$. Therefore, $f$ is graceful.

Theorem 3.12. Let $T$ be the caterpillar obtained by linking one leaf of the star $K_{1, s-1}$ to a leaf of the path $P_{n}$. If $n \geq 3$ and $s \geq\left\lceil\frac{n}{2}\right\rceil$, then $T$ is 0 -rotatable.

Proof. Let $P_{n}=\left(v_{0}, \ldots, v_{n-1}\right)$ and $V\left(K_{1, s-1}\right)=\left\{x_{0}, \ldots, x_{s-1}\right\}$, with $x_{s-1}$ its central vertex. Let $T$ be the caterpillar obtained by linking $x_{0}$ to $v_{0}$. Thus, $T$ has vertex set $V(T)=V\left(K_{1, s-1}\right) \cup V\left(P_{n}\right)$ and edge set $E(T)=E\left(K_{1, s-1}\right) \cup E\left(P_{n}\right) \cup\left\{x_{0} v_{0}\right\}$.

Suppose $n \geq 3$ and $s \geq\left\lceil\frac{n}{2}\right\rceil$. In the following, we prove that, for every $v \in V(T)$, there exists a graceful labelling $f$ of $T$ such that $f(v)=0$. We consider two cases depending on which subgraph, $K_{1, s-1}$ or $P_{n}$, vertex $v$ belongs to.

Case 1. $v \in V\left(K_{1, s-1}\right)$.
By Corollary 2.20, for every $v \in\left\{x_{1}, \ldots, x_{s-1}\right\}$, there exists a graceful labelling $f$ of $T$ such that $f(v)=0$. Therefore, in order to conclude this case, it remains to show that there exists a graceful labelling $f$ of $T$ such that $f\left(x_{0}\right)=0$.

Let $H_{1}$ and $H_{2}$ be subgraphs of $T$ induced by the vertex set $\left\{v_{0}, x_{0}, x_{1}, \ldots, x_{s-1}\right\}$ and $\left\{v_{1}, v_{2}, \ldots, v_{n-1}\right\}=V(T) \backslash V\left(H_{1}\right)$, respectively. Define a graceful labelling $h_{1}: V\left(H_{1}\right) \rightarrow$ $\{0, \ldots, s\}$ as follows: (i) $h_{1}\left(x_{i}\right)=i$, for $0 \leq i \leq s-1$; and (ii) $h_{1}\left(v_{0}\right)=s$. Since $h_{1}\left(x_{s-1}\right)=s-1$ and its neighbours have labels $0,1, \ldots, s-2$, the edges incident with $x_{s-1}$ have induced labels $1,2, \ldots, s-1$. Moreover, since $h_{1}\left(x_{0}\right)=0$ and $h_{1}\left(v_{0}\right)=s$, the edge $x_{0} v_{0}$ has label $s$. Next, we modify $h_{1}$ in order to obtain another labelling $h_{1}^{\prime}$ :

$$
h_{1}^{\prime}(v)= \begin{cases}h_{1}(v), & \text { if } v \in\left\{x_{0}, x_{1}, \ldots, x_{s-2}\right\} ; \\ h_{1}(v)+\left|V\left(H_{2}\right)\right|, & \text { if } v \in\left\{x_{s-1}, v_{0}\right\} .\end{cases}
$$

Since $n \geq 3$ and $\left|V\left(H_{2}\right)\right|=n-1,\left|V\left(H_{2}\right)\right| \geq 2$. By the definition, $h_{1}^{\prime}\left(x_{s-1}\right)=$ $n+s-2$. Moreover, the neighbours of $x_{s-1}$ have labels $0,1, \ldots, s-2$ under $h_{1}^{\prime}$. Therefore, $L_{E\left(K_{1, s-1}\right)}^{h_{1}^{\prime}}=\{n, n+1, \ldots, n+s-2\}$. Also, since $h_{1}^{\prime}\left(x_{0}\right)=0$ and $h_{1}^{\prime}\left(v_{0}\right)=n+s-1$, we have that $\left|h_{1}^{\prime}\left(x_{0}\right)-h_{1}^{\prime}\left(v_{0}\right)\right|=n+s-1$. Thus, we conclude that the vertex labels $s-1, s, \ldots, s+n-3$ are missing, as well as the edge labels $1,2, \ldots, n-1$.

Since $H_{2}$ is a path with $\left|V\left(H_{2}\right)\right| \geq 2$, by Lemma $3.9, H_{2}$ has an injective labelling $h_{2}: V\left(H_{2}\right) \rightarrow\{s-1, s, \ldots, s+n-3\}$ such that $h_{2}\left(v_{1}\right)=s$ and $L_{E\left(H_{2}\right)}^{h_{2}}=\{1,2, \ldots, n-2\}$.

We define labelling $f: V(T) \rightarrow\{0,1, \ldots,|E(T)|\}$ as follows:

$$
f(v)= \begin{cases}h_{1}^{\prime}(v), & \text { if } v \in V\left(H_{1}\right) \\ h_{2}(v), & \text { if } v \in V\left(H_{2}\right)\end{cases}
$$

Labelling $f$ is graceful since: (i) $f$ is an injective function from $V(T)$ to $\{0, \ldots,|E(T)|\}$; and (ii) $L_{E\left(H_{2}\right)}^{f}=\{1, \ldots, n-2\}, L_{E\left(H_{1}\right)}^{f}=\{n, \ldots, n+s-1\}$, and $\left|f\left(v_{1}\right)-f\left(v_{0}\right)\right|=$ $|s-(n+s-1)|=n-1$. Thus, $L_{E(T)}^{f}=\{1, \ldots, n+s-1\}$ and the result follows.

Case 2. $v \in V\left(P_{n}\right)$.
If $n=5$ and $v$ is the central vertex of $P_{5}$, the result follows by Lemma 3.11. Thus, consider $n \neq 5$ or $v$ different from the central vertex of $P_{5}$. By Theorem 2.15, since $v$ is not the central vertex of $P_{5}$, path $P_{n}$ has an $\alpha$-labelling $g$ such that $g(v)=0$. By Corollary 2.7, there exists a bipartition $\{A, B\}$ of $P_{n}$ such that $L_{A}^{g}=\{0,1, \ldots,|A|-1\}$ and $L_{B}^{g}=\{|A|, \ldots,|A|+|B|-1\}$. Using this bipartition, we modify $g$ in order to obtain another labelling $f_{P}$ of $P_{n}$. For each $u \in V\left(P_{n}\right)$, define

$$
f_{P}(u)= \begin{cases}g(u), & \text { if } u \in A \\ g(u)+s, & \text { if } u \in B\end{cases}
$$

Thus, we obtain the labelling $f_{P}: V\left(P_{n}\right) \rightarrow A \cup B^{\prime}$, such that $A=\{0,1, \ldots,|A|-1\}$ and $B^{\prime}=\{|A|+s,|A|+s+1, \ldots,|A|+s+|B|-1\}$. Since each label in $B$ was increased by $s, L_{E\left(P_{n}\right)}^{f_{P}}=\{1+s, 2+s, \ldots, n-1+s=|E(T)|\}$. Note that the vertex labels $|A|,|A|+1, \ldots,|A|+s-1$ are missing in $f_{P}$, as well as the induced edge labels $1,2, \ldots, s$. Let $C=\{|A|,|A|+1, \ldots,|A|+s-1\}$ and let $\ell=f_{P}\left(v_{0}\right)$. Next, we show that there exists an integer $t \in C$, such that $|\ell-t|=|C|=s$.

Consider $X=A, Y=C, Z=B^{\prime}$, and $\ell$ as previously chosen. Since $|C| \geq\left\lceil\frac{n}{2}\right\rceil$, by Lemma 3.8, there exists $t \in Y$, such that $|\ell-t|=|Y|=|C|$, as required. By Lemma 3.9, there exists an injective labelling $f_{K}: V\left(K_{1, s-1}\right) \rightarrow C$, such that: (i) $f_{K}\left(x_{0}\right)=t$; and (ii) $L_{K_{1, s-1}}^{f_{K}}=\{1,2, \ldots, s-1\}$. Thus, define labelling $f: V(T) \rightarrow\{0,1, \ldots,|E(T)|\}$ as follows:

$$
f(u)= \begin{cases}f_{P}(u), & \text { if } u \in V\left(P_{n}\right) \\ f_{K}(u), & \text { if } u \in V\left(K_{1, s-1}\right) .\end{cases}
$$

Labelling $f$ is graceful since: (i) $f$ is an injective function from $V(T)$ to $\{0,1, \ldots,|E(T)|\}$; and (ii) $L_{E\left(K_{1, s-1}\right)}^{f}=\{1, \ldots, s-1\}, L_{E\left(P_{n}\right)}^{f}=\{s+1, \ldots, s+n-1\}$ and $\left|f\left(x_{0}\right)-f\left(v_{0}\right)\right|=s$. Therefore, $L_{E(T)}^{f}=\{1, \ldots,|E(T)|\}$ and the result follows.

### 3.1.1 Caterpillars with diameter five

The main result of this section is Theorem 3.15, which states that every caterpillar $T$ with diameter five is 0 -rotatable. In order to prove it, for each non-leaf vertex $v \in V(T)$, we construct a graceful labelling $f$ of $T$ that assigns label 0 to $v$ and assigns label $|E(T)|$ to any leaf $u \in V(T)$ adjacent to $v$. Consequently, we use its complementary labelling $\bar{f}$
in order to obtain $\bar{f}(u)=0$ and $\bar{f}(v)=|E(T)|$. Since $\bar{f}$ is also a graceful labelling and $f$ is constructed considering an arbitrary non-leaf vertex $v$ of $T$, we conclude that $T$ is 0 -rotatable.

The above mentioned labellings are obtained either directly from Corollary 2.20, or by modifying one of the trees presented in Figure 3.9. These trees are modified by transfer operations and need some properties presented in Lemma 3.13.

An ordered 5 -tuple ( $r, s, n_{-1}, n_{0}, n_{1}$ ) of nonnegative integers is special if:
(i) $r \leq s$;
(ii) $\sum_{i=-1}^{1} n_{i} \leq s-r$; and
(iii) either all of $n_{-1}, n_{0}, n_{1}$ are even or $n_{-1}$ and $n_{1}$ have the same parity as $r+s$ and $n_{0}$ does not.

Lemma 3.13. Let $\left(r, s, n_{-1}, n_{0}, n_{1}\right)$ be a special 5 -tuple. Let $T$ be a tree with a graceful labelling $f$ having a vertex $v$ adjacent to a set $\mathcal{S}$ of leaves such that $L_{\mathcal{S}}^{f}=[r, s]$. If, for $i \in\{-1,0,1\}, T$ has a vertex $w^{i} \notin \mathcal{S}$ such that $f(v)+f\left(w^{i}\right)=r+s+i$, then, for all such $i$, it is possible to simultaneously safely transfer $n_{i}$ vertices in $\mathcal{S}$ from $v$ to $w^{i}$.

Proof. Let $\left(r, s, n_{-1}, n_{0}, n_{1}\right), T, \mathcal{S}, f, L_{\mathcal{S}}^{f}, v$ and $w^{i}$ be as stated in the hypothesis. We exhibit three disjoint subsets $L_{-1}, L_{0}, L_{1}$ of $[r, s]$ such that, for each $i \in\{-1,0,1\},\left|L_{i}\right|=n_{i}$ and every vertex $s \in \mathcal{S}$ with $f(s) \in L_{i}$ can be safely transferred to $w^{i}$.

We consider three cases, depending on the parities of $n_{-1}, n_{0}, n_{1}$. In all cases, we set $n_{\text {min }}=\min \left\{n_{-1}, n_{1}\right\}$. For positive integers $q, t$, with $q \leq t$ and $t-q$ even, the set $[[q, t]]$ is $\left\{q+2 j \left\lvert\, j \in\left[0, \frac{t-q}{2}\right]\right.\right\}$, which is the same as $\{q, q+2, q+4, \ldots, t\}$. We remark that, if $t<q$, then $[q, t]$ and $[[q, t]]$ are both empty.

Case 1. All of $n_{-1}, n_{0}, n_{1}$ are even.
Set $L_{0}^{1}=\left[r, r+\frac{n_{0}}{2}-1\right], L_{0}^{2}=\left[s-\frac{n_{0}}{2}+1, s\right]$, and $L_{0}=L_{0}^{1} \cup L_{0}^{2}$. For $i \in\{-1,1\}, L_{i}$ is the union of the following four sets:
(i) $L_{i}^{1}=\left[\left[r+\frac{n_{0}}{2}+\frac{(i+1)}{2}, r+\frac{n_{0}}{2}+\frac{(i+1)}{2}+n_{\text {min }}-2\right]\right]$;
(ii) $L_{i}^{2}=\left[\left[s-\frac{n_{0}}{2}-\frac{(1-i)}{2}-n_{\text {min }}+2, s-\frac{n_{0}}{2}-\frac{(1-i)}{2}\right]\right]$;
(iii) $L_{i}^{3}=\left[r+\frac{n_{0}}{2}+n_{\text {min }}+\frac{(i+1)}{2}, r+\frac{n_{0}}{2}+n_{\text {min }}+\frac{(i+1)}{2}+\frac{\left(n_{i}-n_{\min }\right)}{2}-1\right]$; and
(iv) $L_{i}^{4}=\left[s-\frac{n_{0}}{2}-n_{\text {min }}-\frac{(1-i)}{2}-\frac{\left(n_{i}-n_{\text {min }}\right)}{2}+1, s-\frac{n_{0}}{2}-n_{\text {min }}-\frac{(1-i)}{2}\right]$.

By construction, $\left|L_{i}^{1}\right|=\left|L_{i}^{2}\right|$, for $i \in\{-1,0,1\}$. Moreover, the sum of the $j$ th smallest element of $L_{i}^{1}$ and of the $j$ th largest element of $L_{i}^{2}$ is independent of $j$ and equals $r+s+i$. Therefore, by Corollary 2.37, this pair of vertices can be safely transferred from $v$ to $w^{i}$ since $f(v)+f\left(w^{i}\right)=r+s+i$. For $i \in\{-1,1\}$, the same statements hold for $L_{i}^{3}$ and $L_{i}^{4}$ (when they are not empty).

In order to conclude the case, it remains to prove the disjointness of the sets $L_{-1}, L_{0}, L_{1}$. Recall from page 18 that, for two sets $A, B$ of positive integers, we write $A<B$ if, for every $a \in A$ and $b \in B, a<b$.

By the definition, the elements of $L_{0}^{1}$ and $L_{0}^{2}$ are consecutive integers. On the other hand, the parities of the elements of $L_{-1}^{1} \cup L_{-1}^{2}$ are all the same and different from the parities of the elements of $L_{1}^{1} \cup L_{1}^{2}$. Moreover, $L_{0}^{1}<L_{i}^{1}<L_{i}^{2}<L_{0}^{2}$, for $i \in\{-1,1\}$.

Now, consider $L_{i}^{p}, i \in\{-1,1\}$ and $p \in\{3,4\}$. Observe that at most one of $L_{1}^{3} \cup L_{1}^{4}$ and $L_{-1}^{3} \cup L_{-1}^{4}$ is nonempty. Also, note that these sets are composed by consecutive integers. Suppose that $L_{1}^{3} \neq \emptyset$ and $L_{1}^{4} \neq \emptyset$. It implies that $n_{\text {min }}=n_{-1}$. Since $n_{-1}+n_{0}+n_{1} \leq s-r$, we conclude that $L_{1}^{3}<L_{1}^{4}$. Moreover, by inspection, $L_{i}^{1}<L_{1}^{3}$ and $L_{1}^{4}<L_{i}^{2}$. Therefore, $L_{0}^{1}<L_{i}^{1}<L_{1}^{3}<L_{1}^{4}<L_{i}^{2}<L_{0}^{2}$. By a similar reasoning, considering $L_{-1}^{3} \neq \emptyset$ and $L_{-1}^{4} \neq \emptyset$, we obtain that $L_{0}^{1}<L_{i}^{1}<L_{-1}^{3}<L_{-1}^{4}<L_{i}^{2}<L_{0}^{2}$. Summing up, we conclude that $L_{i} \cap L_{i^{\prime}}=\emptyset$, for distinct $i, i^{\prime} \in\{-1,0,1\}$, since $L_{0}^{1}<\left(L_{-1}^{1} \cup L_{1}^{1}\right)<\left(L_{-1}^{3} \cup L_{1}^{3}\right)<$ $\left(L_{-1}^{4} \cup L_{1}^{4}\right)<\left(L_{-1}^{2} \cup L_{1}^{2}\right)<L_{0}^{2}$ and also $L_{-1}^{1} \cap L_{1}^{1}=L_{-1}^{2} \cap L_{1}^{2}=\emptyset$.

Further simple checks show that: (i) $\left|L_{0}\right|=n_{0}$; (ii) if $j \in\{-1,1\}$ is such that $n_{j}=n_{\text {min }}$, then $L_{j}^{3}=\emptyset=L_{j}^{4}$ and $\left|L_{j}\right|=n_{j}$; and (iii) if $j \in\{-1,1\}$ is such that $n_{j}>n_{\text {min }}$, then $L_{j}^{3} \neq \emptyset \neq L_{j}^{4}$ and $\left|L_{j}\right|=n_{j}$. An example of this case is illustrated in Figure 3.8, and is described in the text immediately following this proof.

Case 2. $n_{-1}, n_{1}$ and $r+s$ are even, $n_{0}$ is odd.
Set $L_{0}^{1}=\left[\frac{r+s}{2}-\frac{\left(n_{0}-1\right)}{2}, \frac{r+s}{2}-1\right], L_{0}^{2}=\left[\frac{r+s}{2}+1, \frac{r+s}{2}+\frac{\left(n_{0}-1\right)}{2}\right]$, and $L_{0}=L_{0}^{1} \cup L_{0}^{2} \cup\left\{\frac{r+s}{2}\right\}$. For $i \in\{-1,1\}, L_{i}$ is the union of the following four sets:
(i) $L_{i}^{1}=\left[\left[\frac{r+s}{2}-\frac{\left(n_{0}+1\right)}{2}-\frac{(1-i)}{2}-n_{\min }+2, \frac{r+s}{2}-\frac{\left(n_{0}+1\right)}{2}-\frac{(1-i)}{2}\right]\right]$;
(ii) $L_{i}^{2}=\left[\left[\frac{r+s}{2}+\frac{\left(n_{0}+1\right)}{2}+\frac{(1+i)}{2}, \frac{r+s}{2}+\frac{\left(n_{0}+1\right)}{2}+\frac{(1+i)}{2}+n_{\text {min }}-2\right]\right]$;
(iii) $L_{i}^{3}=\left[\frac{r+s}{2}-\frac{\left(n_{0}+1\right)}{2}-\frac{(1-i)}{2}-n_{\text {min }}-\frac{n_{i}-n_{\min }}{2}+1, \frac{r+s}{2}-\frac{\left(n_{0}+1\right)}{2}-\frac{(1-i)}{2}-n_{\text {min }}\right]$; and
(iv) $L_{i}^{4}=\left[\frac{r+s}{2}+\frac{\left(n_{0}+1\right)}{2}+\frac{(1+i)}{2}+n_{\text {min }}, \frac{r+s}{2}+\frac{\left(n_{0}+1\right)}{2}+\frac{(1+i)}{2}+n_{\text {min }}+\frac{n_{i}-n_{\text {min }}}{2}-1\right]$.

The proof that $L_{-1}, L_{0}, L_{1}$ have the required properties is similar to the previous case.
Case 3. $n_{-1}, n_{1}$ and $r+s$ are odd, $n_{0}$ is even.
Set $L_{0}^{1}=\left[\frac{r+s-1}{2}-\frac{n_{0}}{2}, \frac{r+s-1}{2}-1\right], L_{0}^{2}=\left[\frac{r+s+1}{2}+1, \frac{r+s+1}{2}+\frac{n_{0}}{2}\right]$, and $L_{0}=L_{0}^{1} \cup L_{0}^{2}$. For $i \in\{-1,1\}, L_{i}=\left\{\frac{r+s+i}{2}\right\} \cup L_{i}^{1} \cup L_{i}^{2} \cup L_{i}^{3} \cup L_{i}^{4}$, where:
(i) $L_{i}^{1}=\left[\left[\frac{r+s-1}{2}-\frac{n_{0}}{2}-\frac{(3-i)}{2}-n_{\text {min }}+3, \frac{r+s-1}{2}-\frac{n_{0}}{2}-\frac{(3-i)}{2}\right]\right]$;
(ii) $L_{i}^{2}=\left[\left[\frac{r+s+1}{2}+\frac{n_{0}}{2}+\frac{(3+i)}{2}, \frac{r+s+1}{2}+\frac{n_{0}}{2}+\frac{(3+i)}{2}+n_{\text {min }}-3\right]\right]$;
(iii) $L_{i}^{3}=\left[\frac{r+s-1}{2}-\frac{n_{0}}{2}-\frac{(1-i)}{2}-\frac{n_{i}+n_{\min }}{2}+1, \frac{r+s-1}{2}-\frac{n_{0}}{2}-\frac{(1-i)}{2}-n_{\text {min }}\right]$; and
(iv) $L_{i}^{4}=\left[\frac{r+s+1}{2}+\frac{n_{0}}{2}+\frac{(1+i)}{2}+n_{\text {min }}, \frac{r+s+1}{2}+\frac{n_{0}}{2}+\frac{(1+i)}{2}+\frac{n_{i}+n_{\text {min }}}{2}-1\right]$.

The proof that $L_{-1}, L_{0}, L_{1}$ have the required properties is similar to the proof of the first case.

As an example for the first case of the proof of Lemma 3.13, consider a special 5 -tuple $(5,20,6,0,8)$. Figure 3.8(a) shows a tree $T$ with a graceful labelling $f$, such that:
(i) there exists a vertex $v \in V(T)$ with label $f(v)=22$ that is adjacent to a set $\mathcal{S}$ of leaves such that $L_{\mathcal{S}}^{f}=[5,20]$;
(ii) for each $i \in\{-1,0,1\}$, there exists $w^{i} \in V(T)$ such that $f(v)+f\left(w^{i}\right)=r+s+i$ and $w^{i} \notin \mathcal{S}: w^{-1}$ is the vertex with label $2, w^{0}$ is the vertex with label 3 , and $w^{1}$ is the vertex with label 4; and
(iii) all of $n_{-1}, n_{0}, n_{1}$ are even: $n_{-1}=6, n_{0}=0$, and $n_{1}=8$.

As described in the proof of Case 1 , the three disjoint sets $L_{-1}, L_{0}, L_{1}$ are:
(a) $L_{0}=L_{0}^{1} \cup L_{0}^{2}$, where $L_{0}^{1}=L_{0}^{2}=\emptyset$;
(b) $L_{-1}=L_{-1}^{1} \cup L_{-1}^{2} \cup L_{-1}^{3} \cup L_{-1}^{4}$, where
$L_{-1}^{1}=\{5,7,9\}, L_{-1}^{2}=\{15,17,19\}, L_{-1}^{3}=L_{-1}^{4}=\emptyset ;$
(c) $L_{1}=L_{1}^{1} \cup L_{1}^{2} \cup L_{1}^{3} \cup L_{1}^{4}$, where $L_{1}^{1}=\{6,8,10\}, L_{1}^{2}=\{16,18,20\}, L_{1}^{3}=\{12\}, L_{1}^{4}=\{14\}$.

By the construction, the vertices of $\mathcal{S}$ with labels in $L_{i}$ can be safely transferred from $v$ to $w^{i}$, for $i \in\{-1,0,1\}$. The graceful tree $T^{\prime}$, obtained from $T$ after these transfers, is exhibited in Figure 3.8(b).

(a) Tree $T$ with a graceful labelling.

(b) Tree $T^{\prime}$ with a graceful labelling.

Figure 3.8: A tree $T$ with a graceful labelling and a graceful tree $T^{\prime}$ obtained from $T$ after a sequence of transfers.

Before proceeding further, we need to introduce an additional definition: the modeltree $T_{d}\left(c_{1}, \ldots, c_{d-1}\right)$ is the caterpillar with diameter $d$ and spine $P=u_{0} \cdots u_{d}$ such that, for $i \in\{1, \ldots, d-1\}$, vertex $u_{i}$ is adjacent to exactly $c_{i}$ leaves. Figure 3.9 shows three model-trees with special graceful labellings.

Lemma 3.14. Let $T$ be a caterpillar with diameter five and let $w$ be either a central vertex or a leaf neighbour of a central vertex. Then, $T$ has a graceful labelling $f$ such that $f(w)=0$.

(a) Model-tree $T_{5}(a+1,0, b, 1)$. Note that $u_{0}$ is one of the leaves adjacent to $u_{1}$.

(b) Model-tree $T_{5}(1,0,1, a+1)$.

(c) Model-tree $T_{5}(1,0,0, a+1)$. Note that $u_{5}$ is one of the leaves adjacent to $u_{4}$.

Figure 3.9: Three model-trees with a graceful labelling and $m$ edges.

Proof. Let $T$ be a caterpillar with $\operatorname{diam}(T)=5$ and spine $P=u_{0} u_{1} u_{2} u_{3} u_{4} u_{5}$. For each $i \in\{1,2,3,4\}$, define $U_{i}$ as the set of leaves from $V(T) \backslash\left\{u_{0}, u_{5}\right\}$ that are adjacent to $u_{i}$. The central vertices of $T$ are $u_{2}$ and $u_{3}$. We prove the result for $u_{2}$; the result for $u_{3}$ is analogous.

Let $T^{\prime}$ be the subtree $T-U_{2}$. Note that, if $f^{\prime}$ is a graceful labelling of $T^{\prime}$ with $f^{\prime}\left(u_{2}\right)=0$, then adding the labels $\left|E\left(T^{\prime}\right)\right|+1, \ldots,|E(T)|$ to the vertices in $U_{2}$ yields a graceful labelling $f$ of $T$ with $f\left(u_{2}\right)=0$. For the neighbour $v$ of $u_{2}$ with $f(v)=|E(T)|$, the complementary labelling $\bar{f}$ has $\bar{f}(v)=0$, as required. Thus, we may assume $U_{2}=\emptyset$ and prove that $T$ has a graceful labelling $f$ such that $f\left(u_{2}\right)=0$. We consider two main cases depending on the parities of $\left|U_{1}\right|$ and $\left|U_{4}\right|$.

Case 1. $\left(\left|U_{1}\right| \bmod 2,\left|U_{4}\right| \bmod 2\right) \neq(0,1)$.
Let $a=\left|U_{1}\right|+\left|U_{4}\right|, b=\left|U_{3}\right|$, and let $T^{\prime}$ be the model-tree $T_{5}(a+1,0, b, 1)$. We illustrate in Figure 3.9(a) a graceful labelling $f$ of $T^{\prime}$ with $f\left(u_{2}\right)=0$. We show that it is possible to safely transfer $\left|U_{4}\right|$ leaves from $u_{1}$ to $u_{4}$, obtaining a gracefully labelled tree isomorphic to $T$.

Note that $\left|E\left(T^{\prime}\right)\right|=\left|U_{1}\right|+\left|U_{3}\right|+\left|U_{4}\right|+5=a+b+5$. Let $r=b+3, s=a+b+3$, $n_{-1}=n_{1}=0$, and $n_{0}=\left|U_{4}\right|$. Initially, note that $r \leq s$ and $\sum_{i=-1}^{1} n_{i} \leq s-r$. To complete the verification that ( $r, s, n_{-1}, n_{0}, n_{1}$ ) is a special 5 -tuple, also note that: (i) when $\left|U_{4}\right| \equiv 0$ $(\bmod 2)$, each $n_{i}$ is even; (ii) when $\left|U_{1}\right| \equiv\left|U_{4}\right| \equiv 1(\bmod 2), n_{-1} \equiv n_{1} \equiv(r+s)(\bmod 2)$ and $n_{0} \not \equiv(r+s)(\bmod 2)$. Moreover, vertex $u_{1} \in V\left(T^{\prime}\right)$ is adjacent to a set of leaves $\mathcal{S}$ with labels $b+3, \ldots, a+b+3$, vertex $u_{4} \notin \mathcal{S}$, and $f\left(u_{1}\right)+f\left(u_{4}\right)=r+s$. By Lemma 3.13, we can safely transfer $n_{0}=\left|U_{4}\right|$ leaves of $\mathcal{S}$ from $u_{1}$ to $u_{4}$, resulting in a graceful labelling of $T$.

Case 2. $\left(\left|U_{1}\right| \bmod 2,\left|U_{4}\right| \bmod 2\right)=(0,1)$.
Subcase 2.1. $\left|U_{3}\right| \equiv 1(\bmod 2)$.
Let $a=\left|U_{1}\right|+\left|U_{3}\right|+\left|U_{4}\right|-1$ and let $T^{\prime}$ be the model-tree $T^{\prime}=T_{5}(1,0,1, a+1)$. Note that $\left|E\left(T^{\prime}\right)\right|=a+6$. Figure 3.9(b) exhibits a graceful labelling $f$ of $T^{\prime}$ such that $f\left(u_{2}\right)=0$. The verification that the 5 -tuple ( $\left.5, a+4,\left|U_{1}\right|,\left|U_{3}\right|-1,0\right)$ is special is analogous
to that given in Case 1. Lemma 3.13 implies that we can simultaneously safely transfer $\left|U_{1}\right|$ leaves of $\mathcal{S}$ from $u_{4}$ to $u_{1}$ and $\left|U_{3}\right|-1$ leaves of $\mathcal{S}$ from $u_{4}$ to $u_{3}$ since $u_{4} \in V\left(T^{\prime}\right)$ is adjacent to a set of leaves $\mathcal{S}$ with labels $5, \ldots, a+4 ; u_{1}, u_{3} \notin \mathcal{S}, f\left(u_{1}\right)+f\left(u_{4}\right)=r+s-1$, and $f\left(u_{3}\right)+f\left(u_{4}\right)=r+s$.

Subcase 2.2. $\left|U_{3}\right| \equiv 0(\bmod 2)$.
If $|E(T)|=6$, the result follows from the graceful labelling of $T$ depicted in Figure 3.9(c). Thus, suppose $|E(T)| \geq 7$. Let $a=\left|U_{1}\right|+\left|U_{3}\right|+\left|U_{4}\right|$ and let $T^{\prime}$ be the model-tree $T_{5}(1,0,0, a+1)$. Note that $\left|E\left(T^{\prime}\right)\right|=a+5$. Figure 3.9(c) illustrates a graceful labelling $f$ of $T^{\prime}$ such that $f\left(u_{2}\right)=0$. By considering the following two subcases, we show that it is possible to safely transfer $\left|U_{i}\right|$ leaves from $u_{4}$ to $u_{i}$, for $i \in\{1,3\}$.

Subcase 2.2.1. $\left|U_{1}\right|>0$.
Since $f\left(u_{1}\right)+f\left(u_{4}\right)=a+7$ and $u_{4}$ has two leaves with labels $a+4$ and 3 , it is possible to safely transfer this pair of leaves from vertex $u_{4}$ to vertex $u_{1}$. The verification that the 5 -tuple ( $4, a+2,\left|U_{3}\right|, 0,\left|U_{1}\right|-2$ ) is special is analogous to that given in Case 1 . Lemma 3.13 implies we can safely transfer $\left|U_{3}\right|$ leaves from $u_{4}$ to $u_{3}$ and $\left|U_{1}\right|-2$ leaves from $u_{4}$ to $u_{1}$.

Subcase 2.2.2. $\left|U_{1}\right|=0$.
In this subcase, it is sufficient to safely transfer $\left|U_{3}\right|$ leaves from $u_{4}$ to $u_{3}$. Since $f\left(u_{3}\right)+f\left(u_{4}\right)=a+5$, we move the pairs of leaves with labels in the set $\{3+i, a+2-i: 0 \leq$ $\left.i<\left|U_{3}\right| / 2\right\}$ from $u_{4}$ to $u_{3}$. Since $(3+i)+(a+2-i)=a+5$, by Corollary 2.37, the tree obtained after these transfers is graceful.

Theorem 3.15. If $T$ is a caterpillar with diameter five, then $T$ is 0 -rotatable.
Proof. The result follows from Corollary 2.20 and Lemma 3.14.

### 3.1.2 Caterpillars with diameter six

The main result of this section is Theorem 3.18, which states that every caterpillar with diameter six is 0 -rotatable. The technique used to prove this result is the same used to prove Theorem 3.15. Accordingly, Lemma 3.16 and Lemma 3.17 present auxiliary results needed in the proof of Theorem 3.18. Furthermore, Figure 3.10 shows four model-trees of diameter six with graceful labellings $f$ such that $f\left(u_{3}\right)=0$, that are used in Lemma 3.16.

Lemma 3.16. Let $T$ be a caterpillar with diameter six and let $w$ be either the central vertex or a leaf neighbour of the central vertex. Then, $T$ has a graceful labelling $f$ such that $f(w)=0$.

Proof. Let $T$ be a caterpillar with diameter six and spine $P=u_{0} u_{1} u_{2} u_{3} u_{4} u_{5} u_{6}$. For each $i \in\{1,2,3,4,5\}$, define $U_{i}$ as the set of leaves from $V(T) \backslash\left\{u_{0}, u_{6}\right\}$ that are adjacent to $u_{i}$. Note that $u_{3}$ is the unique central vertex of $T$. As shown in the proof of Lemma 3.14, we can assume $\left|U_{3}\right|=0$.

In our proof, we consider five cases depending on the parities of the $\left|U_{i}\right| \mathrm{s}$. In order to do this, we introduce the following definition: given tree $T$, we assign $T$ a 4 -tuple


Figure 3.10: Four model-trees of diameter six with graceful labellings.
$\left(p_{1}, p_{2}, p_{4}, p_{5}\right)$ such that, for each $i \in\{1,2,4,5\}, p_{i}$ is the parity of $\left|U_{i}\right|$. Since $p_{i} \in\{0,1\}$, there exist 16 distinct 4 -tuples.

Case 1. Tree $T$ is assigned one of the following 4-tuples: $(0,0,0,0),(1,0,0,1)$, $(1,0,0,0),(1,1,1,0)$.

Let $a=\left|U_{1}\right|+\left|U_{2}\right|+\left|U_{4}\right|+\left|U_{5}\right|$ and let $T^{\prime}$ be the model-tree $T_{6}(a+1,0,0,0,1)$. Figure 3.10(a) shows a graceful labelling of $T^{\prime}$ with $f\left(u_{3}\right)=0$. We show that it is possible to safely transfer $\left|U_{i}\right|$ leaves from $u_{1}$ to $u_{i}$, for $i \in\{2,4,5\}$, thus obtaining a gracefully labelled tree isomorphic to $T$

Note that $\left|E\left(T^{\prime}\right)\right|=a+6$. Let $r=3, s=a+3, n_{-1}=\left|U_{4}\right|, n_{0}=\left|U_{5}\right|$, and $n_{1}=\left|U_{2}\right|$. In order to verify that ( $r, s, n_{-1}, n_{0}, n_{1}$ ) is a special 5 -tuple, note that: (i) $r \leq s$ and $\sum_{i=-1}^{1} n_{i} \leq s-r$; (ii) if $T^{\prime}$ is assigned $(0,0,0,0)$ or $(1,0,0,0)$, then each $n_{i}$ is even; (iii) if $T^{\prime}$ is assigned $(1,0,0,1)$ or $(1,1,1,0)$, then $n_{-1} \equiv n_{1} \equiv(r+s)(\bmod 2)$ and $n_{0} \not \equiv(r+s)(\bmod 2)$. By Lemma 3.13, we can safely transfer $\left|U_{i}\right|$ leaves from $u_{1}$ to $u_{i}$, for $i \in\{2,4,5\}$, since vertex $u_{1}$ is adjacent to a set $\mathcal{S}$ of leaves with labels in the set $\{3, \ldots, a+3\}, u_{2}, u_{4}, u_{5} \notin \mathcal{S}, f\left(u_{1}\right)+f\left(u_{2}\right)=r+s+1, f\left(u_{1}\right)+f\left(u_{4}\right)=r+s-1$, and $f\left(u_{1}\right)+f\left(u_{5}\right)=r+s$,

Case 2. Tree $T$ is assigned one of the following 4-tuples: $(1,0,1,0),(0,0,1,0)$.
Let $a=\left|U_{1}\right|+\left|U_{2}\right|+\left|U_{4}\right|+\left|U_{5}\right|-1$ and let $T^{\prime}$ be the model-tree $T_{6}(a+1,0,0,1,1)$. Figure 3.10 (b) shows a graceful labelling of $T^{\prime}$ with $f\left(u_{3}\right)=0$. We show that it is possible to safely transfer $\left|U_{i}\right|$ leaves from $u_{1}$ to $u_{i}$, for $i \in\{2,5\}$, and $\left|U_{4}\right|-1$ leaves from $u_{1}$ to $u_{4}$, obtaining a gracefully labelled tree isomorphic to $T$.

Note that $\left|E\left(T^{\prime}\right)\right|=a+7$. As in Case 1, (4, a+4, |U $\left.\left|U_{5}\right|,\left|U_{4}\right|-1,\left|U_{2}\right|\right)$ is a special 5-tuple. Moreover, by Lemma 3.13, we can safely transfer $\left|U_{i}\right|$ leaves from $u_{1}$ to $u_{i}$, for $i \in\{2,5\}$, and $\left|U_{4}\right|-1$ leaves from $u_{1}$ to $u_{4}$ since vertex $u_{1}$ is adjacent to a set $\mathcal{S}$ of leaves with labels in the set $\{4, \ldots, a+4\}, u_{2}, u_{4}, u_{5} \notin \mathcal{S}, f\left(u_{1}\right)+f\left(u_{2}\right)=r+s+1, f\left(u_{1}\right)+f\left(u_{4}\right)=r+s$, and $f\left(u_{1}\right)+f\left(u_{5}\right)=r+s-1$.

Case 3. Tree $T$ is assigned one of the following 4 -tuples: $(0,0,1,1),(1,1,1,1)$, $(1,0,1,1)$.

Let $a=\left|U_{1}\right|+\left|U_{2}\right|+\left|U_{4}\right|+\left|U_{5}\right|-2$ and let $T^{\prime}$ be the model-tree $T_{6}(a+1,0,0,1,2)$. Note that $\left|E\left(T^{\prime}\right)\right|=a+8$. Figure $3.10(\mathrm{c})$ shows a graceful labelling of $T^{\prime}$ with $f\left(u_{3}\right)=0$.

As in Case 1, $\left(4, a+4,\left|U_{5}\right|-1,\left|U_{2}\right|,\left|U_{4}\right|-1\right)$ is a special 5 -tuple. Moreover, by Lemma 3.13, we can safely transfer $\left|U_{2}\right|$ leaves from $u_{1}$ to $u_{2}$ and $\left|U_{i}\right|-1$ leaves from $u_{1}$ to $u_{i}$, for $i \in\{4,5\}$, since $u_{1}$ is adjacent to a set $\mathcal{S}$ of leaves with labels $4, \ldots, a+4$; $u_{2}, u_{4}, u_{5} \notin \mathcal{S}, f\left(u_{1}\right)+f\left(u_{2}\right)=r+s, f\left(u_{1}\right)+f\left(u_{4}\right)=r+s+1$, and $f\left(u_{1}\right)+f\left(u_{5}\right)=r+s-1$.

Case 4. Tree $T$ is assigned the 4 -tuple ( $0,1,1,0$ ).
Let $a=\left|U_{1}\right|+\left|U_{2}\right|+\left|U_{4}\right|+\left|U_{5}\right|-2$ and let $T^{\prime}$ be the model-tree $T_{6}(a+1,1,0,1,1)$. Note that $\left|E\left(T^{\prime}\right)\right|=a+8$. Figure $3.10(\mathrm{~d})$ shows a graceful labelling of $T^{\prime}$ with $f\left(u_{3}\right)=0$. Again, as in Case $1,\left(4, a+4,\left|U_{4}\right|-1,\left|U_{5}\right|,\left|U_{2}\right|-1\right)$ is a special 5 -tuple. By Lemma 3.13, we can safely transfer $\left|U_{5}\right|$ leaves from $u_{1}$ to $u_{5}$ and $\left|U_{i}\right|-1$ leaves from $u_{1}$ to $u_{i}$, for $i \in\{2,4\}$.

Case 5. Tree $T$ is assigned one of the following 4-tuples: $(0,0,0,1),(0,1,1,1)$, $(0,1,0,0),(0,1,0,1),(1,1,0,0),(1,1,0,1)$.

For $a, b, c, d$ nonnegative integers, tree $T_{6}(a, b, 0, c, d)$ is isomorphic to $T_{6}(d, c, 0, b, a)$. Thus, the trees in this case are isomorphic to trees treated in Case 1, Case 2, and Case 3 , and the result follows.

For the next lemma, consider the eight model-trees exhibited in Figure 3.11, each with a graceful labelling $f$ such that $f\left(u_{2}\right)=0$.

(a) Graceful tree $T_{6}(1,0,0,0, a+1)$.

(c) Graceful tree $T_{6}(2,0,1,1, a+1)$.

(g) Graceful tree $T_{6}(2,0,0,1, a+1)$.

(b) Graceful tree $T_{6}(1,0,0,1, a+1)$.

(d) Graceful tree $T_{6}(1,0,1,0, a+1)$.
(e) Graceful tree $T_{6}(1,0,1,1, a+1)$.


(f) Graceful tree $T_{6}(2,0,0,1, a+1)$.

(h) Graceful tree $T_{6}(2,0,1,2, a+1)$.

Figure 3.11: Eight model-trees with a graceful labelling and $m$ edges.

Lemma 3.17. Let $T$ be a caterpillar with diameter six and spine $P=u_{0} u_{1} u_{2} u_{3} u_{4} u_{5} u_{6}$. Let $w$ be either $u_{2}, u_{4}$ or a leaf neighbour of either $u_{2}$ or $u_{4}$. Then, $T$ has a graceful labelling $f$ such that $f(w)=0$.

Proof. Let $T$ and $P$ be as stated in the hypothesis. For each $i \in\{1,2,3,4,5\}$, define $U_{i}$ as the set of leaves from $V(T) \backslash\left\{u_{0}, u_{6}\right\}$ that are adjacent to $u_{i}$. We prove the result for $u_{2}$ and the proof for $u_{4}$ is analogous. As shown in the proof of Lemma 3.14, we can assume $U_{2}=\emptyset$. We consider eight cases depending on the parities of the $\left|U_{i}\right| \mathrm{s}$. In order to do this, we assign $T$ a 4 -tuple ( $p_{1}, p_{3}, p_{4}, p_{5}$ ) such that, for each $i \in\{1,3,4,5\}$, $p_{i}$ is the parity of $\left|U_{i}\right|$.

Case 1. Tree $T$ is assigned one of the following 4 -tuples: $(0,0,0,0),(0,0,0,1)$, $(0,1,0,1),(1,0,1,1)$.

Let $a=\left|U_{1}\right|+\left|U_{3}\right|+\left|U_{4}\right|+\left|U_{5}\right|$ and let $T^{\prime}$ be the tree-model $T_{6}(1,0,0,0, a+1)$. Note that $\left|E\left(T^{\prime}\right)\right|=a+6$. Figure 3.11(a) exhibits a graceful labelling of $T^{\prime}$ with $f\left(u_{2}\right)=0$. Next, we show how to safely transfer $\left|U_{i}\right|$ leaves from $u_{5}$ to $u_{i}$, for $i \in\{1,3,4\}$, obtaining a tree isomorphic to $T$.

Let $r=3, s=a+3, n_{-1}=\left|U_{4}\right|, n_{0}=\left|U_{3}\right|$, and $n_{1}=\left|U_{1}\right|$. In order to verify that ( $r, s, n_{-1}, n_{0}, n_{1}$ ) is a special 5-tuple, note that: (i) $r \leq s$ and $\sum_{i=-1}^{1} n_{i} \leq s-r$; (ii) if $T^{\prime}$ is assigned $(0,0,0,0)$ or $(0,0,0,1)$, then each $n_{i}$ is even; and (iv) if $T^{\prime}$ is assigned ( $0,1,0,1$ ) or $(1,0,1,1)$, then $n_{-1} \equiv n_{1} \equiv(r+s)(\bmod 2)$ and $n_{0} \not \equiv(r+s)(\bmod 2)$. Moreover, since vertex $u_{5}$ is adjacent to a set of leaves $\mathcal{S}$ with labels $3, \ldots, a+3 ; u_{1}, u_{3}, u_{4} \notin \mathcal{S}$, $f\left(u_{1}\right)+f\left(u_{5}\right)=r+s+1, f\left(u_{3}\right)+f\left(u_{5}\right)=r+s$, and $f\left(u_{4}\right)+f\left(u_{5}\right)=r+s-1$, by Lemma 3.13, we can safely transfer $\left|U_{i}\right|$ leaves from $u_{5}$ to $u_{i}$, for $i \in\{1,3,4\}$.

Case 2. Tree $T$ is assigned one of the following 4-tuples: $(0,0,1,0),(0,0,1,1)$, $(1,1,1,1)$.

Let $a=\left|U_{1}\right|+\left|U_{3}\right|+\left|U_{4}\right|+\left|U_{5}\right|-1$ and let $T^{\prime}$ be the model-tree $T_{6}(1,0,0,1, a+1)$. Note that $\left|E\left(T^{\prime}\right)\right|=a+7$. Figure $3.11(\mathrm{~b})$ shows a graceful labelling $f$ of $T^{\prime}$ such that $f\left(u_{2}\right)=0$. We show that it is possible to safely transfer $\left|U_{i}\right|$ leaves from $u_{5}$ to $u_{i}$, for $i \in\{1,3\}$, and $\left|U_{4}\right|-1$ leaves from $u_{5}$ to $u_{4}$.

As in Case $1,\left(4, a+4,\left|U_{3}\right|,\left|U_{4}\right|-1,\left|U_{1}\right|\right)$ is a special 5 -tuple. Thus, by Lemma 3.13, we can safely transfer $\left|U_{i}\right|$ leaves from $u_{5}$ to $u_{i}$, for $i \in\{1,3\}$, and we can also safely transfer $\left|U_{4}\right|-1$ leaves from $u_{5}$ to $u_{4}$ since vertex $u_{5} \in V\left(T^{\prime}\right)$ is adjacent to a set $\mathcal{S}$ of leaves with labels $4, \ldots, a+4 ; u_{1}, u_{3}, u_{4} \notin \mathcal{S}, f\left(u_{1}\right)+f\left(u_{5}\right)=r+s+1, f\left(u_{3}\right)+f\left(u_{5}\right)=r+s-1$, and $f\left(u_{4}\right)+f\left(u_{5}\right)=r+s$.

Case 3. Tree $T$ is assigned the 4 -tuple ( $1,1,1,0$ ).
Let $a=\left|U_{1}\right|+\left|U_{3}\right|+\left|U_{4}\right|+\left|U_{5}\right|-3$ and let $T^{\prime}$ be the model-tree $T_{6}(2,0,1,1, a+1)$. Note that $\left|E\left(T^{\prime}\right)\right|=a+9$. Figure $3.11(\mathrm{c})$ shows a graceful labelling $f$ of $T^{\prime}$ such that $f\left(u_{2}\right)=0$. As in Case 1, $\left(5, a+5,\left|U_{4}\right|-1,\left|U_{1}\right|-1,\left|U_{3}\right|-1\right)$ is a special 5 -tuple. Thus, by Lemma 3.13, we can safely transfer $\left|U_{i}\right|-1$ leaves from $u_{5}$ to $u_{i}$, for $i \in\{1,3,4\}$.

Case 4. Tree $T$ is assigned the 4-tuple ( $0,1,0,0$ ).
Let $a=\left|U_{1}\right|+\left|U_{3}\right|+\left|U_{4}\right|+\left|U_{5}\right|-1$ and let $T^{\prime}$ be the model-tree $T_{6}(1,0,1,0, a+1)$. Note that $\left|E\left(T^{\prime}\right)\right|=a+7$. Figure $3.11(\mathrm{~d})$ shows a graceful labelling $f$ of $T^{\prime}$ such that $f\left(u_{2}\right)=0$. As in Case $1,\left(4, a+4,\left|U_{4}\right|,\left|U_{3}\right|-1,\left|U_{1}\right|\right)$ is a special 5 -tuple. Thus, by Lemma 3.13, we can safely transfer $\left|U_{i}\right|$ leaves from $u_{5}$ to $u_{i}$, for $i \in\{1,4\}$, and we can also safely transfer $\left|U_{3}\right|-1$ leaves from $u_{5}$ to $u_{3}$.

Case 5. Tree $T$ is assigned one of the following 4-tuples: $(0,1,1,0),(0,1,1,1)$.

Let $a=\left|U_{1}\right|+\left|U_{3}\right|+\left|U_{4}\right|+\left|U_{5}\right|-2$ and let $T^{\prime}$ be the model-tree $T_{6}(1,0,1,1, a+1)$. Note that $\left|E\left(T^{\prime}\right)\right|=a+8$. Figure 3.11(e) shows a graceful labelling $f$ of $T^{\prime}$ with $f\left(u_{2}\right)=0$. As in Case $1,\left(4, a+4,\left|U_{4}\right|-1,\left|U_{1}\right|,\left|U_{3}\right|-1\right)$ is a special 5 -tuple. Thus, by Lemma 3.13, we can safely transfer $\left|U_{i}\right|-1$ leaves from $u_{5}$ to $u_{i}$, for $i \in\{3,4\}$, and we can safely transfer $\left|U_{1}\right|$ leaves from $u_{5}$ to $u_{1}$.

Case 6. Tree $T$ is assigned the 4 -tuple ( $1,0,1,0$ ).
Let $a=\left|U_{1}\right|+\left|U_{3}\right|+\left|U_{4}\right|+\left|U_{5}\right|-2$ and let $T^{\prime}$ be the model-tree $T^{\prime}=T_{6}(2,0,0,1, a+1)$. Note that $\left|E\left(T^{\prime}\right)\right|=a+8$. Figure $3.11(\mathrm{f})$ shows a graceful labelling $f$ of $T^{\prime}$ with $f\left(u_{2}\right)=0$. As in Case 1, $\left(5, a+5,\left|U_{4}\right|-1,\left|U_{3}\right|,\left|U_{1}\right|-1\right)$ is a special 5 -tuple. Thus, by Lemma 3.13, we can safely transfer $\left|U_{3}\right|$ leaves from $u_{5}$ to $u_{3}$ and we can also safely transfer $\left|U_{i}\right|-1$ leaves from $u_{5}$ to $u_{i}$, for $i \in\{1,4\}$.

Case 7. Tree $T$ is assigned one of the following 4-tuples: $(1,1,0,0),(1,1,0,1)$.
Subcase 7.1. $\left|U_{4}\right|=0$.
Let $a=\left|U_{1}\right|+\left|U_{3}\right|+\left|U_{5}\right|-2$ and let $T^{\prime}$ be the model-tree $T_{6}(2,0,1,0, a+1)$. Note that $\left|E\left(T^{\prime}\right)\right|=a+8$. Figure $3.11(\mathrm{~g})$ shows a graceful labelling $f$ of $T^{\prime}$ such that $f\left(u_{2}\right)=0$. As in Case $1,\left(5, a+5,0,\left|U_{1}\right|-1,\left|U_{3}\right|-1\right)$ is a special 5 -tuple. Thus, by Lemma 3.13, we can safely transfer $\left|U_{i}\right|-1$ leaves from $u_{5}$ to $u_{i}$, for $i \in\{1,3\}$.

Subcase 7.2. $\left|U_{4}\right| \geq 2$.
Let $a=\left|U_{1}\right|+\left|U_{3}\right|+\left|U_{4}\right|+\left|U_{5}\right|-4$ and let $T^{\prime}$ be the model-tree $T_{6}(2,0,1,2, a+1)$. Note that $\left|E\left(T^{\prime}\right)\right|=a+10$. Figure $3.11(\mathrm{~h})$ shows a graceful labelling $f$ of $T$ such that $f\left(u_{2}\right)=0$. As in Case $1,\left(7, a+7,\left|U_{3}\right|-1,\left|U_{4}\right|-2,\left|U_{1}\right|-1\right)$ is a special 5 -tuple. Thus, by Lemma 3.13, we can safely transfer $\left|U_{4}\right|-2$ leaves from $u_{5}$ to $u_{4}$ and we can also safely transfer $\left|U_{i}\right|-1$ leaves from $u_{5}$ to $u_{i}$, for $i \in\{1,3\}$.

Case 8. Tree $T$ is assigned one of the following 4-tuples: $(1,0,0,0),(1,0,0,1)$.
Let $k=\left|U_{1}\right|+\left|U_{3}\right|+\left|U_{4}\right|+\left|U_{5}\right|+2$ and let $T^{\prime}$ be the tree shown in Figure 3.12 such that vertex $u_{3} \in V\left(T^{\prime}\right)$ has exactly $k$ leaves adjacent to it. Let $m=\left|E\left(T^{\prime}\right)\right|$. Note that $m=k+4$. Figure 3.12 also exhibits $T^{\prime}$ with a graceful labelling $f$ such that $f\left(u_{2}\right)=0$. Next, we show how to perform a sequence of safe transfers in $T^{\prime}$ so as to obtain a gracefully labelled tree isomorphic to $T$.


Figure 3.12: Scheme of a gracefully labelled tree $T^{\prime}$ with $m$ edges.
Since $f\left(u_{3}\right)+f\left(u_{5}\right)=m+1$ and vertex $u_{3}$ is adjacent to vertices with labels $3, \ldots, m-2$, by Corollary 2.37 , we can safely transfer all the $m-4-\left|U_{3}\right|$ leaves with labels in the interval $\left[3+\frac{\left|U_{3}\right|}{2}, m-2-\frac{\left|U_{3}\right|}{2}\right]$ from vertex $u_{3}$ to vertex $u_{5}$. After this transfer, $u_{3}$ is adjacent to $\left|U_{3}\right|$ leaves and $u_{5}$ is adjacent to exactly $\left|U_{1}\right|+\left|U_{4}\right|+\left|U_{5}\right|+2$ leaves.

Now, consider $a=f\left(u_{5}\right)=1, b=f\left(u_{1}\right)=m-1, r_{1}=1, r_{2}=0$, and the set $\mathcal{S}$ of leaves adjacent to $u_{5}$ with labels $3+\frac{\left|U_{3}\right|}{2}, \ldots, m-2-\frac{\left|U_{3}\right|}{2}$. By Lemma 2.40, it is possible
to perform a sequence of safe transfers $u_{5} \rightarrow u_{1} \rightarrow u_{4}$, such that the resulting tree has $\left|U_{5}\right|+1$ leaves at vertex $u_{5},\left|U_{1}\right|+1$ leaves at vertex $u_{1}$, and $\left|U_{4}\right|$ leaves at vertex $u_{4}$. This concludes the proof.

Theorem 3.18. If $T$ is a caterpillar with diameter six, then $T$ is 0 -rotatable.
Proof. The result follows from Corollary 2.20, Lemma 3.16 and Lemma 3.17.

### 3.1.3 Caterpillars with diameter at least seven

The main result of this section is Theorem 3.19, which shows that, for each integer $d$, $d \geq 7$, there exists a family of 0 -rotatable caterpillars with diameter $d$.

Theorem 3.19. Let $T$ be a caterpillar with $\operatorname{diam}(T) \geq 7$. Let $t=2$ if the diameter of $T$ is odd and $t=4$ otherwise. If every non-leaf vertex of $T$ has an even number, at least $t$, of leaf neighbours, then $T$ is 0 -rotatable.

Proof. Let $T$ be a caterpillar as described in the hypothesis. Let $N$ be the set of non-leaf vertices of $T$. Since the diameter of $T$ is at least $7,|N| \geq 6$. The bipartition of $T$ induces a bipartition $\{A, B\}$ of $N$.

Let $v \in N$; we choose the labelling of $A, B$ so that $v \in A$. Since the subtree $T_{N}$ of $T$ induced by $N$ is a path, Theorem 2.15 shows it has an $\alpha$-labelling $g$ such that $g(v)=0$. Moreover, $L_{A}^{g}=\{0,1, \ldots,|A|-1\}$ and $L_{B}^{g}=\{|A|, \ldots,|N|-1\}$.

Let $T^{\prime}$ be the tree obtained from $T$ by deleting: all but two leaf neighbours of $v$, if $|A|<|B|$; or all leaf neighbours of $v$, otherwise. Let $\ell$ be the number of leaves of $T^{\prime}$. Next, we show how to construct a graceful labelling $f$ of $T^{\prime}$ such that $f(v)=0$.

We create a new tree $T_{\ell}^{0}$ by adding to $T_{N}$ a set $L$ of $\ell$ leaves adjacent to vertex $u \in V\left(T_{N}\right)$ that has label $g(u)=|A|$. Let $f$ be the labelling of $T_{\ell}^{0}$ obtained from $g$ by giving labels $|A|, \ldots,|A|+\ell-1$ to the leaves in $L$ and adding $\ell$ to all the $g$-labels of vertices in $B$. In order to see that $f$ is an $\alpha$-labelling of $T_{\ell}^{0}$, first note that the $f$-labels of the vertices in $A \cup L$ are $0,1, \ldots,|A|+\ell-1$ and the $f$-labels of the vertices in $B$ are $|A|+\ell, \ldots,|N|+\ell-1$. Furthermore, the edges incident with vertices of $L$ have $f$-labels $1,2, \ldots, \ell$, while every edge of $T_{N}$ has $f$-label $\ell$ more than in $g$, so these are $\ell+1, \ell+2, \ldots, \ell+|N|-1$, as required.

Let $u \in V\left(T_{\ell}^{0}\right)$ such that $f(u)=|A|+\ell$ (note that $u \in V\left(T_{\ell}^{0}\right)$ is related with $u \in$ $V\left(T_{N}\right)$ with $\left.g(u)=|A|\right)$. Let $u^{\prime} \in V\left(T_{\ell}^{0}\right)$ such that $f\left(u^{\prime}\right)=|A|-1$. Also, let $2 j$ be the number of leaves adjacent to $u$ in tree $T^{\prime}$. By Corollary 2.37, it is possible to make a safe transfer $u \rightarrow u^{\prime}$ of the first type by transferring the leaves with $f$-labels $|A|+j,|A|+j+1, \ldots,|A|+\ell-j-1$ in pairs whose labels sum to $2|A|+\ell-1$. Note that we are transferring a positive even number $\ell-2 j$ of leaves and that the resulting tree, denoted $T_{\ell}^{1}$, is a caterpillar whose vertices $u$ and $u^{\prime}$ have $2 j$ and $\ell-2 j$ leaf neighbours, respectively.

Now, considering $T_{\ell}^{1}$ gracefully labelled with $f$, we apply Lemma 2.39 to make a sequence of safe transfers in $T_{\ell}^{1}$ so as to obtain a gracefully labelled caterpillar isomorphic to $T^{\prime}$. Take $a=|A|-1, b=|A|+\ell+1, s=|A|+j, p=\ell-(2 j+1), r_{2}=|B|-2$,

$$
r_{1}= \begin{cases}|A|-1, & \text { if }|N| \text { is odd and }|A|<|B| ; \\ |A|-2, & \text { otherwise; }\end{cases}
$$

and

$$
z= \begin{cases}a-r_{1}, & \text { if }|N| \text { is odd and }|A|>|B| ; \\ b+r_{2}, & \text { otherwise }\end{cases}
$$

In order to see that Lemma 2.39 applies to $T_{\ell}^{1}$, first recall that $f\left(u^{\prime}\right)=a$ and $u^{\prime}$ is adjacent to the leaf vertices with consecutive labels $s=|A|+j, \ldots,|A|+\ell-(j+1)=s+p$ and that $\ell$ is the number of leaves of $T^{\prime}$. Moreover, note that:
(i) $0 \leq a-r_{1} \leq a<b \leq b+r_{2}=|A|+\ell+|B|-1=|N|+\ell-1$, so hypothesis (i) of Lemma 2.39 is satisfied;
(ii) since $\ell \geq 2 j+8, p=\ell-(2 j+1) \geq 7$ and hypothesis (ii)(a) of Lemma 2.39 is satisfied;
(iii) since $j \geq 1, a=|A|-1<|A|+j=s$ and $s+p=|A|+\ell-j-1<|A|+\ell+1=b$, satisfying hypothesis (ii)(b) of Lemma 2.39;
(iv) finally, $2 s+p=2|A|+\ell-1$, while $a+b=2|A|+\ell$, which implies that hypothesis (ii)(c) is satisfied with $2 s+p+1=a+b$.

In order to conclude that the tree obtained after the sequence of transfers described in Lemma 2.39 is isomorphic to $T^{\prime}$, we need to show that: every vertex of $N \backslash\{u\}$ is in the sequence of transfers, if $|N|$ is odd and $|A|<|B|$; or every vertex of $N \backslash\{u, v\}$ is in the sequence of transfers, otherwise.

For each $i \in\left\{a-r_{1}, \ldots, a, b, \ldots, b+r_{2}\right\}$, let $v_{i}$ be the vertex of $T_{\ell}^{1}$ with $f$-label $i$. By Lemma 2.39, the sequence of transfers starts at vertex $v_{a}=u^{\prime}$, alternating vertices from $A$ and $B$, with the vertices in $A$ occurring as $v_{a}, v_{a-1}, \ldots$ and in $B$ occurring as $v_{b}, v_{b+1}, \ldots$. The sequence finishes at vertex $v_{z}$, which is either $v_{a-r_{1}} \in\left\{v_{0}, v_{1}\right\}$ or $v_{b+r_{2}}=v_{|N|+\ell-1}$.

Recall again that $u$ has label $|A|+\ell$; this is smaller than $b$ and greater than $a$, so $u$ does not occur in any sequence of transfers. In fact, $u$ already has $2 j$ leaf neighbours in $T_{\ell}^{1}$. Thus, we do not need to transfer any vertices to or from $u$. We consider two cases depending on the parity of $|N|$.

Case 1. $|N|$ is even.
In this case, $r_{1}=|A|-2$ and $z=b+r_{2}=|N|+\ell-1$. Moreover, since $|N|$ is even, $|A|=|B|$. Thus, the sequence of transfers is $v_{a} \rightarrow v_{b} \rightarrow v_{a-1} \rightarrow v_{b+1} \rightarrow \ldots \rightarrow v_{1} \rightarrow v_{z}$. These are precisely the vertices in $N \backslash\{u, v\}$, as required.

Case 2. $|N|$ is odd.
Since $N$ is the disjoint union of $A$ and $B,|A| \neq|B|$. Also, $N$ induces a path, so $|A|=|B| \pm 1$.

Subcase 2.1. $|A|<|B|$.
In this case, $r_{1}=|A|-1$ and $z=b+r_{2}$. Thus, the sequence of transfers is $v_{a} \rightarrow v_{b} \rightarrow$ $v_{a-1} \rightarrow v_{b+1} \rightarrow \ldots \rightarrow v_{0} \rightarrow v_{z}$. These are precisely the vertices in $N \backslash\{u\}$, as required.

Subcase 2.2. $|A|>|B|$.
Now, $r_{1}=|A|-2$ and $z=b-r_{1}=1$. In this case, the sequence of transfers is $v_{a} \rightarrow v_{b} \rightarrow v_{a-1} \rightarrow v_{b+1} \rightarrow \ldots \rightarrow v_{b+r_{2}} \rightarrow v_{1}$. These are precisely the vertices in $N \backslash\{u, v\}$, as required.

For all these cases, Lemma 2.39 guarantees that we may leave the appropriate even number of leaves adjacent to each vertex of $N$ so that the resulting caterpillar is isomorphic to $T^{\prime}$.

In order to obtain a graceful labelling $h$ of $T$ such that $h(v)=0$, take $T^{\prime}$ gracefully labelled with $f$ and add $k$ new leaves to $v$, where $k=|E(T)|-\left|E\left(T^{\prime}\right)\right|$, and assign the labels $\left|E\left(T^{\prime}\right)\right|+1,\left|E\left(T^{\prime}\right)\right|+2, \ldots,|E(T)|$ to these new leaves. Note that the resulting tree is isomorphic to $T$ and is gracefully labelled. Furthermore, since the edge of $T$ with the largest label is incident with a new leaf and $v$, the complementary labelling $\bar{h}$ assigns label 0 to one of the leaves adjacent to $v$, completing the proof.

As an additional remark, the requirement that each non-leaf vertex must have at least four leaf neighbours when the diameter of $T$ is even is due to Subcase 2.1. In this subcase, we are obliged to deposit an even positive number of leaves in $v_{0}$. If $v_{0}$ has exactly two leaves adjacent to it in the tree $T$, then both leaves need to be in $T^{\prime}$ so as to be able to apply Lemma 2.39 on $T_{\ell}^{1}$. However, since these two leaves do not have label $|E(T)|$, it is impossible to obtain label 0 in a leaf adjacent to $v_{0}$ by applying the complementary labelling. This is the reason why at least four leaves are required in order to obtain label 0 in a leaf adjacent to $v_{0}$.

### 3.2 Concluding remarks

The results presented in Section 3.1 are positive steps towards settling Conjecture 3.5. One possible next step to pursue is to consider caterpillars in which non-leaf vertices are adjacent to an odd number of leaves. The technique of transfers proved to be very useful and powerful in the proof of Theorem 3.19 and we believe it can be further developed in order to also approach this case.

## Chapter 4

## $\alpha$-labellings of lobsters with $\Delta(G)=3$

In 1967, Rosa [98] proved that all caterpillars have $\alpha$-labellings (see Theorem 2.19) and observed that not all trees have such labellings. In fact, Rosa [98] observed that the smallest tree that does not have $\alpha$-labellings is the lobster obtained from a star $K_{1,3}$ by subdividing each of its edges exactly once, as illustrated in Figure 4.1.


Figure 4.1: The smallest tree that does not have an $\alpha$-labelling.

Later, in 1982, Huang, Kotzig and Rosa [59] proved that all lobsters with diameter four that are not isomorphic to caterpillars do not have $\alpha$-labellings (see Theorem 2.47). Although there are several examples of lobsters that do not have $\alpha$-labellings, no example of lobster without a graceful labelling is known. In fact, in 1979, Jean-Claude Bermond [17] conjectured that all lobsters are graceful. In spite of many efforts to settle Bermond's conjecture [85-89,114], it remains open and only some classes of lobsters are known to be graceful.

In 1989, Zhao [119] proved that all trees with diameter four (which are lobsters) are graceful. In 1994, Wang et al. [114] used the technique of transfers to show that a family of lobsters is graceful. Almost a decade later, Mishra and Panigrahi [85-89] also used transfers as a main tool to exhibit other families of graceful lobsters. Other special families of graceful lobsters have been found by Sethuraman and Jeshinta [61,104] and Chen et al. [35]. In 2002, Morgan [90] published a paper where he claims that all lobsters with a perfect matching are graceful. However, in 2015, Haviar and Ivaška [56] showed that there are mistakes in Morgan's proof and observed that the result that all lobsters with a perfect matching are graceful follows from results presented in Broersma and Hoede's paper [24].

Although it is still unknown whether all lobsters have graceful labellings, it is wellknown that they have $\sigma$-labellings [69]. Since $\sigma$-labellings are also $\rho$-labellings, by Theorem 2.3, we conclude that the Ringel-Kotzig Conjecture is true for all lobsters.

## Corollary 4.1 (Kotzig [69]). Every lobster has a $\sigma$-labelling.

Proof. Let $T$ be a lobster. Note that the subgraph $T^{\prime} \subset T$ obtained from $T$ by removing all of its leaves is a caterpillar. By Theorem 2.19, $T^{\prime}$ has a graceful labelling. Therefore, by Theorem 2.12, $T$ has a $\sigma$-labelling.

While the Ringel-Kotzig Conjecture is true for all lobsters, the search for a proof to Bermond's Conjecture continues. In fact, it is astonishing that the Graceful Tree Conjecture remains open for lobsters since these trees are very close to caterpillars, which have a very simple graceful labelling (see Theorem 2.19).

Given the current difficulty in proving that an arbitrary tree has a graceful labelling, a good way to start investigating these labellings is to impose some restriction on the trees under consideration and hope that such a restriction helps somehow to find a graceful labelling. A possible restriction is to consider only trees with perfect matching. Another very common restriction is to consider trees with small maximum degree. In this vein, since all trees with maximum degree two (paths) are graceful, it is natural to consider trees with maximum degree three. In fact, it is still not known if all trees with maximum degree three have graceful labellings or even which trees with maximum degree three have $\alpha$-labellings. Some results in these directions are known [20-23]. In particular, Brankovic et al. [22] showed that all trees with at most 28 vertices, maximum degree three and a perfect matching have $\alpha$-labellings and, based on this result, they posed the following conjecture.

Conjecture 4.2 (Brankovic et al. [22]). All trees with maximum degree three and a perfect matching have an $\alpha$-labelling.

Computational results obtained by Brinkman et al. [23] suggest that, as the number of vertices grows, many trees with maximum degree three may have $\alpha$-labellings. In fact, Brinkman et al. [23] performed a computer search on all trees with maximum degree three up to 36 vertices and found that the only trees with maximum degree three up to 36 vertices that do not have $\alpha$-labellings are the ones exhibited in Figure 4.2 and Figure 4.3.


Figure 4.2: All trees with maximum degree three up to 15 vertices that do not have $\alpha$-labellings. Vertices with degree two have black colour and all the others have white colour. These trees first appeared in the work of Bonnington and Širáň [20]. Note that all these trees are lobsters.

Let $\mathcal{F}$ be the set of trees that can be obtained from trees $T^{\prime}$ with $4 k$ vertices, all with odd degree, by subdividing each edge of $T^{\prime}$ exactly once. Brinkman et al. [23] observed that the trees in Figure 4.2 and Figure 4.3 with at least 15 vertices belong to $\mathcal{F}$ (by Theorem 2.44, any tree in $\mathcal{F}$ does not have $\alpha$-labellings). Based on their findings, Brinkmann et al. [23] posed the following question.


Figure 4.3: All trees with maximum degree three and number of vertices between 23 to 31 that do not have $\alpha$-labellings. Vertices with degree two have black colour and all the others have white colour. Note that the only lobsters are the first and third trees in the picture, with 23 and 31 vertices, respectively.

Question 4.3 (Brinkmann et al. [23]). Do all trees with at least 15 vertices, maximum degree three, and without $\alpha$-labellings belong to family $\mathcal{F}$ ?

An affirmative answer to Question 4.3 implies a complete and nice characterization of all trees with maximum degree three and without $\alpha$-labellings. To our best knowledge, both Conjecture 4.2 and Question 4.3 remain open, even when restricted to the family of lobsters. In this chapter we prove that two general classes of lobsters with maximum degree three have $\alpha$-labellings, thus reinforcing Conjecture 4.2 and Question 4.3. In order to state our results, we first introduce some definitions.

Let $G$ be a lobster with $\Delta(G)=3$. The legs of $G$ are the non-trivial connected components obtained by removing the edges of its spine. Note that, since $\Delta(G)=3$, the legs of $G$ are isomorphic to a path with two vertices, a path with three vertices, or the bipartite graph $K_{1,3}$, and are called 1-leg, 2-leg and $Y$-leg, respectively. If the spine P is the path $\left(v_{1}, v_{2}, \ldots, v_{t}\right)$, then each leg contains exactly one of $v_{2}, v_{3}, \ldots, v_{t-1}$ as a vertex of degree 1 in the leg. An ending of $G$ consists of a subpath $P^{\prime}$ of $P$ containing either $v_{1}$ or $v_{t}$, together with all the legs having a vertex in $P^{\prime}$. There are six forbidden endings: two have $P^{\prime}=\left(v_{1}, v_{2}, v_{3}\right)$ with a 2-leg containing $v_{3}$, and $v_{2}$ is either not in any leg or it is in a 1 -leg; one has $P^{\prime}=\left(v_{1}, v_{2}, v_{3}, v_{4}\right)$, with $v_{4}$ in a 2 -leg and both $v_{2}$ and $v_{3}$ in 1 -legs. Figure $4.4(\mathrm{a})$ illustrates these first three forbidden endings. The other three are the reflections of these containing $v_{t}$. See Figure 4.4(b) for diagrams of these last three. Figure 4.5 shows examples of lobsters with maximum degree three with and without forbidden endings.

In this chapter, we construct $\alpha$-labellings for a family of lobsters of degree three as established in the next theorem. The proof of this theorem is presented in Section 4.2.

Theorem 4.4. Let $G$ be a lobster with $\Delta(G)=3$ and without $Y$-legs. If $G$ has at most one forbidden ending, then $G$ has an $\alpha$-labelling.

We say that a bipartition $\{A, B\}$ of a tree $T$ is balanced if $|A|$ and $|B|$ differ by at most one unit. Tree $T$ is balanced if it has a balanced bipartition. In this chapter, we also


Figure 4.4: The six forbidden endings.

(a) Lobster with $\Delta(G)=3$, without $Y$-legs and with two forbidden endings.

(c) Lobster with $\Delta(G)=3$, without $Y$-legs and with only one forbidden ending.

(b) Lobster with $\Delta(G)=3$, with one $Y$-leg and with one forbidden ending.

(d) Lobster with $\Delta(G)=3$, without $Y$-legs and without forbidden endings.

Figure 4.5: Examples of lobsters with maximum degree three.
prove results on $\alpha$-labellings of some families of trees with a perfect matching.
Theorem 4.5. Let $T$ be a tree with a perfect matching and let $T^{\prime}$ be its contree. If $T^{\prime}$ is balanced and has an $\alpha$-labelling, then $T$ also has an $\alpha$-labelling.

Theorem 4.6. Let $T$ be a tree with a perfect matching such that its contree $T^{\prime}$ is a balanced caterpillar. Then, $T$ has an $\alpha$-labelling.

The proofs of these last two results are presented in Section 4.3. It is important to remark that the contree of a lobster with a perfect matching is a caterpillar. Thus, by Theorem 4.6, we obtain the following result.

Corollary 4.7. If $G$ is a lobster with a perfect matching such that its contree is balanced, then $G$ has an $\alpha$-labelling.

Our contributions in this topic point towards an affirmative answer to Question 4.3. In the next section, we present additional definitions as well as classic results and techniques that are used in the proofs.

### 4.1 A graphical representation of $\alpha$-labellings

In 1973, Kotzig [69] showed that the existence of an $\alpha$-labelling in a bipartite graph $G$ is equivalent to the existence of a special geometric representation of $G$ that he called a $\pi$-representation. We use this representation to prove Theorem 4.4 and introduce it in the context of trees.

Let $T$ be a tree with bipartition $\left\{V_{1}, V_{2}\right\}$. A $\pi$-representation of $T$ consists of a drawing of $T$ on the plane such that:
(i) the vertices of $V_{1}$ are points on the line $y=1$, while those of $V_{2}$ are points on the line $y=-1$, and consecutive vertices on each of these lines are distance 1 apart;
(ii) the edges of $T$ are straight line segments; and
(iii) if two edges cross, the point of crossing is not on the line $y=0$.

Two $\pi$-representations of the path $P_{8}$ are illustrated in Figure 4.6.



Figure 4.6: Two $\pi$-representations $\theta$ and $\theta^{\prime}$ of $P_{8}$. In each of these, $P_{8}$ is shown with an $\alpha$-labelling.

It should be clear that any other three equally distanced parallel lines may be used in this definition. An important property of a $\pi$-representation $\theta$ of $T$ is the following: let $r, s$ be any two real numbers; then, adding $r$ to the $x$-coordinate of each $v \in V_{1}$, and adding $s$ to the $x$-coordinate of each $v \in V_{2}$, gives another $\pi$-representation of $T$.

Kotzig turns a $\pi$-representation of a tree $T$ into an $\alpha$-labelling as follows: label the leftmost vertex on line $y=1$ with 0 and continue labelling the vertices consecutively along $y=1$ until reaching the rightmost vertex on this line, which receives label $\left|V_{1}\right|-1$; then label the rightmost vertex on line $y=-1$ with $\left|V_{1}\right|$ and continue labelling the vertices consecutively along $y=-1$ until reaching the leftmost vertex on this line, which receives label $|E(T)|$. Figure 4.6 shows two $\alpha$-labellings of $P_{8}$. Kotzig proved that this is an $\alpha$ labelling of $T$ and that, conversely, the inverse function converts an $\alpha$-labelling of $T$ into a $\pi$-representation. That is, Kotzig proved that a tree has an $\alpha$-labelling if and only if it has a $\pi$-representation.

Let $\theta$ be a $\pi$-representation of a tree $T$ with bipartition $\left\{V_{1}, V_{2}\right\}$. For each vertex $v$ of $T$, with $v \in V_{i}$, we let $d_{\theta}^{\leftarrow}(v)$ be the number of vertices of $V_{i}$ that are to the right of $v$ on the line $y=(-1)^{i-1}$ in $\theta$, while $d_{\theta}(v)$ is the number to the left of $v$. Note that $d_{\theta}^{\leftarrow}(v)+d_{\theta}(v)=\left|V_{i}\right|-1$. When the particular $\pi$-representation is clear from context, we will drop the subscript $\theta$. As an example, in the first $\pi$-representation $\theta$ of Figure 4.6, vertex $v_{1}$ has $d_{\theta}\left(v_{1}\right)=0$ and $d_{\theta}^{\leftarrow}\left(v_{1}\right)=3$, and vertex $v_{6}$ has $d_{\theta}\left(v_{6}\right)=2$ and $d_{\theta}^{\leftarrow}\left(v_{6}\right)=1$.

Among other things, Kotzig showed how to link $\pi$-representations of two trees to get a $\pi$-representation of a larger tree. This is our next lemma and is illustrated in Figure 4.7.

Lemma 4.8 (Kotzig [69]). Let $\theta^{\prime}$ and $\theta^{\prime \prime}$ be $\pi$-representations of trees $T^{\prime}$ and $T^{\prime \prime}$, respectively, such that there exist $u \in V\left(T^{\prime}\right)$ and $v \in V\left(T^{\prime \prime}\right)$ for which $d_{\theta^{\prime}}^{\leftarrow}(u)=d_{\theta^{\prime \prime}}^{\vec{\prime}}(v)$. Then, tree $T$ obtained from $T^{\prime} \cup T^{\prime \prime}$ by adding a new edge uv has a $\pi$-representation.


Figure 4.7: Linking two $\pi$-representations $\theta^{\prime}$ and $\theta^{\prime \prime}$ by an edge $u v, u \in V\left(T^{\prime}\right)$ and $v \in V\left(T^{\prime \prime}\right)$. Note that $\underset{d_{\theta^{\prime}}^{\leftarrow}}{\leftarrow}(u)=\underset{d_{\theta^{\prime \prime}}}{\stackrel{(v)}{ }}(v)=2$. Furthermore, the drawing resulting from the addition of edge $u v$ is a $\pi$-representation since no other edge cross $y=0$ at the same point as $u v$.

Note that, in order to link two $\pi$-representations (one for each of $T^{\prime}, T^{\prime \prime}$ ), we add a straight line segment connecting two vertices $u \in V\left(T^{\prime}\right)$ and $v \in V\left(T^{\prime \prime}\right)$ such that (i) $d_{\overleftarrow{\theta^{\prime}}}^{\leftarrow}(u)=d_{\theta^{\prime \prime}}(v)$; and (ii) $u, v$ lie in distinct lines. If $d_{\overleftarrow{\theta^{\prime}}}^{\leftarrow}(u)=d_{\overleftarrow{\theta^{\prime \prime}}}^{\leftarrow}(v)$, that is, condition (i) is satisfied "in the reverse direction", we can perform a vertical reflection on $\theta$ ", as shown in Figure 4.8. An horizontal reflection is also shown in that figure, for the case when it may be necessary to switch the lines of $u$ and $v$. The resulting $\pi$-representation $\theta^{\prime \prime \prime}$ of $T^{\prime \prime}$ now obeys the conditions of Lemma 4.8.


(a) It is not possible to link $u$ and $v$ since they belong to the same line $y=-1$ and $d_{\theta^{\prime}}^{\leftarrow}(u) \neq d_{\theta^{\prime \prime}}^{\vec{\prime}}(v)$.

(b) Vertical and horizontal reflections performed on $\theta^{\prime \prime}$ so as to obtain a new $\pi$ representation $\theta^{\prime \prime \prime}$ of $T^{\prime \prime}$ with $d_{\theta^{\prime \prime \prime}}(v)=0$ and such that $v$ lies on line $y=1$.

Figure 4.8: Vertical and horizontal reflections performed on a $\pi$-representation.

### 4.2 Lobsters with maximum degree three

Let $G$ be a lobster with $\Delta(G)=3$, without $Y$-legs, and with at most one forbidden ending. As previously observed, it suffices to show that $G$ has a $\pi$-representation to conclude that $G$ has an $\alpha$-labelling. In order to construct a $\pi$-representation for $G$, we partition $V(G)$ into subsets $B_{1}, \ldots, B_{k}$, find a suitable $\pi$-representation for each induced
subgraph $G\left[B_{i}\right]$ and, finally, show that these $\pi$-representations can be linked in order to obtain a $\pi$-representation of the original lobster $G$.

Let $G$ be a lobster with maximum degree three and without $Y$-legs. Let $P=$ $\left(s_{1}, \ldots, s_{t}\right)$ be the spine of $G$. Let $\mathcal{B}=\left\{B_{1}, \ldots, B_{k}\right\}$ be a partition of $V(G)$ into blocks $B_{i}$ such that:
(i) for $1 \leq i \leq k, B_{i} \cap V(P)$ is the nonempty set $\left\{s_{j}, \ldots, s_{j^{\prime}}\right\}$ of consecutive vertices of $P$, with $1 \leq j \leq j^{\prime} \leq t$, and we set $\ell_{i}=s_{j}$ and $r_{i}=s_{j^{\prime}}$;
(ii) if $1 \leq i<j \leq k, r_{i}=s_{p}$ and $\ell_{j}=s_{q}$, then $p<q$;
(iii) $E(G)=\left\{\bigcup E\left(G\left[B_{i}\right]\right)\right\} \cup\left\{r_{i} \ell_{i+1}: 1 \leq i \leq k-1\right\}$, that is, $E(G) \backslash\left\{\bigcup E\left(G\left[B_{i}\right]\right)\right\}$ comprises the edges that link consecutive blocks.

Set $\mathcal{B}$ is called a block-partition and it is illustrated in Figure 4.9.

(a) Lobster $G$. Partition of $V(G)$ into blocks $B_{1}=\left\{s_{1}, s_{2}\right\}, B_{2}=\left\{s_{3}, s_{4}, s_{3}^{1}, s_{3}^{2}, s_{4}^{1}, s_{4}^{2}\right\}, B_{3}=\left\{s_{5}, s_{6}, s_{6}^{1}\right\}$, $B_{4}=\left\{s_{7}, s_{8}, s_{9}, s_{7}^{1}, s_{7}^{2}, s_{9}^{1}\right\}, B_{5}=\left\{s_{10}, s_{11}, s_{12}, s_{13}, s_{10}^{1}, s_{10}^{2}\right\}$. Thin edges link consecutive blocks $B_{i}$ and $B_{i+1}$.

(b) Induced subgraphs $G\left[B_{1}\right], G\left[B_{2}\right], G\left[B_{3}\right], G\left[B_{4}\right]$ and $G\left[B_{5}\right]$.

Figure 4.9: Block-partition $\mathcal{B}$ of a lobster $G$ with $\Delta(G)=3$ and without $Y$-legs.
Lemma 4.8 immediately implies the following lemma, which is the core of our proof of Theorem 4.4.

Lemma 4.9. Let $G$ be a lobster with maximum degree three and without $Y$-legs. If $G$ has a block-partition $\mathcal{B}=\left\{B_{1}, \ldots, B_{k}\right\}$ such that, for $1 \leq i \leq k-1$, subgraphs $G\left[B_{i}\right]$ and $G\left[B_{i+1}\right]$ have $\pi$-representations $\theta_{i}$ and $\theta_{i+1}$, respectively, such that $d_{\theta_{i}}^{\leftarrow}\left(r_{i}\right)=d_{\theta_{i+1}}^{\overrightarrow{( }}\left(\ell_{i+1}\right)$, then $G$ has a $\pi$-representation.

Figure 4.10 illustrates the application of Lemma 4.9 for the lobster $G$ of Figure 4.9.
Let $G$ be a lobster with maximum degree three, without $Y$-legs and with at most one forbidden ending. Let $P=\left(s_{1}, \ldots, s_{t}\right)$ be the spine of $G$ and $\mathcal{B}=\left\{B_{1}, \ldots, B_{k}\right\}$ a block-partition of $V(G)$. We say that block $B_{i}$ is a:
(i) $\mathcal{C}$-block, if no vertex in $B_{i} \cap V(P)$ is in a 2-leg of $G$;
(ii) $\mathcal{L}$-block, if

$\theta_{1}$

$\theta_{2}$

$\theta_{3}$

$\theta_{4}$

$\theta_{5}$
(a) $\pi$-representations of subgraphs $G\left[B_{1}\right], G\left[B_{2}\right], G\left[B_{3}\right], G\left[B_{4}\right], G\left[B_{5}\right]$ of Figure $4.9(\mathrm{~b})$, with $d_{\theta_{i}}^{\leftarrow}\left(r_{i}\right)=d_{\theta_{i+1}}^{\overrightarrow{ }}\left(l_{i+1}\right)=0$.

(b) A $\pi$-representation of $G$ obtained by adding edges $r_{i} l_{i+1}$ to the $\pi$-representations of subgraphs $G\left[B_{i}\right]$ and $G\left[B_{i+1}\right]$, for $1 \leq i \leq 4$.

Figure 4.10: Construction of a $\pi$-representation for the lobster $G$ of Figure 4.9.
(a) $\ell_{i}$ is in a 2 -leg of $G$;
(b) $r_{i}$ is in a 1-leg or 2-leg;
(c) and no vertex in $\left(B_{i} \cap V(P)\right) \backslash\left\{\ell_{i}, r_{i}\right\}$ is in a 2-leg of $G$;
(iii) $\mathcal{E}$-block, if
(a) $\ell_{i}$ is in a 2-leg;
(b) and no vertex in $\left(B_{i} \cap V(P)\right) \backslash\left\{\ell_{i}\right\}$ is in a 2-leg of $G$.

As an example, in Figure 4.9, blocks $B_{1}$ and $B_{3}$ are $\mathcal{C}$-blocks, blocks $B_{2}$ and $B_{4}$ are $\mathcal{L}$-blocks, and block $B_{5}$ is an $\mathcal{E}$-block.

Note that, by the definition, the subgraphs induced by $\mathcal{C}$-blocks are caterpillars. Kotzig [69] proved that every caterpillar $T$ with at least two vertices and spine $\left(v_{1}, \ldots, v_{n}\right)$ has a $\pi$-representation $\theta$ such that $\left.d_{\theta} \overrightarrow{( } v_{1}\right)=d_{\theta}\left(v_{2}\right)=d_{\theta}^{\leftarrow}\left(v_{n-1}\right)=d_{\theta}^{\leftarrow}\left(v_{n}\right)=0$. Moreover, note that a trivial graph has a $\pi$-representation $\theta$ such that its unique vertex $v$ has $d_{\theta}(v)=d_{\theta}^{\leftarrow}(v)=0$. Therefore, if $B_{i}$ is a $\mathcal{C}$-block, then $G\left[B_{i}\right]$ has a $\pi$-representation $\theta_{i}$ such that $d_{\theta_{i}}\left(\ell_{i}\right)=d_{\theta_{i}}^{\leftarrow}\left(r_{i}\right)=0$. We remark that the subgraphs induced by $\mathcal{L}$-blocks and $\mathcal{E}$-blocks are also caterpillars.

Lemma 4.10 shows some families of $\mathcal{L}$-blocks and $\mathcal{E}$-blocks which have suitable $\pi$ representations that are used in the proof of Theorem 4.4. In order to present these families, we introduce additional notation.

Let $B_{i} \in \mathcal{B}$. Let $T=G\left[B_{i}\right]$ and $\left(v_{1}, \ldots, v_{n}\right)$ be its spine. We say that $T$ belongs to family $\left(\left\{i_{1}, \ldots, i_{k}\right\}, b\right)$ if its degree 3 vertices are $v_{i_{1}}, \ldots, v_{i_{k}}$ such that $3<i_{1}<i_{2}<\cdots<$ $i_{k}<n-b$, for $b \in\{1,2\}$. Note that $\ell_{i}=v_{3}$ and $r_{i}=v_{n-b}$. Figure 4.11 schematizes fourteen families of the form $\left(\left\{i_{1}, \ldots, i_{k}\right\}, b\right)$, that are presented in Lemma 4.10.

Lemma 4.10. Let $T$ be a caterpillar with spine $\left(v_{1}, \ldots, v_{n}\right)$ such that $T$ belongs to one of these families:

(a) ( $(, 2)$ with $|V(T)| \geq 6$.
(b) $(\emptyset, 1)$ with $|V(T)| \notin\{5,8\}$.
(c) $(\{4\}, 2)$ with $|V(T)| \geq 8$.

(d) (\{4\},1) with $|V(T)| \geq 8$ and $|V(T)| \neq 11$.

(e) $(\{4,5\}, 2)$ with $|V(T)| \geq 10$.

(g) $(\{4,5,9\}, 2)$ with $|V(T)| \geq 15$.
(f) $(\{4,5\}, 1)$ with $|V(T)| \geq 9$ and $|V(T)| \neq 12$.

(h) (\{4,5,9\},1) with $|V(T)| \geq 14$.

(j) (\{7\},1) with $|V(T)| \geq 11$.

(1) $(\{7,8\}, 1)$ with $|V(T)| \geq 12$.

(n) $(\{4,9\}, 1)$ with $|V(T)| \geq 13$.

Figure 4.11: Fourteen families of caterpillars $T=\left(\left\{i_{1}, \ldots, i_{k}\right\}, b\right)$ that are used in our block-decomposition of a lobster $G$ with $\Delta(G)=3$, without $Y$-legs and with at most one forbidden ending. Each member of one of these families is isomorphic to an $\mathcal{L}$-block or an $\mathcal{E}$-block.
(i) $(\emptyset, 2)$ with $|V(T)| \geq 6$;
(ii) $(\{4\}, 2)$ with $|V(T)| \geq 8$;
(iii) $(\{4,5\}, 2)$ with $|V(T)| \geq 10$;
(iv) $(\{7\}, 2)$ with $|V(T)| \geq 11$;
(v) $(\{7,8\}, 2)$ with $|V(T)| \geq 13$;
(vi) $(\{4,5,9\}, 2)$ with $|V(T)| \geq 15$;
(vii) $(\{4,9\}, 2)$ with $|V(T)| \geq 14$;
(viii) $(\emptyset, 1)$ with $|V(T)| \geq 6$ and $|V(T)| \neq 8$;
(ix) $(\{4,5\}, 1)$ with $|V(T)| \geq 9$ and $|V(T)| \neq 12$;
(x) $(\{4\}, 1)$ with $|V(T)| \geq 8$ and $|V(T)| \neq 11$;
(xi) $(\{7\}, 1)$ with $|V(T)| \geq 11$;
(xii) $(\{7,8\}, 1)$ with $|V(T)| \geq 12$;
(xiii) $(\{4,5,9\}, 1)$ with $|V(T)| \geq 14$;
(xiv) $(\{4,9\}, 1)$ with $|V(T)| \geq 13$.

Then, $T$ has a $\pi$-representation $\theta$ such that $d_{\theta}\left(v_{3}\right)=d_{\theta}^{\leftarrow}\left(v_{n-b}\right)=0$.
Proof. Let $T \in\left(\left\{i_{1}, \ldots, i_{k}\right\}, b\right)$ be a caterpillar with spine $\left(v_{1}, \ldots, v_{n}\right)$. There are fourteen cases to consider depending on the set $\left\{i_{1}, \ldots, i_{k}\right\}$ and the value of $b$.

Case 1. $T \in(\emptyset, 2)$ and $|V(T)| \geq 6$.
In this case, we prove, by induction on $|V(T)|$, the stronger property that $T$ has a $\pi$-representation $\theta$ with $d_{\theta} \vec{\theta}\left(v_{3}\right)=d_{\theta}^{\leftarrow}\left(v_{n-2}\right)=0$ and $d_{\theta}^{\leftarrow}\left(v_{n}\right)=1$.

Figure 4.12 exhibits a $\pi$-representation of $T$ satisfying the required properties, for $6 \leq$ $|V(T)| \leq 8$. Now, suppose $|V(T)| \geq 9$. In this case, $T$ is a path. Set $H=\left(v_{1}, \ldots, v_{n-3}\right)$ and $P_{3}=\left(v_{n-2}, v_{n-1}, v_{n}\right)$. By the definition, $H$ is a path and $|V(H)|=|V(T)|-3 \geq 6$, which implies that $H$ is a caterpillar of type ( $\emptyset, 2)$. By the induction hypothesis, $H$ has a $\pi$-representation $\theta_{H}$ such that $\underset{\theta_{\theta_{H}}}{\leftrightarrows}\left(v_{3}\right)=d_{\theta_{H}}^{\leftarrow}\left(v_{n-5}\right)=0$ and $d_{\theta_{H}}^{\leftarrow}\left(v_{n-3}\right)=1$. Consider the $\pi$-representation $\theta_{P_{3}}$ of $P_{3}$ with $d_{\theta_{P_{3}}}^{\leftarrow}\left(v_{n-1}\right)=0$ and $d_{\theta_{P_{3}}}^{\rightarrow}\left(v_{n-2}\right)=d_{\theta_{P_{3}}}^{\leftarrow}\left(v_{n}\right)=1$. Since $d_{\theta_{H}}^{\leftarrow}\left(v_{n-3}\right)=d_{\theta_{P_{3}}}^{\vec{n}}\left(v_{n-2}\right)$, by Lemma 4.8, we obtain a $\pi$-representation of $T$ by adding a new edge $v_{n-3} v_{n-2}$ linking the $\pi$-representations of $H$ and $P_{3}$. Furthermore, the resulting $\pi$-representation $\theta$ of $T$ has $d_{\theta} \overrightarrow{( }\left(v_{3}\right)=d_{\theta}^{\leftarrow}\left(v_{n-2}\right)=0, d_{\theta}^{\leftarrow}\left(v_{n}\right)=1$, and the result follows.

Case 2. $T \in(\{4\}, 2)$ and $|V(T)| \geq 8$.
In this case, for $8 \leq|V(T)| \leq 15$, Figure 4.13 exhibits the required $\pi$-representation of $T$. We can assume $|V(T)| \geq 16$.

For $16 \leq|V(T)| \leq 18$, Figure 4.13 exhibits a $\pi$-representation $\theta$ of $T$ with the stronger property that $d_{\theta} \rightarrow\left(v_{3}\right)=d_{\theta}^{\leftarrow}\left(v_{n-2}\right)=0$ and $d_{\theta}^{\leftarrow}\left(v_{n}\right)=1$. Now, suppose $|V(T)| \geq 19$. Define $H$ and $P_{3}$ as in the previous case. By the definition, $H$ is a caterpillar with $\Delta(H)=3$ and spine $\left(v_{1}, \ldots, v_{n-3}\right)$, with at least 16 vertices, such that the unique vertex with degree three in $H$ is $v_{4}$. Hence, $H$ is a caterpillar of type $(\{4\}, 2)$ with $16 \leq|V(H)|<|V(T)|$. Thus, by induction hypothesis, $H$ has a $\pi$-representation $\theta_{H}$ such that $d_{\theta_{H}}\left(v_{3}\right)=d_{\theta_{H}}^{\leftarrow}\left(v_{n-5}\right)=0$ and $d_{\theta_{H}}^{\overleftarrow{ }}\left(v_{n-3}\right)=1$. Consider the $\pi$-representation $\theta_{P_{3}}$ of $P_{3}$ with $d_{\theta_{P_{3}}}^{\leftarrow}\left(v_{n-1}\right)=0$ and $d_{\theta_{P_{3}}}^{\overrightarrow{ }}\left(v_{n-2}\right)=d_{\overleftarrow{\theta_{3}}}^{\leftarrow}\left(v_{n}\right)=1$. Since $d_{\theta_{H}}^{\leftarrow}\left(v_{n-3}\right)=d_{\overrightarrow{\theta_{P_{3}}}}\left(v_{n-2}\right)$, by Lemma 4.8 , we obtain a $\pi$-representation $\theta$ of $T$ by adding a new edge $v_{n-3} v_{n-2}$ linking the $\pi$-representations of $H$ and $P_{3}$. Furthermore, the resulting $\pi$-representation $\theta$ of $T$ has $d_{\theta}^{\rightarrow}\left(v_{3}\right)=d_{\theta}^{\leftarrow}\left(v_{n-2}\right)=0$, $d_{\theta}^{\leftarrow}\left(v_{n}\right)=1$, and the result follows.

The remaining twelve cases are proved similarly as Case 2. That is, for each Case $i$, with $3 \leq i \leq 14$, there exists a positive number $n_{i}$ such that, for every tree $T$ of Case $i$ with $|V(T)| \leq n_{i}$, the required $\pi$-representation of $T$ is exhibited in one of the figures
numbered from 4.12 to 4.18 . (Table 4.1 presents the values of $n_{i}$ and the number of the figures for each of the remaining cases.) Moreover, the three largest $\pi$-representations exhibited in each of these figures have the stronger property that $d_{\theta}\left(v_{3}\right)=d_{\theta}^{\leftarrow}\left(v_{n-b}\right)=0$ and $d_{\theta}^{\leftarrow}\left(v_{n}\right)=1$. Using this stronger property, we apply induction similarly as applied in Case 2. This concludes the proof.


$$
|V(T)|=6
$$

$|V(T)|=7$

$|V(T)|=8$

$|V(T)|=9$

$|V(T)|=10$

$|V(T)|=11$

Figure 4.12: $\pi$-representations of trees $T \in(\emptyset, 2)$, with $6 \leq|V(T)| \leq 8$, and $\pi$ representations of trees $T \in(\emptyset, 1)$ with $6 \leq|V(T)| \leq 11$ and $|V(T)| \neq 8$.

$|V|=8$

$|V|=9$

$|V(T)|=10$

$|V(T)|=11$


$|V(T)|=15$

$|V(T)|=17$


$$
|V(T)|=16
$$


$|V(T)|=18$

Figure 4.13: $\pi$-representations of trees $T \in(\{4\}, 2)$, with $8 \leq|V(T)| \leq 18$, and $\pi$ representations of trees $T \in(\{4\}, 1)$ with $8 \leq|V(T)| \leq 18$ and $|V(T)| \neq 11$.

$|V(T)|=9$

$|V(T)|=10$

$|V(T)|=11$
$|V(T)|=13$


$|V(T)|=14$

$|V(T)|=15$

$|V(T)|=16$

$|V(T)|=18$

Figure 4.14: $\pi$-representations of trees $T \in(\{4,5\}, 2)$, with $10 \leq|V(T)| \leq 18$, and $\pi$-representations of trees $T \in(\{4,5\}, 1)$ with $9 \leq|V(T)| \leq 18$ and $|V(T)| \neq 12$.

$|V(T)|=16$

$|V(T)|=15$

$|V(T)|=17$

Figure 4.15: $\pi$-representations of trees $T \in(\{4,5,9\}, 2)$, with $15 \leq|V(T)| \leq 17$, and $\pi$-representations of trees $T \in(\{4,5,9\}, 1)$ with $14 \leq|V(T)| \leq 17$.

The next lemma provides particular $\pi$-representations of a few more caterpillars. The caterpillars and the required $\pi$-representations are illustrated in Figure 4.19, which also constitutes the proof.

Lemma 4.11. Let $T$ be one of the caterpillars presented in Figure 4.19, and $\left(v_{1}, \ldots, v_{n}\right)$ be its spine. Then, $T$ is isomorphic to an $\mathcal{E}$-block and has a $\pi$-representation $\theta$ such that vertex $v_{3}$ has $d_{\theta}\left(v_{3}\right)=0$.

Now, we present Lemma 4.12, which is fundamental for proving Theorem 4.4.
Lemma 4.12. Let $G$ be a lobster with $\Delta(G)=3$ and without $Y$-legs. If $G$ has at most one forbidden ending, then there exists a block-partition $\mathcal{B}=\left\{B_{1}, \ldots, B_{k}\right\}$ of $V(G)$ such


Figure 4.16: $\pi$-representations of trees $T \in(\{4,9\}, 2)$, with $14 \leq|V(T)| \leq 16$, and $\pi$-representations of trees $T \in(\{4,9\}, 1)$ with $13 \leq|V(T)| \leq 16$.


Figure 4.17: $\pi$-representations of trees $T$ in $(\{7\}, 2)$ and $(\{7\}, 1)$ with $11 \leq|V(T)| \leq 13$.


Figure 4.18: $\pi$-representations of trees $T \in(\{7,8\}, 2)$, with $13 \leq|V(T)| \leq 15$, and $\pi$-representations of trees $T \in(\{7,8\}, 1)$ with $12 \leq|V(T)| \leq 15$.
that: (i) for $1 \leq i \leq k-1$, subgraph $G\left[B_{i}\right]$ has a $\pi$-representation $\theta_{i}$ such that ${\overrightarrow{\theta_{i}}}\left(\ell_{i}\right)=$ $d_{\theta_{i}}^{\leftarrow}\left(r_{i}\right)=0$; and (ii) $G\left[B_{k}\right]$ has a $\pi$-representation $\theta_{k}$ such that ${\overrightarrow{\theta_{k}}}_{\vec{k}}\left(\ell_{k}\right)=0$.

Proof. Let $G$ be a lobster as stated in the hypothesis. Let $P=\left(s_{1}, \ldots, s_{t}\right)$ be the spine of $G$. We choose the orientation of $P$ so that $s_{t}$ is not in a forbidden ending. For $1 \leq i \leq j \leq t$, let $\left\langle P^{i, j}\right\rangle$ be the subgraph of $G$ consisting of the subpath $\left(s_{i}, s_{i+1}, \ldots, s_{j}\right)$ of $P$, together with all legs having a vertex in $\left(s_{i}, s_{i+1}, \ldots, s_{j}\right)$. Note that an ending is a $\left\langle P^{i, j}\right\rangle$ with either $i=1$ or $j=t$.

Proceeding in order from $s_{1}$, we partition $V(G)$ into blocks $B_{1}, \ldots, B_{k}$ that are $\mathcal{C}$ blocks, $\mathcal{L}$-blocks or $\mathcal{E}$-blocks. This block-partition of $V(G)$ is constructed inductively. If there is no 2 -leg in $G$, then $G$ is a caterpillar and is the only block; it is a $\mathcal{C}$-block and we

| Case | Value $n_{i}$ | Figure |
| :---: | :---: | :---: |
| 8 | 11 | 4.12 |
| 10 | 18 | 4.13 |
| 3 and 9 | 18 | 4.14 |
| 6 and 13 | 17 | 4.15 |
| 7 and 14 | 16 | 4.16 |
| 4 and 11 | 13 | 4.17 |
| 5 and 12 | 15 | 4.18 |

Table 4.1: Values of $n_{i}$ and the number of the figures for each of the remaining cases listed in the proof of Lemma 4.10.


Figure 4.19: Caterpillars and $\pi$-representations of each one of them such that $d^{\rightarrow}\left(v_{3}\right)=0$.
are done. Thus, we may assume $G$ has 2 -legs. Let $j$ be the least index such that $s_{j}$ is in a 2-leg; we set the the first block $B_{1}$ to be $\left\langle P^{1, j-1}\right\rangle$. This is a $\mathcal{C}$-block.

Now, suppose there exists $i \geq 1$ such that the $i^{\text {th }}$ block $B_{i}$ has been determined with $r_{i}=s_{j-1}$ and such that $B_{i}$ has a $\pi$-representation with $d^{\leftarrow}\left(r_{i}\right)=0$. In the first case, suppose that there is no $q>j-1$ such that $s_{q}$ is in a 2 -leg. Then, $\left\langle P^{j, t}\right\rangle$ is the last block $B_{i+1}$; it is a $\mathcal{C}$-block and we are done.

In the remaining case, there is a least $q>j-1$ such that $s_{q}$ is in a 2-leg. If $q>j$, then the next block $B_{i+1}$ is the subgraph $\left\langle P^{j, q-1}\right\rangle$, that is, a $\mathcal{C}$-block. Thus, we may assume $q=j$. If there is an $p>j$ such that subgraph $\left\langle P^{j, p}\right\rangle$ is an $\mathcal{L}$-block, then, choosing the minimal such $p$ yields one of the graphs in Lemma 4.10 as $B_{i+1}$ (this claim is proved below). If no such $p$ exists, subgraph $\left\langle P^{j, t}\right\rangle$ is the last block $B_{k}$, it is an $\mathcal{E}$-block isomorphic to one of the graphs presented in Lemma 4.10 and Lemma 4.11 (this claim is proved below).

If $B_{i+1}$ is a $\mathcal{C}$-block or an $\mathcal{L}$-block presented in Lemma 4.10, then $G\left[B_{i+1}\right]$ has a $\pi$ representation such that $d^{\rightarrow}\left(\ell_{i+1}\right)=0$ and $d^{\leftarrow}\left(r_{i+1}\right)=0$. Moreover, if $B_{i+1}$ is the last block of the block-partition, $B_{i+1}$ is either a $\mathcal{C}$-block or an $\mathcal{E}$-block isomorphic to one of the graphs presented in Lemma 4.10 and Lemma 4.11. In both cases, $G\left[B_{i+1}\right]$ has a
$\pi$-representation in which $d \rightarrow\left(\ell_{i+1}\right)=0$.
Now, we prove the claims above, that is, if the $i^{\text {th }}$ block $B_{i}$ has $r_{i}=s_{j-1}$ and the next spine vertex $s_{j}$ is in a 2 -leg, then block $B_{i+1}$, as previously defined, is isomorphic to one of the graphs presented in Lemma 4.10 and Lemma 4.11.

If no vertex in $\left\langle P^{j+1, t}\right\rangle$ has degree 3 in $G$, then $B_{i+1}$ is the subgraph $\left\langle P^{j, t}\right\rangle$, that is, an $\mathcal{E}$-block $(\emptyset, 2)$, and has $\left|V\left(\left\langle P^{j, t}\right\rangle\right)\right| \geq 6$. Thus, the result follows. Now, assume that there exists a degree 3 vertex in $\left\{s_{j+1}, \ldots, s_{t}\right\}$. Let $p=\min \left\{r: r \in\{j+1, \ldots, t\}\right.$ and $d_{G}\left(s_{r}\right)=$ $3\}$. Note that $\left\langle P^{j, p}\right\rangle$ is an $\mathcal{L}$-block. If $s_{p}$ is in a 2-leg, then $\left\langle P^{j, p}\right\rangle$ has at least six vertices and is isomorphic to $\mathcal{L}$-block $(\emptyset, 2)$. Therefore, the result follows. For the remaining cases we assume that $s_{p}$ is in a 1 -leg. Let $T=\left\langle P^{j, p}\right\rangle$. Note that $T$ is an $\mathcal{L}$-block $(\emptyset, 1)$. Thus, if $|V(T)| \neq 5$ and $|V(T)| \neq 8$, the result follows. Now, we consider the cases $|V(T)|=5$ and $|V(T)|=8$.

Case 1. $|V(T)|=5$.
First, suppose no vertex in $\left\langle P^{p+1, t}\right\rangle$ has degree 3 in $G$. Let $T^{\prime}=\left\langle P^{j, t}\right\rangle$. Since $T^{\prime}$ cannot be isomorphic to a forbidden ending, $\left|V\left(T^{\prime}\right)\right| \geq 7$. If $\left|V\left(T^{\prime}\right)\right|=7, T^{\prime}$ is the graph illustrated in Figure 4.19(d); otherwise, it is isomorphic to an $\mathcal{E}$-block ( $\{4\}, 2$ ), and the result follows. Now, assume that there exists a degree 3 vertex in $\left\{s_{p+1}, \ldots, s_{t}\right\}$. Let $p_{1}=\min \left\{r: r \in\{p+1, \ldots, t\}\right.$ and $\left.d_{G}\left(s_{r}\right)=3\right\}$. Note that $\left\langle P^{j, p_{1}}\right\rangle$ is an $\mathcal{L}$-block. If $s_{p_{1}}$ is in a 2 -leg, then $\left\langle P^{j, p_{1}}\right\rangle$ has at least 8 vertices and is isomorphic to $\mathcal{L}$-block ( $\{4\}, 2$ ). Therefore, the result follows. For the remaining cases, assume that $s_{p_{1}}$ is in a 1-leg.

Let $T_{1}=\left\langle P^{j, p_{1}}\right\rangle$. Note that $T_{1}$ has at least 7 vertices and is an $\mathcal{L}$-block ( $\{4\}, 1$ ). Thus, if $\left|V\left(T_{1}\right)\right| \neq 7$ and $\left|V\left(T_{1}\right)\right| \neq 11$, the result follows. Now, we consider the cases $\left|V\left(T_{1}\right)\right|=7$ and $\left|V\left(T_{1}\right)\right|=11$.

Case 1.1. $\left|V\left(T_{1}\right)\right|=7$.
First, suppose no vertex in $\left\langle P^{p_{1}+1, t}\right\rangle$ has degree 3 in $G$. Let $T_{1}^{\prime}=\left\langle P^{j, t}\right\rangle$. Since $T_{1}^{\prime}$ cannot be isomorphic to a forbidden ending, $\left|V\left(T_{1}^{\prime}\right)\right| \geq 9$. If $\left|V\left(T_{1}^{\prime}\right)\right|=9$, then $T_{1}^{\prime}$ is isomorphic to an $\mathcal{E}$-block $(\{4,5\}, 1)$; otherwise, $T_{1}^{\prime}$ is isomorphic to an $\mathcal{E}$-block $(\{4,5\}, 2)$, and the result follows. Now, assume that there exists a degree 3 vertex in $\left\{s_{p_{1}+1}, \ldots, s_{t}\right\}$.

Let $p_{2}=\min \left\{r: r \in\left\{s_{p_{1}+1}, \ldots, t\right\}\right.$ and $\left.d_{G}\left(s_{r}\right)=3\right\}$. Note that $\left\langle P^{j, p_{2}}\right\rangle$ is an $\mathcal{L}$-block. If $s_{p_{2}}$ is in a 2 -leg, then $\left\langle P^{j, p_{2}}\right\rangle$ has at least 10 vertices and is isomorphic to $\mathcal{L}$-block $(\{4,5\}, 2)$. Therefore, the result follows. For the remaining cases we assume that $s_{p_{2}}$ is in a 1-leg. Let $T_{2}=\left\langle P^{j, p_{2}}\right\rangle$. Note that $T_{2}$ has at least 9 vertices and is an $\mathcal{L}$-block $(\{4,5\}, 1)$. Thus, if $\left|V\left(T_{2}\right)\right| \neq 12$, the result follows.

Now, consider $\left|V\left(T_{2}\right)\right|=12$. First, suppose no vertex in $\left\langle P^{p_{2}+1, t}\right\rangle$ has degree 3 in $G$. Let $T_{2}^{\prime}=\left\langle P^{j, t}\right\rangle$. Note that $\left|V\left(T_{2}^{\prime}\right)\right| \geq 13$. If $\left|V\left(T_{2}^{\prime}\right)\right|=13, T_{2}^{\prime}$ is the graph illustrated in Figure 4.19(e); otherwise, $T_{2}^{\prime}$ is isomorphic to an $\mathcal{E}$-block $(\{4,5,9\}, 1)$ and the result follows. Now, assume that there exists a degree 3 vertex in $\left\{s_{p_{2}+1}, \ldots, s_{t}\right\}$. Let $p_{3}=$ $\min \left\{r: r \in\left\{p_{2}+1, \ldots, t\right\}\right.$ and $\left.d_{G}\left(s_{r}\right)=3\right\}$. Note that $\left\langle P^{j, p_{3}}\right\rangle$ is an $\mathcal{L}$-block. If $s_{p_{3}}$ is in a 2 -leg, then $\left\langle P^{j, p_{3}}\right\rangle$ has at least 15 vertices and is isomorphic to $\mathcal{L}$-block $(\{4,5,9\}, 2)$. If $s_{p_{3}}$ is in a 1-leg, then $\left\langle P^{j, p_{3}}\right\rangle$ has at least 14 vertices and is isomorphic to $\mathcal{L}$-block $(\{4,5,9\}, 1)$. In both cases, the result follows.

Case 1.2. $\left|V\left(T_{1}\right)\right|=11$.

First, suppose no vertex in $\left\langle P^{p_{1}+1, t}\right\rangle$ has degree 3 in $G$. Let $T_{1}^{\prime}=\left\langle P^{j, t}\right\rangle$. Note that $\left|V\left(T_{1}^{\prime}\right)\right| \geq 12$. If $\left|V\left(T_{1}^{\prime}\right)\right|=12, T_{1}^{\prime}$ is the graph illustrated in Figure 4.19(f); otherwise, $T_{1}^{\prime}$ is isomorphic to an $\mathcal{E}$-block $(\{4,9\}, 1)$ with $\left|V\left(T_{1}^{\prime}\right)\right| \geq 13$ and the result follows.

Now, assume that there exists a degree 3 vertex in $\left\{s_{p_{1}+1}, \ldots, s_{t}\right\}$. Let $p_{2}=\min \{r: r \in$ $\left\{p_{1}+1, \ldots, t\right\}$ and $\left.d_{G}\left(s_{r}\right)=3\right\}$. Note that $\left\langle P^{j, p_{2}}\right\rangle$ is an $\mathcal{L}$-block. If $s_{p_{2}}$ is in a 2-leg, then $\left\langle P^{j, p_{2}}\right\rangle$ has at least 14 vertices and is isomorphic to $\mathcal{L}$-block $(\{4,9\}, 2)$. If $s_{p_{2}}$ is in a 1-leg, then $\left\langle P^{j, p_{2}}\right\rangle$ has at least 13 vertices and is isomorphic to $\mathcal{L}$-block $(\{4,9\}, 1)$. In both cases, the result follows.

Case 2. $|V(T)|=8$.
First, suppose no vertex in $\left\langle P^{p+1, t}\right\rangle$ has degree 3 in $G$. Let $T^{\prime}=\left\langle P^{j, t}\right\rangle$. Note that $\left|V\left(T^{\prime}\right)\right| \geq 9$. If $\left|V\left(T^{\prime}\right)\right| \in\{9,10\}, T^{\prime}$ is exhibited in Figure 4.19(a) and Figure 4.19(b); otherwise, it is isomorphic to an $\mathcal{E}$-block $(\{7\}, 1)$ and the result follows. Now, assume that there exists a degree 3 vertex in $\left\{s_{p+1}, \ldots, s_{t}\right\}$. Let $p_{1}=\min \{r: r \in\{p+1, \ldots, t\}$ and $\left.d_{G}\left(s_{r}\right)=3\right\}$. Note that $\left\langle P^{j, p_{1}}\right\rangle$ is an $\mathcal{L}$-block. If $s_{p_{1}}$ is in a 2 -leg, then $\left\langle P^{j, p_{1}}\right\rangle$ has at least 11 vertices and is isomorphic to $\mathcal{L}$-block $(\{7\}, 2)$. Therefore, the result follows. For the remaining cases, assume that $s_{p_{1}}$ is in a 1-leg. Let $T_{1}=\left\langle P^{j, p_{1}}\right\rangle$. Note that $T_{1}$ has at least 10 vertices and is an $\mathcal{L}$-block $(\{7\}, 1)$. Thus, if $\left|V\left(T_{1}\right)\right| \geq 11$, the result follows.

Next, consider the case $\left|V\left(T_{1}\right)\right|=10$. First, suppose no vertex in $\left\langle P^{p_{1}+1, t}\right\rangle$ has degree 3 in $G$. Let $T_{1}^{\prime}=\left\langle P^{j, t}\right\rangle$. Note that $\left|V\left(T_{1}^{\prime}\right)\right| \geq 11$. If $\left|V\left(T_{1}^{\prime}\right)\right|=11, T_{1}^{\prime}$ is the graph illustrated in Figure 4.19(c); otherwise, $T_{1}^{\prime}$ is isomorphic to an $\mathcal{E}$-block $(\{7,8\}, 1)$, and the result follows.

Now, assume that there exists a degree 3 vertex in $\left\{s_{p_{1}+1}, \ldots, s_{t}\right\}$. Let $p_{2}=\min \{r: r \in$ $\left\{p_{1}+1, \ldots, t\right\}$ and $\left.d_{G}\left(s_{r}\right)=3\right\}$. Note that $\left\langle P^{j, p_{2}}\right\rangle$ is an $\mathcal{L}$-block. If $s_{p_{2}}$ is in a 2 -leg, then $\left\langle P^{j, p_{2}}\right\rangle$ has at least 13 vertices and is isomorphic to $\mathcal{L}$-block $(\{7,8\}, 2)$. If $s_{p_{2}}$ is in a 1-leg, then $\left\langle P^{j, p_{2}}\right\rangle$ has at least 12 vertices and is isomorphic to $\mathcal{L}$-block $(\{7,8\}, 1)$. In both cases, the result follows. This case concludes the proof.

Now we are ready to prove Theorem 4.4, which is restated below.
Theorem 4.4. Let $G$ be a lobster with $\Delta(G)=3$ and without $Y$-legs. If $G$ has at most one forbidden ending, then $G$ has an $\alpha$-labelling.

Proof. By Lemma 4.12 and Lemma 4.9 we conclude that every lobster $G$ with maximum degree three, without $Y$-legs and with at most one forbidden ending has a $\pi$ representation. This implies that $G$ has an $\alpha$-labelling.

### 4.3 Trees with a perfect matching

In this section, we prove Theorem 4.5 and Theorem 4.6. Recall that the contree of a tree $T$ with a perfect matching $M$ is the tree obtained from $T$ by contracting the edges of $M$. As previously observed, the contree of a lobster with a perfect matching is a caterpillar. Thus, by Theorem 2.49, every lobster with a perfect matching has a strongly-graceful labelling since every caterpillar has an $\alpha$-labelling (which is also a graceful labelling).

While analysing strongly-graceful labellings of lobsters with a perfect matching, we observed that some strongly-graceful labellings are also $\alpha$-labellings. This observation led
us to the concept of strongly- $\alpha$ labellings: we say that a labelling $f$ of $T$ is strongly- $\alpha$ if $f$ is strongly-graceful and, additionally, $f$ is also an $\alpha$-labelling.

While every lobster with a perfect matching has a strongly-graceful labelling, there are lobsters that do not have strongly- $\alpha$ labellings. Figure 4.20 exhibits an example (remember that, by Theorem 2.47, all lobsters with diameter four that are not caterpillars do not have $\alpha$-labellings).


Figure 4.20: A lobster with a perfect matching that does not have strongly- $\alpha$ labellings.
In Theorem 4.13, we characterize the trees with a perfect matching that have strongly$\alpha$ labellings. In our proof, we use Broersma-Hoede's construction described in Section 2.5.

Theorem 4.13. Let $T$ be a tree with a perfect matching. Then, $T$ has a strongly- $\alpha$ labelling if and only if its contree has a balanced bipartition and an $\alpha$-labelling.

Proof. Let $T$ be a tree with a perfect matching $M$ and let $T^{\prime}$ be the contree of $T$. Let $n_{T}=|V(T)|$ and $n_{T^{\prime}}=\left|V\left(T^{\prime}\right)\right|=n_{T} / 2$. The result is trivial for $n_{T}=2$. Thus, consider $n_{T} \in\{4 p, 4 p+2\}, p \in \mathbb{N}_{>0}$.

First, suppose $T$ has a strongly- $\alpha$ labelling $f$. Since $T$ has a perfect matching, its bipartition $\{X, Y\}$ satisfies $|X|=|Y|$. Because $f$ is an $\alpha$-labelling, $L_{X}^{f}$ is either $\left\{0, \ldots, \frac{n_{T}}{2}-1\right\}$ or $\left\{\frac{n_{T}}{2}, \ldots, n_{T}-1\right\}$ and $L_{Y}^{f}$ is the other. We use labelling $f$ so as to obtain an $\alpha$-labelling $g$ for $T^{\prime}$. Let $v_{x y} \in V\left(T^{\prime}\right)$ be obtained from $T$ by the contraction of edge $x y \in M$. Proposition 2.48 implies that $f(x) \not \equiv f(y)(\bmod 2)$. Let $f^{\prime}\left(v_{x y}\right)$ be the one of $f(x)$ and $f(y)$ that is even. Now, Proposition 2.48 shows that $f^{\prime}: V\left(T^{\prime}\right) \rightarrow\left\{0,2, \ldots, n_{T}-2\right\}=L_{E(T) \backslash M}^{f} \cup\{0\}$.

Let $v_{x y} v_{z w} \in E\left(T^{\prime}\right)$ with $x, z \in X$ and $y, w \in Y$. By the definition of $T^{\prime}$, exactly one of $x w$ and $z y$ belongs to $E(T) \backslash M$. Suppose $x w \in E(T) \backslash M$. Note that this implies that $f(x) \equiv f(w)(\bmod 2)$. Also, if $f(x) \equiv 0(\bmod 2)$, then $\left|f^{\prime}\left(v_{x y}\right)-f^{\prime}\left(v_{z w}\right)\right|=|f(x)-f(w)|$; otherwise, $f(x) \equiv 1(\bmod 2)$ and $\left|f^{\prime}\left(v_{x y}\right)-f^{\prime}\left(v_{z w}\right)\right|=|f(y)-f(z)|=\mid\left(n_{T}-1-f(x)\right)-$ $\left(n_{T}-1-f(w)\right)\left|=|f(x)-f(w)|\right.$. We conclude that $L_{E\left(T^{\prime}\right)}^{f^{\prime}}=L_{E(T) \backslash M}^{f}$. For $v \in V\left(T^{\prime}\right)$, define $g(v)=f^{\prime}(v) / 2$. By the definition of $g, L_{V\left(T^{\prime}\right)}^{g}=\left\{0, \ldots, \frac{n_{T}}{2}-1\right\}=\left\{0, \ldots, n_{T^{\prime}}-1\right\}$ and $L_{E\left(T^{\prime}\right)}^{g}=\left\{1, \ldots, \frac{n_{T}}{2}-1\right\}=\left\{1, \ldots, n_{T^{\prime}}-1\right\}$. Thus, $g$ is graceful.

Now, we prove that $g$ is an $\alpha$-labelling. In order to do this, we show that there exists $k \in L_{V\left(T^{\prime}\right)}^{g}$ such that, either $g\left(v_{x y}\right) \leq k<g\left(v_{z w}\right)$ or $g\left(v_{z w}\right) \leq k<g\left(v_{x y}\right)$, for every edge $v_{x y} v_{z w} \in E\left(T^{\prime}\right)$. By the definition of $f, f(x) \leq n_{T} / 2-1<f(w)$. First, suppose that $f(x)$ and $f(w)$ are both even. Thus, we have

$$
\begin{align*}
& f(x) \leq n_{T} / 2-1<f(w) \\
& f^{\prime}\left(v_{x y}\right) \leq n_{T} / 2-1<f^{\prime}\left(v_{z w}\right) \\
& g\left(v_{x y}\right) \leq n_{T} / 4-1 / 2<g\left(v_{z w}\right) \tag{4.1}
\end{align*}
$$

Now, assume $f(x)$ and $f(w)$ are both odd. Thus, we have

$$
\begin{align*}
& f(x) \leq n_{T} / 2-1<f(w) \\
& n_{T}-1-f(x) \geq n_{T} / 2>n_{T}-1-f(w) \\
& f(y) \geq n_{T} / 2>f(z) \\
& f^{\prime}\left(v_{x y}\right) \geq n_{T} / 2>f^{\prime}\left(v_{z w}\right) \\
& g\left(v_{x y}\right) \geq n_{T} / 4>g\left(v_{z w}\right) \tag{4.2}
\end{align*}
$$

Hence, if $n_{T}=4 p$, then let $k=p-1$; otherwise, $n_{T}=4 p+2$ and we let $k=p$. In both cases, $g\left(v_{x y}\right) \leq k<g\left(v_{z w}\right)$ or $g\left(v_{z w}\right) \leq k<g\left(v_{x y}\right)$. In order to conclude the proof, just observe that when $n_{T^{\prime}}=2 p, T^{\prime}$ has a bipartition with parts of equal size since $k=p-1$. Moreover, when $n_{T^{\prime}}=2 p+1, T^{\prime}$ has a bipartition in which the cardinalities of the parts differ by one since $k=p$. Therefore, $T^{\prime}$ has a balanced bipartition and the result follows.

Now, suppose $T^{\prime}$ is balanced and that $g: V\left(T^{\prime}\right) \rightarrow\left\{0, \ldots, n_{T^{\prime}}-1\right\}$ is an $\alpha$-labelling. Let $\{A, B\}$ be the bipartition of $T^{\prime}$, labelled so that $|A| \geq|B|$. Changing to the complementary labelling if necessary, we may assume $L_{A}^{g}=\{0, \ldots,|A|-1\}$ and $L_{B}^{g}=$ $\left\{|A|, \ldots, n_{T^{\prime}}-1\right\}$. Since $T^{\prime}$ has $\alpha$-labelling $g$, by Theorem 2.49, $T$ has a strongly-graceful labelling $f$ obtained by the Broersma-Hoede's construction.

Next, we show that $f$ is also an $\alpha$-labelling; that is, we prove that there exists an integer $k \in\left\{0, \ldots, n_{T}-1\right\}$ such that, for each edge $u v \in E(T)$, either $f(u) \leq k<f(v)$ or $f(v) \leq k<f(u)$. We claim that $k=2|A|-1$ when $|A|=|B|$ and that $k=2|A|-2$ when $|A|=|B|+1$. Thus, let $k_{1}=2|A|-1, k_{2}=2|A|-2$ and consider an edge $u v \in E(T)$. There are two cases to analyse depending on which set, $M$ or $E(T) \backslash M$, edge $u v$ belongs to.

## Case 1. $u v \in M$.

By the construction of $f$, vertices $u$ and $v$ receive different labels $2 q$ and $\left(n_{T}-1\right)-2 q$, for $q \in L_{A}^{g} \cup L_{B}^{g}$. Without loss of generality, assume that $f(u)=\min \left\{2 q,\left(n_{T}-1\right)-2 q\right\}$ and $f(v)=\max \left\{2 q,\left(n_{T}-1\right)-2 q\right\}$.

First, suppose that $q \in L_{A}^{g}$. In this case, $2 q<\left(n_{T}-1\right)-2 q$. Thus, $f(u)=2 q$ and $f(v)=\left(n_{T}-1\right)-2 q$. Moreover, $f(u)=2 q \leq 2|A|-2$ and $f(v) \geq\left(n_{T}-1\right)-(2|A|-2)=$ $\left(2 n_{T^{\prime}}-1\right)-(2|A|-2)=(2(|A|+|B|)-1)-2|A|+2=2|B|+1$. Therefore, since $f(u) \leq 2|A|-2$ and $f(v) \geq 2|B|+1$, we have:

$$
\begin{align*}
& \text { if }|A|=|B| \text {, then } f(u) \leq 2|A|-2<2|A|-1=k_{1}<2|A|+1 \leq f(v) \text {; }  \tag{4.3}\\
& \text { if }|A|=|B|+1 \text {, then } f(u) \leq 2|A|-2=k_{2}<2|A|-1 \leq f(v) \tag{4.4}
\end{align*}
$$

Now, suppose that $q \in L_{B}^{g}$. In this case, $2 q>\left(n_{T}-1\right)-2 q$. Thus, $f(v)=2 q$ and $f(u)=\left(n_{T}-1\right)-2 q$. Moreover, $f(v)=2 q \geq 2|A|$ and $f(u) \leq\left(n_{T}-1\right)-2|A|=$ $\left(2 n_{T^{\prime}}-1\right)-2|A|=(2(|A|+|B|)-1)-2|A|=2|B|-1$. Therefore, since $f(u) \leq 2|B|-1$
and $f(v) \geq 2|A|$, we have:

$$
\begin{align*}
& \text { if }|A|=|B| \text {, then } f(u) \leq 2|A|-1=k_{1}<2|A| \leq f(v)  \tag{4.5}\\
& \text { if }|A|=|B|+1 \text {, then } f(u) \leq 2|A|-3<2|A|-2=k_{2}<2|A| \leq f(v) \tag{4.6}
\end{align*}
$$

Case 2. $u v \in E(T) \backslash M$.
By the construction of $f, f(u) \equiv f(v)(\bmod 2)$. Without loss of generality, assume that $f(u)<f(v)$. First, suppose that $f(u)$ and $f(v)$ are both even. In this case, $f(u)=2 q$ and $f(v)=2 r$, for $q, r \in L_{A}^{g} \cup L_{B}^{g}$. Since $u v \notin M$, edge $u v$ has a corresponding edge $u^{\prime} v^{\prime}$ in the contree $T^{\prime}$ whose endpoints are in different parts of $\{A, B\}$. Since $f(u)<f(v)$, we have that $q<r$ with $q \in L_{A}^{g}$ and $r \in L_{B}^{g}$. Also, since $L_{A}^{g}=\{0, \ldots,|A|-1\}$ and $L_{B}^{g}=\left\{|A|, \ldots, n_{T^{\prime}}-1\right\}$, we have that $f(u)=2 q \leq 2|A|-2$ and $f(v)=2 r \geq 2|A|$. These inequalities imply that $f(u) \leq 2|A|-2=k_{2}<k_{1}<2|A| \leq f(v)$, and the result follows.

Now, suppose that $f(u)$ and $f(v)$ are both odd. By the construction of $f$, we have that $f(u)=\left(n_{T}-1\right)-2 q$ and $f(v)=\left(n_{T}-1\right)-2 r$, for $q, r \in L_{A}^{g} \cup L_{B}^{g}$. Since $f(u)<f(v), r \in L_{A}^{g}$ and $q \in L_{B}^{g}$. This implies that $2 q \geq 2|A|$ and $2 r \leq 2|A|-2$. Since $n_{T}=2 n_{T^{\prime}}=2(|A|+|B|)$, we obtain that $f(u)=\left(n_{T}-1\right)-2 q=2|A|+2|B|-1-2 q \leq 2|A|+2|B|-1-2|A|=2|B|-1$ and $f(v)=\left(n_{T}-1\right)-2 r=2|A|+2|B|-1-2 r \geq 2|A|+2|B|-1-2|A|+2=2|B|+1$. Since $f(u) \leq 2|B|-1$ and $f(v) \geq 2|B|+1$, we have that

$$
\begin{align*}
& \text { if }|A|=|B| \text {, then } f(u) \leq 2|A|-1=k_{1}<2|A|+1 \leq f(v) \text {; }  \tag{4.7}\\
& \text { if }|A|=|B|+1 \text {, then } f(u) \leq 2|A|-3<k_{2}<2|A|-1 \leq f(v) \text {; } \tag{4.8}
\end{align*}
$$

and the result follows.
Now, we are ready to prove Theorem 4.5 and Theorem 4.6, which are restated below.
Theorem 4.5. Let $T$ be a tree with a perfect matching and let $T^{\prime}$ be its contree. If $T^{\prime}$ is balanced and has an $\alpha$-labelling, then $T$ also has an $\alpha$-labelling.

Proof. Follows directly from Theorem 4.13.
Theorem 4.6. Let $T$ be a tree with a perfect matching such that its contree $T^{\prime}$ is a balanced caterpillar. Then, $T$ has an $\alpha$-labelling.

Proof. The result follows by Theorem 4.5 and by the fact that every caterpillar has an $\alpha$-labelling.

### 4.4 Concluding remarks

In this chapter, we have considered lobsters with $\Delta(G)=3$, without $Y$-legs, that have at most one forbidden ending and lobsters with a perfect matching whose contrees are balanced. These are positive steps towards settling Conjecture 4.2 for lobsters. One possible next step to pursue is to consider lobsters for which neither Theorem 4.4 nor Corollary 4.7 apply, as in the example shown in Figure 4.21.


Figure 4.21: An $\alpha$-labelling of a lobster $G$ with $\Delta(G)=3$ and a perfect matching. Note that $G$ has two forbidden endings and its contree is not balanced.

## Chapter 5

## Neighbour-distinguishing labellings

> Over the years, much research has been done in graph theory concerning concepts dealing with "all things the same." In recent decades, there has been considerable research on concepts of a somewhat opposite nature (all things different). This has led to a number of concepts, results, conjectures, and open questions that have attracted the attention of many graph theorists.

-Gary Chartrand [31]
In the previous three chapters of this thesis we discussed injective vertex-labellings of graphs that induce an edge-labelling of the graph that is also injective. In this chapter, we present and discuss two different types of labellings: edge-labellings and total-labellings of graphs that induce vertex-labellings with the property that any two adjacent vertices have distinct labels. These labellings are called neighbour-distinguishing edge-labellings and neighbour-distinguishing total-labellings and are formally defined below.

Let $G=(V(G), E(G))$ be a simple graph and let $\mathcal{L} \subset \mathbb{R}$. A labelling $\pi: E(G) \rightarrow \mathcal{L}$ is an $\mathcal{L}$-edge-labelling of $G$ and a labelling $\pi: V(G) \cup E(G) \rightarrow \mathcal{L}$ is an $\mathcal{L}$-total-labelling of $G$. Given an $\mathcal{L}$-edge-labelling ( $\mathcal{L}$-total-labelling) $\pi$ of $G$, we define a mapping $C_{\pi}: V(G) \rightarrow \mathbb{R}$ such that, for each $v \in V(G)$,

$$
C_{\pi}(v)= \begin{cases}\sum_{u v \in E(G)} \pi(u v), & \text { if } \pi \text { is an } \mathcal{L} \text {-edge-labelling } \\ \pi(v)+\sum_{u v \in E(G)} \pi(u v), & \text { if } \pi \text { is an } \mathcal{L} \text {-total-labelling } .\end{cases}
$$

The value $C_{\pi}(v)$ is the colour of vertex $v$. Moreover, we define $\mathcal{C}_{\pi}(G)=\operatorname{Im}\left(C_{\pi}\right)$. The mapping $C_{\pi}$ is a proper-vertex-colouring of $G$ if $C_{\pi}(u) \neq C_{\pi}(v)$, for every edge $u v \in E(G)$. We say that the pair $\left(\pi, C_{\pi}\right)$ is a neighbour-distinguishing $\mathcal{L}$-edge-labelling when $\pi$ is an $\mathcal{L}$-edge-labelling and $C_{\pi}$ is a proper-vertex-colouring. Similarly, taking $\pi$ as an $\mathcal{L}$ -total-labelling in the previous definition, we say that $\left(\pi, C_{\pi}\right)$ is a neighbour-distinguishing $\mathcal{L}$-total-labelling. Figure 5.1 illustrates three graphs with a neighbour-distinguishing $\mathcal{L}$ -edge-labelling, for $\mathcal{L}=\{1,2,3\}$, and Figure 5.2 illustrates three graphs with a neighbourdistinguishing $\mathcal{L}^{\prime}$-total-labelling, for $\mathcal{L}^{\prime}=\{1,2\}$.

Neighbour-distinguishing edge-labellings appeared in the literature around 2004 [65] and neighbour-distinguishing total-labellings appeared around 2010 [94]. These two la-


Figure 5.1: Three graphs with a neighbour-distinguishing $\{1,2,3\}$-edge-labelling. The number inside each vertex corresponds to its colour.


Figure 5.2: Three graphs with a neighbour-distinguishing $\{1,2\}$-total-labelling. The number inside each vertex corresponds to its colour and the number next to each vertex is its label.
bellings are variants of a prior type of edge-labelling of graphs, introduced by Chartrand et al. [33], motivated by previous discussions about irregular graphs [9,33,34]. According to Chartrand, Oellerman and Erdös [34], if we want an irregular graph to be the opposite of a regular graph rather than just having the property of being not regular, then, perhaps, the most natural definition of irregularity is that a graph $G$ is irregular if every two vertices of $G$ have distinct degrees. However, as observed by Chartrand et al. [33], the unique simple graph that is irregular according to this definition is the trivial graph $K_{1}$, since every simple graph with at least two vertices has at least two vertices with the same degree [32].

While the trivial graph is the unique simple graph that is irregular, the situation is quite different when considering graphs with multiple edges. In fact, Chartrand et al. [33] observed that it is quite easy to find irregular graphs with multiple edges and, based on this observation, they investigated ways of turning an arbitrary simple graph $G$ into an irregular graph $G^{\prime}$ by replacing each edge $e \in E(G)$ with a given positive integer number $n_{e}$ of multiple edges. For example, the 2-regular graph $K_{3}$ can be transformed into an irregular graph with multiple edges by replacing one of its edges by two multiple edges and a second one by three multiple edges, as illustrated in Figure 5.3.

The main reason for just adding multiple edges to $G$ (one could also add new vertices or add edges linking two nonadjacent vertices of $G$ ) to get $G^{\prime}$ is that this transformation does not alter the adjacencies in the sense that every two adjacent vertices in $G^{\prime}$ are also adjacent in $G$. So the structure of $G^{\prime}$ is representative of the structure of $G$. The main


Figure 5.3: The 2-regular graph $K_{3}$ and the irregular graph $G^{\prime}$ obtained from $K_{3}$ by replacing edge $v_{1} v_{2}$ by two multiple edges and replacing edge $v_{2} v_{3}$ by three multiple edges. Note that every two vertices of $G^{\prime}$ have distinct degrees.
concern of Chartrand et al. [33] was to perform this transformation in such a way that the quantity $\max \left\{n_{e}: e \in E(G)\right\}$ is minimized.

Chartrand et al. [33] formalized this optimization problem using the notion of irregular networks. As defined by the authors, a network is a simple graph $G$ in which, for each edge $e \in E(G)$, is assigned a positive integer label $f(e)$. For each vertex $v \in V(G)$, the (induced) label of vertex $v$, denoted by $f(v)$, is the sum of the labels of its incident edges, that is, $f(v)=\sum_{u v \in E(G)} f(u v)(f(v)=0$ if $v$ has no incident edges). An irregular network is a network $G$ in which $f(u) \neq f(v)$, for any two vertices $u, v \in V(G)$. Note that the function $f: V(G) \rightarrow \mathbb{N}_{\geq 0}$ is an injection. As an illustration, Figure 5.4 exhibits an assignment of positive integers to the edges of the Petersen Graph that results in an irregular network.


Figure 5.4: An edge-labelling of the Petersen Graph resulting in an irregular network.

Note that no graph with two vertices can be an irregular network since, in any edgelabelling of $\bar{K}_{2}$ and $K_{2}$, both vertices of $\bar{K}_{2}$ have induced label 0 and the labels of the vertices of $K_{2}$ are those of its unique edge. Furthermore, note that if a graph $G$ has more than one isolated vertex, then $G$ also cannot be an irregular network since all of its isolated vertices have induced label 0 in any edge-labelling of $G$. However, if $G$ is a connected graph with at least three vertices, then it is always possible to assign positive integer labels to the edges of $G$ so as to obtain an irregular network.

Theorem 5.1 (Chartrand et al. [33]). If $G$ is a connected simple graph with at least three vertices, then there exists an edge-labelling $f: E(G) \rightarrow \mathbb{N}_{>0}$ of $G$ that results in an irregular network.

Proof. Let $G$ be a simple connected graph with at least three vertices. Let $E(G)=$ $\left\{e_{1}, e_{2}, \ldots, e_{m}\right\}$. Since $G$ is connected and $|V(G)| \geq 3$, we have that $m \geq 2$. Define an edge-labelling $f$ of $G$ such that, for each $i \in[1, m], f\left(e_{i}\right)=2^{i-1}$. Now, note that each vertex $v \in V(G)$ can have its label expressed as a binary number $a^{m-1} a^{m-2} \cdots a^{1} a^{0}$ such that each $a^{i} \in\{0,1\}$ and $a^{i}=1$ if and only if $e^{i+1}$ is incident with vertex $v$. Since there are no two vertices in $G$ with the same set of incident edges and since the binary representation of a positive integer is unique, we conclude that any two vertices of $G$ have distinct induced labels and the result follows.

Observe that, if we multiply all edge labels of an irregular network by a fixed positive integer, then the resulting labelled graph is also an irregular network. By the previous observations and by Theorem 5.1, we immediately obtain the following corollary.

Corollary 5.2 (Chartrand et al. [33]). There exists an edge-labelling of a simple graph $G$ that results in an irregular network if and only if $G$ has at most one isolated vertex and has no connected component isomorphic to $K_{2}$.

Let $G$ be an arbitrary connected simple graph with at least three vertices. Chartrand et al. [33] posed the problem of determining the least positive integer $k$ for which there exists an edge-labelling $f: E(G) \rightarrow[1, k]$ that results in an irregular network. The authors called such a number the irregularity strength of $G$, which is denoted by $s(G)$, and also determined the irregularity strength of some classes of graphs such as paths, complete graphs and complete equipartite graphs.

The problem of determining the irregularity strength of families of graphs or finding upper bounds for the irregular strength of arbitrary graphs has attracted the attention of many researchers, as can be noted in the surveys of Gallian and Lehel [49,73]. As examples of upper bounds, Nierhoff [91] proved that every connected simple graph $G, G \neq K_{3}$, with at least three vertices has $s(G) \leq|V(G)|-1$ and Kalkowski et al. [64] proved that every connected simple graph $G$ with at least three vertices has $s(G) \leq 6\lceil|V(G)| / \delta(G)\rceil$.

Over the last decades, many variants of the concept of irregular networks were introduced $[10,14,25,65,92,94]$. The reader will find more information about irregular networks and its variants in the surveys of Gallian [49] and Seamone [103].

In particular, a variant of irregular networks is the neighbour-distinguishing edgelabelling of graphs, defined in the beginning of this chapter. While in an irregular network any two distinct vertices have distinct induced labels, it is natural to consider edgelabellings of graphs where we require only that adjacent vertices have distinct induced labels, that is, the induced vertex-labelling is a proper-vertex-colouring of the graph.

Neighbour-distinguishing edge-labellings were introduced in 2004 by Karónski, Łuczak and Thomason [65], who proposed the problem of determining the least positive integer $k$, denoted by $\chi_{\Sigma}^{\prime}(G)$, needed to obtain a neighbour-distinguishing [ $k$ ]-edge-labelling of an arbitrary simple graph $G$ without connected components isomorphic to $K_{2}$. As with
irregular networks, $K_{2}$ does not have neighbour-distinguishing edge-labellings. However, all other simple graphs have neighbour-distinguishing edge-labellings (see Theorem 5.7).

Karónski et al. [65] observed that all the families of graphs they studied have a neighbour-distinguishing [3]-edge-labelling and, based on this observation, they posed the following conjecture.

Conjecture 5.3 (Karónski et al. [65]). If $G$ is a simple graph without connected components isomorphic to $K_{2}$, then $\chi_{\Sigma}^{\prime}(G) \leq 3$.

Conjecture 5.3 is known as the 1,2,3-Conjecture and the best known result about it is due to Kalkowski et al. [63], who proved that every simple graph $G$ without connected components isomorphic to $K_{2}$ has $\chi_{\Sigma}^{\prime}(G) \leq 5$.

Based on experiments, Karónski et al. [65] suggested that, for almost all graphs, labels 1 and 2 would be sufficient to obtain a neighbour-distinguishing edge-labelling. Nevertheless, Dudek and Wajc [39] proved that deciding whether an arbitrary graph has a neighbour-distinguishing [2]-edge-labelling is $\mathcal{N} \mathcal{P}$-complete. Since then, neighbourdistinguishing [2]-edge-labellings have been investigated for families of graphs [30, 37, 76, 112]. In particular, Thomassen et al. [112] completely characterized the bipartite graphs that have neighbour-distinguishing [2]-edge-labellings and proved that every nonbipartite ( $6 p-7$ )-edge-connected graph of chromatic number at most $p$ (where $p$ is any odd natural number greater than or equal to 3) has a neighbour-distinguishing [2]-edge-labelling.

In 2012, Khatirinejad et al. [66] generalized the neighbour-distinguishing [2]-edgelabelling problem to that of determining whether a graph $G$ has a neighbour-distinguishing $\{a, b\}$-edge-labelling, for $a, b \in \mathbb{R}, a \neq b$. In their article, the authors investigated neighbour-distinguishing $\{a, b\}$-edge-labellings for some families of graphs such as complete graphs, bipartite graphs, unicyclic graphs and cartesian products. Later, Bensmail [15] proved that deciding whether an arbitrary graph has a neighbour-distinguishing $\{a, b\}$-edge-labelling is $\mathcal{N} \mathcal{P}$-complete for every pair of distinct real numbers $a$ and $b$.

Motivated by the neighbour-distinguishing [2]-edge-labelling problem, Przybyło and Woźniak [94] introduced neighbour-distinguishing total-labellings and proposed the problem of determining the least positive integer $k$, denoted by $\chi_{\Sigma}^{\prime \prime}(G)$, needed to obtain a neighbour-distinguishing [k]-total-labelling of an arbitrary simple graph $G$. The authors [94] proved that complete graphs, 3 -colourable graphs and 4-regular graphs all have $\chi_{\Sigma}^{\prime \prime}(G) \leq 2$ and, based on these results, they posed Conjecture 5.4.

Conjecture 5.4 (Przybyło and Woźniak [94]). If $G$ is a simple graph, then $\chi_{\Sigma}^{\prime \prime}(G) \leq 2$.
Conjecture 5.4 is known as the 1,2-Conjecture and the best known result toward this conjecture is due to Kalkowski [62], who proved that every simple graph $G$ has $\chi_{\Sigma}^{\prime \prime}(G) \leq 3$.

Following the natural line of generalizations, Hulgan et al. [60] proved that complete graphs, complete multipartite graphs, 3 -colourable graphs and 4 -regular graphs have neighbour-distinguishing $\{a, b\}$-total-labellings, for any pair of distinct real numbers $a$ and $b$, and based on these results, they asked the following question.

Question 5.5 (Hulgan et al. [60]). Given any two distinct real numbers $a$ and $b$, does every simple graph $G$ have a neighbour-distinguishing $\{a, b\}$-total-labelling?

Question 5.5 remains open for arbitrary simple graphs and, more specifically, it is still unknown whether there exists a graph that requires the three integers $1,2,3$ in order to admit a neighbour-distinguishing total-labelling.

In the next section, we discuss known upper bounds on $\chi_{\Sigma}^{\prime}(G)$ and $\chi_{\Sigma}^{\prime \prime}(G)$ and present proofs for the respective best upper bounds known in the literature. In Section 5.2, we present known results that determine $\chi_{\Sigma}^{\prime}(G)$ and $\chi_{\Sigma}^{\prime \prime}(G)$ for some families of graphs. We also discuss other results used in proofs of Chapter 6.

### 5.1 Upper bounds on $\chi_{\Sigma}^{\prime}(G)$ and $\chi_{\Sigma}^{\prime \prime}(G)$

The most significant progress toward solving the 1,2,3-Conjecture is the determination of upper bounds on $\chi_{\Sigma}^{\prime}(G)$, for every simple graph $G$ without connected components isomorphic to $K_{2}$.

The first constant bound on $\chi_{\Sigma}^{\prime}(G)$ was found by Addario-Berry et al. [7], who proved that $\chi_{\Sigma}^{\prime}(G) \leq 30$. This bound was later improved to $\chi_{\Sigma}^{\prime}(G) \leq 16$ by Addario-Berry et al. [8]. Posteriorly, Wang and Yu [115] improved the previous upper bound to $\chi_{\Sigma}^{\prime}(G) \leq 13$.

Around the same time, Przybyło and Woźniak [94] introduced neighbour-distinguishing total-labellings and, using Addario-Berry et al.'s results [8], they proved that $\chi_{\Sigma}^{\prime \prime}(G) \leq 11$. Later, Przybyło [94] improved this upper bound for the class of regular graphs, showing that $\chi_{\Sigma}^{\prime \prime}(G) \leq 7$ for every regular graph $G$.

However, a great breakthrough on the 1,2,3-Conjecture and 1,2-Conjecture has been achieved with the results presented in Kalkowski's Ph.D thesis [62]. Kalkowski designed an algorithm that processes the vertices of a given simple graph in linear order $v_{1}, \ldots, v_{n}$ such that, at each step, some labels of the edges incident with the current vertex $v_{i}$ are adjusted so as to guarantee that the final colour of $v_{i}$ is different from the colours of its previously considered neighbours in the linear ordering. Kalkowski [62] used this algorithm to prove that every simple graph $G$ has $\chi_{\Sigma}^{\prime \prime}(G) \leq 3$. Kalkowski, Karoński and Pfender [62,64] modified the previous proof in order to show that every simple graph $G$ without connected components isomorphic to $K_{2}$ has $\chi_{\Sigma}^{\prime}(G) \leq 5$. These current best upper bounds are presented in the next two theorems.

Theorem 5.6 (Kalkowski [62]). If $G$ is a simple graph, then there exists a [3]-totallabelling $\pi$ of $G$ using labels 1, 2, 3 on the edges and labels 1,2 on the vertices such that $\left(\pi, C_{\pi}\right)$ is a neighbour-distinguishing [3]-total-labelling of $G$.

Proof. Let $G$ be a simple graph with $n$ vertices. We assume that $G$ is connected since its components can be dealt with separately. Next, we show how to inductively construct a sequence of $n$ total-labellings $\pi_{1}, \ldots, \pi_{n}$ of $G$ that assign labels $1,2,3$ to the edges, labels 1,2 to the vertices and such that, for each $i \in[2, n], \pi_{i}$ is a modification of $\pi_{i-1}$. The final [3]-total-labelling $\pi_{n}$ is defined such that $\left(\pi_{n}, C_{\pi_{n}}\right)$ is a neighbour-distinguishing [3]-total-labelling of $G$. In the remaining of this proof, it should be understood that, for each $i \in[2, n], \pi_{i}(e)=\pi_{i-1}(e)$ and $\pi_{i}(v)=\pi_{i-1}(v)$, for every $e \in E(G)$ and every $v \in V(G)$, unless stated otherwise.

Let $v_{1}, \ldots, v_{n}$ be an arbitrary ordering of the vertices of $G$. Start by defining $\pi_{0}$ as the [2]-total-labelling of $G$ that assigns label 2 to all edges of $G$ and label 1 to all vertices
of $G$. Set $\pi_{1}=\pi_{0}$. Thus, $C_{\pi_{1}}\left(v_{i}\right)=2 d_{G}\left(v_{i}\right)+1$, for every $v_{i} \in V(G)$.
Now, suppose that, for any integer $k \in[2, n]$, labelling $\pi_{k-1}$ satisfies the following four properties:
(i) for every $i<k$, we have that $\pi_{k-1}\left(v_{i}\right) \in\{1,2\}$;
(ii) $\pi_{k-1}\left(v_{i} v_{j}\right) \in\{1,2,3\}$, for all $v_{i} v_{j} \in E(G)$ with $i, j<k$;
(iii) $C_{\pi_{k-1}}\left(v_{i}\right) \neq C_{\pi_{k-1}}\left(v_{j}\right)$, for $v_{i} v_{j} \in E(G)$ with $i, j<k$;
(iv) for each $i \in[k, n]$, every edge $e$ incident with $v_{i}$ has label $\pi_{k-1}(e)=2$.

Note that these properties guarantee that the colour of $v_{i} \in\left\{v_{1}, \ldots, v_{k-1}\right\}$ is different from the colours of its neighbours in this set. Keeping this property is the key idea behind the construction of $\pi_{k}$. When adding $v_{k}$, we may make adjustments to the labels of its neighbours in the set $\left\{v_{1}, \ldots, v_{k-1}\right\}$ and also to some edges linking $v_{k}$ to its neighbours in $\left\{v_{1}, \ldots, v_{k-1}\right\}$. We make these adjustments guaranteeing that the previous colours of the vertices in $\left\{v_{1}, \ldots, v_{k-1}\right\}$ are preserved.

Now, we show how to construct $\pi_{k}$ so that the above properties hold. If $v_{i} v_{k} \notin E(G)$, for all $i<k$, then we are done. Thus, suppose that $v_{k}$ has $d$ neighbours among the vertices $v_{1}, \ldots, v_{k-1}$. For $i \in\{1,2\}$, let $d_{i}$ be the number of vertices in $N_{G}\left(v_{k}\right) \cap\left\{v_{1}, \ldots, v_{k-1}\right\}$ with label $i$. Note that $d=d_{1}+d_{2}$. For each edge $v_{i} v_{k} \in E(G)$, with $i<k$, we have the following possibilities for $\pi_{k}\left(v_{i} v_{k}\right)$ :
(a) if $\pi_{k-1}\left(v_{i}\right)=1$, either $\left(\pi_{k}\left(v_{i}\right)=1, \pi_{k}\left(v_{i} v_{k}\right)=2\right)$ or $\left(\pi_{k}\left(v_{i}\right)=2, \pi_{k}\left(v_{i} v_{k}\right)=1\right)$. Both options preserve the colour of $v_{i}$;
(b) if $\pi_{k-1}\left(v_{i}\right)=2$, either $\left(\pi_{k}\left(v_{i}\right)=2, \pi_{k}\left(v_{i} v_{k}\right)=2\right)$ or $\left(\pi_{k}\left(v_{i}\right)=1, \pi_{k}\left(v_{i} v_{k}\right)=3\right)$. Both options preserve the colour of $v_{i}$.

We claim that there exists an adjustment of the labels, as previously defined, that results in a colour for $v_{k}$ that is different from all the colours of its neighbours in the set $\left\{v_{1}, \ldots, v_{k-1}\right\}$.

By items (a) and (b), at most $d_{1}$ of the edges incident with $v_{k}$ can have their labels decreased by one unit. Let $s$ be the number of decrements, $0 \leq s \leq d_{1}$. Similarly, at most $d_{2}$ of the edges incident with $v_{k}$ can have their labels increased by one unit. Let $t$ be the number of increments, $0 \leq t \leq d_{2}$. By construction, $C_{\pi_{k-1}}\left(v_{k}\right)=2 d_{G}\left(v_{k}\right)+1$. Thus, after $s$ decrements and $t$ increments, the colour of $v_{k}$ is $2 d_{G}\left(v_{k}\right)+1+t-s$, which implies that the range of possibilities for the colour of $v_{k}$ is $\left[2 d_{G}\left(v_{k}\right)+1-d_{1}, 2 d_{G}\left(v_{k}\right)+1+d_{2}\right]$. Note that this set has $d+1$ elements. Since $v_{k}$ has exactly $d$ neighbours in the set $\left\{v_{1}, \ldots, v_{k-1}\right\}$, there exists at least one value that can be chosen for $C_{\pi_{k}}\left(v_{k}\right)$. Therefore, make the necessary adjustments to define $\pi_{k}$ accordingly. Also observe that these adjustments preserve properties (i), (ii), (iii) and (iv). This implies that ( $\pi_{n}, C_{\pi_{n}}$ ) is the required neighbour-distinguishing [3]-total-labelling of $G$, and the result follows.

Theorem 5.7 (Kalkowski et al. [63]). If $G$ is a simple graph without connected components isomorphic to $K_{2}$, then $G$ has a neighbour-distinguishing [5]-edge-labelling.

Proof. Let $G$ be a simple graph without connected components isomorphic to $K_{2}$. We assume that $G$ is connected since its components can be dealt with separately. Let $v_{1}, v_{2}, v_{3}, \ldots, v_{n}$ be an ordering of the vertices of $G$ such that $d_{G}\left(v_{n}\right) \geq 2$ and, for every $i \in[1, n-1]$, $v_{i}$ has a neighbour $v_{j}$ with $j>i$. Such an ordering of $V(G)$ can be obtained as follows: start with some vertex having at least two neighbours and call it $v_{n}\left(v_{n}\right.$ exists since $G$ is connected and $|V(G)| \geq 3)$. Assume that $v_{k}, v_{k+1}, \ldots, v_{n}$ have been defined, for $k \leq n$. Since $G$ is connected, unless the ordering is complete, there is some non-indexed vertex, say $v$, which is adjacent to some already indexed vertex. Define $v_{k-1}=v$. Stop when all vertices are indexed.

Next, we show how to construct a sequence of [5]-edge-labellings $f_{1}, \ldots, f_{n}$ for $G$ such that, for each $i \in[2, n]$, edge-labelling $f_{i}$ is a modification of $f_{i-1}$. Moreover, the final edge-labelling $f_{n}$ is constructed such that $C_{f_{n}}$ is a proper-vertex-colouring of $G$. The idea is to:
(i) assign, for every vertex $v_{i}$ with $i<n$, a set of two colours $W\left(v_{i}\right)=\left\{c_{i}, c_{i}+2\right\}$ with $c_{i} \equiv 0,1(\bmod 4) ;$
(ii) define edge-labellings $f_{1}, \ldots, f_{n-1}$ such that, for every edge $v_{i} v_{j} \in E(G)$ with $1 \leq$ $i<j \leq n-1, W\left(v_{i}\right) \cap W\left(v_{j}\right)=\emptyset$ and $C_{f_{q}}\left(v_{p}\right) \in W\left(v_{p}\right)$, for every $p \leq q ;$
(iii) define $f_{n}$ adjusting the labels of the edges incident with $v_{n}$ so as to make sure that $C_{f_{n}}$ is a proper-vertex-colouring of $G$.

Define $f_{1}$ as the labelling that assigns label 3 to all edges of $G$. Thus, $C_{f_{1}}\left(v_{1}\right)=3 d_{G}\left(v_{1}\right)$ and $c_{1}$ is defined as follows:

$$
c_{1}=\left\{\begin{array}{lll}
C_{f_{1}}\left(v_{1}\right), & \text { if } C_{f_{1}}\left(v_{1}\right) \equiv 0,1 & (\bmod 4) ; \\
C_{f_{1}}\left(v_{1}\right)-2, & \text { if } C_{f_{1}}\left(v_{1}\right) \equiv 2,3 & (\bmod 4) .
\end{array}\right.
$$

Thus, $W\left(v_{1}\right)=\left\{C_{f_{1}}\left(v_{1}\right), C_{f_{1}}\left(v_{1}\right)+2\right\}$, if $C_{f_{1}}\left(v_{1}\right) \equiv 0,1(\bmod 4)$; and $W\left(v_{1}\right)=$ $\left\{C_{f_{1}}\left(v_{1}\right)-2, C_{f_{1}}\left(v_{1}\right)\right\}$, otherwise. Note that it is always possible to define $c_{1} \equiv 0,1$ $(\bmod 4)$, independently of the value of $C_{f_{1}}\left(v_{1}\right)$. For the remaining of this proof, it should be understood that, for each $i \in[2, n], f_{i}(e)=f_{i-1}(e)$ for every edge $e \in E(G)$, unless otherwise stated.

Now, suppose that, for any integer $k \in[2, n-1]$, the following five conditions hold:
(I) for all $i<k, W\left(v_{i}\right)=\left\{c_{i}, c_{i}+2\right\}$ for some positive integer $c_{i} \equiv 0,1(\bmod 4)$;
(II) for every edge $v_{i} v_{j}$, with $i, j<k, W\left(v_{i}\right) \cap W\left(v_{j}\right)=\emptyset$;
(III) $C_{f_{k-1}}\left(v_{i}\right) \in W\left(v_{i}\right)$, for $i<k$;
(IV) $f_{k-1}\left(v_{k} v_{j}\right)=3$, for all edges $v_{k} v_{j}$ with $j>k$;
(V) for $i<k$, if $f_{k-1}\left(v_{i} v_{k}\right) \neq 3$, then either $f_{k-1}\left(v_{i} v_{k}\right)=2$ and $C_{f_{k-1}}\left(v_{i}\right)=c_{i}$, or $f_{k-1}\left(v_{i} v_{k}\right)=4$ and $C_{f_{k-1}}\left(v_{i}\right)=c_{i}+2$.

Define $S_{\ell}=\left\{v_{1}, \ldots, v_{\ell}\right\}$. Note that conditions (II) and (III) guarantee that any two adjacent vertices of $S_{k-1}$ have distinct colours under $C_{f_{k-1}}$. Keeping this property is the main point behind the construction of $f_{k}$. When adding $v_{k}$, we may make adjustments in
the labels of the vertices in $N_{G}\left(v_{k}\right) \cap S_{k-1}$ and in the labels of the edges in $E_{G}\left[\left\{v_{k}\right\}, S_{k-1}\right]$. We may also need to make one (and only one) extra adjustment in the label of an edge linking $v_{k}$ to a vertex in $V(G) \backslash S_{k}$. We make these adjustments guaranteeing that the final colour $C_{f_{k}}\left(v_{i}\right)$ of each vertex $v_{i} \in S_{k-1}$ belongs to set $W\left(v_{i}\right)$.

Now, we show how to construct $f_{k}$ and $W\left(v_{k}\right)$ such that the above five conditions hold. If $v_{k}$ has no neighbours in $S_{k-1}$, then set $f_{k}=f_{k-1}$, define $W\left(v_{k}\right)$ as it was done for $W\left(v_{1}\right)$, and we are done. Then, suppose that $v_{k}$ has neighbours in $S_{k-1}$. For each edge $v_{i} v_{k} \in E(G)$, with $i<k$, recalling that $C_{f_{k-1}}\left(v_{i}\right) \in W\left(v_{i}\right)=\left\{c_{i}, c_{i}+2\right\}$, we have the following three possibilities for $f_{k}\left(v_{i} v_{k}\right)$ :
(a) if $C_{f_{k-1}}\left(v_{i}\right)=c_{i}$, increase $f_{k-1}\left(v_{i} v_{k}\right)$ by 2 - defining $f_{k}$ in this way yields $C_{f_{k}}\left(v_{i}\right)=$ $c_{i}+2 \in W\left(v_{i}\right) ;$
(b) if $C_{f_{k-1}}\left(v_{i}\right)=c_{i}+2$, decrease $f_{k-1}\left(v_{i} v_{k}\right)$ by 2 - this yields $C_{f_{k}}\left(v_{i}\right)=c_{i} \in W\left(v_{i}\right)$;
(c) leave the label of $v_{i} v_{k}$ unchanged, that is, $f_{k}\left(v_{i} v_{k}\right)=f_{k-1}\left(v_{i} v_{k}\right)$.

Let $j=\min \left\{i: k+1 \leq i \leq n\right.$ and $\left.v_{k} v_{i} \in E(G)\right\}$. For $f_{k}\left(v_{k} v_{j}\right)$ we may alter $f_{k-1}\left(v_{k} v_{j}\right)$ by $\pm 1$. By the construction, $f_{k-1}\left(v_{k} v_{j}\right)=3$. As we see later in this proof, we only increase this value by one if $C_{f_{k}}\left(v_{k}\right)=c_{i}+2$ and only decrease this value by one if $C_{f_{k}}\left(v_{k}\right)=c_{i}$, so as to guarantee that condition (V) holds for $\left(f_{k}, C_{f_{k}}\right)$.

We claim that there exists an adjustment of labels as previously defined, that results in a colour $C_{f_{k}}\left(v_{k}\right)$ for $v_{k}$ and a set $W\left(v_{k}\right)$ such that $C_{f_{k}}\left(v_{k}\right) \in W\left(v_{k}\right)$ and $W\left(v_{k}\right) \cap W\left(v_{i}\right)=\emptyset$, for all $v_{i} \in N_{G}\left(v_{k}\right) \cap S_{k-1}$.

Let $d=\left|N_{G}\left(v_{k}\right) \cap S_{k-1}\right|, d_{1}=\left|\left\{v \in N_{G}\left(v_{k}\right) \cap S_{k-1}: C_{f_{k-1}}(v)=c_{i}\right\}\right|$ and $d_{2}=d-d_{1}$. By (a) and (b), at most $d_{1}$ of the edges incident with $v_{k}$ can have their labels increased by two and at most $d_{2}$ of them can have their labels decreased by two. Suppose there were $t$ increments and $s$ decrements in edges linking $v_{k}$ to vertices in $S_{k-1}$. Therefore, the colour of $v_{k}$ is $C_{f_{k-1}}\left(v_{k}\right)+2 t-2 s$. Since $0 \leq t \leq d_{1}$ and $0 \leq s \leq d_{2}$, there are $d_{1}+d_{2}+1=d+1$ different possible values for $C_{f_{k}}\left(v_{k}\right)$ and these values have the same parity. In addition, remember that we also allow label $f_{k-1}\left(v_{k} v_{j}\right)$ to be increased or decreased by one, for one edge $v_{k} v_{j} \in E(G)$ with $j$ previously defined. We conclude that $C_{f_{k}}\left(v_{k}\right)$ can take all integer values contained in $\mathcal{S}=\left[C_{f_{k-1}}\left(v_{k}\right)-2 d_{2}-1, C_{f_{k-1}}\left(v_{k}\right)+2 d_{1}+1\right]$. Note that this set has exactly $2 d+3$ elements.

We claim that at least one of the $2 d+3$ colours of $\mathcal{S}$ is available for $C_{f_{k}}\left(v_{k}\right)$. By the induction hypothesis, for $v_{i} v_{\ell} \in E(G)$ with $i, \ell<k, W\left(v_{i}\right) \cap W\left(v_{\ell}\right)=\emptyset$. This implies that at most $2 d$ colours of $\mathcal{S}$ appear in sets $W\left(v_{i}\right)$ of neighbours of $v_{k}$ and are not available for $C_{f_{k}}\left(v_{k}\right)$. At this point, three colours of $S$ remain available for $C_{f_{k}}\left(v_{k}\right)$. Consider $j=$ $\min \left\{i: k+1 \leq i \leq n\right.$ and $\left.v_{k} v_{i} \in E(G)\right\}$. The minimum colour for $v_{k}, C_{f_{k-1}}\left(v_{k}\right)-2 d_{2}-1$, can only be obtained when $f_{k}\left(v_{k} v_{j}\right)=2$, that has some restriction imposed by condition $(\mathrm{V})$, as previously mentioned. The same is true for colour $C_{f_{k-1}}\left(v_{k}\right)+2 d_{1}+1$, obtained only when $f_{k}\left(v_{k} v_{j}\right)=4$. Therefore, these two colours may be not available for $C_{f_{k}}\left(v_{k}\right)$. On the other hand, note that any value in $\mathcal{S}$ that is different from $C_{f_{k-1}}\left(v_{k}\right)-2 d_{2}-1$ and $C_{f_{k-1}}\left(v_{k}\right)+2 d_{1}+1$ and that is obtained with $f_{k}\left(v_{k} v_{j}\right) \neq 3$, can be obtained with $f_{k}\left(v_{k} v_{j}\right)=2$ or $f_{k}\left(v_{k} v_{j}\right)=4$ (that is, it can be obtained in two ways). These facts imply that at most $2 d+2$ values of $\mathcal{S}$ may not be available for $C_{f_{k}}\left(v_{k}\right)$. However, since
$|\mathcal{S}|=2 d+3$, at least one value $x \in \mathcal{S}$, with $C_{f_{k-1}}\left(v_{k}\right)-2 d_{2}-1<x<C_{f_{k-1}}\left(v_{k}\right)+2 d_{1}+1$, remains free for $C_{f_{k}}\left(v_{k}\right)$. Define $C_{f_{k}}\left(v_{k}\right)=x$.

In the following, we prove that there exists $c_{k} \equiv 0,1(\bmod 4)$ such that $C_{f_{k}}\left(v_{k}\right) \in$ $W\left(v_{k}\right)=\left\{c_{k}, c_{k}+2\right\}$ and $W\left(v_{k}\right) \cap W\left(v_{i}\right)=\emptyset$, for all $v_{i} \in N_{G}\left(v_{k}\right) \cap S_{k-1}$. If $C_{f_{k}}\left(v_{k}\right) \equiv 0,1$ $(\bmod 4)$, then define $c_{k}=C_{f_{k}}\left(v_{k}\right)$, which implies that $W\left(v_{k}\right)=\left\{C_{f_{k}}\left(v_{k}\right), C_{f_{k}}\left(v_{k}\right)+2\right\}$. By the construction of $f_{k},\left\{C_{f_{k}}\left(v_{k}\right)\right\} \cap W\left(v_{i}\right)=\emptyset$, for $v_{i} \in S_{k-1} \cap N_{G}\left(v_{k}\right)$. Therefore, $W\left(v_{k}\right) \cap W\left(v_{i}\right)=\emptyset$. If $C_{f_{k}}\left(v_{k}\right) \equiv 2,3(\bmod 4)$, then define $c_{k}=C_{f_{k}}\left(v_{k}\right)-2 \equiv 0,1$ $(\bmod 4)$. By the same reasoning of the previous case we conclude that $W\left(v_{k}\right) \cap W\left(v_{i}\right)=\emptyset$ for $v_{i} \in S_{k-1} \cap N_{G}\left(v_{k}\right)$. Therefore, we can define $W\left(v_{k}\right)$ and $f_{k}$ with the required properties for $k \leq n-1$.

It remains to consider vertex $v_{n}$ and define $\left(f_{n}, C_{f_{n}}\right)$. For $v_{i} v_{n} \in E(G)$ we can either add or subtract 2 to or from $f_{n-1}\left(v_{i} v_{n}\right)$ while keeping $C_{f_{n}}\left(v_{i}\right) \in W\left(v_{i}\right)$. These possible adjustments give a total of $d_{G}\left(v_{n}\right)+1 \geq 3$ options for $C_{f_{n}}\left(v_{n}\right)$. Let $V_{0}=\left\{v_{i} \in\right.$ $\left.N_{G}\left(v_{n}\right): C_{f_{n-1}}\left(v_{i}\right)=c_{i}\right\}$ and $V_{+2}=\left\{v_{i} \in N_{G}\left(v_{n}\right): C_{f_{n-1}}\left(v_{i}\right)=c_{i}+2\right\}$. The minimum value for $C_{f_{n}}\left(v_{n}\right)$ is $C_{f_{n-1}}\left(v_{n}\right)-2\left|V_{+2}\right|$, which is obtained decreasing by two the label of every edge $v_{i} v_{n}$ for $v_{i} \in V_{+2}$. Let $a=C_{f_{n-1}}\left(v_{n}\right)-2\left|V_{+2}\right|$.

Case 1. $a \equiv 2,3(\bmod 4)$.
In this case, for every $v_{i} \in V_{+2}$, decrease the label of $v_{i} v_{n}$ by two. This yields a proper-vertex-colouring $C_{f_{n}}$ of $G$ in which: (i) $C_{f_{n}}\left(v_{n}\right)=a \equiv 2,3(\bmod 4)$; and (ii) for all $v_{i} \in N_{G}\left(v_{n}\right), C_{f_{n}}\left(v_{i}\right)=c_{i} \equiv 0,1(\bmod 4)$.

Case 2. $a \equiv 0,1(\bmod 4)$ and there exists $v_{i} \in N_{G}\left(v_{n}\right)$ such that $a \neq c_{i} \in W\left(v_{i}\right)$.
If $v_{i} \in V_{+2}$, then $C_{f_{n}}\left(v_{i}\right)=C_{f_{n-1}}\left(v_{i}\right)=c_{i}+2$. Otherwise, increase the label of edge $v_{i} v_{n}$ by two so as to obtain $C_{f_{n}}\left(v_{i}\right)=C_{f_{n-1}}\left(v_{i}\right)+2=c_{i}+2$. For every $v \in V_{+2} \backslash\left\{v_{i}\right\}$, decrease the label of edge $v v_{n}$ by two. Note that $C_{f_{n}}(v)=C_{f_{n-1}}(v)-2=c_{i} \equiv 0,1$ $(\bmod 4)$. After these adjustments, $C_{f_{n}}\left(v_{n}\right)=a+2 \equiv 2,3(\bmod 4)$. Moreover, $C_{f_{n}}\left(v_{n}\right)=$ $a+2 \neq c_{i}+2=C_{f_{n}}\left(v_{i}\right)$. Therefore, $C_{f_{n}}$ is a proper-vertex-colouring of $G$.

Case 3. $a \equiv 0,1(\bmod 4)$ and $a=c_{i} \in W\left(v_{i}\right)$ for all $v_{i} \in N_{G}\left(v_{n}\right)$.
Let $v_{k}, v_{\ell} \in N_{G}\left(v_{n}\right)$ and define the final colours $C_{f_{n}}\left(v_{k}\right)$ and $C_{f_{n}}\left(v_{\ell}\right)$ of these vertices as done for vertex $v_{i}$ in the previous case. For the vertices in $V_{+2} \backslash\left\{v_{k}, v_{\ell}\right\}$, again, proceed as in the previous case for the vertices in $V_{+2} \backslash\left\{v_{i}\right\}$. After these adjustments, $C_{f_{n}}\left(v_{n}\right)=a+4>$ $a+2 \geq C_{f_{n}}(v)$ for every $v \in N_{G}\left(v_{n}\right)$. Therefore, $\left(f_{n}, C_{f_{n}}\right)$ is a neighbour-distinguishing [5]-edge-labelling of $G$.

The technique presented in the proof of Theorem 5.7 inspired some new results toward a solution to the $1,2,3$-Conjecture. For example, in 2016, Gao et al. [50] used a similar technique in order to prove that the $1,2,3$-Conjecture is true if the condition of being a proper-vertex-colouring is relaxed to allow colour classes to induce forests. Furthermore, in 2017, Bensmail [16] used a similar technique to show that every 5 -regular graph $G$ has $\chi_{\Sigma}^{\prime}(G) \leq 4$.

### 5.2 Known results for some families of graphs

We begin this section presenting general results and observations concerning neighbourdistinguishing $\mathcal{L}$-edge-labellings of simple graphs, for any finite set of real numbers $\mathcal{L}$. Later, we present some known results on neighbour-distinguishing edge-labellings and neighbour-distinguishing total-labellings for some classic families of graphs, such as bipartite graphs, cycles and complete graphs.

Khatirinejad et al. [66] observed that the existence of neighbour-distinguishing $\mathcal{L}$-edgelabellings of a graph depends not only on the size of $\mathcal{L}$ but also on the particular elements of $\mathcal{L}$. For example, the graph in Figure 5.5 is shown with a neighbour-distinguishing [2]-edgelabelling but it is not difficult to verify that it does not admit neighbour-distinguishing $\{0,1\}$-edge-labellings (a generalization of this fact is proved in Theorem 6.24).


Figure 5.5: A graph with a neighbour-distinguishing [2]-edge-labelling.
It is natural to think about operations that allow us to turn a neighbour-distinguishing $\{a, b\}$-edge-labelling of a graph $G$ into another with distinct edge labels. A possible operation could be to increase all edge labels of $G$ by a fixed real number $t, t \neq 0$. However, as illustrated in Figure 5.6, depending on the graph and on its neighbour-distinguishing $\{a, b\}$-edge-labelling, this operation may not result in another neighbour-distinguishing edge-labelling.

Another modification on the edge labels one may think of is defined as follows. Let $\mathcal{L}$ and $\mathcal{L}^{\prime}$ be two distinct sets of real numbers such that $|\mathcal{L}|=\left|\mathcal{L}^{\prime}\right| \geq 2$ and let $\varphi: \mathcal{L} \rightarrow \mathcal{L}^{\prime}$ be a bijection. Given a graph $G$ with a neighbour-distinguishing $\mathcal{L}$-edge-labelling $\left(\pi, C_{\pi}\right)$, we can use $\pi$ and $\varphi$ to obtain an $\mathcal{L}^{\prime}$-edge-labelling $\pi^{\prime}$ of $G$ by defining $\pi^{\prime}(u v)=\varphi(\pi(u v))$, for every $u v \in E(G)$. However, it may occur that $C_{\pi^{\prime}}$ is not a proper-vertex-colouring of $G$, as illustrated in Figure 5.7. Despite of this fact, the use of bijections $\varphi: \mathcal{L} \rightarrow \mathcal{L}^{\prime}$ always works if the graph under consideration is regular and $|\mathcal{L}|=\left|\mathcal{L}^{\prime}\right|=2$, as shown in the proof of Lemma 5.8.

Lemma 5.8 (Khatirinejad et al. [66], Hulgan et al. [60]). Let $G$ be a $k$-regular graph. If $G$ has a neighbour-distinguishing \{a,b\}-edge-labelling (neighbour-distinguishing \{a,b\}-totallabelling) for fixed distinct real numbers $a$ and $b$, then $G$ has a neighbour-distinguishing $\left\{a^{\prime}, b^{\prime}\right\}$-edge-labelling (neighbour-distinguishing $\left\{a^{\prime}, b^{\prime}\right\}$-total-labelling) for any two distinct real numbers $a^{\prime}$ and $b^{\prime}$.


Figure 5.6: A neighbour-distinguishing $\{3,8\}$-edge-labelling $\pi$ of a graph $G$ and a $\{5,10\}$ -edge-labelling $\pi^{\prime}$ obtained from $\pi$ by increasing all edge labels of $G$ by 2. Note that $\left(\pi^{\prime}, C_{\pi^{\prime}}\right)$ is not neighbour-distinguishing since two adjacent vertices have colour 25 under $C_{\pi^{\prime}}$.

(a) Graph $G$ with a neighbour-distinguishing $\{1,2\}$-edge-labelling $\left(\pi, C_{\pi}\right)$.

(c) Graph $G$ with a $\{-1,1\}$-edge-labelling $\pi_{2}$ obtained from $\pi$ by defining $\pi_{2}(u v)=\varphi(\pi(u v))$ for all $u v \in E(G)$, where $\varphi(1)=1$ and $\varphi(2)=$ -1 . $\left(\pi_{2}, C_{\pi_{2}}\right)$ is not neighbour-distinguishing.

(b) Graph $G$ with a $\{-1,1\}$-edge-labelling $\pi_{1}$ obtained from $\pi$ by defining $\pi_{1}(u v)=\varphi(\pi(u v))$ for all $u v \in E(G)$, where $\varphi(1)=-1$ and $\varphi(2)=$ 1. $\left(\pi_{1}, C_{\pi_{1}}\right)$ is not neighbour-distinguishing.

(d) Graph $G$ with a neighbour-distinguishing $\{-1,1\}$-edge-labelling $\left(\pi^{\prime}, C_{\pi^{\prime}}\right)$.

Figure 5.7: Example showing that a neighbour-distinguishing $\{1,2\}$-edge-labelling of a graph may not be transformed into a neighbour-distinguishing $\{-1,1\}$-edge-labelling by simply applying a bijection between the two sets of labels $\{1,2\}$ and $\{-1,1\}$.

Proof. Let $G$ be a $k$-regular graph and let $a$ and $b$ be fixed distinct real numbers. Let $\left(\pi, C_{\pi}\right)$ be a neighbour-distinguishing $\{a, b\}$-edge-labelling of $G$. Since every vertex of $G$ has degree $k$, for each $u \in V(G)$, the colour of $u$ is $C_{\pi}(u)=t a+(k-t) b$, for some $t \in[0, k]$. Note that the colour of each vertex of $G$ is in a one-to-one correspondence with the number of incident edges with label $a$. Thus, if a specific choice of $a$ and $b$ gives a proper-vertex-colouring of $G$, then changing all edge labels from $a$ to $a^{\prime}$ and from $b$ to $b^{\prime}$ yields a neighbour-distinguishing $\left\{a^{\prime}, b^{\prime}\right\}$-edge-labelling of $G$. In order to see this, take an
$\left\{a^{\prime}, b^{\prime}\right\}$-edge-labelling $\pi_{2}$ of $G$ as previously defined, that is, for each edge $u v \in E(G)$,

$$
\pi_{2}(u v)= \begin{cases}a^{\prime}, & \text { if } \pi(u v)=a \\ b^{\prime}, & \text { if } \pi(u v)=b\end{cases}
$$

where $a^{\prime}$ and $b^{\prime}$ are any two distinct real numbers. Suppose that there exists an edge $u v \in$ $E(G)$ with $C_{\pi_{2}}(u)=C_{\pi_{2}}(v)$. Thus, $C_{\pi_{2}}(u)=t_{1} a^{\prime}+\left(k-t_{1}\right) b^{\prime}=t_{2} a^{\prime}+\left(k-t_{2}\right) b^{\prime}=C_{\pi_{2}}(v)$, for $t_{1}, t_{2} \in[0, k]$. By rearranging the equation $t_{1} a^{\prime}+\left(k-t_{1}\right) b^{\prime}=t_{2} a^{\prime}+\left(k-t_{2}\right) b^{\prime}$, we obtain that $\left(t_{2}-t_{1}\right)\left(b^{\prime}-a^{\prime}\right)=0$. This last equation implies that $t_{1}=t_{2}$ since $a^{\prime} \neq b^{\prime}$. However, this contradicts the fact that $C_{\pi}(u)=t_{1} a+\left(k-t_{1}\right) b \neq t_{2} a+\left(k-t_{2}\right) b=C_{\pi}(v)$. The proof considering neighbour-distinguishing $\{a, b\}$-total-labellings is analogous.

Given a neighbour-distinguishing $\{a, b\}$-edge-labelling or a neighbour-distinguishing $\{a, b\}$-total-labelling of a graph $G$, another possible operation is to multiply all the labels by a fixed real number $t, t \neq 0$. By Lemma 5.9, the labelling resulting from this operation is neighbour-distinguishing.

Lemma 5.9 (Khatirinejad et al. [66], Hulgan et al. [60]). Let $\mathcal{L} \subset \mathbb{R}$ be a nonempty set and let $t \in \mathbb{R}$ with $t \neq 0$. If a simple graph $G$ has a neighbour-distinguishing $\mathcal{L}$-edge-labelling (neighbour-distinguishing $\mathcal{L}$-total-labelling), then $G$ has a neighbour-distinguishing $\mathcal{L}^{\prime}$ -edge-labelling (neighbour-distinguishing $\mathcal{L}^{\prime}$-total-labelling), where $\mathcal{L}^{\prime}=\{$ at: $a \in \mathcal{L}\}$.

Proof. Let $t, \mathcal{L}$ and $\mathcal{L}^{\prime}$ be as stated in the hypothesis. Let $G$ be a simple graph with a neighbour-distinguishing $\mathcal{L}$-edge-labelling $\left(\pi, C_{\pi}\right)$. Let $\pi^{\prime}$ be an $\mathcal{L}^{\prime}$-edge-labelling of $G$ defined by $\pi^{\prime}(u v)=\pi(u v) \cdot t$, for each edge $u v \in E(G)$. By the definition of $\pi^{\prime}, C_{\pi^{\prime}}(u)=$ $C_{\pi}(u) \cdot t$, for every vertex $u \in V(G)$. Moreover, for each edge $u v \in E(G), C_{\pi^{\prime}}(u)=$ $C_{\pi}(u) \cdot t \neq C_{\pi}(v) \cdot t=C_{\pi^{\prime}}(v)$ since $C_{\pi}(u) \neq C_{\pi}(v)$. Therefore, $\left(\pi^{\prime}, C_{\pi^{\prime}}\right)$ is a neighbourdistinguishing $\mathcal{L}^{\prime}$-edge-labelling of $G$. By a similar reasoning, we obtain a neighbourdistinguishing $\mathcal{L}^{\prime}$-total-labelling from a neighbour-distinguishing $\mathcal{L}$-total-labelling.

As discussed in the introduction of this chapter, there are graphs that do not have neighbour-distinguishing edge-labellings with less than three edge labels [30,65]. This is the case, for example, of all complete graphs (see Corollary 5.11). Hence, the general upper bound on $\chi_{\Sigma}^{\prime}(G)$ proposed by the 1,2,3-Conjecture is tight. However, Addario-Berry et al. [8] proved that almost all graphs $G$ have $\chi_{\Sigma}^{\prime}(G) \leq 2$. Despite these results, Dudek and Wajc [39] proved that deciding whether an arbitrary graph $G$ without connected components isomorphic to $K_{2}$ has a neighbour-distinguishing [2]-edge-labelling is $\mathcal{N P}$ complete. Later, Bensmail [15] proved that deciding whether an arbitrary graph has a neighbour-distinguishing $\{a, b\}$-edge-labelling is $\mathcal{N} \mathcal{P}$-complete for every pair of distinct real numbers $a$ and $b$.

In view of these results, an interesting line of research consists of determining $\chi_{\Sigma}^{\prime}(G)$ for classes of graphs or, even further, investigating the structural conditions that require the presence of a third edge label so as to obtain a neighbour-distinguishing edge-labelling for the graph. One of the first works in these directions is due to Chang et al. [30], who determined $\chi_{\Sigma}^{\prime}(G)$ for cycles, complete graphs, complete bipartite graphs, trees, and posed
the problem of characterizing the bipartite graphs that have neighbour-distinguishing [2]-edge-labellings. Since then, some authors $[37,66,75,76]$ tried to solve this problem and, in 2016, Thomassen, Wu and Zhang [112] completely characterized the bipartite graphs without connected components isomorphic to $K_{2}$ that have $\chi_{\Sigma}^{\prime}(G)=3$. The family $\mathcal{T}$ of all bipartite simple graphs $G$ with $\chi_{\Sigma}^{\prime}(G)=3$ is recursively defined by the following two rules:
(i) $\mathcal{T}$ contains all cycles $C_{4 k+2}, k \geq 1$. We represent cycle $C_{4 k+2}$ with a fixed perfect matching $M$ by $\left(C_{4 k+2} ; M\right)$;
(ii) if $(G ; M) \in \mathcal{T}$, then $\left(G^{\prime} ; M^{\prime}\right) \in \mathcal{T}$, where:
(a) $G^{\prime}$ is obtained from a path $P_{4 r+2}=\left(v_{0}, \ldots, v_{4 r+1}\right), r \geq 1$, disjoint from $G$, by identifying $v_{0}$ with $x$ and $v_{4 r+1}$ with $y$ for some edge $x y \in M$;
(b) $M^{\prime}=M \cup\left\{v_{2 i-1} v_{2 i}: 1 \leq i \leq 2 r\right\}$.

Figure 5.8 illustrates some bipartite graphs that belong to family $\mathcal{T}$.


Figure 5.8: Three connected bipartite graphs that do not have neighbour-distinguishing [2]-edge-labellings. Bold edges form a perfect matching in each graph.

In fact, Davoodi and Omoomi [37] proved that all graphs in $\mathcal{T}$ do not have neighbourdistinguishing $\{a, b\}$-edge-labellings, for any two distinct real numbers $a$ and $b$. However, it is still an open question if every bipartite graph that does not belong to $\mathcal{T}$ does have a neighbour-distinguishing $\{a, b\}$-edge-labelling, for every two distinct $a, b \in \mathbb{R}$ (Note that Thomassen et al.'s characterization refers to neighbour-distinguishing \{1, 2\}-edgelabellings).

Another family of graphs for which neighbour-distinguishing $\{a, b\}$-edge-labellings have been investigated is the family of complete graphs. In fact, the complete graphs have no neighbour-distinguishing $\{a, b\}$-edge-labelling, for any pair of distinct real numbers $a$ and $b$. This result is presented in Corollary 5.11 and its proof uses Lemma 5.10.

Lemma 5.10 (Khatirinejad et al. [66]). Let $G$ be a $k$-regular graph and let $a, b \in \mathbb{R}$, with $a \neq b$. If $G$ has a neighbour-distinguishing $\{a, b\}$-edge-labelling, then $G$ is $k$-colourable.

Proof. Let $G$ be a $k$-regular graph with a neighbour-distinguishing $\{a, b\}$-edge-labelling $\left(\pi, C_{\pi}\right)$. Thus, each vertex $v \in V(G)$ has colour $C_{\pi}(v)=t_{v} a+\left(k-t_{v}\right) b$, for some $t_{v} \in[0, k]$. This implies that there are at most $k+1$ colours occurring in the vertices of $G$ : the colours in the set $\{a i+b(k-i): 0 \leq i \leq k, i \in \mathbb{Z}\}$. If less than $k+1$ colours occur in the vertices
of $G$, then we are done. Thus, suppose that $C_{\pi}$ is a proper $(k+1)$-vertex-colouring of $G$. Note that a vertex with colour $a k$ cannot be adjacent to a vertex with colour $b k$. Thus, putting the vertices with colours $a k$ and $b k$ in the same colour class results in a proper $k$-vertex-colouring of $G$.

Corollary 5.11 (Khatirinejad et al. [66]). Let $a, b \in \mathbb{R}$, with $a \neq b$. If $G$ is an odd cycle or a complete graph, then $G$ does not have a neighbour-distinguishing $\{a, b\}$-edge-labelling.

Proof. Let $G$ be an odd cycle or a complete graph. Note that $\chi(G)=\Delta(G)+1$. Therefore, the result follows by the contrapositive of Lemma 5.10.

Even though complete graphs do not have neighbour-distinguishing edge-labellings with two edge labels, they admit an $\{a, b\}$-edge-labelling that is very close to being neighbour-distinguishing. This specific edge-labelling is used in the proof of some results in Chapter 6.

Lemma 5.12 (Khatirinejad et al. [66]). Given $n \geq 2$ and $a, b \in \mathbb{R}, a<b$, there exist two $\{a, b\}$-edge-labellings of $K_{n}$, such that the colours of all but two vertices are distinct. Furthermore, for any such $\{a, b\}$-edge-labelling, the subgraph induced by the edges with one of the labels has degree sequence $\left(1, \ldots,\left\lfloor\frac{n}{2}\right\rfloor,\left\lfloor\frac{n}{2}\right\rfloor, \ldots, n-1\right)$ and the subgraph induced by the edges with the other label has degree sequence $\left(0, \ldots,\left\lceil\frac{n}{2}\right\rceil-1,\left\lceil\frac{n}{2}\right\rceil-1, \ldots, n-2\right)$.

Proof. First, we prove, by induction on $n$, that $K_{n}$ has two $\{a, b\}$-edge-labellings $\pi_{1}$ and $\pi_{2}$ such that the colours of all but two vertices are distinct and such that $a(n-1) \leq$ $C_{\pi_{1}}(v) \leq a+b(n-2)$ and $a(n-2)+b \leq C_{\pi_{2}}(v) \leq b(n-1)$, for all $v \in V\left(K_{n}\right)$. For $K_{2}$ the required edge-labellings $\pi_{1}$ and $\pi_{2}$ are obtained by assigning either label $a$ or label $b$ to its unique edge.

Now, consider $K_{n}$, with $n \geq 3$, and let $G^{\prime}=K_{n}-v, v \in V\left(K_{n}\right)$. By the induction hypothesis, $G^{\prime}$ has two $\{a, b\}$-edge-labellings $\pi_{1}^{\prime}$ and $\pi_{2}^{\prime}$ such that the colours of all but two vertices are distinct and such that $a(n-2) \leq C_{\pi_{1}^{\prime}}(w) \leq a+b(n-3)$ and $a(n-$ $3)+b \leq C_{\pi_{2}^{\prime}}(w) \leq b(n-2)$, for all $w \in V\left(G^{\prime}\right)$. We construct the required $\{a, b\}$-edgelabelling $\pi_{1}$ for $K_{n}$ by assigning label $a$ to the edges incident with $v$ and assigning $\pi_{2}^{\prime}$ to $G^{\prime} \subset K_{n}$. Since $v$ is adjacent to all vertices of $G^{\prime}$ and its incident edges all have label $a$, we have that $C_{\pi_{1}}(w)=C_{\pi_{2}^{\prime}}(w)+a$, for all $w \in V\left(G^{\prime}\right)$. This implies that $a(n-2)+b \leq$ $C_{\pi_{1}}(w) \leq a+b(n-2)$, for all $w \in V\left(G^{\prime}\right)$, and only two vertices of $G^{\prime}$ have the same colour under $C_{\pi_{1}}$. Since $C_{\pi_{1}}(v)=a(n-1)$ and $a(n-1)<a(n-2)+b$, we conclude that $a(n-1) \leq C_{\pi_{1}}(w) \leq a+b(n-2)$, for all $w \in V\left(K_{n}\right)$, as required. Now, we construct the required $\{a, b\}$-edge-labelling $\pi_{2}$ for $K_{n}$ by assigning label $b$ to the edges incident with $v$ and assigning $\pi_{1}^{\prime}$ to $G^{\prime} \subset K_{n}$. Since $v$ is adjacent to all vertices of $G^{\prime}$ and its incident edges all have label $b$, we have that $C_{\pi_{2}}(w)=C_{\pi_{1}^{\prime}}(w)+b$, for all $w \in V\left(G^{\prime}\right)$. This implies that $a(n-2)+b \leq C_{\pi_{2}}(v) \leq a+b(n-2)$, for all $w \in V\left(G^{\prime}\right)$. Since $C_{\pi_{2}}(v)=b(n-1)$ and $a+b(n-2)<b(n-1)$, we conclude that $a(n-2)+b \leq C_{\pi_{2}}(w) \leq b(n-1)$, for all $w \in V\left(K_{n}\right)$, and the result follows.

Next, we prove the second part of the lemma also by induction on $n$. Let $\pi$ be an $\{a, b\}$-edge-labelling of $K_{n}$ such that the colours of all but two vertices are distinct. Note that the colour of each vertex of $K_{n}$ is a real number $a(n-1-i)+i b$ for some positive
integer $i \in[0, n-1]$. It is not difficult to verify the claim to $K_{2}$ and $K_{3}$. Thus, consider $K_{n}$ with $n \geq 4$ and let $u, v \in V\left(K_{n}\right)$ such that $C_{\pi}(u)=C_{\pi}(v)$.

If, for every vertex $w \in V\left(K_{n}\right), C_{\pi}(w) \notin\{a(n-1), b(n-1)\}$, then $C_{\pi}(w)$ can only take $n-2$ values, which is a contradiction to the choice of $\pi$. If $C_{\pi}(u)=C_{\pi}(v) \in$ $\{a(n-1), b(n-1)\}$, then, by removing $u$ and $v$, the $\{a, b\}$-edge-labelling $\pi$ restricted to $K_{n}-\{u, v\}$ is a neighbour-distinguishing $\{a, b\}$-edge-labelling of $K_{n}-\{u, v\} \cong K_{n-2}$, which is a contradiction to the fact that complete graphs have no neighbour-distinguishing $\{a, b\}$-edge-labelling. Therefore, there exists a vertex $w \in V\left(K_{n}\right)$, such that $w \neq u, w \neq v$ and $C_{\pi}(w) \in\{a(n-1), b(n-1)\}$. Next, we analyse these two possible values for $C_{\pi}(w)$.

First, consider $C_{\pi}(w)=a(n-1)$. Let $\pi^{\prime}$ be the restriction of $\pi$ to the edges of $K_{n}-w$. By the induction hypothesis, there are only two possibilities for the degree sequence of the subgraph of $K_{n}-w$ induced by edges with label $a$ : it is either $\left(1, \ldots,\left\lfloor\frac{n-1}{2}\right\rfloor,\left\lfloor\frac{n-1}{2}\right\rfloor, \ldots, n-\right.$ $2)$ or $\left(0, \ldots,\left\lceil\frac{n-1}{2}\right\rceil-1,\left\lceil\frac{n-1}{2}\right\rceil-1, \ldots, n-3\right)$ (Note that since $K_{n}-w$ is an $(n-1)$-regular graph, each one of these possibilities immediately determines the degree sequence of the subgraph of $K_{n}-w$ induced by edges with label $b$ ). However, since $w$ is the unique vertex in $K_{n}$ with colour $a(n-1)$, the degree sequence of the subgraph of $K_{n}-w$ induced by the edges with label $a$ is $\left(0, \ldots,\left\lceil\frac{n-1}{2}\right\rceil-1,\left\lceil\frac{n-1}{2}\right\rceil-1, \ldots, n-3\right)$. This implies that the subgraph of $K_{n}-w$ induced by the edges with label $b$ is $\left(1, \ldots,\left\lfloor\frac{n-1}{2}\right\rfloor,\left\lfloor\frac{n-1}{2}\right\rfloor, \ldots, n-2\right)$. Therefore, by adding $w$ and all of its incident edges labelled with $a$ to $K_{n}-w$, we obtain the graph $K_{n}$ and we have that the degree sequence of the subgraph of $K_{n}$ induced by edges with label $a$ is $\left(1, \ldots,\left\lfloor\frac{n}{2}\right\rfloor,\left\lfloor\frac{n}{2}\right\rfloor, \ldots, n-2, n-1\right)$ - observe that $\left\lceil\frac{n-1}{2}\right\rceil=\left\lfloor\frac{n}{2}\right\rfloor$. Moreover, the degree sequence of the subgraph of $K_{n}$ induced by edges with label $b$ is $\left(0,1, \ldots,\left\lceil\frac{n}{2}\right\rceil-1,\left\lceil\frac{n}{2}\right\rceil-1, \ldots, n-2\right)-$ also observe that $\left\lfloor\frac{n-1}{2}\right\rfloor=\left\lceil\frac{n}{2}\right\rceil-1$.

The remaining case $C_{\pi}(w)=b(n-1)$ is analogous to the previous one.
As a consequence of Lemma 5.12, we obtain the following corollary.
Corollary 5.13. Let $a, b \in \mathbb{R}, a<b$. Let $K_{n}$ be a complete graph with vertex set $V\left(K_{n}\right)=\left\{v_{1}, \ldots, v_{n}\right\}, n \geq 2$, and $\left(\pi, C_{\pi}\right)$ an $\{a, b\}$-edge-labelling of $K_{n}$ such that the colours of all but two vertices are distinct. Then, it is possible to adjust notation such that the colours of the vertices of $K_{n}$ under $\left(\pi, C_{\pi}\right)$ are either

$$
C_{\pi}\left(v_{i}\right)= \begin{cases}a(n-1-i)+b i, & \text { for } 1 \leq i \leq\left\lfloor\frac{n}{2}\right\rfloor \\ a(n-i)+b(i-1), & \text { for }\left\lfloor\frac{n}{2}\right\rfloor+1 \leq i \leq n\end{cases}
$$

or

$$
C_{\pi}\left(v_{i}\right)= \begin{cases}a(n-i)+b(i-1), & \text { for } 1 \leq i \leq\left\lceil\frac{n}{2}\right\rceil \\ a(n-i+1)+b(i-2), & \text { for }\left\lceil\frac{n}{2}\right\rceil+1 \leq i \leq n\end{cases}
$$

Despite the fact that there are no neighbour-distinguishing $\{a, b\}$-edge-labellings of $K_{n}$, all complete graphs with at least three vertices have neighbour-distinguishing $\{a, b, c\}$ -edge-labellings, for any three distinct $a, b, c \in \mathbb{R}$, as shown in Theorem 5.14.

Theorem 5.14. Let $K_{n}$ be a complete graph, with vertex set $V\left(K_{n}\right)=\left\{v_{1} \ldots, v_{n}\right\}$, and $a, b, c \in \mathbb{R}$, with $a<b<c$. Then, the following statements are true:
(i) $K_{3}$ has a neighbour-distinguishing $\{a, b, c\}$-edge-labelling $\left(\pi_{1}, C_{\pi_{1}}\right)$ such that $\mathcal{C}_{\pi_{1}}\left(K_{3}\right)=$ $\{a+b, a+c, b+c\} ;$
(ii) $K_{4}$ has two neighbour-distinguishing $\{a, b, c\}$-edge-labellings $\left(\pi_{1}, C_{\pi_{1}}\right)$ and $\left(\pi_{2}, C_{\pi_{2}}\right)$ such that $\mathcal{C}_{\pi_{1}}\left(K_{4}\right)=\{a+b+c, a+2 c, b+2 c, 3 c\}$ and $\mathcal{C}_{\pi_{2}}\left(K_{4}\right)=\{3 a, 2 a+b, 2 a+$ $c, a+b+c\} ;$
(iii) for $n \geq 5$, $K_{n}$ has two neighbour-distinguishing $\{a, b, c\}$-edge-labellings $\left(\pi_{1}, C_{\pi_{1}}\right)$ and $\left(\pi_{2}, C_{\pi_{2}}\right)$ such that:
(a) $\min \left\{\mathcal{C}_{\pi_{1}}\left(K_{n}\right)\right\}=(n-2) a+c$ and $\max \left\{\mathcal{C}_{\pi_{1}}\left(K_{n}\right)\right\}=(n-1) c$;
(b) $\min \left\{\mathcal{C}_{\pi_{2}}\left(K_{n}\right)\right\}=(n-1) a$ and $\max \left\{\mathcal{C}_{\pi_{2}}\left(K_{n}\right)\right\}=a+(n-2) c$.

Proof. Let $K_{n}$ be the complete graph with vertex set $V\left(K_{n}\right)=\left\{v_{1}, \ldots, v_{n}\right\}$ and let $a, b, c \in \mathbb{R}$ such that $a<b<c$. The required $\{a, b, c\}$-edge-labelling $\pi_{1}$ of $K_{3}$ is obtained by assigning distinct labels $a, b, c$ to its edges. The colours of the vertices of $K_{3}$ are all distinct under $C_{\pi_{1}}$ since $a+b<a+c<b+c$.

For $t \in\{4,5\}$, we construct two $\{a, b, c\}$-edge-labellings $\pi_{1}$ and $\pi_{2}$ for $K_{t}$ as follows: edge-labelling $\pi_{1}$ is obtained by assigning label $c$ to each edge incident with vertex $v_{t}$ and assigning neighbour-distinguishing $\{a, b, c\}$-edge-labelling $\left(\pi_{2}, C_{\pi_{2}}\right)$ of $K_{t-1}$ to subgraph $G\left[\left\{v_{1}, \ldots, v_{t-1}\right\}\right]$ (when $t-1=3$ make $\pi_{2}=\pi_{1}$.) Additionally, edge-labelling $\pi_{2}$ is obtained by assigning label $a$ to each edge incident with vertex $v_{t}$ and assigning neighbourdistinguishing $\{a, b, c\}$-edge-labelling $\left(\pi_{1}, C_{\pi_{1}}\right)$ of $K_{t-1}$ to subgraph $G\left[\left\{v_{1}, \ldots, v_{t-1}\right\}\right]$. Note that the colours of the vertices of $K_{4}$ are all distinct since $a+b+c<a+2 c<b+2 c<3 c$ and $3 a<2 a+b<2 a+c<a+b+c$. Moreover, the colours of the vertices of $K_{5}$ are all distinct since $3 a+c<2 a+b+c<2 a+2 c<a+b+2 c<4 c$ and $4 a<2 a+b+c<$ $2 a+2 c<a+b+2 c<a+3 c$.

Now, consider $K_{n}$, with $n>5$. Let $G^{\prime}=K_{n}-v_{n}$ and assume that $G^{\prime}$ has two neighbour-distinguishing $\{a, b, c\}$-edge-labellings $\left(\pi_{1}^{\prime}, C_{\pi_{1}^{\prime}}\right)$ and $\left(\pi_{2}^{\prime}, C_{\pi_{2}^{\prime}}\right)$ as described in the hypothesis. In order to construct $\pi_{1}$ for $K_{n}$, assign label $c$ to each edge incident with vertex $v_{n}$ and assign $\pi_{2}^{\prime}$ to $G^{\prime}$. First, note that all vertices of $K_{n}$ are assigned different colours since $C_{\pi_{2}^{\prime}}$ is a vertex colouring of $G^{\prime}$. Also, note that $v_{n}$ has the maximum colour under $C_{\pi_{1}}$ since $C_{\pi_{1}}\left(v_{n}\right)=(n-1) c>a+(n-2) c=\max \left\{\mathcal{C}_{\pi_{2}^{\prime}}\left(G^{\prime}\right)\right\}+c$. Moreover, the minimum colour under $C_{\pi_{1}}$ is $\min \left\{\mathcal{C}_{\pi_{2}^{\prime}}\left(G^{\prime}\right)\right\}+c=(n-2) a+c$, as required. The reasoning for constructing $\pi_{2}$ for $K_{n}$ is analogous, just assigning label $a$ to each edge incident with vertex $v_{n}$ and $\left(\pi_{1}^{\prime}, C_{\pi_{1}^{\prime}}\right)$ to $G^{\prime}$.

As previously discussed, the 1,2-Conjecture and its generalization have also been investigated for classic families of graphs, such as cycles, bipartite graphs and complete graphs. Before presenting the known results on neighbour-distinguishing $\{a, b\}$-totallabellings of these families, it is opportune to observe that every neighbour-distinguishing $\{a, b\}$-edge-labelling of a graph $G$ naturally gives rise to a neighbour-distinguishing $\{a, b\}$ -total-labelling of $G$, as stated in the following proposition.

Proposition 5.15. Let $\mathcal{L} \subset \mathbb{R}$ be a nonempty set and let $a \in \mathcal{L}$. If a graph $G$ has a neighbour-distinguishing $\mathcal{L}$-edge-labelling $\pi$, then $G$ has a neighbour-distinguishing $\mathcal{L}$ -total-labelling $\pi^{\prime}$ defined by $\pi^{\prime}(v)=a$ and $\pi^{\prime}(e)=\pi(e)$ for every $v \in V(G)$ and every $e \in E(G)$.

One can use results on neighbour-distinguishing [2]-edge-labellings along with Proposition 5.15 in order to derive results on neighbour-distinguishing [2]-total-labellings of graphs. On the other hand, not all neighbour-distinguishing [2]-total-labellings can be easily transformed into a neighbour-distinguishing [2]-edge-labelling by simply dropping out the label of the vertex from the calculation of its colour. However, observe that, if all vertices have the same label, this approach always gives rise to a neighbour-distinguishing edge-labelling of the graph.
Proposition 5.16. Let $\mathcal{L} \subset \mathbb{R}$ be a nonempty set and let $a \in \mathcal{L}$. Let $\left(\pi, C_{\pi}\right)$ be a neighbour-distinguishing $\mathcal{L}$-total-labelling of a graph $G$. If $\pi(v)=a$, for every $v \in V(G)$, then $\left(\pi^{\prime}, C_{\pi^{\prime}}\right)$ such that $\pi^{\prime}(u v)=\pi(u v)$ for every $u v \in E(G)$ is a neighbour-distinguishing $\mathcal{L}$-edge-labelling of $G$.

As previously discussed, not all bipartite graphs admit neighbour-distinguishing [2]-edge-labellings. However all of them admit a neighbour-distinguishing [2]-total-labelling, as can be immediately deduced from Lemma 5.17.

Lemma 5.17 (Hulgan et al. [60]). Let $G$ be a bipartite graph and $a, b$ different nonnegative integers. Then, $G$ has a neighbour-distinguishing $\{a, b\}$-total-labelling.

Proof. Let $a, b \in \mathbb{Z}_{\geq 0}$, with $a<b$. Let $G$ be a bipartite graph with parts $V_{1}$ and $V_{2}$. If $a=0$, label the vertices of $V_{1}$ with $b$ and all other elements of $G$ with 0 . Since the vertices in $V_{1}$ have colour $b$ and the vertices in $V_{2}$ have colour 0 , this results in a proper-vertexcolouring of $G$. Now, suppose $a$ and $b$ are nonzero. Label the vertices in $V_{1}$ with $a$ and all other elements with $b$. Note that, different from the previous case, we label the vertices of $V_{1}$ with the smallest label in this case. Moreover, for $u \in V_{1}, C(u) \equiv a(\bmod b)$ and, for $v \in V_{2}, C(v) \equiv 0(\bmod b)$. Thus, for $u v \in E(G), C(u) \neq C(v)$.

A similar result is also true when considering graphs with maximum degree two.
Lemma 5.18 (Hulgan et al. [60]). Every simple graph with maximum degree two has a neighbour-distinguishing $\{a, b\}$-total-labelling for any two distinct $a, b \in \mathbb{Z}_{\geq 0}$.

Proof. Let $a, b$ be two nonnegative integers with $a<b$ and let $G$ be a simple graph with maximum degree two. We can suppose that $G$ is connected. Thus, $G$ is either a path or a cycle. By Lemma 5.17, the result is true for paths and even cycles. Then, suppose $G$ is an odd cycle $\left(v_{0}, v_{1}, v_{2}, \ldots, v_{2 n}\right)$. Let $\pi$ be an $\{a, b\}$-total-labelling of $G$ defined as follows: $\pi\left(v_{0}\right)=\pi\left(v_{0} v_{1}\right)=\pi\left(v_{2 j-1}\right)=a$, for $2 \leq j \leq n$, and label all other elements of $G$ with $b$. This results in a vertex-colouring $C_{\pi}$ of $G$ satisfying $C_{\pi}\left(v_{0}\right)=2 a+b, C_{\pi}\left(v_{2 j-1}\right)=a+2 b$ and $C_{\pi}\left(v_{2 j}\right)=3 b$, for $1 \leq j \leq n$. Since $0 \leq a<b, C_{\pi}$ is a proper-vertex-colouring of $G$.

We conclude this section showing that, differently from the case of neighbour distinguishing edge-labellings, all complete graphs have neighbour-distinguishing $\{a, b\}$-totallabellings, for any two distinct real numbers $a$ and $b$.

Theorem 5.19 (Hulgan et al. [60], Przybylo and Wozniak [94]). Let $K_{n}$ be a complete graph with vertex set $V\left(K_{n}\right)=\left\{v_{1}, \ldots, v_{n}\right\}$ and $a, b \in \mathbb{R}$ with $a<b$. Then, there exist two neighbour-distinguishing $\{a, b\}$-total-labellings $\left(\pi_{1}, C_{\pi_{1}}\right)$ and $\left(\pi_{2}, C_{\pi_{2}}\right)$ of $K_{n}$ such that:
(i) $C_{\pi_{1}}\left(v_{i}\right)=a(n-i+1)+b(i-1)$, for $1 \leq i \leq n$;
(ii) $C_{\pi_{2}}\left(v_{i}\right)=a(n-i)+i b$, for $1 \leq i \leq n$.

Proof. Let $K_{n}$ be a complete graph with vertex set $V\left(K_{n}\right)=\left\{v_{1}, \ldots, v_{n}\right\}$ and $a, b \in \mathbb{R}$ with $a<b$. We prove the result by induction on $n$. First, note that the required neighbour-distinguishing $\{a, b\}$-total-labellings $\left(\pi_{1}, C_{\pi_{1}}\right)$ and $\left(\pi_{2}, C_{\pi_{2}}\right)$ of $K_{1}$ are obtained by assigning $\pi_{1}\left(v_{1}\right)=a$ and $\pi_{2}\left(v_{1}\right)=b$. Thus, consider $K_{n}$ with $n \geq 2$. Let $G^{\prime}=K_{n}-v_{n}$ and assume that $G^{\prime}$ has two neighbour-distinguishing $\{a, b\}$-total-labellings ( $\pi_{1}^{\prime}, C_{\pi_{1}^{\prime}}$ ) and $\left(\pi_{2}^{\prime}, C_{\pi_{2}^{\prime}}\right)$ as described in the hypothesis.

In order to construct $\pi_{2}$ for $K_{n}$, assign label $b$ to $v_{n}$ and to each edge incident with $v_{n}$ and assign $\pi_{1}^{\prime}$ to $G^{\prime}$. First, note that all vertices of $G^{\prime}$ are assigned different colours under $C_{\pi_{2}}$ since $C_{\pi_{1}^{\prime}}$ is a vertex-colouring of $G^{\prime}$ and each vertex $v_{i}$ of $G^{\prime}$ has $C_{\pi_{2}}\left(v_{i}\right)=C_{\pi_{1}^{\prime}}\left(v_{i}\right)+b$. By the induction hypothesis, $C_{\pi_{1}^{\prime}}\left(v_{i}\right)=a((n-1)-i+1)+b(i-1)=a(n-i)+b(i-1)$, for $1 \leq i \leq n-1$. Moreover, by the definition of $\left(\pi_{2}, C_{\pi_{2}}\right), C_{\pi_{2}}\left(v_{i}\right)=C_{\pi_{1}^{\prime}}\left(v_{i}\right)+b=$ $a(n-i)+b(i-1)+b=a(n-i)+i b$, for $1 \leq i \leq n-1$. By this fact and by the fact that $C_{\pi_{2}}\left(v_{n}\right)=b n$, we obtain that any two vertices of $K_{n}$ have distinct colours under $C_{\pi_{2}}$. Therefore, $\left(\pi_{2}, C_{\pi_{2}}\right)$ is a neighbour-distinguishing $\{a, b\}$-total-labelling, satisfying (ii).

In order to construct $\pi_{1}$ for $K_{n}$, assign label $a$ to $v_{n}$ and to each edge incident with $v_{n}$ and assign $\pi_{2}^{\prime}$ to $G^{\prime}$. As in the previous case, all vertices of $G^{\prime}$ are assigned different colours under $C_{\pi_{1}}$ and each $v_{i} \in V\left(G^{\prime}\right)$ has $C_{\pi_{1}}\left(v_{i}\right)=C_{\pi_{2}^{\prime}}\left(v_{i}\right)+a$. By the induction hypothesis, $C_{\pi_{2}^{\prime}}\left(v_{i}\right)=a(n-1-i)+i b$, for $1 \leq i \leq n-1$. Moreover, by the definition of $\left(\pi_{1}, C_{\pi_{1}}\right)$, $C_{\pi_{1}}\left(v_{i}\right)=C_{\pi_{2}^{\prime}}\left(v_{i}\right)+a=a(n-i)+i b$, for $1 \leq i \leq n-1$. By adjusting notation so that $v_{1}=v_{n}$ and $v_{i}=v_{i-1}$, for $2 \leq i \leq n$, we obtain that $C_{\pi_{1}}\left(v_{i}\right)=a(n-i+1)+b(i-1)$, for $1 \leq i \leq n$. Therefore, $\left(\pi_{1}, C_{\pi_{1}}\right)$ is a neighbour-distinguishing $\{a, b\}$-total-labelling, satisfying item $(i)$, and the result follows.

In the next chapter, we present our results on neighbour-distinguishing $\{a, b, c\}$-edgelabellings and neighbour-distinguishing $\{a, b\}$-total-labellings for five families of graphs, namely, the families of powers of paths, powers of cycles, split graphs, regular cobipartite graphs and complete multipartite graphs. As corollaries of our results we obtain that the 1,2 -Conjecture and the 1,2,3-Conjecture are true for these families.

## Chapter 6

## Neighbour-distinguishing labellings of families of graphs

In this chapter, we prove that some families of graphs have neighbour-distinguishing $\{a, b\}$-total-labellings and neighbour-distinguishing $\{a, b, c\}$-edge-labellings, for some real values $a, b, c$.

In the first and second sections, we verify the 1,2-Conjecture and the 1,2,3-Conjecture for the families of powers of paths and powers of cycles. From these results and from lemmas stated in the previous chapter, we obtain other stronger results for these families, such as that every power of cycles has neighbour-distinguishing $\{a, b\}$-total-labellings, for any two distinct real numbers $a$ and $b$.

From Section 6.3 to Section 6.5, we investigate neighbour-distinguishing $\{a, b\}$-totallabellings and neighbour-distinguishing $\{a, b, c\}$-edge-labellings, where $a, b$ and $c$ are distinct nonnegative real numbers, for the families of split graphs, regular cobipartite graphs and complete multipartite graphs. Additional results determining $\chi_{\Sigma}^{\prime}(G)$ for subfamilies of these classes of graphs are also presented.

We conclude the chapter with Section 6.6 discussing the relation between neighbourdistinguishing edge-labellings and another kind of labelling called detectable edge-labelling. We show that our results on neighbour-distinguishing edge-labellings naturally imply similar results on detectable edge-labellings of the previously cited families.

In the construction of neighbour-distinguishing [2]-total-labellings for powers of paths and powers of cycles, we use three specific total-labellings of the complete graph defined as follows. Let $K_{n}$ be a complete graph with $V\left(K_{n}\right)=\left\{x_{0}, \ldots, x_{n-1}\right\}$. A neighbourdistinguishing [2]-total-labelling $\left(\omega^{*}, C_{\omega^{*}}\right)$ of $K_{n}$ is called type-1, if $C_{\omega^{*}}\left(x_{i}\right)=n+i$, for $0 \leq i \leq n-1$; it is called type-2, if $C_{\omega^{*}}\left(x_{i}\right)=n+i+1$, for $0 \leq i \leq n-1$; and it is called type-3, if

$$
C_{\omega^{*}}\left(x_{i}\right)= \begin{cases}n+2 i+1, & \text { for } 0 \leq i \leq\lceil n / 2\rceil-1 ; \\ 3 n-2 i, & \text { for }\lceil n / 2\rceil \leq i \leq n-1\end{cases}
$$

These three special total-labellings of $K_{n}$ are also called canonical total-labellings of $K_{n}$ and are exemplified in Figure 6.1(a), Figure 6.1(b), and Figure 6.1(c). The existence of the canonical total-labellings is guaranteed by the following theorem due to Przybyło and Woźniak [94] (this theorem can also be obtained as a corollary of Theorem 5.19).


Figure 6.1: Illustrations of type-1, type-2 and type-3 [2]-total-labellings of $K_{4}$ and $K_{5}$. The number inside each vertex is its colour.

Theorem 6.1 (Przybyło and Woźniak [94]). If $G$ is a complete graph on $n$ vertices, then $G$ has a neighbour-distinguishing-[2]-total-labelling $\left(\pi, C_{\pi}\right)$, called canonical total-labelling, such that, either $\mathcal{C}_{\pi}(G)=\{n, \ldots, 2 n-1\}$ or $\mathcal{C}_{\pi}(G)=\{n+1, \ldots, 2 n\}$. Moreover, if either $n=3$ and $\mathcal{C}_{\pi}(G)=\{4,5,6\}$ or $n \geq 4$, then labelling $\pi$ assigns label 2 to at least two vertices of $G$ and one of these vertices has the largest colour under $C_{\pi}$.

In order to conclude the introduction of this chapter, we present the following result that is used in some proofs of Section 6.1 and Section 6.2.

Lemma 6.2. Let $G$ be a simple graph with a neighbour-distinguishing [2]-total-labelling $\left(\pi^{\prime}, C_{\pi^{\prime}}\right)$. Also, let $Q \subseteq V(G)$ be a clique of $G$, with $|Q| \geq 3$, such that at least two vertices of $Q$ have label 2 under $\pi^{\prime}$ and such that one of them, called $v_{\text {max }}$, is the vertex of $Q$ that has the largest colour under $C_{\pi^{\prime}}$. Then, $\pi^{\prime}$ can be modified so as to obtain a [3]-total-labelling $\pi: V(G) \cup E(G) \rightarrow\{1,2,3\}$ such that:
(i) every edge $e \in E(G)$ has $\pi(e) \in\{1,2,3\}$;
(ii) if $v \in Q$, then $\pi(v)=1$; otherwise, $\pi(v)=\pi^{\prime}(v)$;
(iii) for $v \in V(G)$, if $v=v_{\max }$, then $C_{\pi}(v) \in\left\{C_{\pi^{\prime}}(v), C_{\pi^{\prime}}(v)+1\right\}$; otherwise, $C_{\pi}(v)=$ $C_{\pi^{\prime}}(v)$.

Proof. Let $G, Q$ and $\left(\pi^{\prime}, C_{\pi^{\prime}}\right)$ be as stated in the hypothesis. Let $S=\left\{v \in Q: \pi^{\prime}(v)=2\right\}$. By the hypothesis, $|S| \geq 2$. Let $M$ be a maximum matching of $G[S]$. Adjust notation so that, if $|S| \equiv 1(\bmod 2)$, the unsaturated vertex $u \in S$ has $C_{\pi^{\prime}}(u)=\min _{v \in S}\left\{C_{\pi^{\prime}}(v)\right\}$.

First, define $\pi(v)=1$, if $v \in S$, and $\pi(v)=\pi^{\prime}(v)$, otherwise. If $e \in M, \pi(e)=\pi^{\prime}(e)+1$, otherwise $\pi(e)=\pi^{\prime}(e)$ (since each edge $e \in M$ has label 1 or 2 under $\pi^{\prime}$, it has label 2 or 3 under $\pi$ ). Note that every vertex $v \in S$ has its label decreased by one unit. Moreover, $v$ has the label of exactly one of its incident edges increased by one unit, except in the case of vertex $u$ when $|S| \equiv 1(\bmod 2)$. Therefore, $C_{\pi}(v)=C_{\pi^{\prime}}(v)$ for $v \neq u$. If $|S| \equiv 1(\bmod 2)$, let $v_{\text {max }} \in S$ such that $C_{\pi^{\prime}}\left(v_{\max }\right)=\max _{v \in Q}\left\{C_{\pi^{\prime}}(v)\right\}$. Define $\pi\left(u v_{\max }\right)=\pi^{\prime}\left(u v_{\max }\right)+1$. This implies that $C_{\pi}(u)=C_{\pi^{\prime}}(u)$ and $C_{\pi}\left(v_{\text {max }}\right)=C_{\pi^{\prime}}\left(v_{\text {max }}\right)+1$, and the result follows.

As an illustration of Lemma 6.2, the canonical total-labellings of $K_{4}$ and $K_{5}$, shown in Figure 6.1(b), can be modified, according to Lemma 6.2, to the [3]-total-labellings exhibited in Figure 6.2. Note that, in these two cases, the resulting total-labellings are neighbour-distinguishing.


Figure 6.2: Neighbour-distinguishing [3]-total-labellings of $K_{4}$ and $K_{5}$ that assign label 1 to all vertices of the graph. The label inside each vertex is its colour.

### 6.1 Powers of paths

The $k$-th power of a simple graph $G$ is the simple graph $G^{k}$ with vertex set $V\left(G^{k}\right)=V(G)$ and such that two distinct vertices $u, v \in V\left(G^{k}\right)$ are adjacent in $G^{k}$ if and only if their distance in $G$ is at most $k$, that is, if $d_{G}(u, v) \leq k$. We say that an edge $u v \in E\left(G^{k}\right)$ has reach $\ell$ if $d_{G}(u, v)=\ell$. Given a simple graph $G$, if $G \cong P_{n}$, then the $k$-th power of $G$ is called power of paths and is denoted by $P_{n}^{k}$. A linear sequence of $P_{n}$ is also a linear sequence of $P_{n}^{k}$. Observe that $P_{n}^{k} \cong P_{n}$, when $k=1$, and that $P_{n}^{k} \cong K_{n}$, when $k \geq n-1$. Figure 6.3 illustrates $P_{10}^{3}$.


Figure 6.3: Graph $P_{10}^{3}$. Bold edges have reach one; continuous thin edges have reach two; and dashed edges have reach three.

In this section, we verify the 1,2 -Conjecture and the $1,2,3$-Conjecture for all powers of paths by constructing neighbour-distinguishing [2]-total-labellings and neighbourdistinguishing [3]-edge-labellings for the graphs belonging to this family. In order to do this, we suitably partition the vertex set of the graph under consideration and use the canonical total-labellings of the complete graph in parts of the partition.

Let $P_{n}^{k}$ be a power of paths and $\left(v_{0}, \ldots, v_{n-1}\right)$ be a linear sequence of $V\left(P_{n}\right)$, with $n \geq 1$ and $1 \leq k \leq n-1$. Suppose $n \geq 2 k+2$. Let $n=\alpha(k+1)+r$, with $\alpha \in \mathbb{N}_{>0}$ and $0 \leq r \leq k$. Then, for $0 \leq i \leq \alpha-2$, let block $B^{i}$ be the set of $k+1$ consecutive vertices, starting from $v_{i(k+1)}$ and following the linear sequence. Define block $B^{\alpha-1}$ as the set of $r$ consecutive vertices starting from $v_{(\alpha-1)(k+1)}$. The remaining $k+1$ vertices comprise block $B^{\alpha}$. An edge whose endpoints are in distinct blocks is called a link-edge. Figure 6.4 shows this partition for $P_{14}^{3}$. We call this partition a multiblock-partition and we denote $u_{i-1}^{j}$ the $i$-th vertex of block $B^{j}, j \in\{0, \ldots, \alpha\}$. Note that each block induces a complete graph. This property follows from the definition of $P_{n}^{k}$ and the fact that each block contains at most $k+1$ vertices.


Figure 6.4: A multiblock-partition of $P_{14}^{3}$. Dashed edges represent the link-edges.

Now, consider $k+2 \leq n \leq 2 k+1$. In this case, we partition $V\left(P_{n}\right)$ into three blocks $B^{0}, B^{1}, B^{2}$. The vertices of $V\left(P_{n}\right)$ are taken consecutively, following the linear sequence, so that $\left|B^{0}\right|=\left|B^{2}\right|=\lfloor(n-k) / 2\rfloor$. This is called a triblock-partition of $V\left(P_{n}^{k}\right)$ and it is illustrated in Figure 6.5. An edge whose endpoints are in distinct blocks is also called a link-edge. Note that, if $k \not \equiv n(\bmod 2)$, then block $B^{1}$ induces a complete graph with $k+1$ vertices. Otherwise, block $B^{1}$ induces a complete graph with $k$ vertices. These properties follow from the definition of $P_{n}^{k}$ and the fact that block $B^{1}$ contains exactly $n-2\lfloor(n-k) / 2\rfloor$ consecutive vertices. Additionally, note that any two vertices $u \in B^{0}$ and $v \in B^{2}$ are always nonadjacent since $d_{P_{n}}(u, v)>k$.


Figure 6.5: Triblock-partitions of $P_{6}^{3}$ and $P_{7}^{3}$, respectively. Dashed edges represent the link-edges.

In Theorem 6.5, it is proved that every power of paths has a neighbour-distinguishing [2]-total-labelling. In order to prove this result, we use the following two lemmas.

Lemma 6.3. Let $G \cong P_{n}^{k}$ with $n \geq 2 k+2$. Let $\left\{B^{0}, \ldots, B^{\alpha}\right\}$ be a multiblock-partition of
G. Then,

$$
d_{G}\left(u_{i}^{j}\right)= \begin{cases}k+i, & \text { if } j=0 ; \\ 2 k, & \text { if } 1 \leq j \leq \alpha-1 ; \\ 2 k-i, & \text { if } \quad j=\alpha\end{cases}
$$

Proof. Consider a multiblock-partition $\left\{B^{0}, \ldots, B^{\alpha}\right\}$ of $G$ and let $u_{i}^{j} \in B^{j}, 0 \leq j \leq \alpha$ and $0 \leq i \leq\left|B^{j}\right|-1$. If $j=0$, then $u_{i}^{0}=v_{i}$. Therefore, vertex $u_{i}^{j}$ is adjacent to its $i$ predecessors and to its first $k$ successors in the linear sequence, that is, $d_{G}\left(u_{i}^{0}\right)=i+k$. If $1 \leq j \leq \alpha-1$, then $d_{G}\left(u_{i}^{j}, u_{0}^{0}\right)>k$ and $d_{G}\left(u_{i}^{j}, u_{k}^{\alpha}\right)>k$. This implies that vertex $u_{i}^{j}$ is adjacent to its first $k$ predecessors and to its first $k$ successors. Hence, $d_{G}\left(u_{i}^{j}\right)=2 k$. As the last case, consider $j=\alpha$. Thus, $u_{i}^{\alpha}$ is adjacent to its first $k$ predecessors and to its $k-i$ successors in the linear sequence. Therefore, $d_{G}\left(u_{i}^{\alpha}\right)=2 k-i$.

Lemma 6.4. Let $G \cong P_{n}^{k}$ with $k+2 \leq n \leq 2 k+1$. Let $\left(v_{0}, \ldots, v_{n-1}\right)$ be a linear sequence of $G$. Then,
(i) $d_{G}\left(v_{i}\right)=n-1$, if $n-1-k \leq i \leq k$;
(ii) $d_{G}\left(v_{i}\right)=d_{G}\left(v_{(n-1)-i}\right)=k+i$, if $0 \leq i \leq(n-1)-k-1$.

Proof. Let $S=\left\{v_{(n-1)-k}, \ldots, v_{k}\right\}$. First note that $(n-1)-k \leq k$ since $n \leq 2 k+1$. Therefore, $v_{(n-1)-k}$ occurs before $v_{k}$ in the linear sequence of $G$. This implies that $v_{k}$ is farther from $v_{0}$ than any other vertex of $S$. Analogously, $v_{(n-1)-k}$ is the farthest from $v_{n-1}$, considering the reverse of the linear sequence. Since $v_{0} v_{k}$ and $v_{(n-1)-k} v_{n-1}$ belong to $E(G)$, we conclude that all elements of $S$ are universal vertices of $G$.

For $0 \leq i \leq(n-1)-k-1$, by the definition of $P_{n}^{k}$, vertex $v_{i}$ is adjacent to the $i$ vertices that are its predecessors in the linear sequence, and to the first $k$ vertices that are its successors in the linear sequence. Thus, $d_{G}\left(v_{i}\right)=k+i$. Similarly, vertex $v_{n-i-1}$ is adjacent to its $i$ successors, and to its first $k$ predecessors. Therefore, $d_{G}\left(v_{n-i-1}\right)=k+i$ and the result follows.

Now, we are ready to verify the 1,2 -Conjecture for powers of paths.
Theorem 6.5. If $G$ is a power of paths, then $\chi_{\Sigma}^{\prime \prime}(G) \leq 2$.
Proof. Let $G=P_{n}^{k}$ be a power of paths with vertex set $V(G)=\left\{v_{0}, \ldots, v_{n-1}\right\}$. Since the result is true for paths and complete graphs (see Lemma 5.17 and Theorem 5.19), we assume that $1<k<n-1$. In order to prove the result, we consider two cases: $n \geq 2 k+2$; and $k+2 \leq n \leq 2 k+1$. In each case, we construct a neighbour-distinguishing [2]-total-labelling $\left(\pi, C_{\pi}\right)$ for $G$.

Case 1. $n \geq 2 k+2$.
Consider a multiblock-partition of $V(G)$. That is, $V(G)$ is partitioned into $\alpha+1$ blocks $B^{0}, \ldots, B^{\alpha}, n=\alpha(k+1)+r$, such that: $\left|B^{i}\right|=k+1$, if $i \neq \alpha-1$; and $\left|B^{\alpha-1}\right|=r$. In this case, labelling $\pi$ is constructed so as to satisfy:

$$
C_{\pi}\left(u_{i}^{j}\right)= \begin{cases}2 k-i+1, & \text { if } j=\alpha  \tag{6.1}\\ 2 k+i+2, & \text { otherwise }\end{cases}
$$

First, we show that $\left(\pi, C_{\pi}\right)$ is a neighbour-distinguishing labelling, which amounts to showing that $C_{\pi}$ is a proper-vertex-colouring of $G$. In fact, two distinct vertices $u_{i}^{j}$ and $u_{l}^{j+1}$ with $C_{\pi}\left(u_{i}^{j}\right)=2 k+i+2$ and $C_{\pi}\left(u_{l}^{j+1}\right)=2 k+l+2$ have the same colour if and only if $i=l$. Since the vertices in the blocks are taken following the linear sequence, $i=l$ only when these vertices have the same relative position in their respective blocks $B^{j}$ and $B^{j+1}$. However, this implies that $d_{P_{n}}\left(u_{i}^{j}, u_{l}^{j+1}\right) \geq k+1$. Since the maximum reach of an edge of $G$ is $k$, we conclude that $u_{i}^{j}$ and $u_{l}^{j+1}$ are nonadjacent. Additionally, the colour of vertex $u_{i}^{\alpha} \in B^{\alpha}$ is distinct from the colour of any other vertex of $G$. In order to see this, suppose $C_{\pi}\left(u_{i}^{\alpha}\right)=C_{\pi}\left(u_{j}^{\alpha-1}\right)$ or $C_{\pi}\left(u_{i}^{\alpha}\right)=C_{\pi}\left(u_{j}^{\alpha-2}\right)$. This means that $2 k-i+1=2 k+j+2$, which implies $i+j=-1$, a contradiction. Therefore, $C_{\pi}$ is a proper-vertex-colouring of $G$ as claimed.

Now, we explain how to construct labelling $\pi$. First, assign label 2 to all the elements of $G\left[B^{0}\right]$. Then, assign label 1 to all the elements of $G\left[B^{\alpha}\right]$ and to all the link-edges. It remains to assign labels to the elements of $G\left[B^{j}\right], 1 \leq j \leq \alpha-1$. Recall that each $G\left[B^{j}\right] \cong K_{\left|B^{j}\right|}$. Therefore, we assign the previously defined type-2 total-labelling $\omega^{*}$ of complete graph $K_{\left|B^{j}\right|}$ to each subgraph $G\left[B^{j}\right]$ as follows:
(i) $\pi\left(u_{i}^{j}\right)=\omega^{*}\left(x_{i}\right)$, for $0 \leq i \leq\left|B^{j}\right|-1$;
(ii) $\pi\left(u_{i}^{j} u_{l}^{j}\right)=\omega^{*}\left(x_{i} x_{l}\right)$, for each edge $u_{i}^{j} u_{l}^{j} \in E\left(G\left[B^{j}\right]\right)$.

Figure 6.6 illustrates $P_{14}^{3}$ with its neighbour-distinguishing [2]-total-labelling $\left(\pi, C_{\pi}\right)$.


Figure 6.6: Illustration of the neighbour-distinguishing [2]-total-labelling $\left(\pi, C_{\pi}\right)$ of $P_{14}^{3}$. Dashed edges and white vertices receive label 1; continuous edges and black vertices receive label 2 . The label inside each vertex is its colour.

By the definition of $\pi$, it is clear that just labels 1 and 2 are used. In order to conclude this case, we show that, for every vertex $v \in V(G), C_{\pi}(v)$ satisfies condition (6.1).

- $u_{i}^{0} \in B^{0}$.

By the definition of $\pi$, all the elements of $G\left[B^{0}\right]$ receive label 2. Since $G\left[B^{0}\right] \cong$ $K_{k+1}$, vertex $u_{i}^{0}$ and exactly $k$ of its incident edges have label 2. By Lemma 6.3, $d\left(u_{i}^{0}\right)=k+i$. The remaining $i$ edges incident with $u_{i}^{0}$ are link-edges and, thus, have label 1. Therefore, we conclude that $C_{\pi}\left(u_{i}^{0}\right)=2(k+1)+i=2 k+i+2$.

- $u_{i}^{j} \in B^{j}, 1 \leq j \leq \alpha-1$.

By the definition of $\pi$, the labelling of subgraph $G\left[B^{j}\right]$ was obtained from type-2 total-labelling $\omega^{*}$ of complete graph $K_{\left|B^{j}\right|}$. Therefore, vertex $u_{i}^{j}$ and exactly $\left|B^{j}\right|-1$
of its incident edges are labelled according to $\omega^{*}$. The remaining $2 k-\left(\left|B^{j}\right|-1\right)$ edges incident with $u_{i}^{j}$ are link-edges and have label 1 . Consequently, $C_{\pi}\left(u_{i}^{j}\right)=$ $C_{\omega^{*}}\left(x_{i}\right)+2 k-\left(\left|B^{j}\right|-1\right)=\left(\left|B^{j}\right|+i+1\right)+2 k-\left(\left|B^{j}\right|-1\right)=2 k+i+2$.

- $u_{i}^{\alpha} \in B^{\alpha}$.

By the definition of $\pi$, all elements of subgraph $G\left[B^{\alpha}\right]$ are assigned label 1. The remaining edges incident with $u_{i}^{\alpha}$ are link-edges and, hence, also have label 1. By Lemma 6.3, $d\left(u_{i}^{\alpha}\right)=2 k-i$. Therefore, $C_{\pi}\left(u_{i}^{\alpha}\right)=2 k-i+1$.

Case 2. $k+2 \leq n \leq 2 k+1$.
For this case, we consider a triblock-partition, which is a partition of $V(G)$ into blocks $B^{0}=\left\{v_{0}, \ldots, v_{\left\lfloor\frac{n-k}{2}\right\rfloor-1}\right\}, B^{1}=\left\{v_{\left\lfloor\frac{n-k}{2}\right\rfloor}, \ldots, v_{n-1-\left\lfloor\frac{n-k}{2}\right\rfloor}\right\}$ and $B^{2}=\left\{v_{n-\left\lfloor\frac{n-k}{2}\right\rfloor}, \ldots, v_{n-1}\right\}$. Labelling $\pi$ is constructed so as to satisfy:

$$
C_{\pi}\left(v_{j}\right)= \begin{cases}k+j+1, & \text { if } v_{j} \in B^{0} ;  \tag{6.2}\\ k+3 j+2-2\lfloor(n-k) / 2\rfloor, & \text { if } v_{j} \in B^{1} \text { and } \\ & \lfloor(n-k) / 2\rfloor \leq j \leq n-k-2 ; \\ n+2 j+1-2\lfloor(n-k) / 2\rfloor, & \text { if } v_{j} \in B^{1} \text { and } \\ & n-k-1 \leq j \leq\lceil n / 2\rceil-1 ; \\ 3 n-2 j-2\lfloor(n-k) / 2\rfloor, & \text { if } v_{j} \in B^{1} \text { and }\lceil n / 2\rceil \leq j \leq k ; \\ 3 n-3 j+k-2\lfloor(n-k) / 2\rfloor, & \text { if } v_{j} \in B^{1} \text { and } \\ & k+1 \leq j \leq n-1-\lfloor(n-k) / 2\rfloor ; \\ k+n-j, & \text { if } v_{j} \in B^{2} .\end{cases}
$$

As in the previous case, first we show that $\left(\pi, C_{\pi}\right)$ is neighbour-distinguishing, and, next, how to construct labelling $\pi$ satisfying these properties.

Initially, note that, in each of the three first lines of (6.2), the colours assigned to the vertices, following the linear sequence, form an increasing sequence without repetitions; and, for each of the last three lines, the colours form a decreasing sequence, also without repetitions. Let $\mathcal{C}\left(B^{i}\right)$ be the set of colours assigned to vertices of $B^{i}, i \in\{0,1,2\}$. Thus, $\mathcal{C}\left(B^{0}\right)=\{k+1, \ldots, k+\lfloor(n-k) / 2\rfloor\}=\mathcal{C}\left(B^{2}\right)$; that is, vertices of $B^{0}$ and of $B^{2}$ are assigned the same set of colours. However, if $v_{i} \in B^{0}$ and $v_{j} \in B^{2}$, then $v_{i}$ and $v_{j}$ are nonadjacent. Therefore, $C_{\pi}$ restricted to the vertices of $B^{0} \cup B^{2}$ is a proper-vertex-colouring.

Determining $\mathcal{C}\left(B^{1}\right)$ is not so straightforward as determining $\mathcal{C}\left(B^{0}\right)$ and $\mathcal{C}\left(B^{2}\right)$ since consecutive vertices in the linear sequence are assigned nonconsecutive numbers as colours. In (6.2), $B^{1}$ is partitioned into four sub-blocks $B_{1}^{1}, B_{2}^{1}, B_{3}^{1}$, and $B_{4}^{1}$, corresponding to the sets of vertices in lines two, three, four and five of (6.2), respectively. Let $c_{i}^{\text {min }}=$ $\min \left\{\mathcal{C}\left(B_{i}^{1}\right)\right\}$. By inspection, we conclude that:

$$
\begin{array}{cc}
c_{1}^{\min }=k+2+\left\lfloor\frac{n-k}{2}\right\rfloor, & c_{2}^{\min }=n-1+2\left\lceil\frac{n-k}{2}\right\rceil, \\
c_{3}^{\min }=n+2\left\lceil\frac{n-k}{2}\right\rceil, & c_{4}^{\min }=k+3+\left\lfloor\frac{n-k}{2}\right\rfloor .
\end{array}
$$

Since the greatest colour that occurs in $B^{0}$ is $k+\lfloor(n-k) / 2\rfloor$, we conclude that $\mathcal{C}\left(B^{i}\right) \cap$ $\mathcal{C}\left(B^{1}\right)=\emptyset, i \in\{0,2\}$. By the definition, for every $v_{j}, v_{j+1} \in B_{1}^{1}\left(\right.$ resp. $\left.B_{4}^{1}\right), \mid C_{\pi}\left(v_{j}\right)-$
$C_{\pi}\left(v_{j+1}\right) \mid=3$. Moreover, $c_{4}^{\min }=c_{1}^{\min }+1$. Therefore, $\mathcal{C}\left(B_{1}^{1}\right) \cap \mathcal{C}\left(B_{4}^{1}\right)=\emptyset$. By similar reasoning, $\mathcal{C}\left(B_{2}^{1}\right) \cap \mathcal{C}\left(B_{3}^{1}\right)=\emptyset$. Let $c_{i}^{\max }=\max \left\{\mathcal{C}\left(B_{i}^{1}\right)\right\}$. Also by inspection, we get that $c_{1}^{\max }=n-4+2\left\lceil\frac{n-k}{2}\right\rceil$ and $c_{4}^{\max }=n-3+2\left\lceil\frac{n-k}{2}\right\rceil$. Therefore, $c_{i}^{\max }<c_{j}^{\min }, i \in\{1,4\}$ and $j \in\{2,3\}$, concluding the proof that $\left(\pi, C_{\pi}\right)$ is neighbour-distinguishing.

Now, we show how to construct $\pi$. First, we assign label 1 to the elements of $G\left[B^{i}\right]$, $i \in\{0,2\}$, and to the link-edges. It remains to assign labels to the elements of $G\left[B^{1}\right]$. Since $G\left[B^{1}\right]$ is isomorphic to a complete graph with $n-2\lfloor(n-k) / 2\rfloor$ vertices, we assign labels to its elements using the type-3 total-labelling $\omega^{*}$ of complete graph $K_{n-2\lfloor(n-k) / 2\rfloor}$ as follows:
(i) $\pi\left(v_{i}\right)=\omega^{*}\left(x_{i-\lfloor(n-k) / 2\rfloor}\right)$, for $\lfloor(n-k) / 2\rfloor \leq i \leq n-1-\lfloor(n-k) / 2\rfloor$;
(ii) $\pi\left(v_{i} v_{j}\right)=\omega^{*}\left(x_{i-\lfloor(n-k) / 2\rfloor} x_{j-\lfloor(n-k) / 2\rfloor}\right)$, for each edge $v_{i} v_{j} \in E\left(G\left[B^{1}\right]\right)$.

Figure 6.7 shows $P_{6}^{3}$ and $P_{7}^{3}$ with their respective neighbour-distinguishing [2]-totallabellings.


Figure 6.7: Neighbour-distinguishing [2]-total-labellings of $P_{6}^{3}$ and $P_{7}^{3}$. Dashed edges and white vertices receive label 1 ; continuous edges and black vertices receive label 2. Each vertex is labelled with its colour.

By construction, $\pi$ uses only labels 1 and 2. Since every element of $G\left[B^{0}\right] \cup G\left[B^{2}\right]$ and the link-edges receive label $1, v_{i} \in B^{0} \cup B^{2}$ has $C_{\pi}\left(v_{i}\right)=d\left(v_{i}\right)+1$. For $v_{i} \in B^{0}$, the result follows directly from Lemma 6.4. For $v_{i} \in B^{2}$, note that $i=(n-1)-j$, $j \in[0,\lfloor(n-k) / 2\rfloor-1]$. Moreover, $i \geq k+1$. Therefore, by Lemma 6.4, $d\left(v_{i}\right)=k+j=$ $k+(n-1)-i$ and, hence, we conclude that $C_{\pi}\left(v_{i}\right)=k+n-i$.

To conclude this case, we have to determine $C_{\pi}\left(v_{i}\right)$ for $v_{i} \in B^{1}$. By construction of $\pi$, the labels of the elements of $G\left[B^{1}\right]$ are determined by $\omega^{*}$. This implies that $v_{i}$ and exactly $n-2\lfloor(n-k) / 2\rfloor-1$ of its incident edges are labelled as determined by $\omega^{*}$. The remaining incident edges of $v_{i}$ are link-edges and have label 1 . We analyse $C_{\pi}\left(v_{i}\right)$ depending on which sub-block of $B^{1} v_{i}$ belongs to.

- $v_{i} \in B_{1}^{1}$.

By Lemma 6.4, $d_{G}\left(v_{i}\right)=k+i$ since $i \leq(n-1)-k-1$. This implies that the number of link-edges incident with $v_{i}$ is $(k+i)-(n-2\lfloor(n-k) / 2\rfloor-1)=k+i-n+1+2\lfloor(n-k) / 2\rfloor$. Therefore, we obtain that $C_{\pi}\left(v_{i}\right)=C_{\omega^{*}}\left(x_{i-\lfloor(n-k) / 2\rfloor}\right)+(k+i-n+1+2\lfloor(n-k) / 2\rfloor)=$ $(n+2 i+1-4\lfloor(n-k) / 2\rfloor)+(k+i-n+1+2\lfloor(n-k) / 2\rfloor)=k+3 i+2-2\lfloor(n-k) / 2\rfloor$.

- $v_{i} \in B_{2}^{1} \cup B_{3}^{1}$.

By Lemma 6.4, we have $d_{G}\left(v_{i}\right)=n-1$ since $n-k-1 \leq i \leq k$. Thus, the number
of link-edges incident with $v_{i}$ is $(n-1)-(n-2\lfloor(n-k) / 2\rfloor-1)=2\lfloor(n-k) / 2\rfloor$. Therefore, $C_{\pi}\left(v_{i}\right)=C_{\omega^{*}}\left(x_{i-\lfloor(n-k) / 2\rfloor}\right)+(2\lfloor(n-k) / 2\rfloor)$. If $v_{i} \in B_{2}^{1}$, then $C_{\pi}\left(v_{i}\right)=$ $(n+2 i+1-4\lfloor(n-k) / 2\rfloor)+(2\lfloor(n-k) / 2\rfloor)=n+2 i+1-2\lfloor(n-k) / 2\rfloor$. Otherwise, e. g. if $v_{i} \in B_{3}^{1}, C_{\pi}\left(v_{i}\right)=(3 n-2 i-4\lfloor(n-k) / 2\rfloor)+(2\lfloor(n-k) / 2\rfloor)=3 n-2 i-2\lfloor(n-k) / 2\rfloor$.

- $v_{i} \in B_{4}^{1}$.

By Lemma 6.4, we have $d_{G}\left(v_{i}\right)=k+(n-1-i)$ since $i \geq k+1$. This implies that $v_{i}$ is incident with $(k+n-i-1)-(n-2\lfloor(n-k) / 2\rfloor-1)=k-i+2\lfloor(n-k) / 2\rfloor$ link-edges. Therefore, we conclude that $C_{\pi}\left(v_{i}\right)=C_{\omega^{*}}\left(x_{i-\lfloor(n-k) / 2\rfloor}\right)+(k-i+2\lfloor(n-k) / 2\rfloor)=$ $(3 n-2 i-4\lfloor(n-k) / 2\rfloor)+(k-i+2\lfloor(n-k) / 2\rfloor)=3 n-3 i+k-2\lfloor(n-k) / 2\rfloor$.

Since, for every $v \in V(G), C_{\pi}(v)$ satisfies (6.2), the result follows.
Theorem 6.6. Let $t \in \mathbb{R}$ with $t \neq 0$. If $G$ is a simple graph such that $G \cong P_{n}^{k}$, then $G$ has a neighbour-distinguishing $\{t, 2 t\}$-total-labelling.

Proof. The result follows by Theorem 6.5 and Lemma 5.9.
From the proof of Theorem 6.5, we immediately obtain the next two lemmas.
Lemma 6.7. Let $P_{n}^{k}$ be a power of paths with $n \geq 2 k+2$ and let $\left\{B^{0}, \ldots, B^{\alpha}\right\}$ be a multiblock-partition of $V\left(P_{n}^{k}\right)$ into $\alpha+1$ blocks, where $n=\alpha(k+1)+r$, for $r \in[0, k]$. Then, $P_{n}^{k}$ has a neighbour-distinguishing [2]-total-labelling $\left(\pi^{\prime}, C_{\pi^{\prime}}\right)$ such that:
(i) labelling $\pi^{\prime}$ assigns label 2 to all elements of $G\left[B^{0}\right]$; moreover, the vertex of $B^{0}$ that has the largest colour is vertex $u_{k}^{0}$;
(ii) labelling $\pi^{\prime}$ assigns label 1 to all elements of $G\left[B^{\alpha}\right]$ and to all link-edges of $P_{n}^{k}$; moreover, the colours of the vertices of $B^{\alpha}$ are smaller than the colours of the remaining vertices of $P_{n}^{k}$;
(iii) for every $i \in[1, \alpha-1]$, the restriction of $\pi^{\prime}$ to complete subgraph $G\left[B^{i}\right]$ is a type-2 total-labelling; moreover, $u_{k}^{i}$ is the vertex of $B^{i}$ that has the largest colour.

Lemma 6.8. Let $P_{n}^{k}$ be a power of paths with $k+2 \leq n \leq 2 k+1$ and let $\left\{B^{0}, B^{1}, B^{2}\right\}$ be a triblock-partition of $V\left(P_{n}^{k}\right)$. Then, $P_{n}^{k}$ has a neighbour-distinguishing [2]-total-labelling ( $\pi^{\prime}, C_{\pi^{\prime}}$ ) such that:
(i) labelling $\pi^{\prime}$ assigns label 1 to all elements of $G\left[B^{0}\right], G\left[B^{2}\right]$, and to all link-edges;
(ii) the restriction of $\pi^{\prime}$ to subgraph $G\left[B^{1}\right]$ is a type-3 total-labelling; moreover, the vertex with largest colour under $C_{\pi^{\prime}}$ belongs to $B^{1}$ and has label 2.

In the next theorem, we verify the $1,2,3$-Conjecture for powers of paths.
Theorem 6.9. If $G$ is a simple graph such that $G \cong P_{n}^{k}$, then $\chi_{\Sigma}^{\prime}(G) \leq 3$.
Proof. Let $G \cong P_{n}^{k}$. We assume that $G$ is not a path or a complete graph, since the result is known for these cases [30]. Thus, $1<k<n-1$. By Proposition 5.16, it suffices to show that $G$ has a neighbour-distinguishing [3]-total-labelling $\left(\pi, C_{\pi}\right)$ such that $\pi(v)=1$
for every $v \in V(G)$. Such a labelling $\left(\pi, C_{\pi}\right)$ is obtained by modifying the neighbourdistinguishing [2]-total-labelling ( $\pi^{\prime}, C_{\pi^{\prime}}$ ) of $G$, defined in the statements of Lemma 6.7 and Lemma 6.8. In order to do this, we apply Lemma 6.2: for each vertex $v \in V(G)$ with $\pi^{\prime}(v)=2$, we assign $\pi(v)=1$ and increase the labels of some selected edges by one unit, keeping the property that any two adjacent vertices have distinct colours. We consider two cases, depending on the values of $n$.

Case 1. $n \geq 2 k+2$.
Consider a multiblock-partition of $V(G)$. That is, $V(G)$ is partitioned into $\alpha+1$ blocks $B^{0}, \ldots, B^{\alpha}$, with $n=\alpha(k+1)+r$, such that: $\left|B^{i}\right|=k+1$, if $i \neq \alpha-1$; and $\left|B^{\alpha-1}\right|=r$. Note that, if $r=0$, then $B^{\alpha-1}=\emptyset$. Considering the multiblock-partition of $V(G)$, let $\left(\pi^{\prime}, C_{\pi^{\prime}}\right)$ be the neighbour-distinguishing [2]-total-labelling of $G$ defined in the statement of Lemma 6.7.

By the definition of ( $\pi^{\prime}, C_{\pi^{\prime}}$ ), subgraph $G\left[B^{\alpha}\right]$ has no elements with label 2. We start the construction of $\left(\pi, C_{\pi}\right)$ by defining $\pi(x)=\pi^{\prime}(x)$ for every element $x$ of subgraph $G\left[B^{\alpha}\right]$ and by defining $\pi(e)=\pi^{\prime}(e)$ for every link-edge $e$ of $G$.

Now, we analyse blocks $B^{0}, \ldots, B^{\alpha-2}$. Block $B^{\alpha-1}$ is analysed later since it is a special case. By the definition of ( $\pi^{\prime}, C_{\pi^{\prime}}$ ), all vertices of $B^{0}$ have label 2 under $\pi^{\prime}$. Since $\left|B^{0}\right| \geq 3$, $B^{0}$ has at least three vertices with label 2 . Moreover, for each $i \in[1, \alpha-2]$, labelling $\pi^{\prime}$, restricted to subgraph $G\left[B^{i}\right]$, is a type- 2 total-labelling and, by Theorem $6.1, B^{i}$ has at least two vertices with label 2 under $\pi^{\prime}$. Thus, we modify $\left(\pi^{\prime}, C_{\pi^{\prime}}\right)$ so as to obtain $\left(\pi, C_{\pi}\right)$ as follows. For $0 \leq i \leq \alpha-2$, taking $Q=B^{i}$, we apply Lemma 6.2 to $G\left[B^{i}\right]$ obtaining: (i) for every $v \in B^{i}, \pi(v)=1$; and (ii) if $v$ is the vertex of $B^{i}$ that has the largest colour under $C_{\pi^{\prime}}$, then $C_{\pi}(v) \in\left\{C_{\pi^{\prime}}(v), C_{\pi^{\prime}}(v)+1\right\}$; otherwise, $C_{\pi}(v)=C_{\pi^{\prime}}(v)$. Note that the only vertex of $B^{i}$ that eventually had its colour increased by one unit is vertex $u_{k}^{i}$, since it is the vertex with largest colour under $C_{\pi^{\prime}}$.

In order to conclude $\left(\pi, C_{\pi}\right)$, it remains to analyse block $B^{\alpha-1}$. If the number of vertices with label 2 in $B^{\alpha-1}$ is even, apply Lemma 6.2 to $G\left[B^{\alpha-1}\right]$ taking $Q=B^{\alpha-1}$. Note that, after these modifications, all vertices of $B^{\alpha-1}$ have label 1 under $\pi$ and $C_{\pi}(v)=C_{\pi^{\prime}}(v)$ for every $v \in B^{\alpha-1}$. On the other hand, if the number of vertices with label 2 in $B^{\alpha-1}$ is odd, then: (i) choose any $w \in B^{\alpha-1}$ with $\pi^{\prime}(w)=2$ and apply Lemma 6.2 to $G\left[B^{\alpha-1} \backslash\{w\}\right]$ by taking $Q=B^{\alpha-1} \backslash\{w\}$; (ii) set $\pi(w v)=\pi^{\prime}(w v)$ for every $v \in B^{\alpha-1}$; and (iii) let $\pi(w)=1$ and $\pi\left(w u_{k}^{\alpha-2}\right)=\pi^{\prime}\left(w u_{k}^{\alpha-2}\right)+1$ (note that edge $w u_{k}^{\alpha-2}$ exists in $G$ since $d_{P_{n}}\left(w, u_{k}^{\alpha-2}\right) \leq k$; moreover, recall that $u_{k}^{\alpha-2}$ is the vertex of $B^{\alpha-2}$ that has the largest colour under $C_{\pi^{\prime}}$ ). After these modifications, all vertices of $B^{\alpha-1}$ have label 1 under $\pi$ and $C_{\pi}(v)=C_{\pi^{\prime}}(v)$ for every $v \in B^{\alpha-1}$. Moreover, the colour of vertex $u_{k}^{\alpha-2} \in B^{\alpha-2}$ was increased by one unit and is such that $C_{\pi}\left(u_{k}^{\alpha-2}\right) \in\left\{C_{\pi^{\prime}}\left(u_{k}^{\alpha-2}\right)+1, C_{\pi^{\prime}}\left(u_{k}^{\alpha-2}\right)+2\right\}$.

As an example, Figure 6.8 illustrates neighbour-distinguishing [3]-total-labelling ( $\pi, C_{\pi}$ ) of $P_{14}^{3}$, obtained from the neighbour-distinguishing [2]-total-labelling ( $\pi^{\prime}, C_{\pi^{\prime}}$ ) of $P_{14}^{3}$ exhibited in Figure 6.6.

In order to see that $\left(\pi, C_{\pi}\right)$ is a neighbour-distinguishing [3]-total-labelling, note that:
(i) $\pi(e) \in\{1,2,3\}$ for every edge $e \in E(G)$;
(ii) $\pi(v)=1$ for every $v \in V(G)$;


Figure 6.8: Neighbour-distinguishing [3]-total-labelling $\left(\pi, C_{\pi}\right)$ of $P_{14}^{3}$ obtained from neighbour-distinguishing [2]-total-labelling $\left(\pi^{\prime}, C_{\pi^{\prime}}\right)$ of $P_{14}^{3}$ exhibited in Figure 6.6. All vertices and dashed edges receive label 1 ; continuous edges receive label 2, and bold edges receive label 3 . The label inside each vertex is its colour.
(iii) $C_{\pi}(v)=C_{\pi^{\prime}}(v)$ for every vertex $v \in V(G) \backslash\left\{u_{k}^{i}: 0 \leq i \leq \alpha-2\right\}$; and
(iv) $C_{\pi^{\prime}}(v) \leq C_{\pi}(v) \leq C_{\pi^{\prime}}(v)+2$ for $v \in\left\{u_{k}^{i}: 0 \leq i \leq \alpha-2\right\}$.

Note that the last mentioned vertices, that eventually had their colours changed, are pairwise nonadjacent and have colours under $C_{\pi^{\prime}}$ that are greater than the colours of their neighbours. Therefore, any two adjacent vertices have distinct colours under $C_{\pi}$, and the result follows.

Case 2. $k+2 \leq n \leq 2 k+1$.
For this case, we consider a triblock-partition, which is a partition of $V(G)$ into blocks $B^{0}=\left\{v_{0}, \ldots, v_{\left\lfloor\frac{n-k}{2}\right\rfloor-1}\right\}, B^{1}=\left\{v_{\left\lfloor\frac{n-k}{2}\right\rfloor}, \ldots, v_{n-1-\left\lfloor\frac{n-k}{2}\right\rfloor}\right\}$ and $B^{2}=\left\{v_{n-\left\lfloor\frac{n-k}{2}\right\rfloor}, \ldots, v_{n-1}\right\}$. Let $\left(\pi^{\prime}, C_{\pi^{\prime}}\right)$ be the neighbour-distinguishing [2]-total-labelling of $G$ defined in the statement of Lemma 6.8.

By the definition of ( $\pi^{\prime}, C_{\pi^{\prime}}$ ), blocks $B^{0}$ and $B^{2}$ have no vertices with label 2. We start the construction of $\left(\pi, C_{\pi}\right)$ by defining $\pi(x)=\pi^{\prime}(x)$ for every element $x$ of subgraph $G\left[B^{0} \cup B^{2}\right]$ and by defining $\pi(e)=\pi^{\prime}(e)$ for every link-edge $e$ of $G$.

Now, we analyse block $B^{1}$. Recall that $\pi^{\prime}$ restricted to $G\left[B^{1}\right]$ is a type- 3 total-labelling. Thus, by Theorem 6.1, we have that: (i) if $\left|B^{1}\right| \geq 3$, then at least two vertices of $B^{1}$ have label 2 under $\pi^{\prime}$ and the vertex of $B^{1}$ with largest colour has label 2 ; and (ii) if $\left|B^{1}\right|=2$, then only one vertex of $B^{1}$ has label 2 under $\pi^{\prime}$. First, we consider the case when $\left|B^{1}\right|=2$. Since $\left|B^{1}\right|=n-2\left\lfloor\frac{n-k}{2}\right\rfloor$ and $k+2 \leq n \leq 2 k+1$, we have that $\left|B^{1}\right|=2$ only if $k=2$ and $n=4$, that is, when $G \cong P_{4}^{2}$. The required neighbour-distinguishing [3]-total-labelling of $P_{4}^{2}$ is exhibited in Figure 6.9.


Figure 6.9: A neighbour-distinguishing [3]-total-labelling of $P_{4}^{2}$.
Now, consider the case when $\left|B^{1}\right| \geq 3$. By Theorem 6.1, the complete subgraph $G\left[B^{1}\right]$ has at least two vertices with label 2 . Thus, we modify $\left(\pi^{\prime}, C_{\pi^{\prime}}\right)$ so as to obtain
$\left(\pi, C_{\pi}\right)$ as follows: taking $Q=B^{1}$, we apply Lemma 6.2 , obtaining: (i) for every $v \in$ $B^{1}, \pi(v)=1$; and (ii) if $v$ is the vertex of $B^{1}$ that has the largest colour under $C_{\pi^{\prime}}$, then $C_{\pi}(v) \in\left\{C_{\pi^{\prime}}(v), C_{\pi^{\prime}}(v)+1\right\}$; otherwise, $C_{\pi}(v)=C_{\pi^{\prime}}(v)$. Figure 6.10 shows $P_{7}^{5}$ with a neighbour-distinguishing [2]-total-labelling ( $\pi^{\prime}, C_{\pi^{\prime}}$ ) as defined in the statement of Lemma 6.8 and also shows the neighbour-distinguishing [3]-total-labelling $\left(\pi, C_{\pi}\right)$ of $P_{7}^{5}$, as previously described.

(a) Neighbour-distinguishing $\{1,2\}$-total-labelling of $P_{7}^{5}$ as defined in the statement of Lemma 6.8. Dashed edges and white vertices have label 1 and continuous edges and black vertices have label 2. The label inside each vertex is its colour.

(b) Neighbour-distinguishing $\{1,2,3\}$-total-labelling of $P_{7}^{5}$ obtained from the labelling presented in item (a), as described in the proof of Theorem 6.9. All vertices and dashed edges receive label 1; continuous edges receive label 2, and bold edges receive label 3 .

Figure 6.10: Two neighbour-distinguishing total-labellings of $P_{7}^{5}$.
In order to see that the resulting labelling $\left(\pi, C_{\pi}\right)$ is the required neighbour-distinguishing [3]-total-labelling of $G$, note that: (i) $\pi(e) \in\{1,2,3\}$ for every edge $e \in E(G)$; (ii) all vertices of $G$ have label 1 under $\pi$; and (iii) the only vertex that eventually had its colours increased by one unit has a colour in $\pi^{\prime}$ that is greater than the colours of all the other vertices of the graph. Therefore, any two adjacent vertices have distinct colours under $C_{\pi}$, and the result follows.
Theorem 6.10. Let $t \in \mathbb{R}$ with $t \neq 0$. If $G$ is a simple graph such that $G \cong P_{n}^{k}$, then $G$ has a neighbour-distinguishing $\{t, 2 t, 3 t\}$-edge-labelling.

Proof. The result follows by Theorem 6.9 and Lemma 5.9.
In addition to the result of Theorem 6.9, we know some powers of paths that have $\chi_{\Sigma}^{\prime}(G) \leq 2$. For example, $\chi_{\Sigma}^{\prime}\left(P_{n}^{n-2}\right) \leq 2$ when $n \geq 3$. Based on this result, we pose the following conjecture.

Conjecture 6.11. If $G \cong P_{n}^{k}$ and $G \not \not K_{n}$, then $\chi_{\Sigma}^{\prime}(G) \leq 2$.

### 6.2 Powers of cycles

Let $G$ be a simple graph. If $G \cong C_{n}$, the $k$-th power ${ }^{1}$ of $G$ is called power of cycles and is denoted by $C_{n}^{k}$. A cyclic sequence $\left(v_{0}, \ldots, v_{n-1}\right)$ of $C_{n}$ is also a cyclic sequence of $C_{n}^{k}$. Figure 6.11 illustrates the graph $C_{16}^{2}$.


Figure 6.11: The power of cycles $C_{16}^{2}$. Continuous edges have reach one and dashed edges have reach two.

Observe that $C_{n}^{k} \cong C_{n}$ when $k=1$ and that $C_{n}^{k} \cong K_{n}$ when $k \geq\lfloor n / 2\rfloor$. Furthermore, note that $C_{n}^{k}$ is $2 k$-regular when $k<\lfloor n / 2\rfloor$.

In this section, we verify the 1,2 -Conjecture and the $1,2,3$-Conjecture for powers of cycles by constructing neighbour-distinguishing [2]-total-labellings and neighbour-distinguishing [3]-edge-labellings for the graphs belonging to this family. As in the case of the powers of paths, we construct the neighbour-distinguishing [2]-total-labellings by suitably partitioning the vertex set of the graph under consideration and using the canonical labellings of the complete graph in parts of the partition.

We define a block-partition of $V\left(C_{n}^{k}\right)$ into $\alpha+1$ blocks, $B^{0}, \ldots, B^{\alpha}$. Let $n=\alpha(k+1)+r$ with $0 \leq r \leq k$. We take $\left(v_{0}, \ldots, v_{n-1}\right)$ to be the cyclic sequence of $V\left(C_{n}^{k}\right)$. Then, starting at vertex $v_{0}$ and proceeding in cyclic sequence, we partition $V\left(C_{n}^{k}\right)$ into the first $\alpha$ blocks $B^{0}, \ldots, B^{\alpha-1}$, each one with $k+1$ consecutive vertices; these are called standard blocks. The last block, $B^{\alpha}$, contains the remaining $r$ vertices and is called residual block. Note that $B^{\alpha}=\emptyset$ when $r=0$. We denote $u_{i-1}^{j}$ the $i$-th vertex of block $B^{j}, j \in\{0, \ldots, \alpha\}$. Also in this case, an edge whose endpoints are in different blocks is called link-edge. Figure 6.12 illustrates this definition. It is important to notice that, again, each block induces a complete graph. This property follows from the definition of $C_{n}^{k}$ and from the fact that each block comprises at most $k+1$ vertices.

Now, we are ready to prove that all powers of cycles have a neighbour-distinguishing [2]-total-labelling.

Theorem 6.12. If $G$ is a simple graph such that $G \cong C_{n}^{k}$, then $\chi_{\Sigma}^{\prime \prime}(G)=2$.
Proof. Let $G \cong C_{n}^{k}$. Since $G$ has two adjacent vertices with the same degree, we have that $\chi_{\Sigma}^{\prime \prime}(G) \geq 2$. Since the result is known for cycles and complete graphs (see Lemma 5.18 and Theorem 5.19), we assume $1<k<\lfloor n / 2\rfloor$. Let $n=\alpha(k+1)+r$, with $0 \leq r \leq k$. Let

[^2]

Figure 6.12: Block-partition of $V\left(C_{8}^{2}\right)$. Dashed edges represent the link-edges.
$\left(v_{0}, \ldots, v_{n-1}\right)$ be a cyclic sequence of $G$ and $\left\{B^{0}, \ldots, B^{\alpha}\right\}$ be a block-partition of $V(G)$. In order to prove the result, we construct a [2]-total-labelling $\pi$ for $G$ such that, for each vertex $u_{i}^{j} \in V(G)$ :

$$
C_{\pi}\left(u_{i}^{j}\right)= \begin{cases}3 k+r-i+1, & \text { if } 0 \leq i \leq r-1 \text { and } j=0 \\ 2 k+i+1, & \text { otherwise }\end{cases}
$$

We claim that $C_{\pi}$ is a proper-vertex-colouring of $G$. In fact, two distinct vertices $u_{i}^{j}$ and $u_{l}^{p}$ with $C_{\pi}\left(u_{i}^{j}\right)=2 k+i+1$ and $C_{\pi}\left(u_{l}^{p}\right)=2 k+l+1$ have the same colour if and only if $i=l$. By the cyclic sequence imposed on $G$, these vertices have $i=l$ only when they have the same relative position in their respective blocks $B^{j}$ and $B^{p}$, which implies that $d_{C_{n}}\left(u_{i}^{j}, u_{l}^{p}\right) \geq k+1$. Note that in this case $\{j, p\} \neq\{0, \alpha\}$. Since $d_{C_{n}}\left(u_{i}^{j}, u_{l}^{p}\right) \geq k+1$ and the maximum reach of an edge of $G$ is $k$, we conclude that $u_{i}^{j}$ and $u_{l}^{p}$ are nonadjacent. Additionally, for $i \in\{0, \ldots, r-1\}, C_{\pi}\left(u_{i}^{0}\right)$ is distinct from the colour of any other vertex of $G$. Therefore, $C_{\pi}$ is a proper-vertex-colouring of $G$ as claimed. Figure 6.13 exhibits the [2]-total-labelling $\pi$ for $C_{12}^{4}$.


Figure 6.13: Illustration of the neighbour-distinguishing [2]-total-labelling ( $\pi, C_{\pi}$ ) of $C_{12}^{4}$. Dashed edges and white vertices receive label 1; continuous edges and black vertices receive label 2. Each vertex is labelled with its colour.

Now, it is shown how to construct $\pi$, considering each standard block $B^{j}, 1 \leq j \leq \alpha-1$. Remember that each standard block $B^{j}$ induces a complete graph with $k+1$ vertices. Thus, for each $B^{j}, 1 \leq j \leq \alpha-1$, we use the type- 1 total-labelling $\omega^{*}$ of complete graph $K_{k+1}$ to assign labels to the elements of $G\left[B^{j}\right]$ as follows:
(i) $\pi\left(u_{i}^{j}\right)=\omega^{*}\left(x_{i}\right)$, for $0 \leq i \leq k$;
(ii) $\pi\left(u_{i}^{j} u_{l}^{j}\right)=\omega^{*}\left(x_{i} x_{l}\right)$, for each edge $u_{i}^{j} u_{l}^{j} \in E\left(G\left[B^{j}\right]\right)$.

Additionally, we assign label 1 to each link-edge of $G$ that has at least one endpoint in $B^{j}, 1 \leq j \leq \alpha-1$.

Note that each vertex $u_{i}^{j} \in G\left[B^{1} \cup \ldots \cup B^{\alpha-1}\right]$ and its incident edges received a label. In fact, vertex $u_{i}^{j}$ and exactly $k$ of its incident edges were labelled according to the type-1 total-labelling $\omega^{*}$. Moreover, the other $k$ remaining edges incident with $u_{i}^{j}$ received label 1. Therefore, $C_{\pi}\left(u_{i}^{j}\right)=C_{\omega^{*}}\left(x_{i}\right)+k=(k+1+i)+k=2 k+i+1$.

In order to complete $\pi$, it remains to assign labels to the elements of subgraph $G\left[B^{0} \cup B^{\alpha}\right]$. We start with the labels of the vertices of $B^{\alpha}$ and its incident edges. For every vertex $u_{i}^{\alpha}, \pi\left(u_{i}^{\alpha}\right)=1$. At this point, all link-edges between $B^{\alpha}$ and $B^{\alpha-1}$ have label 1 , that is $\pi\left(u_{i}^{\alpha} u_{j}^{\alpha-1}\right)=1$, for $i+1 \leq j \leq k$. The remaining edges incident with $u_{i}^{\alpha}$ are labelled as follows:

$$
\pi\left(u_{i}^{\alpha} u_{j}^{l}\right)= \begin{cases}1, & \text { if }(i \leq j \leq k-r+i \text { and } l=0) \text { or }(0 \leq j \leq r-1, l=\alpha \text { and } j \neq i) \\ 2, & \text { if } 0 \leq j \leq i-1 \text { and } l=0 .\end{cases}
$$

Remember that there exist $2 k$ edges incident with $u_{i}^{\alpha}$. Since exactly $i$ of these edges received label 2 , the remaining $2 k-i$ incident edges received label 1 . Therefore, $C_{\pi}\left(u_{i}^{\alpha}\right)=$ $1+2 i+(2 k-i)=2 k+i+1$.

It remains to label the elements of subgraph $G\left[B^{0}\right]$. First, we partition block $B^{0}$ into two new blocks $B_{\text {prefix }}$ and $B_{\text {suffix }}$, such that $B_{\text {prefix }}=\left\{u_{0}^{0}, \ldots, u_{r-1}^{0}\right\}$ and $B_{\text {suffix }}=$ $\left\{u_{r}^{0}, \ldots, u_{k}^{0}\right\}$. Figure 6.14 illustrates such a partition of $B^{0}$. Note that when $r=0$, $B_{\text {prefix }}=\emptyset$ and $B^{0}=B_{\text {suffix }}$.


Figure 6.14: Partition of block $B^{0}$ into two new blocks $B_{\text {prefix }}$ and $B_{\text {suffix }}$.
Start by considering block $B_{\text {prefix }}$. Let $u_{i}^{0} \in B_{\text {prefix }}$. There exist $2 k$ edges incident with $u_{i}^{0}$, and $k$ of them have already been labelled as follows:
(i) $r-i-1$ edges received label 2 (link-edges between $B^{0}$ and $B^{\alpha}$ );
(ii) $k-r+1$ edges received label 1 (link-edges between $B^{0}$ and $B^{\alpha} \cup B^{\alpha-1}$ );
(iii) $i$ edges received label 1 (link-edges between $B^{0}$ and $B^{1}$ ).

Vertex $u_{i}^{0}$ and its remaining $k$ unlabelled incident edges receive label 2. Therefore, $C_{\pi}\left(u_{i}^{0}\right)=2(r-i-1)+(k-r+1)+i+2+2 k=3 k+r-i+1$.

Now, we analyse block $B_{\text {suffix }}$. Let $u_{i}^{0} \in B_{\text {suffix }}$. There exist $2 k$ edges incident with $u_{i}^{0}$ and $k+r$ of them have already been labelled as follows:
(i) $r$ edges received label 2 (edges with the other endpoint in $B_{\text {prefix }}$ );
(ii) $i$ edges received label 1 (link-edges between $B^{0}$ and $B^{1}$ );
(iii) $k-i$ edges received label 1 (link-edges between $B^{0}$ and $B^{\alpha} \cup B^{\alpha-1}$ ).

It remains to label the elements of subgraph $G\left[B_{\text {suffix }}\right]$. Note that $B_{\text {suffix }}$ induces a complete graph with $k-r+1$ vertices. Then, we use the type- 1 total-labelling $\omega^{*}$ of complete graph $K_{k-r+1}$ to assign labels to the elements of $G\left[B_{\text {suffix }}\right]$ as follows:
(i) $\pi\left(u_{i}^{0}\right)=\omega^{*}\left(x_{i-r}\right)$, for $r \leq i \leq k$;
(ii) $\pi\left(u_{i}^{0} u_{j}^{0}\right)=\omega^{*}\left(x_{i-r} x_{j-r}\right)$, for each edge $u_{i}^{0} u_{j}^{0} \in E\left(G\left[B_{\text {suffix }}\right]\right)$.

Therefore, $C_{\pi}\left(u_{i}^{0}\right)=C_{\omega^{*}}\left(x_{i-r}\right)+2 r+i+(k-i)=(k-r+1+i-r)+2 r+i+(k-i)=$ $2 k+i+1$, concluding the proof.

Theorem 6.13. Let $a, b \in \mathbb{R}$ such that $a \neq b$. If $G$ is a power of cycles, then $G$ has $a$ neighbour-distinguishing $\{a, b\}$-total-labelling.

Proof. The result follows by Lemma 5.8, Theorem 6.12 and by the fact that every power of cycles $C_{n}^{k}$ is a regular graph.

From the proof of Theorem 6.12 we obtain the following lemma.
Lemma 6.14. Let $G \cong C_{n}^{k}$ be a power of cycles with $n=\alpha(k+1)+r$, for $0 \leq r \leq k$ and $1<k<\lfloor n / 2\rfloor$. Also, let $\left\{B^{0}, \ldots, B^{\alpha}\right\}$ be a block-partition of $V(G)$. Then, $G$ has a neighbour-distinguishing $[2]$-total-labelling $\left(\pi^{\prime}, C_{\pi^{\prime}}\right)$ such that:
(i) labelling $\pi^{\prime}$ assigns label 1 to all elements of $G\left[B^{\alpha}\right]$ and to all link-edges of $G$ that have at least one endpoint in $B^{j}$, for $1 \leq j \leq \alpha-1$;
(ii) for every $i \in[1, \alpha-1]$, the restriction of $\pi^{\prime}$ to complete subgraph $G\left[B^{i}\right]$ is a type- 1 total-labelling; moreover, $u_{k}^{i}$ is the vertex of $B^{i}$ with largest colour under $C_{\pi^{\prime}}$ and has label 2 under $\pi^{\prime}$;
(iii) block $B^{0}$ can be partitioned into two blocks $B_{\text {prefix }}$ and $B_{\text {suffix }}$, such that $B_{\text {prefix }}=$ $\left\{u_{0}^{0}, \ldots, u_{r-1}^{0}\right\}$ and $B_{\text {suffix }}=\left\{u_{r}^{0}, \ldots, u_{k}^{0}\right\}$ (when $r=0, B_{\text {prefix }}=\emptyset$ and $B^{0}=$ $\left.B_{\text {suffix }}\right)$. All vertices of $B_{\text {prefix }}$ and all edges with one endpoint in $B_{\text {prefix }}$ and the other in $B_{\text {suffix }}$ receive label 2. Moreover, the complete subgraph $G\left[B_{\text {suffix }}\right]$ receives a type-1 total-labelling;
(iv) the vertex of $B^{0}$ with largest colour has label 2 under $\pi^{\prime}$. This vertex is $u_{k}^{0}$, if $r=0$, and $u_{0}^{0}$, otherwise. Moreover, when $u_{0}^{0}$ is the vertex of $B^{0}$ with largest colour, its colour is the largest among all colours of vertices of $G$.

Next, we verify the 1,2,3-Conjecture for powers of cycles.
Theorem 6.15. If $G$ is a simple graph such that $G \cong C_{n}^{k}$, then $\chi_{\Sigma}^{\prime}(G) \leq 3$.
Proof. Let $G \cong C_{n}^{k}$ with $n=\alpha(k+1)+r$, for $r \in[0, k]$. We assume that $G$ is not a cycle or a complete graph since the result is known for these cases [30]. By Proposition 5.16, it suffices to show that $G$ has a neighbour-distinguishing [3]-total-labelling $\left(\pi, C_{\pi}\right)$ such that $\pi(v)=1$ for every $v \in V(G)$. Such a labelling $\left(\pi, C_{\pi}\right)$ is obtained by modifying the neighbour-distinguishing [2]-total-labelling $\left(\pi^{\prime}, C_{\pi^{\prime}}\right)$ of $G$ constructed in the proof of Theorem 6.12.

Consider a block-partition $\left\{B^{0}, \ldots, B^{\alpha}\right\}$ of $V(G)$. Recall that $\left|B^{i}\right|=k+1$ if $i \neq \alpha$; and $\left|B^{\alpha}\right|=r$. Note that, if $r=0$, then $B^{\alpha}=\emptyset$. Let $\left(\pi^{\prime}, C_{\pi^{\prime}}\right)$ be the neighbour-distinguishing [2]-total-labelling of $G$, defined in the statement of Lemma 6.14. We consider two cases depending on the value of $k$.

Case 1. $k \geq 3$.
By the definition of ( $\pi^{\prime}, C_{\pi^{\prime}}$ ), subgraph $B^{\alpha}$ has no vertices with label 2. We start the construction of $\left(\pi, C_{\pi}\right)$ by defining $\pi(x)=\pi^{\prime}(x)$ for every element $x$ of subgraph $G\left[B^{\alpha}\right]$, and by defining $\pi(e)=\pi^{\prime}(e)$ for every link-edge $e$ of $G$.

First, we analyse blocks $B^{1}, \ldots, B^{\alpha-1}$ and then block $B^{0}$ since it is a special case: its labelling is constructed in a unique way. Remember that ( $\pi^{\prime}, C_{\pi^{\prime}}$ ) restricted to subgraph $G\left[B^{i}\right], i \in[1, \alpha-1]$, is a type- 1 total-labelling. Hence, by Theorem 6.1 and by the fact that $\left|B^{i}\right| \geq 4$ for each $i \in[1, \alpha-1]$, the complete subgraph $G\left[B^{i}\right]$ has at least two vertices with label 2. Moreover, the vertex with largest colour has label 2. Therefore, by taking $Q=B^{i}$, we apply Lemma 6.2 to $G\left[B^{i}\right]$, for each $i \in[1, \alpha-1]$, obtaining: (i) for every $v \in B^{i}, \pi(v)=1$; and (ii) if $v \in B^{i}$ has the largest colour under $C_{\pi^{\prime}}$, then $C_{\pi}(v) \in\left\{C_{\pi^{\prime}}(v), C_{\pi^{\prime}}(v)+1\right\}$; otherwise, $C_{\pi}(v)=C_{\pi^{\prime}}(v)$.

In order to conclude the result, it remains to analyse block $B^{0}$. We claim that $B^{0}$ has at least two vertices with label 2 under $\pi^{\prime}$ and that one of these vertices is the vertex of $B^{0}$ that has the largest colour under $C_{\pi^{\prime}}$. Remember that $B^{0}=B_{\text {prefix }} \cup B_{\text {suffix }}$ with $B_{\text {prefix }} \cap B_{\text {suffix }}=\emptyset$. By Lemma 6.14, when $r \geq 1$, at least one vertex in both $B_{\text {prefix }}$ and $B_{\text {suffix }}$ has label 2; when $r=0, B_{\text {prefix }}=\emptyset, B^{0}=B_{\text {suffix }}$ and $B_{\text {suffix }}$ has at least two vertices with label 2 since $\left|B_{\text {suffix }}\right| \geq 4$. We conclude that block $B^{0}$ also has at least two vertices with label 2 under $\pi^{\prime}$ and one of these vertices is the vertex of $B^{0}$ that has the largest colour under $C_{\pi^{\prime}}$. Therefore, by taking $Q=B^{0}$, we apply Lemma 6.2 obtaining: (i) for every $v \in B^{0}, \pi(v)=1$; and (ii) if $v \in B^{0}$ has the largest colour under $C_{\pi^{\prime}}$, then $C_{\pi}(v) \in\left\{C_{\pi^{\prime}}(v), C_{\pi^{\prime}}(v)+1\right\}$; otherwise, $C_{\pi}(v)=C_{\pi^{\prime}}(v)$.

As an example, Figure 6.15 illustrates the neighbour-distinguishing [3]-total-labelling $\left(\pi, C_{\pi}\right)$ of $C_{12}^{4}$, obtained from the neighbour-distinguishing [2]-total-labelling exhibited in Figure 6.13.

In order to see that $\left(\pi, C_{\pi}\right)$ is a neighbour-distinguishing [3]-total-labelling of $G$, first note that:
(i) $\pi(e) \in\{1,2,3\}$ for every edge $e \in E(G)$;
(ii) $\pi(v)=1$, for every $v \in V(G)$;


Figure 6.15: Illustration of the neighbour-distinguishing [3]-total-labelling ( $\pi, C_{\pi}$ ) of $C_{12}^{4}$. All vertices and dashed edges receive label 1; continuous edges receive label 2; and bold edges receive label 3. Each vertex is labelled with its colour.
(iii) for every $i \in[1, \alpha-1], C_{\pi}\left(u_{j}^{i}\right)=C_{\pi^{\prime}}\left(u_{j}^{i}\right)$ if $j \neq k$, and $C_{\pi}\left(u_{k}^{i}\right) \in\left\{C_{\pi^{\prime}}\left(u_{k}^{i}\right), C_{\pi^{\prime}}\left(u_{k}^{i}\right)+1\right\}$;
(iv) if $v \in B^{0}$ has the largest colour under $C_{\pi^{\prime}}$, then $C_{\pi}(v) \in\left\{C_{\pi^{\prime}}(v), C_{\pi^{\prime}}(v)+1\right\}$. Every other vertex $u \in B^{0} \backslash\{v\}$ has $C_{\pi}(v)=C_{\pi^{\prime}}(v)$.

Note that the vertices belonging to $B^{i}$, for $i \in[1, \alpha-1]$, that eventually had their colours changed, are pairwise nonadjacent and have colours under $C_{\pi}$ that are greater than the colours of their neighbours. Regarding $B^{0}$, we have two cases: (i) if $r=0$, then $u_{k}^{0}$ is the vertex of $B^{0}$ that has the largest colour and the previous reasoning applies here; (ii) if $r>0$, then the vertex of $B^{0}$ with largest colour is $u_{0}^{0}$ and its colour is the largest among all colours of vertices of $G$, as can be verified in Lemma 6.14. Therefore, any two adjacent vertices have distinct colours under $C_{\pi}$, and the result follows.

Case 2. $k=2$.
By the definition of $\left(\pi^{\prime}, C_{\pi^{\prime}}\right), G\left[B^{\alpha}\right]$ has no elements with label 2 and, again, we define $\pi(x)=\pi^{\prime}(x)$ for every element $x$ of subgraph $G\left[B^{\alpha}\right]$. Also, define $\pi(e)=\pi^{\prime}(e)$ for every link-edge $e$ of $G$. By the definition of $\left(\pi^{\prime}, C_{\pi^{\prime}}\right)$ : if $\left|B^{\alpha}\right|=1$, then $C_{\pi}\left(u_{0}^{\alpha}\right)=5$; and, if $\left|B^{\alpha}\right|=2$, then $C_{\pi}\left(u_{0}^{\alpha}\right)=5$ and $C_{\pi}\left(u_{1}^{\alpha}\right)=6$.

Now, we analyse blocks $B^{1}, \ldots, B^{\alpha-1}$. Block $B^{0}$ is analysed later since its labelling is also constructed in a different way. By the definition of ( $\pi^{\prime}, C_{\pi^{\prime}}$ ), for each $i \in[1, \alpha-1]$ we have that $C_{\pi^{\prime}}\left(u_{0}^{i}\right)=5, C_{\pi^{\prime}}\left(u_{1}^{i}\right)=6, C_{\pi^{\prime}}\left(u_{2}^{i}\right)=7$, and $\pi^{\prime}\left(u_{2}^{i}\right)=2$. Thus, for each $i \in[1, \alpha-1]$, we define $\pi\left(u_{0}^{i} u_{1}^{i}\right)=\pi^{\prime}\left(u_{0}^{i} u_{1}^{i}\right)+1, \pi\left(u_{1}^{i} u_{2}^{i}\right)=\pi^{\prime}\left(u_{1}^{i} u_{2}^{i}\right)+1, \pi\left(u_{0}^{i} u_{2}^{i}\right)=\pi^{\prime}\left(u_{0}^{i} u_{2}^{i}\right)$, $\pi\left(u_{2}^{i}\right)=1$, and all other labels of elements of $G\left[B^{i}\right]$ remain the same. Therefore, for each $i \in[1, \alpha-1]$ we have that $C_{\pi}\left(u_{0}^{i}\right)=6, C_{\pi}\left(u_{1}^{i}\right)=8$ and $C_{\pi}\left(u_{2}^{i}\right)=7$. Moreover, when $r=2, C_{\pi}\left(u_{1}^{\alpha}\right)=C_{\pi}\left(u_{0}^{\alpha-1}\right)$ but these vertices are nonadjacent. Therefore, any two adjacent vertices belonging to $B^{1} \cup \cdots \cup B^{\alpha}$ have distinct colours.

In order to conclude the construction, it remains to analyse block $B^{0}$. Recall that
$B^{0}=B_{\text {prefix }} \cup B_{\text {suffix }}$ such that $B_{\text {prefix }} \cap B_{\text {suffix }}=\emptyset$. We consider three subcases depending on the cardinality of $B_{\text {prefix }}$.

Subcase 2.1. $\left|B_{\text {prefix }}\right|=0$. In this case $B^{\alpha}=\emptyset$. By the definition of $\left(\pi^{\prime}, C_{\pi^{\prime}}\right)$, $\pi^{\prime}\left(u_{0}^{0}\right)=\pi^{\prime}\left(u_{1}^{0}\right)=1, \pi^{\prime}\left(u_{2}^{0}\right)=2, C_{\pi^{\prime}}\left(u_{0}^{0}\right)=5, C_{\pi^{\prime}}\left(u_{1}^{0}\right)=6$ and $C_{\pi^{\prime}}\left(u_{2}^{0}\right)=7$. Thus, we define $\pi\left(u_{0}^{0}\right)=\pi\left(u_{1}^{0}\right)=1, \pi\left(u_{2}^{0}\right)=1, \pi\left(u_{0}^{0} u_{1}^{0}\right)=\pi^{\prime}\left(u_{0}^{0} u_{1}^{0}\right)+1, \pi\left(u_{1}^{0} u_{2}^{0}\right)=\pi^{\prime}\left(u_{1}^{0} u_{2}^{0}\right)+1$ and $\pi\left(u_{0}^{0} u_{2}^{0}\right)=\pi^{\prime}\left(u_{0}^{0} u_{2}^{0}\right)$. Therefore, we obtain $C_{\pi}\left(u_{0}^{0}\right)=6, C_{\pi}\left(u_{1}^{0}\right)=8$ and $C_{\pi}\left(u_{2}^{0}\right)=7$. By inspection, it is possible to verify that every vertex of $B^{0}$ has a colour that is different from the colours of its neighbours.

Subcase 2.2. $\left|B_{\text {prefix }}\right|=1$. By the definition of $\left(\pi^{\prime}, C_{\pi^{\prime}}\right), \pi^{\prime}\left(u_{0}^{0}\right)=\pi^{\prime}\left(u_{2}^{0}\right)=2$, $\pi^{\prime}\left(u_{1}^{0}\right)=1, C_{\pi^{\prime}}\left(u_{0}^{0}\right)=8, C_{\pi^{\prime}}\left(u_{1}^{0}\right)=6, C_{\pi^{\prime}}\left(u_{2}^{0}\right)=7$. Moreover, recall that vertex $u_{0}^{\alpha}$ has colour 5 under $C_{\pi}$. Thus, we define $\pi\left(u_{0}^{0}\right)=\pi\left(u_{2}^{0}\right)=1, \pi\left(u_{0}^{0} u_{1}^{0}\right)=\pi^{\prime}\left(u_{0}^{0} u_{1}^{0}\right)+1$, $\pi\left(u_{0}^{0} u_{2}^{0}\right)=\pi^{\prime}\left(u_{0}^{0} u_{2}^{0}\right)+1$ and all other labels of elements of $G\left[B^{0}\right]$ remain the same. We also increase the label of edge $u_{0}^{\alpha} u_{0}^{0}$ by one unit, thus redefining the colour of vertex $u_{0}^{\alpha}$ to $C_{\pi}\left(u_{0}^{\alpha}\right)=6$. Therefore, $C_{\pi}\left(u_{0}^{0}\right)=9, C_{\pi}\left(u_{1}^{0}\right)=8$ and $C_{\pi}\left(u_{2}^{0}\right)=7$. In this case, colour 9 only appears in vertex $u_{0}^{0}$. By inspection, it is possible to verify that $u_{0}^{\alpha}$ and vertices $u_{1}^{0}, u_{2}^{0} \in B^{0}$ have a colour that is different from the colours of its neighbours.

Subcase 2.3. $\left|B_{\text {prefix }}\right|=2$. By the definition of $\left(\pi^{\prime}, C_{\pi^{\prime}}\right), \pi^{\prime}\left(u_{0}^{0}\right)=\pi^{\prime}\left(u_{1}^{0}\right)=2$, $\pi^{\prime}\left(u_{2}^{0}\right)=1, C_{\pi^{\prime}}\left(u_{0}^{0}\right)=9, C_{\pi^{\prime}}\left(u_{1}^{0}\right)=8$ and $C_{\pi^{\prime}}\left(u_{2}^{0}\right)=7$. Thus, we define $\pi\left(u_{0}^{0}\right)=\pi\left(u_{1}^{0}\right)=1$, $\pi\left(u_{0}^{0} u_{1}^{0}\right)=\pi^{\prime}\left(u_{0}^{0} u_{1}^{0}\right)+1$ and all other labels of elements of $G\left[B^{0}\right]$ remain the same. As a consequence, $C_{\pi}\left(u_{0}^{0}\right)=9, C_{\pi}\left(u_{1}^{0}\right)=8$ and $C_{\pi}\left(u_{2}^{0}\right)=7$. In this case, colour 9 only appears in vertex $u_{0}^{0}$. By inspection, it is possible to verify that vertices $u_{1}^{0}, u_{2}^{0} \in B^{0}$ have colours that are different from the colours of its neighbours.

Theorem 6.16. Let $t \in \mathbb{R}$ with $t \neq 0$. If $G$ is a simple graph such that $G \cong C_{n}^{k}$, then $G$ has a neighbour-distinguishing $\{t, 2 t, 3 t\}$-edge-labelling.

Proof. The result follows by Theorem 6.15 and Lemma 5.9.
In addition to the result of Theorem 6.15, we know some powers of cycles that have neighbour-distinguishing [2]-edge-labellings. For example, from results obtained by Escuadro et al. [42], it is possible to derive the following theorems.

Theorem 6.17 (Escuadro et al. [42]). Let $k, n \in \mathbb{Z}$ such that $n \geq 3$ and $2 \leq k \leq 6$. If $C_{n}^{k} \not \neq K_{n}$, then $\chi_{\Sigma}^{\prime}\left(C_{n}^{k}\right)=2$.

Theorem 6.18 (Escuadro et al. [42]). Let $k, n \in \mathbb{Z}$ such that $k \geq 2$ and $n \geq k(k+1)$. Then, $\chi_{\Sigma}^{\prime}\left(C_{n}^{k}\right)=2$.

Motivated by these results, we pose the following conjecture.
Conjecture 6.19. If $G$ is a simple graph with $n$ vertices, such that $G \cong C_{n}^{k}, G \not \equiv K_{n}$ and $G \not \not C_{n}$, then $G$ has a neighbour-distinguishing [2]-edge-labelling.

### 6.3 Split graphs

A split graph is a simple graph $G$ whose vertex set $V(G)$ can be partitioned into a disjoint union of an independent set $S$ and a clique $Q$. The next theorem shows that split graphs
have neighbour-distinguishing total-labellings with two distinct nonnegative real labels. For the remaining of this section, we take $Q=\left\{v_{1}, \ldots, v_{|Q|}\right\}$ and $S=\left\{u_{1}, \ldots, u_{|S|}\right\}$.

Theorem 6.20. Let $a, b \in \mathbb{R}$, with $0 \leq a<b$. If $G$ is a split graph, then $G$ has $a$ neighbour-distinguishing $\{a, b\}$-total-labelling.

Proof. Let $G$ be a split graph and $a, b \in \mathbb{R}$, with $0 \leq a<b$. Let $Q$ be a maximal clique of $G$ and $S=V(G) \backslash Q$ an independent set. Define $q=|Q|$. Adjust notation such that $d_{G}\left(v_{i}\right) \leq d_{G}\left(v_{i+1}\right)$, for $i \in\{1, \ldots, q-1\}$. Since $G[Q]$ is a complete graph, by Theorem 5.19, it has a neighbour-distinguishing $\{a, b\}$-total-labelling $\left(\pi_{2}, C_{\pi_{2}}\right)$ such that $C_{\pi_{2}}\left(v_{i}\right)=a(q-i)+i b$, for $1 \leq i \leq q$. This implies that $C_{\pi_{2}}\left(v_{i}\right)<C_{\pi_{2}}\left(v_{i+1}\right)$, for $1 \leq i \leq q-1$, and $\min \left\{C_{\pi_{2}}(v): v \in Q\right\}=a(q-1)+b$. To define an $\{a, b\}$-total-labelling $\pi$ for $G$ we assign $\left(\pi_{2}, C_{\pi_{2}}\right)$ for $G[Q]$ and assign label $a$ to each vertex in $S$ and to each edge in $E_{G}[Q, S]$.

Now, we prove that $\left(\pi, C_{\pi}\right)$ is neighbour-distinguishing by showing, initially, that any two vertices $v_{i}, v_{j}$, with $i<j$, have $C_{\pi}\left(v_{i}\right)<C_{\pi}\left(v_{j}\right)$. By the definition of $\pi, C_{\pi}\left(v_{\ell}\right)=$ $C_{\pi_{2}}\left(v_{\ell}\right)+a\left(d_{G}\left(v_{\ell}\right)-q+1\right)$, for $1 \leq \ell \leq q$. Since $d_{G}\left(v_{i}\right) \leq d_{G}\left(v_{j}\right), d_{G}\left(v_{i}\right)-q+1 \leq$ $d_{G}\left(v_{j}\right)-q+1$. Moreover, by the definition of $\pi_{2}, C_{\pi_{2}}\left(v_{i}\right)<C_{\pi_{2}}\left(v_{j}\right)$. Since $a \geq 0$, we obtain that $C_{\pi}\left(v_{i}\right)=C_{\pi_{2}}\left(v_{i}\right)+a\left(d_{G}\left(v_{i}\right)-q+1\right)<C_{\pi_{2}}\left(v_{j}\right)+a\left(d_{G}\left(v_{j}\right)-q+1\right)=C_{\pi}\left(v_{j}\right)$.

Now, consider $u_{i}$ and $v_{j}$ such that $C_{\pi}\left(v_{j}\right)=\min \left\{C_{\pi}(v): v \in Q\right.$ and $\left.N_{G}(v) \cap S \neq \emptyset\right\}$ and $C_{\pi}\left(u_{i}\right)=\max \left\{C_{\pi}(u): u \in S\right\}$. In order to conclude the proof, we show that $C_{\pi}\left(u_{i}\right)<$ $C_{\pi}\left(v_{j}\right)$. Since $C_{\pi}\left(u_{i}\right)=d_{G}\left(u_{i}\right) a+a$ and $d_{G}\left(u_{i}\right) \leq q-1$, we have that $C_{\pi}\left(u_{i}\right) \leq a q$. By the definition of $\pi, C_{\pi}\left(v_{j}\right)=C_{\pi_{2}}\left(v_{j}\right)+a\left(d_{G}\left(v_{j}\right)-q+1\right)$. Therefore, $C_{\pi}\left(v_{j}\right) \geq$ $(a(q-1)+b)+a\left(d_{G}\left(v_{j}\right)-q+1\right)=d_{G}\left(v_{j}\right) a+b \geq a q+b>a q \geq C_{\pi}\left(u_{i}\right)$. Therefore, any two adjacent vertices of $G$ have distinct colours under $C_{\pi}$.

Theorem 6.21. Let $a, b, c \in \mathbb{R}$, with $0 \leq a<b<c$. Let $G$ be a split graph without connected components isomorphic to $K_{2}$, with a maximal clique $Q$ and an independent set $S=V(G) \backslash Q$. If $|Q|=2$, then $G$ has a neighbour-distinguishing $\{a, b\}$-edge-labelling. Moreover, if $|Q| \geq 3$, then $G$ has a neighbour-distinguishing $\{a, b, c\}$-edge-labelling.

Proof. Let $a, b, c$, and $G$ be as stated in the hypothesis. Let $Q$ be a maximal clique of $G$ and $S=V(G) \backslash Q$ an independent set. Define $q=|Q|$. Adjust notation such that $d_{G}\left(v_{i}\right) \leq d_{G}\left(v_{i+1}\right)$, for $i \in\{1, \ldots, q-1\}$.

First, assume that $q=2$. We construct an $\{a, b\}$-edge-labelling $\pi$ for $G$ by choosing an arbitrary edge $v_{2} u_{i} \in E_{G}[Q, S]$, assigning label $b$ to edges $v_{2} u_{i}$ and $v_{1} v_{2}$, and assigning label $a$ to all other edges of $G$. By the definition of $\pi, C_{\pi}\left(v_{1}\right)=b+a\left(d_{G}\left(v_{1}\right)-1\right)$ and $C_{\pi}\left(v_{2}\right)=2 b+a\left(d_{G}\left(v_{2}\right)-2\right)$. Since $d_{G}\left(v_{1}\right) \leq d_{G}\left(v_{2}\right)$ and $0 \leq a<b, C_{\pi}\left(v_{1}\right)<C_{\pi}\left(v_{2}\right)$. Since $Q$ is a maximal clique, every vertex of $S$ has degree one. Thus, $C_{\pi}\left(u_{i}\right)=b$ and every vertex $u_{j} \in S \backslash\left\{u_{i}\right\}$ has colour $C_{\pi}\left(u_{j}\right)=a$. Therefore, $C_{\pi}\left(u_{i}\right)<C_{\pi}\left(v_{2}\right)$ and $C_{\pi}\left(u_{j}\right)<\min \left\{C_{\pi}\left(v_{1}\right), C_{\pi}\left(v_{2}\right)\right\}$.

Now, suppose $q \geq 3$. By Theorem 5.14, $G[Q]$ has a neighbour-distinguishing $\{a, b, c\}$ -edge-labelling $\left(\pi_{1}, C_{\pi_{1}}\right)$ such that $C_{\pi_{1}}\left(v_{i}\right)<C_{\pi_{1}}\left(v_{i+1}\right)$, for $1 \leq i \leq q-1$; moreover, $\min \left\{C_{\pi_{1}}(v): v \in Q\right\} \geq a(q-2)+c$, if $q \geq 4$, and $\min \left\{C_{\pi_{1}}(v): v \in Q\right\}=a+b$, if $q=3$. Define an $\{a, b, c\}$-edge-labelling $\pi$ of $G$ by assigning $\left(\pi_{1}, C_{\pi_{1}}\right)$ for $G[Q]$ and label $a$ to every edge in $E_{G}[Q, S]$.

We prove that $\left(\pi, C_{\pi}\right)$ is neighbour-distinguishing as follows. By the definition of $\pi$, for every $v \in Q, C_{\pi}(v)=C_{\pi_{1}}(v)+a\left(d_{G}(v)-q+1\right)$. For $v_{i}, v_{j}$ with $i<j, d_{G}\left(v_{i}\right)-$ $q+1 \leq d_{G}\left(v_{j}\right)-q+1$ since $d_{G}\left(v_{i}\right) \leq d_{G}\left(v_{j}\right)$. Moreover, by the definition of $\pi_{1}$, we also have that $C_{\pi_{1}}\left(v_{i}\right)<C_{\pi_{1}}\left(v_{j}\right)$. Since $a \geq 0$, we conclude that $C_{\pi}\left(v_{i}\right)<C_{\pi}\left(v_{j}\right)$. Now, consider $u_{i}$ and $v_{j}$ such that $C_{\pi}\left(v_{j}\right)=\min \left\{C_{\pi}(v): v \in Q\right.$ and $\left.N_{G}(v) \cap S \neq \emptyset\right\}$ and $C_{\pi}\left(u_{i}\right)=\max \left\{C_{\pi}(u): u \in S\right\} \leq a(q-1)$. By a similar reasoning used in the proof of Theorem 6.20, we conclude that: (i) $C_{\pi}\left(u_{i}\right) \leq a(q-1)<a(q-2)+c+a \leq C_{\pi}\left(v_{j}\right)$, when $q \geq 4$; and (ii) $C_{\pi}\left(u_{i}\right) \leq 2 a<2 a+b \leq C_{\pi}\left(v_{j}\right)$, when $q=3$. Thus, $C_{\pi}\left(u_{i}\right)<C_{\pi}\left(v_{j}\right)$ and the result follows.

Recall that the neighbour-distinguishing $\{a, b, c\}$-edge-labellings of $K_{n}$, constructed in the proof of Theorem 5.14, assign label $b$ to exactly one edge of $K_{n}$. This implies that the neighbour-distinguishing $\{a, b, c\}$-edge-labelling $\pi$ constructed in the proof of Theorem 6.21 also assigns label $b$ to exactly one edge of split graph $G$. Thus, it is reasonable to think that most split graphs have neighbour-distinguishing edge-labellings with only two real labels. Theorem 6.22 and Theorem 6.23 reinforce this conjecture by presenting families of split graphs that have a neighbour-distinguishing $\{a, b\}$-edgelabelling, for $a, b \in \mathbb{R}$ with $0<a<b$. Nevertheless, Theorem 6.24 presents a family of split graphs that do not have a neighbour-distinguishing $\mathcal{L}$-edge-labelling, for $\mathcal{L}=\{0, a\}$ and $\mathcal{L}=\{a, 2 a\}, a \in \mathbb{R} \backslash\{0\}$.

Theorem 6.22. Let $a, b \in \mathbb{R}$ with $0<a<b$. Let $G$ be a connected split graph with $a$ maximal clique $Q,|Q| \geq 3$, and an independent set $S=V(G) \backslash Q$. Also, let $Q^{\prime} \subset Q$ such that $Q^{\prime}$ comprises $\lceil|Q| / 2\rceil$ vertices of $Q$ that have the largest degrees in $G$. If there exists a matching in $E_{G}\left[Q^{\prime}, S\right]$ that saturates all the vertices of $Q^{\prime}$, then $G$ has a neighbourdistinguishing $\{a, b\}$-edge-labelling.

Proof. Let $a, b$ and $G$ be as stated in the hypothesis. Define $q=|Q|$. Adjust notation such that $d_{G}\left(v_{i}\right) \leq d_{G}\left(v_{i+1}\right)$, for $1 \leq i \leq q-1$. Let $Q^{\prime}=\left\{v_{\lfloor q / 2\rfloor+1}, \ldots, v_{q}\right\}$. Suppose that there exists a matching $M$ in $E_{G}\left[Q^{\prime}, S\right]$ that saturates all the vertices of $Q^{\prime}$. This implies that $|S| \geq\left|Q^{\prime}\right|=\lceil q / 2\rceil$. In order to construct an $\{a, b\}$-edge-labelling $\pi$ for $G$, first, assign to $G[Q]$ an $\{a, b\}$-edge-labelling $\pi^{\prime}$ such that

$$
C_{\pi^{\prime}}\left(v_{i}\right)= \begin{cases}a(q-1-i)+b i, & \text { for } 1 \leq i \leq\lfloor q / 2\rfloor ; \\ a(q-i)+b(i-1), & \text { for }\lfloor q / 2\rfloor+1 \leq i \leq q\end{cases}
$$

Labelling $\pi^{\prime}$ exists by Corollary 5.13. Then, assign label $b$ to the edges of the matching $M$ and label $a$ to the remaining unlabelled edges.

Next, we prove that $\left(\pi, C_{\pi}\right)$ is neighbour-distinguishing. By the definition of $\left(\pi, C_{\pi}\right)$,

$$
C_{\pi}\left(v_{i}\right)= \begin{cases}C_{\pi^{\prime}}\left(v_{i}\right)+a\left(d_{G}\left(v_{i}\right)-q+1\right), & \text { for } 1 \leq i \leq\lfloor q / 2\rfloor \\ C_{\pi^{\prime}}\left(v_{i}\right)+a\left(d_{G}\left(v_{i}\right)-q\right)+b, & \text { for }\lfloor q / 2\rfloor+1 \leq i \leq q\end{cases}
$$

If $1 \leq k<\ell \leq\lfloor q / 2\rfloor$, then $C_{\pi}\left(v_{k}\right)=C_{\pi^{\prime}}\left(v_{k}\right)+a\left(d_{G}\left(v_{k}\right)-q+1\right)<C_{\pi^{\prime}}\left(v_{\ell}\right)+a\left(d_{G}\left(v_{\ell}\right)-q+\right.$ $1)=C_{\pi}\left(v_{\ell}\right)$ since $C_{\pi^{\prime}}\left(v_{k}\right)<C_{\pi^{\prime}}\left(v_{\ell}\right), d_{G}\left(v_{k}\right) \leq d_{G}\left(v_{\ell}\right)$ and $a>0$. Similarly, if $\lfloor q / 2\rfloor+1 \leq$ $k<\ell \leq q$, then $C_{\pi}\left(v_{k}\right)=C_{\pi^{\prime}}\left(v_{k}\right)+a\left(d_{G}\left(v_{k}\right)-q\right)+b<C_{\pi^{\prime}}\left(v_{\ell}\right)+a\left(d_{G}\left(v_{\ell}\right)-q\right)+b=C_{\pi}\left(v_{\ell}\right)$.

Now, consider $k \leq\lfloor q / 2\rfloor$ and $\ell \geq\lfloor q / 2\rfloor+1$. Since $C_{\pi^{\prime}}\left(v_{k}\right) \leq C_{\pi^{\prime}}\left(v_{\ell}\right), d_{G}\left(v_{k}\right) \leq d_{G}\left(v_{\ell}\right)$ and $0<a<b$, we have that $C_{\pi}\left(v_{k}\right)=C_{\pi^{\prime}}\left(v_{k}\right)+a\left(d_{G}\left(v_{k}\right)-q+1\right)<C_{\pi^{\prime}}\left(v_{\ell}\right)+a\left(d_{G}\left(v_{\ell}\right)-q\right)+b=$ $C_{\pi}\left(v_{\ell}\right)$, as required.

Next, take $u_{i}$ and $v_{j}$ such that $C_{\pi}\left(v_{j}\right)=\min \left\{C_{\pi}(v): v \in Q\right.$ and $\left.N_{G}(v) \cap S \neq \emptyset\right\}$ and $C_{\pi}\left(u_{i}\right)=\max \left\{C_{\pi}(u): u \in S\right\}$. In order to conclude the proof, we show that $C_{\pi}\left(u_{i}\right)<$ $C_{\pi}\left(v_{j}\right)$. Since $C_{\pi}\left(u_{i}\right) \in\left\{d_{G}\left(u_{i}\right) a, a\left(d_{G}\left(u_{i}\right)-1\right)+b\right\}$ and $d_{G}\left(u_{i}\right) \leq q-1$, we have that $C_{\pi}\left(u_{i}\right) \leq a(q-2)+b$. By the definition of $\pi, C_{\pi}\left(v_{j}\right) \geq C_{\pi^{\prime}}\left(v_{j}\right)+a\left(d_{G}\left(v_{j}\right)-q+1\right)$. Therefore, $C_{\pi}\left(v_{j}\right) \geq C_{\pi^{\prime}}\left(v_{j}\right)+a\left(d_{G}\left(v_{j}\right)-q+1\right) \geq(a(q-2)+b)+a\left(d_{G}\left(v_{j}\right)-q+1\right)=$ $a\left(d_{G}\left(v_{j}\right)-1\right)+b \geq a(q-1)+b>a(q-2)+b \geq C_{\pi}\left(u_{i}\right)$ since $a>0$. Therefore, any two adjacent vertices of $G$ have distinct colours under $C_{\pi}$.

Theorem 6.23. Let $a, b \in \mathbb{R}$ with $0<a<b$. Let $G$ be a connected split graph with a clique $Q,|Q| \geq 3$, and an independent set $S=V(G) \backslash Q$, with $|S| \geq\lceil|Q| / 2\rceil$. If each vertex in $S$ is adjacent to every vertex in $Q$, then $G$ has a neighbour-distinguishing $\{a, b\}$-edge-labelling.

Proof. Let $G$ be as stated in the hypothesis. Define $q=|Q|$. Let $Q^{\prime} \subset Q$ such that $Q^{\prime}=\left\{v_{\lfloor q / 2\rfloor+1}, \ldots, v_{q}\right\}$. Note that the bipartite subgraph induced by $E_{G}\left[Q^{\prime}, S\right]$ is complete since each vertex in $S$ is adjacent to every vertex in $Q^{\prime}$. Moreover, since $|S| \geq\lceil q / 2\rceil=\left|Q^{\prime}\right|$, there exists a matching $M$ in $E_{G}\left[Q^{\prime}, S\right]$ that saturates all vertices of $Q^{\prime}$. In order to construct an $\{a, b\}$-edge-labelling $\pi$ for $G$, first, assign to $G[Q]$ the following $\{a, b\}$-edgelabelling $\pi^{\prime}$, whose existence is guaranteed by Corollary 5.13:

$$
C_{\pi^{\prime}}\left(v_{i}\right)= \begin{cases}a(q-1-i)+b i, & \text { for } 1 \leq i \leq\lfloor q / 2\rfloor \\ a(q-i)+b(i-1), & \text { for }\lfloor q / 2\rfloor+1 \leq i \leq q\end{cases}
$$

Then, assign label $b$ to the edges of the matching $M$ and, to conclude, assign label $a$ to the remaining unlabelled edges. By a similar reasoning used in the proof of Theorem 6.22, we conclude that: (i) any two vertices $v_{k}$ and $v_{\ell}$, with $k<\ell$, have $C_{\pi}\left(v_{k}\right)<C_{\pi}\left(v_{\ell}\right)$; (ii) by taking $u_{i}$ and $v_{j}$ such that $C_{\pi}\left(u_{i}\right)=\max \left\{C_{\pi}(u): u \in S\right\}$ and $C_{\pi}\left(v_{j}\right)=\min \left\{C_{\pi}(v): v \in\right.$ $Q$ and $\left.N_{G}(v) \cap S \neq \emptyset\right\}$, we have that $C_{\pi}\left(u_{i}\right)=a(q-1)+b<a q+b \leq C_{\pi}\left(v_{j}\right)$. Therefore, any two adjacent vertices of $G$ have distinct colours under $C_{\pi}$.

Theorem 6.24. Let $a \in \mathbb{R} \backslash\{0\}$. Let $G$ be a split graph with a maximal clique $Q$ and an independent set $S=V(G) \backslash Q=\left\{u_{1}\right\}$ with $d_{G}\left(u_{1}\right)=1$. The following statements are true:
(i) if $|Q| \geq 4$, then $G$ does not have a neighbour-distinguishing $\{0, a\}$-edge-labelling;
(ii) if $|Q| \geq 6$, then $G$ does not have a neighbour-distinguishing $\{a, 2 a\}$-edge-labelling.

Proof. Let $G$ and $a$ be as stated in the hypothesis and define $q=|Q|$.
Case 1. Proof of statement (i).
Let $q \geq 4$ and suppose that $G$ has a neighbour-distinguishing $\{0, a\}$-edge-labelling $\left(\pi, C_{\pi}\right)$. Take $\pi^{\prime}$ as the restriction of $\pi$ to $E(G[Q])$ and define $C_{\pi^{\prime}}$ for $G[Q]$ as usual. Let $v_{i} \in Q$ be the vertex adjacent to $u_{1}$ in $G$. Note that $C_{\pi}\left(v_{i}\right)=C_{\pi^{\prime}}\left(v_{i}\right)+\pi\left(v_{i} u_{1}\right)$ and, for every vertex $v \in Q \backslash\left\{v_{i}\right\}, C_{\pi}(v)=C_{\pi^{\prime}}(v)$. By Corollary 5.11, $G[Q]$ does not
have a neighbour-distinguishing $\{a, b\}$-edge-labelling, for any two distinct $a, b \in \mathbb{R}$. Thus, $\left(\pi^{\prime}, C_{\pi^{\prime}}\right)$ is not neighbour-distinguishing. Since $\left(\pi, C_{\pi}\right)$ is neighbour-distinguishing, we conclude that: (i) there exists exactly two vertices in $Q$ with the same colour under $C_{\pi^{\prime}}$ and $v_{i}$ is one of them; and (ii) edge $v_{i} u_{1}$ has label $\pi\left(v_{i} u_{1}\right)=a$. By Lemma 5.12, either $C_{\pi^{\prime}}\left(v_{i}\right)=a\lfloor q / 2\rfloor$ or $C_{\pi^{\prime}}\left(v_{i}\right)=a(\lceil q / 2\rceil-1)$.

First, suppose that $C_{\pi^{\prime}}\left(v_{i}\right)=a\lfloor q / 2\rfloor$. By Lemma 5.12, the degree sequence of the subgraph of $G[Q]$ induced by the edges with label $a$ is $(1, \ldots,\lfloor q / 2\rfloor,\lfloor q / 2\rfloor, \ldots, q-1)$. Since $q \geq 4$, there exists a vertex $v_{j} \in Q, v_{j} \neq v_{i}$, with colour $C_{\pi}\left(v_{j}\right)=C_{\pi^{\prime}}\left(v_{j}\right)=$ $a(\lfloor q / 2\rfloor+1)$. Moreover, since $C_{\pi}\left(v_{i}\right)=C_{\pi^{\prime}}\left(v_{i}\right)+a=a(\lfloor q / 2\rfloor+1)$, we conclude that $C_{\pi}\left(v_{i}\right)=C_{\pi}\left(v_{j}\right)$, which is a contradiction.

Now, consider $C_{\pi^{\prime}}\left(v_{i}\right)=a(\lceil q / 2\rceil-1)$. In this subcase, the degree sequence of the subgraph of $G[Q]$ induced by the edges with label $a$ is $(0, \ldots,\lceil q / 2\rceil-1,\lceil q / 2\rceil-1, \ldots, q-2)$ and we reach a contradiction by applying a similar reasoning as the one applied to the previous subcase. Therefore, we conclude that $G$ does not have a neighbour-distinguishing $\{0, a\}$-edge-labelling.

Case 2. Proof of statement (ii).
Let $q \geq 6, b=2 a$ and suppose that $G$ has a neighbour-distinguishing $\{a, b\}$-edgelabelling $\left(\pi, C_{\pi}\right)$. Take $\pi^{\prime}$ as the restriction of $\pi$ to $E(G[Q])$ and define $C_{\pi^{\prime}}$ for $G[Q]$ as usual. Adjust notation so that $C_{\pi^{\prime}}\left(v_{i}\right) \leq C_{\pi^{\prime}}\left(v_{i+1}\right)$ for $1 \leq i \leq q-1$. By Corollary 5.11, $G[Q]$ does not have a neighbour-distinguishing $\{a, b\}$-edge-labelling and, again, there are exactly two vertices $v_{j}, v_{j+1} \in Q$ such that $C_{\pi^{\prime}}\left(v_{j}\right)=C_{\pi^{\prime}}\left(v_{j+1}\right)$. Since $\left(\pi, C_{\pi}\right)$ is neighbourdistinguishing, one of these two vertices, say $v_{j}$, is adjacent to $u_{1}$ in $G$. Moreover, by Corollary 5.13, either $j=\lfloor q / 2\rfloor$ and $C_{\pi^{\prime}}\left(v_{j}\right)=a(q-1-\lfloor q / 2\rfloor)+b\lfloor q / 2\rfloor$, or $j=\lceil q / 2\rceil$ and $C_{\pi^{\prime}}\left(v_{j}\right)=a(q-\lceil q / 2\rceil)+b(\lceil q / 2\rceil-1)$.

Suppose that $j=\lfloor q / 2\rfloor$ and $C_{\pi^{\prime}}\left(v_{j}\right)=a(q-1-\lfloor q / 2\rfloor)+b\lfloor q / 2\rfloor$. We prove that, if $\pi\left(v_{j} u_{1}\right)=a$, then $C_{\pi}\left(v_{j}\right)=C_{\pi}\left(v_{j+2}\right)$; and, if $\pi\left(v_{j} u_{1}\right)=b$, then $C_{\pi}\left(v_{j}\right)=C_{\pi}\left(v_{j+3}\right)$. First, suppose that $\pi\left(v_{j} u_{1}\right)=a$. In this case, $C_{\pi}\left(v_{j}\right)=C_{\pi^{\prime}}\left(v_{j}\right)+a=a(q-\lfloor q / 2\rfloor)+b\lfloor q / 2\rfloor=$ $C_{\pi}\left(v_{j+2}\right)$ since $b=2 a$. Now, suppose that $\pi\left(v_{j} u_{1}\right)=b$. In this case, $C_{\pi}\left(v_{j}\right)=C_{\pi^{\prime}}\left(v_{j}\right)+b=$ $a(q-1-\lfloor q / 2\rfloor)+b(\lfloor q / 2\rfloor+1)=C_{\pi}\left(v_{j+3}\right)$ since $b=2 a$.

The case when $j=\lceil q / 2\rceil$ and $C_{\pi^{\prime}}\left(v_{j}\right)=a(q-\lceil q / 2\rceil)+b(\lceil q / 2\rceil-1)$ is similar to the previous subcase, with the only difference being in the values of $C_{\pi}\left(v_{j+2}\right)$ and $C_{\pi}\left(v_{j+3}\right)$; that is, if $\pi\left(v_{j} u_{1}\right)=a$, then $C_{\pi}\left(v_{j}\right)=C_{\pi}\left(v_{j+2}\right)=a(q+1-\lceil q / 2\rceil)+b(\lceil q / 2\rceil-1)-2 a+b$; and, if $\pi\left(v_{j} u_{1}\right)=b$, then $C_{\pi}\left(v_{j}\right)=C_{\pi}\left(v_{j+3}\right)=a(q-\lceil q / 2\rceil)+b\lceil q / 2\rceil-2 a+b$.

In order to conclude the proof, note that the indices of the vertices $v_{j+2}$ and $v_{j+3}$ are smaller than or equal to $q$ since $q \geq 6$. Therefore, in both cases, $G$ does not have a neighbour-distinguishing $\{a, 2 a\}$-edge-labelling.

Based on Theorem 6.24 and on computational results, we pose the following conjectures.

Conjecture 6.25. Let $G$ be a connected split graph with a maximal clique $Q$ and an independent set $S=V(G) \backslash Q$. Also, let $Q^{\prime}=Q \cap N_{G}(S)$. If $|Q| \geq 4,1 \leq\left|Q^{\prime}\right| \leq\lfloor|Q| / 2\rfloor-1$ and every vertex $v \in Q^{\prime}$ has degree $d_{G}(v)=|Q|$, then $G$ does not have a neighbourdistinguishing $\{0, a\}$-edge-labelling, for $a \in \mathbb{R} \backslash\{0\}$.

Conjecture 6.26. Let $G$ be a connected split graph with a maximal clique $Q$ and an independent set $S=V(G) \backslash Q$. Also, let $Q^{\prime}=Q \cap N_{G}(S)$. If $|Q| \geq 6,1 \leq\left|Q^{\prime}\right| \leq$ $\lfloor(\lfloor|Q| / 2\rfloor-1) / 2\rfloor$ and every vertex $v \in Q^{\prime}$ has degree $d_{G}(v)=|Q|$, then $G$ does not have a neighbour-distinguishing $\{a, 2 a\}$-edge-labelling, for $a \in \mathbb{R} \backslash\{0\}$.

### 6.4 Regular cobipartite graphs

A simple graph $G$ is cobipartite if its complement is bipartite. Next, we present neighbourdistinguishing $\{a, b\}$-total-labellings and neighbour-distinguishing $\{a, b, c\}$-edge-labellings for $k$-regular cobipartite graphs. It is well known that if $G$ is a $k$-regular cobipartite graph non-isomorphic to a complete graph, then $V(G)$ can be partitioned into two cliques $X$ and $Y$ such that $|X|=|Y|$.

The next result shows that $k$-regular cobipartite graphs have neighbour-distinguishing $\{a, b\}$-total-labellings, for $a, b \in \mathbb{R}, a<b$.

Theorem 6.27. Let $a, b \in \mathbb{R}, a<b$. If $G$ is a $k$-regular cobipartite graph, then $G$ has $a$ neighbour-distinguishing $\{a, b\}$-total-labelling.

Proof. Let $G$ be a $k$-regular cobipartite graph and $a, b \in \mathbb{R}, a<b$. If $G$ is a complete graph, then the result follows by Theorem 5.19. Thus, assume $G$ is not complete. Therefore, $V(G)$ can be partitioned into two cliques $X$ and $Y$ such that $|X|=|Y|$. Let $X=$ $\left\{x_{1}, \ldots, x_{q}\right\}$ and $Y=\left\{y_{1}, \ldots, y_{q}\right\}$, where $q=|V(G)| / 2$. Note that the subgraph induced by edge set $E_{G}[X, Y]$ is a $(k-q+1)$-regular bipartite graph. Moreover, $k-q+1<q=$ $|X|=|Y|$. Therefore, it is possible to add a perfect matching $M$ to graph $G$ such that each edge of $M$ link a vertex of $X$ to a vertex of $Y$, resulting in a simple graph $G^{\prime}$ that is $(k+1)$-regular. Adjust notation so that $M=\left\{x_{i} y_{i}: 1 \leq i \leq q\right\}$.

Define an $\{a, b\}$-total-labelling $\pi^{\prime}$ for $G^{\prime}$ as follows: assign to $G^{\prime}[X]$ and $G^{\prime}[Y]$ the neighbour-distinguishing $\{a, b\}$-total-labelling $\pi_{1}$ defined in Theorem 5.19. Note that $C_{\pi_{1}}\left(x_{i}\right)=C_{\pi_{1}}\left(y_{i}\right)$ if and only if $x_{i} y_{i} \in M$. Then, assign label $a$ to every edge in $E_{G^{\prime}}[X, Y]$. Let $\pi$ be the restriction of $\pi^{\prime}$ to $E(G)$ and define $C_{\pi}$ for $G$ as usual. Note that $\pi$ is an $\{a, b\}$-total-labelling of $G$.

Now, we prove that $\left(\pi, C_{\pi}\right)$ is neighbour-distinguishing. First, note that $C_{\pi}(v)=$ $C_{\pi_{1}}(v)+a(k-q+1)$ for every $v \in V(G)$. Thus, since $C_{\pi_{1}}$ is injective, any two distinct vertices in $X$ have distinct colours under $C_{\pi}$; the same is true for $Y$. By the definition, $C_{\pi}\left(x_{i}\right)=C_{\pi}\left(y_{j}\right)$ if and only if $x_{i} y_{j} \in M$. Since $M \nsubseteq E(G)$, for each edge $x_{i} y_{j} \in E(G)$, we have that $C_{\pi}\left(x_{i}\right) \neq C_{\pi}\left(y_{j}\right)$.

Theorem 6.28. Let $a, b, c \in \mathbb{R}, a<b<c$. If $G$ is a $k$-regular cobipartite graph without connected components isomorphic to $K_{2}$, then $G$ has a neighbour-distinguishing $\{a, b, c\}$ -edge-labelling.

Proof. Let $a, b, c$ and $G$ be as stated in the hypothesis. Since the result is known for complete graphs (see Theorem 5.14), assume $G$ is not complete and partition $V(G)$ into two cliques $X=\left\{x_{1}, \ldots, x_{q}\right\}$ and $Y=\left\{y_{1}, \ldots, y_{q}\right\}$ with $q=|V(G)| / 2$. When $q=2$, $G \cong C_{4}$ and the result is known [66]. For $q \geq 3$, the result follows by the same reasoning
of the proof of Theorem 6.27 except that, in this case, $\pi_{1}$ is the edge-labelling defined in Theorem 5.14.

The neighbour-distinguishing $\{a, b, c\}$-edge-labelling constructed in the proof of Theorem 6.28 assigns label $b$ to exactly two edges of the $k$-regular cobipartite graph $G$. Thus, we conjecture that most of the $k$-regular cobipartite graphs have neighbour-distinguishing edge-labellings with two distinct real labels. In fact, Theorem 6.29 and Theorem 6.30 present two families of $k$-regular cobipartite graphs that have neighbour-distinguishing $\{a, b\}$-edge-labellings, for $a, b \in \mathbb{R}, a \neq b$.

Theorem 6.29. Let $a, b \in \mathbb{R}, a<b$. If $G$ is a $k$-regular cobipartite graph with $2 k$ vertices, $k \geq 4$, then $G$ has a neighbour-distinguishing $\{a, b\}$-edge-labelling.

Proof. Let $a, b$ and $G$ be as stated in the hypothesis. Let $\{X, Y\}$ be a bipartition of $V(G)$ into cliques $X$ and $Y$ such that $|X|=|Y|$. Let $X=\left\{x_{1}, \ldots, x_{k}\right\}$ and $Y=\left\{y_{1}, \ldots, y_{k}\right\}$. Since $G$ is $k$-regular, $E_{G}[X, Y]$ is a perfect matching. Let $E_{1}, E_{2} \subset E_{G}[X, Y]$ such that $\left|E_{1}\right|=\lfloor k / 2\rfloor$ and $E_{2}=E_{G}[X, Y] \backslash E_{1}$. Let $G_{1}=G\left[E_{1}\right]$ and $G_{2}=G\left[E_{2}\right]$. Since $G_{1}$ and $G_{2}$ are 1-regular and have at least four vertices, it is possible to add perfect matchings $M_{1}$ and $M_{2}$ to $G_{1}$ and $G_{2}$, respectively, such that $G\left[E_{1} \cup M_{1}\right]$ and $G\left[E_{2} \cup M_{2}\right]$ are simple. Adjust notation so that $M_{1}=\left\{x_{i} y_{i}: 1 \leq i \leq\lfloor k / 2\rfloor\right\}$ and $M_{2}=\left\{x_{i} y_{i}:\lfloor k / 2\rfloor+1 \leq i \leq k\right\}$.

In order to construct a neighbour-distinguishing $\{a, b\}$-edge-labelling $\left(\pi, C_{\pi}\right)$ for $G$, first, assign to $G[X]$ and $G[Y]$ a neighbour-distinguishing $\{a, b\}$-edge-labelling $\left(\pi^{\prime}, C_{\pi^{\prime}}\right)$ such that

$$
C_{\pi^{\prime}}\left(x_{i}\right)=C_{\pi^{\prime}}\left(y_{i}\right)= \begin{cases}a(k-1-i)+b i, & \text { for } 1 \leq i \leq\lfloor k / 2\rfloor ; \\ a(k-i)+b(i-1), & \text { for }\lfloor k / 2\rfloor+1 \leq i \leq k\end{cases}
$$

Note that ( $\pi^{\prime}, C_{\pi^{\prime}}$ ) exists by Corollary 5.13. Then, assign label $a$ to every edge in $E_{1}$ and assign label $b$ to every edge in $E_{2}$. By removing the matching $M_{1} \cup M_{2}$, we obtain an $\{a, b\}$-edge-labelling $\pi$ of $G$.

Now, we prove that $\left(\pi, C_{\pi}\right)$ is neighbour-distinguishing. By the definition of $\pi$, any vertex $x_{i}$ in $X \cap V\left(G\left[E_{1}\right]\right)$ has colour $C_{\pi}\left(x_{i}\right)=C_{\pi^{\prime}}\left(x_{i}\right)+a=a(k-i)+b i$, and any vertex $x_{i}$ in $X \cap V\left(G\left[E_{2}\right]\right)$ has colour $C_{\pi}\left(x_{i}\right)=C_{\pi^{\prime}}\left(x_{i}\right)+b=a(k-i)+b i$. Therefore, the colour of the vertices in $X$ under $C_{\pi}$ are distinct. By a similar reasoning, we conclude that any two vertices in $Y$ have distinct colours. By the definition, $C_{\pi}\left(x_{i}\right)=C_{\pi}\left(y_{j}\right)$ if and only if $i=j$. Since $x_{i} y_{i} \notin E(G)$, we have that $C_{\pi}\left(x_{i}\right) \neq C_{\pi}\left(y_{j}\right)$, for every edge $x_{i} y_{j} \in E(G)$, and the result follows.

Theorem 6.30. Let $a, b \in \mathbb{R}, a<b$. If $G$ is $a(2 k-2)$-regular cobipartite graph with $2 k$ vertices, $k \geq 3$, then $G$ has a neighbour-distinguishing $\{a, b\}$-edge-labelling.

Proof. Let $a, b$ and $G$ be as stated in the hypothesis. In this case, $G \cong K_{2 k} \backslash M$, where $M$ is a perfect matching. Adjust notation so that $M=\left\{x_{i} y_{i}: 1 \leq i \leq k\right\}, X=\left\{x_{1}, \ldots, x_{k}\right\}$ and $Y=\left\{y_{1}, \ldots, y_{k}\right\}$. Consider the nonempty regular bipartite graph $G^{\prime}=G\left[E_{G}[X, Y]\right]$. Let $M_{b}$ be a matching of $G^{\prime}$ such that the saturated vertices belong to $\left\{x_{i}, y_{i}:\lfloor k / 2\rfloor+1 \leq\right.$ $i \leq k\}$. Note that $M_{b}$ exists since $G\left[\left\{x_{\lfloor k / 2\rfloor+1}, \ldots, x_{k}, y_{\lfloor k / 2\rfloor+1}, \ldots, y_{k}\right\}\right]$ is regular.

In order to construct a neighbour-distinguishing $\{a, b\}$-edge-labelling $\left(\pi, C_{\pi}\right)$ for $G$, assign to $G[X]$ and $G[Y]$ a neighbour-distinguishing $\{a, b\}$-edge-labelling ( $\pi^{\prime}, C_{\pi^{\prime}}$ ) such that

$$
C_{\pi^{\prime}}\left(x_{i}\right)=C_{\pi^{\prime}}\left(y_{i}\right)= \begin{cases}a(k-1-i)+b i, & \text { for } 1 \leq i \leq\lfloor k / 2\rfloor ; \\ a(k-i)+b(i-1), & \text { for }\lfloor k / 2\rfloor+1 \leq i \leq k\end{cases}
$$

The existence of $\left(\pi^{\prime}, C_{\pi^{\prime}}\right)$ is guaranteed by Corollary 5.13. Then, assign label $b$ to every edge in $M_{b}$ and assign label $a$ to the remaining unlabelled edges.

Now, to prove that $\left(\pi, C_{\pi}\right)$ is neighbour-distinguishing, we initially consider two vertices $x_{i}, x_{j} \in X, i<j$. By the definition of $\pi$, any vertex $x_{i}$ with $i \leq\lfloor k / 2\rfloor$ has colour $C_{\pi}\left(x_{i}\right)=C_{\pi^{\prime}}\left(x_{i}\right)+a(k-1)=a(2 k-i-2)+b i$, and any vertex $x_{i}$ with with $i \geq\lfloor k / 2\rfloor+1$ has colour $C_{\pi}\left(x_{i}\right)=C_{\pi^{\prime}}\left(x_{i}\right)+b+a(k-2)=a(2 k-i-2)+b i$. Therefore, the colours of the vertices in $X$ are distinct. Analogously, this is true for the vertices of $Y$. We conclude the proof remembering that $C_{\pi}\left(x_{i}\right)=C_{\pi}\left(y_{j}\right)$ if and only if $i=j$ and that $x_{i} y_{i} \notin E(G)$.

Figure 6.16 shows a 3 -regular cobipartite graph that does not have a neighbourdistinguishing $\{a, b\}$-edge-labelling. Furthermore, we also know that complete graphs do not have neighbour-distinguishing $\{a, b\}$-edge-labellings. These are the only examples of regular cobipartite graphs that do not have neighbour-distinguishing $\{a, b\}$-edge-labellings that we know. Thus, based on these observations and our results, we pose the following conjecture.

Conjecture 6.31. With exception of the graph shown in Figure 6.16, every regular connected cobipartite graph $G$ non-isomorphic to a complete graph has a neighbour-distinguishing $\{a, b\}$-edge-labelling, for $a, b \in \mathbb{R}, a \neq b$.


Figure 6.16: A 3-regular graph that does not have a neighbour-distinguishing $\{a, b\}$-edgelabelling.

### 6.5 Complete multipartite graphs

The last class considered in this chapter is the class of complete multipartite graphs, which are defined in page 18. Hulgan et al. [60] proved that any complete multipartite graph has a neighbour-distinguishing $\{a, b\}$-total-labelling for any two distinct real numbers $a$ and $b$. Therefore, in this work, we just consider the edge version of the problem. The next result shows that complete multipartite graphs with at least three parts have a neighbour-distinguishing edge-labelling with any three distinct nonnegative real labels.

Theorem 6.32. Let $G$ be a complete multipartite graph with $r \geq 3$ parts and let $a, b, c \in \mathbb{R}$, $0 \leq a<b<c$. Then $G$ has a neighbour-distinguishing $\{a, b, c\}$-edge-labelling.

Proof. Let $a, b, c \in \mathbb{R}, 0 \leq a<b<c$, and $G$ be a complete multipartite graph with $r \geq 3$ parts denoted $V_{1}, \ldots, V_{r}$. Let $G_{i}$ be the subgraph of $G$ induced by the vertex set $V_{1} \cup \ldots \cup V_{i}$, for $1 \leq i \leq r$. Let $v^{i}$ denote a vertex of $V_{i}$. Without loss of generality, assume that $\left|V_{1}\right| \leq\left|V_{2}\right|$ and, for $i \in\{3, \ldots, r\},\left|V_{i-2}\right| \leq\left|V_{i}\right|$ if $i$ is even, and $\left|V_{i}\right| \leq\left|V_{i-2}\right|$ otherwise.

In order to construct an $\{a, b, c\}$-edge-labelling $\pi_{3}$ for $G_{3}$, assign label $a$ to every edge in $E_{G}\left[V_{1}, V_{2}\right]$, assign label $c$ to every edge in $E_{G}\left[V_{1}, V_{3}\right]$, and assign label $b$ to every edge in $E_{G}\left[V_{2}, V_{3}\right]$. Note that $C_{\pi_{3}}\left(v^{2}\right)<C_{\pi_{3}}\left(v^{1}\right)<C_{\pi_{3}}\left(v^{3}\right)$ since $0 \leq a<b<c$ and $a\left|V_{1}\right|+b\left|V_{3}\right|<$ $a\left|V_{2}\right|+c\left|V_{3}\right|<b\left|V_{2}\right|+c\left|V_{1}\right|$. Therefore, $\left(\pi_{3}, C_{\pi_{3}}\right)$ is a neighbour-distinguishing $\{a, b, c\}$ -edge-labelling of $G_{3}$. Moreover, note that $\min \left\{\mathcal{C}_{\pi_{3}}\left(G_{3}\right)\right\}=C_{\pi_{3}}\left(v^{2}\right)=a\left|V_{1}\right|+b\left|V_{3}\right|$ and $\max \left\{\mathcal{C}_{\pi_{3}}\left(G_{3}\right)\right\}=C_{\pi_{3}}\left(v^{3}\right)=b\left|V_{2}\right|+c\left|V_{1}\right|$.

In order to construct an $\{a, b, c\}$-edge-labelling $\pi_{4}$ for $G_{4}$, assign neighbour-distinguishing $\{a, b, c\}$-edge-labelling $\pi_{3}$ for subgraph $G_{3} \subset G_{4}$ and label $a$ to every edge in the set $E_{G}\left[V_{4}, V\left(G_{3}\right)\right]$. Since $\left(\pi_{3}, C_{\pi_{3}}\right)$ is a neighbour-distinguishing $\{a, b, c\}$-edge-labelling of $G_{3}$ with $C_{\pi_{3}}\left(v^{2}\right)<C_{\pi_{3}}\left(v^{1}\right)<C_{\pi_{3}}\left(v^{3}\right)$ and the value $a\left|V_{4}\right|$ was added to the label of every vertex of $G_{3}$, we conclude that $C_{\pi_{4}}\left(v^{2}\right)<C_{\pi_{4}}\left(v^{1}\right)<C_{\pi_{4}}\left(v^{3}\right)$. Moreover, we have that $C_{\pi_{4}}\left(v^{4}\right)<C_{\pi_{4}}\left(v^{2}\right)$ since $a\left|V_{1}\right|+a\left|V_{2}\right|+a\left|V_{3}\right|<a\left|V_{1}\right|+a\left|V_{4}\right|+b\left|V_{3}\right|$. Therefore, we conclude that $\left(\pi_{4}, C_{\pi_{4}}\right)$ is a neighbour-distinguishing $\{a, b, c\}$-edge-labelling of $G_{4}$ with $\min \left\{\mathcal{C}_{\pi_{4}}\left(G_{4}\right)\right\}=C_{\pi_{4}}\left(v^{4}\right)=a\left|V\left(G_{3}\right)\right|$ and $\max \left\{\mathcal{C}_{\pi_{4}}\left(G_{4}\right)\right\}=a\left|V_{4}\right|+b\left|V_{2}\right|+c\left|V_{1}\right|$.

For $i \geq 5$, we prove that $G_{i}$ has a neighbour-distinguishing $\{a, b, c\}$-edge-labelling $\pi_{i}$ such that: (i) if $i$ is odd, then $\min \left\{\mathcal{C}_{\pi_{i}}\left(G_{i}\right)\right\}=a\left|V\left(G_{i-2}\right)\right|+c\left|V_{i}\right|$ and $\max \left\{\mathcal{C}_{\pi_{i}}\left(G_{i}\right)\right\}=$ $c\left|V\left(G_{i-1}\right)\right|$; and (ii) if $i$ is even, then $\min \left\{\mathcal{C}_{\pi_{i}}\left(G_{i}\right)\right\}=a\left|V\left(G_{i-1}\right)\right|$ and $\max \left\{\mathcal{C}_{\pi_{i}}\left(G_{i}\right)\right\}=$ $c\left|V\left(G_{i-2}\right)\right|+a\left|V_{i}\right|$.

In order to construct an $\{a, b, c\}$-edge-labelling $\pi_{5}$ for $G_{5}$, assign neighbour-distinguishing $\{a, b, c\}$-edge-labelling $\pi_{4}$ for subgraph $G_{4} \subset G_{5}$ and label $c$ to every edge in the set $E_{G}\left[V_{5}, V\left(G_{4}\right)\right]$. Note that $C_{\pi_{5}}\left(v^{5}\right)>C_{\pi_{5}}\left(v^{3}\right)$ since $c\left|V_{1}\right|+c\left|V_{2}\right|+c\left|V_{3}\right|+c\left|V_{4}\right|>$ $c\left|V_{1}\right|+b\left|V_{2}\right|+c\left|V_{5}\right|+a\left|V_{4}\right|$. Moreover, since $\left(\pi_{4}, C_{\pi_{4}}\right)$ is a neighbour-distinguishing $\{a, b, c\}-$ edge-labelling of $G_{4}$ with $C_{\pi_{4}}\left(v^{4}\right)<C_{\pi_{4}}\left(v^{2}\right)<C_{\pi_{4}}\left(v^{1}\right)<C_{\pi_{4}}\left(v^{3}\right)$ and the value $c\left|V_{5}\right|$ was added to the label of every vertex of $G_{4}$, we conclude that $C_{\pi_{5}}\left(v^{4}\right)<C_{\pi_{5}}\left(v^{2}\right)<C_{\pi_{5}}\left(v^{1}\right)<$ $C_{\pi_{5}}\left(v^{3}\right)$ and $\left(\pi_{5}, C_{\pi_{5}}\right)$ is a neighbour-distinguishing $\{a, b, c\}$-edge-labelling of $G_{5}$. Furthermore, note that $\min \left\{\mathcal{C}_{\pi_{5}}\left(G_{5}\right)\right\}=C_{\pi_{5}}\left(v^{4}\right)=C_{\pi_{4}}\left(v^{4}\right)+c\left|V_{5}\right|=a\left|V\left(G_{3}\right)\right|+c\left|V_{5}\right|$ and $\max \left\{\mathcal{C}_{\pi_{5}}\left(G_{5}\right)\right\}=C_{\pi_{5}}\left(v^{5}\right)=c\left|V\left(G_{4}\right)\right|$.

Now, consider $G_{i}$ with $i>5$. First, suppose that $i$ is even. In order to construct an $\{a, b, c\}$-edge-labelling $\pi_{i}$ for $G_{i}$, assign neighbour-distinguishing $\{a, b, c\}$-edge-labelling $\pi_{i-1}$ for subgraph $G_{i-1} \subset G_{i}$ and label $a$ to every edge in the set $E_{G}\left[V_{i}, V\left(G_{i-1}\right)\right]$. Since $\left(\pi_{i-1}, C_{\pi_{i-1}}\right)$ is a neighbour-distinguishing $\{a, b, c\}$-edge-labelling of $G_{i-1}$ and the value $a\left|V_{i}\right|$ was added to the label of every vertex of $G_{i-1}$, we have that $C_{\pi_{i}}\left(v^{l}\right)=C_{\pi_{i-1}}\left(v^{l}\right)+a\left|V_{i}\right|$, for $l \in\{1, \ldots, i-1\}$. Since $i-1$ is odd, $\min \left\{\mathcal{C}_{\pi_{i-1}}\left(G_{i-1}\right)\right\}=a\left|V\left(G_{i-3}\right)\right|+c\left|V_{i-1}\right|$ and $\max \left\{\mathcal{C}_{\pi_{i-1}}\left(G_{i-1}\right)\right\}=c\left|V\left(G_{i-2}\right)\right|$. Thus, we have that $C_{\pi_{i}}\left(v^{i}\right)=a\left|V\left(G_{i-1}\right)\right|=a\left|V\left(G_{i-3}\right)\right|+$ $a\left|V_{i-1}\right|+a\left|V_{i-2}\right|<a\left|V\left(G_{i-3}\right)\right|+c\left|V_{i-1}\right|+a\left|V_{i}\right|=\min \left\{\mathcal{C}_{\pi_{i-1}}\left(G_{i-1}\right)\right\}+a\left|V_{i}\right|$. Therefore, we conclude that $\left(\pi_{i}, C_{\pi_{i}}\right)$ is a neighbour-distinguishing $\{a, b, c\}$-edge-labelling of $G_{i}$ such that $\max \left\{\mathcal{C}_{\pi_{i}}\left(G_{i}\right)\right\}=\max \left\{\mathcal{C}_{\pi_{i-1}}\left(G_{i-1}\right)\right\}+a\left|V_{i}\right|=c\left|V\left(G_{i-2}\right)\right|+a\left|V_{i}\right|$ and $\min \left\{\mathcal{C}_{\pi_{i}}\left(G_{i}\right)\right\}=$ $C_{\pi_{i}}\left(v^{i}\right)=a\left|V\left(G_{i-1}\right)\right|$.

For the case when $i$ is odd, we construct an $\{a, b, c\}$-edge-labelling $\pi_{i}$ for $G_{i}$ by assigning
the neighbour-distinguishing $\{a, b, c\}$-edge-labelling $\pi_{i-1}$ for subgraph $G_{i-1} \subset G_{i}$ and assigning label $c$ to every edge in the set $E_{G}\left[V_{i}, V\left(G_{i-1}\right)\right]$. The proof that $\left(\pi_{i}, C_{\pi_{i}}\right)$ is neighbour-distinguishing is similar to the proof of the previous case with the difference that, in this case, $C_{\pi_{i}}\left(v^{i}\right)=c\left|V\left(G_{i-1}\right)\right|$ and $v^{i}$ has the maximum colour under $C_{\pi_{i}}$.

It is known that a complete bipartite graph with parts of different cardinalities has a neighbour-distinguishing $\{a\}$-edge-labelling [66], for $a \in \mathbb{R} \backslash\{0\}$. In fact, this result can be extended to some complete multipartite graphs, as stated in Proposition 6.33.

Proposition 6.33. Let $a \in \mathbb{R} \backslash\{0\}$. Let $G$ be a complete multipartite graph with at least three vertices and $r \geq 2$ parts $V_{1}, \ldots, V_{r}$. If $\left|V_{1}\right|<\cdots<\left|V_{r}\right|$, then $G$ has a neighbourdistinguishing $\{a\}$-edge-labelling.

Proof. Assign label $a$ to every edge of $G$. Since $\left|V_{1}\right|<\cdots<\left|V_{r}\right|$, any two adjacent vertices of $G$ have distinct degrees. Therefore, $C(u)=d_{G}(u) a \neq d_{G}(v) a=C(v)$, for every edge $u v \in E(G)$.

Davoodi and Omoomi [37] proved that every complete equipartite graph ${ }^{2} K(r, n)$ with $r \geq 2$ and $n \geq 2$ has a neighbour-distinguishing [2]-edge-labelling. Since every complete equipartite graph is regular, this result, along with Lemma 5.8, implies that every complete equipartite graph $K(r, n)$ with $r \geq 2$ and $n \geq 2$ has a neighbour-distinguishing $\{a, b\}$ -edge-labelling, for any two distinct real numbers $a$ and $b$. In view of these results, we pose the following question:

Question 6.34. Does every complete multipartite graph not isomorphic to a complete graph have a neighbour-distinguishing $\{a, b\}$-edge-labelling, for any two distinct real numbers a and b?

### 6.6 The related problem of detectable edge-labellings

Let $G$ be a connected simple graph and let $\pi: E(G) \rightarrow\{1, \ldots, k\}$ be a [k]-edge-labelling of $G$, for some positive integer $k$. The code of a vertex $v \in V(G)$ is the ordered $k$-tuple $\operatorname{code}_{\pi}(v)=\left(\ell_{1}, \ldots, \ell_{k}\right)$, where $\ell_{i}$ is the number of edges incident with $v$ that have label $i$, for $1 \leq i \leq k$. Labelling $\pi$ is a detectable $[k]$-edge-labelling of $G$ if $\operatorname{code}_{\pi}(u) \neq \operatorname{code}_{\pi}(v)$ for every edge $u v \in E(G)$. The detectable number $\operatorname{det}(G)$ of $G$ is the minimum positive integer $k$ for which $G$ has a detectable [ $k]$-edge-labelling. Figure 6.17 exhibits detectable edge-labellings of two graphs.

The concept of detectable edge-labelling was introduced in 2004 by Karoński et al. [65] motivated by the neighbour-distinguishing edge-labelling problem. In 2008, Escuadro et al. [43], independently, introduced and investigated the same problem. In both works, the authors pose the following conjecture.

Conjecture 6.35 (Karoński et al. [65], Escuadro et al. [43]). Every connected simple graph $G$ with at least three vertices has $\operatorname{det}(G) \leq 3$.

[^3]

Figure 6.17: A detectable [3]-edge-labelling of cycle $C_{5}$ and a detectable [2]-edge-labelling of a tree. The label of each vertex is its code.

Escuadro et al. [43] verified Conjecture 6.35 for cycles, complete graphs, complete multipartite graphs, trees, bipartite graphs and unicyclic graphs. Addario-Berry et al. [6] proved that every simple graph $G$ without connected components isomorphic to $K_{2}$ has $\operatorname{det}(G) \leq 4$ and also proved that if $G$ has $\delta(G) \geq 1000$, then $G$ has $\operatorname{det}(G) \leq 3$. Paramaguru and Sampathkumar [93] verified Conjecture 6.35 for cartesian products and tensor products of some graphs. In 2014, Havet et al. [55] proved that it is $\mathcal{N} \mathcal{P}$-complete to decide whether a cubic graph has a detectable [2]-edge-labelling and characterized all cubic graphs up to ten vertices according to their detectable number.

As observed by some authors [6, 65, 93], detectable edge-labellings and neighbourdistinguishing edge-labellings are closely related. In fact, a neighbour-distinguishing edgelabelling is also a detectable edge-labelling.

Proposition 6.36 (Paramaguru and Sampathkumar [93]). If a graph $G$ has a neighbourdistinguishing $[k]$-edge-labelling $\pi$, then $\pi$ is also a detectable $[k]$-edge-labelling of $G$.

Proof. Let $G$ be a graph with a neighbour-distinguishing [k]-edge-labelling $\pi$. For any edge $u v \in E(G)$, let $\ell_{i}, \ell_{i}^{\prime}$, respectively, be the number of edges incident with $u, v$ that have label $i$ under $\pi$. Then, $C_{\pi}(u)=1 \ell_{1}+2 \ell_{2}+\ldots+k \ell_{k} \neq 1 \ell_{1}^{\prime}+2 \ell_{2}^{\prime}+\ldots+k \ell_{k}^{\prime}=C_{\pi}(v)$. This implies that $\left(\ell_{1}, \ell_{2}, \ldots, \ell_{k}\right) \neq\left(\ell_{1}^{\prime}, \ell_{2}^{\prime}, \ldots, \ell_{k}^{\prime}\right)$, which in turn implies $\operatorname{code}_{\pi}(u) \neq \operatorname{code}_{\pi}(v)$. Therefore, $\pi$ is also a detectable [ $k$ ]-edge-labelling of $G$.

From Proposition 6.36, Theorem 6.9, Theorem 6.15, Theorem 6.21 and Theorem 6.28, we obtain the following corollary.

Corollary 6.37. If $G$ is a power of paths, a power of cycles, a split graph or a $k$-regular cobipartite graph without connected components isomorphic to $K_{2}$, then $\operatorname{det}(G) \leq 3$.

Although every neighbour-distinguishing edge-labelling is also detectable, the converse is not true. In fact, in 2016, Paramaguru and Sampathkumar [93] asked the following question.

Question 6.38 (Paramaguru and Sampathkumar [93]). Is there a positive integer $k$ for which there exists a graph $G$ with $\operatorname{det}(G)=k$ but no neighbour-distinguishing [ $k$ ]-edgelabelling?

Remember that the split graphs presented in the second item of Theorem 6.24 do not have a neighbour-distinguishing [2]-edge-labelling. However, they have a detectable [2]-edge-labelling, as shown in Theorem 6.40 below. Therefore, Theorem 6.40 answers

Question 6.38 in the affirmative. The following proposition is used in the proof of Theorem 6.40.

Proposition 6.39 (Escuadro et al. [43]). Let $\pi$ be a $[k]$-edge-labelling of a graph $G$ and let $u, v \in V(G)$. If $d_{G}(u) \neq d_{G}(v)$, then $\operatorname{code}_{\pi}(u) \neq \operatorname{code}_{\pi}(v)$.

Proof. Let $G$ be a graph with a $[k]$-edge-labelling $\pi$ and let $u, v \in V(G)$. Let $\ell_{i}, \ell_{i}^{\prime}$, respectively, be the number of edges incident with $u, v$ that have label $i$ under $\pi$. Suppose that $\operatorname{code}_{\pi}(u)=\operatorname{code}_{\pi}(v)$. Thus, $\left(\ell_{1}, \ell_{2}, \ldots, \ell_{k}\right)=\left(\ell_{1}^{\prime}, \ell_{2}^{\prime}, \ldots, \ell_{k}^{\prime}\right)$. Since $\ell_{1}+\ell_{2}+\ldots+\ell_{k}=$ $d_{G}(u)$ and $\ell_{1}^{\prime}+\ell_{2}^{\prime}+\ldots+\ell_{k}^{\prime}=d_{G}(v)$, we conclude that $d_{G}(u)=d_{G}(v)$.

Theorem 6.40. If $G$ is a split graph with a maximal clique $Q,|Q| \geq 3$, and an independent set $S=V(G) \backslash Q=\left\{u_{1}\right\}$ with $d_{G}\left(u_{1}\right)=1$, then $\operatorname{det}(G)=2$.

Proof. Let $G$ be as stated in the hypothesis and $q=|Q|$. Since $G$ has at least two adjacent vertices of the same degree, $\operatorname{det}(G) \geq 2$. In order to construct a [2]-edge-labelling $\pi$ for $G$, first, assign label 2 to the edge incident with vertex $u_{1}$. By Corollary 5.13, there exists a [2]-edge-labelling $\pi^{\prime}$ of $G[Q]$ such that

$$
\operatorname{code}_{\pi^{\prime}}\left(v_{i}\right)= \begin{cases}(q-1-i, i), & \text { for } 1 \leq i \leq\lfloor q / 2\rfloor \\ (q-i, i-1), & \text { for }\lfloor q / 2\rfloor+1 \leq i \leq q\end{cases}
$$

Then, assign $\pi^{\prime}$ to $G[Q]$ and adjust notation so that vertex $v_{\lfloor q / 2\rfloor}$ is adjacent to vertex $u_{1}$. Now, we prove that $\pi$ is detectable. By the definition of $\pi, \operatorname{code} e_{\pi}\left(v_{i}\right)=\operatorname{code} e_{\pi^{\prime}}\left(v_{i}\right)$, for $i \neq\lfloor q / 2\rfloor$. Thus, by the definition, any two distinct vertices in $Q \backslash\left\{v_{\lfloor q / 2\rfloor}\right\}$ have distinct codes under $\operatorname{code} e_{\pi}$. Since the degree of $v_{\lfloor q / 2\rfloor}$ is different from the degrees of its neighbours, by Proposition 6.39, the code of $v_{\lfloor q / 2\rfloor}$ is different from the code of all of its neighbours, and the result follows.

As previously discussed, a detectable edge-labelling may not be neighbour-distinguishing. However, a detectable [2]-edge-labelling of a regular graph $G$ is a neighbour-distinguishing [2]-edge-labelling of $G$, as can be seen in the proof of Lemma 6.41.

Lemma 6.41 (Paramaguru and Sampathkumar [93]). Let $G$ be a $k$-regular graph with $k \geq 2$. Then, $\operatorname{det}(G)=2$ if and only if $\chi_{\Sigma}^{\prime}(G)=2$.

Proof. Let $G$ be a $k$-regular graph with $k \geq 2$. Since $G$ has two adjacent vertices with the same degree, $\operatorname{det}(G) \geq 2$ and $\chi_{\Sigma}^{\prime}(G) \geq 2$. Suppose that $G$ has a detectable [2]-edge-labelling $\pi$. For any edge $u v \in E(G)$, let $\ell_{i}, \ell_{i}^{\prime}$, respectively, be the number of edges incident with $u, v$ that have label $i$ under $\pi$. Then, $\ell_{1}+\ell_{2}=k=\ell_{1}^{\prime}+\ell_{2}^{\prime}$ and $\left(\ell_{1}, \ell_{2}\right) \neq\left(\ell_{1}^{\prime}, \ell_{2}^{\prime}\right)$. These last two facts imply that $C_{\pi}(u)=1 \ell_{1}+2 \ell_{2} \neq 1 \ell_{1}^{\prime}+2 \ell_{2}^{\prime}=C_{\pi}(v)$. Therefore, $\pi$ is also a neighbour-distinguishing [2]-edge-labelling of $G$. The converse follows by Proposition 6.36.

From Lemma 6.41, Theorem 6.29 and Theorem 6.30, we obtain the following corollaries.

Corollary 6.42. If $G$ is a $k$-regular cobipartite graph with $2 k$ vertices, $k \geq 4$, then $\operatorname{det}(G)=2$.

Corollary 6.43. If $G$ is a $(2 k-2)$-regular cobipartite graph with $2 k$ vertices, $k \geq 3$, then $\operatorname{det}(G)=2$.

## Chapter 7

## Conclusions and future work

This thesis addresses three graph labelling problems: graceful labellings, neighbourdistinguishing edge-labellings, and neighbour-distinguishing total-labellings.

In Chapter 3, we investigate the 0 -rotatability of caterpillars. Our main target in that chapter is Conjecture 3.5, which states that every caterpillar with diameter at least five is 0 -rotatable. With that in mind, we prove that the following families of caterpillars are 0 -rotatable:
(i) caterpillars with a perfect matching (Theorem 3.7);
(ii) caterpillars obtained by identifying a central vertex of a path $P_{n}$ with a vertex of $K_{2}$ (Theorem 3.10);
(iii) caterpillars obtained by linking one leaf of the star $K_{1, s-1}$ to a leaf of a path $P_{n}$, $n \geq 3$ and $s \geq\left\lceil\frac{n}{2}\right\rceil$ (Theorem 3.12);
(iv) caterpillars with diameter five or six (Theorem 3.15 and Theorem 3.18); and
(v) caterpillars $T$ with $\operatorname{diam}(T) \geq 7$ such that, for every non-leaf vertex $v \in V(T)$, the number of leaves adjacent to $v$ is even and is at least $2+2((\operatorname{diam}(T)-1) \bmod 2)$ (Theorem 3.19).

These results reinforce Conjecture 3.5. In particular, the last three families show that, for each integer $d \geq 5$, there exist 0 -rotatable caterpillars with diameter $d$ and arbitrary number of vertices. It is also worth noting that we use a variety of techniques and previous known results in order to attack Conjecture 3.5. For example, Broersma-Hoede's construction ${ }^{1}$ is used in the proof of Theorem 3.7; on the other hand, Theorem 2.15 and Theorem 3.2 on 0 -rotatability of paths are crucial in the proofs of Theorems 3.10 and 3.12; and the technique of transfers is used in the proofs of Theorems 3.15, 3.18 and 3.19.

We believe that the construction of the model-tree $T_{\ell}^{1}$ in the proof of Theorem 3.19, as well as the way the technique of transfers is applied there, can be modified in order to find other families of 0-rotatable caterpillars with diameter at least seven. A natural extension of this work would be to consider caterpillars in which non-leaf vertices are adjacent to an odd number of leaves. We hope that some modification of the construction of the model-tree $T_{\ell}^{1}$ would help to settle this new case or even Conjecture 3.5. Thus, as future work, we would like to investigate the following question:

[^4]Question 7.1. Are all caterpillars with diameter at least seven, whose all non-leaf vertices are adjacent to an odd number of leaves, 0-rotatable?

In Chapter 4, we investigate $\alpha$-labellings of lobsters with maximum degree three. Our main targets in this topic are: Conjecture 4.2, which states that all trees with maximum degree three and a perfect matching have $\alpha$-labellings; and Question 4.3, which asks if all trees with at least 15 vertices, maximum degree three, and without $\alpha$-labellings belong to family $\mathcal{F}$ (family $\mathcal{F}$ is defined in page 91). In Chapter 4, we prove the following main results:
(i) If $G$ is a lobster with maximum degree three, without $Y$-legs, and with at most one forbidden ending, then $G$ has an $\alpha$-labelling (Theorem 4.4);
(ii) If $G$ is a lobster with a perfect matching such that its contree is balanced, then $G$ has an $\alpha$-labelling (Corollary 4.7).

Theorem 4.4 points towards an affirmative answer to Question 4.3 and Corollary 4.7 reinforces Conjecture 4.2. It is also important to emphasize the role of $\pi$-representations in the analysis of $\alpha$-labellings as graphical drawings, which made possible to devise an approach to find $\alpha$-labellings for the family of lobsters of Theorem 4.4. We believe that the same technique can be used to extend Theorem 4.4 so as to include the case in which lobster $G$ has two forbidden ends or, even further, the case in which $G$ has $Y$-legs. Therefore, the following questions arise as extensions of this work:

Question 7.2. Consider any lobster $G$ with at least 15 vertices, with $\Delta(G)=3$, without $Y$-legs, with two forbidden ends and such that $G$ does not belong to family $\mathcal{F}$. Does $G$ have an $\alpha$-labelling?

Question 7.3. Does every lobster $G$ with at least 15 vertices, maximum degree three and with $Y$-legs have an $\alpha$-labelling?

A research topic that was not approached in this thesis is the search for better upper bounds on the parameter $\operatorname{grac}(G)$ for trees ${ }^{2}$. Another interesting family for which this parameter could be investigated is the family of multipartite graphs. As discussed in page 41, it was conjectured by Beutner and Harborth [18] that the only graceful multipartite graphs are $K_{p, q}, K_{1, p, q}, K_{2, p, q}$, and $K_{1,1, p, q}$, for $p, q \in \mathbb{Z}_{>0}$. Thus, presenting upper bounds on $\operatorname{grac}(G)$ for this class could be a starting point to approach this conjecture. Moreover, the study of the parameter $\operatorname{grac}(G)$ for families of graphs is an important topic per se.

We would like to remark that the results presented in Chapter 3 and Chapter 4 were obtained in co-authorship with professor R. Bruce Richter, during our one-year stay at University of Waterloo, Canada. The results of Chapter 3 were presented in the workshop Primeiro Encontro de Teoria da Computação, XXXVI Congresso da Sociedade Brasileira de Computação, that occurred in July 2016, in Porto Alegre, Brazil. Moreover, the extended abstract [80] submitted to the workshop was awarded as the best article of the event. Another extended abstract [79], this time containing Theorem 4.4, was presented

[^5]in the Fourth Bordeaux Graph Workshop, that occurred in November 2016, in Bordeaux, France. In addition, two articles containing the results of Chapter 3 and Chapter 4, respectively, are submitted to periodicals of the area.

In Chapter 6, we investigate neighbour-distinguishing edge-labellings and neighbourdistinguishing total-labellings for five classes of graphs, namely: powers of paths, powers of cycles, split graphs, regular cobipartite graphs, and complete multipartite graphs. More specifically, we prove the following main results.
(a) Let $G$ be a simple graph and $a, b, c, t \in \mathbb{R}$, with $t \neq 0$ and $a<b<c$. Then:
(i) if $G \cong P_{n}^{k}$, then $G$ has a neighbour-distinguishing $\{t, 2 t\}$-total-labelling and a neighbour-distinguishing $\{t, 2 t, 3 t\}$-edge-labelling (Theorems 6.6 and 6.10);
(ii) if $G \cong C_{n}^{k}$, then $G$ has a neighbour-distinguishing $\{a, b\}$-total-labelling and a neighbour-distinguishing $\{t, 2 t, 3 t\}$-edge-labelling (Theorems 6.13 and 6.16);
(iii) if $G$ is a $k$-regular cobipartite graph, then $G$ has a neighbour-distinguishing $\{a, b\}$-total-labelling (Theorem 6.27);
(iv) if $G$ is a $k$-regular cobipartite graph, without connected components isomorphic to $K_{2}$, then $G$ has a neighbour-distinguishing $\{a, b, c\}$-edge-labelling (Theorem 6.28).
(b) Let $a, b, c \in \mathbb{R}$, with $0 \leq a<b<c$. Then:
(i) if $G$ is a split-graph, then $G$ has a neighbour-distinguishing $\{a, b\}$-total-labelling (Theorem 6.20);
(ii) if $G$ is a split-graph without connected components isomorphic to $K_{2}$, then $G$ has a neighbour-distinguishing $\{a, b, c\}$-edge-labelling (Theorem 6.21);
(iii) if $G$ is a complete multipartite graph with at least three parts, then $G$ has a neighbour-distinguishing $\{a, b, c\}$-edge-labelling (Theorem 6.32).

We point out that the proofs of Theorems 6.20, 6.21 and 6.32 can be adjusted in order to deal with the case where $a, b$ and $c$ are distinct negative real numbers. On the other hand, the methods used in these proofs certainly do not work for all cases where the set $\{a, b, c\}$ is allowed to have real numbers with different signals.

We also determine the minimum number of real labels in $\mathcal{L}$ in order to obtain neighbourdistinguishing $\mathcal{L}$-edge-labellings for two families of regular cobipartite graphs:
(i) if $G$ is a $k$-regular cobipartite graph with $2 k$ vertices, $k \geq 4$, then $G$ has a neighbourdistinguishing $\{a, b\}$-edge-labelling (Theorem 6.29);
(ii) if $G$ is a $(2 k-2)$-regular cobipartite graph with $2 k$ vertices, $k \geq 3$, then $G$ has a neighbour-distinguishing $\{a, b\}$-edge-labelling (Theorem 6.30).

It is worth noting that the results obtained for neighbour-distinguishing $\{a, b\}$-totallabellings of these classes point towards an affirmative answer to Question 5.5, which asks if every simple graph $G$ has a neighbour-distinguishing $\{a, b\}$-total-labelling, for distinct $a, b \in \mathbb{R}$. Furthermore, based on our results on neighbour-distinguishing edge-labellings, we pose Conjecture 7.4.

Conjecture 7.4. If $G$ is a simple graph without connected components isomorphic to $K_{2}$, then $G$ has a neighbour-distinguishing $\{a, b, c\}$-edge-labelling, for any three distinct real numbers $a, b, c$.

Let $\mathcal{L} \subset \mathbb{R}$. We remark that there are graphs that do not have neighbour-distinguishing $\mathcal{L}$-edge-labellings for any $\mathcal{L}$ with cardinality two. Examples of such graphs are the complete graphs and all members of family $\mathcal{T}$ defined on page 123. Therefore, the settlement of Conjecture 7.4 would imply that, in order to obtain a neighbour-distinguishing $\mathcal{L}$-edgelabelling of an arbitrary simple graph without connected components isomorphic to $K_{2}$, we must have $|\mathcal{L}| \geq 3$.

In Chapter 6, we also pose some conjectures on neighbour-distinguishing $\{a, b\}$-edgelabellings of the families we worked on. Two of these conjectures are very appealing to us, so that we intend to work on them in a near future:
(i) if $G$ is a simple graph with $n$ vertices such that $G \cong P_{n}^{k}, G \not \not K_{n}$, then $G$ has a neighbour-distinguishing [2]-edge-labelling (Conjecture 6.11);
(ii) if $G$ is a simple graph with $n$ vertices such that $G \cong C_{n}^{k}, G \nsubseteq K_{n}$ and $G \not \not C_{n}$, then $G$ has a neighbour-distinguishing [2]-edge-labelling (Conjecture 6.19);

The proof of Theorem 6.12 was presented in the VIII Latin-American Algorithms, Graphs and Optimization Symposium (LAGOS 2015), that occurred on May 2015, in Beberibe, Brazil, and an extended abstract is published in the Electronic Notes in Discrete Mathematics [78]. Another extended abstract, sketching the proofs of Theorem 6.9 and Theorem 6.15, was presented in the workshop Segundo Encontro de Teoria da Computação, XXXVII Congresso da Sociedade Brasileira de Computação [77], which occurred in July 2017, in São Paulo, Brazil. An article containing all these results is submitted to a periodical in the area. Most of these results were obtained in co-authorship with professors Simone Dantas and Diana Sasaki.

The results of Sections 6.3, 6.4, 6.5, and 6.6 were presented in the 17 th Colourings, Independence and Domination Workshop on Graph Theory, that occurred in September 2017, in Piechowice, Poland [81]. These results were obtained in co-authorship with professor Sheila M. de Almeida. An article detailing them is submitted to a periodical in the area.

Graceful labellings and neighbour-distinguishing labellings have different definitions. In particular, we would like to emphasize the difference in the property that the induced labelling must satisfy in each of these problems. For example, in a graceful labelling, the induced edge-labelling has to satisfy a global property: all edges of the graph must have distinct labels. However, in a neighbour-distinguishing labelling, the induced vertexlabelling has to satisfy a local property: it must be a vertex-colouring, that is, only adjacent vertices must have distinct colours. In fact, the local property makes it easier to devise inductive constructions for neighbour-distinguishing labellings of graphs, or to reuse neighbour-distinguishing labellings of specific subgraphs in order to compose the final neighbour-distinguishing labelling of the whole graph under consideration. Unfortunately, such a favourable context is rare when considering graceful labellings. In fact, despite
much effort done on the Graceful Tree Conjecture, no general technique is known that allows us to combine two or more gracefully labelled trees in arbitrary ways so as to generate a larger gracefully labelled tree. When considering graceful labellings of other classes of graphs, the situation can be even worse since many techniques developed to deal with trees may not work for other families. The difficulty in settling the Graceful Tree Conjecture remains and, despite the apparent ease in finding neighbour-distinguishing edge-labellings and neighbour-distinguishing total-labellings for specific graphs, the 1,2,3Conjecture and the 1,2-Conjecture remain open for arbitrary graphs.

## Bibliography

[1] J. Abrham. Graceful 2-regular graphs and Skolem sequences. Discrete Mathematics, 93(2-3):115-121, 1991.
[2] J. Abrham and A. Kotzig. On the missing value in a graceful numbering of a 2-regular graph. Congressus Numerantium, 65:261-266, 1988.
[3] J. Abrham and A. Kotzig. Graceful valuations of 2-regular graphs with two components. Discrete Mathematics, 150(1-3):3-15, 1996.
[4] B. D. Acharya, S. Arumugam, and A. Rosa, editors. Labelings of Discrete Structures and its Applications. Narosa Publishing House, 2008.
[5] B. D. Acharya, S. B. Rao, and S. Arumugam. Embeddings and NP-complete problems for graceful graphs. In B. D. Acharya, A. Arumugam, and A. Rosa, editors, Labelings of discrete structures and applications, page 57-62. Narosa Publishing House, 2008.
[6] L. Addario-Berry, R. E. L. Aldred, K. Dalal, and B. A. Reed. Vertex colouring edge partitions. Journal of Combinatorial Theory, Series B, 94(2):237-244, 2005.
[7] L. Addario-Berry, K. Dalal, C. McDiarmid, B. A. Reed, and A. Thomason. Vertexcolouring edge-weightings. Combinatorica, 21(1):1-12, 2007.
[8] L. Addario-Berry, K. Dalal, and B. A. Reed. Degree constrained subgraphs. Discrete Applied Mathematics, 156(7):1168-1174, 2008.
[9] Y. Alavi, G. Chartrand, F. R. K. Chung, P. Erdös, R. L. Graham, and O. R. Oellermann. Highly irregular graphs. Journal of Graph Theory, 11(2):235-249, 1987.
[10] M. Anholcer. Product irregularity strength of graphs. Discrete Mathematics, 309(22):6434-6439, 2009.
[11] D. Anick. Counting graceful labelings of trees: A theoretical and empirical study. Discrete Applied Mathematics, 198:65-81, 2016.
[12] C. Balbuena, P. García-Vázquez, X. Marcote, and J. C. Valenzuela. Trees having an even or quasi even degree sequence are graceful. Applied Mathematics Letters, 20(4):370-375, 2007.
[13] C. Barrientos and E. Krop. Improved bounds for relaxed graceful trees. Graphs and Combinatorics, 33(2):287-305, 2017.
[14] M. Bača, S. Jendrol, M. Miller, and J. Rian. On irregular total labellings. Discrete Mathematics, 307(11-12):1378-1388, 2007.
[15] J. Bensmail. Partitions et décompositions de graphes. PhD thesis, École Doctorale de Mathématiques et Informatique de Bordeaux, Université de Bordeaux, Bordeaux, France, 2014.
[16] J. Bensmail. A 1-2-3-4 result for the 1-2-3 conjecture in 5-regular graphs. Available at: https://hal.archives-ouvertes.fr/hal-01509365/document, April 2017.
[17] J.-C. Bermond. Graceful graphs, radio antennae and french windmills. In R. J. Wilson, editor, Graph Theory and Combinatorics, Research Notes in Mathematics, pages 18-37. Pitman Publishing, 1979.
[18] D. Beutner and H. Harborth. Graceful labelings of nearly complete graphs. Results in Mathematics, 41(1-2):34-39, 2002.
[19] G. S. Bloom and S. W. Golomb. Numbered complete graphs, unusual rulers, and assorted applications. In Y. Alavi and D. R. Lick, editors, Theory and Applications of Graphs, volume 642 of Lecture Notes in Mathematics, page 53-65. Springer, Berlin, Heidelberg, 1978.
[20] C. P. Bonnington and J. Širáň. Bipartite labeling of trees with maximum degree three. Journal of Graph Theory, 31(1):7-15, 1999.
[21] L. Brankovic, C. Murch, J. Pond, and A. Rosa. Alpha-size of trees with maximum degree three and perfect matching. In Proccedings of AWOCA 2005, Ballarat, Australia, page 47-56, 2005.
[22] L. Brankovic, A. Rosa, and J. Širáň. Labellings of trees with maximum degree three - an improved bound. Journal of Combinatorial Mathematics and Combinatorial Computing, 55:159-169, 2005.
[23] G. Brinkmann, S. Crevals, H. Mélot, L. Rylands, and E. Steffen. $\alpha$-labellings and the structure of trees with nonzero $\alpha$-deficit. Discrete Mathematics and Theoretical Computer Science, 14(1):159-174, 2012.
[24] H. J. Broersma and C. Hoede. Another equivalent of the graceful tree conjecture. Ars Combinatoria, 51:183-192, 1999.
[25] A. C. Burris and R. H. Schelp. Vertex-distinguishing proper edge-colorings. Journal of Graph Theory, 26(2):73-82, 1997.
[26] F. V. Bussel. Relaxed graceful labellings of trees. The Eletronic Journal of Combinatorics, 9 (R4):1-12, 2002.
[27] F. V. Bussel. 0-centred and 0-ubiquitously graceful trees. Discrete Mathematics, 277(1-3):193-218, 2004.
[28] N. Cairnie and K. Edwards. The computational complexity of cordial and equitable labelling. Discrete Mathematics, 216(1-3):29-34, 2000.
[29] R. Cattell. Graceful labellings of paths. Discrete Mathematics, 307(24):3161-3176, 2007.
[30] G. J. Chang, C. Lu, J. Wu, and Q. Yu. Vertex-coloring edge-weightings of graphs. Taiwanese Journal of Mathematics, 15(4):1807-1813, 2011.
[31] G. Chartrand. Highly irregular. In R. Gera, S. Hedetniemi, and C. Larson, editors, Graph Theory - Favorite conjectures and open problems - 1, page 1-16. Springer, 2010.
[32] G. Chartrand and M. Behzad. No graph is perfect. American Mathematical Monthly, 74:962-963, 1967.
[33] G. Chartrand, M. S. Jacobson, J. Lehel, O. R. Oellermann, S. Ruiz, and F. Saba. Irregular networks. Congressus Numerantium, 64:197-210, 1988.
[34] G. Chartrand, O. R. Oellermann, and P. Erdös. How to define an irregular graph. The College Mathematics Journal, 19(1):36-42, 1988.
[35] W. C. Chen, H. I. Lu, and Y. N. Yeh. Operations of interlaced trees and graceful trees. Southeast Asian Bulletin of Mathematics, 21:337-348, 1997.
[36] F. Chung and F. Hwang. Rotatable graceful graphs. Ars Combinatoria, 11:239-250, 1981.
[37] A. Davoodi and B. Omoomi. On the 1-2-3-conjecture. Discrete Mathematics and Theoretical Computer Science, 17(1):67-78, 2015.
[38] distributed.net. Project OGR. Available at: http://www.distributed.net/ogr, 2017.
[39] A. Dudek and D. Wajc. On the complexity of vertex-coloring edge-weightings. Discrete Mathematics and Theoretical Computer Science, 13(3):45-50, 2011.
[40] R. A. Duke. Can the complete graph with $2 n+1$ vertices be packed with copies of an arbitrary tree having $n$ edges? The American Mathematical Monthly, 76(10):1128-1130, 1969.
[41] P. Erdös, A. Rubin, and H. Taylor. Choosability in graphs. Congressus Numerantium, 26:125-157, 1979.
[42] H. Escuadro, F. Okamoto, and P. Zhang. Circulants and a three-color conjecture. In Proceedings of the Thirty-Seventh Southeastern International Conference on Combinatorics, Graph Theory and Computing. Congressus Numerantium, volume 178, pages 33-55, 2006.
[43] H. Escuadro, F. Okamoto, and P. Zhang. A three-color problem in graph theory. Bulletin of the Institute of Combinatorics and its Applications, 52:68-82, 2008.
[44] K. Eshghi. Construction of $\alpha$-labeling of 2-regular graphs with three components. PhD thesis, University of Toronto, 1997.
[45] K. Eshghi. Extension of $\alpha$-labelings of quadratic graphs. International Journal of Mathematics and Mathematical Sciences, 2004(11):571-578, 2004.
[46] W. Fang. A computational approach to the graceful tree conjecture. Available at: http://arxiv.org/pdf/1003.3045v2.pdf, 2010.
[47] E. Flandrin, I. Fournier, and A. Germa. Numerotations gracieuses des chemins. Ars Combinatoria, 16:149-181, 1983.
[48] H. Fleischner. Eulerian Graphs and Related Topics. Annals of Discrete Mathematics 45. North-Holland, 1990.
[49] J. A. Gallian. A dynamic survey of graph labeling. The Eletronic Journal of Combinatorics, DS6:1-432, 2017.
[50] Y. Gao, G. Wang, and J. Wu. A relaxed case on the 1-2-3 conjecture. Graphs and Combinatorics, 32(4):1415-1421, 2016.
[51] M. Gardner. Mathematical games: A miscellany of transcendental problens simple to state but not at all easy to solve. Scientific American, 226(6):114-121, June 1972.
[52] S. W. Golomb. How to number a graph. In R. C. Read, editor, Graph Theory and Computing, page 23-27. Academic Press, New York, 1972.
[53] R. L. Graham and N. J. A. Sloane. On additive bases and harmonious graphs. SIAM Journal on Algebraic Discrete Methods, 1(4):382-404, 1980.
[54] H Hajiabolhassan, M. L. Mehrabadi, and R. Tusserkani. Minimal coloring and strength of graphs. Discrete Mathematics, 215(1-3):265-270, 2000.
[55] F. Havet, N. Paramaguru, and R. Sampathkumar. Detection number of bipartite graphs and cubic graphs. Discrete Mathematics and Theoretical Computer Science, 16(3):333-342, 2014.
[56] M. Haviar and M. Ivaška. Vertex Labellings of Simple Graphs. Research and exposition in mathematics. Heldermann Verlag, 2015.
[57] K. Heinrich and P. Hell. On the problems of bandsize. Graphs and Combinatorics, 3(1):279-284, 1987.
[58] P. Hrnčiar and A. Haviar. All trees of diameter five are graceful. Discrete Mathematics, 233(1-3):133-150, 2001.
[59] C. Huang, A. Kotzig, and A. Rosa. Further results on tree labellings. Utilitas Mathematica, 21:31-48, 1982.
[60] J. Hulgan, J. Lehel, K. Ozeki, and K. Yoshimoto. Vertex coloring of graphs by total 2-weightings. Graphs and Combinatorics, 32(6):2461-2471, 2016.
[61] J. J. Jesintha and G. Sethuraman. All arbitrarily fixed generalized banana trees are graceful. Mathematics in Computer Science, 5(1):51-62, 2011.
[62] M. Kalkowski. Metody algorytmiczne w badaniach sily nieregularności grafów. PhD thesis, WydziałMatematyki i Informatyki, Uniwersytet im. Adama Mickiewicza, 2010. (in Polish).
[63] M. Kalkowski, M. Karoński, and F. Pfender. Vertex-coloring edge-weightings: towards the 1-2-3-conjecture. Journal of Combinatorial Theory, Series B, 100(3):347-349, 2010.
[64] M. Kalkowski, M. Karoński, and F. Pfender. A new upper bound for the irregularity strength of graphs. SIAM Journal on Discrete Mathematics, 25(3):1319-1321, 2011.
[65] M. Karoński, T. Łuczak, and A. Thomason. Edge weights and vertex colours. Journal of Combinatorial Theory, Series B, 91(1):151-157, 2004.
[66] M. Khatirinejad, R. Naserasr, M. Newman, B. Seamone, and B. Stevens. Vertexcolouring edge-weightings with two edge weights. Discrete Mathematics and Theoretical Computer Science, 14(1):1-20, 2012.
[67] K. M. Koh, D. G. Rogers, and T. Tan. On graceful trees. Nanta Mathematica, 10:207-211, 1977.
[68] K. M. Koh, T. Tan, and D. G. Rogers. Two theorems on graceful trees. Discrete Mathematics, 25(2):141-148, 1979.
[69] A. Kotzig. On certain vertex-valuations of finite graphs. Utilitas Mathematica, 4:261-290, 1973.
[70] A. Kotzig. $\beta$-valuations of quadratic graphs with isomorphic components. Utilitas Mathematica, 7:263-279, 1975.
[71] A. Kotzig. Recent results and open problems in graceful graphs. Congressus Numerantium, 44:197-219, 1984.
[72] E. Kubicka. The chromatic sum and efficient tree algorithms. PhD thesis, Western Michigan University, 1989.
[73] J. Lehel. Facts and quests on degree irregular assignments. In Y. Alavi, G. Chartrand, A. J. Schwenk, and O. R. Oellermann, editors, Graph Theory, Combinatorics, and Applications, volume 2, page 765-781. John Willey, New York, 1988.
[74] L. Lovász. On decomposition of graphs. Studia Scientiarum Mathematicarum Hungarica, 1:237-238, 1966.
[75] H. Lu. Vertex-coloring edge-weighting of bipartite graphs with two edge weights. Discrete Mathematics and Theoretical Computer Science, 17(3):1-12, 2015.
[76] H. Lu, Q. Yu, and C.-Q. Zhang. Vertex-coloring 2-edge-weighting of graphs. European Journal of Combinatorics, 32(1):21-27, 2011.
[77] A. G. Luiz and C. N. Campos. The 1,2,3-conjecture for powers of paths and powers of cycles. In Anais do XXXVII Congresso da Sociedade Brasileira de Computação, pages 95-98, São Paulo, Brazil, July 2017. Available at: http://www.sbc.org.br/csbc2017/.
[78] A. G. Luiz, C. N. Campos, S. Dantas, and D. Sasaki. The 1,2-conjecture for powers of cycles. Electronic Notes in Discrete Mathematics, 50:83-88, 2015.
[79] A. G. Luiz, C. N. Campos, and R. B. Richter. $\alpha$-labellings of lobsters with maximum degree three. In Bordeaux Graph Workshop 2016, pages 70-73, Bordeaux, France, November 2016. Available at: http://bgw.labri.fr/2016/.
[80] A. G. Luiz, C. N. Campos, and R. B. Richter. Some families of 0-rotatable graceful caterpillars. In Anais do XXXVI Congresso da Sociedade Brasileira de Computação, pages 812-815, Porto Alegre, Brazil, July 2016. Available at: http://ebooks.pucrs.br/edipucrs/anais/csbc/assets/2016/anais-csbc-2016.pdf.
[81] A. G. Luiz, S. M. de Almeida, and C. N. Campos. Neighbour-distinguishing edgelabellings and total-labellings of families of graphs. In $1^{7}$ th Colourings, Independence and Domination Workshop on Graph Theory (booklet of abstracts), pages 72-73, Piechowice, Poland, September 2017.
[82] S. C. López and F. A. Muntaner-Batle. Graceful, Harmonious and Magic Type Labelings - Relations and Techniques. Springer Briefs in Mathematics. Springer International Publishing, 2017.
[83] A. M. Marr and W. D. Wallis. Magic graphs. Birkhäuser Basel, 2013.
[84] M. Mavronicolas and L. Michael. A substitution theorem for graceful trees and its applications. Discrete Mathematics, 309(12):3757-3766, 2009.
[85] D. Mishra and P. Panigrahi. Graceful lobsters obtained by partitioning and component moving of branches of diameter four trees. Computers $\mathcal{B}$ Mathematics with Applications, 50(3-4):367-380, 2005.
[86] D. Mishra and P. Panigrahi. Graceful lobsters obtained by component moving of diameter four trees. Ars Combinatoria, 81:129-146, 2006.
[87] D. Mishra and P. Panigrahi. Some graceful lobsters with both odd and even degree vertices on the central path. Utilitas Mathematica, 74:155-177, 2007.
[88] D. Mishra and P. Panigrahi. Some graceful lobsters with all three types of branches incident on the vertices of the central path. Computers $8 \mathcal{B}$ Mathematics with Applications, 56(5):1382-1394, 2008.
[89] D. Mishra and P. Panigrahi. Some new classes of graceful lobsters obtained from diameter four trees. Mathematica Bohemica, 135(3):257-278, 2010.
[90] D. Morgan. All lobsters with perfect matchings are graceful. Electronic Notes in Discrete Mathematics, 11:503-508, 2002.
[91] T. Nierhoff. A tight bound on the irregularity strength of graphs. SIAM Journal on Discrete Mathematics, 13(3):313-323, 2000.
[92] F. Okamoto, E. Salehi, and P. Zhang. On multiset colorings of graphs. Discussiones Mathematicae Graph Theory, 30(1):137-153, 2010.
[93] N. Paramaguru and R. Sampathkumar. Graphs with vertex-coloring and detectable 2-edge-weighting. AKCE International Journal of Graphs and Combinatorics, 13(2):146-156, 2016.
[94] J. Przybylo and M. Wozniak. On a 1, 2 conjecture. Discrete Mathematics and Theoretical Computer Science, 12(1):101-108, 2010.
[95] W. T. Rankin. Optimal golomb rulers: An exhaustive parallel search implementation. Master's thesis, Duke University, 1993.
[96] G. Ringel. Problem 25. In Theory of graphs and its applications, Proceedings of Symposium Smolenice, volume 162, 1964.
[97] G. Ringel and N. Hartsfield. Pearls in Graph Theory. Academic Press, Cambridge, 1990.
[98] A. Rosa. On certain valuations of the vertices of a graph. Theory of Graphs (Internat. Sympos., Rome, 1966) Gordon and Breach, New York; Dunod, Paris, page 349-355, 1967.
[99] A. Rosa. Labeling snakes. Ars Combinatoria, 3:67-74, 1977.
[100] A. Rosa. A discourse on three combinatorial diseases, 1991.
[101] A. Rosa and A. Kotzig. Magic valuations of finite graphs. Canadian Mathematical Bulletin, 13(4):451-461, 1970.
[102] A. Rosa and J. Širáň. Bipartite labelings of trees and the gracesize. Journal of Graph Theory, 19(2):201-215, 1995.
[103] B. Seamone. The 1-2-3 conjecture and related problems: a survey. Available at: https://arxiv.org/abs/1211.5122, 2012.
[104] G. Sethuraman and J. J. Jesintha. A new class of graceful lobsters. Journal of Combinatorial Mathematics and Combinatorial Computing, 67:99-109, 2008.
[105] G. Sethuraman, P. Ragukumar, and P. J. Slater. Any tree with $m$ edges can be embedded in a graceful tree with less than $4 m$ edges and in a graceful planar graph. Discrete Mathematics, 340:96-106, 2017.
[106] J. B. Shearer. Best known golomb rulers. Available at: http://www.research.ibm.com/people/s/shearer/grtab.html, 2017.
[107] G. J. Simmons. Synch-sets: a variant of difference sets. Congressus Numerantium, 10:625-645, 1974.
[108] R. G. Stanton and C. R. Zarnke. Labelling of balanced trees. In Proceedings of the Fourth South Eastern Conference on Combinatorics, Graph Theory and Computing, page 479-495, Boca Raton, 1973.
[109] R. A. Sucupira, S. Klein, and L. Faria. Grafos half cut. In Anais do XXXVII Congresso da Sociedade Brasileira de Computação, page 115-118, 2017.
[110] M. C. Superdock. The graceful tree conjecture: a class of graceful diameter-6 trees. Available at: https://arxiv.org/abs/1403.1564, May 2013.
[111] C. Thomassen, P. Erdös, Y. Alavi, P. J. Malde, and A. J. Schwenk. Tight bounds on the chromatic sum of a connected graph. Journal of Graph Theory, 13(3):353-357, 1989.
[112] C. Thomassen, Y. Wu, and C.-Q. Zhang. The 3-flow conjecture, factors modulo k, and the 1-2-3-conjecture. Journal of Combinatorial Theory, Series B, 121:308-325, 2016.
[113] O. Veblen. An application of modular equations in analysis situs. Annals of Mathematics, 14(2):86-94, 1912.
[114] J.-G. Wang, D.-J. Jin, X.-G. Lu, and D. Zhang. The gracefulness of a class of lobster trees. Mathematical and Computer Modelling, 20(9):105-110, 1994.
[115] T. Wang and Q. Yu. On vertex-colouring 13-edge-weighting. Frontier of Mathematics in China, 3(4):581-587, 2008.
[116] T-M. Wang, C-C. Yang, L-H. Hsu, and E. Cheng. Infinitely many equivalent versions of the graceful tree conjecture. Applicable Analysis and Discrete Mathematics, $9(1): 1-12,2015$.
[117] B. Yao, H. Cheng, M. Yao, and M. Zhao. A note on strongly graceful trees. Ars Combinatoria, 92:155-169, 2009.
[118] P. Zhang. Color-Induced Graph Colorings. SpringerBriefs in Mathematics. Springer International Publishing, 2015.
[119] S. L. Zhao. All trees with diameter four are graceful. Annals of the New York Academy of Sciences, 576(1):700-706, 1989.

## Index of definitions

Symbols
$\alpha$-labelling ..... 28
$\beta$-labelling ..... 28
$\pi$-representation ..... 94
$\rho$-labelling ..... 28
$\sigma$-labelling ..... 28
B
balanced bipartition ..... 92
balanced tree ..... 92
bipartition of a graph ..... 18
C
caterpillar ..... 20
central vertex ..... 19
clique ..... 16
closed trail ..... 41
cobipartite graph ..... 152
complement of a graph ..... 16
connected component ..... 17
contree ..... 67
copy of a graph ..... 17
cycle ..... 19
D
decomposition ..... 21
cyclic decomposition ..... 22
degree
degree of a vertex ..... 16
degree sequence ..... 16
maximum degree ..... 16
minimum degree ..... 16
detectable number ..... 156
diameter ..... 19
disjoint ..... 17
distance ..... 19
E
edge cut ..... 17
edge parity ..... 60
edge-disjoint ..... 17
element ..... 14
endpoints ..... 14
even graph ..... 41
F
family $\mathcal{F}$ ..... 91
family $\mathcal{T}$ ..... 123
G
gracefulness ..... 47
gracesize ..... 51
graph ..... 14
bipartite ..... 18
complete ..... 18
connected ..... 17
empty ..... 18
equipartite ..... 19
multipartite ..... 18
r-partite ..... 18
simple ..... 15
trivial ..... 14
H
half cut ..... 44
I
identification ..... 20
independent set ..... 16
isomorphic graphs ..... 16
isomorphism ..... 17
L
labelling
bipartite labelling ..... 60
complementary labelling ..... 32
edge-relaxed graceful labelling ..... 51
range relaxed graceful labelling ..... 47
reverse labelling ..... 32
strongly- $\alpha$ labelling ..... 106
vertex-relaxed graceful labelling ..... 51
leaf ..... 19
length of path or cycle ..... 19
level ..... 19
lobster ..... 20
M
matching ..... 16
perfect matching ..... 16
model-tree ..... 80
O
order of a graph ..... 14
P
parity property ..... 62
part ..... 18
path ..... 19, 20
power of a graph ..... 131
power of cycles ..... 141
power of paths ..... 131
R
reach of an edge ..... 22, 131
Ringel's Conjecture ..... 27
Ringel-Kotzig Conjecture ..... 27
rotation of a subgraph ..... 22
S
size of a graph ..... 14
spine ..... 19
split graph ..... 147
star ..... 18
subgraph
forbidden subgraph ..... 52
T
transfer ..... 53
first type ..... 55
second type ..... 55
tree
rooted ..... 19
V
vertex
isolated ..... 16
saturated ..... 16
universal ..... 16
vertex parity ..... 60


[^0]:    ${ }^{1}$ Reach of an edge of $K_{n}$ is defined in page 22.

[^1]:    ${ }^{2}$ The separator of an $\alpha$-labelling was defined in page 27.

[^2]:    ${ }^{1}$ The $k$-th power of graph was defined on page 131.

[^3]:    ${ }^{2}$ Complete equipartite graph is defined in page 19.

[^4]:    ${ }^{1}$ Broersma-Hoede's construction is described in Section 2.5.

[^5]:    ${ }^{2}$ The parameter $\operatorname{grac}(G)$ is defined in page 47 .

