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Optimal Sampled-Data State Feedback Control Applied to Markov Jump Linear Systems

Controle Amostrado Ótimo de Sistemas Lineares com Saltos Markovianos Através de Realimentação de Estados

Campinas

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Orientador: Prof. Dr. José Cláudio Geromel

Este exemplar corresponde à versão final da Tese defendida pela aluna Gabriela Werner Gabriel,e orientada pelo Prof. Dr. José Cláudio Geromel.

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"No great thing is suddenly created."...

..."It is difficulties that show what men are."

(Epictetos)

Abstract

This work is entirely devoted to develop an optimal sampled-data control law applied to Markov jump linear systems, whose main usage is Networked Control Systems (NCS). In this context, two network characteristics are simultaneously considered: the bandwidth limitation addressed by the existence of sampled-data signals in the system, and the packet dropouts modeled by a continuous-time Markov chain. In order to accomplish this goal, the general adopted approach is broken in four steps: stability analysis and norm evaluation based on the $\mathcal{H}_2$ norm; stability analysis and norm evaluation in the $\mathcal{H}_\infty$ context; the optimal sampled-data control design that minimizes a $\mathcal{J}_2$ performance index based on the $\mathcal{H}_2$ norm, which can be expressed in a convex formulation based on LMIs; the optimal sampled-data control design that minimizes a certain $\mathcal{J}_\infty$ performance index based on the $\mathcal{H}_\infty$ norm, which also admits a convex formulation based on LMIs, even though a deeper mathematical analysis is required. Each step has the same structure described in the sequel. First, the theoretical results are mathematically developed and proved. Second, some particular cases are derived from these theoretical results. Third, a convergent algorithm is proposed to solve each of the mentioned cases. The convergence of the algorithms are also proved. Finally, a numerical example illustrates the main developments in each step. The theory developed here is new and there is no similar result in the current literature. For a practical view of the outcomes, three practical examples are borrowed and adapted from available works: two of them are physical systems controlled through an NCS, where one is originally stable and the other unstable, and the third one is an economical system whose policy is applied in a discrete-time basis.

Keywords: Control theory; Markov processes; Optimal control systems; Hybrid systems.

Resumo

Este trabalho é inteiramente dedicado ao desenvolvimento de uma lei de controle ótimo amostrado aplicada a sistemas lineares com saltos markovianos, cujo principal uso são os sistemas controlados através da rede (NCS - *Networked Control System*). Neste contexto, duas características da rede são consideradas simultaneamente: a limitação da largura de banda, tratada através da existência de sinais amostrados no sistema, e a perda de pacotes, modelada através de uma cadeia de Markov a tempo contínuo. A fim de alcançar este objetivo, a abordagem geral adotada é dividida em quatro etapas: análise de estabilidade e cálculo de norma no contexto da norma $\mathcal{H}_2$; análise de estabilidade e cálculo de norma no contexto da norma $\mathcal{H}_\infty$; projeto de controle amostrado ótimo que minimiza o índice de desempenho $\mathcal{J}_2$ baseado na norma $\mathcal{H}_2$, o qual pode ser expresso em uma formulação convexa baseada em LMIs; projeto de controle amostrado ótimo que minimiza um certo índice de desempenho $\mathcal{J}_\infty$ baseado na norma $\mathcal{H}_\infty$, o qual também admite uma formulação convexa baseada em LMI, embora uma análise matemática mais aprofundada seja necessária. Cada uma destas etapas possui a mesma estrutura descrita a seguir. Primeiro, os resultados teóricos são matematicamente desenvolvidos e provados. Segundo, alguns casos particulares são derivados a partir destes resultados teóricos. Terceiro, um algoritmo convergente é proposto para resolver cada um dos casos mencionados. As convergências também são provadas. Finalmente, um exemplo teórico ilustra os principais desenvolvimentos em cada caso. A teoria aqui desenvolvida é nova, não havendo resultado similar na literatura atual. Para uma visão prática dos resultados desta dissertação, três exemplos são considerados e adaptados de trabalhos disponíveis: dois deles correspondem a sistemas físicos controlados através de uma rede sendo um originalmente estável e o outro instável, e o terceiro corresponde a um sistema econômico cujas políticas de controle são aplicadas a tempo discreto.

Palavras-chaves: Teoria de controle; Processos de Markov; Controle ótimo; Sistemas híbridos.

Preface

Around a decade ago when I was finalizing my studies to achieve my master of science degree, I was trying to control a robot that I made by myself. The main idea was that given a structured environment with initial, final, free and forbidden cells, the robot should travel through the defined area from the start to the final cell avoiding the forbidden ones (see Gabriel, Nascimento-Jr. & Yagyu (2006)). This robot, named ROMEO III (*http://www.ele.ita.br/romeo/romeoiii/, accessed in November 25th, 2015, in Portuguese*), had three embedded computational boards interconnected through a linear wired point-to-point communication network. They were responsible for controlling the robot leading it to follow exactly the path previously computed. The communication among all computational boards was based on interruptions, but without any rigorous protocol to control synchronization among receiving, transmitting, and processing tasks. The general structure of the communication system of ROMEO III is presented in Figure 1. Evidently, the linear network structure among all boards could cause communication delays between the main computer board (Embedded Board I) and the board used to primarily control the trajectory (Embedded Board III). This is just to name one of all communication problems it indeed had. However,

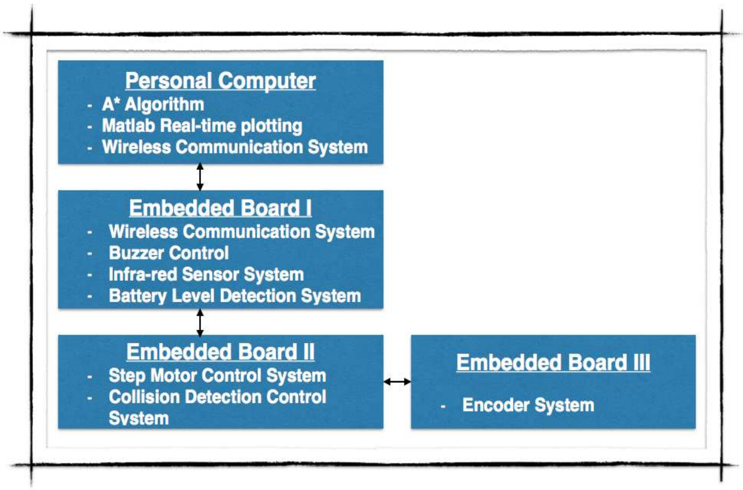

Figure 1 – ROMEO III Communication Network.

at that time, it was interesting and somewhat curious for me that sometimes the platform did not followed the trajectory as expected! Consequently, it did not reach the final cell!

That was my first observation on the effects of communication intrinsic characteristics on controlled systems!

The following eight years I worked in the telecommunication area developing equipment for optical communication including, and mainly, the ones devoted to manage the communication process either as development engineer, designing hardware and firmwares, or latter as a project or a product manager. During these years I surely could experience different problems and aspects of communication process over optical fiber. Even in this very fast environment, due to the available technology, optical fiber has also lots of limitations such as: bandwidth limitation, finite packet length, packet dropout, interference among signals traveling in a same fiber, finite capacity of different frequency channels in a same fiber, scheduling necessity, among others. The most important problems were bypassed using proprietary communication protocols.

Curiously, when I was back to the university, I started my doctorate at Networked Control Systems (NCS), which merges concepts about Telecommunication and System Control theories. Here, I could study mathematically the effects of the telecommunication environment over the control process. Among all important aspects, I could face the mathematical difficulties to put all of them together in a same model. Evidently, the solution of this problem is not trivial. Proof of this is the very low quantity of works devoted to implement it. That said, considering that it is mathematically challenging to treat all problems derived from network characteristics in a same model, this work is a mathematical study about two important aspects of NCS: bandwidth limitation and packet dropouts. In the following pages, it will be used a continuous-time Markov jump linear plant controlled by a sampled-data control law to address both issues simultaneously. As many other authors, I will use Markov Jump Linear System (MJLS) to model an imperfect network, and a hybrid system, also known as a Hybrid Control System, to model the sampled-data control. I must tell you that this dissertation is a very small part of the evolution of our working group in the understanding of the NCS design.

G. W. Gabriel

List of Abbreviations and Acronyms

CARE	Continuous-time Algebraic Riccati Equation
DALE	Discrete-time Algebraic Lyapunov Equation
DARE	Discrete-time Algebraic Riccati Equation
DLE	Differential Lyapunov Equation
DRE	Differential Riccati Equation
HMJLS	Hybrid Markov Jump Linear System
$\inf$	Infimum mathematical operator
$\lim$	Limit mathematical operator
LMI	Linear Matrix Inequality
LTI	Linear Time-Invariant
$\max$	Maximum mathematical operator
MJLS	Markov Jump Linear System
MSS	Mean Square Stable
NCS	Networked Control System
ODE	Ordinary Differential Equation
PWC	Piecewise Continuous
$\sup$	Supremum mathematical operator
TPBVP	Two-Point Boundary Value Problem
ZCD	Zero-Crossing-Detector

List of Some Important Symbols

e_l	Each of the l-th column of the identity matrix of compatible dimension
$\mathfrak{F}(\cdot)$, $\mathfrak{G}(\cdot)$	Transfer function
$\mathcal{H}_2$	Hardy space $\mathcal{RH}_2$ (Definition II.16)
$\mathcal{H}_\infty$	Hardy space $\mathcal{RH}_\infty$ (Definition II.17)
$\mathcal{J}_2$	Performance index based on $\mathcal{H}_2$ Hardy space
$\mathcal{J}_\infty$	Performance index based on $\mathcal{H}_\infty$ Hardy space
$\mathbb{K}$	Set of the modes of the Markov chain
$\mathcal{L}$	Infinitesimal generator operator
ℓ	Iteration counter of an algorithm
N	Number of the Markov modes
$\mathcal{P}(\cdot)$	Probability of event $(\cdot)$
$Q(\cdot)$	Probability transition matrix
$\mathcal{S}_C$	Continuous-time system
$\mathcal{S}_D$	Discrete-time system
T	Sampling period
γ	Upper bound of the $\mathcal{H}_\infty$ norm
$\delta(\cdot)$	Dirac delta function
$\theta(t)$	Continuous-time Markov variable, $\theta(t) \in \mathbb{K}$
Λ	Transition rate matrix with λ_{ij} the i, j-th element
π	Vector of probabilities (corresponds also to the number $3.14\cdots$)
$\bar{\sigma}(\cdot)$	Largest singular value of $(\cdot)$

Contents

CHAPTER I

Introduction

In the last decades, a new approach has been largely studied and carefully applied to industrial, academic and commercial environments: Networked Control Systems (NCS). This approach not only intended to reduce the operational and the installation costs of system control structures but also to increase their flexibility and maintenance since it allows to interconnect different processes and different controllers in a same physical structure. Clearly, the possibility of an asynchronous structure is much more useful than a synchronous one. As a practical example, Galloway & Hancke (2013) presented in their paper an interesting introduction on Industrial Control Networks, which is a networked environment to control industrial plants. In this scenario, some characteristics like reliable and secure networks require very special attention mainly when implementing industrial safety systems, necessary for instance in nuclear, oil, and gas plants environments. Hence, different protocols have been developed or adapted to be used in industrial plants like CAN, Profibus, Foundation Fieldbus, or DeviceNet. These protocols increase reliability, flexibility and efficiency of industrial networks, so reinforcing the development of NCS theory.

Figure I.1 shows an example of NCS, where a computer controls different actuators interconnected through a network. In this example, the measured and control signals are transmitted through the same network. It is interesting to notice that there is no specific network structure such that the system is indeed considered a NCS. In other words, there are different possible structures and all of them are valid. Considering each kind of topology will depend on each specific use, as written in Zhang, Gao & Kaynak (2013). This work considers the network structure shown in Figure I.2, where a continuous-time Markov Jump Linear System (MJLS) models the plant to be controlled, and sensors and actuators communicate with the control device through a network.

Although it seems to be easy to use networks in a vast range of applications, based on either classical or modern theory of control, it is not. Due to network intrinsic characteristics, the signals traveling through wires, air, or even optical fibers can be irreversibly degraded. Consequently, the stability of the controlled systems can be seriously damaged. So, the NCS area is the union of the Control System and the Telecommunication areas

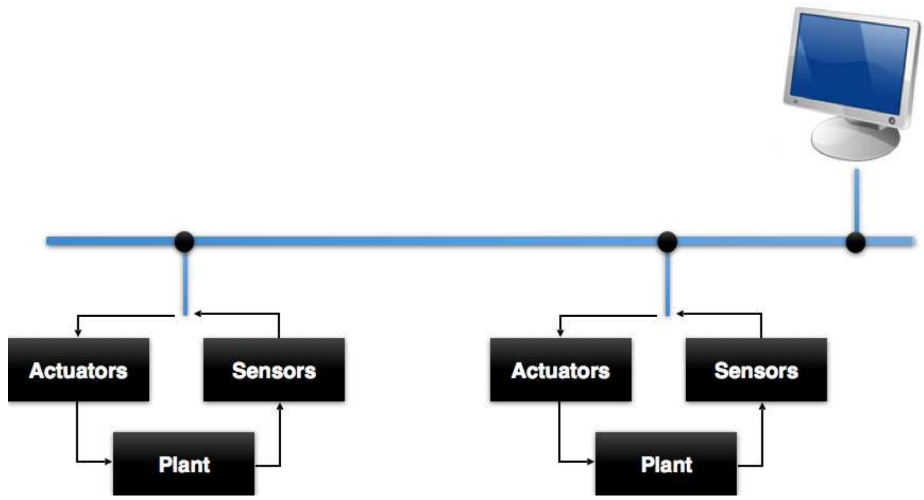

Figure I.1 – NCS where a computer controls two continuous-time systems through a network.

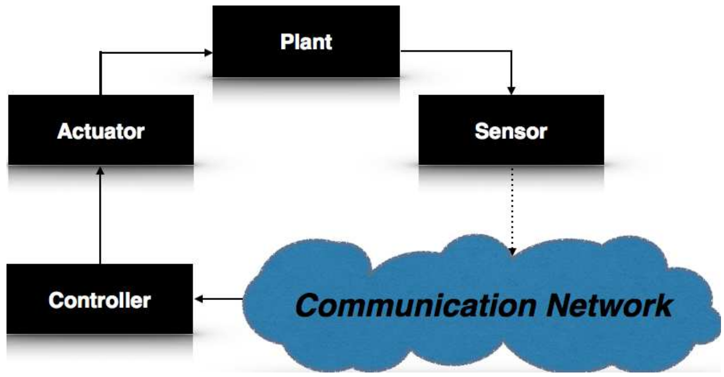

Figure I.2 – General structure of the NCS used throughout this work.

allowing a proper study of the influence networks have on control systems. Several works can be found in literature about this new topic. Specifically, Hespanha, Naghshtabrizi & Xu (2007) and Zhang, Gao & Kaynak (2013) report a general view and the state of the art of NCS up to 2013. These two papers are very complete in the sense of listing the main intrinsic characteristics that networked controllers should consider. Moreover, they gather the main techniques and offer a vast bibliography on each subject. The most relevant characteristics, which produce visible effects in control systems, are listed in the sequel:

$a.$ bandwidth limitations,

$b.$ interference of different sources on traveling signals, and

$c.$ transmission over finite length packet.

These issues are common to every physical network, independent of its structure or even of the environment signals are traveling through. Add to the previous list the item:

d. packet transmission concurrence of packets sharing the same network resource, but with different sources and/or sinks,

that is an issue of mesh networks. These items produce control design constraints, named network induced constraints. Zhang, Gao & Kaynak (2013) synthesizes the main of them which can degrade control signals traveling through a network:

a. packet delay;

b. packet loss ;

c. sampled-data signals;

d. node competition in data transmission process over a multiple node network; and,

e. data quantization due to the finite word length in sampling process.

As mentioned before, because of mathematical complexity of dealing with all constraints together, usually, tractable combinations of them are found in literature. Techniques to handle each constraint are described in Zhang, Gao & Kaynak (2013) and references therein. Concerning the present dissertation, issues b and c of the latter list are addressed, that is packet loss and sampled-data control. In order to accomplish this purpose, hybrid systems theory is used to handle sampled-data control systems applied to a continuous-time plant as in Hara, Fijioka & Kabamba (1994), Goebel, Sanfelice & Teel (2009), and Souza, Gabriel & Geromel (2014); and Markov chain theory is used to handle packet loss as in Farias (1998), Marcondes (2005), Seiler & Segupta (2005), and Huang (2013).

Regarding sampled-data signals, notice that bandwidth limitation occurs due to the maximum transmission rate of the communication channel, leading to sampled-data signals in a feedback control system, that is a sampled-data control system. Consequently, there is a minimum refresh interval of the variables transmitted through a network. This characteristic is commonly interpreted as a sampling process over some transmitted signal with the sampling period obeying

$$T_\star \le T_k \le T^\star, \; k \in \mathbb{N}, \tag{I.1}$$

where $T_\star$ is the minimum refresh interval of the network, and $T^\star$ is the maximum sampling period with which the system remains stable, i.e., $1/T^\star$ corresponds to the minimum admissible sampling rate. It is interesting to mention that the sampled-data theory development started with the use of digital systems applied to control systems, so it predates the study of NCS. Sampled-data systems can be described as a continuous-time dynamical system in which some signals are discrete-time – with constant or variable sampling period – and others, continuous-time valued signals, Chen & Francis (1995).

Considering the historical context, one of the first approaches to deal with mixed systems was to discretize a continuous-time plant and to design a discrete-time controller (*aggregation*). Another possible, but less used, approach was to interpolate discrete-time signals in order to have the whole feedback system as a continuous-time system (*continuation*); nonetheless, these techniques were only good approximations used to solve specific cases. The term approximation is used because one of both is necessary whenever using these techniques: either suppression of any signal characteristic between successive samples of time – for aggregation – or arbitrariness of the interpolated signal – for continuation – to mention a few of many problems listed by Antsaklis, Stiver & Lemmon (1993) and Branicky (1995). Thus, these models did not reflect the exact behavior of the plant. As a consequence, for them, there were well-known sets of control procedures that lead to optimal theoretical results. Yet, the optimality was lost when applied to the practical system. Particularly, the books written by Ragazzini & Franklin (1958), Chen & Francis (1995), Franklin, Powell & Workman (1997), and the references therein address most results related to digital systems, specially when considering the optimal design problems based on $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms. For the continuous-time classical and modern control references, see contributions from Luenberger (1979), Zhou, Doyle & Glover (1996), Geromel & Korogui (2011), and the references therein. However, how to design an exact model to control this kind of hybrid system?

First, and extensively used in literature, the well-established lifting technique can be used. This technique mainly "lifts" a continuous-time signal to a discrete-time signal preserving all norms, as $\mathcal{H}_2$, $\mathcal{H}_\infty$, and others, as described in Bamieh & Pearson (1992). A very good reference about sampled-data systems, where this technique is addressed, is Chen & Francis (1995). Applications and uses of this technique to handle sampled-data systems can be seen in Toivonen (1992), Bamieh & Pearson (1992), and in the recent paper written by Ramezanifar, Mohammadpour & Grigoriadis (2014).

Next, after Witsenhausen (1966), a hybrid proposal was introduced and became a largely studied subject in the literature nowadays. The theory of hybrid systems is a richer approach that uses a mathematical formulation to describe the original mixed real system. Many books and papers address hybrid systems specially Goebel, Sanfelice & Teel (2009) shows a very useful overview on hybrid systems and an extensive stability and robustness analysis of them. Another work, Souza (2015), analyses and synthesizes a state feedback sampled-data control applied to deterministic continuous-time linear system. In this work, the author uses a totally equivalent hybrid approach reaching optimal $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm based results, which are also expressed by Linear Matrix Inequalities (LMIs). The main advantage of this technique is the mathematical simplicity, which is needed for the purpose of the present work.

Returning to the NCS context and based on well-accepted works on the topic, different strategies to deal with sampled-data control can be found; and all of them can be used with both constant or variable sampling periods (Zhang, Gao & Kaynak (2013) uses transmission interval, in the context of NCS) with widely available literature. In the case of constant sampling intervals, stability and the $\mathcal{H}_2$ sampled-data control design problem are tackled in Chen (1999) and Souza (2015) using a totally equivalent discrete-time system, leading to optimal results. On the other hand, dealing with variable transmission intervals, more general strategies are available, according to Mazo-Jr. & Tabuada (2008), Meng & Chen (2014) and Souza (2015). The last one uses a fixed interval where the period can vary and is precisely the approach that will be adopted here. In addition, as stated earlier, there is an interval where the period can vary. Defining it is essentially important to maintain system stability. Curiously, for the deterministic systems, pathological periods of transmission can be observed even inside a well-defined interval $[T_\star, T^\star]$, according to Souza (2015).

Addressing packet dropout, network data transmission sometimes fails due to packet losses during the transmission process, which generally has a burst behavior. Losses in the system can be implemented in an online or an offline way. Offline techniques implement a controller previously designed that remains unchangeable during all plant execution. A more flexible structure is the online implementation that considers the actual state of the system to predict the best controller structure. Transmission success or failure depends on the previous transmission state – failure or success – as seen in Marcondes (2005), which indicates a stochastic behavior. Here, a Markov process is used because it allows good mathematical properties. The main characteristic of a Markov process is the fact that the probability of the future event depends only on the occurrence of the present event, that means, all past occurrences can be abandoned.

It is relatively common to use a Markov process to model packet dropout as in Marcondes (2005) and Xie & Xie (2009), besides it becomes better if it is possible to join some linearity properties to it. In the next chapters, a continuous-time MJLS is used to model packet loss, as in Farias (1998) and others. Indeed, modeling a plant as an MJLS, one mode can be associated to transmission success and the other to transmission failure. The books Costa, Fragoso & Marques (2005) and Costa, Fragoso & Todorov (2013) show interesting results and techniques available up to date about using discrete-time and continuous-time MJLS, respectively, with optimal control theory. Some basic results and definitions about MJLS, that will be useful to develop Chapters III– VI, will be addressed in Section II.4.

At this point, considering both constraints together, sampled-data signals and packet loss modeled by a continuous-time MJLS, a few works can be found to study the stability and to present an optimal design procedure. All results are only sufficient. (See Hu, Shi & Frank (2006), Gao, Wu & Shi (2009), and Mao (2013).) Hu, Shi & Frank (2006) obtains an $\mathcal{H}_2$ static output feedback sampled-data control applied to MJLS, but only sufficient conditions are achieved and they are not expressed in terms of LMIs. The conservatism, in this case, is due to two main reasons: first, Lyapunov matrices, $P_i(t)$, $\forall i \in \mathbb{K}$, where $\mathbb{K}$ is a set of possible Markov modes, must satisfy a more restrictive constraint than the conditions introduced by the Two-Point Boundary Value Problem (TPBVP) used in the present work, in Chapters III and IV; and second, these Lyapunov matrices are of the form $P_i(t) = P_{0i} + tP_{1i}$, $\forall i \in \mathbb{K}$, $\forall t \in [0, T]$, for a fixed transmission period, $T > 0$. Similarly, Gao, Wu & Shi (2009) proposes stability and robust control design, both of them expressed by sufficient conditions based on LMIs and Riccati equation. Recently, Mao (2013) presented also sufficient conditions to the sampled-data state feedback control applied to MJLS subject to a Brownian motion. Numerically speaking, this problem is not easy to be solved even considering a fixed period of time, $T > 0$.

In this sense, the main purpose of the present work is to fill this gap providing necessary and sufficient conditions to the optimal controller design of a state feedback control system when a continuous-time MJLS is taken into account. This project is based on some performance index of interest. Specifically, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms will be considered. These conditions are rewritten in a convex formulation in terms of LMIs. In order to accomplish these purposes, it is assumed throughout a constant sampling period $T_k = T > 0$ for all $k \in \mathbb{N}$. This dissertation is structured as follows:

Chapter II – Some basic concepts and theorems about hybrid systems, Markov jump linear systems, and optimal control theory are presented. Specifically, the system structure to be used is defined and some stability concepts are presented. In the framework of optimal control, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms are defined.

Chapter III and IV – Necessary and sufficient conditions to assure stability and determine $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms, respectively, of an state feedback sampled-data control system are provided. These results will lead to optimal control design in the following chapters. Moreover, specific algorithms to handle these norms are proposed.

Chapter V and VI – Using the previous results, the necessary and sufficient conditions are changed to obtain the state feedback sampled-data control law that minimizes the performances indexes based on $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms, respectively. LMI conditions are also derived. Algorithms to determine the optimal control law are proposed and proved to be convergent.

Chapter VII – Some practical examples, borrowed from the literature, illustrate the theoretical results obtained so far.

Finally, some conclusions are given and some topics to future works are proposed. Papers derived from this work are also listed.

CHAPTER II

Preliminaries

This chapter is devoted to present the notation and the system used throughout. Besides, some fundamentals for the next developments are also presented. First of all, some basic concepts about equilibrium point and stability of continuous-time and discrete-time systems are reminded. Then, a hybrid control system is defined. In the sequel, a Markov jump linear system with a sampled-data control is obtained, which leads to the concept of a Hybrid Markov Jump Linear System (HJMLS). For these systems, the stability definition used hereafter is presented. Optimality concepts for MJLS are also addressed. As all results listed here can be easily found in the current literature, proofs are omitted or only the more relevant steps are mentioned.

II.1 Notation

The notation used is standard. Upper Arabic or Greek letters represent real matrices. Exceptions are the letter T, that represents the sampling time period, and N, that represents the cardinality of the countable set $\mathbb{K} = \{1, \cdots, N\}$. Small Arabic letters represent vectors or functions, easily identified by the context. Exceptions are the matrix dimensions n, m, r, p and q. Small Greek letters represent constant numbers. Particular sets are represented by capital blackboard-bold letters; specially the symbols $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{N}$ are the sets of complex, real, and natural numbers, respectively. The notation $\mathbb{A}\backslash\{\cdot\}$ means the complement of $\{\cdot\}$ in the set $\mathbb{A}$. For any complex number, $\bar{v}$ is the conjugate of v. $(')$ means the transpose of a real vector or matrix, while $(^{\sim})$ is the conjugate transpose of a complex vector or matrix. For symmetric matrices, (•) is each of their symmetric blocks. Moreover, for any symmetric matrix A, $A > 0$ $(A \geq 0)$ means that A is positive (semi-)definite; and for square matrices $B \in \mathbb{R}^{n\times n}$, $\mathbf{Tr}(B) = \sum_{i=1}^{n} b_{ii}$ is the trace of B. The symbol $|x|$ means the module of the vector x and $\|x\|_p$ is a p-norm of x. For the next chapters, consider $\|\cdot\|_p$ the stochastic p-norm associated to the Hardy spaces $\mathcal{H}_2$ and $\mathcal{H}_\infty$. The mathematical expected value operator is denoted by $\mathcal{E}[\cdot]$, the conditioned mathematical expected value operator is $\mathcal{E}^{\nu}[(\cdot)] = \mathcal{E}[(\cdot) \mid \nu]$, and $\mathcal{P}[\cdot]$ is the probability of the event $[\cdot]$.

The set of matrices $P_1, \cdots, P_N$ are shortly denoted as $P = (P_1, \cdots, P_N)$ and $P > 0$ (≥ 0) indicates that $P_1 > 0, \cdots, P_N > 0$ (≥ 0). The notation $\xi(t_k^-)$ for $t_k \geq 0$, $k \in \mathbb{N}$, indicates the limit of $\xi(t)$ as $t \to t_k$ from the left. A square matrix is Hurwitz stable if all eigenvalues are inside the region $\mathrm{Re}(s) < 0$, and is Schur stable if its eigenvalues are inside the open unit circle $|z| < 1$ of the complex plane. The space of all random processes $w : \mathbb{R}^+ \to \mathbb{R}^r$ such that $\|w\|_2^2 = \int_0^\infty \mathcal{E}(w(t)'w(t))dt < \infty$ is denoted by $\mathbb{L}_2$, and $\mathbb{L}_2^* = \mathbb{L}_2 \setminus \{0\}$.

II.2 Stability of Nonlinear Systems

First of all, consider a continuous-time dynamical system and a discrete-time dynamical system, respectively defined by the field vectors $f_C(x(t), u(t), w(t))$ for all $t \in \mathbb{R}^+$ and $f_D(x[k], u[k], w[k])$ for all $k \in \mathbb{N}$, such that

$$\mathcal{S}_C : \begin{cases} \dot{x}(t) &= f_C(x(t), u(t), w(t)) \\ z(t) &= g_C(x(t), u(t), w(t)) \end{cases} \tag{II.1}$$

is an open-loop continuous-time system, where $x(t)$ is a continuously differentiable function; and

$$\mathcal{S}_D : \begin{cases} x[k+1] &= f_D(x[k], u[k], w[k]) \\ z[k] &= g_D(x[k], u[k], w[k]) \end{cases} \tag{II.2}$$

is an open-loop discrete-time system. For them,

- $x(t) : \mathbb{R}^+ \to \mathbb{R}^n$ ($x[k] : \mathbb{N} \to \mathbb{R}^n$) is the state vector,
- $u(t) : \mathbb{R}^+ \to \mathbb{R}^m$ ($u[k] : \mathbb{N} \to \mathbb{R}^m$) is the control input,
- $w(t) : \mathbb{R}^+ \to \mathbb{R}^r$ ($w[k] : \mathbb{N} \to \mathbb{R}^r$) is the exogenous input, and
- $z(t) : \mathbb{R}^+ \to \mathbb{R}^p$ ($z[k] : \mathbb{N} \to \mathbb{R}^p$) is the controlled output.

Global equilibrium points are defined as follows. (See the references Luenberger (1979), Vidyasagar (1993), and Geromel & Korogui (2011).)

Definition II.1 *x_e is an equilibrium point for* (II.1) *if $x(t) = x_e$ for all $t \geq t_e \geq 0$ whenever $x(t_e) = x_e$. If, moreover, x_e is unique, then x_e is a global equilibrium point.*

Definition II.2 *x_e is an equilibrium point for* (II.2) *if $x[k] = x_e$ for all $k \geq k_e \geq 0$ whenever $x[k_e] = x_e$. If, moreover, x_e is unique, then x_e is a global equilibrium point.*

In other words, an equilibrium point is the point x_e at which the system stays once it reaches x_e. As a variation, instead of defining a point at which the system stays, it is possible to define a region $\mathbb{I} \subset \mathbb{R}^n$ inside which the system stays. This difference defines the stability and asymptotic stability concepts. These definitions are stated in the sequel.

Definition II.3 *Consider a continuous-time system of the form* (II.1)*.*

1. *An equilibrium point x_e is stable if, for each $\epsilon > 0$, there exists $\rho > 0$ such that*[a] *$\|x(t_e) - x_e\| < \rho$ implies that $\|x(t) - x_e\| < \epsilon$ for all $t_e < t \in \mathbb{R}^+$.*

2. *An equilibrium point x_e is asymptotically stable whenever it is stable and, in addition, there exists $\rho > 0$ such that $\|x(t_e) - x_e\| < \rho$ implies that $\|x(t) - x_e\| \to 0$ as $t \to \infty$.*

3. *An equilibrium point x_e is globally asymptotically stable whenever it is asymptotically stable and $\|x(t) - x_e\| \to 0$ as $t \to \infty$ regardless of the initial state $x(t_e) \in \mathbb{R}^n$.*

[a] The operator $\|\cdot\|$ follows from the norm definition presented later in this chapter.

Definition II.4 *Consider a discrete-time system of the form* (II.2)*.*

1. *An equilibrium point x_e is stable if, for each $\epsilon > 0$, there exists $\rho > 0$ such that $\|x[k_e] - x_e\| < \rho$ implies that $\|x[k] - x_e\| < \epsilon$ for all $k_e < k \in \mathbb{N}$.*

2. *An equilibrium point x_e is asymptotically stable whenever it is stable and, in addition, there exists $\rho > 0$ such that $\|x[k_e] - x_e\| < \rho$ implies that $\|x[k] - x_e\| \to 0$ as $k \to \infty$.*

3. *An equilibrium point x_e is globally asymptotically stable whenever it is asymptotically stable and $\|x[k] - x_e\| \to 0$ as $k \to \infty$ regardless of the initial state $x[k_e] \in \mathbb{R}^n$.*

In the context of nonlinear systems, the Lyapunov direct method to assure stability plays a fundamental role since almost all the following theorems, available in the literature, are based on it. So, it is important to remember Lyapunov theory in both scenarios: continuous-time, (II.1), and discrete-time systems, (II.2). Theorems II.1 and II.2 summarize the Lyapunov direct method for stability.

Definition II.5 *Let $V(x(t), t)$ be a real functional of the continuous-time state $x(t)$ defined by* (II.1)*. Then, $V(x(t), t)$ is a Lyapunov functional if $V(x(t), t)$ is continuously positive definite for all $t \in \mathbb{R}^+$ around the equilibrium point. Mathematically, $V(x_e, t) = 0$ and $V(x(t), t) > 0$ for $x(t) \neq x_e$ and for all $t \in \mathbb{R}^+$ with continuous partial derivatives for all $t \in \mathbb{R}^+$. Moreover, for any $t \in \mathbb{R}^+$, $V(x(t), t)$ is such that*

$$\frac{d}{dt} V(x(t), t) \leq 0. \tag{II.3}$$

Definition II.6 *Let $V(x[k], k)$ be a real function of the discrete-time state $x[k]$ defined y* (II.2)*. Then, $V(x[k], k)$ is a Lyapunov functional if $V(x[k], k)$ is positive definite for all $k \in \mathbb{N}$ around the equilibrium point. Mathematically, $V(x_e, k) = 0$ and $V(x[k], k) > 0$ for $x[k] \neq x_e$ and for all $k \in \mathbb{N}$. Moreover, for any $k \in \mathbb{N}$, $V(x[k], k)$ is such that*

$$V(x[k+1], k+1) - V(x[k], k) \leq 0. \tag{II.4}$$

Theorem II.1 *The Lyapunov second method applied to the continuous-time system* (II.1)*.*

1. *If there is a Lyapunov functional $V(x(t), t)$ in the sense of Definition II.5, for $x(t)$ satisfying* (II.1)*, then the equilibrium point x_e is stable.*
2. *If item 1 is satisfied and* (II.3) *is a strict inequality for $x(t) \neq x_e$, then the equilibrium point is asymptotically stable.*
3. *If item 2 holds and, additionally, for all $t \in \mathbb{R}^+$, $V(x(t), t)$ is such that*

$$\lim_{x \to \infty} V(x, t) \to \infty, \tag{II.5}$$

then the equilibrium point is globally asymptotically stable.

Theorem II.2 *The Lyapunov second method applied to the discrete-time system* (II.2)*.*

1. *If there is a Lyapunov functional $V(x[k], k)$ in the sense of Definition II.6, for $x[k]$ satisfying* (II.2)*, then the equilibrium point x_e is stable.*
2. *If item 1 is satisfied and* (II.4) *is a strict inequality for $x[k] \neq x_e$, then the equilibrium point is asymptotically stable.*
3. *If item 2 holds and, additionally, for all $k \in \mathbb{N}$, $V(x[k], k)$ is such that*

$$\lim_{x \to \infty} V(x, k) \to \infty \tag{II.6}$$

then the equilibrium point is globally asymptotically stable.

For a deeper study, a further research on this topic and the detailed proofs of Theorems II.1 and II.2, see Luenberger (1979), Vidyasagar (1993), and the references therein.

II.3 Sampled-Data System as a Hybrid System

As largely studied from second half of last century, hybrid systems combine continuous-valued and discrete-valued dynamics in a same system. In a very simple way and as described in many works in this area, a hybrid system can be represented by two different dynamics in a single formulation: one of them describing the continuous-valued part, and the other describing the discrete-valued part of the system. Different kinds of systems are well-modeled by a hybrid approach: biological systems as occurs in swarms of fireflies, groups of crickets, ensembles of neuronal oscillators, and groups of heart muscle cells; mechanical systems modeling colliding masses; and electronic systems such as Zero-Crossing-Detector (ZCD), aircraft control, and sample-data control. As can be seen in Goebel, Sanfelice & Teel (2009), each of them needs a specific mathematical formulation.

Considering the scope of sampled-data control systems, each of both equations is defined inside some specific domain with proper mapping: an equation that represents the continuity and an equation that represents the jumps. The continuous-time equation is the dynamic flow of the system, that is, the plant dynamics. This is usually expressed as $\dot{\xi} = f_C(\xi)$ for $\xi \in \mathbb{A} \subset \mathbb{R}^n$ evolving from a given initial condition $\xi_0 \in \mathbb{A}$. On the other hand, the discrete-time process describes the behavior of the jumps, which is represented by $\xi(t_{k+1}) = f_D(\xi(t_k))$, for $\xi \in \mathbb{B} \subset \mathbb{R}^n$ and $\xi_0 \in \mathbb{B}$. In this case, the jump process is the sampled-data control: a constant by parts control law defined by $u(t) = u(t_k)$ for each time interval $t \in [t_k, t_{k+1})$, with $k \in \mathbb{N}$. Notice that boundary constraints are necessary to describe the domain that represents the real system with necessary complexity. This is the usual formulation of hybrid systems as adopted by many authors including Antsaklis, Stiver & Lemmon (1993), Branicky (1995), Goebel, Sanfelice & Teel (2009), as well as Souza (2015).

Thus, the hybrid system description of a sampled-data control system is composed by a dynamic equation, usually expressed in state space representation, and the jump law that, here, is a sampled-data state feedback control law. At this point, consider a general Linear Time-Invariant (LTI) system, which depends on time-varying functions $u(t)$ and $w(t)$,

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t) \tag{II.7}$$

$$z(t) = Cx(t) + Du(t), \tag{II.8}$$

where $x(t) : \mathbb{R}^+ \to \mathbb{R}^n$ is the state, $u(t) : \mathbb{R}^+ \to \mathbb{R}^m$ is the control input, $w(t) : \mathbb{R} \to \mathbb{R}^r$ is the exogenous input, which is defined in some set $\mathbb{L}$ to be determined, and $z : \mathbb{R} \to \mathbb{R}^p$ is the controlled output. This system evolves from $x(0) = 0$. Moreover, the class of admissible control signals is defined as

$$u \in \mathbb{U} = \{u(t) = Lx(t_k),\ t \in [t_k, t_{k+1})\ \forall k \in \mathbb{N}\}. \tag{II.9}$$

The sequence $\{t_k\}_{k\in\mathbb{N}}$ is composed by successive sampling instants such that $t_0 = 0$, $t_{k+1} > t_k$ and $\lim_{k\to\infty} t_k = \infty$. As mentioned before, notice that defining a control law in the form shown in (II.9) is fundamental to rewrite the system (II.7)–(II.8) in a hybrid approach.

Indeed, defining $\xi(t)' = [x(t)'\ u(t)']'$, this system becomes

$$\dot{\xi}(t) = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \xi(t) + \begin{bmatrix} E \\ 0 \end{bmatrix} w(t), \tag{II.10}$$

$$z(t) = \begin{bmatrix} C & D \end{bmatrix} \xi(t), \tag{II.11}$$

$$\xi(t_k) = \begin{bmatrix} I & 0 \\ L & 0 \end{bmatrix} \xi(t_k^-), \tag{II.12}$$

subject to initial conditions $\xi(0^-) = \xi_0 = 0$, and valid for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$. This formulation is used by Souza (2015) and other authors as a totally equivalent approach. Indeed, the second component of (II.10) imposes $\xi_2(t) = \xi_2(t_k)$ constant as a consequence of $\dot{\xi}_2(t) = 0$, which together with (II.12) provides $\xi_2(t) = L\xi_1(t_k)$. In addition, plugging this solution in the first equation of (II.10) yields $x(t) = \xi_1(t)$, first component of $\xi(t)$, and consequently, $u(t) = \xi_2(t)$. Thus, the controlled output $z(t) = Cx(t) + DLx(t_k)$, for all $t \in [t_k, t_{k+1})$, is exactly the closed-loop $z(t)$, (II.8), controlled by the sampled-data state feedback law (II.9). This formulation to describe sampled-data systems was firstly suggested by Yamamoto (1990).

Remark II.1 *For the system defined by* (II.10)–(II.12)*, the only constraint on the length of the sampling interval T_k is $T_k \in \mathbb{R}^+$. For the formulas described here, T_k is not required to be a periodic interval. However, as already mentioned, it is assumed throughout that $T_k = t_{k+1} - t_k = T > 0$, $k \in \mathbb{N}$, a constant period of time.* □

A more general representation of this class of hybrid systems is expressed by

$$\mathcal{S}_H : \begin{cases} \dot{\xi}(t) &= F\xi(t) + Jw(t) \\ z(t) &= G\xi(t) \\ \xi(t_k) &= H\xi(t_k^-) \end{cases} \tag{II.13}$$

evolving from arbitrary initial conditions $\xi(0^-) = \xi_0$. This model contains, as particular case, the one given in (II.10)-(II.12), which has special matrix structures.

Remark II.2 *The hybrid system* (II.13) *is a piecewise continuous-time linear system. As a consequence, all definitions and theorems from Section II.2 are applicable. Intuitively, Theorem II.1 can be used to assure stability of a hybrid system since* (II.13) *is defined in the continuous set $\mathbb{R}^{n+m}$ of the augmented state $\xi(t)$.* □

II.4 Continuous-Time Markov Jump Linear Systems

A Markov process is, before all, a stochastic process. In such a way, consider the following definitions to characterize it formally.

Definition II.7 *Given a set $\mathbb{O}$, a σ-algebra $\mathcal{F}$ on $\mathbb{O}$ is a collection of subsets of $\mathbb{O}$ that contains the empty set and is closed under countable operations of union, intersection, and complement.*

Remark II.3 *The σ-algebra concept is useful for defining measures on $\mathbb{O}$.* □

Definition II.8 *Using Coculescu & Nikeghbali (2007) definition, consider the complete probability space $(\mathbb{O}, \mathcal{F}, \mathcal{P})$. A filtration on $(\mathbb{O}, \mathcal{F}, \mathcal{P})$ is an increasing family $\mathcal{F}_{t\geq 0}$ of sub-σ-algebras on $\mathbb{O}$. In other words, for each t, $\mathcal{F}_t$ is a σ-algebra included in $\mathcal{F}$ and if $s \leq t$, then $\mathcal{F}_s \subseteq \mathcal{F}_t$. A probability space $(\mathbb{O}, \mathcal{F}, \mathcal{P})$ endowed with a filtration $\mathcal{F}_t$ is called a filtered probability space.*

Definition II.9 *If the triple $(\mathbb{O}, \mathcal{F}, \mathcal{P})$ is a complete probabilistic space, then the filtration $\mathcal{F}_t$ is complete if $\mathcal{F}_0$ contains all the $\mathcal{P}$-null sets.*

Definition II.10 *A random variable $\theta = \{(\theta(t), \mathcal{F}_t); t \in \mathbb{R}^+\}$, where $\theta(t) \ : \ \mathbb{R}^+ \to \mathbb{K}$, is a measurable function. The probability of $\theta(t) = i$ is expressed by*

$$\pi_i = \mathcal{P}[\theta(t) = i], \ i \in \mathbb{K}. \tag{II.14}$$

As mentioned before, Markov processes are used to well represent the stochastic characteristic of the packet loss since they are tractable mathematical formulations. Markov process is a random process where the prediction of the future depends only on the information available in the present, according to Leon-Garcia (2007). The continuous-time Markov chain main characteristic is that it makes finitely many jumps in any finite time interval. The continuous variable representing the instants of time is denoted by $t \in \mathbb{R}^+$. For arbitrary instants of time $s_1 < s_2 < \cdots < s < t$, the Markov property is expressed on the probabilities associated to the random variable $\theta(t)$, that is

$$\mathcal{P}[\theta(t) = j|\theta(s) = i, \cdots, \theta(s_1) = i_1] = P[\theta(t) = j|\theta(s) = i], \tag{II.15}$$

where $i_1, \cdots, i$ and j are any possible element of the state space of $\theta(t)$. Additionally, the continuous-time Markov chain is time-homogeneous if for every $i, j \in \mathbb{K}$ and $s < t$, in other words

$$\mathcal{P}[\theta(t) = j|\theta(s) = i] = P[\theta(t - s) = j|\theta(0) = i]. \tag{II.16}$$

Hereafter, without explicitly writing, the continuous-time Markov chain used is a continuous-time homogeneous Markov chain.

Now, consider a generic open-loop system with more than one possible state space representation, each of them selected from $\mathbb{K} = \{1, \cdots, N\}$, N a positive natural number. Consider also that, at some instant of time[1] t, the Markov chain jumps from state $i \in \mathbb{K}$ to the next state $j \in \mathbb{K}$. Then, the time-varying function $\theta = \{(\theta(t), \mathcal{F}_t); t \in \mathbb{R}^+\}$ describes the state of the random variable, which is governed by a continuous-time Markov process.

Due to the previous definitions, a continuous-time Markov process can be well defined by an initial state $\theta(0)$ and a transition rate matrix $\{\lambda_{ij}\} = \Lambda \in \mathbb{R}^{N \times N}$. The transition matrix $Q(t)$, that describes the probability of jumps inside the set $\mathbb{K}$, is intrinsically related to the transition rate matrix and is given by

$$Q(t) = e^{\Lambda t}. \tag{II.17}$$

This is the unique solution of the forward and backward Kolmogorov differential equations with $Q(0) = I$, as described in Costa, Fragoso & Todorov (2013). The vector of probabilities $\pi(t) = [\pi_1 \ \cdots \ \pi_N]'$, where $\pi_i(t) = \mathcal{P}[\theta(t) = i]$ for all $i \in \mathbb{K}$ and $t \in \mathbb{R}^+$, verifies

$$\pi(t) = Q(t)'\pi(0). \tag{II.18}$$

This statement together with (II.17) implies that

$$\pi(t + h) = e^{\Lambda' h}\pi(t). \tag{II.19}$$

On the other hand, considering an arbitrarily small time interval $h > 0$, expanding (II.19) in Taylor series, and using (II.15), each element of matrix $Q(t)$ is such that

$$Q_{ij}(h) = \mathcal{P}[\theta(t+h) = j | \theta(t) = i] = \begin{cases} 1 + \lambda_{ij} h + o(h) & \text{, for } i = j \\ \lambda_{ij} h + o(h) & \text{, for } i \neq j \end{cases} \tag{II.20}$$

holds for all $t \in \mathbb{R}^+$, where $o(h)$ represents high order terms which goes to zero faster than the others. In other words, $\lim_{h \to 0^+} o(h)/h = 0$. Moreover, the elements of $\Lambda \in \mathbb{R}^{N \times N}$ are such that $\lambda_{ij} \geq 0$ for all $i \neq j$ and $\sum_{j \in \mathbb{K}} \lambda_{ij} = 0$ for all $i \in \mathbb{K}$. This implies that $\lambda_{ii} \leq 0$ for all $i \in \mathbb{K}$. Notice that this probability depends uniquely on the difference between two consecutive jump instants $t + h$ and t.

Finally, the first moment of the expected value of some stochastic process $X_{\theta(t)}(t)$, where $\theta(t) \in \mathbb{K}$ and $t \in \mathbb{R}^+$, is readily calculated from

$$\mathcal{E}[X_{\theta(t)}(t)] = \sum_{j \in \mathbb{K}} \pi_j(t) X_j(t). \tag{II.21}$$

In the context of the control problems to be solved in the sequel, consider an homogeneous continuous-time MJLS defined by

$$\dot{x}(t) = A_{\theta(t)} x(t) + E_{\theta(t)} w(t) \tag{II.22}$$

$$z(t) = C_{\theta(t)} x(t), \tag{II.23}$$

[1] In Chapter VII, the time interval between consecutive jumps is defined according to Leon Garcia's procedure. (See Leon-Garcia (2007).)

which evolves from initial conditions $x(0) = 0$ and $\theta(0) = \theta_0$, with $\mathcal{P}[\theta_0 = i] = \pi_{i0} > 0$ for all $i \in \mathbb{K}$. In (II.22)–(II.23), once more, $x(t) : \mathbb{R}^+ \to \mathbb{R}^n$ is the state, $w(t) : \mathbb{R}^+ \to \mathbb{R}^r$ is the exogenous input, which is defined in some set $\mathbb{L}$, and $z(t) : \mathbb{R}^+ \to \mathbb{R}^p$ is the output. For this system, some stability results have been largely used in literature. (See Fang & Loparo (2002), Costa, Fragoso & Todorov (2013), and references therein.) The stability definition used throughout is stated in the sequel.

Definition II.11 *Let $\mathcal{F}_t$ be a filtration that defines the random variable $\theta(t)$. System* (II.22)–(II.23) *with $w(t) \equiv 0$ is Mean Square Stable (MSS) with respect to $\mathcal{F}_t$ if, for any initial state $x(0) = x_0$ and θ_0,*

$$\lim_{t\to\infty} \mathcal{E}[\|x(t)\|^2] = 0. \tag{II.24}$$

Once again the stability analysis of system (II.22)–(II.23) can be based on the Lyapunov direct method. For MJLS, it is usual to use multiple Lyapunov functions as in Fang & Loparo (2002) and Costa, Fragoso & Todorov (2013). Thus, in Definition II.5, instead of $V(x(t), t)$, consider N Lyapunov-like functionals indexed by $i \in \mathbb{K}$, $V_i(x(t), t)$, each of them associated to one Markov mode. Moreover, it has been proved that if a Continuous Algebraic Riccati Equation (CARE) is satisfied, then the MSS is assured. Notice that, the concept of multiple Lyapunov-like functionals is usual in the context of switched systems.

Theorem II.3 *(From Costa, Fragoso & Todorov (2013).) Consider the MJLS*

$$\dot{x} = A_{\theta(t)} x(t),\ \theta(0) = \theta_0,\ x(0) = x_0, \tag{II.25}$$

where $x(t) \in \mathbb{R}^n$, $\theta(t) \in \mathbb{K}$, and $t \in \mathbb{R}^+$, with initial distribution $\pi_{i0} = \mathcal{P}[\theta_0 = i]$. System (II.25) *is MSS if and only if, for any symmetric matrices $U = (U_1, \cdots, U_N) > 0$, there exist symmetric matrices $P = (P_1, \cdots, P_N) > 0$ satisfying*

$$A_i' P_i + P_i A_i + \sum_{j\in\mathbb{K}} \lambda_{ij} P_j + U_i = 0,\ i \in \mathbb{K}. \tag{II.26}$$

Remark II.4 *Only continuous-time systems have been presented in this section due to the fact that this is the only case treated in this work.* □

II.5 $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms for MJLS

Certainly, an important feature of control systems is the possibility of optimizing some specific performance index. Generally, a performance index represents some system characteristic of interest, for instance, the length or energy of some signal, a measure of the system uncertainty, or the signal gain from disturbances input to error outputs as well exposed by Zhou, Doyle & Glover (1996). These measures are in essence norms, which attribute a real number to a vector or matrix, making possible the comparison among different closed-loop systems. So, the next lines are devoted to define the performance indexes used throughout. First of all, consider the norm, the normed space, and the Hilbert space definitions. Then, the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ spaces can be introduced. With these tools, it will be possible to consider the performance indexes used to guarantee the closed-loop system stability and to design the optimal control law, which are the purposes of the following chapters.

Definition II.12 *Consider a vector field $\mathbb{V} \subset \mathbb{C}^q$ such that $q \in \mathbb{N} \setminus \{0\}$ with an one-dimensional subspace $\mathbb{X} \subset \mathbb{V}$. A norm $\|\cdot\|$ is a function $\|\cdot\| : \mathbb{V} \to \mathbb{R}^+$ such that the next three axioms hold for $u, v \in \mathbb{V}$ and $\alpha \in \mathbb{X}$.*

1. *Axiom of the null element: $\|v\| \geq 0$ for all $v \in \mathbb{V}$, and $\|v\| = 0$ if and only if $v = 0$.*
2. *Axiom of homogeneity: $\|\alpha v\| = |\alpha| \cdot \|v\|$.*
3. *Axiom of triangle inequality: $\|u + v\| \leq \|u\| + \|v\|$.*

Definition II.12 is available in Vidyasagar (1993). As a consequence of the previous definition, a norm on the set $\mathbb{V}$ defines a metric such as $d(u, v) = \|u - v\|$ induced by the norm. Thus, a normed space is defined (see Kreyszig (1978) for reference).

Definition II.13 *(Normed space, Banach space). A normed space $\mathbb{V}$, also called normed vector space or normed linear space, is a vector space with a norm defined on it. A Banach space is a complete normed space (completeness in the metric defined by the norm). Then, the normed space just defined is denoted by $(\mathbb{V}, \|\cdot\|)$ or simply by $\mathbb{V}$.*

Some known spaces are Banach spaces, for example, $\mathbb{C}^q$, or $\mathbb{R}^q$, respectively the set of complex and real vectors with dimension $q \in \mathbb{N} \setminus \{0\}$. But, among all Banach spaces, the Hilbert space is of particular interest. So, consider the definition of inner product, the Hilbert Space, and the special Hardy spaces $\mathcal{H}_2$ and $\mathcal{H}_\infty$ (all of them extracted from Zhou, Doyle & Glover (1996) and Colaneri, Geromel & Locatelli (1997)).

Definition II.14 *Let $\mathbb{V}$ be a vector space on $\mathbb{C}^q$. An inner product on $\mathbb{V}$ is a complex valued function $\langle \cdot, \cdot \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{C}$ such that for any $u, v, y \in \mathbb{V}$ and $\alpha, \beta \in \mathbb{C}$*

1. *$\langle u, \alpha v + \beta y \rangle = \alpha \langle u, v \rangle + \beta \langle u, y \rangle$.*
2. *$\langle u, v \rangle = \overline{\langle v, u \rangle}$.*
3. *$\langle u, u \rangle > 0$ if $u \neq 0$.*

A vector space with an inner product is called an inner product space.

Definition II.15 *A Hilbert space is a complete inner product space with the norm induced by its inner product.*

Consequently, the Hilbert space is also a Banach space. $\mathbb{R}^q$ with the usual inner product is a Hilbert space. Obviously, many other spaces are Hilbert spaces. The next Definitions II.16 and II.17 stand for spaces that conduct to the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms. Specifically, consider the transfer function from the exogenous input w to the controlled output z of the LTI system (II.7)–(II.8) with zero control input $u(t) = 0$, that is

$$\mathfrak{F}(\zeta) = C(\zeta I - A)^{-1}E \tag{II.27}$$

Although this transfer function of interest is strictly proper, the next definitions hold for any transfer function.

Definition II.16 *Hardy Space $\mathcal{RH}_2$ (here, just $\mathcal{H}_2$): $\mathcal{H}_2$ is a subspace of the Hilbert space consisting of all matrix-valued functions $\mathfrak{F}(\zeta) : \mathbb{C} \to \mathbb{C}^{p\times r}$ for which the following integral is bounded*

$$\int_{-\infty}^{\infty} \textbf{\textit{Tr}}(\mathfrak{F}(j\omega)^{\sim}\mathfrak{F}(j\omega))d\omega < \infty \tag{II.28}$$

such that $\mathfrak{F}(\zeta)$ is analytic in the open right-half plane. The inner product for this Hilbert space is defined as

$$\langle \mathfrak{F}, \mathfrak{G} \rangle := \frac{1}{2\pi}\int_{-\infty}^{\infty} \textbf{\textit{Tr}}(\mathfrak{F}(j\omega)^{\sim}\mathfrak{G}(j\omega))d\omega \tag{II.29}$$

for $\mathfrak{F}, \mathfrak{G} \in \mathbb{V}$ and the inner product induced norm is given by $\|\mathfrak{F}\|_2^2 := \langle \mathfrak{F}, \mathfrak{F} \rangle$.

Definition II.17 *Hardy Space $\mathcal{RH}_\infty$ (here, just $\mathcal{H}_\infty$): $\mathcal{H}_\infty$ is a subspace of a Banach space consisting of all matrix-valued functions $\mathfrak{F}(\zeta) : \mathbb{C} \to \mathbb{C}^{p\times r}$ with norm*

$$\|\mathfrak{F}\|_\infty := \sup_{\omega\in\mathbb{R}} \bar{\sigma}[\mathfrak{F}(j\omega)] \tag{II.30}$$

such that $\mathfrak{F}(\zeta)$ is analytic and bounded in the open right-half plane. The function $\bar{\sigma}(\cdot)$ is the largest singular value of $(\cdot)$.

The $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms can be alternatively calculated in time domain. First, notice that only strictly proper transfer functions satisfy the condition (II.28). Using Parseval theorem and trace properties, the $\mathcal{H}_2$ norm can be rewritten as

$$\|\mathfrak{F}\|_2^2 = \sum_{l=1}^{r} \|z_l\|_2^2, \tag{II.31}$$

where

$$\|z_l\|_2^2 = \int_0^\infty \|z_l(t)\|_2^2 dt \tag{II.32}$$

is the norm of the entire trajectory $z_l(t) : R^+ \to \mathbb{R}^p$ and $\|z_l(t)\|_2^2 = z_l(t)'z_l(t)$ is the Euclidian norm evaluated at the time instant $t \in \mathbb{R}^+$. The signal $z_l(t)$ is the output of system (II.7)–(II.8) corresponding to the exogenous impulsive input $w(t) = e_l\delta(t)$, where $e_l \in \mathbb{R}^r$ is th l-th column of the identity matrix with compatible dimension. On the other hand, the $\mathcal{H}_\infty$ norm can also be determined in time domain by

$$\|\mathfrak{F}\|_\infty = \sup_{w \in \mathbb{L}_2^*} \frac{\|z\|_2}{\|w\|_2}, \tag{II.33}$$

where $z(t)$ is the output of the system (II.7)–(II.8) corresponding to the exogenous input $w \in \mathbb{L}_2^* == \mathbb{L}_2 \setminus \{0\}$ where $\mathbb{L}_2$ is the space of all signals $w(t) : \mathbb{R}^+ \to \mathbb{R}^r$ such that $\|w\|_2^2 = \int_0^\infty \|w(t)\|dt < \infty$ (see Colaneri, Geromel & Locatelli (1997)). Observe that the existence of a finite $\mathcal{H}_\infty$ norm does not require the transfer function to be strictly proper.

Obviously, these definitions are restricted to deterministic approaches. From the stochastic point of view, for the system in (II.22)–(II.23), equivalent definitions for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms follow from the appropriate norm of a generic trajectory as being

$$\|z\|_2^2 = \int_0^\infty \mathcal{E}[z(t)'z(t)]dt. \tag{II.34}$$

These norm definitions are the same used in Costa, Fragoso & Todorov (2013).

At this point, consider the open-loop system represented in (II.22)–(II.23). Respectively, and in such a way defined by Costa, Fragoso & Todorov (2013), the performance indexes to be considered are

$$\mathcal{J}_2 = \sum_{l=1}^{r} \int_0^\infty \mathcal{E}[z_l(t)'z_l(t)]dt \tag{II.35}$$

for z_l the output corresponding to the impulsive input defined previously and

$$\mathcal{J}_\infty = \sup_{w \in \mathbb{L}_2} \int_0^\infty \mathcal{E}[z(t)'z(t) - \gamma^2 w(t)'w(t)]dt, \tag{II.36}$$

for a given $\gamma > 0$. The usual manner to evaluate the performance indexes (II.35) and (II.36) is to consider a multiple Lyapunov functional, as mentioned before in Section II.4, together with the Lyapunov direct method from Theorem II.1. To illustrate this approach, let the set of quadratic Lyapunov functionals be $V_i(x(t), t) = x(t)'P_i(t)x(t)$ associated to the i-th Markov mode of the open-loop system (II.22)–(II.23), where $\theta(t) = i \in \mathbb{K}$. Consider $x(t) = x$ and $w(t) = w$ at $t \in \mathbb{R}^+$. Consider also the set of positive definite solutions $P_i(t)$, $i \in \mathbb{K}$, to the coupled Differential Riccati Equations (DRE)

$$\dot{P}_i(t) + A_i'P_i(t) + P_i(t)A_i + \gamma^{-2}P_i(t)E_iE_i'P_i(t) + \sum_{j \in \mathbb{K}} \lambda_{ij}P_j(t) + C_i'C_i = 0 \tag{II.37}$$

subject to the final boundary conditions $P_i(t_f) \geq 0, i \in \mathbb{K}$, whose existence will be discussed later. Define the infinitesimal generator as

$$\mathcal{L}^w V_i(x,t) = \lim_{h \to 0} \frac{\mathcal{E}^{i,x,t}\left[V_{\theta(t+h)}(x(t+h), t+h)\right] - V_i(x,t)}{h}, \tag{II.38}$$

in the same way proposed by Costa, Fragoso & Todorov (2013), where

$$\begin{aligned}\mathcal{E}^{i,x,t}\left[V_{\theta(t+h)}(x(t+h), t+h)\right] = \ & \mathcal{E}^{i,x,t}\left[\frac{\partial V_{\theta(t+h)}}{\partial t}(x,t)h\right] \\ & + \mathcal{E}^{i,x,t}\left[\frac{\partial V'_{\theta(t+h)}}{\partial x}(x,t)\dot{x}(t)h\right] \\ & + \mathcal{E}^{i,x,t}\left[V_{\theta(t+h)}(x,t)\right] + o(h).\end{aligned} \tag{II.39}$$

Consequently, the definition (II.20) yields

$$\lim_{h \to 0} \mathcal{E}^{i,x,t}\left[\frac{\partial V_{\theta(t+h)}}{\partial t}(x,t)\right] = \frac{\partial V_i}{\partial t}, \tag{II.40}$$

$$\lim_{h \to 0} \mathcal{E}^{i,x,t}\left[\frac{\partial V'_{\theta(t+h)}}{\partial x}(x,t)\dot{x}(t)\right] = \frac{\partial V'_i}{\partial x}\dot{x}(t), \tag{II.41}$$

and

$$\lim_{h \to 0} \frac{\mathcal{E}^{i,x,t}\left[V_{\theta(t+h)}(x,t)\right] - V_i(x,t)}{h} = \sum_{j \in \mathbb{K}} \lambda_{ij} V_j, \tag{II.42}$$

which holds for all $i \in \mathbb{K}$. Thus, in the more general case of the $\mathcal{H}_\infty$ norm, taking into account the dynamic system (II.22)–(II.23), the infinitesimal generator becomes

$$\begin{aligned}\mathcal{L}^w V_i(x,t) &= \frac{\partial V_i}{\partial t} + \frac{\partial V'_i}{\partial x}\dot{x}(t) + \sum_{j \in \mathbb{K}} \lambda_{ij} V_j \\ &= x'\left(\dot{P}_i(t) + A'_i P_i(t) + P_i(t) A_i + \sum_{j \in \mathbb{K}} \lambda_{ij} P_j(t)\right) x + 2x' P_i(t) E_i w.\end{aligned} \tag{II.43}$$

This equality together with (II.37) and the final condition $P_i(t_f) \geq 0$ leads to

$$\begin{aligned}\mathcal{L}^w V_i(x,t) &= -x'\left(C'_i C_i + \gamma^{-2} P_i(t) E_i E'_i P_i(t)\right) x + 2x' P_i(t) E_i w \\ &= -z(t)'z(t) + \gamma^2 w'w - \gamma^2 \left\|w - \gamma^{-2} E'_i P_i(t) x\right\|_2^2.\end{aligned} \tag{II.44}$$

Making use of Dinkyn's formula (see Costa, Fragoso & Todorov (2013)), it follows that

$$\begin{aligned}V_{\theta_0}(0,0) - \mathcal{E}^{\theta_0,0,0}[V_{\theta(t_f)}(x(t_f), t_f)] &= -\int_0^{t_f} \mathcal{E}^{\theta_0,0,0}\left[\mathcal{L}^w V_{\theta(\tau)}(x(\tau),\tau)\right] d\tau \\ &= \int_0^{t_f} \mathcal{E}^{\theta_0,0,0}\left[z(\tau)'z(\tau) - \gamma^2 w'w\right] d\tau \\ &\quad + \gamma^2 \int_0^{t_f} \mathcal{E}^{\theta_0,0,0}\left[\left\|w - \gamma^{-2} E'_{\theta(\tau)} P_{\theta(\tau)}(\tau) x\right\|_2^2\right] d\tau.\end{aligned} \tag{II.45}$$

Hence,

$$V_{\theta_0}(0,0) - \mathcal{E}^{\theta_0,0,0}[V_{\theta(t_f)}(x(t_f),t_f)] \geq \int_0^{t_f} \mathcal{E}^{\theta_0,0,0}\left[z(\tau)'z(\tau) - \gamma^2 w'w\right] d\tau \tag{II.46}$$

for all $w \in \mathbb{L}_2$. The equality in (II.46) holds for the worst case disturbance, which is given by $w(t) = \gamma^{-2} E'_{\theta(t)} P_{\theta(t)}(t)x(t)$. Calculating the mathematical expectation with respect to $\theta_0 \in \mathbb{K}$, then

$$\sup_{w\in\mathbb{L}_2} \mathcal{E}\left[\int_0^{t_f} (z(\tau)'z(\tau) - \gamma^2 w'w)d\tau + x(t_f)'P_{\theta(t_f)}x(t_f)\right] = 0 \tag{II.47}$$

due to $x(0) = 0$. Finally, the existence of a solution of (II.37) implies, from Theorem II.3, that the system is MSS, then the limit of the left hand side of (II.47) as t_f goes to infinity is well defined and gives

$$\sup_{w\in\mathbb{L}_2} \mathcal{E}\left[\int_0^{\infty} (z(\tau)'z(\tau) - \gamma^2 w'w)d\tau\right] = 0. \tag{II.48}$$

As a consequence, the $\mathcal{H}_\infty$ norm of the system (II.22)–(II.23) equals the minimum value of $\gamma > 0$ such that the coupled DRE (II.37) becomes a coupled CARE, which admits a unique set of positive definite stabilizing solutions corresponding to $\mathcal{J}_\infty = 0$.

There are two more important results. One of them concerns the fact that the solution of the DRE (II.37) is positive semi-definite inside the time interval $[0, t_f]$ provided that $P(t_f) \geq 0$. The other, concerns the existence and uniqueness of a stationary solution to the associated CARE.

Theorem II.4 *If there exists a set of N continuous matrices $P_i(t) \in \mathbb{R}^{n\times n}$ in the time interval $0 \leq t \leq t_f$ satisfying the coupled DRE*

$$\dot{P}_i(t) + A_i'P_i(t) + P_i(t)A_i + \gamma^{-2}P_i(t)E_iE_i'P_i(t) + \sum_{j\in\mathbb{K}} \lambda_{ij}P_j(t) + C_i'C_i = 0 \tag{II.49}$$

with boundary condition $P(t_f) \geq 0$, then $P(t) \geq 0$ for all $0 \leq t \leq t_f$.

Proof: Assuming that a solution exists, it can be expressed as

$$\begin{aligned} P_i(t) = \; & e^{\bar{A}_i'(t_f-t)}P_i(t_f)e^{\bar{A}_i(t_f-t)} \\ & + \int_t^{t_f} e^{\bar{A}_i'(\tau-t)}\left(\gamma^{-2}P_i(\tau)E_iE_i'P_i(\tau) + \sum_{j\neq i\in\mathbb{K}} \lambda_{ij}P_j(\tau) + C_i'C_i\right)e^{\bar{A}_i(\tau-t)}d\tau, \end{aligned} \tag{II.50}$$

where $\bar{A}_i = A_i + (\lambda_{ii}/2)I$ for all $i \in \mathbb{K}$. The proof follows because the backwards integration of the integral in (II.50), from $t = t_f$, considering a positive semi-definite final boundary condition leads to a positive semi-definite result. ■

The stationary solution obtained by setting $P_i(t) \equiv P_i$ for all $t \in \mathbb{R}^+$ and all $i \in \mathbb{K}$ is fully characterized by the next result due to Costa, Fragoso & Todorov (2013).

Theorem II.5 *Let $\gamma > 0$ be given such that $\mathcal{J}_\infty < 0$. There exists a unique set of N positive semi-definite and stabilizing solutions $P = (P_1, \cdots, P_N)$, $P_i \in \mathbb{R}^{n\times n}$, satisfying the coupled CAREs*

$$A_i'P_i + P_iA_i + \gamma^{-2}P_iE_iE_i'P_i + \sum_{j\in\mathbb{K}} \lambda_{ij}P_j + C_i'C_i = 0 \tag{II.51}$$

for each $i \in \mathbb{K}$.

Similarly, for the $\mathcal{H}_2$ norm calculation, the same path can be followed to obtain

$$\begin{aligned}\mathcal{E}\left[\int_0^\infty z(\tau)'z(\tau)d\tau\right] =& \mathcal{E}[V_{\theta_0}(x_0,0)] \\ =& \sum_{i\in\mathbb{K}} \pi_{i0}x_0'P_ix_0\end{aligned} \tag{II.52}$$

whenever (II.22)–(II.23) evolves from the initial condition $x(0) = x_0$ and $w(t) \equiv 0$. Since the system trajectory corresponding to $x(0) = 0$ and $w(t) = e_l\delta(t)$ equals the system trajectory corresponding to $x(0) = E_{\theta_0}e_l$ and $w(t) \equiv 0$, for each $l = 1, \cdots, r$, then

$$\begin{aligned}\sum_{l=1}^r \int_0^\infty \mathcal{E}\left[z_l(\tau)'z_l(\tau)\right] d\tau =& \sum_{i\in\mathbb{K}}\sum_{l=1}^r \pi_{i0}e_l'E_i'P_iE_ie_l \\ =& \sum_{i\in\mathbb{K}} \pi_{i0}\mathbf{Tr}(E_i'P_iE_i),\end{aligned} \tag{II.53}$$

where the positive semi-definite matrices $P_i \in \mathbb{R}^{n\times n}, i \in \mathbb{K}$, satisfy the coupled Lyapunov equations obtained from the CAREs (II.51) by setting $\gamma \to \infty$. Under mild assumptions, the CAREs in (II.51) admit a set of positive definite solutions. (See Costa, Fragoso & Todorov (2013).)

II.6 Continuous-Time Hybrid MJLS Definition

Finally, consider the MJLS defined by the state space realization

$$\dot{x}(t) = A_{\theta(t)}x(t) + B_{\theta(t)}u(t) + E_{\theta(t)}w(t) \tag{II.54}$$

$$z(t) = C_{\theta(t)}x(t) + D_{\theta(t)}u(t), \tag{II.55}$$

which evolves from initial conditions $x(0) = 0$ and $\theta(0) = \theta_0$, with $\mathcal{P}[\theta_0 = i] = \pi_{i0} > 0$ for all $i \in \mathbb{K}$. Once more, $x(t) : \mathbb{R}^+ \to \mathbb{R}^n$ is the state, $u(t) : \mathbb{R}^+ \to \mathbb{R}^m$ is the control input, $w(t) : \mathbb{R}^+ \to \mathbb{R}^r$ is the exogenous input, which is defined in some set $\mathbb{L}$ to be defined, and $z(t) : \mathbb{R}^+ \to \mathbb{R}^p$ is the output. Moreover, this system is subject to a sampled-data control law $u(t)$ defined by a set of N state feedback sampled-data control laws analogous to (II.9) expressed by

$$u \in \mathbb{U} = \{u(t) = L_{\theta(t_k)}x(t_k),\ t \in [t_k, t_{k+1})\ \forall k \in \mathbb{N}\} \tag{II.56}$$

with $\theta(t_k) \in \mathbb{K}$. As defined before, the sequence $\{t_k\}_{k\in\mathbb{N}}$ are successive sampling instants of time and $T = t_{k+1} - t_k$, $k \in \mathbb{N}$. Thus, the hybrid system composed by (II.54)–(II.55) subject to the control law (II.56) can be rewritten as

$$\dot{\xi}(t) = \begin{bmatrix} A_{\theta(t)} & B_{\theta(t)} \\ 0 & 0 \end{bmatrix} \xi(t) + \begin{bmatrix} E_{\theta(t)} \\ 0 \end{bmatrix} w(t) \tag{II.57}$$

$$z(t) = \begin{bmatrix} C_{\theta(t)} & D_{\theta(t)} \end{bmatrix} \xi(t) \tag{II.58}$$

$$\xi(t_k) = \begin{bmatrix} I & 0 \\ L_{\theta(t_k)} & 0 \end{bmatrix} \xi(t_k^-) \tag{II.59}$$

evolving from initial conditions $\xi(0^-) = \xi_0 = 0$ and $\theta(0^-) = \theta(0) = \theta_0$, valid for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$. This is called a Hybrid Markov Jump Linear System (HMJLS) and the rationale behind its definition follows the same reason as exposed in Section II.3. Again, the second component of (II.57) imposes $\xi_2(t) = \xi_2(t_k)$ for all $t \in [t_k, t_{k+1})$ as a consequence of $\dot{\xi}_2(t) = 0$, which together with (II.59) provides $\xi_2(t) = L_{\theta(t_k)}\xi_1(t_k)$. In addition, plugging this solution in the first equation of (II.57) yields $x(t) = \xi_1(t)$. Thus, the controlled output $z(t) = C_{\theta(t)}x(t) + D_{\theta(t)}L_{\theta(t_k)}x(t_k)$, which is valid for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, is exactly the closed-loop system $z(t)$, (II.55), controlled by the sampled-data state feedback law (II.56). In a more general representation

$$\mathcal{S}_H : \begin{cases} \dot{\xi}(t) &= F_{\theta(t)}\xi(t) + J_{\theta(t)}w(t) \\ z(t) &= G_{\theta(t)}\xi(t) \\ \xi(t_k) &= H_{\theta(t_k)}\xi(t_k^-) \end{cases}, \tag{II.60}$$

which evolves from arbitrary initial conditions $\xi(0^-) = \xi_0$ and $\theta(0^-) = \theta(0) = \theta_0$. Clearly, this model contains, as particular case, the one given in (II.57)-(II.59).

Remark II.5 *Jump and sampling processes are totally independent. Consequently there is no relationship between the time when the system "jumps", that is a random variable with exponential distribution (see Leon-Garcia (2007)), and the instants of time t_k, $k \in \mathbb{N}$, of the sampling process.* □

Stability conditions for the system (II.60) have already been determined, as mentioned in Chapter I. Among them, Hu, Shi & Frank (2006) obtained sufficient conditions, based also on multiple Lyapunov-like functionals. This is a similar hypothesis as used by Fang & Loparo (2002) and Costa, Fragoso & Todorov (2013), which encourages the usage of an analogous approach to the problem addressed by this work.

As a final remark, notice that the same performance indexes (II.35) and (II.36) can be adopted to the HMJLS (II.57)–(II.59) in the same way used for the MJLS (II.22)–(II.23). So, considering the set of the state feedback gains $L = (L_1, \cdots, L_N)$ in (II.56), the main goal is to determine the optimal gains L_i, $i \in \mathbb{K}$, that minimizes the $\mathcal{H}_2$ performance

$$\inf_L \mathcal{J}_2(L) \tag{II.61}$$

subject to (II.57)–(II.59) or minimizes the $\mathcal{H}_\infty$ performance

$$\inf_{L,\gamma} \left\{ \gamma^2 : \mathcal{J}_\infty(L) < 0 \right\} \tag{II.62}$$

subject to (II.57)–(II.59) as well. Thereby, these design problems are in the general form of any optimal control problem. However, they have never been solved until now. The purpose of this work is to solve these problems in the context of sampled-data control as previously discussed. Moreover, the existence of a general design procedure allows future applications in other problems of interest, as for instance, filtering and dynamic output feedback.

CHAPTER III

Stability and $\mathcal{H}_2$ Norm Evaluation of Hybrid MJLS

This chapter is dedicated to state necessary and sufficient conditions to assure the HMJLS mean-square stability. At the same time the exact value of the $\mathcal{H}_2$ norm is determined. In order to accomplish these goals, evenly spaced sampling instants, $T = t_{k+1} - t_k \geq 0$ for all $k \in \mathbb{N}$, are considered. Theoretical results on the stability analysis in the context of $\mathcal{H}_2$ norm are derived from a TPBVP with initial and final boundary conditions. Moreover, an algorithm based on an iterative procedure is suggested and its convergence proved whenever the TPBVP admits a positive definite solution. A numerical example shows that the procedure is suitable for the purpose of calculating the $\mathcal{H}_2$ norm of an HMJLS.

The feedback system used in this chapter is the open-loop continuous-time MJLS (II.54)–(II.55) subject to a sampled-data state feedback control law, (II.56). However, since no particularity of the system matrices is required, the general hybrid structure (II.60) can conveniently be used. Hence, consider the HMJLS in the form of

$$\mathcal{S}_H : \left\{ \begin{array}{rcl} \dot{\xi}(t) & = & F_{\theta(t)}\xi(t) + J_{\theta(t)}w(t) \\ z(t) & = & G_{\theta(t)}\xi(t) \\ \xi(t_k) & = & H_{\theta(t_k)}\xi(t_k^-) \end{array} \right. \tag{III.1}$$

evolving from initial conditions $\xi(0^-) = \xi_0 = 0$ and $\theta(0^-) = \theta(0) = \theta_0$. For this system, $\xi(t) : \mathbb{R}^+ \to \mathbb{R}^{n+m}$ is the augmented state variable, $w(t) : \mathbb{R}^+ \to \mathbb{R}^r$ is the exogenous input, $z(t) : \mathbb{R}^+ \to \mathbb{R}^p$ is the controlled output, and $\{\theta(t) \in \mathbb{K}\}$, $\mathbb{K} = \{1, \cdots, N\}$, is a continuous-time Markov process with a transition rate matrix $\{\lambda_{ij}\} = \Lambda \in \mathbb{R}^{N\times N}$. As defined in Chapter II, the conditional probability associated to Λ depends only on the time interval h between successive jumps of the Markov chain and is given by

$$Q_{ij}(h) = \mathcal{P}[\theta(t+h) = j | \theta(t) = i] = \left\{ \begin{array}{ll} 1 + \lambda_{ij}h + o(h) & \text{, for } i = j \\ \lambda_{ij}h + o(h) & \text{, for } i \neq j \end{array} \right. , \tag{III.2}$$

where $o(h)$ is high order terms such that $\lim_{h\to 0^+} o(h)/h = 0$. Moreover, the elements of $\Lambda \in \mathbb{R}^{N\times N}$ satisfy $\lambda_{ij} \geq 0$ for all $i \neq j$ and $\sum_{j\in\mathbb{K}} \lambda_{ij} = 0$ for all $i \in \mathbb{K}$. Thus, $\lambda_{ii} \leq 0$ for all $i \in \mathbb{K}$. Consider also that $\pi_i(0) = \pi_{i0} = \mathcal{P}[\theta_0 = i]$ is the initial distribution of the variable $\theta(t)$, that is, π_{i0} is the distribution of $\theta(0) = \theta_0$.

III.1 Theoretical Results

The main result of this section is to verify the stability of the HMJLS (III.1) at the same time the $\mathcal{J}_2$ performance index is evaluated. In order to accomplish that, remember that an impulsive exogenous input is a consequence of Definition II.16. Moreover, the system trajectory evaluated in (II.22)–(II.23) corresponding to $w(t) = e_l\delta(t)$, $l = 1, \cdots, r$, and initial state $x_0 = 0$ is equivalent to the system trajectory corresponding to $w(t) \equiv 0$ and $x_0 = E_{\theta_0}e_l$, $l = 1, \cdots, r$. Remember also that the performance index $\mathcal{J}_2$ applied to the MJLS (II.22)-(II.23) has been appropriately defined by

$$\mathcal{J}_2 = \sum_{l=1}^{r} \int_0^\infty \mathcal{E}[z_l(t)'z_l(t)]dt. \tag{III.3}$$

Then, consider the equivalent hybrid system (III.1) and observe that it is a Piecewise Continuous (PWC) MJLS since at each instant t_k, $k \in \mathbb{N}$, occurs a discontinuity imposed by the matrices H_i, $i \in \mathbb{K}$. The performance index (III.3) can be rewritten as

$$\mathcal{J}_2 = \sum_{l=1}^{r} \sum_{k\in\mathbb{N}} \int_{t_k}^{t_{k+1}} \mathcal{E}[z_l(t)'z_l(t)]dt \tag{III.4}$$

subject to the system (III.1) with $w(t) \equiv 0$, which evolves from $\theta(0^-) = \theta(0) = \theta_0$ and $\xi(0^-) = \xi_0 = J_{\theta_0}e_l$, for each $l = 1, \cdots, r$, where e_l is the l-th column of the identity matrix with compatible dimensions.

Since multiple Lyapunov-like functionals can be used to assure the MJLS stability, consider a collection of continuous nonnegative cost-to-go functionals $V_{\theta(t)}(\xi(t), t) = \xi(t)'P_{\theta(t)}(t)\xi(t)$, where $\theta(t) \in \mathbb{K}$ and $t \in [t_k, t_{k+1})$ for each $k \in \mathbb{N}$ and $P_i(t)$ solves the coupled Differential Lyapunov Equations (DLE)

$$\dot{P}_i(t) + F_i'P_i(t) + P_i(t)F_i + \sum_{j\in\mathbb{K}} \lambda_{ij}P_j(t) + G_i'G_i = 0, \tag{III.5}$$

for all $i \in \mathbb{K}$ inside the time interval defined by $[t_k, t_{k+1})$, $k \in \mathbb{N}$. Notice that $i \in \mathbb{K}$ is the value of $\theta(t)$ at the instant of time $t \in [t_k, t_{k+1})$. The solution of (III.5) always exists and is uniquely determined by

$$\begin{aligned} P_i(t) = \; & e^{\bar{F}_i'(t_{k+1}-t)}P_i(t_{k+1}^-)e^{\bar{F}_i(t_{k+1}-t)} \\ & + \int_t^{t_{k+1}} e^{\bar{F}_i'(\tau-t)} \left(G_i'G_i + \sum_{j\neq i\in\mathbb{K}} \lambda_{ij}P_j(\tau) \right) e^{\bar{F}_i(\tau-t)}d\tau \end{aligned} \tag{III.6}$$

for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, provided a set of final boundary conditions $P_i(t_{k+1}^-)$. The new matrices $\bar{F}_i$ are defined by $\bar{F}_i = F_i + (\lambda_{ii}/2)I$ for each $i \in \mathbb{K}$. Hence, imposing $P_i(t_{k+1}^-) \geq 0$ for all $i \in \mathbb{K}$, a positive semi-definite solution $P(t)$ can always be obtained since the second right hand term of (III.6) is positive semi-definite for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, and all $i \in \mathbb{K}$.

Moreover, due to the time invariance of the coupled Lyapunov equations (III.5), the solutions defined by (III.6) admit a periodic extension in any subsequent time interval $1 \leq k \in \mathbb{N}$ whenever the final boundary constraints remain unchanged for all $k \in \mathbb{N}$ and each $i \in \mathbb{K}$. Mathematically, $P_i(t) = P_i(t - t_k)$ for all $i \in \mathbb{K}$ and all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$. Consequently, defining the sampling period $T = t_{k+1} - t_k > 0$ and the boundary conditions $P_i(t_k) = P_i(0)$ and $P_i(t_{k+1}^-) = P_i(T)$ for all $i \in \mathbb{K}$ and all $k \in \mathbb{N}$, the solutions $P_i(t)$ evaluated at the beginning of the time interval $[0, T)$ are

$$P_i(0) = e^{\bar{F}_i' T} P_i(T) e^{\bar{F}_i T} + R_i(P, T). \tag{III.7}$$

Matrices $R_i(P, T)$ are positive semi-definite functions that depend on matrices $P_j(t)$, $j \neq i \in \mathbb{K}$, evaluated inside the interval $[0, T)$ and on the sampling period T. From (III.6) and (III.7), they are expressed as

$$R_i(P, T) = \int_0^T e^{\bar{F}_i' \tau} \left(G_i' G_i + \sum_{j \neq i \in \mathbb{K}} \lambda_{ij} P_j(\tau) \right) e^{\bar{F}_i \tau} d\tau. \tag{III.8}$$

As a consequence, the behavior of the system trajectory described by the hybrid approach (III.1) can be entirely analysed by evaluating (III.5) inside the time interval $[0, T)$ and considering at each $k \in \mathbb{N}$ the discontinuity imposed by the third equation of (III.1). Since this discontinuity occurs at the end of the time interval defined by two consecutive jumps, it can be efficiently regarded in the boundary conditions of a TPBVP. This understanding is fundamental to state the next theorem, which is the key result to solve the $\mathcal{H}_2$ optimal control problem for an HMJLS in the form of (III.1).

Theorem III.1 *Let $T > 0$ be given. If there exist matrices $S_i > 0$, $i \in \mathbb{K}$, satisfying the TPBVP composed by the coupled Lyapunov equations* (III.5) *subject to the initial $P_i(0) < S_i$ and final $P_i(T) > H_i' S_i H_i$ boundary conditions for all $i \in \mathbb{K}$, then the HMJLS* (III.1) *is MSS and the performance index* (III.4) *satisfies*

$$\mathcal{J}_2 < \sum_{i \in \mathbb{K}} \pi_{i0} \, \boldsymbol{Tr}(J_i' H_i' S_i H_i J_i). \tag{III.9}$$

Proof: In order to prove Theorem III.1, consider that there exist matrices $S_i > 0$ for all $i \in \mathbb{K}$ such that the TPBVP composed by the coupled differential Lyapunov equations (III.5) and initial $P_i(0) < S_i$ and final $P_i(T) > H_i' S_i H_i$ boundary conditions holds. Then, for $\nu(t) = (\xi(t), \theta(t), t)$, define the quadratic functional $V : \mathbb{R}^{n+m} \times \mathbb{K} \times \mathbb{R}^+ \to \mathbb{R}^+$ as

$$V(\nu(t)) := \xi(t)' P_{\theta(t)}(t) \xi(t) \tag{III.10}$$

for each $\theta(t) \in \mathbb{K}$ and all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$. Notice that $P(t) \geq 0$ for all $t \in [0, T]$ provided that $P(T) \geq 0$. Additionally, the possibility to reproduce the solutions of the TPBVP in a periodic way, where $T = t_{k+1} - t_k > 0$ for all $k \in \mathbb{N}$, makes possible to do the same to the functional (III.10). As a consequence, the boundary conditions of the TPBVP can be rewritten as $P_i(t_k) = P_i(0) < S_i$ and $P_i(t_{k+1}^-) = P_i(T) > H_i' S_i H_i$ for all $i \in \mathbb{K}$. The functional $V(\nu(t))$ evaluated at the beginning of each time interval $[t_k, t_{k+1})$ yields

$$V(\nu(t_k)) < \xi(t_k)' S_{\theta(t_k)} \xi(t_k) \tag{III.11}$$

for all $k \in \mathbb{N}$. Notice that (III.11) states an upper bound to the initial condition of (III.10) defined inside the interval $[t_k, t_{k+1})$. Furthermore, due to (III.2), the stochastic process imposes $\theta(t_{k+1}^-) = \theta(t_{k+1})$ with probability one (almost surely). Hence, due to the discontinuity of the state variable $\xi(t)$ at the time instant $t_{k+1}^- \to t_{k+1}$, similarly, at the end of the time interval $[t_k, t_{k+1})$

$$\begin{aligned}
\mathcal{E}^{\nu(t_k)} \left\{ V(\nu(t_{k+1}^-)) \right\} &= \mathcal{E}^{\nu(t_k)} \left\{ \xi(t_{k+1}^-)' P_{\theta(t_{k+1}^-)}(t_{k+1}^-) \xi(t_{k+1}^-) \right\} \\
&= \mathcal{E}^{\nu(t_k)} \left\{ \xi(t_{k+1}^-)' P_{\theta(t_{k+1})}(t_{k+1}^-) \xi(t_{k+1}^-) \right\} \\
&> \mathcal{E}^{\nu(t_k)} \left\{ \xi(t_{k+1}^-)' H_{\theta(t_{k+1})}' S_{\theta(t_{k+1})} H_{\theta(t_{k+1})} \xi(t_{k+1}^-) \right\} \\
&= \mathcal{E}^{\nu(t_k)} \left\{ \xi(t_{k+1})' S_{\theta(t_{k+1})} \xi(t_{k+1}) \right\},
\end{aligned} \tag{III.12}$$

which provides a lower bound to the mean of the final condition of the functional $V(\nu(t))$ inside the interval $[t_k, t_{k+1})$.

On the other hand, as a consequence of the Dynkin's formula (see Section II.5) applied to the functional $V(\nu(t))$ and using the definition of $\mathcal{J}_2$, (III.4), it is possible to write

$$\begin{aligned}
V(\nu(t_k)) - \mathcal{E}^{\nu(t_k)} \left[V(\nu(t_{k+1}^-)) \right] &= -\mathcal{E}^{\nu(t_k)} \left[\int_{t_k}^{t_{k+1}} \mathcal{L}V(\nu(\tau)) d\tau \right] \\
&= \mathcal{E}^{\nu(t_k)} \left[\int_{t_k}^{t_{k+1}} z(\tau)' z(\tau) d\tau \right]
\end{aligned} \tag{III.13}$$

for each $i \in \mathbb{K}$ and all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, where $\xi(t)$ evolves according to (III.1) with $w(t) \equiv 0$ and initial conditions $\theta(0^-) = \theta(0) = \theta_0$ and $\xi(0^-) = \xi_0$.

Notice that $\mathcal{E}[V(\nu(t))]$ can be considered a valid Lyapunov functional in terms of Definition II.5 since the time-varying matrices $P_i(t)$ are positive definite for all $t \in [t_k, t_{k+1})$. Even though, define a new quadratic functional as $v(\nu(t_k)) \triangleq \xi(t_k)' S_{\theta(t_k)} \xi(t_k)$, which is valid for all $k \in \mathbb{N}$ and depends on constant matrices S_i, $i \in \mathbb{K}$. Due to $S_i > 0, \forall i \in \mathbb{K}$, then $v(\nu(t_k))$ is positive definite and $\mathcal{E}[v(\nu(t_k))]$ can be considered a valid Lyapunov functional associated to the discrete-time stochastic process $\xi(t_k) \to \xi(t_{k+1}^-) \to \xi(t_{k+1})$ for all $k \in \mathbb{N}$. Plugging $v(\nu(t))$ into (III.11) and (III.12), and using (III.13) thus

$$v(\nu(t_k)) - \mathcal{E}^{\nu(t_k)} \left[v(\nu(t_{k+1})) \right] > \mathcal{E}^{\nu(t_k)} \left[\int_{t_k}^{t_{k+1}} z(\tau)' z(\tau) d\tau \right] \tag{III.14}$$

for all $k \in \mathbb{N}$. Hence, two consequences can be drawn. First, due to the strict inequality in (III.14) and by the fact that the right hand term of (III.14) is positive definite for all $k \in \mathbb{N}$, there exists $\varepsilon > 0$ sufficiently small such that $\mathcal{E}^{\nu(t_k)}[v(\nu(t_{k+1}))] \leq (1-\varepsilon)v(\nu(t_k))$ for all $i \in \mathbb{K}$. This implies that $\mathcal{E}[v(\nu(t_{k+1}))] \to 0$ as $k \in \mathbb{N}$ goes to infinity. Consequently, $\mathcal{E}[\|\xi(t)\|^2] \to 0$ as $t \to \infty$, that is, mean square stability (see Definition II.11) holds.

In addition, (III.14) also yields

$$\begin{aligned}\mathcal{E}\left[\int_0^\infty z(t)'z(t)dt\right] &= \mathcal{E}\left[\sum_{k\in\mathbb{N}} \mathcal{E}^{\nu(t_k)}\left[\int_{t_k}^{t_{k+1}} z(t)'z(t)dt\right]\right] \\ &< \mathcal{E}\left[\sum_{k\in\mathbb{N}} \left\{v(\nu(t_k)) - \mathcal{E}^{\nu(t_k)}[v(\nu(t_{k+1}))]\right\}\right] \\ &= \mathcal{E}\left[v(\nu(0))\right] \\ &= \sum_{i\in\mathbb{K}} \pi_{i0}\xi_0' H_i' S_i H_i \xi_0 \end{aligned} \tag{III.15}$$

where it has been used the fact that $\xi(0) = H_{\theta_0}\xi_0$ and $\pi_{i0} = \mathcal{P}(\theta_0 = i)$, $i \in \mathbb{K}$. Then, remembering that the initial condition $\xi(0^-) = \xi_0 = J_{\theta(0^-)}e_l$, $l = 1, \cdots, r$, the Markov chain model (III.2) imposes $\theta(0) = \theta(0^-)$ with probability one (almost surely),

$$\mathcal{J}_2 < \sum_{l=1}^{r}\sum_{i\in\mathbb{K}} \pi_{i0} e_l' J_i' H_i' S_i H_i J_i e_l = \sum_{i\in\mathbb{K}} \pi_{i0} \textbf{Tr}(J_i' H_i' S_i H_i J_i), \tag{III.16}$$

completing thus the proof. ■

Theorem III.1 puts in evidence that an upper bound to the quadratic cost of interest can be evaluated by adding the contribution of the cost corresponding to each time interval $[t_k, t_{k+1})$ for all $k \in \mathbb{N}$. Notice that the key issue to state this result is the existence of a solution to the TPBVP introduced in Theorem III.1, which is ensured by the expression (III.6). Hence, one question remains about the possibility to calculate exactly the $\mathcal{J}_2$ performance index (III.3). The following remark aims to answer this question.

Remark III.1 *By the fact that the functional* (III.10) *encompasses all solutions of the coupled Lyapunov equation* (III.5)*, this choice* (III.10) *has a straightforward and strong consequence: the exact value of the performance index* $\mathcal{J}_2$*,* (III.3)*, can be obtained using Theorem III.1. Clearly, this statement holds whenever* $P_i(0) \to S_i$ *and* $P_i(T) \to H_i' S_i H_i$ *for all* $i \in \mathbb{K}$ *as a consequence of the fact that, in the limit, inequality* (III.14) *reduces to*

$$v(\nu(t_k)) - \mathcal{E}^{\nu(t_k)}\left[v(\nu(t_{k+1}))\right] = \mathcal{E}^{\nu(t_k)}\left[\int_{t_k}^{t_{k+1}} z(\tau)'z(\tau)d\tau\right] \tag{III.17}$$

for all $k \in \mathbb{N}$*. Then, the performance index is given by* $\mathcal{J}_2 = \sum_{i\in\mathbb{K}} \pi_{i0} \textbf{Tr}(J_i' H_i' S_i H_i J_i)$*.* □

Even though the equality can be stated, the formulation considered in Theorem III.1, with an upper bound to the performance index, is essential to numerically solve the optimal control problem. This is handled in the next chapters by means of a convex programming formulation. Moreover, equations (III.7) induce a way to calculate iteratively the solution of the coupled Lyapunov equations (III.5). Hence, the result from Theorem (III.1) together with (III.7) solves completely and numerically the problem of evaluating the quadratic cost $\mathcal{J}_2$ for the HMJLS (III.1) whenever a sampling interval $T > 0$ is given.

Remark III.2 *The HMJLS mean square stability depends strongly on the discontinuity imposed by the jump equation of* (III.1) *since the TPBVP admits a solution even if the continuous time system defined by the first equation of* (III.1) *is not MSS. In other words, the effect of closing the loop by means of the state feedback sampled-data control law* (II.56) *is the discontinuity imposed by the third equation of* (III.1) *by means of the matrices H_i, $i \in \mathbb{K}$.* □

Remark III.3 *From Theorem III.1, the HMJLS* (III.1) *is MSS whenever a solution to the TPBVP exists. This condition implies that the infinitesimal generator* (**??**) *is negative definite for all $i \in \mathbb{K}$ once $P_i(t)$ are determined from the existence of $P_i(0) < S_i$ such that $P_i(T) > H_i' S_i H_i$ for all $i \in \mathbb{K}$. This result reflects the Lyapunov theory (see Definition II.5 and Theorem II.1) adapted to the stochastic scenario.* □

III.2 Reduction to the Pure MJLS Case

An important analysis to validate the result from Theorem III.1 is to verify what happens when the sampling period $T \to 0$, that is, when the HMJLS (III.1) collapses to a mean square stable MJLS. In this case, it is expected that the result (II.53) can be recovered. Indeed, impose $H_i = I$ with compatible dimensions for all $i \in \mathbb{K}$ in the system (III.1). Applying these matrices to the TPBVP introduced in Theorem III.1, the boundary conditions of the TPBVP become $P_i(0) = P_i(T) = S_i$ for all $i \in \mathbb{K}$ and, consequently, the solution is expressed by

$$e^{\bar{F}_i' T} S_i e^{\bar{F}_i T} = S_i - R_i(S, T) \tag{III.18}$$

with $R_i(S, T)$ defined by (III.8) for all $i \in \mathbb{K}$ and a given $T > 0$. In fact, (III.18) is a discrete-time Lyapunov equation, whose solution is

$$S_i = \sum_{k \in \mathbb{N}} \left(e^{\bar{F}_i' T} \right)^k R_i(S, T) \left(e^{\bar{F}_i T} \right)^k \tag{III.19}$$

for all $i \in \mathbb{K}$. Then, it becomes evident that the time invariant matrices $P_i(t) \equiv S_i$ solve completely the TPBVP of Theorem III.1.

On the other hand, notice that, according to Theorem II.5, there exist matrices $S_i > 0$, $i \in \mathbb{K}$, such that the coupled algebraic Lyapunov equations

$$F_i' S_i + S_i F_i + \sum_{j \in \mathbb{K}} \lambda_{ij} S_j = -G_i' G_i \tag{III.20}$$

hold for all $i \in \mathbb{K}$ since matrices $\bar{F}_i = F_i + (\lambda_{ii}/2)I$ are Hurwitz stable. The solution of (III.20) is

$$\begin{aligned} S_i &= \int_0^\infty e^{\bar{F}_i'\tau}\left(G_i'G_i + \sum_{j\neq i\in\mathbb{K}}\lambda_{ij}S_j\right)e^{\bar{F}_i\tau}d\tau \\ &= \sum_{k\in\mathbb{N}} e^{\bar{F}_i'kT}\left[\int_{t_k}^{t_{k+1}} e^{\bar{F}_i'(\tau-kT)}\left(G_i'G_i + \sum_{j\neq i\in\mathbb{K}}\lambda_{ij}S_j\right)e^{\bar{F}_i(\tau-kT)}d\tau\right]e^{\bar{F}_ikT} \\ &= \sum_{k\in\mathbb{N}}\left(e^{\bar{F}_i'T}\right)^k R_i(S,T)\left(e^{\bar{F}_iT}\right)^k \end{aligned} \tag{III.21}$$

which is exactly the expression (III.18). Hence, the new matrices $H_i = I$, $i \in \mathbb{K}$, represent consistently the hypothesis of the mean square stability of the MJLS and the performance index becomes

$$\mathcal{J}_2 = \sum_{i\in\mathbb{K}} \pi_{i0}\mathbf{Tr}(J_i'S_iJ_i). \tag{III.22}$$

This is exactly the result expected from (II.53).

III.3 Iterative Procedure to Solve the TPBVP

Due to intrinsic mathematical difficulties to solve analytically the TPBVP presented in Theorem III.1, an iterative procedure is necessary. As previously stated, equations (III.7)–(III.8) suggest a way to solve that problem. Indeed, (III.7)–(III.8) together with (III.16) completely describe the problem to calculate the $\mathcal{J}_2$ performance index in a convex programming formulation easily solved by the numerical machinery available to date. In this case, the $\mathcal{J}_2$ performance index can be evaluated by solving

$$\inf_{S_i>0}\left\{\sum_{i\in\mathbb{K}}\pi_{i0}\mathbf{Tr}(J_i'H_i'S_iH_iJ_i) \ : \ e^{\bar{F}_i'T}H_i'S_iH_ie^{\bar{F}_iT} < S_i - R_i(P,T)\right\}. \tag{III.23}$$

This formulation is easily solved since it is expressed as N uncoupled subproblems once the coupling matrices $R_i(P,T)$ are fixed. In order to solve (III.23), it is necessary to consider that matrices $H_ie^{\bar{F}_iT}$ are Schur stable for all $i \in \mathbb{K}$. Otherwise a solution does not exist. The following iterative procedure converges to the exact value of the $\mathcal{J}_2$ performance index, (III.3). In other words, the exact value of the $\mathcal{H}_2$ norm is determined.

Algorithm III.1

1. *Define a constant sampling interval $T > 0$. Considering $t \in [0,T)$, initialize $\ell = 0$, and set $S_\ell = 0$ and $\mathcal{J}_{2\ell} = 0$.*

2. *Determine the solutions $P_{i\ell}(0)$ of the coupled Lyapunov equations*

$$\dot{P}_{i\ell} + \bar{F}_i' P_{i\ell} + P_{i\ell}\bar{F}_i + \sum_{j\neq i\in\mathbb{K}} \lambda_{ij} P_{j\ell} + G_i' G_i = 0 \tag{III.24}$$

subject to the final boundary conditions $P_{i\ell}(T) = H_i' S_{i\ell} H_i \geq 0$ for each $i \in \mathbb{K}$. From (III.7)*, determine*

$$R_i(P_\ell, T) = P_{i\ell}(0) - e^{\bar{F}_i' T} P_{i\ell}(T) e^{\bar{F}_i T}, \tag{III.25}$$

where $\bar{F}_i = F_i + \frac{1}{2}\lambda_{ii} I$ for each $i \in \mathbb{K}$.

3. *Determine $S_{(\ell+1)} > 0$ and the current value of the cost $\mathcal{J}_{2(\ell+1)}$ by solving*

$$\inf_{S_{i(\ell+1)}>0} \left\{ \sum_{i\in\mathbb{K}} \pi_{i0}\, \textbf{Tr}(J_i' H_i' S_{i(\ell+1)} H_i J_i) \ : \right. \\ \left. e^{\bar{F}_i' T} H_i' S_{i(\ell+1)} H_i e^{\bar{F}_i T} < S_{i(\ell+1)} - R_i(P_\ell, T) \right\}, \tag{III.26}$$

which can be decomposed into N uncoupled convex programming subproblems expressed by LMIs.

4. *Set $(\ell+1) \to \ell$ and iterate until the cost variation $\mathcal{J}_{2(\ell+1)} - \mathcal{J}_{2\ell}$ becomes sufficiently small.*

Consider the following remarks concerning Algorithm III.1.

Remark III.4 *Since $R_i(P,T) \geq 0$ and matrices $S_i > 0$ for all $i \in \mathbb{K}$, problem* (III.23) *is feasible if matrices $H_i e^{\bar{F}_i T}$ are Schur stable for all $i \in \mathbb{K}$. These conditions are in accordance with the strong dependence of the solution of* (III.23) *on the matrices H_i for all $i \in \mathbb{K}$ and on the choice of the sampling period $T > 0$. Obviously, large sampling periods can cause an unstable closed-loop system.* □

Remark III.5 *Notice that the iterative process of the Algorithm III.1 stems from the fact that, for all $i \in \mathbb{K}$, the matrices $R_i(P_\ell, T)$,* (III.26)*, depend on the matrices $P_{i\ell}$ calculated in the previous iteration. Furthermore, with $R_i(P_\ell, T)$ fixed for all $i \in \mathbb{K}$, the optimal solution of each subproblem* (III.26) *is arbitrarily close to the positive definite solution of the corresponding Discrete-time Algebraic Lyapunov Equation (DALE). Clearly, in this case, a proof of the convergence of the Algorithm III.1 to a stationary solution $(S_i^*, P_i^*(t))$ for all $i \in \mathbb{K}$ and all $t \in [0,T)$ becomes necessary.* □

It is important to mention that similar algorithms are adopted to solve numerically optimal control problems in the context of MJLS for a fixed final condition $P(T)$, as in (COSTA; FRAGOSO; TODOROV, 2013). However, the TPBVP from Theorem III.1 is constrained on both extremes of the interval of definition of the coupled differential Lyapunov equations (III.5). Notice that both boundary conditions are taken into account in problem (III.23) since the DALE is the solution (III.6) evaluated at the closure of the time interval $[0,T)$. In other words, the main difference between the present context and those usually found in literature is that all matrices $P(0)$ and $P(T)$ vary in each iteration. Next theorem certificates the convergence of Algorithm III.1.

Theorem III.2 *Assume that the TPBVP defined in Theorem III.1 has a bounded solution* $(S_i^*, P_i^*(t))$ *for all* $i \in \mathbb{K}$ *and all* $t \in [0,T)$. *The algorithm III.1 is uniformly convergent, and any two subsequent iterations are such that* $S^* \geq S_{(\ell+1)} \geq S_\ell \geq 0$ *and* $\mathcal{J}_2^* \geq \mathcal{J}_{2(\ell+1)} \geq \mathcal{J}_{2(\ell)}$.

Proof: First, consider that matrices $H_i e^{\bar{F}_i T}$ for each $i \in \mathbb{K}$ are Schur stable. Thus, problem (III.23) is feasible, and there exist matrices $S_i^* > 0$ for all $i \in \mathbb{K}$ such that the TPBVP from Theorem III.1 holds. From (III.24), in Step 2, it is possible to write that in two subsequent iterations

$$\dot{\Delta}_{i(\ell+1)} + F_i' \Delta_{i(\ell+1)} + \Delta_{i(\ell+1)} F_i + \sum_{j \in \mathbb{K}} \lambda_{ij} \Delta_{j(\ell+1)} = 0, \tag{III.27}$$

where $\Delta_{i(\ell+1)}(t) = P_{i(\ell+1)}(t) - P_{i\ell}(t)$ for all $i \in \mathbb{K}$. Assuming that $\Gamma_{i(\ell+1)} = S_{i(\ell+1)} - S_{i\ell} \geq 0$ for all $i \in \mathbb{K}$, then due to the final boundary conditions, $\Delta_{i(\ell+1)}(T) = H_i' \Gamma_{i(\ell+1)} H_i \geq 0$. Consequently, $\Delta_{i(\ell+1)}(t) \geq 0$ for all $i \in \mathbb{K}$ and all $t \in [0,T)$, which implies, from (III.8), that

$$R_i(P_{\ell+1}, T) - R_i(P_\ell, T) = \int_0^T e^{\bar{F}_i' \tau} \Big(\sum_{j \neq i \in \mathbb{K}} \lambda_{ij} \Delta_{i(\ell+1)}(\tau) \Big) e^{\bar{F}_i \tau} d\tau \geq 0 \tag{III.28}$$

for all $i \in \mathbb{K}$. From (III.26), in Step 3 of the Algorithm III.1, together with Remark III.5, the solutions on the border of the LMI are such that

$$e^{\bar{F}_i' T} H_i' \Gamma_{i(\ell+2)} H_i e^{\bar{F}_i T} - \Gamma_{i(\ell+2)} + (R_i(P_{\ell+1}, T) - R_i(P_\ell, T)) = 0 \tag{III.29}$$

for all $i \in \mathbb{K}$. Since matrices $H_i e^{\bar{F}_i T}$ for each $i \in \mathbb{K}$ are Schur stable, then $\Gamma_{i(\ell+2)} \geq 0$ for all $i \in \mathbb{K}$. In other words, $S_{i(\ell+1)} \geq S_{i\ell}$ implies that $P_{i(\ell+1)}(t) \geq P_{i\ell}(t)$, which yields $S_{i(\ell+2)} \geq S_{i(\ell+1)}$. Initializing the algorithm with $\ell = 0$, the first step imposes $S_{i0} = 0$ for all $i \in \mathbb{K}$ and $\mathcal{J}_{2(0)} = 0$. Then, the solution of (III.24) provides $P_{i0}(0) \geq 0$ for all $i \in \mathbb{K}$. As a consequence, $R_i(P_0, T) \geq 0$, which produces $S_{i1} \geq 0$ in the third step for all $i \in \mathbb{K}$. Due to the previous property, each new iteration produces

$$0 = S_{i0} \leq S_{i1} \leq S_{i2} \leq \cdots \tag{III.30}$$

for all $i \in \mathbb{K}$.

On the other hand, consider the stationary solution of (III.24), characterized by $(S_i^*, P_i^*(t))$ for all $i \in K$, which satisfies

$$e^{\bar{F}_i' T} H_i' S_i^* H_i e^{\bar{F}_i T} - S_i^* + R_i(P_i^*, T) = 0. \tag{III.31}$$

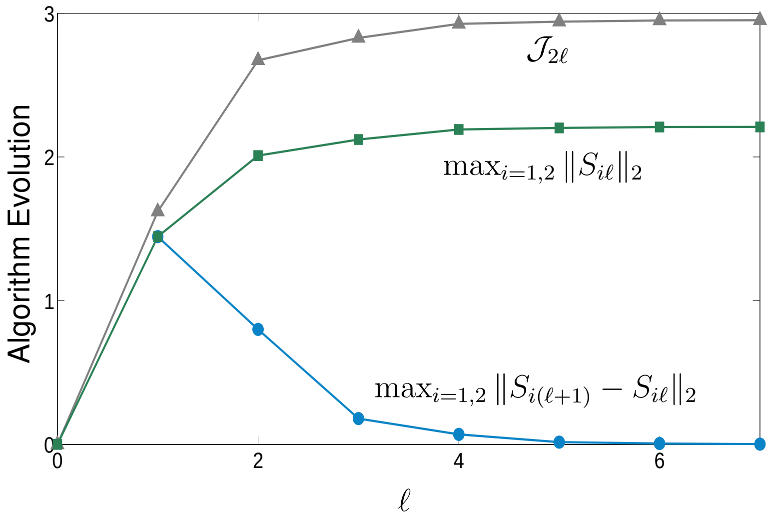

Figure III.1 – Evolution of the Algorithm III.1.

Suppose that $S_i^* \geq S_{i\ell} > 0$ for all $i \in K$. Adopting the same reasoning as before, $P_i^*(t) \geq P_{i\ell}(t) \geq 0$ and $R_i(P^*, T) \geq R_i(P_\ell, T) \geq 0$ for all $i \in \mathbb{K}$ and all $t \in [0, T)$. As a consequence, $S_i^* \geq S_{i(\ell+1)} > 0$ for all $i \in K$, which is valid for all $\ell \in \mathbb{N}$. Due to the $S_{i0} = 0$ for all $i \in \mathbb{K}$, the algorithm generates a sequence of matrices $\{S_{i\ell}\}_{\ell=0}^{\infty}$ bounded by S_i^* for all $i \in \mathbb{K}$. The hypothesis on the existence of a set of bounded solutions S_i^* for each $i \in \mathbb{K}$ is sufficient to assure that, see the monotone convergence result (Lemma 2.17, page 24) of Costa, Fragoso & Todorov (2013),

$$\lim_{\ell \to \infty} S_{i\ell} \to S_i^* \qquad \text{(III.32)}$$

for all $i \in \mathbb{K}$. This indicates that the algorithm monotonically converges to the solution of S_i^* for all $i \in \mathbb{K}$. Consequently, $\mathcal{J}_{2\ell} \to \mathcal{J}_2^*$, concluding thus the proof. ■

A characteristic of Algorithm III.1 is that at each iteration it adds a new cost to a non decreasing sequence $\mathcal{J}_2^* \geq \mathcal{J}_{2(\ell+1)} \geq \mathcal{J}_{2(\ell)}$ for all $\ell \in \mathbb{N}$. This is calculated by minimizing the same objective function in a more constrained feasible set. For this reason, the convergence is expected to be fast. The most important aspects are illustrated by means of the next academical example.

III.4 Illustrative Numerical Example

This numerical example illustrates the theoretical result of Theorem III.1 implemented by the iterative procedure described in Algorithm III.1. The same system was adopted in Gabriel, Souza & Geromel (2014).

Example III.1 *Consider an MJLS with $N = 2$ modes with the system matrices given by*

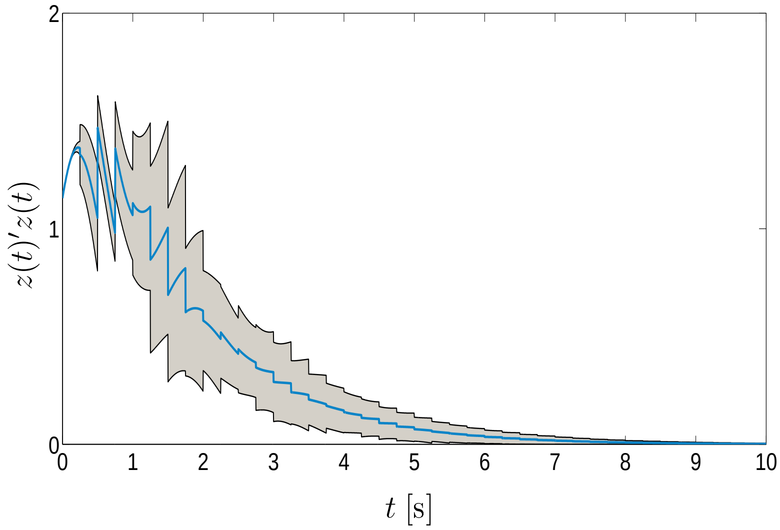

Figure III.2 – Evolution of the Monte Carlo simulation.

$$F_1 = \begin{bmatrix} 0 & 1 & 0 \\ -4 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; F_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$J_1 = J_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; G_1' = G_2' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.1791 & -0.5561 & 0 \end{bmatrix}, \; H_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.2972 & -0.8691 & 0 \end{bmatrix}.$$

Consider also the sampling period $T = 250\ [ms]$ and suppose that the transition rate matrix $\Lambda \in \mathbb{R}^{2\times 2}$ is given by

$$\Lambda = \begin{bmatrix} -0.5 & 0.5 \\ 0.2 & -0.2 \end{bmatrix}$$

with initial probability $\pi_0 = [1\ 0]'$. Then, the computed $\mathcal{H}_2$ cost is

$$\mathcal{J}_2^* = \int_0^\infty \mathcal{E}[z(t)'z(t)]dt = 2.96.$$

Figure III.1 shows the fast convergence of the iterative procedure proposed in Algorithm III.1 that took seven iterations to converge. The gray curve (with triangular markers) shows the convergence of the performance index $\mathcal{J}_{2\ell}$ to the true value (verified numerically *that* $\max_{i\in\mathbb{K}} \|S_i^* - P_i^*(0)\|_2$ *is sufficiently small) of the square of the* $\mathcal{H}_2$ *norm. The green curve (with squared markers) shows the evolution of the norm* $\max_{i\in\mathbb{K}} \|S_{i\ell}\|_2$ *in order to illustrate the convergence of the sequence of matrices* $S_{i\ell}$ *to the stationary solution* S_i^**. The blue curve (with rounded markers) shows how fast the convergence occurs through the evolution of* $\max_{i=1,2} \|S_{i(\ell+1)} - S_{i\ell}\|_2$.

On the other hand, using the efficient numerical procedure proposed in the reference Leon-Garcia (2007), a Monte Carlo simulation of $2,000$ *samples provides the value of* $\mathcal{J}_2 = 2.96$ *for the* $\mathcal{H}_2$ *norm. This result puts in evidence the quality of the calculated index. The Monte Carlo simulation is shown in Figure III.2, where the blue solid curve is the mean value of the square norm* $\|z(t)\|^2$*, and the shaded area corresponds to one standard deviation from the mean value curve. The convergence of all trajectories of the output towards zero is a characteristic of a MSS system.*

CHAPTER IV

Stability and $\mathcal{H}_\infty$ Norm Evaluation of Hybrid MJLS

This chapter is the counterpart of Chapter III for the $\mathcal{H}_\infty$ context. Thus, it is devoted to state necessary and sufficient conditions to assure the HMJLS mean-square stability at the same time the exact value of the $\mathcal{H}_\infty$ norm is determined. As before, in order to accomplish these goals, evenly spaced sampling instants, $T = t_{k+1} - t_k \geq 0$ for all $k \in \mathbb{N}$, are considered. Theoretical results on the stability analysis in the context of $\mathcal{H}_\infty$ norm are derived from a TPBVP with initial and final boundary conditions. Moreover, an algorithm based on an iterative procedure is suggested and its convergence proved whenever the TPBVP admits a positive definite solution. A numerical example shows that the procedure is suitable for the purpose of calculating the $\mathcal{H}_\infty$ norm of an HMJLS.

The $\mathcal{H}_\infty$ analysis is a robustness study since the $\mathcal{H}_\infty$ norm is a measure of the influence of the worst case disturbance on the system. The disturbance is represented by the exogenous input $w(t)$, which must belong to the set $\mathbb{L}_2 \setminus \{0\}$. As before, the HMJLS is expressed by

$$\mathcal{S}_H : \begin{cases} \dot{\xi}(t) &= F_{\theta(t)}\xi(t) + J_{\theta(t)}w(t) \\ z(t) &= G_{\theta(t)}\xi(t) \\ \xi(t_k) &= H_{\theta(t_k)}\xi(t_k^-) \end{cases} \tag{IV.1}$$

evolving from initial conditions $\xi(0^-) = \xi_0 = 0$ and $\theta(0^-) = \theta(0) = \theta_0$. For this system, $\xi(t) : \mathbb{R}^+ \to \mathbb{R}^{n+m}$ is the augmented state variable, $w(t) : \mathbb{R}^+ \to \mathbb{R}^r$ is the exogenous input, $z(t) : \mathbb{R}^+ \to \mathbb{R}^p$ is the controlled output, and $\{\theta(t) \in \mathbb{K}\}$, $\mathbb{K} = \{1, \cdots, N\}$, is a continuous-time Markov process with a transition rate matrix $\{\lambda_{ij}\} = \Lambda \in \mathbb{R}^{N\times N}$. According to Chapter II, the conditional probability associated to Λ depends only on the time interval h between successive jumps of the Markov chain and is given by

$$Q_{ij}(h) = \mathcal{P}[\theta(t+h) = j | \theta(t) = i] = \begin{cases} 1 + \lambda_{ij}h + o(h) & \text{, for } i = j \\ \lambda_{ij}h + o(h) & \text{, for } i \neq j \end{cases}, \tag{IV.2}$$

where $o(h)$ is high order terms such that $\lim_{h\to 0^+} o(h)/h = 0$. The elements of $\Lambda \in \mathbb{R}^{N\times N}$ satisfy $\lambda_{ij} \geq 0$ for all $i \neq j$ and $\sum_{j\in\mathbb{K}} \lambda_{ij} = 0$ for all $i \in \mathbb{K}$. Thus, $\lambda_{ii} \leq 0$ for all $i \in \mathbb{K}$.

For the purposes of this chapter, consider again that $\pi_i(0) = \pi_{i0} = \mathcal{P}[\theta_0 = i]$ is the initial distribution of the variable $\theta(t)$. Notice that the discontinuity introduced by the sampled-data control law (II.55) is represented by the matrices H_i, $i \in \mathbb{K}$, in the third equation of (IV.1). Thus, the system (IV.1) is valid for all $t \in \mathbb{R}^+$ since all $t \in [t_k, t_{k+1})$ for each $k \in \mathbb{N}$ are considered.

IV.1 Theoretical Results

Similar results as stated in Chapter III can be obtained for the $\mathcal{H}_\infty$ norm context considering the HMJLS described in (IV.1). The procedure uses also a TPBVP to guarantee the stability of the hybrid system (III.1) and to compute the $\mathcal{H}_2$ norm. The main difference is the complexity of the mathematical developments in the $\mathcal{H}_\infty$ case. Indeed, considering the adequate performance index $\mathcal{J}_\infty$ allows obtaining the expected results. In this work, it implies in a DRE, whose solution is not trivial. However, some workarounds conduct to a solution computationally feasible. Hence, consider the performance index given by

$$\mathcal{J}_\infty \triangleq \sup_{w \in \mathbb{L}_2^*} \int_0^\infty \mathcal{E}\left[z(t)'z(t) - \gamma^2 w(t)'w(t)\right] dt. \tag{IV.3}$$

As before, because (IV.1) is a PWC MJLS, it is convenient to rewrite the performance index (IV.3) as

$$\mathcal{J}_\infty = \sup_{w \in \mathbb{L}_2^*} \sum_{k \in \mathbb{N}} \int_{t_k}^{t_{k+1}} \mathcal{E}\left[z(t)'z(t) - \gamma^2 w(t)'w(t)\right] dt \tag{IV.4}$$

subject to (IV.1).

Similarly to the $\mathcal{H}_2$ case, multiple Lyapunov-like functionals can be used to assure the HMJLS stability. Thus, consider a collection of cost-to-go functionals of the form $V_{\theta(t)}(\xi(t), t) = \xi(t)'P_{\theta(t)}(t)\xi(t)$ with $\theta(t) \in \mathbb{K}$ and $t \in [t_k, t_{k+1})$ for each $k \in \mathbb{N}$, and $P_i(t)$ solves the coupled DRE

$$\dot{P}_i(t) + F_i'P_i + P_iF_i + \gamma^{-2}P_iJ_iJ_i'P_i + \sum_{j \in \mathbb{K}} \lambda_{ij}P_j + G_i'G_i = 0, \tag{IV.5}$$

for all $t \in [t_k, t_{k+1})$ and all $i \in \mathbb{K}$. Notice that $i \in \mathbb{K}$ is the value of $\theta(t)$ at the instant of time $t \in [t_k, t_{k+1})$. Equation (IV.5) does not admit an explicit solution. However, it has been proved the existence and uniqueness of a solution in Costa, Fragoso & Todorov (2013), see also Wonham (1968). This result, besides the developments included in Appendix A, suggests a way to evaluate (IV.5) using an iterative procedure. In order to obtain it, some mathematical analysis on the solution of (IV.5) are necessary. From (A.9) of Appendix A[1],

$$P_{i(\ell+1)}(t_k) - P_{i\ell}(t_k) \geq \Phi_{i\ell}(t_{k+1}^-)'\Big(P_{i(\ell+1)}(t_{k+1}^-) - P_{i\ell}(t_{k+1}^-)\Big)\Phi_{i\ell}(t_{k+1}^-) \tag{IV.6}$$

[1] Matrices $\Phi_{i\ell}(T)$ are the solutions of $\dot{\Phi}_{i\ell}(t) = M_{i\ell}(t)\Phi_{i\ell}(t)$ subject to $\Phi_{i\ell}(0) = I$ for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$ evaluated at the instant of time $t = T > 0$. Additionally, $M_{i\ell}(t) = F_i + \frac{1}{2}\lambda_{ii}I + \gamma^{-2}J_iJ_i'P_{i\ell}(t)$.

holds for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. Moreover, notice that equation (A.6) is time invariant with respect to the matrices $\Delta_{i(\ell+1)}(t)$ for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. As a consequence, its solution evaluated in the first time interval $[0, T)$ stays exactly the same in the subsequent intervals $[t_k, t_{k+1})$ for all $k \geq 1$ provided that $P_{i(\ell+1)}(t_k) = P_{i(\ell+1)}(0)$ and $P_{i(\ell+1)}(t_{k+1}^-) = P_{i(\ell+1)}(T)$ for all $i \in \mathbb{K}$ and each $\ell \in \mathbb{N}$. The same property that makes $P_i(t) = P_i(t - t_k)$ for all $i \in \mathbb{K}$ and all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, holds for matrices $M_{i\ell}(t)$ and $\Phi_{i\ell}(t)$, defined in Appendix A, as well. Thus, reorganizing the terms of (IV.6), using the mentioned periodic extension property, and imposing the initial $P_{i(\ell+1)}(0) < S_{i(\ell+1)}$ and final $P_{i(\ell+1)}(T) > H_i' S_{i(\ell+1)} H_i \geq 0$ boundary conditions for each $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$, inequality (IV.6) yields

$$\Phi_{i\ell}(T)' H_i' S_{i(\ell+1)} H_i \Phi_{i\ell}(T) - S_{i(\ell+1)} + P_{i\ell}(0) - \Phi_{i\ell}(T)' P_{i\ell}(T) \Phi_{i\ell}(T) < 0 \tag{IV.7}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{K}$. Setting $R_{i\ell}(T) = P_{i\ell}(0) - \Phi_{i\ell}(T)' P_{i\ell}(T) \Phi_{i\ell}(T)$ for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$, equation (IV.6) produces

$$\Phi_{i\ell}(T)' H_i' S_{i(\ell+1)} H_i \Phi_{i\ell}(T) - S_{i(\ell+1)} + R_{i\ell}(T) < 0 \tag{IV.8}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. Equation (IV.8) is a discrete-time algebraic Lyapunov inequality that must be feasible for $S_{\ell+1} > 0$ even though, in general, matrices $R_\ell(T)$ for some $\ell \in \mathbb{N}$ are not positive definite. Actually, this is a well known property that comes to light in the context of classical $\mathcal{H}_\infty$ theory. (See Colaneri, Geromel & Locatelli (1997).)

From the uncoupled Lyapunov inequalities (IV.8), matrices $S_{i(\ell+1)}$ are determined once matrices $\Phi_{i\ell}(T)$, $P_{i\ell}(T)$, and $P_{i\ell}(0)$ for all $i \in \mathbb{K}$ are evaluated. This can be done by enforcing $P_{i\ell}(T)$ arbitrarily close to $H_i' S_{i\ell} H_i$ and by considering a backward integration in (IV.5) since matrices $S_{i\ell}$ are known for all $i \in \mathbb{K}$ and each $\ell \in \mathbb{N}$. This procedure suggests a way to iteratively determine the solution of the TPBVP composed by the DRE (IV.5) subject to the boundary conditions $P_i(T) > H_i' S_i H_i \geq 0$ and $P_i(0) < S_i$, $i \in \mathbb{K}$.

Once again, the possibility to analyse the behavior of the system (IV.1) performance evaluating only the first time interval $[0, T)$ of equation (IV.5) that may admit (depending on the value of $\gamma > 0$) a unique stabilizing solution, yields the result of the next theorem. Moreover, the discontinuity imposed by the third equation of (IV.1) can conveniently be expressed through the boundary conditions of the TPBVP since the discontinuity occurs at the end of the time interval defined by $[t_k, t_{k+1})$ for all $k \in \mathbb{N}$.

Theorem IV.1 *Let $T > 0$ and $\gamma > 0$ be given. If there exist $S_i > 0$, $i \in \mathbb{K}$, satisfying the TPBVP composed by the coupled DRE* (IV.5) *subject to the initial $P_i(0) < S_i$ and final $P_i(T) > H_i' S_i H_i$ boundary conditions for each $i \in \mathbb{K}$, then the HMJLS* (IV.1) *is MSS and the performance index* (IV.4) *satisfies*

$$\mathcal{J}_\infty < \sum_{i \in \mathbb{K}} \pi_{i0} \xi_0' H_i' S_i H_i \xi_0. \tag{IV.9}$$

Proof: The proof is essentially the same as the proof of the Theorem III.1 except by the fact that the solution of the coupled Riccati equations, (IV.5), cannot be analytically but iteratively determined. Then, consider a set of matrices $S_i > 0$ for all $i \in \mathbb{K}$ such that the TPBVP of Theorem IV.1 is satisfied for a given sampling interval $T > 0$ and for a large enough $\gamma > 0$. Since multiple Lyapunov-like functionals can also be used in this case, define

$$V(\nu(t)) = \xi(t)' P_{\theta(t)}(t) \xi(t), \tag{IV.10}$$

where $\nu(t) = (\xi(t), \theta(t), t)$ for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$. The functional (IV.10) evaluated on the end of the time interval $[0, T)$ remains valid in this case and is expressed by

$$V(\nu(t_k)) < \xi(t_k)' S_{\theta(t_k)} \xi(t_k), \tag{IV.11}$$

where the initial boundary conditions $P_i(t_k)$ for all $i \in \mathbb{K}$ and each $k \in \mathbb{N}$ were used. Notice that inequality (IV.11) provides an upper bound to the functional (IV.10) at the time instant $t = t_k$. Analogously, for the final boundary conditions $P_i(t_{k+1})$ for all $i \in \mathbb{K}$ and each $k \in \mathbb{N}$,

$$\begin{aligned}
\mathcal{E}^{\nu(t_k)} \left\{ V(\nu(t_{k+1}^-)) \right\} &= \mathcal{E}^{\nu(t_k)} \left\{ \xi(t_{k+1}^-)' P_{\theta(t_{k+1}^-)}(t_{k+1}^-) \xi(t_{k+1}^-) \right\} \\
&= \mathcal{E}^{\nu(t_k)} \left\{ \xi(t_{k+1}^-)' P_{\theta(t_{k+1})}(t_{k+1}^-) \xi(t_{k+1}^-) \right\} \\
&> \mathcal{E}^{\nu(t_k)} \left\{ \xi(t_{k+1}^-)' H_{\theta(t_{k+1})}' S_{\theta(t_{k+1})} H_{\theta(t_{k+1})} \xi(t_{k+1}^-) \right\} \\
&= \mathcal{E}^{\nu(t_k)} \left\{ \xi(t_{k+1})' S_{\theta(t_{k+1})} \xi(t_{k+1}) \right\},
\end{aligned} \tag{IV.12}$$

for all $k \in \mathbb{N}$. Furthermore, due to (IV.2), the stochastic process imposes $\theta(t_{k+1}^-) = \theta(t_{k+1})$ with probability one (almost surely). This determines a lower bound to the functional (IV.10) at the time instant $t = t_{k+1}$.

Using Dynkin's formula (see Section II.5) applied to the performance index (IV.4), it follows that

$$\begin{aligned}
V(\nu(t_k)) - \mathcal{E}^{\nu(t_k)} \left[V(\nu(t_{k+1}^-)) \right] = \;& \mathcal{E}^{\nu(t_k)} \left[\int_{t_k}^{t_{k+1}} [z(t)'z(t) - \gamma^2 w(t)'w(t)] dt \right] \\
&+ \mathcal{E}^{\nu(t_k)} \left[\int_{t_k}^{t_{k+1}} \| \gamma^{-1} J_{\theta(t)}' P_{\theta(t)}(t) \xi(t) - \gamma w(t) \|_2^2 dt \right] \\
\geq \;& \mathcal{E}^{\nu(t_k)} \left[\int_{t_k}^{t_{k+1}} [z(t)'z(t) - \gamma^2 w(t)'w(t)] dt \right],
\end{aligned} \tag{IV.13}$$

which is valid for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, and for all $w \in \mathbb{L}_2$. Notice that, according to Theorem II.4, the set of matrices $P_i(t)$, satisfying the coupled DRE (IV.5), is positive definite. Consequently, $V(\nu(t))$ is a valid Lyapunov functional associated to the HMJLS (IV.1). Moreover, whenever a solution exists, it is bounded and unique for each $i \in \mathbb{K}$ and all $t \in [t_k, t_{k+1})$. On the other hand, define the quadratic function $v(\nu(t)) \triangleq \xi(t)' S_{\theta(t)} \xi(t)$ for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, and for each $\theta(t) \in \mathbb{K}$, which depends on constant matrices S_i,

$i \in \mathbb{K}$. Plugging (IV.11) and (IV.12) in (IV.13), thus

$$v(\nu(t_k)) - \mathcal{E}^{\nu(t_k)}\left[v(\nu(t_{k+1}))\right] > \mathcal{E}^{\nu(t_k)}\left[\int_{t_k}^{t_{k+1}} [z(t)'z(t) - \gamma^2 w(t)'w(t)]dt\right] \tag{IV.14}$$

holds for all $k \in \mathbb{N}$ and for all $w \in \mathbb{L}_2$. Because $S_i > 0$ for all $i \in \mathbb{K}$, $v(\nu(t_k))$ is positive definite and $\mathcal{E}[v(\nu(t_k))]$ can be considered a valid Lyapunov functional associated to the discrete-time stochastic process $\xi(t_k) \rightarrow \xi(t_{k+1}^-) \rightarrow \xi(t_{k+1})$ for all $k \in \mathbb{N}$. Hence, two consequences can be drawn. First, due to the strict inequality in (IV.14), imposing $w \equiv 0 \in \mathbb{L}_2$, there exists $\varepsilon > 0$ sufficiently small such that $\mathcal{E}[v(\nu(t_{k+1}))|\nu(t_k)] \leq (1-\varepsilon)v(\nu(t_k))$, which implies that $\mathcal{E}[v(\nu(t_{k+1}))] \rightarrow 0$ as $k \in \mathbb{N}$ goes to infinity. Consequently $\mathcal{E}[\|\xi(t)\|^2] \rightarrow 0$ as $t \rightarrow \infty$, that is, mean square stability holds.

In addition, another consequence of (IV.14) is that

$$\begin{aligned}
\mathcal{J}_\infty &= \sup_{w\in\mathbb{L}_2^*} \mathcal{E}\left[\sum_{k\in\mathbb{N}} \mathcal{E}^{\nu(t_k)}\left[\int_{t_k}^{t_{k+1}} [z(t)'z(t) - \gamma^2 w(t)'w(t)]dt\right]\right] \\
&< \mathcal{E}\left[\sum_{k\in\mathbb{N}} \left\{v(\nu(t_k)) - \mathcal{E}^{\nu(t_k)}\left[v(\nu(t_{k+1}))\right]\right\}\right] \\
&= \mathcal{E}\left[v(\nu(0))\right] = \sum_{i\in\mathbb{K}} \pi_{i0}\xi_0' H_i' S_i H_i \xi_0
\end{aligned} \tag{IV.15}$$

due to $\xi(0) = H_{\theta_0}\xi_0$ and $\pi_{i0} = \mathcal{P}(\theta_0 = i)$, $i \in \mathbb{K}$. The proof is concluded. ■

Four important comments about Theorem IV.1 are in order. First, Theorem IV.1 puts in evidence that an upper bound to the performance index of interest can be obtained by adding the cost corresponding to each time interval $[t_k, t_{k+1})$, $k \in \mathbb{N}$. Indeed, this property can be verified by changing inequality (IV.14) to

$$\sup_{w\in\mathbb{L}_2^*} \mathcal{E}^{\nu(t_k)}\left[\int_{t_k}^{t_{k+1}} [z(t)'z(t) - \gamma^2 w(t)'w(t)]dt + v(\nu(t_{k+1}))\right] < v(\nu(t_k)), \tag{IV.16}$$

which is valid for all $k \in \mathbb{N}$. Notice that, for this result, the discontinuity between successive jumps of the state variable are imposed by the jump equation in (IV.1). Second, the mean square stability of the discrete-time stochastic process $\xi(t_k) \rightarrow \xi(t_{k+1})$, $k \in \mathbb{N}$, is essential for the convergence of the sum indicated in (IV.15) since only the extremes of the time interval $[t_k, t_{k+1})$ are considered. Third, analysing (IV.8), the existence of a solution to the TPBVP requires that the matrices $H_i\Phi_{i\ell}(T)$, $i \in \mathbb{K}$, are Schur stable. This result states a very strong relation between the possibility of solving the TPBVP presented in Theorem IV.1 and the mean square stability of the hybrid MJLS (IV.1). Finally, due to the periodic extension property of the coupled DRE, the solution of the TPBVP in each time interval reduces to the solution in the first time interval. As mentioned before, the main challenge to solve it is the mathematical complexity, which is addressed in the next sections.

Remark IV.1 *Following the proof of Theorem IV.1, enforcing $P_i(0) \to S_i$ and $P_i(T) \to H_i' S_i H_i$ for all $i \in \mathbb{K}$, and considering that the exogenous input assumes the worst case disturbance value, $w(t) = \gamma^{-2} J_{\theta(t)}' P_{\theta(t)}(t)\xi(t)$ for all $\theta(t) \in \mathbb{K}$, then inequality* (IV.14) *becomes*

$$v(\nu(t_k)) - \mathcal{E}^{\nu(t_k)}\left[v(\nu(t_{k+1}))\right] = \mathcal{E}^{\nu(t_k)}\left[\int_{t_k}^{t_{k+1}} [z(t)'z(t) - \gamma^2 w(t)'w(t)]dt\right]. \tag{IV.17}$$

As a consequence, (IV.9) *provides the exact value of the performance index given by $\mathcal{E}[v(\nu(0))]$. Then, Theorem IV.1 encompasses all stabilizing solutions including the optimal one as far as the $\mathcal{H}_\infty$ norm is considered. The next corollary states this result.* □

Corollary IV.1 *Let $T > 0$ and $\gamma > 0$ be given. The HMJLS* (IV.1) *with initial condition $\xi_0 = 0$ is MSS and the performance index satisfies*

$$\mathcal{J}_\infty \triangleq \sup_{w \in \mathbb{L}_2^*} \int_0^\infty \mathcal{E}\left[z(t)'z(t) - \gamma^2 w(t)'w(t)\right] dt < 0 \tag{IV.18}$$

if and only if there exists matrices $S_i > 0$ for all $i \in \mathbb{K}$ satisfying the TPBVP of Theorem IV.1.

Proof: Considering that the upper bound of the Theorem IV.1 can be determined by a proper choice of the matrices that define the initial and final boundary conditions and setting $\xi_0 = 0$, then $\mathcal{J}_\infty < 0$. The claim is proved. ■

Remark IV.2 *Notice that the exact value of the $\mathcal{J}_\infty$ performance index can be determined from the choices described in Remark IV.1. However, the inequalities in Theorem IV.1 and Corollary IV.1 are fundamental to solve the control design problem in the next chapters. Furthermore, the necessity of Corollary IV.1 becomes evident when the equality holds.* □

Remark IV.3 *By providing different values to the parameter $\gamma > 0$ in the TPBVP, the exact value of the $\mathcal{H}_\infty$ norm can be easily obtained since, from Corollary IV.1, the optimal value for the parameter γ is determined by the feasibility of* (IV.5). □

IV.2 Reduction to the Pure MJLS Case

In order to validate the theoretical results from Section IV.1, consider the case $T \to 0$, that is, when the HMJLS (IV.1) collapses to a mean square stable MJLS. As before, notice that a pure MJLS system can be defined by the first and second equations of (IV.1). As a consequence, considering that the discontinuity in the HMJLS is imposed by matrices H_i, $i \in \mathbb{K}$, the pure MJLS can be conveniently characterized by $H_i = I$ for each $i \in \mathbb{K}$. This means that the initial and final boundary conditions become $P_i(0) = P_i(T) = S_i$ for all $i \in \mathbb{K}$ in the TPBVP of Theorem IV.1. Thus, assume that $P_i(t) \equiv S_i$ for all $i \in \mathbb{K}$, which satisfies the TPBVP and produces the coupled CARE

$$F_i' S_i + S_i F_i + \gamma^{-2} S_i J_i J_i' S_i + \sum_{j \in \mathbb{K}} \lambda_{ij} S_j + G_i' G_i = 0 \tag{IV.19}$$

for all $i \in \mathbb{K}$ instead of the coupled DRE (IV.5). From Theorem II.4, the existence and uniqueness of a set of solutions $S_i \geq 0$ for $0 \leq t \leq t_f$ and all $i \in \mathbb{K}$ satisfying (IV.19) are assured. Then, for initial conditions $\xi_0 = 0$ and θ_0, due to Theorem IV.1, $\mathcal{J}_\infty = 0$. This is the same result obtained in (II.48).

On the other hand, this choice of $P_i(t) \equiv S_i$ for all $i \in \mathbb{K}$ implies that the solution of (IV.19) is such that

$$\begin{aligned} S_i &= \int_0^\infty e^{M_i'\tau} \Big(G_i' G_i - \gamma^{-2} S_i J_i J_i' S_i + \sum_{j \neq i \in \mathbb{K}} \lambda_{ij} S_j \Big) e^{M_i \tau} d\tau \\ &= \sum_{k \in \mathbb{N}} (e^{M_i' T})^k R_i(T) (e^{M_i T})^k. \end{aligned} \tag{IV.20}$$

where $M_i = F_i + (\lambda_{ii}/2) + \gamma^{-2} J_i J_i S_i$ for all $i \in \mathbb{K}$. This means that $S_i > 0$ solves the Lyapunov equation

$$e^{M_i' T} S_i e^{M_i T} = S_i - R_i(T), \tag{IV.21}$$

whose solution lies on the closure of the feasible set of inequalities (IV.8) since M_i becomes constant with respect to time, which implies that $\Phi_i(t) = e^{M_i t}$ for $i \in \mathbb{K}$ and all $t \in [0, T)$. These algebraic manipulations put in evidence that, for a pure MJLS, the conditions provided by Theorem IV.1 do not depend on the sampling period $T > 0$ and are solvable whenever the coupled CARE (IV.19) admits a stabilizing positive definite solution. For this reason, it is clear that no conservatism of any kind has been included in the calculations done so far.

IV.3 Iterative Procedure to Solve the TPBVP

As mentioned before, an iterative procedure is necessary to solve the coupled DRE (IV.5) and, therefore, to solve the TPBVP of Theorem IV.1. According to Appendix A, the minimal feasible solution of the coupled algebraic Lyapunov inequality (IV.8) is arbitrarily close to the solution of the uncoupled DALE

$$\Phi_{i\ell}(T)' H_i' S_i H_i \Phi_{i\ell}(T) - S_i + R_{i\ell}(T) = 0 \tag{IV.22}$$

for each $i \in \mathbb{K}$ and some $\ell \in \mathbb{N}$. The $\mathcal{H}_\infty$ norm is obtained by a linear search in $\gamma > 0$. In fact, the $\mathcal{H}_\infty$ norm is the lowest value of $\gamma > 0$ such that equations (IV.22) admit positive definite solutions.

Then, considering the developments in Appendix A, which induce a way to iteratively obtain a set of solutions to the TPBVP of Theorem IV.1, the following algorithm determines it (if one exists) by solving the uncoupled DALE (IV.22). It is important to stress that these uncoupled DALE take into account the initial and final boundary conditions of the TPBVP.

Algorithm IV.1

1. *Define the constant sampling period* $T > 0$ *and the* $\mathcal{H}_\infty$ *level* $\gamma > 0$*, large enough, such that* $\mathcal{J}_\infty < 0$*. Consider the time interval* $t \in [0, T)$*, initialize* $\ell = 0$*, and set* $S_\ell = 0$*.*

2. *Determine the value of* $P_\ell(0)$ *by solving the coupled DRE*

$$\dot{P}_{i\ell}(t) + F_i' P_{i\ell} + P_{i\ell} F_i + \gamma^{-2} P_{i\ell} J_i J_i' P_{i\ell} + \sum_{j \in \mathbb{K}} \lambda_{ij} P_{j\ell} + G_i' G_i = 0 \qquad \text{(IV.23)}$$

subject to the final boundary condition $P_{i\ell}(T) = H_i' S_{i\ell} H_i \geq 0$ *for all* $i \in \mathbb{K}$ *through a backward integration. Using a forward integration, determine the value of* $\Phi_{i\ell}(T)$ *by solving the Ordinary Differential Equations (ODEs)*

$$\begin{cases} \dot{\Phi}_{i\ell}(t) & = & M_{i\ell}(t)\Phi_{i\ell}(t) \\ \Phi_{i\ell}(0) & = & I \end{cases} \qquad \text{(IV.24)}$$

for each $i \in \mathbb{K}$*, where* $M_{i\ell}(t) = F_i + (\lambda_{ii}/2)I + \gamma^{-2} J_i J_i' P_{i\ell}(t)$*.*

3. *Determine* $S_{i(\ell+1)} > 0$ *for each* $i \in \mathbb{K}$*, a feasible set of solutions to the* N *uncoupled DALE*

$$\Phi_{i\ell}(T)' H_i' S_{i(\ell+1)} H_i \Phi_{i\ell}(T) - S_{i(\ell+1)} + R_{i\ell}(T) = 0, \qquad \text{(IV.25)}$$

$i \in \mathbb{K}$*, where* $R_{i\ell}(T) = P_{i\ell}(0) - \Phi_{i\ell}(T)' H_i' S_{i\ell} H_i \Phi_{i\ell}(T)$*.*

4. *Set* $(\ell + 1) \to \ell$ *and iterate until* $\|S_{(\ell+1)} - S_\ell\|_2$ *becomes sufficiently small.*

Remark IV.4 *In view of the previous results, if* $\gamma > 0$ *is chosen small enough such that* $\mathcal{J}_\infty \geq 0$*, then the DRE may not admit a solution in the entire time interval* $[0, T)$ *in Step* 2*, or the uncoupled Lyapunov inequality may not admit a positive definite solution* $S_i > 0$ *in Step* 3 *due to the fact that the matrix* $H_i \Phi_{i\ell}(T)$ *is not Schur stable for some* $i \in \mathbb{K}$*.* □

As mentioned before, the main idea of Algorithm IV.1 is to use the value of the previously determined matrices $S_\ell > 0$ to evaluate matrices $P_\ell(0)$, $\Phi_\ell(T)$, and $R_{i\ell}(T)$ for each $i \in \mathbb{K}$. Then, matrices $S_{(\ell+1)}$ are obtained by solving the equation (IV.25). Again, the difficulty to assure the convergence of the proposed method stems from the fact that both boundary conditions vary in each iteration, namely $P(0)$ and $P(T)$. Despite this challenge and based on the developments in Appendix A, the global convergence of Algorithm IV.1 is established in the next theorem.

Theorem IV.2 *Assume that the TPBVP defined in Theorem IV.1 has a bounded solution* $(S_i^*, P_i^*(t))$ *for all* $i \in \mathbb{K}$ *and all* $t \in [0, T)$. *Algorithm IV.1 is uniformly convergent and any two subsequent iterations are such that* $S^* \geq S_{(\ell+1)} \geq S_\ell \geq 0$.

Proof: In order to prove the global convergence, first, the sequence of $\{S\}_{\ell=0}^\infty$ is proved to be monotonically non-decreasing. Then, due to the boundedness assumption, the convergence is shown. To accomplish this purpose, initially, consider the uncoupled Lyapunov equation (IV.22) rewritten as

$$\begin{aligned}\Phi_{i\ell}(T)'H_i'(S_{i(\ell+1)} - S_{i\ell})H_i\Phi_{i\ell}(T) =& S_{i(\ell+1)} - P_{i\ell}(0) \\ =& (S_{i(\ell+1)} - S_{i\ell}) - (P_{i\ell}(0) - S_{i\ell})\end{aligned} \tag{IV.26}$$

for all $i \in \mathbb{K}$. Consider that matrices $H_i\Phi_{i\ell}(T)$ are Schur stable for each $\ell \in \mathbb{N}$ and all $i \in \mathbb{K}$. Thus, there exists matrices $S_i^* > 0$ for all $i \in \mathbb{K}$ such that the TPBVP from Theorem IV.1 holds. Assume also that $P_{i\ell}(0) \geq S_{i\ell} \geq 0$ for all $i \in \mathbb{K}$ and some $\ell \in \mathbb{N}$, a hypothesis that will be discussed later. As a consequence, the second equality in (IV.26) yields $S_{i(\ell+1)} \geq S_{i\ell}$ for all $i \in \mathbb{K}$, while the first equality implies that $S_{i(\ell+1)} \geq P_{i\ell}(0)$ for all $i \in \mathbb{K}$. Together they lead to $S_{i(\ell+1)} \geq P_{i\ell}(0) \geq S_{i\ell} \geq 0$ for each $i \in \mathbb{K}$ and some $\ell \in \mathbb{N}$.

Furthermore, equations (IV.26) together with inequalities (A.9) gives

$$\begin{aligned}P_{i(\ell+1)}(0) - P_{i\ell}(0) &\geq \Phi_{i\ell}(T)'H_i'(S_{i(\ell+1)} - S_{i\ell})H_i\Phi_{i\ell}(T) \\ &\geq S_{i(\ell+1)} - P_{i\ell}(0)\end{aligned} \tag{IV.27}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$, which holds whenever the condition $P_{i\ell}(0) \geq S_{i\ell} \geq 0$ is verified for all $i \in \mathbb{K}$ and some $\ell \in \mathbb{N}$. Thus, the previous relations imply that the algorithm exhibits the interlacing property $P_{i(\ell+1)}(0) \geq S_{i(\ell+1)} \geq P_{i\ell}(0) \geq S_{i\ell} \geq 0$. Then, consider $\ell = 0$ and set $S_{i0} = 0$ for all $i \in \mathbb{K}$ as the first step of the algorithm indicates. From Step 2, due to Theorem II.4, $P_{i0}(t) \geq 0$. Consequently, $P_{i0}(0) \geq S_{i0} = 0$ for all $i \in \mathbb{K}$. Applying the interlacing property successively, it follows that $S_{i(\ell+1)} - S_{i\ell} \geq 0$ for all $\ell \geq 0$ and all $i \in \mathbb{K}$, that is, the sequence $\{S_{i\ell}\}_{\ell=0}^\infty$ is such that

$$0 = S_{i0} \leq S_{i1} \leq S_{i2} \leq \cdots \tag{IV.28}$$

for each $i \in \mathbb{K}$.

Since the TPBVP is solved, the stationary solution $(S_i^*, P_i^*(t))$ satisfies $S_i^* = P_i^*(0)$ for all $i \in \mathbb{K}$ whenever $P_i^*(T) = H_i'S_i^*H_i$. Assume that $S_i^* \geq S_{i\ell} \geq 0$ for some $\ell \in \mathbb{N}$. Inequality (A.9) applied to the stationary solution yields

$$\begin{aligned}P_i^*(0) - P_{i\ell}(0) &\geq \Phi_{i\ell}(T)'(P_i^*(T) - P_{i\ell}(T))\Phi_{i\ell}(T) \\ &\geq \Phi_{i\ell}(T)'H_i'(S_i^* - S_{i\ell})H_i\Phi_{i\ell}(T)\end{aligned} \tag{IV.29}$$

for all $i \in \mathbb{K}$, where the final boundary values are $P_i^*(T) = H_i' S_i^* H_i$ and $P_{i\ell}(T) = H_i' S_{i\ell} H_i$, $\ell \in \mathbb{N}$, for each $i \in \mathbb{K}$. On the other hand, subtracting the first equality of (IV.26) from (IV.29), then

$$\begin{aligned}\Phi_{i\ell}(T)' H_i'(S_i^* - S_{i(\ell+1)}) H_i \Phi_{i\ell}(T) &\leq P_i^*(0) - S_{i(\ell+1)} \\ &\leq S_i^* - S_{i(\ell+1)},\end{aligned} \tag{IV.30}$$

which implies that $S_i^* \geq S_{i(\ell+1)} \geq 0$ for all $i \in \mathbb{K}$ and $\ell \in \mathbb{N}$. Due to the fact that $S_{i0} = 0$ for all $i \in \mathbb{K}$, the algorithm generates a sequence of matrices $\{S_{i\ell}\}_{\ell=0}^{\infty}$ bounded by S_i^* for each $i \in \mathbb{K}$. The hypothesis on the existence of a set of bounded solutions S_i^* for all $i \in \mathbb{K}$ is sufficient to assure that

$$\lim_{\ell \to \infty} S_{i\ell} \to S_i^* \tag{IV.31}$$

for all $i \in \mathbb{K}$, which indicates that the algorithm monotonically converges to the solution of S_i^* for all $i \in \mathbb{K}$, concluding thus the proof. ■

Remark IV.5 *Even though the condition $P_{i\ell}(0) \geq S_{i\ell}$ for all $i \in \mathbb{K}$ is a result of the backward integration in the first step of Algorithm IV.1, a feasible solution on the border of the initial boundary condition of the TPBVP, $P_i(0) < S_i$ for all $i \in \mathbb{K}$, is enforced in the third step of the proposed algorithm, where a new $S_{i(\ell+1)} \geq P_{i\ell}(0)$ for each $i \in \mathbb{K}$ is obtained.* □

As remarked before, the parameters (γ, T) must be chosen such that $\mathcal{J}_\infty < 0$. Otherwise, the algorithm can not achieve any result because the problem to be solved does not admit a solution. Fortunately, due to continuity, the existence of a solution $P_\ell(t)$ to the coupled DRE (IV.5) in the time interval $t \in [0, T)$ and a solution $S_{(\ell+1)} > 0$ lying on the closure of the uncoupled Lyapunov inequalities (IV.8) are assured for some $\gamma > 0$ whenever the matrices $H_i \Phi_i(T) = H_i e^{(F_i + (\lambda_{ii}/2)I)T}$ are Schur stable for all $i \in \mathbb{K}$. In other words, under this condition, there always exists a large enough $\gamma > 0$ assuring the existence of a solution to the TPBVP defined in Theorem IV.1 for the given sampling period $T > 0$. In the particular case corresponding to $\gamma \to +\infty$, the coupled DRE (IV.5) collapses to a coupled DLE, which obviously admits an unique and bounded solution. As a consequence, Algorithm IV.1 can be used to obtain the $\mathcal{H}_2$ norm by imposing $\gamma \to +\infty$.

IV.4 Illustrative Numerical Example

Using the same example as in Chapter III and in Gabriel, Souza & Geromel (2014), the theoretical results of Theorem IV.1 implemented by the iterative procedure described in Algorithm IV.1 is now illustrated.

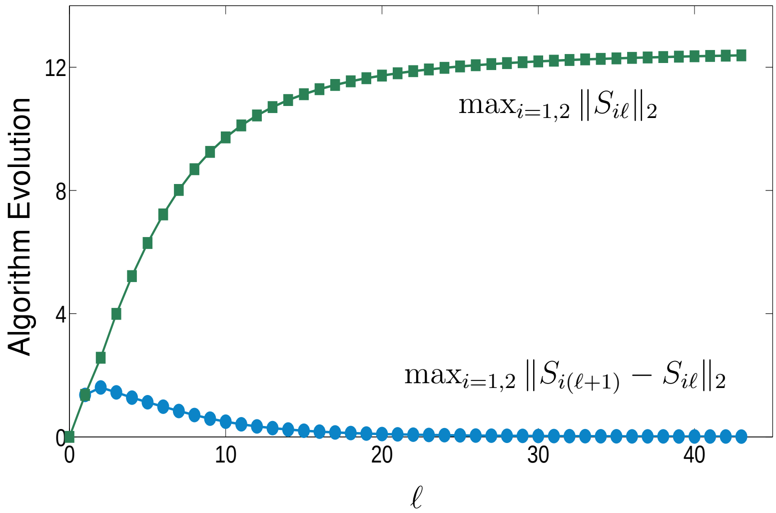

Figure IV.1 – Evolution of the Algorithm IV.1.

Example IV.1 *Consider an MJLS with $N = 2$ modes with the system matrices given by*

$$F_1 = \begin{bmatrix} 0 & 1 & 0 \\ -4 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; F_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$J_1 = J_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; G_1' = G_2' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.1791 & -0.5561 & 0 \end{bmatrix}, \; H_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.2972 & -0.8691 & 0 \end{bmatrix}.$$

Consider also that the sampling period is $T = 250\ [ms]$ and the transition rate matrix $\Lambda \in \mathbb{R}^{2\times 2}$ is given by

$$\Lambda = \begin{bmatrix} -0.5 & 0.5 \\ 0.2 & -0.2 \end{bmatrix}$$

with initial probability $\pi_0 = [1\ 0]'$. Then, running Algorithm IV.1 for decreasing values of $\gamma > 0$ until the TPBVP becomes unfeasible, the computed $\mathcal{H}_\infty$ cost is $\gamma = 2.12$. Figure IV.1 shows the evolution of one run of the iterative procedure from Algorithm IV.1 for $\gamma = \gamma_{opt} = 2.12$. The green curve (with squared markers) shows the convergence of the algorithm through the measure of the maximum value of $\|S_{i\ell}\|_2$ for $i = \{1, 2\}$, while the blue curve (with circular markers) shows the evolution of the stopping criterion, that is, the maximum value of $\|S_{i(\ell+1)} - S_{i\ell}\|_2$ for $i = \{1, 2\}$.

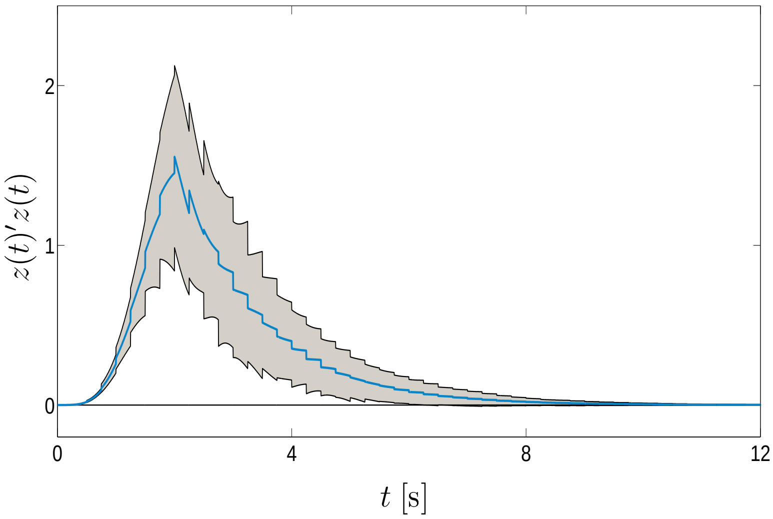

Figure IV.2 – Evolution of the Monte Carlo simulation.

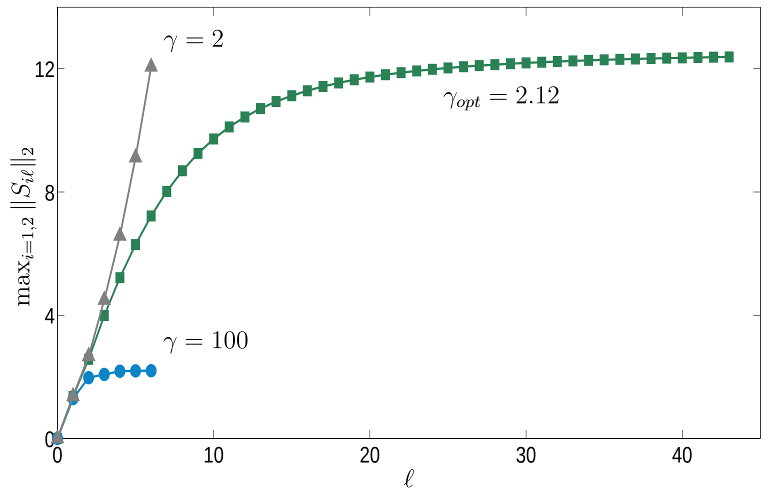

Figure IV.3 – Comparison among algorithm evolutions with different γ values.

On the other hand, using the procedure proposed by Leon-Garcia (2007) and performing a Monte Carlo simulation of 2,000 samples with a time interval of $[0, 12]$ $[s]$, the value of the computed $\mathcal{H}_\infty$ norm is 1.61. For this simulation, even though the worst perturbation is not deterministic, the exogenous input of $w(t) = sin(\pi t/3)$ for $t \in [0, 2]$ $[s]$ and $w(t) \equiv 0$ elsewhere is considered. The frequency for the sinusoidal signal is defined in a previous search such that the worst gain is obtained. The significant difference between the calculated and the simulated $\mathcal{H}_\infty$ norm allows the conclusion that the considered exogenous input is not close to the worst perturbation that, in this particular example, is not known. Figure IV.2 shows the Monte Carlo simulation, where the solid curve in the middle is the mean value of the index $z(t)'z(t)$ and the shaded area corresponds to one standard deviation from the mean value.

Additionally, Figure IV.3 shows the convergence of the algorithm for three dif-

ferent values of γ. The convergence occurs within 6 iterations for $\gamma = 100 > \gamma_{opt}$, while the optimal value is reached after 43 iterations for $\gamma = \gamma_{opt} = 2.12$. For $\gamma = 2 < \gamma_{opt}$ the algorithm diverges as expected because a bounded solution does not exist. Furthermore, assuming $\gamma = 1,000$, the $\mathcal{J}_2$ performance index evaluated through the iterative procedure from Algorithm IV.1 is 2.95, close enough to the result obtained in Chapter III evaluated using the Algorithm III.1. This shows that Algorithm IV.1 is also suitable for the $\mathcal{J}_2$ performance index evaluation by adopting a proper choice (large enough) of the parameter γ. Notice that Algorithms III.1 and IV.1 uses different stopping criteria.

CHAPTER V

$\mathcal{H}_2$ Optimal Sampled-Data Control

Since the indexes $\mathcal{J}_2$ and $\mathcal{J}_\infty$ were determined in Chapters III and IV, the next step is to obtain the optimal control law that minimizes each of these performance indexes. In this chapter, it is considered the $\mathcal{J}_2$ index defined by (III.3), which is based on $\mathcal{H}_2$ norm. In other words, the purpose is to solve the $\mathcal{H}_2$ state feedback sampled-data optimal control applied to an MJLS with constant sampled data intervals, that is, $t_{k+1} - t_k = T > 0$ for all $k \in \mathbb{N}$. In order to accomplish this, first, the result from Theorem III.1 is rewritten to obtain a convex formulation for the optimal control problem based on LMIs. Then, some particular cases are analysed and validate the results obtained so far. An algorithm similar to Algorithm III.1 is proposed and proved to be convergent. The theoretical results are illustrated by means of a numerical example.

Initially, recall that the continuous-time MJLS is defined by

$$\dot{x}(t) = A_{\theta(t)}x(t) + B_{\theta(t)}u(t) + E_{\theta(t)}w(t) \tag{V.1}$$

$$z(t) = C_{\theta(t)}x(t) + D_{\theta(t)}u(t) \tag{V.2}$$

for all $t \in \mathbb{R}^+$ and each $\theta(t) \in \mathbb{K}$. As used before, this system evolves from $x(0) = 0$ and $\theta(0) = \theta_0$, where $\theta(t)$ is an homogeneous continuous-time Markov chain with initial distribution π_{i0} and transition rate matrix Λ satisfying the definitions from Chapter II. Moreover, system (V.1)–(V.2) is controlled by the state feedback sampled-data control law in the form of

$$u(t) = L_{\theta(t_k)}x(t_k) \tag{V.3}$$

for all $t \in [t_k, t_{k+1})$, each $k \in \mathbb{N}$, and all $\theta(t) \in \mathbb{K}$. The set of events $\{t_k\}_{k=0}^{\infty}$ describes the sequence of sampling instants. Due to the sampling process, the original MJLS system (V.1)–(V.2) controlled by (V.3) can be rewritten as an HMJLS expressed by

$$\dot{\xi}(t) = F_{\theta(t)}\xi(t) + J_{\theta(t)}w(t) \tag{V.4}$$

$$z(t) = G_{\theta(t)}\xi(t) \tag{V.5}$$

$$\xi(t_k) = H_{\theta(t_k)}\xi(t_k^-) \tag{V.6}$$

for all $\theta(t) \in \mathbb{K}$ and all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, evolving from the initial conditions $\xi(0^-) = \xi_0 = 0$ and $\theta(0^-) = \theta(0) = \theta_0$.

For the purposes of this chapter, the special structures of the system matrices

$$F_{\theta(t)} = \begin{bmatrix} A_{\theta(t)} & B_{\theta(t)} \\ 0 & 0 \end{bmatrix}, \tag{V.7}$$

$$J_{\theta(t)} = \begin{bmatrix} E_{\theta(t)} \\ 0 \end{bmatrix}, \tag{V.8}$$

$$G_{\theta(t)} = \begin{bmatrix} C_{\theta(t)} & D_{\theta(t)} \end{bmatrix}, \tag{V.9}$$

and

$$H_{\theta(t_k)} = \begin{bmatrix} I & 0 \\ L_{\theta(t_k)} & 0 \end{bmatrix} \tag{V.10}$$

for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, and $\theta(t) \in \mathbb{K}$ are fundamental. Furthermore, the $\mathcal{H}_2$ optimization problem is formulated as

$$\inf_{L_1,\cdots,L_N} \sum_{l=1}^{r} \int_0^\infty \mathcal{E}\left[z_l(t)' z_l(t)\right] dt. \tag{V.11}$$

The signal $z_l(t)$ is the controlled output corresponding to each component of the exogenous input $w(t) = e_l \delta(t^-)$, where $e_l \in \mathbb{R}^r$ is the l-th column of the identity matrix with compatible dimensions (see Chapter II). Clearly, this is a $\mathcal{H}_2$ state feedback sampled-data control problem for the continuous-time MJLS (see Section II.4) that should be optimized by taking into account the sampled-data control constraint (V.3). (See Levis, Schluete & Athans (1971).)

V.1 Theoretical Results

Due to the developments and assumptions adopted in Chapter III, according to (III.23), the infimum problem defined by (V.11) can be rewritten by

$$\inf_{L_1,\cdots,L_N,S_i>0} \left\{ \sum_{i\in\mathbb{K}} \pi_{i0} \mathbf{Tr}(J_i' H_i' S_i H_i J_i) \ : \ e^{\bar{F}_i' T} H_i' S_i H_i e^{\bar{F}_i T} < S_i - R_i(P,T) \right\}, \tag{V.12}$$

where $\bar{F}_i = F_i + (\lambda_{ii}/2)I$ for each $i \in \mathbb{K}$. As before, matrices $R_i(P,T) \geq 0$, $i \in \mathbb{K}$, are fixed by construction of the iterative procedure defined in Algorithm III.1. As a consequence, this problem is composed by N uncoupled subproblems. To solve each of them in terms of LMIs, verify that matrices H_i, $i \in \mathbb{K}$, change at each iteration and notice that a discrete-time equivalent system can be obtained and used to formulate a feasibility problem since (V.12) is a discrete-time algebraic Lyapunov inequality. Indeed, consider the realization

$$\left\{ \begin{array}{rcl} x[k+1] & = & A_{di} x[k] + B_{di} u[k] \\ z[k] & = & C_{di} x[k] + D_{di} u[k] \end{array} \right. \tag{V.13}$$

for each $i \in \mathbb{K}$ and $k \in \mathbb{N}$, which evolves from initial conditions $x[0] \in \mathbb{R}^n$. Matrices C_{di} and D_{di} are obtained by factorizing $R_i(P,T)$, $i \in \mathbb{K}$, in the form of

$$R_i(P,T) = \begin{bmatrix} C_{di}' \\ D_{di}' \end{bmatrix} \begin{bmatrix} C_{di} & D_{di} \end{bmatrix} \geq 0. \tag{V.14}$$

On the other hand, matrices A_{di} and B_{di} are determined from

$$H_i e^{\bar{F}_i T} = \begin{bmatrix} I & 0 \\ L_i & 0 \end{bmatrix} \begin{bmatrix} e^{A_i T} & \int_0^T e^{A_i \tau} d\tau B_i \\ 0 & I \end{bmatrix} e^{(\lambda_{ii}/2)T} = \begin{bmatrix} I \\ L_i \end{bmatrix} \begin{bmatrix} A_{di} & B_{di} \end{bmatrix} \tag{V.15}$$

for each $i \in \mathbb{K}$. Then, it is immediate that $A_{di} = e^{(\lambda_{ii}/2)T} e^{A_i T}$ and $B_{di} = e^{(\lambda_{ii}/2)T} \int_0^T e^{A_i \tau} d\tau B_i$ for all $i \in \mathbb{K}$. Finally, by partitioning matrices $S_i > 0$, $i \in \mathbb{K}$, such as

$$S_i^{-1} = \begin{bmatrix} X_i & Y_i \\ \bullet & Z_i \end{bmatrix} \tag{V.16}$$

where $X_i \in \mathbb{R}^{n \times n}$, $Y_i \in \mathbb{R}^{n \times m}$, and $Z_i \in \mathbb{R}^{m \times m}$, necessary and sufficient conditions can be stated in order to assure mean square stability and to determine the optimal control law that minimizes the performance index $\mathcal{J}_2$. The next theorem formalizes these results.

Theorem V.1 *Problem* (V.12) *is feasible if and only if there exist positive definite matrices X_i, Z_i and matrices Y_i, $i \in \mathbb{K}$, with compatible dimensions such that*

$$\begin{bmatrix} X_i & Y_i \\ \bullet & Z_i \end{bmatrix} > 0 \tag{V.17}$$

and

$$\begin{bmatrix} X_i & 0 \\ 0 & I \end{bmatrix} > \begin{bmatrix} A_{di} & B_{di} \\ C_{di} & D_{di} \end{bmatrix} \begin{bmatrix} X_i & Y_i \\ \bullet & Z_i \end{bmatrix} \begin{bmatrix} A_{di} & B_{di} \\ C_{di} & D_{di} \end{bmatrix}' \tag{V.18}$$

hold for each $i \in \mathbb{K}$. In the affirmative case, a feasible solution is given by $L_i = Y_i' X_i^{-1}$ and S_i^{-1} of the form (V.16) *for each $i \in \mathbb{K}$.*

Proof: For the sufficiency, assume that inequalities (V.17) and (V.18) hold. Calculating the inverse of (V.16) and adopting $L_i = Y_i' X_i^{-1}$ for all $i \in \mathbb{K}$ yields

$$S_i = \begin{bmatrix} I \\ 0 \end{bmatrix} X_i^{-1} \begin{bmatrix} I & 0 \end{bmatrix} + \begin{bmatrix} -L_i' \\ I \end{bmatrix} (Z_i - L_i X_i L_i')^{-1} \begin{bmatrix} -L_i & I \end{bmatrix} \tag{V.19}$$

for all $i \in \mathbb{K}$, which implies that

$$\begin{bmatrix} I & L_i' \end{bmatrix} S_i \begin{bmatrix} I \\ L_i \end{bmatrix} = X_i^{-1} \tag{V.20}$$

for all $i \in \mathbb{K}$. On the other hand, the Schur complement applied to (V.18) leads to

$$\begin{bmatrix} X_i & Y_i \\ \bullet & Z_i \end{bmatrix}^{-1} > \begin{bmatrix} A_{di} & B_{di} \\ C_{di} & D_{di} \end{bmatrix}' \begin{bmatrix} X_i^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{di} & B_{di} \\ C_{di} & D_{di} \end{bmatrix} \tag{V.21}$$

for all $i \in \mathbb{K}$. By applying some algebraic manipulations, (V.21) produces

$$\begin{bmatrix} A'_{di} \\ B'_{di} \end{bmatrix} X_i^{-1} \begin{bmatrix} A_{di} & B_{di} \end{bmatrix} + \begin{bmatrix} C'_{di} \\ D'_{di} \end{bmatrix} \begin{bmatrix} C_{di} & D_{di} \end{bmatrix} < S_i \tag{V.22}$$

for all $i \in \mathbb{K}$. Using (V.20), thus

$$\begin{bmatrix} A'_{di} \\ B'_{di} \end{bmatrix} \begin{bmatrix} I & L'_i \end{bmatrix} S_i \begin{bmatrix} I \\ L_i \end{bmatrix} \begin{bmatrix} A_{di} & B_{di} \end{bmatrix} < S_i - R_i(P,T) \tag{V.23}$$

for all $i \in \mathbb{K}$, which combined with (V.14) and (V.15) yields

$$e^{\bar{F}'_i T} H'_i S_i H_i e^{\bar{F}_i T} < S_i - R_i(P,T) \tag{V.24}$$

for all $i \in \mathbb{K}$ and is exactly the constraint of problem (V.12), providing thus the sufficiency.

Conversely, assume that problem (V.12) is feasible for some pairs of matrices $(L_i, S_i > 0)$ for all $i \in \mathbb{K}$. Analogously, partitioning matrices S_i such as in (V.16), it is immediate from the existence of a solution of problem (V.12) that

$$\begin{bmatrix} X_i & Y_i \\ \bullet & Z_i \end{bmatrix} > 0 \tag{V.25}$$

for each $i \in \mathbb{K}$. On the other hand, from (V.12), (V.14), and (V.15), it can be verified that

$$\begin{bmatrix} A_{di} & B_{di} \end{bmatrix}' \begin{bmatrix} I \\ L_i \end{bmatrix}' S_i \begin{bmatrix} I \\ L_i \end{bmatrix} \begin{bmatrix} A_{di} & B_{di} \end{bmatrix} + \begin{bmatrix} C'_{di} \\ D'_{di} \end{bmatrix} \begin{bmatrix} C_{di} & D_{di} \end{bmatrix} - S_i < 0, \tag{V.26}$$

which is valid for all $i \in \mathbb{K}$. Calculating again the inverse of (V.16), this time, without any assumption about matrices L_i, $i \in \mathbb{K}$, yields

$$\begin{aligned} 0 > & \begin{bmatrix} A_{di} & B_{di} \end{bmatrix}' \begin{bmatrix} I \\ L_i \end{bmatrix}' \begin{bmatrix} I \\ 0 \end{bmatrix} X_i^{-1} \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} I \\ L_i \end{bmatrix} \begin{bmatrix} A_{di} & B_{di} \end{bmatrix} \\ & + \begin{bmatrix} C'_{di} \\ D'_{di} \end{bmatrix} \begin{bmatrix} C_{di} & D_{di} \end{bmatrix} - S_i \\ & + \begin{bmatrix} A_{di} & B_{di} \end{bmatrix}' \left(L_i - Y'_i X_i^{-1}\right)' \left(Z_i - Y'_i X_i^{-1} Y_i\right)^{-1} \left(L_i - Y'_i X_i^{-1}\right) \begin{bmatrix} A_{di} & B_{di} \end{bmatrix} \end{aligned} \tag{V.27}$$

for all $i \in \mathbb{K}$. Since the last term of the right hand side of (V.27) is positive definite due to $S_i > 0$, then

$$\begin{bmatrix} A_{di} & B_{di} \\ C_{di} & D_{di} \end{bmatrix}' \begin{bmatrix} X_i^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{di} & B_{di} \\ C_{di} & D_{di} \end{bmatrix} - S_i < 0 \tag{V.28}$$

for all $i \in \mathbb{K}$. Hence, using the Schur complement, equation (V.18) is recovered for any matrices L_i , $i \in \mathbb{K}$, concluding the proof. ■

Remark V.1 *From the proof of Theorem V.1 (specially inequalities* (V.27)*) the feasible gains are such that* $L_i = Y_i' X_i^{-1}$ *for each* $i \in \mathbb{K}$*. In this case, the inequalities in problem* (V.12) *and* (V.28) *become exactly the same. This is a guarantee that the optimal solution is covered by Theorem V.1.* □

Theorem V.1 puts in evidence that problem (V.12) is jointly convex in the decision matrix variables $S_i > 0$ and L_i for all $i \in \mathbb{K}$ inside the feasible set for which (V.12) is defined. Moreover, as already discussed, it determines the optimal sampled-data control law applied to an HMJLS. In this way, problem (V.12) can be redefined in terms of the new set of variables introduced in Theorem V.1. So, the upper bound of the performance index $\mathcal{J}_2$, (III.9), can be expressed by

$$\begin{aligned} \mathcal{J}_2 &< \sum_{i\in\mathbb{K}} \pi_{i0} \mathbf{Tr}(J_i' H_i' S_i H_i J_i) \\ &= \sum_{i\in\mathbb{K}} \pi_{i0} \mathbf{Tr} \left(E_i' \begin{bmatrix} I & L_i' \end{bmatrix} S_i \begin{bmatrix} I \\ L_i \end{bmatrix} E_i \right) \\ &= \sum_{i\in\mathbb{K}} \pi_{i0} \mathbf{Tr} \left(E_i' X_i^{-1} E_i \right) \end{aligned} \tag{V.29}$$

since (V.20) and

$$H_i J_i = \begin{bmatrix} I & 0 \\ L_i & 0 \end{bmatrix} \begin{bmatrix} E_i \\ 0 \end{bmatrix} = \begin{bmatrix} E_i \\ L_i E_i \end{bmatrix} = \begin{bmatrix} I \\ L_i \end{bmatrix} E_i \tag{V.30}$$

are valid for all $i \in \mathbb{K}$. Consequently, problem (V.12) can be rewritten as N uncoupled subproblems defined by

$$\inf_{X_i, Y_i, Z_i} \left\{ \sum_{i\in\mathbb{K}} \pi_{i0} \mathbf{Tr} \left(E_i' X_i^{-1} E_i \right) \ : \ \text{(V.17)} - \text{(V.18)} \right\}, \tag{V.31}$$

each of them associated to a specific mode of the Markov chain. In this case, the state feedback sampled-data control law in the form of (V.3) is given by

$$L_i = Y_i' X_i^{-1} \tag{V.32}$$

for each $i \in \mathbb{K}$. An important remark can be done about the formulation (V.31).

Remark V.2 *Problem* (V.31) *is the convex formulation of* (V.12) *and it can be solved by the computational tools available to date. Additionally, the iterative procedure presented in Chapter III can be used to solve it. In order to accomplish this task, simple modifications are necessary, which are addressed afterwards.* □

Finally, notice that problem (V.31) can be simplified. Indeed, suppose that (V.31) (as a consequence, (V.12)) has a definite positive solution S_i for all $i \in \mathbb{K}$. Thus, from definition (V.16), it can be verified that $X_i > 0$ and $Z_i > Y_i' X_i^{-1} Y_i$, for all $i \in \mathbb{K}$. These together with inequalities (V.17)–(V.18) imply that

$$\begin{aligned} \begin{bmatrix} X_i & 0 \\ 0 & I \end{bmatrix} &> \begin{bmatrix} A_{di} & B_{di} \\ C_{di} & D_{di} \end{bmatrix} \begin{bmatrix} X_i & Y_i \\ \bullet & Y_i' X_i^{-1} Y_i \end{bmatrix} \begin{bmatrix} A_{di} & B_{di} \\ C_{di} & D_{di} \end{bmatrix}' \\ &= \begin{bmatrix} A_{di} & B_{di} \\ C_{di} & D_{di} \end{bmatrix} \begin{bmatrix} X_i & X_i L_i' \\ \bullet & L_i X_i L_i' \end{bmatrix} \begin{bmatrix} A_{di} & B_{di} \\ C_{di} & D_{di} \end{bmatrix}' \\ &= \begin{bmatrix} A_{di} + B_{di} L_i \\ C_{di} + D_{di} L_i \end{bmatrix} X_i \begin{bmatrix} A_{di} + B_{di} L_i \\ C_{di} + D_{di} L_i \end{bmatrix}' \end{aligned} \tag{V.33}$$

for all $i \in \mathbb{K}$, where the fact that $L_i = Y_i' X_i^{-1}$ for all $i \in \mathbb{K}$ has been used. Once again, the Schur complement yields

$$X_i^{-1} - \begin{bmatrix} A_{di} + B_{di} L_i \\ C_{di} + D_{di} L_i \end{bmatrix}' \begin{bmatrix} X_i^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{di} + B_{di} L_i \\ C_{di} + D_{di} L_i \end{bmatrix} > 0, \tag{V.34}$$

which is valid for all $i \in \mathbb{K}$. Defining a new set of matrices $V_i = X_i^{-1}$ for all $i \in \mathbb{K}$, then inequalities (V.34) become

$$(A_{di} + B_{di} L_i)' V_i (A_{di} + B_{di} L_i) - V_i < -(C_{di} + D_{di} L_i)'(C_{di} + D_{di} L_i) \tag{V.35}$$

for all $i \in \mathbb{K}$. Hence, the optimal solution is given by $\mathcal{J}_2 = \sum_{i \in \mathbb{K}} \pi_{i0} \mathbf{Tr}(E_i' V_i E_i)$, where $V_i > 0$ for each $i \in \mathbb{K}$ is arbitrarily close to the stabilizing solution of the Discrete-time Algebraic Riccati Equation (DARE)

$$A_{di}' V_i A_{di} - V_i + C_{di}' C_{di} - L_i'(B_{di}' V_i B_{di} + D_{di}' D_{di}) L_i = 0, \tag{V.36}$$

and $L_i = -(B_{di}' V_i B_{di} + D_{di}' D_{di})^{-1} (B_{di}' V_i A_{di} + D_{di}' C_{di})$ is the optimal gain. Thus, choosing $Z_i \to Y_i' X_i^{-1} Y_i$, then $S_i^{-1} > 0$ is such that $S_i^{-1} \to [I \ L_i']' V_i^{-1} [I \ L_i'] \geq 0$ for all $i \in \mathbb{K}$. The alternative problem is rewritten in the form of

$$\inf_{V_i > 0} \left\{ \sum_{i \in \mathbb{K}} \pi_{i0} \mathbf{Tr} \left(E_i' V_i E_i \right) \ : \ \text{(V.36)} \right\} \tag{V.37}$$

for each $i \in \mathbb{K}$. Once again, matrices V_i and L_i can be easily determined for all $i \in \mathbb{K}$ by the computational tools available to date. This is an alternative problem to the one shown in (V.31) since (V.35) is equivalently rewritten as (V.36).

Remark V.3 *The exact value of the $\mathcal{H}_2$ norm is obtained according to Chapter III when the inequality in* (V.12) *collapses to an equality. Here, the optimal solution is obtained by enforcing also that matrices $Z_i \to Y_i' X_i^{-1} Y_i$. This means that matrices S_i, $i \in \mathbb{K}$, become the lowest admissible values in the feasibility set. In other words, the optimal solution S_i^* is arbitrarily close to $S_i^* \to [I \ L_i']' V_i^{-1} [I \ L_i'] \geq 0$ for all $i \in \mathbb{K}$. This fact allows the use of equations* (V.36) *instead of inequalities* (V.35). □

V.2 Special Case Analysis

In this section, some particular cases derived from Theorem V.1 are discussed. First, the mode independent case is addressed. Second, by making $T \to 0^+$ it is shown that the result of a pure MJLS is recovered, as expected. Finally, the reduction to a deterministic linear hybrid system is analysed. All these analysis recover results available in the current literature, what validates the theoretical results for the $\mathcal{H}_2$ sampled-data optimal control problem provided in Theorem V.1.

The mode independent case

In practical systems, the Markov modes $\theta(t)$ are frequently not available and to implement the online measurement of this parameter is, in general, a difficult task. This implies that the possibility to have the control law independent of the Markov mode is more suitable for a large number of practical systems. In this work, the dependence of the Markov mode on the control law can be ruled out by enforcing matrices $L_i = L$ for all $i \in \mathbb{K}$, which can be done by setting matrices $Y_i = Y$ and $X_i = X$ for all $i \in \mathbb{K}$. It follows that the problem (V.31) becomes

$$\inf_{X,Y,Z_i} \left\{ \sum_{i\in\mathbb{K}} \pi_{i0} \mathbf{Tr}\left(E_i' X^{-1} E_i\right) \ : \ \begin{bmatrix} X & Y \\ \bullet & Z_i \end{bmatrix} > 0 \right.$$
$$\left. \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} > \begin{bmatrix} A_{di} & B_{di} \\ C_{di} & D_{di} \end{bmatrix} \begin{bmatrix} X & Y \\ \bullet & Z_i \end{bmatrix} \begin{bmatrix} A_{di} & B_{di} \\ C_{di} & D_{di} \end{bmatrix}' \right\} \tag{V.38}$$

for all $i \in \mathbb{K}$. This still is a convex formulation and, thus, can be solved using the same procedure adopted to solve the mode dependent problem. Notice that the LMI approach in the form presented by Theorem V.1 is fundamental to implement the mode independent case since neither Lyapunov nor Riccati equations can be derived for this case.

By doing this, only a guaranteed cost is obtained, which, in general, does not reach the minimum value. Moreover, although problem (V.38) is convex, it cannot be decomposed into N uncoupled subproblems anymore. Naturally, the computational effort involved is much higher than that spent to solve the N uncoupled subproblems from Theorem V.1. The difference becomes more evident by increasing the number of Markov modes belonging to the set $\mathbb{K}$.

The limit case $T \to 0^+$

Another important analysis derived from Theorem V.1 is to make the sampling interval arbitrarily small. This equals to analyse the limit of problem (V.31) as $T > 0$ goes to zero. In this case, two consequences can be drawn:

1) the limit solution of the TPBVP is such that $P_i(0) = \lim_{T\to 0^+} P_i(T)$ for all $i \in \mathbb{K}$. As a consequence,

$$S_i \to \begin{bmatrix} I \\ 0 \end{bmatrix} V_i \begin{bmatrix} I & 0 \end{bmatrix} \tag{V.39}$$

for each $i \in \mathbb{K}$, where the initial $P_i(0) \to S_i$ and final $P_i(T) \to H_i' S_i H_i$ boundary conditions from Theorem III.1 and the fact that $H_i' S_i H_i = [I\ 0]' V_i [I\ 0]$ for all $i \in \mathbb{K}$ have been used;

2) due to their definition, matrices A_{di}, B_{di}, C_{di}, and D_{di} can be approximated by their first order terms. Hence, by making $T > 0$ sufficiently small, (V.14) and (V.15) yield

$$A_{di} = e^{(\lambda_{ii}/2)T} e^{A_i T} \approx I + ((\lambda_{ii}/2)I + A_i)T, \tag{V.40}$$

$$B_{di} = e^{(\lambda_{ii}/2)T} \int_0^T e^{A_i \tau} d\tau B_i \approx B_i T, \tag{V.41}$$

and

$$\begin{aligned} \begin{bmatrix} C_{di}' \\ D_{di}' \end{bmatrix} \begin{bmatrix} C_{di} & D_{di} \end{bmatrix} &= \lim_{T\to 0^+} R_i(P,T) \approx \left(\sum_{j\neq i \in \mathbb{K}} \lambda_{ij} S_j + G_i' G_i \right) T \\ &\approx \begin{bmatrix} C_i' C_i + \sum_{j \neq i \in \mathbb{K}} \lambda_{ij} V_j & C_i' D_i \\ \bullet & D_i' D_i \end{bmatrix} T \end{aligned} \tag{V.42}$$

for all $i \in \mathbb{K}$, where (V.39) has been used. Plugging these approximations into equation (V.36), the first order terms produce

$$(A_i + B_i L_i)' V_i + V_i (A_i + B_i L_i) + \sum_{j \in \mathbb{K}} \lambda_{ij} V_j + (C_i + D_i L_i)'(C_i + D_i L_i) = 0 \tag{V.43}$$

for all $i \in \mathbb{K}$. Then, the optimal gain is obtained by adopting the same approximations, which imply that

$$\begin{aligned} L_i &= -\lim_{T\to 0^+} (B_i' V_i B_i T + D_i' D_i)^{-1} (B_i' V_i + B_i' V_i ((\lambda_{ii}/2)I + A_i)T + D_i' C_i) \\ &= -(D_i' D_i)^{-1} (B_i' V_i + D_i' C_i) \end{aligned} \tag{V.44}$$

for each $i \in \mathbb{K}$ and the associated cost is such that $\mathcal{J}_2 = \sum_{i\in\mathbb{K}} \pi_{i0} \mathbf{Tr}(E_i' V_i E_i)$.

As expected, the sampled-data control problem addressed in this work reduces to the pure MJLS case whenever the constant sampling interval $T \to 0^+$. For details in the pure MJLS case, see Costa, Fragoso & Todorov (2013). Once more, this limit case validates the results obtained so far in this chapter.

The deterministic case

As a final analysis of the results from Theorem V.1, consider the case where $N = 1$. As a consequence, $\mathbb{K} = \{1\}$, $\Lambda = 0$, and $\pi_{i0} = 1$. Because $\Lambda = 0$, the TPBVP reduces to

$$\dot{P}_i + F_i' P_i + P_i F_i + G_i' G_i = 0, \tag{V.45}$$

subject to initial $P_i(0) = S_i$ and final $P_i(T) = H_i' S_i H_i$ boundary conditions for $i = 1$. This implies that, in (V.12), the inequality to be solved becomes

$$e^{F_i' T} H_i' S_i H_i e^{F_i T} - S_i < \int_0^T e^{F_i' \tau} G_i' G_i e^{F_i T} d\tau, \tag{V.46}$$

which leads to a $R_i(P, T)$ constant for $i = 1$. This is in a complete accordance with the results from Souza, Gabriel & Geromel (2014) and can be obtained by solving numerically the LMIs in Theorem V.1 or the problem in (V.37). It is an alternative solution of the linear quadratic sampled-data control problem solved in Levis, Schluete & Athans (1971).

V.3 Iterative Procedure to Solve the $\mathcal{H}_2$ Control Problem

As already mentioned, Algorithm III.1 can be adopted to solve problem (V.31) or, alternatively, problem (V.37). Specifically, all necessary changes in Algorithm III.1 to address the control problem are concentrated in Steps 2 and 3. Indeed, matrices H_i, $i \in \mathbb{K}$, change at each iteration. As a consequence, it is necessary to update the final boundary condition of the TPBVP for each $\ell \in \mathbb{N}$. Hence, in Step 2 of the proposed algorithm, the final boundary conditions to be considered are given by

$$P_{i\ell}(T) = H_{i\ell}' S_{i\ell} H_{i\ell} \geq 0 \tag{V.47}$$

for all $i \in \mathbb{K}$. Matrices $P_{i\ell}(T) > 0$ and the value of the gain matrices are produced in Step 3 by solving (V.31) or, equivalently, (V.37). An iterative procedure for the control problem is summarized in the next algorithm, where problem (V.37) was considered. Again, the convergence of the algorithm must be proved since the control law $u(t) \in \mathbb{R}^m$ is also a variable to be determined.

Algorithm V.1

1. *Define a constant sampling period $T > 0$. Consider $t \in [0, T)$ and initialize $\ell = 0$. Set $V_\ell = 0$, $L_\ell = 0$, and $\mathcal{J}_{2\ell} = 0$.*

2. *Determine the solutions $P_{i\ell}(0)$ of the coupled Lyapunov equations*

$$\dot{P}_{i\ell} + F_i' P_{i\ell} + P_{i\ell} F_i + \sum_{j \in \mathbb{K}} \lambda_{ij} P_{j\ell} + G_i' G_i = 0 \tag{V.48}$$

for each $i \in \mathbb{K}$, which is subject to the final boundary conditions $P_{i\ell}(T) = [I\ 0]'V_{i(\ell)}[I\ 0] \geq 0$ for all $i \in \mathbb{K}$, and determine $R_i(P_\ell, T)$ using

$$R_i(P_\ell, T) = P_{i\ell}(0) - e^{\bar{F}_i'T} P_{i\ell}(T) e^{\bar{F}_i T} \tag{V.49}$$

for each $i \in \mathbb{K}$, where $\bar{F}_i = F_i + (\lambda_{ii}/2)I$, $i \in \mathbb{K}$. Thus, matrices $A_{di\ell}$, $B_{di\ell}$, $C_{di\ell}$, and $D_{di\ell}$ can be obtained for each $i \in \mathbb{K}$ using the relations

$$R_i(P, T) = \begin{bmatrix} C'_{di\ell} \\ D'_{di\ell} \end{bmatrix} \begin{bmatrix} C_{di\ell} & D_{di\ell} \end{bmatrix} \geq 0 \tag{V.50}$$

and

$$\begin{bmatrix} I & 0 \end{bmatrix} e^{\bar{F}_i T} = \begin{bmatrix} A_{di\ell} & B_{di\ell} \end{bmatrix}. \tag{V.51}$$

3. *For each $i \in \mathbb{K}$, determine the stabilizing solution $V_{(\ell+1)} \geq 0$ and the gain $L_{i(\ell+1)}$ by solving the DARE*

$$A'_{di\ell} V_{i(\ell+1)} A_{di\ell} - V_{i(\ell+1)} - L'_{i(\ell+1)} (B'_{di\ell} V_{i(\ell+1)} B_{di\ell} + D'_{di\ell} D_{di\ell}) L_{i(\ell+1)} \\ + C'_{di\ell} C_{di\ell} = 0 \tag{V.52}$$

together with

$$L_{i(\ell+1)} = -(B'_{di\ell} V_{i(\ell+1)} B_{di\ell} + D'_{di\ell} D_{di\ell})^{-1} (B'_{di\ell} V_{i(\ell+1)} A_{di\ell} + D'_{di\ell} C_{di\ell}). \tag{V.53}$$

Determine the current cost $\mathcal{J}_{2(\ell+1)} = \sum_{i\in\mathbb{K}} \pi_{i0} \mathbf{Tr}(E_i' V_{i(\ell+1)} E_i)$.

4. *Set $(\ell + 1) \to \ell$ and iterate until $\mathcal{J}_{2(\ell+1)} - \mathcal{J}_{2\ell}$ becomes small enough.*

As mentioned before, problem (V.31) can be alternatively used in Step 3. In this case, the variables to be determined at each iteration are $X_{i(\ell+1)}$, $Y_{i(\ell+1)}$, and $Z_{i(\ell+1)}$ for each $i \in \mathbb{K}$. As a consequence, the values of matrices $P_{i(\ell+1)}$ and $L_{i(\ell+1)}$ are such that

$$P_{i(\ell+1)}(T) = \begin{bmatrix} I \\ 0 \end{bmatrix} X^{-1}_{i(\ell+1)} \begin{bmatrix} I & 0 \end{bmatrix}, \tag{V.54}$$

$$L_{i(\ell+1)} = Y'_{i(\ell+1)} X^{-1}_{i(\ell+1)}, \tag{V.55}$$

for all $i \in \mathbb{K}$, and the performance index $\mathcal{J}_{2(\ell+1)}$ is given by

$$\mathcal{J}_{2(\ell+1)} = \sum_{i\in\mathbb{K}} \pi_{i0} \mathbf{Tr} \left(E_i' X^{-1}_{i(\ell+1)} E_i \right). \tag{V.56}$$

Due to the differences between the iterative procedures in Algorithms III.1 and V.1, the convergence of Algorithm V.1 must be proved. The next theorem addresses this issue.

Theorem V.2 *Assume that problem V.11 has a bounded solution $(V_i^*, P_i^*(t), L_i^*)$ for all $i \in \mathbb{K}$ and all $t \in [0,T)$. Algorithm V.1 is uniformly convergent and any two subsequent iterations are such that $V^* \geq V_{(\ell+1)} \geq V_\ell \geq 0$ and $\mathcal{J}_2^* \geq \mathcal{J}_{2(\ell+1)} \geq \mathcal{J}_{2(\ell)}$.*

Proof: From the assumption of the existence of a bounded solution, there exist matrices L_i^* and $V_i^* = X_i^* > 0$ for all $i \in \mathbb{K}$ such that the TPBVP of Theorem V.1 holds. From the DLE (V.48) in Step 2, it is possible to verify that any two subsequent iterations satisfy

$$\dot{\Delta}_{i(\ell+1)} + F_i'\Delta_{i(\ell+1)} + \Delta_{i(\ell+1)}F_i + \sum_{j\in\mathbb{K}} \lambda_{ij}\Delta_{j(\ell+1)} = 0, \tag{V.57}$$

where $\Delta_{i(\ell+1)}(t) = P_{i(\ell+1)}(t) - P_{i\ell}(t)$ for each $i \in \mathbb{K}$. Assuming that $\Gamma_{i(\ell+1)} = V_{i(\ell+1)} - V_{i\ell} \geq 0$, $i \in \mathbb{K}$, the final boundary conditions yield

$$\begin{aligned}\Delta_{i(\ell+1)}(T) &= H_{i(\ell+1)}'S_{i(\ell+1)}H_{i(\ell+1)} - H_{i\ell}'S_{i\ell}H_{i\ell} \\ &= \begin{bmatrix} I \\ 0 \end{bmatrix}\Gamma_{i(\ell+1)}\begin{bmatrix} I & 0 \end{bmatrix} \geq 0\end{aligned} \tag{V.58}$$

for each $i \in \mathbb{K}$. As a consequence, $\Delta_{i(\ell+1)}(t) \geq 0$ for all $i \in \mathbb{K}$ and all $t \in [0,T)$, which together with the definition of $R_i(P,T)$, (III.8), implies that

$$R_i(P_{\ell+1},T) - R_i(P_\ell,T) = \int_0^T e^{\bar{F}_i'\tau}\Big(\sum_{j\neq i\in\mathbb{K}} \lambda_{ij}\Delta_{i(\ell+1)}(\tau)\Big)e^{\bar{F}_i\tau}d\tau \geq 0 \tag{V.59}$$

for all $i \in \mathbb{K}$. On the other hand, the stabilizing solution of the DARE in Step 3 and the discrete-time equivalent system (V.13) verify that

$$\begin{aligned}\min_{u[k]}\sum_{k\in\mathbb{N}} \begin{bmatrix} x[k] \\ u[k] \end{bmatrix}' R_i(P_\ell,T)\begin{bmatrix} x[k] \\ u[k] \end{bmatrix} &= \min_{u[k]}\sum_{k\in\mathbb{N}} z[k]'z[k] \\ &= x[0]'V_{i(\ell+1)}x[0],\end{aligned} \tag{V.60}$$

which is valid for all $i \in \mathbb{K}$. Hence, from (V.59), $V_{i(\ell+2)} - V_{i(\ell+1)} \geq 0$ for all $i \in \mathbb{K}$. Thus, initializing the algorithm with $\ell = 0$, the first step imposes that $V_{i0} = 0$ for all $i \in \mathbb{K}$ and $\mathcal{J}_{2(0)} = 0$. Then, the solution of (V.48) provides $P_{i0}(0) \geq 0$ for all $i \in \mathbb{K}$. Consequently, $R_i(P_0,T) \geq 0$, which produces $V_{i1} \geq 0$ for all $i \in \mathbb{K}$ in the third step. Due to the previous property, each new iteration produces, for all $i \in \mathbb{K}$,

$$0 = V_{i0} \leq V_{i1} \leq V_{i2} \leq \cdots . \tag{V.61}$$

Once more, consider the stabilizing solution characterized by $(V_i^*, P_i^*(t), L_i^*)$ for each $i \in K$, which satisfies

$$x[0]'V_i^*x[0] = \min_{u[k]}\sum_{k\in\mathbb{N}} \begin{bmatrix} x[k] \\ u[k] \end{bmatrix}' R_i(P^*,T)\begin{bmatrix} x[k] \\ u[k] \end{bmatrix}. \tag{V.62}$$

Suppose that $V_i^* \geq V_{i\ell} > 0$ for all $i \in K$. Adopting the same reasoning as before, $P_i^*(t) \geq P_{i\ell}(t) \geq 0$ for all $t \in [0, T)$ and $R_i(P^*, T) \geq R_i(P_\ell, T) \geq 0$ for all $i \in K$. As a consequence, $V_i^* \geq V_{i(\ell+1)} > 0$ for all $i \in K$ and all $\ell \in \mathbb{N}$. Due to the fact that $V_{i0} = 0$ for all $i \in \mathbb{K}$, the algorithm generates a sequence of matrices $\{V_{i\ell}\}_{\ell=0}^{\infty}$ bounded by V_i^* for all $i \in \mathbb{K}$. The hypothesis on the existence of a set of bounded solutions V_i^* for all $i \in \mathbb{K}$ is sufficient to assure that

$$\lim_{\ell \to \infty} V_{i\ell} \to V_i^* \tag{V.63}$$

for all $i \in \mathbb{K}$. This means that the algorithm monotonically converges to the solution of V_i^* for all $i \in \mathbb{K}$ and, consequently, $L_{i\ell} \to L_i^*$ and $\mathcal{J}_{2\ell} \to \mathcal{J}_2^*$, concluding thus the proof. ■

Remark V.4 *Due to the equivalence between problems* (V.31) *and* (V.37)*, Theorem V.2 also assures the convergence of the method when* (V.37) *is used instead of* (V.31) *in Step* 3 *of the proposed Algorithm V.1. Moreover, all special cases from Section V.2 can be solved using the same algorithm. The main difference is the problem to be considered in Step* 3*:* (V.38) *for the mode independent case,* (V.37) *with* (V.43) *instead of* (V.36) *for the limit case* $T \to 0^+$*, or* (V.37) *with* $N = 1$ *for the deterministic case. Notice that in the first case the convergence may not be monotonic because all Markov modes are coupled.* □

V.4 Illustrative Numerical Example

In order to illustrate the theoretical results of this chapter, the same numerical example considered in Chapters III and IV is adopted, this time instead, the optimal $\mathcal{H}_2$ sampled-data control law is also evaluated.

Example V.1 *Consider the MJLS with* $N = 2$ *with the state space realization in the form of Example III.1 or IV.1 with the following system matrices*

$$F_1 = \begin{bmatrix} 0 & 1 & 0 \\ -4 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; F_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$J_1 = J_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; G_1' = G_2' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

This time, matrices H_i*,* $i \in \{1, 2\}$*, are unknown since they contain the gain matrices* L_1 *and* L_2 *to be evaluated. Consider also the sampling period* $T = 250\ [ms]$ *and suppose that the transition rate matrix* $\Lambda \in \mathbb{R}^{2\times 2}$ *is such that*

$$\Lambda = \begin{bmatrix} -0.5 & 0.5 \\ 0.2 & -0.2 \end{bmatrix}$$

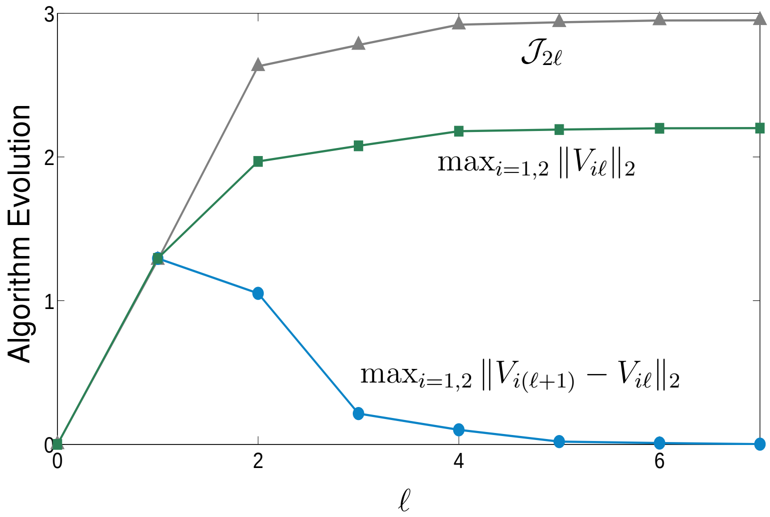

Figure V.1 – Evolution of Algorithm V.1.

and the initial probability of θ_0 is $\pi_0 = [1\ 0]'$. Then, the computed minimum $\mathcal{J}_2^*$ cost equals 2.96, which is assured by the gain matrices

$$L_1^* = \begin{bmatrix} 0.1787 & -0.5559 \end{bmatrix}, \; L_2^* = \begin{bmatrix} -0.2973 & -0.8690 \end{bmatrix}.$$

In this case, the algorithm takes seven iterations to converge. Figure V.1 shows its evolution. The gray curve (with triangular markers) shows the convergence of the performance index $\mathcal{J}_{2\ell}$ to the true value of the square of $\mathcal{H}_2$ norm. The green curve (with squared markers) shows the convergence of $V_\ell \to V^*$ through the measure of the maximum value of $\|V_{i\ell}\|_2$ for $i = \{1, 2\}$. The blue curve (with circular markers) shows how fast this convergence occurs through the measure of the maximum value of $\|V_{i(\ell+1)} - V_{i\ell}\|_2$ for $i = \{1, 2\}$.

A Monte Carlo simulation of $2,000$ samples, which uses the Leon-Garcia's procedure (see Leon-Garcia (2007)), considering the gain matrices L_1^* and L_2^* just calculated provides $\mathcal{J}_2 = 2.96$. Once more, the result shows the quality of the calculated index and, consequently, the efficiency of the proposed Algorithm V.1. Moreover, a mode independent control law can be also designed. It is possible to determine a feasible solution corresponding to $T = 250\ [ms]$. The proposed algorithm gives the state feedback gain

$$L = \begin{bmatrix} -0.1522 & -1.8695 \end{bmatrix},$$

which ensures the guaranteed $\mathcal{J}_2$ cost of

$$\int_0^\infty \mathcal{E}(z(t)'z(t))dt < 9.22$$

within six iterations. For values of the sample period greater than $T = 1.76\ [s]$, it seems that a mode independent sampled-data control that stabilizes the MJLS does not exist. This fact can be numerically verified.

CHAPTER VI

$\mathcal{H}_\infty$ Optimal Sampled-Data Control

In order to complement the theoretical results developed in previous chapters, the present one is devoted to determine the $\mathcal{H}_\infty$ state feedback sampled-data control applied to MJLS with fixed sampling intervals, that is, $t_{k+1} - t_k = T > 0$ for all $k \in \mathbb{N}$. The overall process to accomplish this goal is the same adopted in Chapter V. First, the result from Theorem IV.1 is modified to obtain a convex formulation based on LMI and a theoretical solution is determined. Clearly, in the $\mathcal{H}_\infty$ context, the solution of the control problem depends on the existence of an iterative procedure. Second, some special cases are derived from the results produced so far. An algorithm is proposed to solve the $\mathcal{H}_\infty$ sampled-data control problem, which is proved to be globally convergent. Finally, the theoretical results are illustrated by means of a numerical example.

Once again, remember that the system to be controlled through a state feedback sampled-data control law is the MJLS described by

$$\dot{x}(t) = A_{\theta(t)}x(t) + B_{\theta(t)}u(t) + E_{\theta(t)}w(t) \tag{VI.1}$$

$$z(t) = C_{\theta(t)}x(t) + D_{\theta(t)}u(t) \tag{VI.2}$$

for all $t \in \mathbb{R}^+$, where $\theta(t)$ is a stochastic parameter governed by an homogeneous continuous-time Markov chain. The state feedback sampled-data control law is in the form of

$$u(t) = L_{\theta(t_k)}x(t_k) \tag{VI.3}$$

for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, and all $\theta(t) \in \mathbb{K}$. The set of events $\{t_k\}_{k=0}^{\infty}$ describes the sequence of the sampling instants. Furthermore, this system evolves from initial conditions $x(0) = 0$ and $\theta(0) = \theta_0$ with initial distribution π_{i0} and transition rate matrix Λ previously defined in Chapter II. Then, a totally equivalent hybrid system, HMJLS, can be written as

$$\dot{\xi}(t) = F_{\theta(t)}\xi(t) + J_{\theta(t)}w(t) \tag{VI.4}$$

$$z(t) = G_{\theta(t)}\xi(t) \tag{VI.5}$$

$$\xi(t_k) = H_{\theta(t_k)}\xi(t_k^-) \tag{VI.6}$$

for all $\theta(t) \in \mathbb{K}$ and all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, evolving from initial conditions $\xi(0^-) = \xi_0 = 0$ and $\theta(0^-) = \theta(0) = \theta_0$. Once again, the structure of the system matrices is fundamental for the purposes of the chapter and it is such that

$$F_{\theta(t)} = \begin{bmatrix} A_{\theta(t)} & B_{\theta(t)} \\ 0 & 0 \end{bmatrix}, \tag{VI.7}$$

$$J_{\theta(t)} = \begin{bmatrix} E_{\theta(t)} \\ 0 \end{bmatrix}, \tag{VI.8}$$

$$G_{\theta(t)} = \begin{bmatrix} C_{\theta(t)} & D_{\theta(t)} \end{bmatrix}, \tag{VI.9}$$

and

$$H_{\theta(t_k)} = \begin{bmatrix} I & 0 \\ L_{\theta(t_k)} & 0 \end{bmatrix}. \tag{VI.10}$$

The main goal is to solve the infimum problem described by

$$\inf_{L_1,\cdots,L_N,\gamma} \left\{ \gamma^2 : \int_0^\infty \mathcal{E}\left[z(t)'z(t) - \gamma^2 w(t)'w(t) \right] dt < 0, \forall w \in \mathbb{L}_2^* \right\} \tag{VI.11}$$

subject to (VI.4)–(VI.6). Notice that this is exactly the $\mathcal{H}_\infty$ state feedback control problem from the continuous-time MJLS (see Section II.4) that should be solved by considering the sampled-data control constraint (VI.3).

VI.1 Theoretical Results

Taking into account the developments in Chapter IV, the $\mathcal{H}_\infty$ norm can be evaluated by solving the uncoupled Lyapunov inequalities

$$\Phi_{i\ell}(T)' H_i' S_i H_i \Phi_{i\ell}(T) - S_i + R_{i\ell}(T) < 0 \tag{VI.12}$$

with $S_i > 0$, $i \in \mathbb{K}$, in Step 3 of the iterative procedure defined in Algorithm IV.1. Matrices $R_{i\ell}(T)$ are defined by

$$R_{i\ell}(T) = P_{i\ell}(0) - \Phi_{i\ell}(T)' P_{i\ell}(T) \Phi_{i\ell}(T) \tag{VI.13}$$

for each $i \in \mathbb{K}$, where matrices $P_{i\ell}(0)$ are the solutions of the boundary condition problems defined in Step 2 of the mentioned Algorithm IV.1 evaluated in the beginning of the time interval $[0, T)$ for each $i \in \mathbb{K}$ and each $\ell \in \mathbb{N}$. Moreover, matrices $\Phi_{i\ell}(t)$ are obtained by solving the ODEs $\dot{\Phi}_{i\ell}(t) = M_{i\ell}(t)\Phi_{i\ell}(t)$ with $\Phi_{i\ell}(0) = I$ and $M_{i\ell}(t) = F_{i\ell} + (\lambda_{ii}/2)I + \gamma^{-2} J_i J_i' P_{i\ell}(t)$ for each $i \in \mathbb{K}$. Then, the actual value of the $\mathcal{H}_\infty$ norm is obtained as the lowest value of $\gamma > 0$ for which (VI.12) admit positive definite solutions. However, in the context the $\mathcal{H}_\infty$ control problem, as already analysed in Chapter V for the $\mathcal{H}_2$ context, matrices H_i are also variables to be determined for each $i \in \mathbb{K}$. In this case, inequalities VI.12 are rewritten as

$$S_i - P_{i\ell}(0) > \Phi_{i\ell}(T)'(H_i' S_i H_i - H_{i\ell}' S_{i\ell} H_{i\ell})\Phi_{i\ell}(T) \tag{VI.14}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. Fortunately, again these inequalities can be converted into LMIs. For this, consider the special structures of matrices (VI.7)–(VI.10) and the decomposition of matrices S_i such that, for each $i \in \mathbb{K}$,

$$S_i = \begin{bmatrix} I \\ 0 \end{bmatrix} X_i \begin{bmatrix} I \\ 0 \end{bmatrix}' + \begin{bmatrix} -L_i' \\ I \end{bmatrix} Z_i^{-1} \begin{bmatrix} -L_i' \\ I \end{bmatrix}'. \tag{VI.15}$$

Notice that the decomposition (VI.15) is different from that considered in Chapter V, (V.19).

Remark VI.1 *Matrices S_i and L_i are parameterized with three, not four, independent variables, namely, X_i, Z_i, and L_i for all $i \in \mathbb{K}$. As can be seen in the proof of the next theorem, this choice can be imposed without introducing any kind of conservatism to the solutions of* (VI.14) *in the closure of feasibility, where the variables are matrices $S_i > 0$ and L_i, $i \in \mathbb{K}$.* □

The decomposition (VI.15) has two easily verified properties:

1) $S_i > 0$, if and only if, $X_i > 0$ and $Z_i > 0$ for all $i \in \mathbb{K}$; and

2) considering the gain matrices $\bar{L}_i$ for all $i \in \mathbb{K}$, then

$$H_i' S_i H_i = \begin{bmatrix} I \\ 0 \end{bmatrix} X_i \begin{bmatrix} I \\ 0 \end{bmatrix}' + \begin{bmatrix} L_i' - \bar{L}_i' \\ 0 \end{bmatrix} Z_i^{-1} \begin{bmatrix} L_i' - \bar{L}_i' \\ 0 \end{bmatrix}', \tag{VI.16}$$

which is valid for all $i \in \mathbb{K}$. By imposing $L_i = \bar{L}_i$, (VI.16) is such that

$$H_i' S_i H_i = \begin{bmatrix} I \\ 0 \end{bmatrix} X_i \begin{bmatrix} I \\ 0 \end{bmatrix}' \leq S_i \tag{VI.17}$$

for all $i \in \mathbb{K}$.

In order to solve the control problem, initially, consider that $P_{i\ell}(0) - S_{i\ell} \geq 0$ for all $i \in \mathbb{K}$. Thus, by using this hypothesis and the second property of (VI.15), a similar factorization as written for the $\mathcal{H}_2$ control problem in Chapter V can be defined:

$$0 \leq P_{i\ell}(0) - S_{i\ell} \leq P_{i\ell}(0) - \begin{bmatrix} I \\ 0 \end{bmatrix} X_{i\ell} \begin{bmatrix} I \\ 0 \end{bmatrix}' = \begin{bmatrix} C_{di\ell}' \\ D_{di\ell}' \end{bmatrix} \begin{bmatrix} C_{di\ell}' \\ D_{di\ell}' \end{bmatrix}' \tag{VI.18}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. Due to the structure of matrices depicted in (VI.7)–(VI.10), then

$$\begin{aligned} M_{i\ell}(t) &= F_i + (\lambda_{ii}/2)I + \gamma^{-2} J_i J_i' P_{i\ell}(t) \\ &= \begin{bmatrix} A_i & B_i \\ 0 & 0 \end{bmatrix} + (\lambda_{ii}/2)I + \gamma^{-2} \begin{bmatrix} E_i \\ 0 \end{bmatrix} \begin{bmatrix} E_i \\ 0 \end{bmatrix}' \begin{bmatrix} P_{11i\ell}(t) & P_{12i\ell}(t) \\ \bullet & P_{22i\ell}(t) \end{bmatrix} \\ &= \begin{bmatrix} \bar{A}_{i\ell}(t) & \bar{B}_{i\ell}(t) \\ 0 & (\lambda_{ii}/2)I \end{bmatrix} \end{aligned} \tag{VI.19}$$

for all $i \in \mathbb{K}$ and $\ell \in \mathbb{N}$, where matrices $P_{i\ell}$, $i \in \mathbb{K}$ and $\ell \in \mathbb{N}$, have been decomposed into block structures. Additionally, blocks $\bar{A}_{i\ell}(t)$ and $\bar{B}_{i\ell}(t)$ are defined by $\bar{A}_{i\ell}(t) = A_i + (\lambda_{ii}/2)I + \gamma^{-2}E_iE_i'P_{11i\ell}(t)$ and $\bar{B}_{i\ell}(t) = B_i + \gamma^{-2}E_iE_i'P_{12i\ell}(t)$ for all $i \in \mathbb{K}$ and $\ell \in \mathbb{N}$. As a consequence, matrices $\Phi_\ell(T)$ are given by

$$\Phi_{i\ell}(T) = \begin{bmatrix} A_{di\ell} & B_{di\ell} \\ 0 & e^{(\lambda_{ii}/2)T}I \end{bmatrix} \tag{VI.20}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. From (VI.20), matrices $A_{di\ell} \in \mathbb{R}^{n\times n}$ and $B_{di\ell} \in \mathbb{R}^{n\times m}$ are easily extracted by any available computational tool and are such that

$$A_{di\ell} = e^{\int_0^T \bar{A}_{i\ell}(\tau)d\tau} \tag{VI.21}$$

$$B_{di\ell} = \int_0^T e^{\int_0^{T-\eta} \bar{A}_{i\ell}(\tau)d\tau}\bar{B}_{i\ell}(\eta)e^{(\lambda_{ii}/2)\eta}d\eta \tag{VI.22}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. Notice that these relations are not useful for numerical purposes since $\Phi_{i\ell}(T)$ are readily calculated by direct integration of the differential equation $\dot{\Phi}_{i\ell}(t) = M_{i\ell}(t)\Phi_{i\ell}(t)$ with the initial condition $\Phi_{i\ell}(0) = I$ for each $i \in \mathbb{K}$ and $\ell \in \mathbb{N}$. Then, the next theorem formalises the solution of the $\mathcal{H}_\infty$ state feedback sampled-data control problem by means of a convex formulation expressed by LMIs.

Theorem VI.1 *Inequalities* (VI.14) *are feasible if and only if there exist symmetric matrices W_i, Z_i, and matrix Y_i such that the LMIs*

$$\begin{bmatrix} W_i & 0 & A_{i\ell}W_i + B_{i\ell}Y_i & B_{i\ell}Z_i \\ \bullet & I & C_{i\ell}W_i + D_{i\ell}Y_i & D_{i\ell}Z_i \\ \bullet & \bullet & W_i & 0 \\ \bullet & \bullet & \bullet & Z_i \end{bmatrix} > 0 \tag{VI.23}$$

hold for all $i \in \mathbb{K}$. In the affirmative case, a feasible solution is given by $L_i = Y_iW_i^{-1}$ and S_i of the form (VI.15) *where $X_i = X_{i\ell} + W_i^{-1}$ for each $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$.*

Proof: For the necessity, consider by assumption that there exist matrices $S_i > 0$ in the form of (VI.15) satisfying inequalities (VI.14) such that $S_i > S_{i\ell}$ for all $i \in \mathbb{K}$ and each $\ell \in \mathbb{N}$. Consider also a state feedback control law governed by the gain matrices $\bar{L}_i$ not necessarily equal to L_i, $i \in \mathbb{K}$. Define matrices Υ_i such that

$$\Upsilon_i = \begin{bmatrix} I & 0 \\ L_i & I \end{bmatrix} \tag{VI.24}$$

for all $i \in \mathbb{K}$. Applying Υ_i as a similarity transformation in the inequalities (VI.14) for all

$i \in \mathbb{K}$ yields

$$\begin{aligned}
&\Upsilon_i'\Phi_{i\ell}(T)'H_{i\ell}'S_{i\ell}H_{i\ell}\Phi_{i\ell}(T)\Upsilon_i = \\
&= \begin{bmatrix} I & L_i' \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{di\ell}' & 0 \\ B_{di\ell}' & e^{(\lambda_{ii}/2)T}I \end{bmatrix}' \begin{bmatrix} I \\ 0 \end{bmatrix} X_{i\ell} \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} A_{di\ell} & B_{di\ell} \\ 0 & e^{(\lambda_{ii}/2)T}I \end{bmatrix} \begin{bmatrix} I & 0 \\ L_i & I \end{bmatrix} \\
&= \begin{bmatrix} A_{di\ell}' + L_i'B_{di\ell}' \\ B_{di\ell}' \end{bmatrix} X_{i\ell} \begin{bmatrix} A_{di\ell}' + L_i'B_{di\ell}' \\ B_{di\ell}' \end{bmatrix}'
\end{aligned} \tag{VI.25}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$, where the identity $L_{i\ell} = \bar{L}_{i\ell}$ for all $i \in \mathbb{K}$ in the ℓ-th iteration has been assumed. Moreover, since this hypothesis has been assumed not valid for the subsequent iterations, then

$$\begin{aligned}
&\Upsilon_i'\Phi_{i\ell}(T)'H_i'S_iH_i\Phi_{i\ell}(T)\Upsilon_i \\
&= \begin{bmatrix} I & L_i' \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{di\ell}' \\ B_{di\ell}' \end{bmatrix} \left(X_i + (L_i' - \bar{L}_i')Z_i^{-1}(L_i - \bar{L}_i)\right) \begin{bmatrix} A_{di\ell}' \\ B_{di\ell}' \end{bmatrix}' \begin{bmatrix} I & 0 \\ L_i & I \end{bmatrix} \\
&\geq \begin{bmatrix} A_{di\ell}' + L_i'B_{di\ell}' \\ B_{di\ell}' \end{bmatrix} X_i \begin{bmatrix} A_{di\ell}' + L_i'B_{di\ell}' \\ B_{di\ell}' \end{bmatrix}'
\end{aligned} \tag{VI.26}$$

for all $i \in \mathbb{K}$. Furthermore,

$$\begin{aligned}
\Upsilon_i'P_{i\ell}(0)\Upsilon_i &= \begin{bmatrix} I & L_i' \\ 0 & I \end{bmatrix} \left\{ \begin{bmatrix} C_{di\ell}' \\ D_{di\ell}' \end{bmatrix} \begin{bmatrix} C_{di\ell}' \\ D_{di\ell}' \end{bmatrix}' + \begin{bmatrix} I \\ 0 \end{bmatrix} X_{i\ell} \begin{bmatrix} I \\ 0 \end{bmatrix}' \right\} \begin{bmatrix} I & 0 \\ L_i & I \end{bmatrix} \\
&= \begin{bmatrix} C_{di\ell}' + L_i'D_{di\ell}' \\ D_{di\ell}' \end{bmatrix} \begin{bmatrix} C_{di\ell}' + L_i'D_{di\ell}' \\ D_{di\ell}' \end{bmatrix}' + \begin{bmatrix} X_{i\ell} & 0 \\ \bullet & 0 \end{bmatrix}
\end{aligned} \tag{VI.27}$$

and

$$\begin{aligned}
\Upsilon_i'S_i\Upsilon_i &= \begin{bmatrix} I & L_i' \\ 0 & I \end{bmatrix}' \left\{ \begin{bmatrix} I \\ 0 \end{bmatrix} X_i \begin{bmatrix} I \\ 0 \end{bmatrix}' + \begin{bmatrix} -L_i' \\ I \end{bmatrix} Z_i^{-1} \begin{bmatrix} -L_i' \\ I \end{bmatrix}' \right\} \begin{bmatrix} I & 0 \\ L_i & I \end{bmatrix} \\
&= \begin{bmatrix} X_i & 0 \\ 0 & Z_i^{-1} \end{bmatrix}
\end{aligned} \tag{VI.28}$$

for all $i \in \mathbb{K}$. As a consequence, plugging (VI.25)–(VI.28) into (VI.14), produces

$$\begin{aligned}
&\begin{bmatrix} A_{di\ell}' + L_i'B_{di\ell}' \\ B_{di\ell}' \end{bmatrix} (X_i - X_{i\ell}) \begin{bmatrix} A_{di\ell}' + L_i'B_{di\ell}' \\ B_{di\ell}' \end{bmatrix}' \\
&\leq \Upsilon_i'\Phi_{i\ell}(T)'\left(H_i'S_iH_i - H_{i\ell}'S_{i\ell}H_{i\ell}\right)\Phi_{i\ell}(T)\Upsilon_i \\
&< S_i - P_{i\ell}(0) \\
&= \begin{bmatrix} X_i & 0 \\ 0 & Z_i^{-1} \end{bmatrix} - \begin{bmatrix} C_{di\ell}' + L_i'D_{di\ell}' \\ D_{di\ell}' \end{bmatrix} \begin{bmatrix} C_{di\ell}' + L_i'D_{di\ell}' \\ D_{di\ell}' \end{bmatrix}' - \begin{bmatrix} X_{i\ell} & 0 \\ \bullet & 0 \end{bmatrix}
\end{aligned} \tag{VI.29}$$

for all $i \in \mathbb{K}$ and $\ell \in \mathbb{N}$. Finally, introducing the new variables $W_i = (X_i - X_{i\ell})^{-1}$, the Schur complement applied twice to (VI.29) gives, for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$,

$$\begin{bmatrix} A_{di\ell} + B_{di\ell}L_i & B_{di\ell} \\ C_{di\ell} + D_{di\ell}L_i & D_{di\ell} \end{bmatrix} \begin{bmatrix} W_i & 0 \\ \bullet & Z_i \end{bmatrix} \begin{bmatrix} A_{di\ell} + B_{di\ell}L_i & B_{di\ell} \\ C_{di\ell} + D_{di\ell}L_i & D_{di\ell} \end{bmatrix}' < \begin{bmatrix} W_i & 0 \\ \bullet & I \end{bmatrix}. \tag{VI.30}$$

Applying the Schur complement once more, produces

$$\begin{bmatrix} W_i & 0 & A_{di\ell} + B_{di\ell}L_i & B_{di\ell} \\ \bullet & I & C_{di\ell} + D_{di\ell}L_i & D_{di\ell} \\ \bullet & \bullet & W_i^{-1} & 0 \\ \bullet & \bullet & \bullet & Z_i^{-1} \end{bmatrix} > 0 \tag{VI.31}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$, which multiplied both sides by $\mathrm{diag}(I, I, W_i, Z_i)$ becomes

$$\begin{bmatrix} W_i & 0 & A_{di\ell}W_i + B_{di\ell}L_iW_i & B_{di\ell}Z_i \\ \bullet & I & C_{di\ell}W_i + D_{di\ell}L_iW_i & D_{di\ell}Z_i \\ \bullet & \bullet & W_i & 0 \\ \bullet & \bullet & \bullet & Z_i \end{bmatrix} > 0 \tag{VI.32}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$, which corresponds exactly to (VI.23) provided that $Y_i = L_iW_i$ for all $i \in \mathbb{K}$. Notice that, in this case, without loss of generality the assumption $L_i = \bar{L}_i$ can be done for all $i \in \mathbb{K}$.

Conversely, for the sufficiency, consider that there exist symmetric matrices X_i, Z_i and matrices Y_i, $i \in \mathbb{K}$, such that the LMIs (VI.23) hold for all $i \in \mathbb{K}$ and $\ell \in \mathbb{N}$. In this case, $X_i = W_i + X_{i\ell}$ and $L_i = Y_iW_i^{-1}$ for each $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. Multiplying both sides of (VI.23) by $\mathrm{diag}(I, I, W_i^{-1}, Z_i^{-1})$ and applying the Schur complement twice, (VI.23) can be rewritten as

$$\begin{aligned} &\begin{bmatrix} A_{di\ell} + B_{di\ell}L_i & B_{di\ell} \\ C_{di\ell} + D_{di\ell}L_i & D_{di\ell} \end{bmatrix}' \begin{bmatrix} X_i - X_{i\ell} & 0 \\ \bullet & I \end{bmatrix} \begin{bmatrix} A_{di\ell} + B_{di\ell}L_i & B_{di\ell} \\ C_{di\ell} + D_{di\ell}L_i & D_{di\ell} \end{bmatrix} \\ &< \begin{bmatrix} X_i - X_{i\ell} & 0 \\ 0 & Z_i^{-1} \end{bmatrix} \end{aligned} \tag{VI.33}$$

which is equivalent to

$$\begin{aligned} &\begin{bmatrix} A'_{di\ell} + L'_iB'_{di\ell} \\ B'_{di\ell} \end{bmatrix} (X_i - X_{i\ell}) \begin{bmatrix} A'_{di\ell} + L'_iB'_{di\ell} \\ B'_{di\ell} \end{bmatrix}' + \begin{bmatrix} C'_{di\ell} + L'_iD'_{di\ell} \\ D'_{di\ell} \end{bmatrix} \begin{bmatrix} C'_{di\ell} + L'_iD'_{di\ell} \\ D'_{di\ell} \end{bmatrix}' \\ &< \begin{bmatrix} I \\ 0 \end{bmatrix} X_{i\ell} \begin{bmatrix} I \\ 0 \end{bmatrix}' + \begin{bmatrix} X_i & 0 \\ 0 & Z_i^{-1} \end{bmatrix} \end{aligned} \tag{VI.34}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. Choosing $L_i = \bar{L}_i$ for all $i \in \mathbb{K}$, (VI.25), (VI.27), and (VI.28) remain unchanged and the inequality (VI.26) becomes an equality. Hence, inequality (VI.33) yields

$$\begin{aligned}
&\Upsilon_i' \Phi_{i\ell}(T)' \left(H_i' S_i H_i - H_{i\ell}' S_{i\ell} H_{i\ell}\right) \Phi_{i\ell}(T) \Upsilon_i \\
&= \begin{bmatrix} A_{di\ell}' + L_i' B_{di\ell}' \\ B_{di\ell}' \end{bmatrix} (X_i - X_{i\ell}) \begin{bmatrix} A_{di\ell}' + L_i' B_{di\ell}' \\ B_{di\ell}' \end{bmatrix}' \\
&< \Upsilon_i' \left(S_i - P_{i\ell}(0)\right) \Upsilon_i
\end{aligned} \tag{VI.35}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. As a consequence, (VI.14) can be verified with matrices $S_i > 0$, $i \in \mathbb{K}$, assuming the decomposition described in (VI.15), thus completing the proof. ■

Remark VI.2 *The hypothesis of $L_i = \bar{L}_i$, $i \in \mathbb{K}$, is reasonable and can be stated along the proof of Theorem VI.1 without loss of generality. Indeed, in the proof of the necessity part, it is also proved that since $L_{i\ell} = \bar{L}_{i\ell}$ for some $\ell \in \mathbb{K}$, the subsequent iteration exhibits the same property without introducing any conservatism. This fact can be verified by substituting matrices S_i, W_i, X_i, Y_i, Z_i, L_i, and $\bar{L}_i$ by its correspondent matrices $S_{i(\ell+1)}$, $W_{i(\ell+1)}$, $X_{i(\ell+1)}$, $Y_{i(\ell+1)}$, $Z_{i(\ell+1)}$, $L_{i(\ell+1)}$, and $\bar{L}_{i(\ell+1)}$ for all $i \in \mathbb{K}$ and $\ell \in \mathbb{N}$. Then, the property $L_{i\ell} = \bar{L}_{i\ell}$ is valid for all $\ell \in \mathbb{N}$ and each $i \in \mathbb{K}$.* □

The result of Theorem VI.1 provides a way to parametrize all feasible solutions of a nonlinear inequality in the form of (VI.14) in terms of LMIs. This means that the original inequality defines a convex feasible set after a proper change of variables is performed. Moreover, a feasible solution on the border of inequality (VI.14) can be calculated by imposing $Z_i > 0$ arbitrarily small for all $i \in \mathbb{K}$. Then, applying the Schur complement in the last two columns and rows of (VI.23) provides

$$\begin{aligned}
\begin{bmatrix} W_i & 0 \\ \bullet & I \end{bmatrix} &> \begin{bmatrix} A_{di\ell} + B_{di\ell} L_i & B_{i\ell} \\ C_{di\ell} + D_{i\ell} L_i & D_{i\ell} \end{bmatrix} \begin{bmatrix} W_i & 0 \\ \bullet & Z_i \end{bmatrix} \begin{bmatrix} A_{di\ell} + B_{di\ell} L_i & B_{i\ell} \\ C_{di\ell} + D_{i\ell} L_i & D_{i\ell} \end{bmatrix}' \\
&> \begin{bmatrix} A_{di\ell} + B_{di\ell} L_i \\ C_{di\ell} + D_{di\ell} L_i \end{bmatrix} W_i \begin{bmatrix} A_{di\ell} + B_{di\ell} L_i \\ C_{di\ell} + D_{di\ell} L_i \end{bmatrix}'
\end{aligned} \tag{VI.36}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. From the definition of the new matrix variable $V_i = W_i^{-1} > 0$, $i \in \mathbb{K}$, inequality (VI.36) is equivalent to

$$(A_{di\ell} + B_{di\ell} L_i)' V_i (A_{di\ell} + B_{di\ell} L_i) - V_i < -(C_{di\ell} + D_{di\ell} L_i)' (C_{di\ell} + D_{di\ell} L_i) \tag{VI.37}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$, that is a similar result as obtained for the $\mathcal{H}_2$ state feedback control problem in Chapter V. Then, by taking matrices $V_i > 0$ arbitrarily close to the stabilizing solution of (VI.37), it can be rewritten as a DARE in the form of

$$A_{di\ell}' V_i A_{di\ell} - V_i + C_{di\ell}' C_{di\ell} - L_i' (B_{di\ell}' V_i B_{di\ell} + D_{di\ell}' D_{di\ell}) L_i = 0, \tag{VI.38}$$

where $L_i = -(B_{di\ell}' V_i B_{di\ell} + D_{di\ell}' D_{di\ell})^{-1} (B_{di\ell}' V_i A_{di\ell} + D_{di\ell}' C_{di\ell})$ for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. These results establishes the basis to apply a similar algorithm as implemented in Chapter IV to solve the TPBVP defined in Theorem IV.1 in the case of the $\mathcal{H}_\infty$ state feedback control design problem.

VI.2 Special Case Analysis

This section is devoted to analyse the theoretical results for the $\mathcal{H}_\infty$ state feedback sampled-data control design obtained so far in order to validate the previous outcome. First, the mode independent case is obtained. Then, the limit case $T \to 0^+$ recovers the pure MJLS $\mathcal{H}_\infty$ state feedback control design. Finally, the deterministic $\mathcal{H}_\infty$ state feedback sampled-data control case is shown to be included in the theoretical results from Section VI.1, as expected.

The mode independent case

In the present context, the Markov chain state $\theta(t)$ must be known at each sampling instant t_k, $k \in \mathbb{N}$, in order to implement the sampled-data control law (VI.3). In practice, this necessity increases the computational burden to control the MJLS (VI.1)–(VI.2). Fortunately, once more, this dependence can be ruled out by imposing the additional design constrain $L_i = L$ for all $i \in \mathbb{K}$, which is translated in the next corollary.

Corollary VI.1 *Inequalities* (VI.14) *are feasible for some matrices L and $S_i > 0$ for all $i \in \mathbb{K}$ if there exist matrices U, Y, and symmetric matrices W_i and Z_i such that the LMI*

$$\begin{bmatrix} W_i & 0 & A_{di\ell}U + B_{di\ell}Y & B_{di\ell}Z_i \\ \bullet & I & C_{di\ell}U + D_{di\ell}Y & D_{di\ell}Z_i \\ \bullet & \bullet & U + U' - W_i & 0 \\ \bullet & \bullet & \bullet & Z_i \end{bmatrix} > 0 \qquad \text{(VI.39)}$$

hold for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. In the affirmative case, a feasible solution is given by $L = YU^{-1}$ and S_i of the form (VI.15)*, where $X_i = X_{i\ell} + W_i^{-1}$ and $Y_i = Y$ for each $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$.*

Proof: Assume that (VI.39) hold for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. Replace the third diagonal element by the upper bound $U'W_i^{-1}U \geq U + U' - W_i$, which is valid for all $i \in \mathbb{K}$, that is,

$$\begin{bmatrix} W_i & 0 & A_{di\ell}U + B_{di\ell}Y & B_{di\ell}Z_i \\ \bullet & I & C_{di\ell}U + D_{di\ell}Y & D_{di\ell}Z_i \\ \bullet & \bullet & U'W_i^{-1}U & 0 \\ \bullet & \bullet & \bullet & Z_i \end{bmatrix} > 0. \qquad \text{(VI.40)}$$

Then, by multiplying the resultant inequalities to the right by $\mathrm{diag}\{I, I, U^{-1}W_i, I\}$ and to the left by its transpose, thus, (VI.23) hold for the triple (W_i, Y_i, Z_i) with $Y_i = LW_i$ for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$, concluding the proof. ■

Once the mode independent sampled-data control design is adopted, the necessity cannot be proved anymore. This fact is due to the conservatism introduced by the matrices U and Y, unique for all $i \in \mathbb{K}$. Thus, only an upper bound to the performance index is guaranteed and the optimal solution may not be reached. Moreover, the LMIs (VI.39) remain convex, but they cannot be decomposed into N uncoupled inequalities. Clearly, the computational effort involved is much higher than that spent to solve the N uncoupled LMIs (VI.23). Finally, notice that this problem cannot be solved by the DARE (VI.38), which puts in evidence the importance of the LMI result from Theorem VI.1.

The limit case $T \to 0^+$

An important analysis of the theoretical results obtained in Section VI.1 is to determine the limit of inequalities (VI.14) when the sampling interval $T > 0$ goes to zero. In this case, the boundary conditions of the TPBVP from Theorem IV.1 becomes $P_i(0) = \lim_{T\to 0^+} P_i(T)$ for all $i \in \mathbb{K}$, which implies that

$$S_i \to H_i' S_i H_i = \begin{bmatrix} I \\ 0 \end{bmatrix} X_i \begin{bmatrix} I \\ 0 \end{bmatrix}' \tag{VI.41}$$

for all $i \in \mathbb{K}$ once $T \to 0^+$ and since L_i has been proved to be equal to $\bar{L}_i$ for all $i \in \mathbb{K}$. On the other hand, the approximations of the matrices $A_{di\ell}$ and $B_{di\ell}$ for $T \to 0^+$ are obtained by taking the first order terms of the Taylor series expansion applied to (VI.21) and (VI.22), which are, respectively,

$$A_{di\ell} \approx I + \big(A_i + (\lambda_{ii}/2)I + \gamma^{-2}E_iE_i'X_{i\ell}\big)\, T \tag{VI.42}$$

and

$$B_{di\ell} \approx B_i T \tag{VI.43}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. Analogously, matrices $C_{di\ell}$ and $D_{di\ell}$ are obtained by adopting the same process over definition (VI.18) and are such that

$$\begin{bmatrix} C_{di\ell}' \\ D_{di\ell}' \end{bmatrix} \begin{bmatrix} C_{di\ell}' \\ D_{di\ell}' \end{bmatrix}' = P_{i\ell}(0) - \begin{bmatrix} I \\ 0 \end{bmatrix} X_{i\ell} \begin{bmatrix} I \\ 0 \end{bmatrix}' \tag{VI.44}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$, where

$$P_{i\ell}(0) = \lim_{T\to 0^+} \{P_{i\ell}(T) - \dot{P}_{i\ell}(T)T\} \tag{VI.45}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. The value of $\dot{P}_{i\ell}(T)$ is calculated from the DRE (IV.5), that is,

$$\begin{aligned} \dot{P}_{i\ell}(T) &= -\left\{ F_i'P_{i\ell}(T) + P_{i\ell}(T)F_i + \gamma^{-2}P_{i\ell}(T)J_iJ_i'P_{i\ell}(T) + \sum_{j\in\mathbb{K}} \lambda_{ij}P_{j\ell}(T) + G_i'G_i \right\} \\ &= -\begin{bmatrix} Q_{i\ell} + C_i'C_i & X_{i\ell}B_{i\ell} + C_i'D_i \\ \bullet & D_i'D_i \end{bmatrix} \end{aligned} \tag{VI.46}$$

for each $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$, where $Q_{i\ell} = A_i' X_{i\ell} + X_{i\ell} A_i + \gamma^{-2} X_{i\ell} E_i E_i' X_{i\ell} + \sum_{j\in\mathbb{K}} \lambda_{ij} X_{j\ell}$, $i \in \mathbb{K}$ and $\ell \in \mathbb{N}$. Hence,

$$\begin{bmatrix} C_{di\ell}' \\ D_{di\ell}' \end{bmatrix} \begin{bmatrix} C_{di\ell}' \\ D_{di\ell}' \end{bmatrix}' \approx \begin{bmatrix} Q_{i\ell} + C_i' C_i & X_{i\ell} B_{i\ell} + C_i' D_i \\ \bullet & D_i' D_i \end{bmatrix} T \tag{VI.47}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. Plugging (VI.42), (VI.43) and (VI.47) into the DARE (VI.38), it follows that

$$\begin{aligned} (A_i + B_i L_i)' X_i + X_i (A_i + B_i L_i) + (C_i + D_i L_i)'(C_i + D_i L_i) \\ + \gamma^{-2} X_i E_i E_i' X_i + \sum_{j\in\mathbb{K}} \lambda_{ij} X_j = \Theta_{i\ell} \end{aligned} \tag{VI.48}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$, where matrices $\Theta_{i\ell}$ are in the form of

$$\Theta_{i\ell} = \sum_{j\neq i\in\mathbb{K}} \lambda_{ij}(X_i - X_{i\ell}) + \gamma^{-2}(X_i - X_{i\ell}) E_i E_i' (X_i - X_{i\ell}) \geq 0 \tag{VI.49}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. Provided that the initial assumption $S_i > S_{i\ell} > 0$ holds – it will be proved in the next section – it implies that

$$X_i = \begin{bmatrix} I \\ L_i \end{bmatrix}' S_i \begin{bmatrix} I \\ L_i \end{bmatrix} > \begin{bmatrix} I \\ L_i \end{bmatrix}' S_{i\ell} \begin{bmatrix} I \\ L_i \end{bmatrix} \geq X_{i\ell} \tag{VI.50}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. Additionally, $\lambda_{ij} \geq 0$ for all $j \neq i \in \mathbb{K}$. Furthermore, by applying the same approximations to matrices L_i, it follows that

$$L_i = -(D_i' D_i)^{-1}(B_i' X_i + D_i' C_i) \tag{VI.51}$$

for all $i \in \mathbb{K}$. Thus, the limit case $T \to 0^+$ recovers exactly the $\mathcal{H}_\infty$ state feedback control applied to MJLS, as expected. (See Costa, Fragoso & Todorov (2013).)

The deterministic case

As a final important analysis of the results from Theorem VI.1, consider the case where $N = 1$. As a consequence, $\mathbb{K} = \{1\}$, $\Lambda = 0$, and $\pi_{i0} = 1$. Because $\Lambda = 0$, the TPBVP reduces to

$$\dot{P}_i + F_i' P_i + P_i F_i + \gamma^{-2} P_i J_i J_i' P_i + G_i' G_i = 0 \tag{VI.52}$$

subject to initial $P_i(0) = S_i$ and final $P_i(T) = H_i' S_i H_i$ boundary conditions for $i = 1$. In this case, the inequality (VI.14) is such that

$$\Phi_{i\ell}(T)' H_i' S_i H_i \Phi_{i\ell}(T) - S_i < R_{i\ell}(T) \tag{VI.53}$$

for all $\ell \in \mathbb{N}$ and $i = 1$, where $\Phi_{i\ell}(T)$ follows from the solution of the linear differential equation $\dot{\Phi}_{i\ell}(t) = M_{i\ell}(t)\Phi_{i\ell}(t)$ with $\Phi_{i\ell}(0) = I$ and $M_{i\ell}(t) = F_i + \gamma^{-2}J_iJ_i'P_{i\ell}(t)$ for each $\ell \in \mathbb{N}$ and $i = 1$. Moreover, matrix $R_{i\ell}(T)$ is defined by

$$R_{i\ell}(T) = \Phi_{i\ell}(T)'H_{i\ell}'S_{i\ell}H_{i\ell}\Phi_{i\ell}(T) - P_{i\ell}(0) \tag{VI.54}$$

for all $\ell \in \mathbb{N}$ and $i = 1$. On the other hand, (VI.52) yields

$$R_{i\ell}(T) = \int_0^T \Phi_{i\ell}(\tau)'\left(G_i'G_i - \gamma^{-2}P_{i\ell}(\tau)J_iJ_i'P_{i\ell}(\tau)\right)\Phi_{i\ell}(\tau)d\tau \tag{VI.55}$$

for all $\ell \in \mathbb{N}$ and $i = 1$. Hence, matrices $\Phi_{i\ell}(T)$ and $R_{i\ell}(T)$ for $i = 1$ and $\ell \in \mathbb{N}$ whenever plugged into inequality (VI.53) provide the optimal solution for the next iteration. This is an alternative result equivalent to Souza, Gabriel & Geromel (2014), where iterations are not necessary because, in this case, the DRE (VI.52) admits a closed-form solution. The next algorithm can be applied to the deterministic case, but the inverse is not possible.

VI.3 Iterative Procedure to Solve the $\mathcal{H}_\infty$ Control Problem

As before, the iterative procedure presented in Algorithm IV.1 can be used to determine the optimal sampled-data control law applied to an MJLS in the $\mathcal{H}_\infty$ context. Indeed, in order to accomplish this goal, only two changes in Steps 2 and 3 of the mentioned algorithm must be adequately modified and implemented. First, in Step 2, the final boundary condition to the DRE (IV.23) becomes

$$P_{i\ell}(T) = H_{i\ell}'S_{i\ell}H_{i\ell} = \begin{bmatrix} I \\ 0 \end{bmatrix} X_{i\ell} \begin{bmatrix} I \\ 0 \end{bmatrix}' \tag{VI.56}$$

for all $i \in \mathbb{K}$ and each $\ell \in \mathbb{N}$, which depends only on matrices $X_{i\ell}$. Second, in Step 3, a feasible solution to the LMIs (VI.23) or the solution to the DARE (VI.38) needs to be determined. This solution provides the current state feedback gain matrices

$$L_{i(\ell+1)} = L_i \tag{VI.57}$$

and matrices

$$X_{i(\ell+1)} = X_{i\ell} + V_{i(\ell+1)} \geq X_{i\ell} \tag{VI.58}$$

for all $i \in \mathbb{K}$. An iterative procedure to solve the $\mathcal{H}_\infty$ state feedback sampled-data control problem applied to an MJLS, according to the theoretical results presented in Section VI.1, is summarized in the next algorithm, where the DARE (VI.38) has been considered.

Algorithm VI.1

1. *Define the constant sampling period* $T > 0$ *and the* $\mathcal{H}_\infty$ *level* $\gamma > 0$, *large enough, such that* $\mathcal{J}_\infty < 0$. *Consider* $t \in [0, T)$ *and initialize* $\ell = 0$. *Set* $X_\ell = 0$ *and* $L_{i\ell} = 0$.

2. *Determine the value of* $P_\ell(0)$ *from the coupled DRE*

$$\dot{P}_{i\ell}(t) + F_i' P_{i\ell} + P_{i\ell} F_i + \gamma^{-2} P_{i\ell} J_i J_i' P_{i\ell} + \sum_{j \in \mathbb{K}} \lambda_{ij} P_{j\ell} + G_i' G_i = 0 \quad \text{(VI.59)}$$

subject to the final boundary condition $P_{i\ell}(T) = [I\ 0]' X_{i\ell} [I\ 0] \geq 0$ *for all* $i \in \mathbb{K}$ *through a backward integration. Using a forward integration, determine the value of* $\Phi_{i\ell}(T)$ *by solving the ODE*

$$\begin{cases} \dot{\Phi}_{i\ell}(t) & = & M_{i\ell}(t) \Phi_{i\ell}(t) \\ \Phi_{i\ell}(0) & = & I \end{cases} \quad \text{(VI.60)}$$

for all $i \in \mathbb{K}$, *where* $M_{i\ell}(t) = F_i + (\lambda_{ii}/2) I + \gamma^{-2} J_i J_i' P_{i\ell}(t)$, $i \in \mathbb{K}$. *Thus, matrices* $A_{di\ell}$, $B_{di\ell}$, $C_{di\ell}$, *and* $D_{di\ell}$ *can be obtained, for each* $i \in \mathbb{K}$, *from*

$$P_{i\ell}(0) - \begin{bmatrix} I \\ 0 \end{bmatrix} X_{i\ell} \begin{bmatrix} I \\ 0 \end{bmatrix}' = \begin{bmatrix} C_{di\ell}' \\ D_{di\ell}' \end{bmatrix} \begin{bmatrix} C_{di\ell}' \\ D_{di\ell}' \end{bmatrix}', \quad \text{(VI.61)}$$

$$\Phi_{i\ell}(T) = \begin{bmatrix} A_{di\ell} & B_{di\ell} \\ 0 & e^{(\lambda_{ii}/2)T} I \end{bmatrix}. \quad \text{(VI.62)}$$

3. *For each* $i \in \mathbb{K}$, *determine the stabilizing solution* $V_{i(\ell+1)} \geq 0$ *and the gain* $L_{i(\ell+1)}$, *by solving the DARE*

$$A_{di\ell}' V_{i(\ell+1)} A_{di\ell} - V_{i(\ell+1)} - L_{i(\ell+1)}' (B_{di\ell}' V_{i(\ell+1)} B_{di\ell} + D_{di\ell}' D_{di\ell}) L_{i(\ell+1)} + C_{di\ell}' C_{di\ell} = 0, \quad \text{(VI.63)}$$

together with $L_{i(\ell+1)} = -(B_{di\ell}' V_{i(\ell+1)} B_{di\ell} + D_{di\ell}' D_{di\ell})^{-1} (B_{di\ell}' V_{i(\ell+1)} A_{di\ell} + D_{di\ell}' C_{di\ell})$. *Set the current value of* $X_{i(\ell+1)} = V_{i(\ell+1)} + X_{i\ell}$ *for each* $i \in \mathbb{K}$.

4. *Set* $(\ell + 1) \to \ell$ *and iterate until* $\|V_\ell\|_2$ *becomes sufficiently small.*

Remark VI.3 *Once again, notice that the* $\mathcal{H}_\infty$ *norm is given by the lowest value of* $\gamma > 0$ *such that the overall problem solved by Algorithm VI.1 remains feasible. As a consequence, to determine the optimal solution in the* $\mathcal{H}_\infty$ *context, a single search on parameter* $\gamma > 0$ *is necessary in the same terms stated in Chapter IV.* □

As already mentioned, the convex problem based on LMIs (VI.23) could conveniently be used instead of the DARE (VI.63). In this alternative solution, matrices $W_i = W_{i(\ell+1)}$ and $Y_i = Y_{i(\ell+1)}$ should be determined in Step 3 of the previous algorithm at each iteration $\ell \in \mathbb{N}$ and for all $i \in \mathbb{K}$. Then, the current value of matrices L_i and X_i would be given by $L_{i(\ell+1)} = Y_{i(\ell+1)}W_{i(\ell+1)}^{-1}$ and $X_{i(\ell+1)} = W_{i(\ell+1)}^{-1} + X_{i\ell}$ for each $i \in \mathbb{K}$ and each $\ell \in \mathbb{N}$. Both problems are asymptotically convergent to the stationary solution of the TPBVP defined by Theorem IV.1. The following theorem is necessary to assure this convergence since an important change is introduced in the control design problem, that is the non constant characteristic of matrices $H_{i\ell}$ for all $i \in \mathbb{K}$ and $\ell \in \mathbb{N}$.

Remark VI.4 *The same iterative procedure from Algorithm VI.1 can be used for the particular cases discussed in the previous section. The main difference is the problem to be solved, which can be expressed in terms of LMIs or in the form of a DARE. For the mode independent case, the problem defined in Corollary VI.1 should be solved; for the limit case $T \to 0^+$, the DARE defined in* (VI.48) *with matrices L_i defined by* (VI.51) *should be considered; and for the deterministic case, the same problem defined in Theorem VI.1 but with only one Markov mode, $N = 1$, should be evaluated.* □

Theorem VI.2 *Assume that problem* (VI.11) *has a bounded solution $(X_i^*, P_i^*(t), L_i^*)$ for all $i \in \mathbb{K}$ and all $t \in [0, T)$. Algorithm VI.1 is uniformly convergent and any two subsequent iterations are such that $X^* \geq X_{(\ell+1)} \geq X_\ell \geq 0$.*

Proof: From the assumption of the existence of a bounded solution, there exist matrices L_i^* and $X_i^* > 0$ for all $i \in \mathbb{K}$ such that the TPBVP of Theorem VI.1 holds. By imposing $P_{i\ell}(0) \geq S_{i\ell}$, inequalities (VI.14) evaluated at its boundaries yield

$$\Phi_{i\ell}(T)'(H'_{i(\ell+1)}S_{i(\ell+1)}H_{i(\ell+1)} - H'_{i\ell}S_{i\ell}H_{i\ell})\Phi_{i\ell}(T) = S_{i(\ell+1)} - P_{i\ell}(0) \tag{VI.64}$$

for each $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. This inequality multiplied to the left by $[I \ L'_{i(\ell+1)}]$, $\ell \in \mathbb{N}$, and to the right by its transpose leads to equation (VI.63), which can be rewritten as

$$\begin{aligned}(A_{di\ell} + B_{di\ell}L_{i(\ell+1)})'(X_{i(\ell+1)} - X_{i\ell})(A_{di\ell} + B_{di\ell}L_{i(\ell+1)}) - (X_{i(\ell+1)} - X_{i\ell})\\ +(C_{di\ell} + D_{di\ell}L_{i(\ell+1)})(C_{di\ell} + D_{di\ell}L_{i(\ell+1)}) = 0\end{aligned} \tag{VI.65}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. Then, $V_{i(\ell+1)} = X_{i(\ell+1)} - X_{i\ell} \geq 0$ is the stabilizing solution for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. By initializing Algorithm VI.1 with $X_{i0} = 0$ for all $i \in \mathbb{K}$, it leads to an interlacing property of matrices X_i for each $i \in \mathbb{K}$:

$$0 = X_{i0} \leq X_{i1} \leq X_{i2} \leq \cdots . \tag{VI.66}$$

Additionally, consider the stabilizing solution characterized by $(X_i^*, P_i^*(t), L_i^*)$ for all $i \in \mathbb{K}$, which satisfies

$$P_i^*(0) = S_i^* = \begin{bmatrix} I \\ 0 \end{bmatrix} X_i^* \begin{bmatrix} I \\ 0 \end{bmatrix}' + \begin{bmatrix} -L_i^* \\ I \end{bmatrix} Z_i^{*-1} \begin{bmatrix} -L_i^* \\ I \end{bmatrix}' \tag{VI.67}$$

whenever $P_i^*(T) = H_i^{*\prime} S_i^* H_i^*$ for all $i \in \mathbb{K}$. Assume that $X_i^* \geq X_{i\ell}$ for all $i \in \mathbb{K}$ and some $\ell \in \mathbb{N}$. Then, inequalities (A.9) applied to the stationary solution produces

$$\begin{aligned} P_i^*(0) - P_{i\ell}(0) &\geq \Phi_{i\ell}(T)'(P_i^*(T) - P_{i\ell}(T))\Phi_{i\ell}(T) \\ &\geq \Phi_{i\ell}(T)'(H_i^{*'} S_i^* H_i^* - H_{i\ell}' S_{i\ell} H_{i\ell})\Phi_{i\ell}(T) \end{aligned} \tag{VI.68}$$

for all $i \in \mathbb{K}$, where the final boundary values are $P_i^*(T) = H_i^{*\prime} S_i^* H_i^*$ and $P_{i\ell}(T) = H_{i\ell}' S_{i\ell} H_{i\ell}$ for all $i \in \mathbb{K}$ and each $\ell \in \mathbb{N}$. Inequalities (VI.68) multiplied to left by $[I \ L_i^{*\prime}]$, $i \in \mathbb{K}$, and to the right by its transpose produces

$$\begin{aligned} (A_{di\ell} + B_{di\ell} L_i^*)' \left(X_i^* - X_{i\ell}\right) (A_{di\ell} + B_{di\ell} L_i^*) - (X_i^* - X_{i\ell}) \\ + (C_{di\ell} + D_{di\ell} L_i^*)' \left(C_{di\ell} + D_{di\ell} L_i^*\right) \leq 0, \end{aligned} \tag{VI.69}$$

which is valid for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. This indicates that matrices $A_{di\ell} + B_{di\ell} L_i^*$ are Schur stable for all $i \in \mathbb{K}$. On the other hand, multiplying (VI.64), once again, to left by $[I \ L_i^{*\prime}]$, $i \in \mathbb{K}$, and to the right by its transpose yields

$$\begin{aligned} (A_{di\ell} + B_{di\ell} L_i^*)' \left(X_{i(\ell+1)} - X_{i\ell}\right) (A_{di\ell} + B_{di\ell} L_i^*) - \left(X_{i(\ell+1)} - X_{i\ell}\right) \\ + (C_{di\ell} + D_{di\ell} L_i^*)' \left(C_{di\ell} + D_{di\ell} L_i^*\right) - (L_i^* - L_{i(\ell+1)})' Z_{i(\ell+1)}^{-1} (L_i^* - L_{i(\ell+1)}) = 0 \end{aligned} \tag{VI.70}$$

for all $i \in \mathbb{K}$ and $\ell \in \mathbb{N}$. Subtracting (VI.70) from (VI.69), it can be verified that

$$\begin{aligned} (A_{di\ell} + B_{di\ell} L_i^*)' \left(X_i^* - X_{i(\ell+1)}\right) (A_{di\ell} + B_{di\ell} L_i^*) - \left(X_i^* - X_{i(\ell+1)}\right) \\ + (L_i^* - L_{i(\ell+1)})' Z_{i(\ell+1)}^{-1} (L_i^* - L_{i(\ell+1)}) \leq 0 \end{aligned} \tag{VI.71}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. Due to matrices $A_{di\ell} + B_{di\ell} L_i^*$ are Schur stable for all $i \in \mathbb{K}$, inequalities (VI.71)imply that $X_i^* \geq X_{i(\ell+1)}$ for all $i \in \mathbb{K}$. Consequently, the algorithm generates a sequence of matrices $\{X_{i\ell}\}_{\ell=0}^{\infty}$ bounded by X_i^* for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. The hypothesis on the existence of a set of solutions X_i^*, bounded for all $i \in \mathbb{K}$, is sufficient to assure that

$$\lim_{\ell \to \infty} X_{i\ell} \to X_i^* \tag{VI.72}$$

for all $i \in \mathbb{K}$. Then, the algorithm monotonically converges to the stationary solution of $(X_i^*, P_i^*(t), L_i^*)$ for all $i \in \mathbb{K}$, concluding thus the proof. ■

Since the proof of Algorithm VI.1 is based on the fundamental inequalities (A.9), the convergence is assured for determining matrices $X_i > 0$ and the gain matrices L_i, $i \in \mathbb{K}$, for both cases: using the DARE (VI.38) or the LMI (VI.23) for each $i \in \mathbb{K}$ in its third step.

Remark VI.5 *In the same way noticed in Algorithm V.1, and already mentioned in this section, each of the particular cases from Section VI.2 can be solved by Algorithm VI.1 provided that each specific problem is considered in Step* 3*. For all special cases, the uniform convergence is also assured.* □

Remark VI.6 *The sequence $V_{i(\ell+1)} - V_{i\ell}$ does not exhibits a monotone behavior. Only $V_{i\ell}$ has a definite signal for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. Hence, it is only expected that $V_{i\ell} \to 0$ as ℓ goes to infinity due to $X_{i\ell} \to X_i^*$ as ℓ goes to infinity for all $i \in \mathbb{K}$. This property becomes evident in the next numerical example.* □

VI.4 Illustrative Numerical Example

The purpose of this section is to put in evidence the numerical behavior of the proposed algorithm and the theoretical results obtained so far. To this end, consider the example already treated in the previous chapters.

Example VI.1 *Consider the MJLS with $N = 2$ and with the state space realization in the form of Example IV.1. The system matrices are given by*

$$F_1 = \begin{bmatrix} 0 & 1 & 0 \\ -4 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; F_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$J_1 = J_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; G_1' = G_2' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Again, matrices H_i, $i \in \{1, 2\}$, are unknown since it contains the gain matrices L_1 and L_2 to be evaluated. Consider also the sampling period $T = 250\ [ms]$ and suppose that the transition rate matrix $\Lambda \in \mathbb{R}^{2\times 2}$ is such that

$$\Lambda = \begin{bmatrix} -0.5 & 0.5 \\ 0.2 & -0.2 \end{bmatrix}$$

and the initial probability is $\pi_0 = [1\ 0]'$. Then, the computed minimum γ_{opt} cost equals 2.12 and is assured by the gain matrices

$$L_{1(opt)} = \begin{bmatrix} -2.3116 & -4.2210 \end{bmatrix}, \; L_{2(opt)} = \begin{bmatrix} -2.3548 & -3.7444 \end{bmatrix}.$$

For this evaluation, it is necessary to consider a line search in $\gamma > 0$ as mentioned in Remark VI.3. Figure VI.1 shows the evolution of the iterative algorithm for $\gamma = \gamma_{opt} = 2.12$. The green curve (with squared markers) shows the convergence of the matrices $X_{i\ell}$ to the stationary value X_i^* represented by $\max_{i=1,2} \|X_{i\ell}\|_2$. The blue curve (with rounded markers) shows the evolution of the stopping criterion $\max_{i=1,2} \|V_{i(\ell+1)}\|_2$, which puts in evidence a fast convergence rate. Notice that, in this case, the convergence of matrices $V_{i\ell}$ to zero are not monotone, as expected and discussed in Remark VI.6.

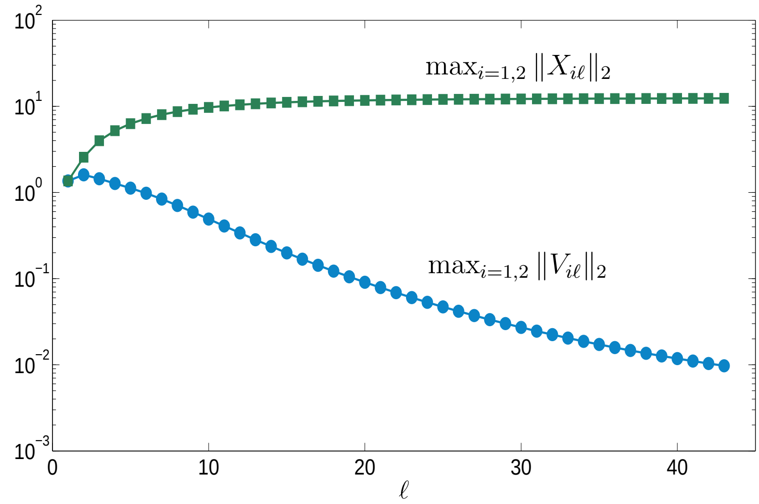

Figure VI.1 – Algorithm evolution for $\gamma_{opt} = 2.12$

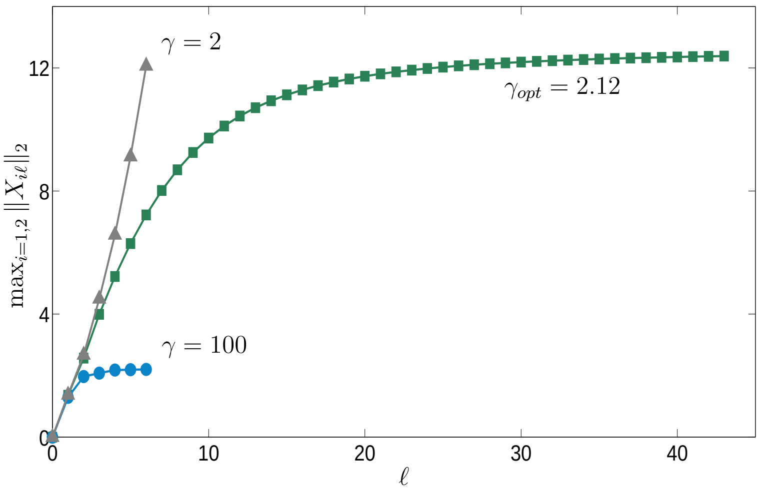

Figure VI.2 – Algorithm evolution for $\gamma = 2$, $\gamma_{opt} = 2.12$ and $\gamma = 100$

In addition, as in Chapter IV, Figure VI.2 shows the convergence of the algorithm for three different values of γ. The convergence occurs within six iterations for $\gamma = 100 > \gamma_{opt}$, while the optimal value is reached after 43 iterations for $\gamma = \gamma_{opt} = 2.12$. For $\gamma = 2 < \gamma_{opt}$, the algorithm diverges because a bounded solution does not exist, as expected. Furthermore, for a γ large enough, the result recovers the values of the matrices L_i, $i = 1, 2$, and the square of $\mathcal{H}_2$ norm obtained by using Algorithm V.1, in Chapter V.1. Indeed, using $\gamma = 1,000$, Algorithm VI.1 converges to $\mathcal{J}_2^* = 2.95$ and to the gain matrices $L_1^* = [0.1786 - 0.5555]$ and $L_2^* = [-0.2970 - 0.8684]$ within six iterations. The minimal difference between this values and those from Example V.1 is due to the difference in the stopping criteria adopted for Algorithms V.1 and VI.1.

A Monte Carlo simulation of $2,000$ samples, which uses the Leon-Garcia's pro-

cedure (see Leon-Garcia (2007)), considering the gain matrices $L_{1(opt)}$ *and* $L_{2(opt)}$ *just calculated provides* $\mathcal{H}_\infty$ *norm of* 2.06. *For this result, even though the worst perturbation is not deterministic, the exogenous input of* $w(t) = sin(\pi t/3)$ *for* $t \in [0, 2]$ $[s]$ *and* $w(t) \equiv 0$ *elsewhere is considered. The frequency for the sinusoidal signal is defined in a previous search such that the worst gain is obtained. The simulation occurs in the time interval* $[0, 12]$ $[s]$. *The difference between the calculated and the simulated* $\mathcal{H}_\infty$ *norm indicates the quality of the proposed method. Moreover, notice that the exogenous input is not the worst perturbation, but the approximation is very close in this particular example. Finally, it seems that there is not a mode independent gain matrix that stabilizes this HMJLS.*

CHAPTER VII

Practical Applications

This chapter is a relevant part of this work since it shows the results obtained in the previous chapters in a practical sense. In order to accomplish this goal, some systems borrowed from the literature are analysed. First, two practical systems are controlled through an NCS, that is, an originally stable mechanical mass-spring-damper system borrowed from the PhD dissertation written by Lutz (2014) and an originally unstable Furuta pendulum extracted from the master's thesis written by Oliveira (2015). Then, an example modified from Costa, Fragoso & Todorov (2013) shows another application of HMJLS: an example on economics.

For examples in Sections VII.1 and VII.2, the general adopted structure is presented in Figure VII.1, where the control signal flows through a network. Among all actual characteristics of the network, packet dropout and bandwidth limitation are simultaneously considered, as already mentioned in Chapter I. In this case, the following tools are used to model each of these network characteristics: Markov chain and sampled-data signals, respectively. Thus, consider the state space realization in the form of

$$\dot{x}(t) = A_{\theta(t)}x(t) + B_{\theta(t)}u(t) + E_{\theta(t)}w(t) \tag{VII.1}$$

$$z(t) = C_{\theta(t)}x(t) + D_{\theta(t)}u(t) \tag{VII.2}$$

evolving from initial conditions $x(0^-) = x(0) = 0$ and $\theta(0^-) = \theta(0) = \theta_0$. This system is subject to the state feedback sampled-data control law $u(t)$ expressed by

$$u \in \mathbb{U} = \{u(t) = L_{\theta(t_k)}x(t_k),\ t \in [t_k, t_{k+1})\ \forall k \in \mathbb{N}\} \tag{VII.3}$$

where $\theta(t_k) \in \mathbb{K}$ represents each of the network states: success or fail of the transmitted signal. The sequence of $\{t_k\}_{k\in\mathbb{N}}$ represents the successive sampling instants of time such that $T = t_{k+1} - t_k$, $k \in \mathbb{N}$, as defined before. Thus, the HMJLS composed by (VII.1)–(VII.2)

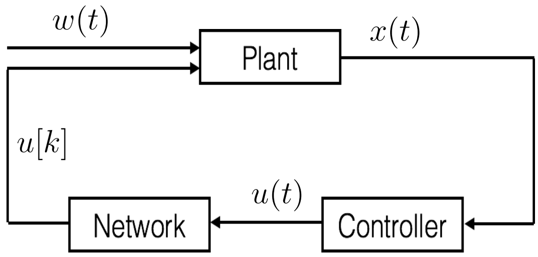

Figure VII.1 – Closed loop structure for Examples VII.1, VII.2.

subject to the control law (VII.3) can be rewritten as

$$\dot{\xi}(t) = \begin{bmatrix} A_{\theta(t)} & B_{\theta(t)} \\ 0 & 0 \end{bmatrix} \xi(t) + \begin{bmatrix} E_{\theta(t)} \\ 0 \end{bmatrix} w(t) \tag{VII.4}$$

$$z(t) = \begin{bmatrix} C_{\theta(t)} & D_{\theta(t)} \end{bmatrix} \xi(t) \tag{VII.5}$$

$$\xi(t_k) = \begin{bmatrix} I & 0 \\ L_{\theta(t_k)} & 0 \end{bmatrix} \xi(t_k^-) \tag{VII.6}$$

for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, and all $\theta(t) \in \mathbb{K}$, which evolves from initial conditions $\xi(0^-) = \xi_0 = 0$ and $\theta(0^-) = \theta(0) = \theta_0$. The time-varying function $\theta = \{(\theta(t), \mathcal{F}_t); t \in \mathbb{R}^+\}$ describes the state of the random variable, which is governed by a continuous-time Markov process characterized by the transition rate matrix $\Lambda \in \mathbb{R}^{N\times N}$ and an initial probability distribution $\pi_0 = \pi(0) = [\pi_{10} \ \cdots \ \pi_{N0}]'$.

Additionally, it is important to notice that all the next examples, whose theoretical background was developed in the previous chapters, are obtained also considering temporal simulations. For that, the method proposed by Leon-Garcia (2007) is adopted. It consists of a Monte Carlo simulation of a time-varying system depending on a given parameter $\theta(t) \in \mathbb{K}$. This parameter spends, in the mode $i \in \mathbb{K}$, a period of time d_i defined by an exponential distribution with mean $1/|\lambda_{ii}|$ and jumps to another state $j \in \mathbb{K}$ according to the probability $\mathcal{P}[\theta(d_i+h) = j|\theta(d_i) = i] = \lambda_{ij}/|\lambda_{ii}|$ for $j \neq i$ and $\mathcal{P}[\theta(d_i+h) = i|\theta(d_i) = i] = 0$ with $h > 0$ arbitrarily small. Additionally, for the $\mathcal{H}_2$ context, an impulse in the exogenous input $w(t)$ at time instant $t = 0$ is simulated by adopting an initial condition $x_0 \neq 0$. On the other hand, for the $\mathcal{H}_\infty$ case, even though the worst perturbation is not deterministic, a sinusoidal signal is applied in $w(t)$, whose frequency is defined in a previous search such that the worst gain is obtained. The value of this cost, in general, is smaller than the actual $\mathcal{H}_\infty$ norm. Clearly, this signal is interrupted when the resultant controlled output $z(t)$ starts oscillating, which means that the transitory period has already been finished.

VII.1 Mass-spring-damper System

In this section, a mass-spring-damper system borrowed from Lutz (2014) is controlled using both approaches developed in previous chapters: the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal con-

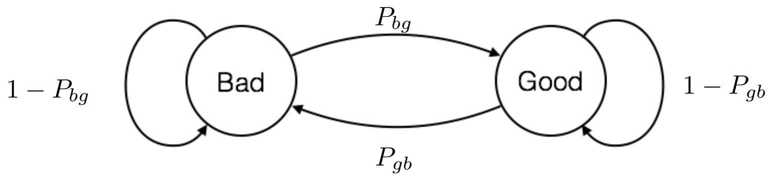

Figure VII.2 – Gilbert model for a simple network modeling.

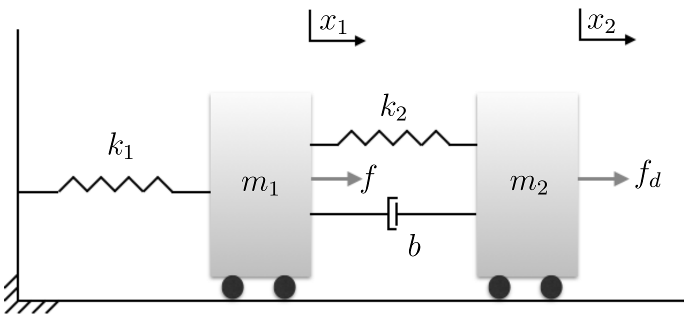

Figure VII.3 – Mass-spring-damper system.

trol design. Additionally, a physical meaning is attributed to the results. The network used in the next example is modeled based on a two-mode Markov chain corresponding to a Gilbert network with data extracted from Lutz (2014). It consists of a first order Markov process which indicates if packets are successfully received (one mode) or not (another mode) as in Figure VII.2. The quantities P_{bg} and P_{gb} are, respectively, the probabilities of the system jumps from mode "Bad" to "Good" and from "Good" to "Bad". This Markov chain is represented by the set of Markov modes $\mathbb{K} = \{1, 2\}$, where "1" stands for the "Bad" mode and "2" for the "Good" one. The transition rate matrix Λ is computed using one of the transition matrices Q defined in Lutz (2014). The chosen one is given by

$$Q = \begin{bmatrix} 1 - P_{bg} & P_{bg} \\ P_{gb} & 1 - P_{gb} \end{bmatrix} = \begin{bmatrix} 0.85 & 0.15 \\ 0.1 & 0.9 \end{bmatrix} \tag{VII.7}$$

with initial distribution such that $\pi_0 = [0.4 \;\; 0.6]'$ and discrete-time period of $h_d = 0.02[s]$, as defined in the same work. Consequently, Λ is generated such that

$$Q = e^{\Lambda h_d}. \tag{VII.8}$$

Example VII.1 *The mass-spring-damper system, shown in Figure VII.3, consists of two friction-less cars with masses* $m_1 = 0.5$ $[kg]$ *and* $m_2 = 1.0$ $[kg]$ *connected with a damper with* $b = 0.2$ $[Ns/m]$ *and two springs such that* $\kappa_1 = 12.0$ $[N/m]$ *and* $\kappa_2 = 7.0$ $[N/m]$*. The force* $u(t) = f(t)$ *is the control input acting in the second car and the force* $w(t) = f_d(t)$ *is* *the exogenous input acting in the first car. The dynamical equation of the system is*

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ (-k_2-k_1)/m_1 & k_2/m_1 & -b/m_1 & b/m_1 \\ k_2/m_2 & -k_2/m_2 & b/m_2 & -b/m_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{bmatrix} w(t). \quad \text{(VII.9)}$$

This mechanical system is connected through a network supposed to have packet dropouts and bandwidth limitation. As already mentioned, a Markov chain with two modes $\mathbb{K} = \{1, 2\}$ *represent packet loss and transmission success, respectively. The transition rate matrix is obtained from* (VII.8)*. The effect of the bandwidth limitation implies in a sampled-data control law given by* (VII.3)*, which is constant inside each time interval* $[t_k, t_{k+1})$ *such that* $T = t_{k+1} - t_k = 200\ [ms]$*. Thus, closing the loop with* (VII.3)*, system* (VII.9) *is rewritten as a hybrid system with augmented matrices* F_i *and* J_i*,* $i = \{1, 2\}$*, described by*

$$F_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ (-k_2-k_1)/m_1 & k_2/m_1 & -b/m_1 & b/m_1 & 0 \\ k_2/m_2 & -k_2/m_2 & b/m_2 & -b/m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$F_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ (-k_2-k_1)/m_1 & k_2/m_1 & -b/m_1 & b/m_1 & 0 \\ k_2/m_2 & -k_2/m_2 & b/m_2 & -b/m_2 & 1/m_2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$J_1 = J_2 = \begin{bmatrix} 0 & 0 & 1/m_1 & 0 & 0 \end{bmatrix}'.$$

Moreover, the controlled output is defined such that

$$z(t) = Cx(t) + Du(t) \quad \text{(VII.10)}$$

for all $t \in \mathbb{R}^+$*. As a consequence, matrices* G_i*,* $i = \{1, 2\}$*, are such that* $G_1 = G_2 = [C\ D]$*. Notice that the packet loos is enforced by setting the last row of the augmented dynamic matrix to zero,* F_1*.*

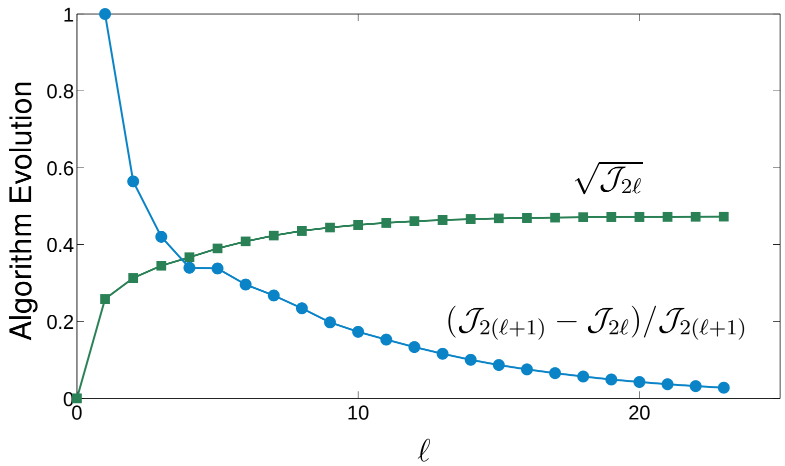

Figure VII.4 – Evolution of Algorithm V.1 to compute the stationary value $\mathcal{J}_2^*$.

$\mathcal{H}_2$ *optimal control design*

The optimal project developed in Chapter V allows to determine a control law in the form of (VII.3) *such that the* $\mathcal{H}_2$ *norm is minimized. For a physical interpretation and to facilitate understanding, consider Figure VII.3 and the performance index* $\mathcal{J}_2$ *that expresses the total dissipated energy of the system, which is minimized while the closed loop system is kept mean square stable. Then,*

$$\mathcal{J}_2 = \int_0^\infty \mathcal{E}\left[b(\dot{x}_1 - \dot{x}_2)^2\right] dt, \tag{VII.11}$$

where the expectation is necessary since system (VII.9)–(VII.10) *is controlled through an NCS. In this case,*

$$C = \begin{bmatrix} 0 & 0 & \sqrt{b} & -\sqrt{b} \end{bmatrix}, \; D = \begin{bmatrix} 0 \end{bmatrix}.$$

The minimum value of the index $\mathcal{J}_2$ *is evaluated by running Algorithm V.1 which converges to* $\sqrt{\mathcal{J}_2^*} = 0.48$ $[J]$ *within* 23 *iterations. This value is assured by the stationary gain matrices* L_1^* *and* L_2^* *such that*

$$L_1^* = \begin{bmatrix} -40.6913 & 19.3200 & 5.5442 & -7.5072 \end{bmatrix},$$
$$L_2^* = \begin{bmatrix} -28.8234 & 13.6792 & 4.9919 & -6.4295 \end{bmatrix}.$$

Figure VII.4 shows the convergence of the proposed method to the optimal result. The green curve (with squared markers) shows the convergence of the $\sqrt{\mathcal{J}_{2\ell}}$ *performance index to the stationary value* $\sqrt{\mathcal{J}_2^*}$*. The blue curve (with rounded markers) shows the convergence of the stopping criterion* $(\mathcal{J}_{2(\ell+1)} - \mathcal{J}_{2\ell})/\mathcal{J}_{2(\ell+1)}$*. Notice that only the convergence of* $\sqrt{\mathcal{J}_{2\ell}}$ *is*

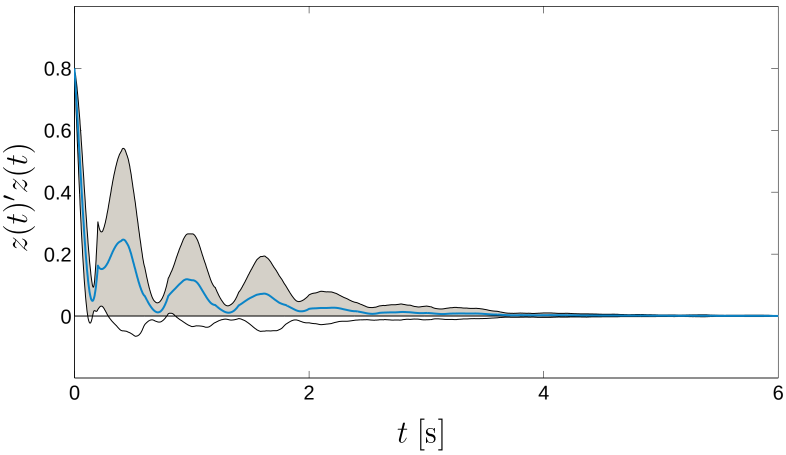

Figure VII.5 – Temporal evolution for the $\mathcal{H}_2$ optimal control.

monotone, as expected.

Additionally, implementing a Monte Carlo simulation of 500 *samples with the stationary gain matrices* L_1^* *and* L_2^**, initial condition numerically equals to* $\xi_0 = J_1 = J_2$*, and exogenous input* $w(t) \equiv 0$*, the value of the computed* $\sqrt{\mathcal{J}_2}$ *performance index is* 0.47 $[J]$*. The relative difference of about* 3% *indicates that the quality of the result obtained by using the proposed method in Chapter V. Figure VII.5 shows the behavior of the temporal evolution of the mass-spring-damper system*[1]*. At this point, changing the gain matrices to*

$$L_1 = \begin{bmatrix} -5.0620 & 6.7414 & -0.5561 & -2.6060 \end{bmatrix},$$
$$L_2 = \begin{bmatrix} -3.7921 & 5.4461 & -0.3098 & -2.5537 \end{bmatrix}$$

for comparison purposes, which also keep the system stability, the temporal simulation gives a total $\sqrt{\mathcal{J}_2}$ *of* 0.86 $[J]$*. This value is greater than the value obtained by using the optimal gain matrices, that is, it confirms that the minimum is obtained for the gain matrices* L_1^* *and* L_2^**, as expected. Moreover, for the mode independent case, the algorithm converges within* 21 *iterations and the gain matrix* $L = [-17.2167 \quad 7.0669 \quad 3.5009 \quad -7.0507]$ *assures the guaranteed cost of* $\sqrt{\mathcal{J}_2} = 0.65$ $[J]$*. This result shows the conservatism associated to the fact that the Markov mode is unknown during the system simulation. By running Algorithm III.1, the* $\sqrt{\mathcal{J}_2}$ *performance index of a system given the feedback control gain matrices* $L_1 = L_2 = L$ *for the mode independent case is* 0.51 $[J]$*, for which the value of* 0.65 $[J]$ *still is an upper bound.*

$\mathcal{H}_\infty$ optimal control design

On the other hand, consider the $\mathcal{H}_\infty$ *scenario. The* $\mathcal{H}_\infty$ *norm can be physically interpreted as a measure of the robustness of the system regarding to an exogenous*

[1] One standard deviation above and bellow the mean trajectory is shown even though the quantity $z(t)'z(t)$ is obviously nonnegative for all $t \geq 0$.

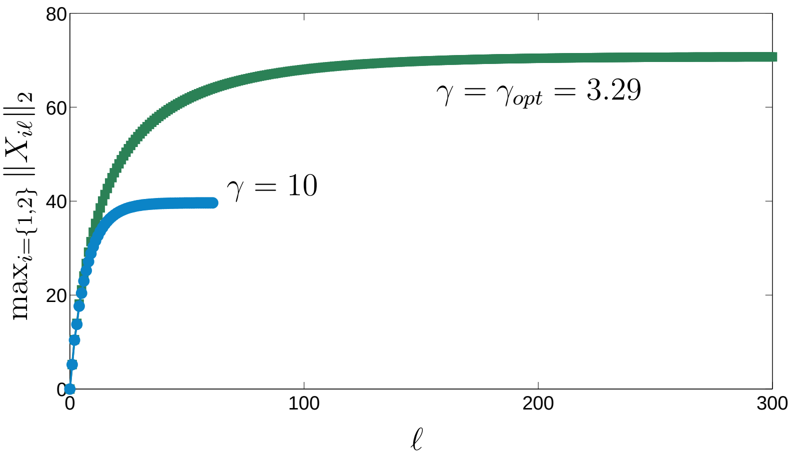

Figure VII.6 – Algorithm VI.1 evolution for $\gamma = \gamma_{opt}$ and $\gamma > \gamma_{opt}$.

input (see the references Zhou, Doyle & Glover (1996) and Colaneri, Geromel & Locatelli (1997)). Hence, consider an exogenous input $w(t) \in \mathbb{L}_2^*$ *applied to the system in the form of the additional force* $w(t) = f_d(t)$ *on mass* m_2 *(see Figure VII.3). As a consequence, the* $\mathcal{H}_\infty$ *project determines the gain matrices* L_1 *and* L_2 *such that the minimum disturbance due to force* f_d *is "felt" by the system. Thus, by considering matrices* C *and* D *such as*

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and performing a line search in parameter γ*, Algorithm VI.1 together with system* (VII.9)–(VII.10) *produces the* $\mathcal{H}_\infty$ *norm* $\gamma_{opt} = 3.29$ *which is ensured by the stationary gain matrices*

$$L_1^* = \begin{bmatrix} -0.4643 & 0.5180 & 0.0772 & -0.6386 \end{bmatrix},$$
$$L_2^* = \begin{bmatrix} -0.4097 & 0.5978 & 0.1429 & -1.0802 \end{bmatrix}.$$

Figure VII.6 shows the convergence of Algorithm VI.1 for a given γ *in terms of the measure of matrices* $X_{i\ell}$*,* $i \in \mathbb{K}$*, since the* $\mathcal{H}_\infty$ *control problem designed in Chapter VI is a feasibility problem and no performance index is evaluated. The green curve (with squared markers) shows the convergence of matrices* $X_{i\ell}$*,* $i = \{1, 2\}$*, to the stationary value through the measure of* $\max_{i=\{1,2\}} \|X_{i\ell}\|_2$ *for* $\gamma = \gamma_{opt}$*, which corresponds to the lowest* γ *such that the DAREs* (VI.63) *are feasible. The blue curve (with rounded markers) shows also the convergence of* $\max_{i=\{1,2\}} \|X_{i\ell}\|_2$ *but this time for* $\gamma = 10$*. The algorithm takes* 299 *iterations to converge for* $\gamma = \gamma_{opt}$ *and* 61 *iterations for* $\gamma = 10 > \gamma_{opt}$*. Clearly, the closer to the feasibility limit the algorithm runs, the slower it becomes in its performance.*

Additionally, consider the case of the mass-spring-damper system being controlled in the classical sense, that is, without the need of a network, which can be enforced by doing $\mathbb{K} = \{1\}$ *and* $T \to 0$*. In practice, consider the period* $T = 0.001$ $[s]$*. Thus, the* *value of $\mathcal{H}_\infty$ norm computed by a linear search is $\gamma = 2.88$ with the associated gain matrix $L = [1.5079\ -0.5546\ 0.3633\ -1.1910]$. Using the Matlab®function norm to compute the $\mathcal{H}_\infty$ norm, this value is 2.87, which verifies the possibility to use the proposed Algorithm VI.1 to compute the classical $\mathcal{H}_\infty$ norm. Moreover, this number makes possible to compare the robustness of the mass-spring-damper system controlled through an NCS or without it. As expected, the system controlled through an NCS is less robust than the same system controlled in a classical way.*

Finally, by imposing $\gamma \to \infty$, the quadratic term of the DRE of Theorem IV.1 vanishes and the TPBVP must solve a DLE instead. In this case, it should be possible to recover the $\mathcal{H}_2$ result. Indeed, by setting $\gamma = 100$ and matrices C and D as in the $\mathcal{H}_2$ analysis, the computed squared value of

$$\int_0^\infty \mathcal{E}\left[z(t)'z(t)\right] dt = \sum_{i\in\{1,2\}} \pi_{i0}\, \boldsymbol{Tr}(J_i'H_i'X_iH_iJ_i) \qquad \text{(VII.12)}$$

is $0.48\ [J]$, which is very close to the same value previously calculated using Algorithm V.1.

Remark VII.1 *From Example VII.1, an interesting property of the $\mathcal{H}_2$ optimal control problem is confirmed. Indeed, the $\mathcal{H}_2$ optimal gain L_i^*, $i \in \mathbb{K}$, associated to the packet dropout mode is not null because the final cost depends on this gain through the matrix H_i present in the initial condition, which is used to reflect the impulsive exogenous input. This fact is not observed in the $\mathcal{H}_\infty$ optimal control problem because the initial condition is zero.* □

VII.2 Furuta Pendulum

Due to the existence of a damper in the previous example, the system mass-spring-damper is originally stable. In order to analyse the possibility to control an originally unstable system using the controllers designed by the methods proposed in this work, consider a Furuta Pendulum. It is a rotary inverted pendulum borrowed from Oliveira (2015). Once more, both contexts are considered: the $\mathcal{H}_2$ and the $\mathcal{H}_\infty$ state-feedback sampled-data control projects developed in Chapters V and VI. As mentioned before, the Markov chain and the sampled-data control are used to model the network characteristics of packet dropouts and bandwidth limitation, respectively and simultaneously.

The network considered in Oliveira (2015) is a Gilbert-Elliot network, which is a 4-state Markov chain. In this case, for each state of the network transmission, “Good" or “Bad", there are two different associated probabilities: one indicating a “Successful" transmission and the other a “Failed" transmission. This model implies that packet dropouts may occur even in a “Good" state. The Gilbert-Elliot model is represented by Figure VII.7, where P_{gb} is the probability of the transition from the “Good" state to the “Bad" state and P_{bg} is the transition from the “Bad" state to the “Good" state. In the same way, P_{fg} and P_{fb} are the

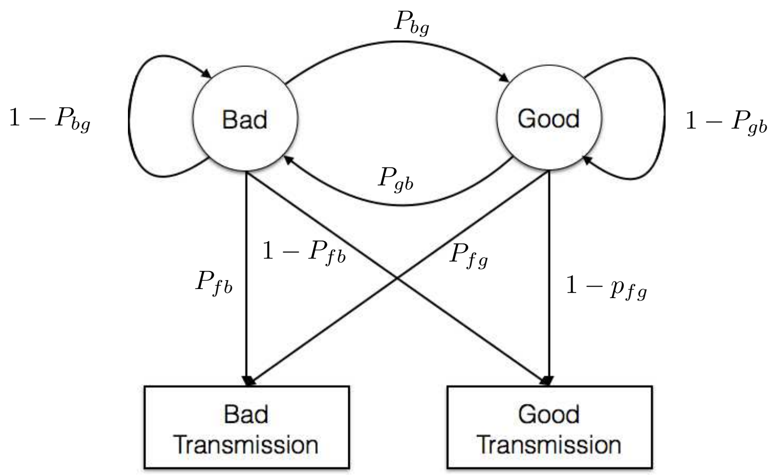

Figure VII.7 – Gilbert-Elliot Model for a simple Network modeling.

probabilities of a “Failed" transmission in the “Good" and in the “Bad" state, respectively. Thus, the Gilbert-Elliot transmission model is such that $\mathbb{K} = \{1, 2, 3, 4\}$, where “$1$" stands for the “Good" state with a “Failed” transmission, “2" for the “Good" state with “Successful" transmission, “3" for the “Bad" state with “Failed" transmission, and “4" for the “Bad" state with “Successful" transmission. Thus, the probability transition matrix Q defined in Oliveira (2015) is given by

$$Q = \begin{bmatrix} (1-P_{gb})P_{fg} & (1-P_{gb})(1-P_{fg}) & P_{gb}P_{fb} & P_{gb}(1-P_{fb}) \\ (1-P_{gb})P_{fg} & (1-P_{gb})(1-P_{fg}) & P_{gb}P_{fb} & P_{gb}(1-P_{fb}) \\ P_{bg}P_{fg} & P_{bg}(1-P_{fg}) & (1-P_{bg})P_{fb} & (1-P_{bg})(1-P_{fb}) \\ P_{bg}P_{fg} & P_{bg}(1-P_{fg}) & (1-P_{bg})P_{fb} & (1-P_{bg})(1-P_{fb}) \end{bmatrix}$$
$$= \begin{bmatrix} 0.0348 & 0.8352 & 0.0377 & 0.0923 \\ 0.0348 & 0.8352 & 0.0377 & 0.0923 \\ 0.0100 & 0.2400 & 0.2175 & 0.5325 \\ 0.0100 & 0.2400 & 0.2175 & 0.5325 \end{bmatrix}. \quad \text{(VII.13)}$$

The initial distribution is such that $\pi_0 = [0.0263\ 0.6316\ 0.0992\ 0.2429]'$ and the discrete-time period $h_d = 0.02[\mathrm{s}]$, as defined in the same work. The relationship (VII.8) does not provide the transition rate matrix Λ because matrix Q has two null eigenvalues. For this reason, the first order approximation $\Lambda = (Q - I)/h_d$ has been adopted in this case.

Example VII.2 *This system consists of one motor connected to a rotational rod, which is connected to a pendulum. Thus, the overall system has two degrees of freedom, that is, the angles α and ϕ. This means that this system is sub-actuated. The main objective of the control action is to take the pendulum to the open-loop unstable equilibrium point. Figure VII.8 shows the pendulum system. According to Oliveira (2015), the dynamic equation for this system is*

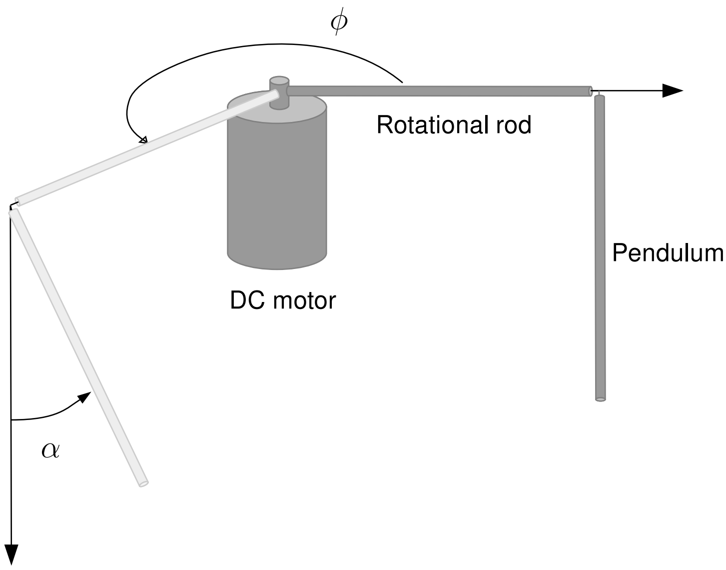

Figure VII.8 – The Furuta pendulum and state variables.

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 34.16 & -18.62 & -0.035 \\ 0 & 76.74 & -17.96 & -0.079 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 18.31 \\ 17.65 \end{bmatrix} u(t), \quad \text{(VII.14)}$$

where $x = [\phi\ \alpha\ \dot{\phi}\ \dot{\alpha}]'$ and $u(t)$ is the voltage applied to the DC motor. In order to prevent the nonlinear behavior of the motor saturation, the matrices of the controlled output and the initial condition have been empirically defined by Oliveira (2015) such as

$$z(t) = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1.2 \end{bmatrix} u(t)$$

$$x_0 = \begin{bmatrix} 0.0175 & 0.2618 & 0 & 0 \end{bmatrix}', \quad \text{(VII.15)}$$

where the initial condition physically corresponds to $\alpha = 15°$ and $\phi = 1°$.

$\mathcal{H}_2$ optimal control design

For the $\mathcal{H}_2$ context, consider matrix E equals to the initial condition x_0. The sampling period adopted is $T = h_d = 0.02$ $[s]$. Hence, running the Algorithm V.1, the computed $\mathcal{J}_2$ performance index is 10.09 and the stationary gain matrices are such that

$$L_1^* = \begin{bmatrix} 1.5947 & -16.1510 & 1.6824 & -2.2731 \end{bmatrix},$$

$$L_2^* = \begin{bmatrix} 3.5069 & -34.6228 & 3.6148 & -4.8762 \end{bmatrix},$$

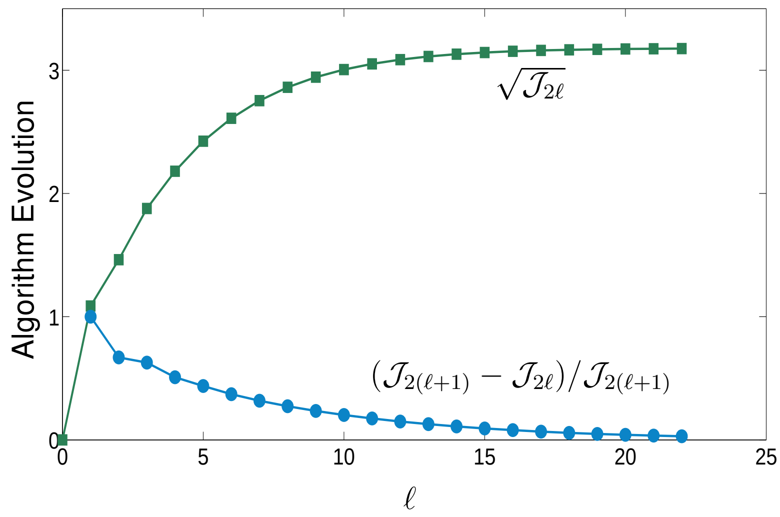

Figure VII.9 – Evolution of Algorithm V.1 to compute the stationary value $\mathcal{J}_2^*$.

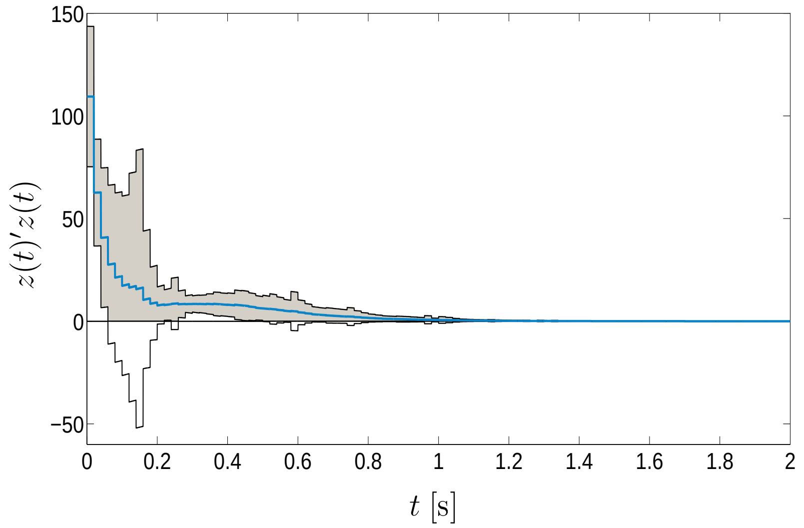

Figure VII.10 – Temporal evolution for the $\mathcal{H}_2$ optimal control.

$$L_3^* = \begin{bmatrix} 1.4452 & -15.2949 & 1.5871 & -2.1501 \end{bmatrix},$$

and

$$L_4^* = \begin{bmatrix} 3.5244 & -37.2767 & 3.8685 & -5.2402 \end{bmatrix}.$$

Algorithm V.1 has taken 23 iterations to converge and its evolution is shown in Figure VII.9. The green curve (with squared markers) shows the convergence of the $\sqrt{\mathcal{J}_{2\ell}}$ performance index to the stationary value $\sqrt{\mathcal{J}_2^*}$. The blue curve (with rounded markers) shows the convergence of the stopping criterion $(\mathcal{J}_{2(\ell+1)} - \mathcal{J}_{2\ell})/\mathcal{J}_{2(\ell+1)}$. Using matrices L_i^*, $i = \{1, 2, 3, 4\}$ to implement a Monte Carlo simulation with 500 samples, the computed $\mathcal{J}_2$ index is 10.46, which confirms the result just calculated. Figure VII.10 shows the temporal evolution.

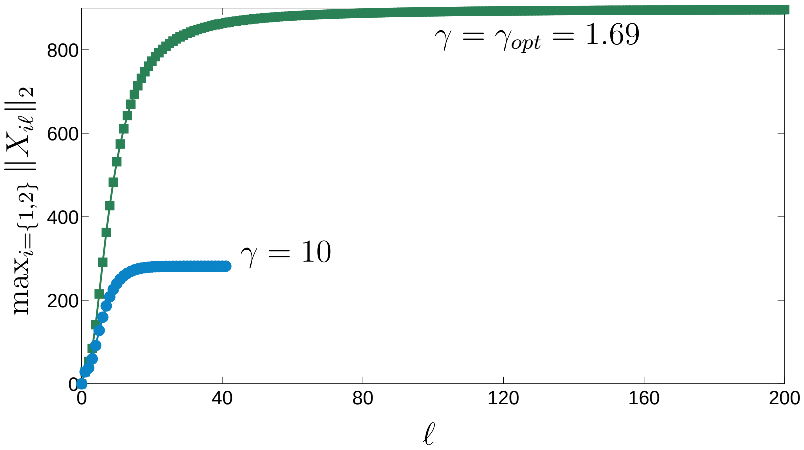

Figure VII.11 – Algorithm VI.1 evolution for $\gamma = \gamma_{opt}$ and $\gamma > \gamma_{opt}$.

$\mathcal{H}_\infty$ *optimal control design*

The purpose of the $\mathcal{H}_\infty$ analysis is to obtain the optimal sampled-data control law that minimizes the maximum energy gain from the exogenous input $w(t)$ to the controlled output $z(t)$. For this purpose, the matrices E_i from the dynamic equation (VII.1)*, $i \in \{1,2,3,4\}$, are chosen in order to introduce the perturbation on the control input $u(t)$ in the same way defined by Oliveira (2015). This means doing $E_i = B_i$ for all $i \in \{1,2,3,4\}$. The initial condition is $x_0 = 0$. Running Algorithm VI.1 with $T = 0.02$ $[s]$, the $\mathcal{H}_\infty$ norm is $\gamma_{opt} = 1.69$ assured by the gain matrices*

$$L_1^* = \begin{bmatrix} 3.7286 & -58.1047 & 5.8662 & -8.1004 \end{bmatrix},$$
$$L_2^* = \begin{bmatrix} 5.1835 & -77.7485 & 7.8677 & -10.8466 \end{bmatrix},$$
$$L_3^* = \begin{bmatrix} 4.2276 & -71.0665 & 7.1438 & -9.8941 \end{bmatrix},$$
$$L_4^* = \begin{bmatrix} 5.4428 & -92.1684 & 9.2612 & -12.8303 \end{bmatrix}.$$

Figure VII.11 shows the algorithm evolution for the $\mathcal{H}_\infty$ analysis. The graphic shows the convergence of the matrices $X_{i\ell}$ to the stationary value X_i^* through the measure of the maximum value of $\|X_{i\ell}\|_2$ for $i = \{1,2,3,4\}$. The green curve (with squared markers) corresponds to the evolution for the optimal $\gamma = \gamma_{opt} = 1.69$, which takes 200 iterations to converge. The blue curve (with rounded markers) shows the same evolution for a larger parameter $\gamma = 10 > \gamma_{opt}$, which takes 41 iterations to reach the stationary gain.

Implementing a Monte Carlo simulation of 500 samples gives the $\mathcal{H}_\infty$ norm of 1.52. Notice that the exogenous input is set as the sinusoidal signal $w(t) = sin(\omega t)$ with frequency $\omega = \pi/3$ $[rad/s]$ determined by simple inspection, which is applied to the system during 2 $[s]$. The effect of turning off the exogenous perturbation at $t = 2$ $[s]$ is clearly seen in Figure VII.12. The value of the $\mathcal{H}_\infty$ norm obtained in both cases are not exactly the same, which means that $w(t)$ is not the worst exogenous perturbation. However, it can provide

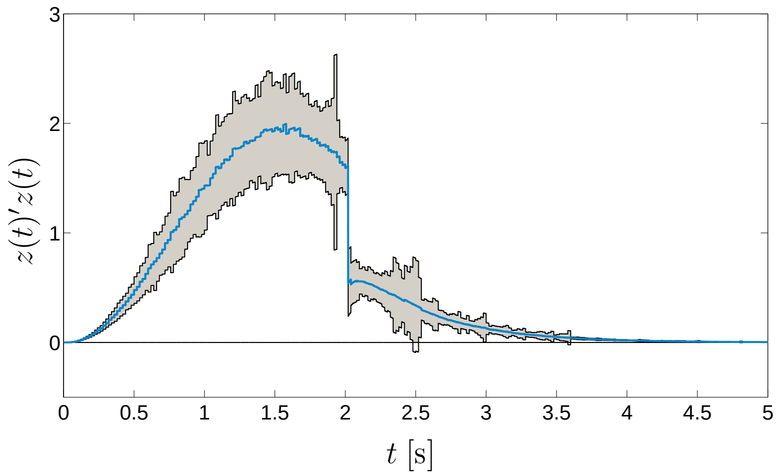

Figure VII.12 – Temporal evolution of pendulum system for the $\mathcal{H}_\infty$ optimal control.

a good felling about the real value of the simulated performance index. As expected, the associated $\mathcal{H}_\infty$ cost is bound above by $\gamma_{opt} = 1.69$ in this case.

VII.3 Economics

It is interesting to notice that the theory developed in the previous chapters is a general result in the sense that it can be applied to other contexts. This is the case, for example, of the macroeconomic model of the U.S.A. national economy as described by Blair & Sworder (1975) and used by Costa, Fragoso & Todorov (2013). Due to many economic variables are subject constantly to exogenous variations, the continuous-time model is more suitable to define this kind of systems than the discrete-time ones even though the measured data is sampled, as pointed out by Blair & Sworder (1975). In this sense, in the referred work, the authors developed a continuous-time Markov jump model for economic systems and applied it to the Samuelson's multiplier-accelerator model. After that, Costa, Fragoso & Todorov (2013) have used it in a continuous-time MJLS. The next example uses data borrowed from the last work. Clearly, a sampled-data control design is suitable for that example since the decision and monetary policy on this kind of model and data is not continuous-time changed, as mentioned also by Blair & Sworder (1975).

Example VII.3 *The system is a 3-mode Markov chain, where "1" corresponds to the "normal" operation, "2" to the "boom" operation, and "3" to the "slump" operation. The system matrices are given in Costa, Fragoso & Todorov (2013). Matrices A_i and B_i are*

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -0.545 & 0.626 \\ 0 & -1.570 & 1.465 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 \\ -0.283 \\ 0.333 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -0.106 & 0.087 \\ 0 & -3.810 & 3.861 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 0 \\ 0.087 \end{bmatrix},$$

and

$$A_3 = \begin{bmatrix} 1.80 & -0.3925 & 4.52 \\ 3.14 & 0.100 & -0.28 \\ -19.06 & -0.148 & 1.56 \end{bmatrix}, \; B_3 = \begin{bmatrix} -0.064 \\ 0.195 \\ -0.080 \end{bmatrix}.$$

Notice that the subsystems 1 and 2 are not stabilizable. Moreover, subsystem 3 is open-loop unstable. The output and input matrices C_i, D_i, and E_i are

$$C_1 = C_2 = C_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; D_1 = D_2 = D_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

and

$$E_1 = E_2 = E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which indicates equal importance of each state in the composition of the final cost. The initial distribution π_0 and the transition rate matrix Λ are, respectively, $\pi_0 = [1/3\ 1/3\ 1/3]'$ and

$$\Lambda = \begin{bmatrix} -0.53 & 0.32 & 0.21 \\ 0.50 & -0.88 & 0.38 \\ 0.40 & 0.13 & -0.53 \end{bmatrix}.$$

Moreover, the time basis of this system is annual according to Blair & Sworder (1975).

$\mathcal{H}_2$ *optimal control design*

The $\mathcal{H}_2$ optimal control design determined by applying Algorithm V.1 is such that, for a monthly sampling interval $T = t_{k+1} - t_k = 1/(12)\ [year]$, it takes 15 iterations to obtain the stationary value of the performance index $\sqrt{\mathcal{J}_2^*} = 56.51$, assured by the optimal gain matrices

$$L_1^* = \begin{bmatrix} 0.6496 & 11.2134 & -17.3529 \end{bmatrix},$$
$$L_2^* = \begin{bmatrix} -0.7714 & 65.0687 & -70.2546 \end{bmatrix},$$
$$L_3^* = \begin{bmatrix} 61.5123 & -10.0325 & 22.2275 \end{bmatrix}.$$

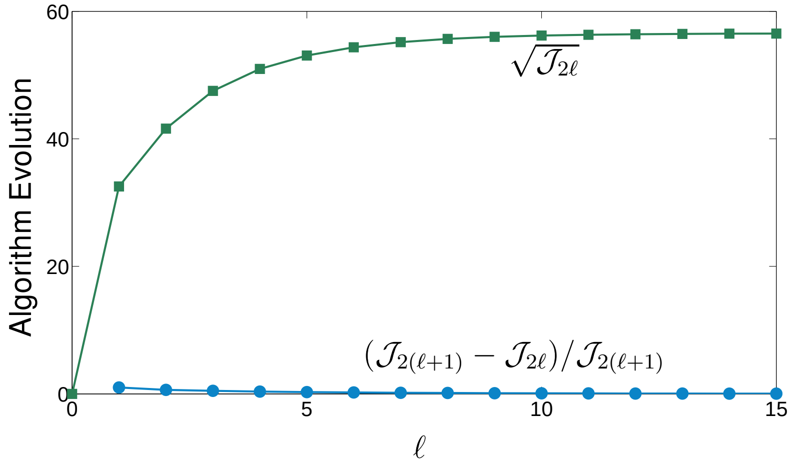

Figure VII.13 – Evolution of Algorithm V.1 to compute the stationary value $\mathcal{J}_2^*$.

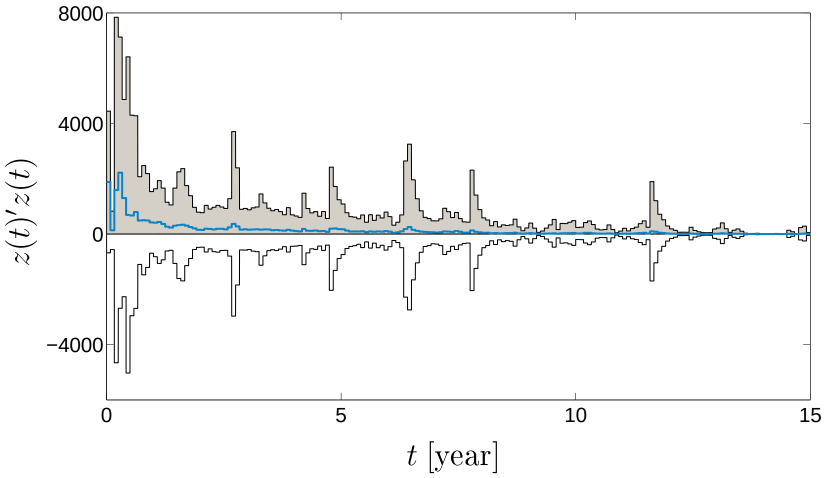

Figure VII.14 – Temporal evolution for the $\mathcal{H}_2$ optimal control.

Figure VII.13 shows the fast convergence of the algorithm for this system. The green curve (with squared markers) is the convergence of the $\sqrt{\mathcal{J}_{2\ell}}$ *index with the iteration* ℓ*. The blue curve (with rounded markers) shows how fast this convergence occurs. For the temporal simulation, the initial state* $\xi_0 = \sum_{l=\{1,2,3\}}[E_i' e_l \; 0]' = [1 \; 1 \; 1 \; 0]'$ *for each* $i = \{1, 2, 3\}$ *is considered. Figure VII.14 shows the Monte Carlo simulation with* 500 *samples performed inside the time interval* $[0, 15]$ $[year]$*. The blue curve is the mean value of the index* $z(t)'z(t)$ *and the shaded area corresponds to one standard deviation. The value of the computed* $\sqrt{\mathcal{J}_2^*}$ *performance index in this case is* 46.32*. This value is smaller than the theoretical optimal one because the simulation horizon considered is not enough for complete stabilization. Due to the two first closed-loop unstable subsystems a large scattering around the mean value is found until the system actually converges. In other words, this implies that the economical system under consideration is very susceptible to variations when a white noise is introduced.*

Considering a minute-to-minute sampling period, that is $T = 1/(12 \times 30 \times 24 \times$

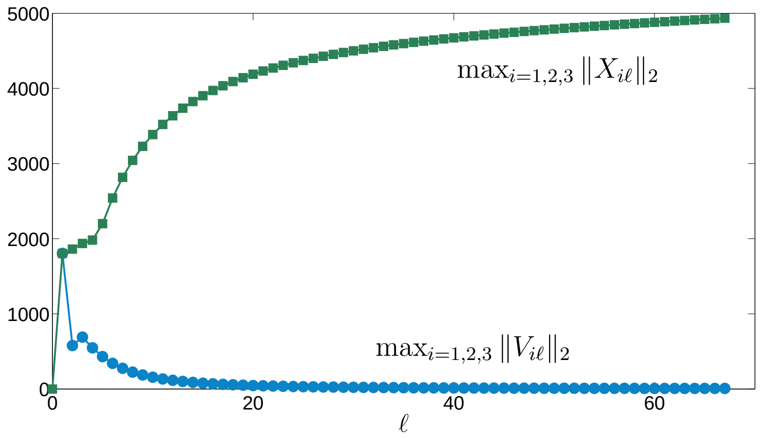

Figure VII.15 – Algorithm evolution for $\gamma_{opt} = 199.45$.

$60)$ $[year]$, *the result from Costa, Fragoso & Todorov (2013) is recovered since the optimal cost* $\sqrt{\mathcal{J}_2^*} = 48.50$ *and the optimal gain matrices*

$$L_1^* = \begin{bmatrix} 2.0349 & 14.5189 & -23.5896 \end{bmatrix},$$
$$L_2^* = \begin{bmatrix} 1.0192 & 73.0969 & -78.7585 \end{bmatrix},$$
$$L_3^* = \begin{bmatrix} 93.6574 & -11.4896 & 11.6732 \end{bmatrix}$$

are recovered. This fact illustrates that the method developed so far reproduces the pure MJLS as a particular case. This important aspect has been pointed out in Section V.2.

$\mathcal{H}_\infty$ *optimal control design*

Using again the period of $T = 1/12$ $[year]$, *running Algorithm VI.1, and performing a line search in the parameter* γ, *the* $\mathcal{H}_\infty$ *norm is* 199.45. *This norm is assured by the stationary optimal gain matrices*

$$L_1^* = \begin{bmatrix} 0.9739 & 11.1716 & -17.5518 \end{bmatrix},$$
$$L_2^* = \begin{bmatrix} -0.9938 & 65.5330 & -70.8408 \end{bmatrix},$$
$$L_3^* = \begin{bmatrix} 65.4855 & -10.0509 & 23.4405 \end{bmatrix}.$$

Figure VII.15 shows the convergence of Algorithm VI.1. The green curve (with squared markers) shows the convergence of the matrices $X_{i\ell}$ *to the stationary value* X_i^* *through the measure of* $\max_{i=1,2,3} \|X_{i\ell}\|_2$. *The blue curve (with circular markers) shows the evolution of the stopping criterion* $\max_{i=1,2,3} \|V_{i(\ell+1)}\|_2$, *which puts in evidence a fast convergence rate. The algorithm takes* 67 *iterations to converge. In complement of reference Costa, Fragoso & Todorov (2013), the minimum value of the* $\mathcal{H}_\infty$ *norm allows to determine* *an upper bound to the parametric uncertainty, namely* $\|\Delta\|_\infty < 1/\gamma_{opt} \approx 0.0050$*, such that* $w = \Delta z$ *preserves the closed-loop system mean square stability. As commented before, this calculation makes clear that this macroeconomic model is extremely sensitive to parameter variations.*

These practical applications show that the developed methods are suitable and useful to sampled-data control design in the context of Markov jump linear systems. The comparison with other similar results is limited due to the lack in the current literature on this topic. However, it has been possible to validate the proposed procedures by comparing the results obtained here with those obtained using other methods in some particular cases.

Conclusions

This work was motivated by networked control systems. In this case the network was imperfect environment that took into account bandwidth limitations and transmission losses due to packet dropout. These characteristics have been modelled by an adequate Markov chain and sampled-data signals. The theory necessary to handle these mathematical models that generalizes the ones of classical Markov jump and sampled-data control systems was developed. Hence, this work has determined optimal control laws in the specific context of a sampled-data control applied to Markov jump linear systems. The optimality lies on the possibility of rewriting the original system in a very specific hybrid formulation as well as on the possibility of deriving necessary and sufficient conditions to obtain the exact value of some considered performance index, which can lead to the exact value of the associated norm. In this case, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms have been considered. The necessary and sufficient conditions are based on a TPBVP composed by coupled differential Lyapunov or Riccati equations depending on the context considered: $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norms, respectively.

Moreover, it has been also determined the optimal sampled-data control law for each of the mentioned cases. In order to emphasize the feasibility of the theoretical results, algorithms have been proposed and proved to be globally convergent. Moreover, all numerical and practical examples show that the proposed method is suitable for any MJLS subject to a state feedback sampled-data control. Additionally, the convergence of these algorithms leads to a complex in building but fast in running method. Although the proposed theory has been developed in a very specific context, many other outcomes can be derived from this work. This is the case, for instance, of expanding the idea to the nonlinear case since all TPBVP addressed here are based on the Hamilton-Jacobi-Bellman equation. Certainly, a much deeper mathematical analysis should be performed. This opens many possibilities to develop sampled-data control for MJLS, for deterministic systems, or even for nonlinear systems since all cases can be enclosed by the last one. Future reserach effort will also deal with topics related to filtering and dynamic output feedback design.

Another important point is concerned about the hybrid approach to model the feedback system. This formulation is essential for the existence of the optimal control. Analogously, it seems to be possible to derive a complete filter, also optimal, for which most probably the separation principle holds. Furthermore, it is important to notice that the results from the sampled-data control applied to MJLS enclose the deterministic sampled-data optimal control, as shown in the examples throughout this dissertation. This is also the case of the pure MJLS and the mode independent control law. The last one is possible due to the convex formulation developed in Chapters V and VI, which are expressed by LMIs and consequently can be solved by the computational tools available to date.

Finally, some papers derived from this work are listed in the sequel. All of them addresses the state feedback sampled-data control design problem with uniform transmission interval.

1) J. C. Geromel, and G. W. Gabriel, “Optimal $\mathcal{H}_2$ State Feedback Sampled-data Control Design of Markov Jump Linear Systems”, *Automatica*, vol. 54, pp. 182–188, 2014.

2) G. W. Gabriel, M. Souza, e J. C. Geromel, “Controle $\mathcal{H}_2$ Amostrado de Sistemas Lineares com Saltos Markovianos via Realimentação de Estados”, *Anais do XX Congresso Brasileiro de Automática*, pp. 739–746, 2014.

3) G. W. Gabriel, M. Souza, and J. C. Geromel, “$\mathcal{H}_2$ State Feedback Sampled-Data Control for Markov Jump Linear Systems”, *Proceedings of the 53rd IEEE Conference on Decision and Control*, pp. 4355–4360, 2014.

4) G. W. Gabriel, J. C. Geromel, and K. M. Grigoriadis, “Optimal $\mathcal{H}_\infty$ State Feedback Sampled-data Control Design for Markov Jump Linear Systems”, *submitted*, 2015.

5) G. W. Gabriel, J. C. Geromel, and K. M. Grigoriadis, “Optimal $\mathcal{H}_\infty$ State Feedback Sampled-Data Control of Markov Jump Linear Systems”, *2016 European Control Conference*, pp. 2489–2494, 2016.

6) G. W. Gabriel, e J. C. Geromel, “Teoria Unificada de Sistemas de Controle Amostrado”, *submitted*, 2016.

7) G. W. Gabriel, and J. C. Geromel, “Unified Approach of Sampled-Data Control Systems”, *submitted*, 2016.

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Appendix

APPENDIX A

Mathematical Analysis of the Coupled DRE

This appendix is devoted to present some mathematical analysis related to the coupled DRE in the form of

$$\dot{P}_i(t) + F_i' P_i(t) + P_i(t) F_i + \gamma^{-2} P_i(t) J_i J_i' P_i(t) + \sum_{j \in \mathbb{K}} \lambda_{ij} P_j(t) + G_i' G_i = 0, \tag{A.1}$$

which is defined in the time interval $[0, T)$ with a given final condition $P_i(T) \geq 0$ for each $i \in \mathbb{K}$. The main reason behind this is to solve the TPBVP of Theorem IV.1 to obtain the value of $\mathcal{H}_\infty$ norm by means of the Algorithm IV.1. Notice that the coupled DREs in (A.1) are different (with a minus signal in the quadratic term) from those studied in Wonham (1968) and Costa, Fragoso & Todorov (2013), for which the existence and uniqueness of a bounded solution $P_i(t) \geq 0$ is assured for all $i \in \mathbb{K}$. On the other hand, only results from Theorem II.4 are available for the coupled DRE (A.1), what justifies the next analysis.

First of all, assume that $\gamma > 0$ is sufficiently large in order to assure that a unique solution such that $0 \leq P_i(t) \leq cI$ for all $i \in \mathbb{K}$, all $t \in [0, T)$, and some finite scalar $c > 0$ exists. Indeed, due to the continuity of the solution with respect to $\gamma > 0$, this hypothesis is enough to assure the existence of a positive semi-definite solution of (A.1) because for γ large enough the coupled DRE becomes a coupled DLE, already solved in Chapter III.

Since equations (A.1) do not admit an explicit solution, an iterative procedure becomes necessary to solve it. For the purpose of constructing such a procedure, define the sequence of matrices $P_{i\ell}(t)$ such that

$$\begin{aligned} \dot{P}_{i\ell} + F_i' P_{i\ell} + P_{i\ell} F_i + \gamma^{-2} P_{i(\ell-1)} J_i J_i' P_{i\ell} + \gamma^{-2} P_{i\ell} J_i J_i' P_{i(\ell-1)} \\ -\gamma^{-2} P_{i(\ell-1)} J_i J_i' P_{i(\ell-1)} + \sum_{j \in \mathbb{K}} \lambda_{ij} P_{j\ell} + G_i' G_i = 0 \end{aligned} \tag{A.2}$$

subject to the final boundary condition $P_{i\ell}(T) = P_i(T) \geq 0$ for all $i \in \mathbb{K}$. Notice that the fixed point of these sequences provides the exact solution of (A.1). Moreover, if the sequence $\{P_{i\ell}(t)\}_{\ell=0}^{\infty}$ is bounded and monotonically non-decreasing from the initial iteration $P_{i0}(t) \equiv 0$ for all $i \in \mathbb{K}$, then it can be concluded that it converges to the fixed-point $\lim_{\ell \to \infty} P_\ell(t) = P^*(t)$, see Lemma 2.17, page 24 of reference Costa, Fragoso & Todorov (2013).

Indeed, consider the identity

$$P_{i(\ell-1)}J_iJ_i'P_{i(\ell-1)} = P_{i\ell}J_iJ_i'P_{i\ell} - P_{i\ell}J_iJ_i'(P_{i\ell} - P_{i(\ell-1)}) - (P_{i\ell} - P_{i(\ell-1)})J_iJ_i'P_{i\ell} + (P_{i\ell} - P_{i(\ell-1)})J_iJ_i'(P_{i\ell} - P_{i(\ell-1)}) \quad \text{(A.3)}$$

and plug this result into (A.2). Then,

$$\dot{P}_{i\ell} + F_i'P_{i\ell} + P_{i\ell}F_i + \gamma^{-2}P_{i\ell}J_iJ_i'P_{i\ell} - \gamma^{-2}(P_{i\ell} - P_{i(\ell-1)})J_iJ_i'(P_{i\ell} - P_{i(\ell-1)}) + \sum_{j\in\mathbb{K}}\lambda_{ij}P_{j\ell} + G_i'G_i = 0 \quad \text{(A.4)}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. Subtracting the ℓ-th iteration of (A.4) from the $(\ell+1)$-th iteration of (A.2) and defining $\Delta_{i(\ell+1)}(t) = P_{i(\ell+1)}(t) - P_{i\ell}(t)$ and

$$M_{i\ell}(t) = F_i + \frac{1}{2}\lambda_{ii}I + \gamma^{-2}J_iJ_i'P_{i\ell}(t) \quad \text{(A.5)}$$

for each $i \in \mathbb{K}$, each $(\ell) \in \mathbb{N}$, and all $t \in [0, T)$ yields

$$\dot{\Delta}_{i(\ell+1)} + M_{i\ell}'\Delta_{i(\ell+1)} + \Delta_{i(\ell+1)}M_{i\ell} + \sum_{j\neq i\in\mathbb{K}}\lambda_{ij}\Delta_{j(\ell+1)} + \gamma^{-2}\Delta_{i\ell}J_iJ_i'\Delta_{i\ell} = 0. \quad \text{(A.6)}$$

Equation (A.6) is a coupled DLE valid for all $i \in \mathbb{K}$, all $\ell \in \mathbb{N}$, and all $t \in [0, T)$ and it is subject to final boundary conditions $\Delta_{i(\ell+1)}(T) = P_{i(\ell+1)}(T) - P_{i\ell}(T) = 0$ for all $i \in \mathbb{K}$ and $\ell \in \mathbb{N}$. Its solution is such that $\Delta_{i(\ell+1)}(t) \geq 0$ for all $i \in \mathbb{K}$, all $t \in [0, T)$, and each $\ell \in \mathbb{N}$ due to Theorem II.4. This implies that $P_{i(\ell+1)}(t) \geq P_{i\ell}(t) \geq 0$ for all $i \in \mathbb{K}$, all $\ell \in \mathbb{N}$, and all $t \in [0, T)$. Moreover, adopting $\Delta_i^*(t) = P_i^*(t) - P_{i\ell}(t)$ and the same algebraic manipulations as before, it can be verified that $P_i^*(t) \geq P_{i\ell}(t) \geq 0$ for all $i \in \mathbb{K}$, all $t \in [0, T)$, and all $\ell \in \mathbb{N}$. This indicates that this matrix sequence converges to the positive semi-definite solution of (A.1), if one exists.

Now, another situation of interest is revisited. Suppose that $P_{i(\ell+1)}(T) \geq P_{i\ell}(T) \geq 0$ holds for two subsequent iterations. Evaluating the solution of (A.6) at the beginning of the time interval $[0, T)$, yields

$$\Delta_{i(\ell+1)}(0) = \Phi_{i\ell}(T)'\Delta_{i(\ell+1)}(T)\Phi_{i\ell}(T) + U_{i\ell}(T), \quad \text{(A.7)}$$

where $U_{i\ell}(T)$ is a positive semi-definite functional defined by

$$U_{i(\ell+1)}(T) = \int_0^T \Phi_{i\ell}(\tau)'\Big(\sum_{j\neq i\in\mathbb{K}}\lambda_{ij}\Delta_{j(\ell+1)}(\tau) + \gamma^{-2}\Delta_{i\ell}(\tau)J_iJ_i'\Delta_{i\ell}(\tau)\Big)\Phi_{i\ell}(\tau)d\tau \quad \text{(A.8)}$$

for all $i \in \mathbb{K}$, all $\ell \in \mathbb{N}$, and all $t \in [0, T)$. Matrices $\Phi_{i\ell}(t)$ are the fundamental matrices associated to $M_{i\ell}(t)$ evaluated at the end of the time interval. Mathematically, $\Phi_{i\ell}(T)$ are the solutions of $\dot{\Phi}_{i\ell}(t) = M_{i\ell}(t)\Phi_{i\ell}(t)$ subject to $\Phi_{i\ell}(0) = I$ for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$ evaluated at the instant of time $t = T > 0$. Taking into account that $\Delta_{i(\ell+1)}(t) \geq 0$ the function $U_{i(\ell+1)}(T)$ is positive semi-definite. As a consequence,

$$P_{i(\ell+1)}(0) - P_{i\ell}(0) \geq \Phi_{i\ell}(T)'\Big(P_{i(\ell+1)}(T) - P_{i\ell}(T)\Big)\Phi_{i\ell}(T) \quad \text{(A.9)}$$

for all $i \in \mathbb{K}$ and all $\ell \in \mathbb{N}$. This inequality is essential for the study of HMJLS in the $\mathcal{H}_\infty$ context.

Remark A.1 *Since the last term of equation* (A.6) *is nonnegative definite, it can be verified that* $\Delta_i(t) = 0$ *defines a minimal solution in* (A.9). *This means that the stationary solution, if one exists, leads to the equality in* (A.9). □