UNIVERSIDADE ESTADUAL DE CAMPINAS
Instituto de Matemática, Estatística
e Computação Científica

Kamila da Silva Andrade

On Degenerate Cycles in Discontinuous Vector Fields and the Dulac's Problem

Sobre Ciclos Degenerados em Campos Vetoriais Descontínuos e o Problema de Dulac

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Thesis presented to the Institute of Mathematics, Statistics and Scientific Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Doctor in mathematics.

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"... Many places I have been
Many sorrows I have seen
But I don't regret
Nor will I forget
All who took the road with me

Night is now falling
So ends this day
The road is now calling
And I must away
Over hill and under tree
Through lands where never light has shone
By silver streams that run down to the sea

To these memories I will hold
With your blessing I will go
To turn at last to paths that lead home
And though where the road then takes me
I cannot tell
We came all this way
But now comes the day
To bid you farewell

I bid you all a very fond farewell"

(Billy Boyd, *The Last Goodbye* in *The Hobbit: The Battle of Five Armies*)

Resumo

Neste trabalho, estuda-se ciclos que ocorrem tipicamente em campos vetoriais descontínuos, planares definidos em duas zonas, $Z = (X, Y)$, com variedade de descontinuidade dada pela imagem inversa do 0 por uma função suave h, definida no plano e assumindo valores reais, para a qual 0 é um valor regular. Primeiramente, mostra-se que, se X e Y são campos vetoriais analíticos e C é um policiclo de Z, então, genericamente, não existem ciclos limite se acumulando em C. Depois disso, o objetivo é estudar bifurcações de ciclos típicos contendo um ponto do tipo sela-regular. Mais especificamente, considera-se ciclos compostos por um segmento de órbita regular de Z, que cruza a variedade de descontinuidade transversalmente, e um ponto do tipo sela-regular resultando numa conexão quase-homoclínica. São apresentados diagramas de bifurcação para o caso onde o raio de hiperbolicidade do ponto de sela é um número irracional, o caso onde o raio de hiperbolicidade da sela é um número racional é ilustrado com alguns modelos. Finalmente, dois modelos comuns em aplicações e que apresentam tal ciclo são estudados por meio de cálculos numéricos.

Palavras-chave: Campos vetoriais descontínuos, Teoria de bifurcação, Problema de Dulac, Ciclos.

Abstract

In this work, a study is performed on cycles occurring typically in planar discontinuous vector fields in two zones, $Z = (X, Y)$, with switching manifold being the inverse image of 0 by a smooth function h, defined on the plane and assuming real values, for which 0 is a regular value. Firstly, it is shown that if X and Y are analytic vector fields and C is a polycycle of Z, then, generically, C cannot have limit cycles accumulating onto it. After that, the objective is to study the bifurcations of typical cycles through a saddle-regular point. More specifically, we consider a cycle composed by one segment of a regular orbit of Z, which crosses the switching manifold transversally, and a saddle-regular point, resulting in a homoclinic-like connection. Bifurcation diagrams are presented for the case where the hyperbolicity ratio of the saddle point is a irrational number, the case where hyperbolicity ratio is a rational number is illustrated with models. Finally, two application models presenting cycles through saddle-regular points are studied by means of numeric calculations.

Keywords: Discontinuous vector fields, Bifurcation theory, Dulac's problem, Cycles.

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Introduction

The concept of limit cycles has arisen in 1882, introduced by the French mathematician Jules Henri Poincaré, in his work "Mémoire sur ler courbes définies par une équation différentielle (II)", see [32]. Few years later, in 1900, during the 2^{nd} International Conference of Mathematicians, in Paris, the German mathematician David Hilbert proposed a collection of twenty-three problems that could influence the development of the mathematics of that century. The Hilbert's sixteenth problem is split in two parts, the first one is related with algebraic topology and the second one with limit cycles in polynomial vector fields. This second part can be written as:

- Determine the maximum number of limit cycles admitted by a polynomial vector field of degree n on $\mathbb{R}^2$.

A preliminary step towards the solution of the second part of Hilbert's sixteenth problem is proving the following finiteness result:

- A polynomial vector field on $\mathbb{R}^2$ has at most a finite number of limit cycles.

This last question can be extended to analytic vector fields and it can be reduced to the problem of non-accumulation of limit cycles, see [35]:

- An elementary polycycle of an analytic vector fields cannot have limit cycles accumulating onto it.

In 1923, the French mathematician Henri Dulac, who was the first to study this non-accumulation problem, gave an incomplete proof which was noticed much later, thus the problem turned out to be called the Dulac's Problem. In 1986, a correct proof was given for quadratic vector fields by R. Bamon [4]. The finiteness result was estated independently by Yu Il'Yashenko [18] (see [19] for more details) in 1984 and by J. Ecalle [12] in 1992. Finally, the complete proof of the second part of the Hilbert's sixteenth problem was recently announced by J. Llibre and P. Pedregal, see [26]. In general, investigations on cycles in smooth systems have always motivated many researchers and plenty of mathematical tools are developed from these studies.

In addition to the study in smooth systems, cycles can be studied in the context of discontinuous vector fields. Let U be an open subset of $\mathbb{R}^2$ with compact closure, i.e., the set $\overline{U}$ is compact. Consider a smooth embedded submanifold $\Sigma = h^{-1}(0) \cap U$, where $f : U \to \mathbb{R}$ is a smooth function for which 0 is a regular value. In this way, Σ splits U in two open regions

$$\Sigma^+ = \{p \in U; f(p) > 0\} \quad \text{and} \quad \Sigma^- = \{p \in U; f(p) < 0\}.$$

A discontinuous vector field in U is of the form

$$Z(p) = \begin{cases} X(p), & p \in \Sigma^+, \\ Y(p), & p \in \Sigma^-, \end{cases}$$

where X and Y are vector fields of class C^r in U. The Russian mathematician Filippov has formalized some aspects of the theory of discontinuous vector fields and presented a convention for the study of the dynamics in the discontinuity manifold Σ, [14]. For this reason, discontinuous vector fields, defined as above with Filippov's convention, are commonly refereed as Filippov vector fields. Many authors also refer to discontinuous vector fields as piecewise smooth vector fields. In the course of this work, we have adopted the terminology "discontinuous vector fields" and we assume Filippov's convention for solutions through the switching manifold Σ.

In 1984, Koslova worked with generic discontinuous vector fields, [21]. In the last years, the theory of discontinuous vector fields has become stronger with growing importance at the frontier between mathematics, physics, engineering, and the life sciences. Interest stems, particularly, from discontinuous dynamical models in control theory [5], nonlinear oscillations [2, 28], impact and friction mechanics [8], economics [20, 16], biology [6], etc. Also, in [27] the authors present a review, focused on bifurcation theory of discontinuous dynamical systems, an introductory state-of-the-art and some open problems are also presented.

The present work concerns the study of degenerate cycles that appear typically for planar discontinuous vector field, see [24, 31, 42]. There exist lots of different typical degenerate cycle and we focus on the particular case where the cycle contains a saddle-regular point. Also, this type of cycle is a particular class of hyperbolic polycycles, i.e., cycles composed by finitely many hyperbolic singular points and regular orbits, it is an extension of the same concept existent for smooth vector fields. Therefore, one natural question arrives by questioning if, or under which conditions, the Dulac's problem can be extended to discontinuous vector fields. This work is composed by two parts: the first one concerns the Dulac's problem in the discontinuous context and, in the second one, it is performed a study of the bifurcation diagrams of a degenerate cycle through a saddle-regular point.

Setting Problems and Main Goals

I. Dulac's Problem

As seen above, Dulac's problem was originally proposed for smooth vector fields. It has arisen from a mistake made for Dulac by trying to prove the finiteness part of the Hilbert's sixteenth problem. In this context, polycycles admit hyperbolic and elementary singularities. A next step in this direction is to replace analytic by piecewise-analytic vector fields. In this direction, we propose a version of this problem for discontinuous vector fields considering polycycles with hyperbolic singularities.

Consider a planar discontinuous vector field $Z = (X, Y)$ where X and Y are analytic vector fields in $\mathbb{R}^2$ and assume Z admits a hyperbolic polycycle Γ, Definition 2.1. The question we want to answer is

- *Can a hyperbolic polycycle* Γ *of a piecewise analytic vector field have limit cycles accumulating onto it?*

In other words, we want to know if the scenario outlined in Figure 1 is realizable for a piecewise analytic vector field. If the polycycle is hyperbolic, we prove that the answer for this question is no. Moreover, we show an example of class a C^k vector field, $1 < k < \infty$, having a limit cycle that admits two sequences of limit cycles accumulating onto it.

If a polycycle Γ does not intersect the switching manifold, then the problem can be reduced to the smooth one. Thus, we also suppose $\Gamma \cap \Sigma \neq \emptyset$. In order to answer this question we follow the same steps for the smooth case in [35]. Our objective is to give an extension for discontinuous vector fields of all concepts and results existent for hyperbolic

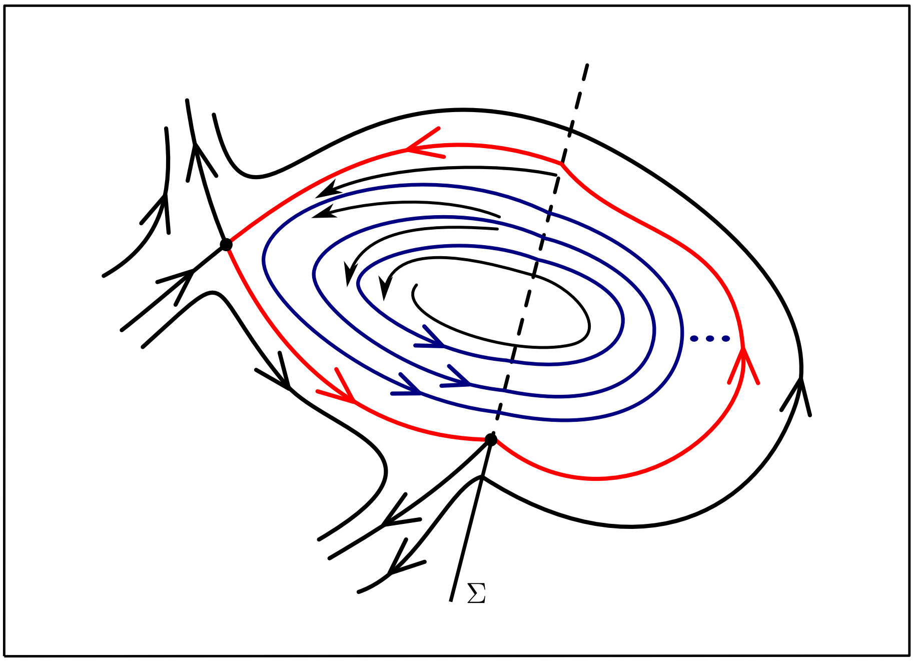

Figure 1: Γ is the polycycle in red, the blue cycles are limit cycles accumulating onto Γ.

polycycles of smooth vector fields. It is worthwhile to emphasize the extension performed is not a particular case of the original problem, there exist essential differences in the construction of first return maps and forms of transition maps. Moreover, we intend to allow a polycycle to have typical singularities of discontinuous vector fields and to check if the result remains the same as the case where polycycles just have hyperbolic polycycles.

II. Degenerate Cycles

The study of cycles in smooth vector fields is a recurrent topic. Cycles (or graphs) are continuous closed curves composed by a finite union of regular orbits and singular points. The simplest kind of cycles are the limit cycles, i.e., periodic (closed) trajectories of a vector field X which are isolated in the set of all periodic orbits of X. The simplest kind of cycles having equilibrium points happens when the cycle is composed by one regular orbit γ and one singular point p. In this case, p is α and $\omega-$limit of any point in γ, and the cycle is also called homoclinic connection or homoclinic loop. If we consider cycles of discontinuous vector fields, there exists different cycles composed by a regular orbit γ (whether it is or not a concatenation of two regular orbits of the smooth vector fields composing the discontinuous one) and a singular point p. For instance if p is on the switching manifold, then γ can reach p in finite time (positive or negative), sometimes it is called homoclinic-like connection. In Figure 2 one can find some examples of cycles.

The main objective here is to study a specific type of a homoclinic-like connection. Consider a planar discontinuous vector field $Z = (X, Y)$, with X and Y being sufficiently smooth vector fields, where the origin is a hyperbolic saddle for X, with eigenspaces being transversal to Σ at the origin, and a regular point for Y. In addition, suppose that the unstable (resp. stable) manifold of the saddle crosses Σ transversely at a point $P \neq 0$ and the positive trajectory (resp. negative trajectory) of Y through P crosses Σ transversely at the origin. Observe that this cycle provides a homoclinic-like connection of the saddle point, given any point in the loop, the trajectory through this point reaches the origin in a finite positive time and infinite negative time or vice-versa. In this work we refer to this

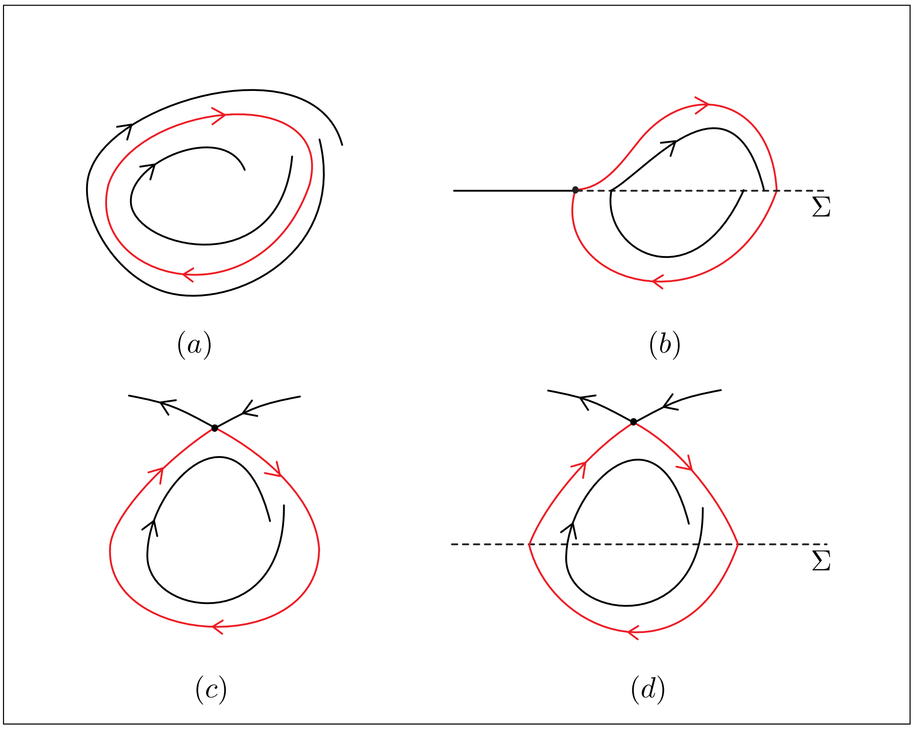

Figure 2: Examples of cycles for smooth vector fields, (a) limit cycle and (c) homoclinic connection, and for discontinuous vector fields, $(b) - (d)$ homoclinic-like connections.

kind of cycle as degenerate cycle through a saddle-regular point. The problem we want to solve is

- *Let Z_0 be a discontinuous vector field having a degenerate cycle, Γ_0, through a saddle-regular point. Given a discontinuous vector field Z sufficiently near Z_0, how is its qualitative behavior in a sufficiently small neighborhood of Γ_0?*

Thus, the main objective of this work is, under suitable generic conditions, to describe the bifurcation diagram of a degenerate cycle Γ_0.

Notice there exist different types of degenerate cycles through a boundary saddle point. For instance, if stable and unstable manifolds do not cross Σ up to the saddle point, in this case there exists a homoclinic connection contained in one of the open regions delimited by Σ. Piecewise Hamiltonian systems having such cycle were studied in [41] by means of Melnikov functions. Piecewise Hamiltonian systems having double homoclinic loops through boundary saddle points, where the loops do not cross the switching manifold, are studied in [25] also by means of Melnikov functions. The usage of Melnikov functions is possible in those works because the authors use an expression for the vector field (given by the Hamiltonian condition) which is perturbed with convenient maps. This approach is not possible in our study, the reason of this is the lack of assumption of any specific form for the vector fields, we only assume some generic conditions. Thus, our objective is also to develop a mechanism to study any degenerate cycle that does not depend on a specific class of vector fields, performing a study as general as possible.

Organization of the Thesis, Methodology, and Contributions

In Chapter 1, some of the basic concepts, definitions and results used through this thesis are briefly presented, the main objective is to establish some notations. None of these results are new and most part of them are presented without proof but the respective references are given. The reader familiar with the theory of discontinuous vector fields can skip it without risk of losing any information.

Chapter 2 is devoted to Dulac's problem. The extension of the concept of polycycle is presented as well as the construction of transition maps at hyperbolic saddle points. Firstly, by assuming transition maps at hyperbolic saddle points are quasi-regular, Definition 2.3, we prove that hyperbolic polycycles of piecewise analytic vector fields cannot have limit cycles accumulating onto it. Our contribution in this part is the extension of the concepts of polycycles and transition maps and to observe that, up to quasi-regularity, nothing needs to be changed in the proof presented in [35]. The second part consists in the proof of quasi-regularity of transition maps at saddle points, whether it is or not on the boundary. The approach employed is very similar to the one used for analytic vector fields. Our contribution in this part is to show that, generically, transition maps at boundary hyperbolic saddle points remain quasi-regular. A consequence of these contributions is to give an answer to Dulac's problem, i.e., we prove that hyperbolic polycycles cannot have limit cycles accumulation onto it. Finally, the last part of this chapter consists on a generalization of the previous study by allowing polycycles with fold points. In order to prove the Dulac's problem for this new kind of polycycle, it is necessary to prove two properties for transition maps at fold points: quasi-regularity and quasi-analyticity, Definition 2.4. The main contribution in this part is to prove these properties and to conclude that by allowing fold points in the definition of polycycles it does not change the result obtained for hyperbolic polycycles.

Chapter 3 concerns the study of bifurcations of typical cycles through fold-regular points. More specifically, suppose a discontinuous vector field $Z = (X, Y)$ has a visible fold-regular point, the trajectory of X which is tangent to the switching manifold Σ also crosses Σ transversally at a point P. After that, the trajectory of Y through P crosses Σ transversally at P and at the fold point of X, see Figure 3. This cycle has already appeared in literature, [9, 24], but its bifurcation diagram was not precisely studied in these works. In [9] the cycle appears in the bifurcation of a cusp-fold singularity while in [24] the interest is to show the birth of a limit cycle under small perturbations of the system having the degenerate cycle. Our contribution in this line is to establish generic conditions for the existence of a cycle, to present a bifurcation diagram for two topologically distinct cycles, and to present the study of two models realizing these topologically distinct cycles. By means of a detailed analysis, we emphasize that a system presenting this kind of cycle, which is relatively simple, can have a very rich unfolding. The main reason for the study of such cycles is the fact they naturally appear in bifurcations of cycles through saddle-regular points.

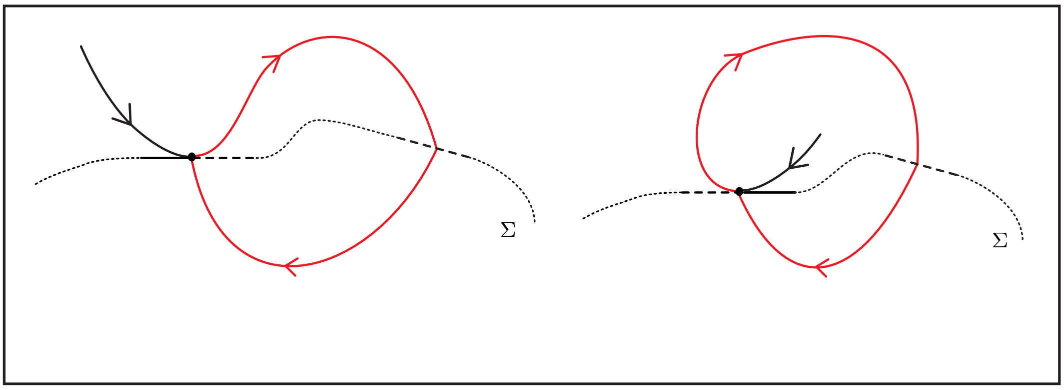

Figure 3: A degenerate cycles passing through a visible fold-regular point.

In Chapter 4, a study of bifurcations of a degenerate cycle through a saddle-regular point discussed above is presented, see Figure 4. This specific kind of cycle typically appears in the bifurcation of a Bogdanov-Takens-fold singularity, i.e., a boundary point which is a Bogdanov-Takens singularity for one vector field and a fold point for the other one. Such a cycle has not received proper attention in literature yet, so it is important to perform a detailed analysis of it. The most difficult part in this study is the absence of an algebraic expression for the vector field. In order to deal with this lack of information, we use local features on the hyperbolic saddle given by Theorem 1.2, which is obtained by means of the Poincaré-Dulac normal form Theorem and a technical result due to Bonckaert, see [35]. The normal form depends on the hyperbolicity ratio of the saddle point, i.e., the quotient $r = -\lambda_2/\lambda_1$, where $\lambda_2 < 0 < \lambda_1$ are the eigenvalues of the saddle point. Another important local study concerns on bifurcations of a saddle-regular point, [15, 22]. After using the local analysis, it is necessary to understand the first return map existent near the cycle. This map depends strongly on the hyperbolicity ratio of the saddle point and whether it is real, virtual or on the boundary. Due to the difficulty in performing a study when the hyperbolicity ratio $r \in \mathbb{Q}$, our study is restricted to the case $r \notin \mathbb{Q}$, a total of six different bifurcation diagrams (DSC_{11}, DSC_{12}, DSC_{21}, DSC_{22}, DSC_{31}, and DSC_{32}) is presented. The case $r \in \mathbb{Q}$ is illustrated with some models. Firstly, we consider a class of systems having hyperbolicity ratio equal to 1, it is shown the results previously found, for $r \notin \mathbb{Q}$, still hold. Second model also corresponds to a family of vector fields having hyperbolicity ratio equal to 1. Similarly to the previous chapter, a detailed study of the model is presented to show how rich and complex a system presenting such degenerate cycle can be. Finally, the last model involves a family having generic hyperbolicity ratio $r > 0$, in this case the analyses on the first return maps and existence of limit cycles are performed numerically.

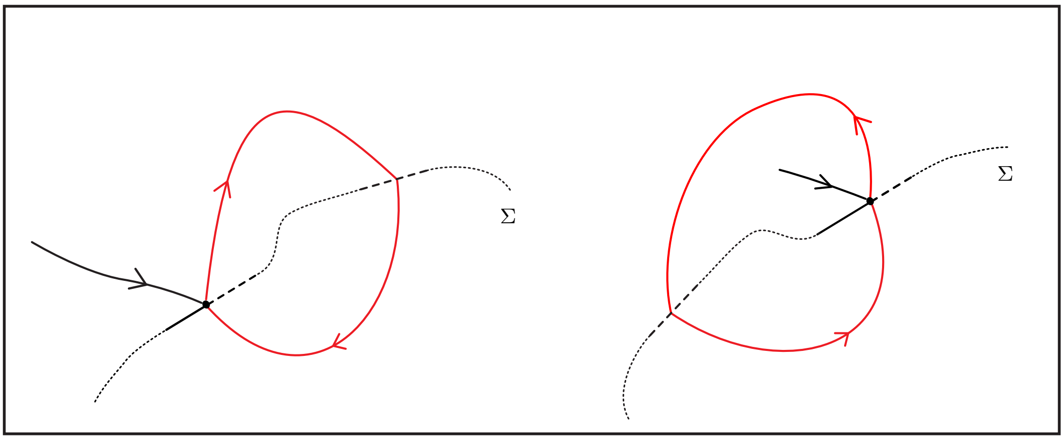

Figure 4: A degenerate cycles passing through a saddle-regular point.

Since the analysis of degenerate cycles through hyperbolic saddle-regular points is the main interest of this thesis, two application models realizing such system are presented in Chapter 5. We perform basic numerical analysis of the models by using the software Mathematica. Firstly, a model for a discontinuous pendulum is studied. We show that the model presents a cycle of type DSC_{11} and realizes the entire bifurcation diagram of this cycle. This is an interesting fact since models not necessarily unfold completely a singularity. The second model is a piecewise Hamiltonian model. For this model, we show it realizes a cycle of type DSC_{31} and we give intervals where there exists a limit cycle.

Chapter 6 addresses future directions. A study of a new type of cycle is proposed as well as proposals of improvements and completeness of the results obtained in this work. Appendix A is a one page proof of a specific Cauchy's inequality, used in Chapter 2, that is not so obvious. In order to simplify the exposition, since it is just a technical tool, it was not included in that chapter.

In what follows, we summarize the main results obtained in this thesis:

- Transition maps at saddle points, with invariant manifolds transversal in Σ, are quasi-regular homeomorphisms: Theorem 2.5 in Chapter 2.
- Transition maps at fold points are quasi-regular homeomorphisms and quasi-analytic maps: Theorems 2.6 and 2.7 in Chapter 2.
- Description of the bifurcation diagrams of degenerate cycles through fold-regular points: Theorem 3.1 in Chapter 3.
- Description of the bifurcation diagrams of degenerate cycle through saddle-regular points, in Chapter 4,
  - Cases DSC_{11} and DSC_{12}: Theorems A, B, and C;
  - Cases DSC_{21} and DSC_{22}: Theorems D and E;
  - Cases DSC_{31} and DSC_{32}: Theorems F, G, and H.
- Study of application models realizing degenerate cycles through saddle-regular points in Chapter 5.

Chapter 1

Preliminaries

In this chapter, some basic concepts, results, and tools necessary to the development of this thesis are presented. The main objective is to fix the notation and the definitions that will be used through this thesis. We assume that the most part of the concepts are already known by the reader, so the exposition is short and direct. Most part of the results are given without proof, but references where they can be found are included.

Firstly, some facts on smooth vector fields in manifolds without boundary are established. After that, some concepts are extended to manifolds with boundary. Finally, the theory of discontinuous vector fields is explored.

1.1 Smooth Vector Fields

Let U be an open subset of $\mathbb{R}^2$. A $\mathcal{C}^r$ vector field in U, $1 \leq r \leq \infty$ or $r = \omega$ ($r = \omega$ means the vector field is analytic), is a map $X : U \to \mathbb{R}^2$ which is of class $\mathcal{C}^r$ in U. A vector field X is associated with an ordinary differential equation (ODE)

$$\dot{x} = X(x). \tag{1.1.1}$$

Analogously, an ODE, as the one in equation (1.1.1), is associated with a vector field. A solution of the equation (1.1.1) is a map $\varphi : I \subset \mathbb{R} \to U$ such that, for all $t \in I$,

$$\frac{d}{dt}\varphi(t) = X(\varphi(t)).$$

This solution is called trajectory (or integral curve) of the vector field X or even of the ODE (1.1.1). The association of the vector field X with the ODE (1.1.1) can be geometrically interpreted as following: $\varphi : I \to U$ is a trajectory of X if and only if the tangent vector $\frac{d}{dt}\varphi(t)$ coincides with X at $\varphi(t)$, for all $t \in I$, see Figure 1.1. For this kind of differential equation there are plenty of basic results, see [3], [17], [30], [36], and [37] among others.

Definition 1.1. *A point $p \in U$ for which $X(p) = 0$ is called a singular point of X, otherwise it is called a regular point. Moreover, a singular point p of X is said to be hyperbolic if all eigenvalues of $DX(p)$ has nonzero real part.*

Given $p \in U$, denote by $\varphi(t, p)$ the trajectory of X satisfying $\varphi(0, p) = \varphi(0) = p$, defined in the interval I_p. If p is considered as a variable we call $\varphi(t, p)$ the flow of X. The set $\gamma_p = \{\varphi(t, p); t \in I_p\}$ is called orbit of X through p, the orbits are orientated

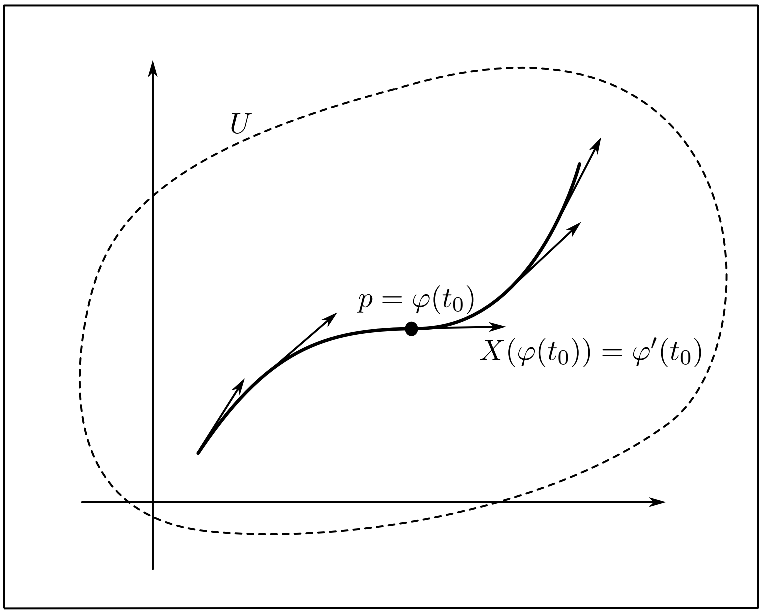

Figure 1.1: Illustration of the trajectory $\varphi(t)$ of the vector field X through $p = \varphi(t_0) \in U$.

according to increasing time in I_p. Observe that two orbits of X either coincide or are disjoint, so the set U can be decomposed as a disjoint union of all orbits of X. To this decomposition of U we give the name of phase portrait of X. Now, we are ready to present the notions of equivalences between vector fields, these concepts allow us to compare the phase portraits of two vector fields.

Definition 1.2. *Consider $\mathcal{C}^r$ vector fields $X : U_1 \to \mathbb{R}^2$ and $Y : U_2 \to \mathbb{R}^2$. X and Y are said to be topologically equivalent (resp. $\mathcal{C}^k-$equivalent) if there exists a homeomorphism (resp. diffeomorphism of class $\mathcal{C}^k$) $h : U_1 \to U_2$ that sends orbits of X in orbits of Y preserving their orientation, see Figure 1.2.*

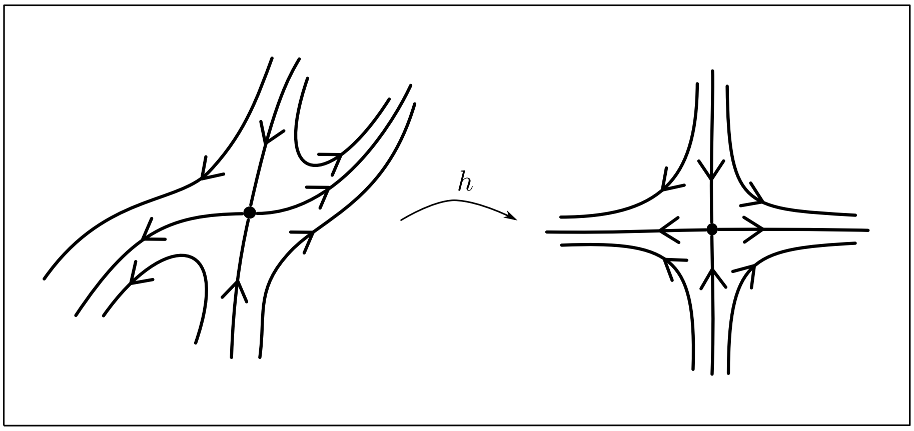

Figure 1.2: h is a topological equivalence between two vector fields having a hyperbolic saddle.

There exists an even more restrictive concept of equivalence.

Definition 1.3. *Two $\mathcal{C}^r$ vector fields $X : U_1 \to \mathbb{R}^2$ and $Y : U_2 \to \mathbb{R}^2$, with corresponding flows $\varphi_X(t,p)$ and $\varphi_Y(t,p)$, are said to be topologically conjugated (resp. $\mathcal{C}^k-$conjugated)* *if there exists a homeomorphism (resp. $\mathcal{C}^k-$diffeomorphism) $h : U_1 \to U_2$ such that $h(\varphi_X(t,p)) = \varphi_Y(t,h(p))$.*

Denote by χ^r the set of all $\mathcal{C}^r$ vector fields defined in $\mathbb{R}^2$ endowed with the $\mathcal{C}^r-$topology. A vector field $X \in \chi^r$ is structurally stable in χ^r if there exists a neighborhood V, of X in χ^r, such that all $Y \in V$ is topologically equivalent to X.

Definition 1.4. *Let $\varphi(t,p) : \mathbb{R} \to \mathbb{R}^2$ be the trajectory of X through p. The set $\omega(p) = \{q \in \mathbb{R}^2; \exists\,(t_n)$ with $t_n \to +\infty$ and $\varphi(t_n,q) \to p,$ when $n \to \infty\}$ is called the $\omega-$limit set of p. Analogously, the set $\alpha(p) = \{q \in \mathbb{R}^2; \exists\,(t_n)$ with $t_n \to -\infty$ and $\varphi(t_n,q) \to p,$ when $n \to \infty\}$ is called the $\alpha-$limit set of p. The $\omega-$limit set (resp. the $\alpha-$limit set) of an orbit γ is the set $\omega(p)$ (resp. $\alpha(p)$) for any $p \in \gamma$.*

For $X \in \chi^r$ with singular point $p \in \mathbb{R}^2$, the set $W^s(p)$ (resp. $W^u(p)$) of all points in $\mathbb{R}^2$ having p as $\omega-$limit (resp. $\alpha-$limit) is called stable (resp. unstable) manifold of p. Thus, $W^s(p)$ and $W^u(p)$ are invariant by the flow of X. Given $\beta > 0$, let $B_\beta(p)$ denote the open ball of $\mathbb{R}^2$ with center at p and radius β.

Definition 1.5. *Consider $X \in \chi^r$ having p as a singular point and let $\varphi(t,q)$ be the flow of X. The sets*

$$W^s_\beta(p) = \{q \in B_\beta(p); \varphi(t,q) \in B_\beta(p), \forall t \geq 0 \text{ and } \varphi(t,q) \to p \text{ as } t \to +\infty\}$$
$$W^u_\beta(p) = \{q \in B_\beta(p); \varphi(t,q) \in B_\beta(p), \forall t \leq 0 \text{ and } \varphi(t,q) \to p \text{ as } t \to -\infty\},$$

are called stable and unstable local manifolds (of size β) of X at p.

Remark 1.1. *Observe that, for $\beta > 0$, $W^s(p) = \bigcup\limits_{t\leq 0} \varphi_t(W^s_\beta(p))$ and $W^u(p) = \bigcup\limits_{t\geq 0} \varphi_t(W^u_\beta(p))$, where $\varphi_t(\cdot) = \varphi(t,\cdot)$. For more details see [29, 30, 36].*

For further references, if necessary, we use $W^j(p) = W^j(p,X)$, $j = s,\, u$, to specify that we are considering the stable or unstable manifold at p related to the vector field X.

1.1.1 Normal Form at a Hyperbolic Saddle Point

Consider the vector field $X \in \chi^\infty$ and suppose that X has a hyperbolic saddle $s \in \mathbb{R}^2$, i.e., s is a hyperbolic singular point of X and $\det(DX(s)) < 0$. The main interest here is to study X in a neighborhood of s so, without loss of generality, we suppose $s = (0,0)$ and X is defined in a neighborhood of s, $\mathcal{V}_0 \subset \mathbb{R}^2$. Due to the hyperbolicity we also assume s is the unique singular point of X in $\mathcal{V}_0$.

Assume the local unstable and stable manifolds of X at s are given by $W^u = 0_x \cap \mathcal{V}_0$ and $W^s = 0_y \cap \mathcal{V}_0$ (see [29], pages $79-81$). Let λ_1 and λ_2 be the eigenvalues of $DX(s)$ with $\lambda_2 < 0 < \lambda_1$ and let $r = -\dfrac{\lambda_2}{\lambda_1}$ be the ratio of hyperbolicity of X at s.

Notice that one can assume $\lambda_1 = 1$ so $\lambda_2 = -r$, thus the $1-$jet (or the linear part) of X, at s, is equal to

$$j^1X(0) = x\frac{\partial}{\partial x} - ry\frac{\partial}{\partial y}. \tag{1.1.2}$$

In fact, the hyperbolicity of s implies that, by means of a linear change of coordinates, the linear part of X at s becomes diagonal. After that it is enough to consider the change of coordinates $(x,y) \mapsto (x/\lambda_1, y)$. Moreover, since $\lambda_1 > 0$, both change of coordinates can be taken as orientation preserving diffeomorphisms.

For hyperbolic singular points we give an important result which is proved by Bonckaert in [7].

Proposition 1.1. *Let X be a germ of a vector field with the conventions above. There exists a function $K : \mathbb{N} \to \mathbb{N}$, with $K(k) \to \infty$ as $k \to \infty$, and such that, if $\bar{X}$ is any germ of an analytic vector field, at s, with the property*

$$j^{K(k)}(\bar{X} - X)(0) = 0, \tag{1.1.3}$$

then the two germs X and $\bar{X}$ are C^k-conjugate.

This result allows us to replace X by a polynomial vector field up to C^k-conjugacy. Parallel to this result, we have the Dulac-Poincaré normal form theorem, see [34, 35], which proof can be easily extended from the prove done in [34] for the case $p = q = 1$. This proof is really extensive, so we will just give a sketch of it, pointing the main parts. See [10] for an introductory theory of normal forms.

Theorem 1.1. *Let X be a $\mathcal{C}^\infty$ vector field having a hyperbolic saddle at $s = 0$ as defined above, with hyperbolicity ratio r.*

1. *Suppose that $r \notin \mathbb{Q}$. Then, for any $N \in \mathbb{N}$,*

$$j^{N+1}X(s) \sim x\frac{\partial}{\partial x} - ry\frac{\partial}{\partial y}.$$

2. *Suppose that $r = \frac{p}{q} \in \mathbb{Q}$, with p and q without common factors. Then, for any $N \in \mathbb{N}$,*

$$j^{(p+q)N+1}X(s) \sim x\frac{\partial}{\partial x} + \left(-r + \frac{1}{q}\sum_{i=1}^{N} \alpha_{i+1}\left(x^p y^q\right)^i\right) y\frac{\partial}{\partial y},$$

with $\alpha_i \in \mathbb{R}$, $i \in \mathbb{R}$. Where the sign $\sim$ means equivalence of jets.

Proof. As seen above, there exists a $C^\infty-$conjugation that brings

$$j^1X(0) \sim x\frac{\partial}{\partial x} - ry\frac{\partial}{\partial y}.$$

If $r \notin \mathbb{Q}$ then, there is no resonance and the result follows from the Poincaré Linearization Theorem, see [10].

If $r = \frac{p}{q} \in \mathbb{Q}$, all resonance relations $\lambda_i - \sum_{k=1}^{2} n_k\lambda_k = 0$, $i = 1, 2$ and $n_k \in \mathbb{N}$, are generated by the unique relation $p\lambda_1 + q\lambda_2 = 0$. Now, the prove is completed by using induction on N. Suppose that we have found $\alpha_2, \alpha_3, \dots, \alpha_N \in \mathbb{R}$ such that

$$j^{(p+q)N+1}X(0) \sim x\frac{\partial}{\partial x} + \left(-r + \frac{1}{q}\sum_{i=1}^{N} \alpha_{i+1}\left(x^p y^q\right)^i\right) y\frac{\partial}{\partial y} = X^N. \tag{1.1.4}$$

So, we have to prove that the equivalence for $j^{(p+q)(N+1)+1}X(0)$. Notice that

$$j^{(p+q)(N+1)+1}X(0) = j^{(p+q)N+1}X(0) + X_{(p+q)N+2} + \cdots + X_{(p+q)N+s+1} + \cdots + X_{(p+q)(N+1)+1},$$

where $X_{(p+q)N+s+1} \in H_2^{(p+q)N+s+1}$ for all $1 \leq s \leq p + q$ (H_2^k is the space of homogeneous polynomials of order k in two variables). Then,

$$j^{(p+q)(N+1)+1}X(0) \sim X^N + \sum_{s=1}^{p+q} Y_{(p+q)N+s+1},$$

where $Y_{(p+q)N+s+1} \in H_2^{(p+q)N+s+1}$ for $1 \leq s \leq p+q$.

In order reach our objective, we prove the equivalence for $j^{(p+q)N+1+i}X(0)$, for $i = 1,\dots,p+q$. The Lie bracket in H_2^k, defined as

$$\begin{array}{rccc} \mathcal{L}^k: & H_2^k & \to & H_2^k \\ & \xi(x,y) & \mapsto & D\xi(x,y)(X^1) - X^1(\xi(x,y)), \end{array}$$

is the key to finish the proof. Normal form theory and the remark done above on the resonance relations provide a basis of $H_2^{(p+q)N+s+1}$,

$$\mathcal{B}_s = \left\{ \begin{pmatrix} x^\alpha y^\beta \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^\alpha y^\beta \end{pmatrix}; \alpha+\beta = (p+q)N+s+1 \right\},$$

for which $\mathcal{L}^{(p+q)N+s+1}$ is diagonal, $1 \leq s \leq p+q-1$, and it has no zero eigenvalues. By means of an inductive process, successive conjugations bring

$$j^{(p+q)(N+1)+1}X(0) \sim X^N + \tilde{Y}_{(p+q)(N+1)+1},$$

with $\tilde{Y}_{(p+q)(N+1)+1} \in H_2^{(p+q)(N+1)+1}$. Finally, by using the resonance relation we obtain

$$\mathcal{K} = \mathrm{Ker}(\mathcal{L}^{(p+q)(N+1)+1}) = \left\langle \left\{ (x^p y^q)^{N+1} x\frac{\partial}{\partial x}, (x^p y^q)^{N+1} y\frac{\partial}{\partial y} \right\} \right\rangle \text{ and}$$

$$\mathcal{I} = \mathrm{Im}(\mathcal{L}^{(p+q)(N+1)+1}) = \left\langle \left\{ \begin{pmatrix} x^\alpha y^\beta \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^\gamma y^\delta \end{pmatrix}; \begin{array}{c} \alpha+\beta = (p+q)(N+1)+1 = \gamma+\delta, \\ \beta \neq q(N+1), \text{ and } \gamma \neq p(N+1) \end{array} \right\} \right\rangle.$$

Thus, $\mathcal{L}^{(p+q)(N+1)+1}|_{\mathcal{I}}\colon \mathcal{I} \to \mathcal{I}$ is a linear map that is diagonal with relation to the basis of $\mathcal{I}$ given above and no zero eigenvalues. It implies that

$$j^{(p+q)(N+1)+1}X(0) \sim X^N + B,$$

where $B \in \mathcal{K}$. From this, some algebraic calculation provides α_{N+2} such that

$$j^{(p+q)(N+1)+1}X(0) \sim X^N + \alpha_{N+2}(x^p y^q)^{N+1} y = x\frac{\partial}{\partial x} + \left(-r + \frac{1}{q}\sum_{i=1}^{N+1} \alpha_{i+1}(x^p y^q)^i y \right).$$

Therefore, by induction we have that equation 1.1.4 holds for any $N \in \mathbb{N}$. It completes the proof. □

Now, by combining Proposition 1.1 and Theorem 1.1 we get the following useful result.

Theorem 1.2. *Let X be a C^∞ vector field as above. Then, there exists a function $N:\mathbb{N} \to \mathbb{N}$ such that, in some neighborhood of the saddle point s, X is C^k-equivalent to the polynomial vector field*

$$x\frac{\partial}{\partial x} + \left(-r + \frac{1}{q}\sum_{i=1}^{N(k)} \alpha_{i+1}(x^p y^q)^i \right) y\frac{\partial}{\partial y}, \tag{1.1.5}$$

if $r = \frac{p}{q} \in \mathbb{Q}$. If $r \notin \mathbb{Q}$, X is C^k-equivalent to the linear vector field

$$x\frac{\partial}{\partial x} - ry\frac{\partial}{\partial y}. \tag{1.1.6}$$

Proof. Define a sequence $N(k)$ satisfying $(p+q)N(k)+1 > K(k)$ if $r = \frac{p}{q} \in \mathbb{Q}$ and $N(k)+1 > K(k)$ if $r \notin \mathbb{Q}$, where $K(k)$ is given in Proposition 1.1. Now, to obtain the result, it is enough to apply Proposition 1.1 and Theorem 1.1. $\square$

Remark 1.2. *Under some simple adaptations, Proposition 1.1 and Theorem 1.1 still hold for $C^\infty-$families of vector fields. So, Theorem 1.2 also holds for $C^\infty-$families of vector fields. Since we are not interested in families, we have presented a simpler versions of these results. A more general exposition was given by Roussarie in Chapter* 5 *of [35].*

1.2 Vector Fields in Manifolds with Boundary

Let M be an embedded submanifold of $\mathbb{R}^2$ with boundary. Two vector fields, X and Y, in $\mathbb{R}^2$ are germ equivalent in M if there exists a neighborhood of M in $\mathbb{R}^2$ where X and Y coincide.

Definition 1.6. *A vector field X in M is a class of vector fields in $\mathbb{R}^2$ that are germ equivalent in M. X is said to be of class $\mathcal{C}^r$ if it has a representative of class $\mathcal{C}^r$ in $\mathbb{R}^2$.*

Let X be a vector field in M and let $\tilde{X}$ be a representative of X with flow $\varphi_{\tilde{X}}$, then the flow of X is the restriction to M of the flow of $\tilde{X}$, i.e., $\varphi_X = \varphi_{\tilde{X}}|_M$. It is clear that this definition does not depends on the choice of the representative of X.

Definition 1.7. *Two vector fields, X and Y, in M are topologically equivalent if there exist a homeomorphism $h : M \to M$ that sends trajectories of X in trajectories of Y, preserving their orientation.*

Denote by $\chi^r(M)$ the set of all vector fields of class $\mathcal{C}^r$ in M. A vector field $X \in \chi^r(M)$ is structurally stable in $\chi^r(M)$ if there exists a neighborhood V of X in $\chi^r(M)$ such that any $Y \in V$ is topologically equivalent to X.

Assume $\partial M = h^{-1}(0)$ where $h : \mathbb{R}^2 \to \mathbb{R}$ is a smooth map having 0 as a regular value and $\text{Int}(M) = \{p \in \mathbb{R}^2; h(p) > 0\}$.

Definition 1.8. *A vector field X in M has a fold singularity (or a quadratic tangency) at $p \in \partial M$ if $Xh(p) = 0$ and $X^2h(p) \neq 0$. Moreover, a fold p of X is visible if $X^2h(p) > 0$ and invisible if $X^2h(p) < 0$, see Figure 1.3.*

Where $Xh(p) = \langle X, \nabla h\rangle(p)$, is the Lie derivative of h with respect to X at p, and $X^jh(p) = \langle X, \nabla X^{j-1}h\rangle(p)$ for $j \geq 2$.

Suppose $p \in \partial M$ is a fold point of X, without loss of generality assume that p is at the origin. In [40] it is proved that, by means of a $C^\infty-$change of coordinates, near the fold point $p = 0 \in \partial M$, X is given by $X(x,y) = (a, bx)^T$, with $a, b \neq 0$.

1.2.1 Transition Map at a Fold Point

Consider $X \in \chi^r(M)$, $r \geq 2$, having a visible fold point at $p \in \Sigma = \partial M$. We want to study X in a neighborhood of p, so, without loss of generality, assume $p = (0,0)$ and $h(x,y) = y$. For $\varepsilon > 0$ sufficiently small such that, for some $\delta_0 > 0$, $\tau = \{(x,\varepsilon) \in \mathbb{R}^2; 0 \leq x < \delta_0\}$ is contained in the neighborhood of p where the local assumption above holds and τ is a transversal section for the flow of X up to $x = 0$. Our objective is to calculate the transition map, from Σ to τ, given by the flow of X. In order to do that, assume

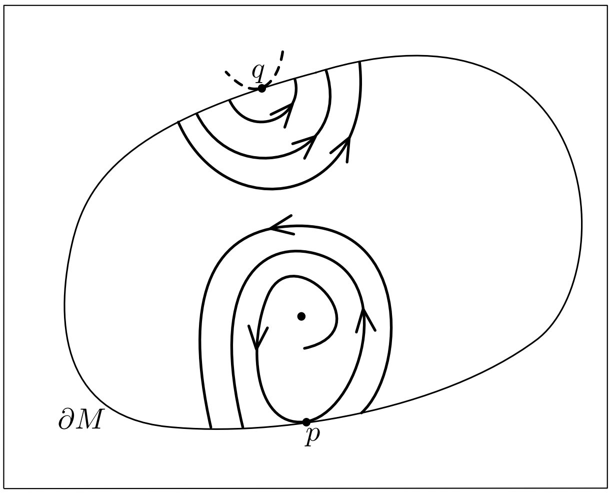

Figure 1.3: Examples of fold points: p is a visible fold point and q is an invisible one.

$X(p) = (X_{01}(p), X_{02}(p))$ where $X_{01}(0) > 0$ and $X_{02}(0) = 0$. Then for $x > 0$ there exists $t(x) > 0$ for which $\varphi(t(x), x, 0) \in \tau$, where $\varphi(t, p)$ is the flow of X, see Figure 1.4. From $\varphi(t(x), x, 0) \in \tau$, we can write $\varphi(t(x), x, 0) = (\rho(x), \varepsilon)$, then define $\rho(x)$ as being the transition map from Σ to τ at 0.

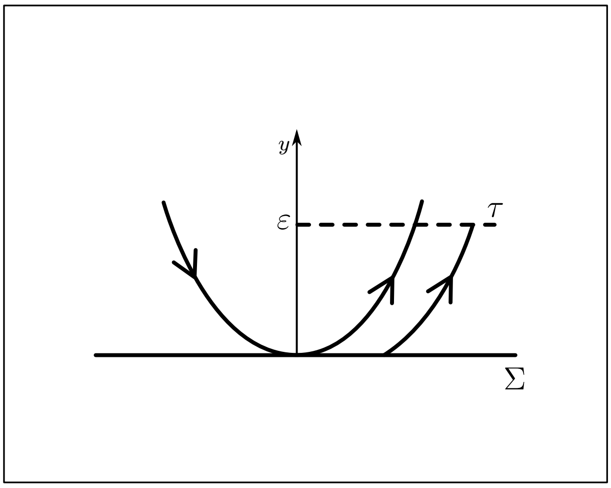

Figure 1.4: Transition map at a fold point from Σ to τ.

Proposition 1.2. *There exists $\delta > 0$ sufficiently small such that*

$$\rho(x) = x_0 + \alpha_2 x^2 + O(x^3) \tag{1.2.1}$$

for all $0 \leq x < \delta$, where $\alpha_2 > 0$ and $x_0 = \rho(0)$.

Proof. Observe that ρ can be C^r-extend to an open neighborhood of the origin by means of the flow of a representative $\tilde{X}$, of X, defined in $\mathbb{R}^2$. Then, we can calculate the Taylor expansion of ρ, around $x = 0$, is

$$\rho(x) = \rho(0) + \frac{d}{dx}\rho(0)x + \frac{1}{2!}\frac{d^2}{dx^2}\rho(0)x^2 + O(x^3).$$

Therefore, it is enough to show that $\frac{d}{dx}\rho(0) = 0$ and $\frac{d^2}{dx^2}\rho(0) > 0$. From the definition of ρ, if $\varphi_{\tilde{X}}(t, x, 0) = (\varphi_1(t, x), \varphi_2(t, x))$, with $\varphi_2(t(x), x) \equiv \varepsilon$, then $\rho(x) = \varphi_1(t(x), x)$. Thus, we obtain

$$\frac{d}{dx}\rho(0) = \frac{\partial}{\partial x}[\varphi_1(t(x),x)]\mid_{x=0} = X_{01}(0)\frac{d}{dx}t(0) + \frac{\partial}{\partial x}\varphi_1(t(0),0).$$

and

$$\frac{d}{dx}t(0) = -\frac{\frac{\partial}{\partial x}\varphi_2(t(0),0)}{X_{01}(0)}.$$

Observe that $\omega(t) = \frac{\partial}{\partial x}\varphi_{\tilde{X}}(t,0)$ and $\bar{\omega}(t) = \frac{1}{X_{01}(0)}X(\varphi_{\tilde{X}}(t,0))$ are both solutions of the Cauchy's problem

$$\left\{\begin{array}{rcl} \dot{\omega} & = & \mathrm{D}X_0(\varphi_{\tilde{X}}(t,0))\omega \\ \omega(0) & = & (1,0)^T \end{array}\right. .$$

By means of the Theorem of Existence and Uniqueness we obtain, for $i = 1,2$, $\frac{\partial}{\partial x}\varphi_i(t(0),0)$ $= \frac{X_{0i}(0)}{X_{01}(0)}$. Therefore, $\frac{d}{dx}\rho(0) = 0$. In [33] it is proved that $\dfrac{d^2}{dx^2}\rho(0) > 0$, which follows from theorems on uniqueness of solutions and it concludes the proof. □

1.3 Discontinuous Vector Fields

In this work we concern about planar discontinuous vector fields defined in two zones, for this reason we just present definitions for this type of vector fields. More general definitions and concepts can be found in [11, 14, 15] among others.

Consider two connected, open, and disjoint sets, Σ^+, $\Sigma^- \subset \mathbb{R}^2$, such that the common frontier of these sets is a codimension 1 submanifold $\Sigma \subset \mathbb{R}^2$ and $\Sigma^+ \cup \Sigma^- \cup \Sigma = \mathbb{R}^2$. Let X and Y be vector fields in χ^r restricted to Σ^+ and Σ^-, respectively. Under these conditions, a discontinuous vector field is defined as following

$$Z(p) = \left\{\begin{array}{ll} X(p), & p \in \Sigma^+, \\ Y(p), & p \in \Sigma^-, \end{array}\right. \tag{1.3.1}$$

Σ is called switching manifold or even discontinuity manifold.

Denote by Ω^r the set of all discontinuous vector fields in $\mathbb{R}^2$ defined as above. For each $Z \in \Omega^r$ write $Z = (X,Y)$ so, we can identify Ω^r with $\chi^r \times \chi^r$. We consider this set endowed with the product topology, where χ^r is endowed with the $\mathcal{C}^r-$topology. In this context, the definition of trajectories of such a discontinuous vector field follows the Filippov's convention, see [14].

Throughout this work, we suppose $\Sigma = h^{-1}(0)$ is an embedded submanifold of $\mathbb{R}^2$, where $h : \mathbb{R} \to \mathbb{R}$ is a smooth function for which 0 is a regular value. In addition, we assume, without loss of generality, $\Sigma^+ = \{p \in \mathbb{R}^2; h(p) > 0\}$ and $\Sigma^- = \{p \in \mathbb{R}^2; h(p) < 0\}$. So, the definition in equation 1.3.1 becomes

$$Z(p) = \left\{\begin{array}{lll} X(p) & \text{if} & h(p) > 0, \\ Y(p) & \text{if} & h(p) < 0, \end{array}\right. \tag{1.3.2}$$

where $p \in \mathbb{R}^2$, X, $Y \in \chi^r$. Observe that we can think on X and Y as being vector fields defined in the manifolds with boundary $\Sigma^+ \cup \Sigma$ and $\Sigma^- \cup \Sigma$, respectively.

Having these assumptions established, on the switching manifold Σ the following regions can be distinguished, depending on the directions of X and Y,

- the crossing (or sewing) region: $\Sigma^c = \{p \in \Sigma;\ Xh \cdot Yh(p) > 0\}$, Figure 1.5$-(a)-(b)$;

- the sliding region: $\Sigma^s = \{p \in \Sigma;\ Xh(p) < 0 \text{ and } Yh(p) > 0\}$, Figure 1.5$-(c)$;
- the escaping region: $\Sigma^e = \{p \in \Sigma;\ Xh(p) > 0 \text{ and } Yh(p) < 0\}$, Figure 1.5$-(d)$.

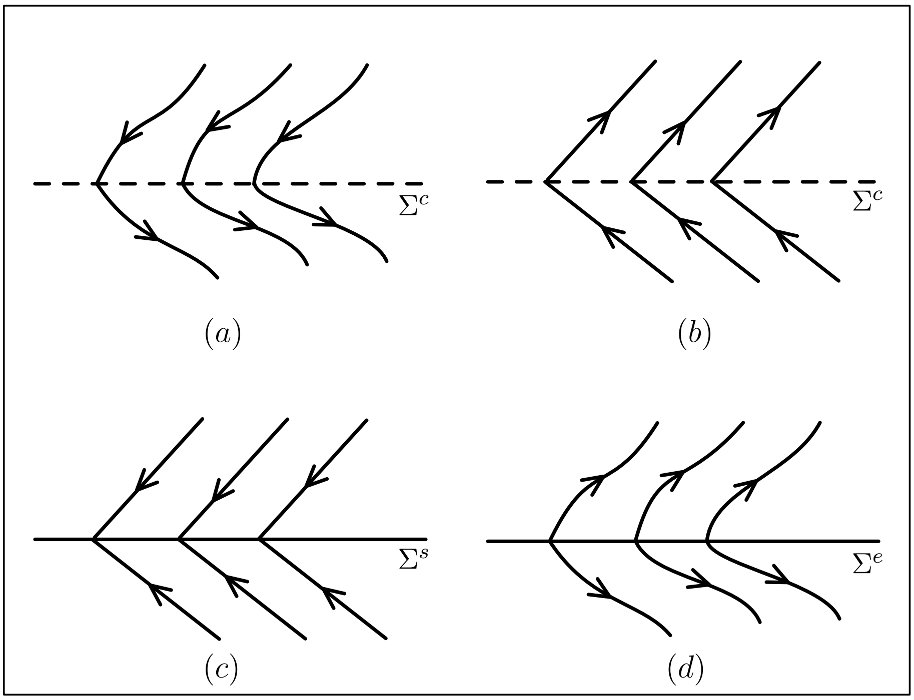

Figure 1.5: Illustrations of Σ^c in (a) and (b), Σ^s in (c) and Σ^e in (d).

The regions Σ^c, Σ^s, and Σ^e are relatively open in Σ and the complementary set of them in Σ is composed by points satisfying $Xh(p) \cdot Yh(p) = 0$, called tangency points. A smooth vector field X is transversal to Σ at $p \in \Sigma$ if it is not a tangency point.

For points in Σ^c it is natural to think that the trajectories of $Z = (X, Y) \in \Omega^r$ are given by concatenations of trajectories of X and Y. So, a preliminary step before discussing about all possible trajectories of such a vector field is to understand the dynamics in regions Σ^s and Σ^e. In $\Sigma^s \cup \Sigma^e$ we define a vector field Z^s given by the unique convex combination of X and Y which is tangent to Σ at p, i.e., given $p \in \Sigma^s \cup \Sigma^e$,

$$Z^s(p) = \frac{1}{Yf(p) - Xf(p)} \left(Yf(p)X(p) - Xf(p)Y(p) \right). \tag{1.3.3}$$

Denoted by $\varphi_X(t, p)$ the flow of the smooth vector field $X \in \chi^r$. This flow is defined for $t \in I = I(p, X) \subset \mathbb{R}$ and, by simplicity, we use I instead of $I(p, X)$.

Definition 1.9. *Consider $Z = (X, Y)$ a discontinuous vector field as defined in the equation* (1.3.2)*. The local trajectory of Z through a point $p \in \mathbb{R}^2 - \overline{\Sigma^s \cup \Sigma^e}$ is defined as*

- *if $p \in \Sigma^+$ or if $p \in \Sigma^-$, then the trajectory through p is $\varphi_Z(t, p) = \varphi_X(t, p)$ or $\varphi_Z(t, p) = \varphi_Y(t, p)$ respectively, for $t \in I \subset \mathbb{R}$. The interval I is such that $\varphi_Z(t, p) \notin \Sigma$ for all $t \in I$;*
- *if $p \in \Sigma^c$, the trajectory is the concatenation of the trajectories $\varphi_X(t, p)$ and $\varphi_Y(t, p)$.*

For simplicity, we consider a trajectory of $Z = (X, Y)$ through $p \in \overline{\Sigma^s \cup \Sigma^e}$ as being multivalued, i.e., it can be any trajectory of X, Y or Z^s through p. More details can be found in [39].

Those singular points of X (resp. Y) that lie on Σ^+ (resp. Σ^-) are called *real singular points*. Conversely, those singular points of X (resp. Y) that lie on Σ^- (resp. Σ^+) are called *virtual singular points* and, although they are called "singular points" they are not singularities of the discontinuous vector field Z. Also, those singular points of X or Y that lie on Σ are called *boundary singular points*. The singularities of $Z = (X, Y)$ are split in two classes

- *distinguished singularities:* real and boundary singular points, pseudo-equilibria or singular tangency points;
- *non-distinguished singularities:* regular tangency points.

The points that are not singularities are called *regular points*.

Definition 1.10. *A discontinuous vector field $Z = (X, Y)$ has a fold singularity at $p \in \Sigma$ if p is a fold singularity for X or Y seen as vector fields defined in the manifold with boundary $\Sigma^+ \cup \Sigma$ or $\Sigma^- \cup \Sigma$, respectively. If p is a fold for X and a regular point for Y or vice-versa then p is a fold-regular point for Z. Moreover, if p is a fold for both, X and Y, then p is a fold-fold point for Z.*

Observe that, from Definition 1.8, a fold point $p \in \Sigma$ is visible if it is a fold for X (resp. Y) and $X^2h(p) > 0$ (resp. $Y^2h(p) < 0$) and it is invisible provided it is a fold for X (resp. Y) and $X^2h(p) < 0$ (resp. $Y^2h(p) > 0$).

Definition 1.11. *A discontinuous vector field Z has a saddle-regular point at $p \in \Sigma$ if p is a saddle point for X (resp. Y) and Y (resp. X) is transversal to Σ at p.*

Given a trajectory $\varphi_Z(t, q) \in \Sigma^+ \cup \Sigma^-$ and $p \in \Sigma$, p is said to be a departing point (resp. arriving point) of $\varphi_Z(t, q)$ if there exists $t_0 < 0$ (resp. $t_0 > 0$) such that $\lim_{t \to t_0^+} \varphi_X(t, q) = p$ (resp. $\lim_{t \to t_0^-} \varphi_Z(t, q) = p$). With these definitions if $p \in \Sigma^c$, then it is a departing point (resp. arriving point) of $\varphi_X(t, q)$ for any $q \in \gamma^+(p)$ (resp. $q \in \gamma^-(p)$), where

$$\gamma^+(p) = \{\varphi_Z(t, p); t \in I \cap \{t \geq 0\}\} \quad \text{and} \quad \gamma^-(p) = \{\varphi_Z(t, p); t \in I \cap \{t \leq 0\}\},$$

are the positive and negative trajectories through p, respectively. Now, we are able to give definitions of cycles, connections and separatrices.

Definition 1.12. *Let $Z = (X, Y) \in \Omega^r$ be a discontinuous vector field. A continuous closed curve Γ is said to be a cycle of the vector field Z if it is composed by a finite union of segments of regular orbits and singularities of Z, $\gamma_1, \gamma_2, \ldots, \gamma_n$. For a cycle Γ there are the following possibilities:*

- *Γ is a simple cycle, i.e, none of γ_i's are singular points and the set $\gamma_i \cap \Sigma$ is either empty or composed only by points of Σ^c, $\forall\, i = 1, \ldots, n$. If such a cycle is isolated in the set of all simple cycles of Z, then it is called limit cycle. See Figure 1.6(a);*
- *Γ is a regular polycycle, i.e., for all $i = 1, \ldots, n$, the set $\gamma_i \cap \Sigma$ is empty and at least one of γ_i's is a singular point or, for some $i = 1, \ldots, n$, $\gamma_i \cap \Sigma \subset \overline{\Sigma^c}$ is nonempty. See figures 1.6(b) and 1.6(c);*

- *Γ is a sliding cycle, i.e., there exists $i \in \{1, 2, \ldots, n\}$ such that γ_i is a segment of sliding orbit and, for any two consecutive curves, the departing or arriving points in Σ are not the same. See Figure 1.6(d);*

- *Γ is a pseudo-cycle, i.e., for some $i \in \{1, 2, \ldots, n\}$, the arriving or departing points of γ_i and γ_{i+1} coincide. See Figure 1.6(e). If the pseudo-cycle contains a segment of a sliding orbit, then it is a sliding pseudo-cycle, see Figure 1.6(f).*

A cycle is said to be degenerate if it does not persist under small perturbations of the system, i.e., if it is not structurally stable.

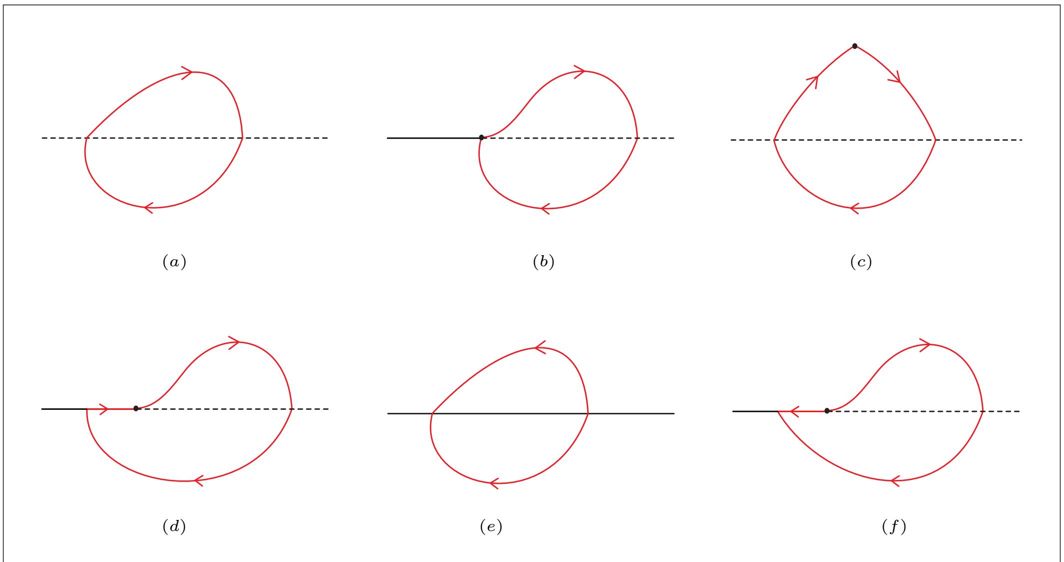

Figure 1.6: Illustration of different types of cycles: (a) simple cycle; (b), (c) regular polycycles; (d) sliding cycle; (e) pseudo-cycle and (f) sliding pseudo-cycle.

Definition 1.13. *An unstable (resp. a stable) separatrix is*

- *a regular orbit Γ which is the unstable (resp. stable) manifold of a saddle point of X , $p \in \overline{\Sigma^+}$, or of Y, $p \in \overline{\Sigma^-}$. It is denoted by $W^u(p)$ (resp. $W^s(p)$). Or,*

- *a regular orbit which has a distinguished singularity, $p \in \Sigma$, as departing (resp. arriving) point. It is denoted by $W^u_{\pm}(p)$ (resp. $W^s_{\pm}(p)$) and $\pm$ means that it leaves (resp. arrives) from $\Sigma^{\pm}$.*

If necessary, we will use the notation $W^{s,u}_{\pm}(X, p)$ to indicate which vector field is being considered. If a separatrix is, at the same time, unstable and stable then it is a separatrix connection. When a orbit Γ is connecting two distinguished singularities of Z, p and q, it will be called homoclinic connection if $p = q$ and heteroclinic connection if $p \neq q$.

Definition 1.14. *A hyperbolic pseudo-equilibrium point p is said to be a*

- *pseudonode if $p \in \Sigma^s$ (resp. $p \in \Sigma^e$) and it is an attractor (resp. a repellor) for the sliding vector field;*

- *pseudosaddle if $p \in \Sigma^s$ (resp. $p \in \Sigma^e$) and it is a repellor (resp. an attractor) for the sliding vector field.*

Now, it is necessary to discuss about equivalences in Ω^r. For discontinuous vector fields there are different notions of topological equivalence.

Definition 1.15. *Consider two discontinuous vector fields Z and $\tilde{Z}$ defined in open sets, U and $\tilde{U}$, of $\mathbb{R}^2$ with discontinuity curves Σ and $\tilde{\Sigma}$, respectively. Then,*

- *Z and $\tilde{Z}$ are $\Sigma-$equivalent if there exists an orientation preserving homeomorphism $h : U \to \tilde{U}$ that sends Σ to $\tilde{\Sigma}$ and sends orbits of Z to orbits of $\tilde{Z}$;*
- *Z and $\tilde{Z}$ are topologically equivalent if there exists an orientation preserving homeomorphism $h : U \to \tilde{U}$ that sends orbits of Z to orbits of $\tilde{Z}$.*

Observe that $\Sigma-$equivalence implies topologically equivalence but the reverse is not true, for an example illustrating that these definitions are not equivalent see [15].

Through this work we consider $\Sigma-$equivalences. Thus, a vector field $Z_0 \in \Omega^r$ is structurally stable if there exists a neighborhood $\mathcal{V}_{Z_0}$, of Z_0 in Ω^r, such that, any $Z \in \mathcal{V}_{Z_0}$ is $\Sigma-$equivalent to Z_0.

Chapter 2

Dulac's Problem

In this chapter we present a version of the Dulac's problem for discontinuous vector fields. The original problem due to Dulac concerns about vector fields that are analytic in $\mathbb{R}^2$, piecewise analytic ones are not considered. The main difference between the discontinuous and the continuous cases is the analysis of the transition map near the singularity. We emphasize that the extension obtained here is not a particular case of the original problem, there are some considerable differences in the structure of the first return maps.

It is worthwhile to emphasize the original Dulac's problem is true for elementary singular points but here we just consider hyperbolic singularities. Moreover, as a generalization, we show the result remains true if we change hyperbolic saddles by fold points, this kind of cycle does not exist for smooth vector fields.

2.1 The Problem

Let $Z = (X, Y)$ be a discontinuous vector field, where X and Y are smooth enough for our purposes. The first thing to do is to extend the concept of hyperbolic polycycle.

Definition 2.1. *A continuous closed curve Γ is said to be a hyperbolic polycycle of the vector field Z if it is composed by a finite union of segments of regular orbits of Z, $\gamma_1, \gamma_2, \ldots, \gamma_n$, and hyperbolic saddles, $p_1, p_2, \ldots, p_n$, such that for each $1 \leq i \leq n$, the $\omega, \alpha-$limit sets of γ_i are p_i and p_{i+1}, respectively. Moreover,*

- *$\gamma_i \cap \Sigma \subset \Sigma^c$, for all $i = 1, \ldots, n$;*
- *if $p_i \in \Sigma$ then $p_i \in \overline{\Sigma^c}$ and p_i is a saddle-regular or saddle-saddle point of Z for which the invariant manifolds of the saddle are transversal to Σ at this point;*
- *there exists a first return map defined in one of the two regions (bounded or unbounded) delimited by Γ.*

Remark 2.1. *If $\Gamma \cap \Sigma = \emptyset$ then Γ is a hyperbolic polycycle for one of the smooth vector fields X or Y. In this case, the problem is reduced to the hyperbolic version of the classic Dulac's Problem*
- A hyperbolic polycycle of an analytic vector field cannot have limit cycles accumulating onto it.

Theorem 2.1 (Dulac's Problem). *A hyperbolic polycycle of a piecewise analytic vector field $Z \in \Omega^\omega$ cannot have limit cycles accumulated onto it.*

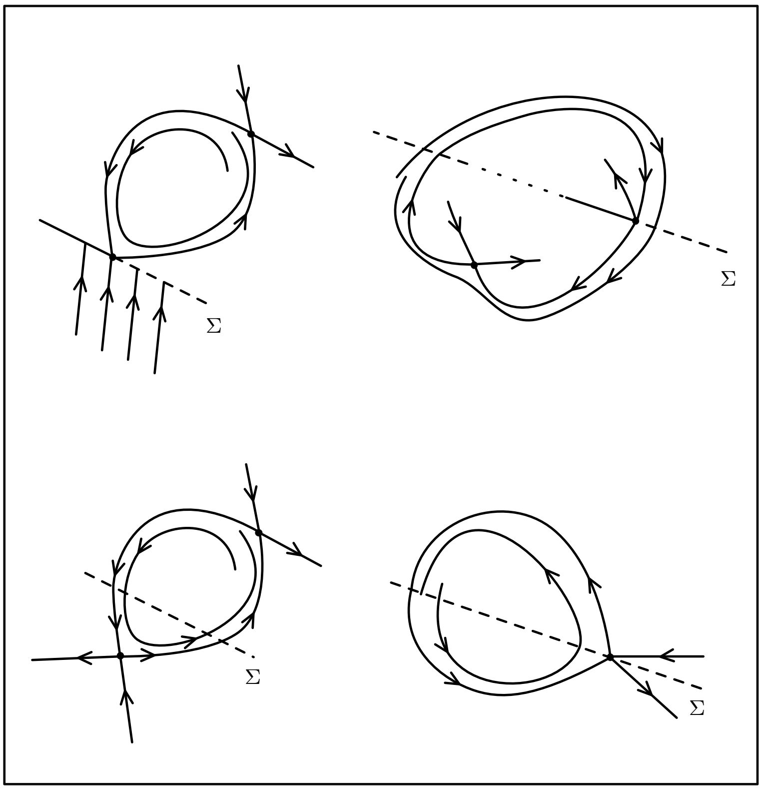

Figure 2.1: Examples of hyperbolic polycycles.

In order to prove the Theorem 2.1, it is necessary to construct the first return map in such a way we are able to analyze it. Let Γ be a hyperbolic polycycle and let $p_1, \ldots, p_n$ be the vertices of Γ, in some cyclic order. Let $\sigma' \simeq [a, b)$ be a transversal section to Γ at $a \in \Gamma$ such that the first return map P of Γ is defined from $\sigma \subset \sigma' \to \sigma'$.

At each vertex p_i choose a local system of coordinates (x_i, y_i) in such way the axis 0_{x_i} and 0_{y_i} are the local unstable and stable manifolds, respectively. If $p_i \in \Sigma$ we assume that Σ is a curve crossing transversely both axes at the origin, denote by Σ_i^+ the half-plane that contains the positive part of axis 0_{y_i} and by Σ_i^- the half-plane that contains the negative part of axis 0_{y_i}. Moreover, the first return map P is defined in the first quadrant of the $(x_i, y_i)-$plane. Consider transversal sections σ_i and τ_i defined as following:

C_1- If $p_i \notin \Sigma$ or $p_i \in \Sigma$ and Σ does not lie on the first quadrant: let σ_i and τ_i be transversal sections to 0_{y_i} at $(0, \varepsilon_i)$ and to 0_{x_i} at $(\varepsilon_i, 0)$, respectively with $\varepsilon_i > 0$. See Figures 2.2(a) and 2.2(b);

C_2- If $p_i \in \Sigma$, Σ lies on the first quadrant and p_i is a saddle of the vector field defined in Σ_i^+: let σ_i be a transversal section to 0_{y_i} at $(0, \varepsilon_i)$ with $\varepsilon_i > 0$ and let τ_i be a segment contained in Σ with $0 \in \tau_i$. See Figure 2.2(c);

C_3- If $p_i \in \Sigma$, Σ lies on the first quadrant and p_i is a saddle of the vector field defined in Σ_i^-: let τ_i be a transversal section to 0_{x_i} at $(\varepsilon_i, 0)$ with $\varepsilon_i > 0$ and let σ_i be a segment contained in Σ with $0 \in \sigma_i$. See Figure 2.2(d).

In each one of these cases, one defines a transition map D_i, near the saddle point p_i, from σ_i^+ to τ_i, where $\sigma_i^+ \subset \sigma_i$ with $x_i \geq 0$ and $D_i(0) = 0$. As we will see later, there is no loss of generality in considering σ_i and τ_i orthogonal to the axis when they are not subsets of Σ. Hence, in the cases C_1 and C_2, consider σ_i as the image of the

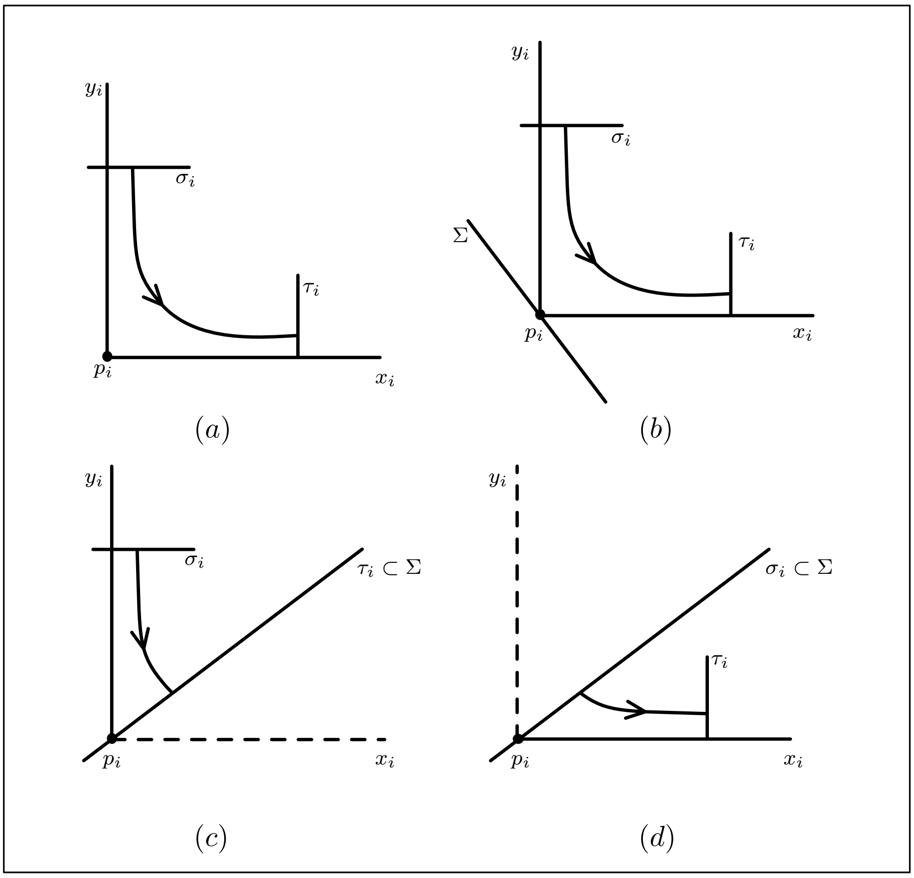

Figure 2.2: Transition maps from σ_i to τ_i: $(a) - (b)$ Case C_1, (c) Case C_2 and (d) Case C_3.

map $x \in (-\delta_i, \delta_i) \mapsto (x, \varepsilon_i) \in \mathbb{R}^2$, where δ_i, $\varepsilon_i > 0$ are small enough. In the case C_3 we consider σ_i given by the parametrization of Σ. Let $\varphi_i(t, x, y)$ be the flow near the saddle and let $t_i(x)$ be the smallest positive time spent by the trajectory passing through $(x, y(x)) \in \sigma_i^+$ to intersect τ_i. Then, define $D_i(x_i) = \pi_2(\varphi_i(t_i(x_i), x_i, \varepsilon_i))$ where π_2 is the canonical projection on the second coordinate.

After crossing τ_i, the trajectory of Z by $p \in \sigma_i$ will cross σ_{i+1}. If $\tau_i \cap \sigma_{i+1} = \emptyset$, since a regular orbit makes the connection between p_i and p_{i+1}, there exists an analytic diffeomorphism $R_i : \tau_i \to \sigma_{i+1}$ such that $R_i(q)$ is the point in σ_{i+1} where the positive trajectory of Z passing through $q \in \tau_i$ intersects σ_{i+1}. If $\tau_i \cap \sigma_{i+1} \neq \emptyset$, the transition from τ_i to σ_{i+1} will be just the identity map and for questions of completeness we also denote it by R_i. See Figure 2.3 as an example. Finally, the first return map can be written as a composition

$$P(x) = R_n \circ D_n \circ \cdots \circ R_1 \circ D_1(x). \tag{2.1.1}$$

This characterization for the first return map allows us to come back to the analysis of the Theorem 2.1. Suppose that there exits a sequence of limit cycles accumulating onto the polycycle Γ. It implies that the equation $P(x) - x = 0$ has infinitely many roots near to $x = 0$ (for $x > 0$), i.e., for any real number $\epsilon > 0$ there exists a root of this equation in $(0, \epsilon)$. Therefore, proving the Theorem 2.1 is equivalent to proving that 0 is an isolated solution of $P(x) - x = 0$.

The most difficult part in the analysis of the map P is to know the structure of each transition map in its composition. For this reason, a careful study is necessary. In order

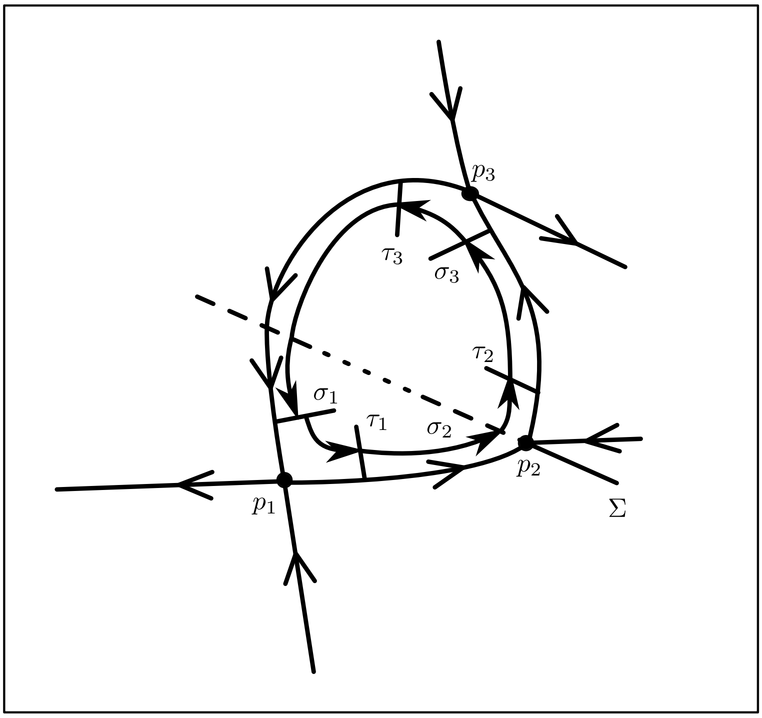

Figure 2.3: First return map for a polycycle with 3 saddle points. Here $\sigma_2 \subset \Sigma$.

to do this we need some technical definitions and results.

Definition 2.2. *The Dulac series of a map D, defined in a half open interval $[0, \delta)$ is a formal series*

$$\widehat{D}(x) = \sum_{i=1}^{\infty} x^{\lambda_i} P_i(\ln x),$$

where $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ is an increasing sequence of positive numbers tending to infinity, P_i are polynomials. Moreover, this series must be asymptotic to D in the following sense: for any $n \in \mathbb{N}$, there is $s \in \mathbb{N}$ such that

$$\left| D(x) - \sum_{i=1}^{s} x^{\lambda_i} P_i(\ln x) \right| = O(x^{\lambda_n}). \tag{2.1.2}$$

Definition 2.3. *A germ of a map f at $0 \in \mathbb{R}^+$ is called quasi-regular if*

1. *f has a representative on $[0, \delta)$ which is C^∞ on $(0, \delta)$, where δ is a positive constant;*
2. *f has a Dulac series (f is asymptotic to a Dulac series $\widehat{f}$).*

f is a quasi-regular homeomorphism if it is quasi-regular and $P_1 \equiv A$ (a positive constant).

It follows from the definition that the set $\mathcal{D}$ of the quasi-regular homeomorphisms is a group (with the composition of maps) which contains the group Diff_0 of germs of diffeomorphisms fixing the origin.

Assume, for a moment, that the transition maps near hyperbolic saddle points are quasi-regular and we will conclude the proof of the Theorem 2.1. The proof of this assumption is quite difficult and extensive so, to facilitate readability, we will prove it later.

Proposition 2.1. *The first return map P is quasi-regular.*

Proof. We have seen that $\mathcal{D}$ is a group with respect to the composition of maps. Moreover, diffeomorphisms fixing the origin and transition maps near a hyperbolic saddle are quasi-regular. Therefore, the result follows from the expression of P given in equation 2.1.1. □

Now, we proceed to show that P being quasi-regular is enough to give an affirmative answer for Theorem 2.1.

Lemma 2.1. *Suppose that $\widehat{P}(x) \not\equiv x$, then the Dulac series of $P(x) - x$ is non-zero.*

Proof. By definition, it is possible to write $\widehat{P}(x) = \sum_{i=1}^{\infty} x^{\lambda_i} P_i(\ln x)$ with $0 < \lambda_1 < \lambda_2 < \cdots$ and $\lambda_n \to \infty$ when $n \to \infty$. In order to write a Dulac series for $P - Id$, let $i_0 \geq 1$ be the smaller positive integer for which $\lambda_{i_0} \geq 1$. If $\lambda_{i_0} = 1$, define $\bar{\lambda}_i = \lambda_i$ for all $i \geq 1$, $\bar{P}_i \equiv P_i$ for $i \neq i_0$, and $\bar{P}_{i_0}(t) = P_{i_0}(t) - 1$. If $\lambda_{i_0} > 1$ then it follows from the definition of i_0 that $\lambda_{i_0 - 1} < 1 < \lambda_{i_0}$, in this case define

- $\bar{\lambda}_i = \lambda_i$, $\bar{P}_i \equiv P_i$ for all $1 \leq i \leq i_0 - 1$,
- $\bar{\lambda}_{i_0} = 1$, $\bar{P}_{i_0} \equiv -1$, and
- $\bar{\lambda}_i = \lambda_{i-1}$, $\bar{P}_i \equiv P_{i-1}$ for all $i \geq i_0$.

Then, we have a Dulac series for $P - \mathrm{Id}$ given by

$$\widehat{P - \mathrm{Id}}(x) = \sum_{i=1}^{\infty} x^{\bar{\lambda}_i} \bar{P}_i(\ln x),$$

in fact the asymptotic condition is trivially satisfied from the construction of this series. Notice that this series is identically zero if and only if $\bar{P}_i \equiv 0$ for all $i \geq 1$. But, since the Dulac series of the identity map coincides with its Taylor series, if the series above is identically zero, from the construction above, $\widehat{P}(x) \equiv \widehat{\mathrm{Id}}(x) \equiv x$. Hence, from hypothesis $\widehat{P}(x) \not\equiv x$ we have $\widehat{P - \mathrm{Id}}(x) \not\equiv 0$. □

Now, a result about the accumulation of limit cycles.

Proposition 2.2. *If there exists a sequence of limit cycles accumulating onto the hyperbolic polycycle Γ, then $\widehat{P}(x) \equiv x$.*

Proof. Suppose by contradiction that $\widehat{P}(x) \not\equiv x$ then, from Lemma 2.1 the Dulac series of $P(x) - x$ is non-zero. Thus, $\widehat{P - \mathrm{Id}}(x) = x^{\bar{\lambda}_k} \bar{P}_k(\ln x) + \sum_{i=k+1}^{\infty} x^{\bar{\lambda}_i} \bar{P}_i(\ln x)$, with $\bar{P}_k(t) \not\equiv 0$. Then, there exists $x_0 > 0$ such that $x^{\bar{\lambda}_k} \bar{P}_k(\ln x) \neq 0$ for all $x \in (0, x_0)$ and, consequently, $\widehat{P - \mathrm{Id}}(x) \neq 0$ for all $x \in (0, x_0)$. Since $P(x) - x$ is asymptotic to its Dulac series, x_0 can be chosen such that $P(x) - x$ has no roots in $(0, x_0)$. In fact, otherwise it should exists a sequence $x_n \to 0$ such that $P(x_n) - x_n \equiv 0$ and

$$\left| x_n^{\bar{\lambda}_k} \bar{P}_k(\ln x_n) \right| = O(x_n^{\bar{\lambda}_k}), \quad \text{consequently} \quad |\bar{P}_k(\ln x_n)| = \frac{O(x_n^{\bar{\lambda}_k})}{x_n^{\bar{\lambda}_k}}.$$

Now, $\bar{P}_k$ is non-zero so $\lim_{n \to \infty} \bar{P}_k(\ln x_n) \neq 0$ (because $\ln x \to -\infty$ when $x \to 0^+$ and $\bar{P}_k$ is a polynomial) while the factor on the right-hand side of the last equality tends to 0 when $n \to \infty$, so it is a contradiction. Thus, $P(x) - x = 0$ has no roots in $(0, x_0)$ and, consequently, Γ is not accumulated by limit cycles contradicting the initial hypothesis. Therefore, $\widehat{P}(x) \equiv x$. □

With these results, to prove Theorem 2.1, it is enough to show that if $\widehat{P} \equiv x$ then $P \equiv x$. In order to do that we need more technical results.

Definition 2.4. *A germ of a map $f : [0, \delta) \to \mathbb{R}$ is quasi-analytic if*

1. *f is quasi-regular;*
2. *the map $x \to f \circ \exp(-x)$ has a bounded holomorphic extension $F(z)$ in some domain Δ_b of $\mathbb{C}$, defined by $\Delta_b = \{z = u + iv;\ u > b(1 + v^2)^{1/4}\}$ where b is a positive real number.*

For quasi-analytic functions the wanted result holds. The following result is proved in Chapter 3 of [35] (pages $45 - 49$).

Theorem 2.2. *If a quasi-analytic function f is such that $\widehat{f} \equiv 0$, then $f \equiv 0$. Moreover, the transition map, D, at a hyperbolic saddle singularity is quasi-analytic. Consequently, the first return map P of a hyperbolic polycycle is quasi-analytic.*

Now, we are ready to prove the Theorem 2.1.

Theorem 2.3. *If the transition maps at hyperbolic saddle points are quasi-regular then a hyperbolic polycycle of $Z \in \Omega^\omega$ cannot have limit cycles accumulating onto it.*

Proof. The proof follows by contradiction. From Theorem 2.2 we know that the first return map P is quasi-analytic. Proposition 2.2 gives that, if Γ has limit cycles accumulating onto it, then $\widehat{P}(x) \equiv x$ and, again from Theorem 2.2, we obtain $P(x) \equiv x$. Therefore, all trajectories of Z are closed in the half-open neighborhood where P is defined. It contradicts the existence of limit cycles (which are isolated closed orbits) accumulating onto Γ. Hence, Γ cannot have limit cycles accumulating onto it. □

Remark 2.2. *It follows from Theorem 2.3 that, we just need to prove that transition maps at hyperbolic saddles are quasi-regular, to prove the Theorem 2.1. Moreover, the only difference from the classical case for smooth vector fields is the construction of the first return map.*

Now, we proceed to prove the quasi-regularity of transition maps at hyperbolic saddle points, what first requires technical results. The case C_1, when the saddle point does not belong to Σ or it belongs to Σ and Σ do not cross the region where the first return map is defined, is done in Chapter 5 of [35]. Now, we will adapt that proof to the case C_3 where the saddle is in Σ, Σ cross the region where the first return map is defined and we have a saddle for the vector field defined in Σ^-. Having proved thi result, the case C_2 follows directly from the fact that the set $\mathcal{D}$ of quasi-regular maps is a group with composition.

2.2 Transition Maps

Without loss of generality we suppose the saddle point is at the origin of the system of coordinates (x, y) and consider V_0 be the neighborhood of the saddle point where the normal form given in Chapter 1 holds. Let $\varepsilon > 0$ be a real number small enough such that the transversal sections σ_ε, where the transition map is defined, and τ_ε, which is the counter-domain of the transition map, are contained in V_0. Denote by $\varphi(t, x, y) = (\varphi_1(t, x, y), \varphi_2(t, x, y))$ the flow of the vector field X defined in Σ^-, then, from Theorem

1.2, we have $\varphi_1(t,x,y) = e^t x$. Remember that we denote by r the hyperbolicity radius of the saddle point, i.e., $r = -\mu_2/\mu_1$ where $\mu_2 < 0 < \mu_1$ are the eigenvalues of $DX(0)$. Since we are focused in just one saddle point we do not need to use the index (to specify the point) above. For further references we will denote by D_j the transition map and σ_j and τ_j the transversal sections in the case C_j, for $j = 1, 2, 3$.

Remark 2.3. *The quasi-regularity of the transition maps does not depend on the choice of the transversal sections σ_j, for $j = 1, 2$, and τ_j, for $j = 1, 3$. In fact, if we choose different transversal sections at the correspondent branch of the invariant manifold, the transition between these two transversal sections is done by means of a diffeomorphism. As we have seen above $\mathcal{D}$ is a group, then the new transition map is quasi-regular provided the old one is quasi-regular. This idea is illustrated in Figure 2.4. For σ_3 and τ_2 it is important to keep them contained in Σ but, once Σ does not depend on the vector field, these assumptions are not problematic.*

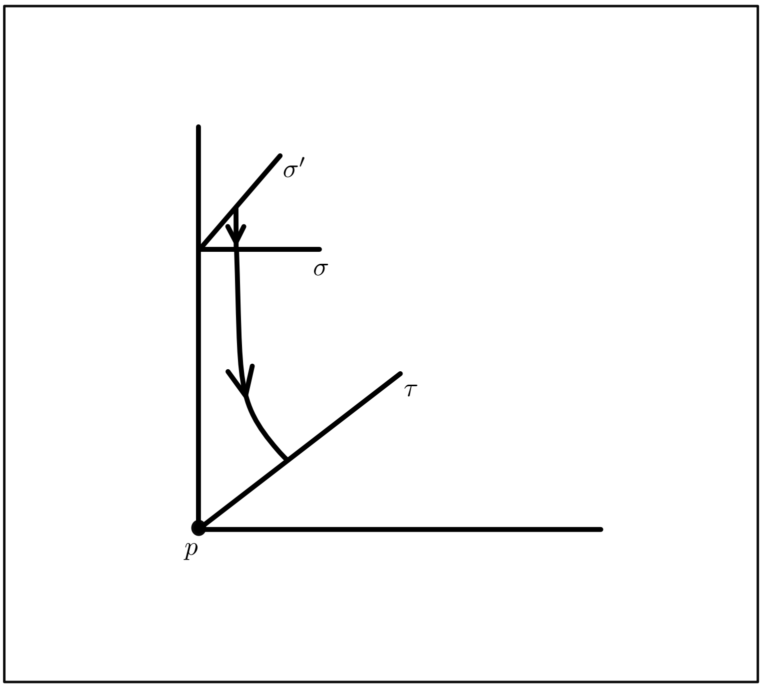

Figure 2.4: The transitions from σ to σ' is a diffeomorphism. Being D (from σ to τ) quasi-regular, then the transition D' (from σ' to τ) is also quasi-regular.

Remark 2.4. *Since the analysis is local we can suppose that Σ is locally a straight line. Moreover, under a linear change of coordinates $(x, y) \mapsto (ax, by)$ with $a, b > 0$ such that in these new coordinates $\Sigma = \{(x, y) \in \mathbb{R}^2; y = x\}$. Furthermore, the linear change of coordinates $(x, y) \mapsto (\varepsilon x, \varepsilon y)$ allows us to assume that $\varepsilon = 1$. In fact, this change will just modify the coefficients of the resonant terms in the normal form given in Chapter 1.*

Now, in order to study the transition maps, assume $\Sigma = \{(x, x) \in \mathbb{R}^2\}$, $\varepsilon = 1$ and fix transversal sections as following:

C_1- We have $\sigma_1 = \{(x, 1); 0 \le x < \delta_1\}$ and $\tau_1 = \{(1, y); |y| < \delta_2\}$ for some $\delta_1, \delta_2 > 0$. To find the transition time, i.e., the time spent by trajectories of X to go from σ_1 to τ_1, it is just necessary to solve, in t, the following equation

$$\varphi_1(t, x, 1) = 1,$$

and we obtain $t = t_1(x) = -\ln x$. In this case the transition map is $D_1(x) = \varphi_2(t_1(x), x, 1)$.

C_2- We have $\sigma_2 = \sigma_1$ and $\tau_2 = \{(x, x) \in \mathbb{R}^2; x \in (-\delta, \delta)\} \subset \Sigma$, for some $\delta > 0$. Let $t_2(x)$ be transition time from σ_2 to τ_2, then $D_2(x) = \varphi_1(t_2(x), x, 1) = \varphi_2(t_2(x), x, 1)$. We will not study this case now. We will show that D_1 and D_3 (to be defined below) are quasi-regular, so, as we have seen above, it is enough to obtain the quasi-regularity of D_2.

C_3- We have $\sigma_3 = \tau_2$ and $\tau_3 = \{(1, y) \in \mathbb{R}^2; |y|< \delta_3\} \subset \Sigma$, for some $\delta_3 > 0$. So, if $t_3(x)$ is the transition time from σ_3 to τ_3, then

$$\varphi_1(t_3(x), x, x) = 1,$$

so $t_3(x) = -\ln x$ and $D_3(x) = \varphi_2(t_3(x), x, x)$.

2.2.1 Quasi-Regularity of Transition Maps

The normal form considered has two different expressions provided $r \in \mathbb{Q}$ or $r \notin \mathbb{Q}$. For this reason we have two different situations to explore. Moreover, from Remark 2.3 we have $D_1(x) = D_3 \circ D_2(x)$. From the fact that $\mathcal{D}$ is a group with the composition of maps it is enough to prove that two of them are quasi-regular, and we are going to prove that D_1, $D_3 \in \mathcal{D}$.

I) Case: $r \notin \mathbb{Q}$

If $r \notin \mathbb{Q}$ then $\varphi_2(t, x, y) = e^{-rt}y$. We obtain $D_1(x) = x^r$ which is an analytic map that coincides with its Dulac series

$$\hat{D}_1(x) = \sum_{i=1}^{\infty} x^{\lambda_i^1} P_i^1(\ln x),$$

where $\lambda_1^1 < \lambda_2^1 < \cdots$ is an increasing sequence tending to infinity with $\lambda_1^1 = r > 0$, $P_1^1(u) \equiv 1$ and $P_i^1(u) \equiv 0$ for all $i > 1$. Hence, $D_1(x)$ is a quasi-regular homeomorphism.

Also, $D_3(x) = e^{-rt_3(x)}x = x^{1+r}$ which is an analytic map that coincides with the Dulac series

$$\hat{D}_3(x) = \sum_{i=1}^{\infty} x^{\lambda_i^3} P_i^3(\ln x),$$

where $\lambda_1^3 < \lambda_2^3 < \cdots$ is an increasing sequence tending to infinity with $\lambda_1^3 = 1 + r$, $P_1^3(u) \equiv 1$ and $P_i^3(u) \equiv 0$ for all $i > 1$. Hence, $D_3(x)$ is a quasi-regular homeomorphism.

Hence, D_1, D_2 and D_3 are quasi-regular homeomorphisms.

II) Case: $r = \dfrac{p}{q} \in \mathbb{Q}$, p and q without common factors

If $r \in \mathbb{Q}$ then we cannot find the flow of X explicitly and it makes a huge difference in the analysis. It is not possible to obtain an algebraically expression for $D_i(x)$ however it is possible to prove that D_i is a quasi-regular homeomorphism for $i = 1$, 3 and, consequently, for $i = 2$.

When $r = \frac{p}{q} \in \mathbb{Q}$ we have one characterization of X, up to C^k-conjugacy, given in the Theorem 1.1.

More generally, we can consider the analytic vector field

$$X = x\frac{\partial}{\partial x} + \frac{1}{q}\left(-p + \sum_{i=0}^{\infty} \alpha_{i+1}(x^p y^q)^i\right) y\frac{\partial}{\partial y}, \tag{2.2.1}$$

where $P_\alpha(z) = \sum_{i=1}^{\infty} \alpha_i z^i$ is an analytic function of $z \in \mathbb{R}$, and $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathcal{A}$, where $\mathcal{A}$ is the set

$$\mathcal{A} = \left\{ \alpha = (\alpha_1, \alpha_2, \ldots);\ |\alpha_1| < \frac{1}{2}, |\alpha_i| < M \text{ for } i \geq 2 \right\} \tag{2.2.2}$$

and $M > 0$ is a fixed constant. Now, we need to study the properties of $D_i(x)$ at $x = 0$ for $i = 1, 3$.

Consider the singular change of coordinates $x = x$, $u = x^p y^q$, so the differential equation associated with X becomes

$$\begin{cases} \dot{x} = & x \\ \dot{u} = & P_\alpha(u) = \sum_{i=1}^{\infty} \alpha_i u^i. \end{cases} \tag{2.2.3}$$

Now, system (2.2.3) has separable variables and $\varphi_1(t, x) = e^t x$ is the solution for the first equation satisfying $\varphi_1(0, x) = x$. For the second one, P_α is analytic for $\alpha \in \mathcal{A}$. Let $\varphi_2(t, u)$ be the solution of this equation, which is analytic, verifying $\varphi_2(0, u) = u$.

We can expand $\varphi(t, u)$ in series in u for each t,

$$\varphi_2(t, u) = \sum_{i=1}^{\infty} g_i(t) u^i. \tag{2.2.4}$$

Using this expression we obtain:

Lemma 2.2. *Consider $\varphi_2(t, u)$ as defined above, then $g_1(t) = e^{\alpha_1 t}$ and $g_i(0) = 0$ for $i \geq 2$.*

Proof. From equation 2.2.4 we obtain

$$\frac{\partial}{\partial t} \varphi_2(t, u) = \frac{\partial}{\partial t} \left(\sum_{i=1}^{\infty} g_i(t) u^i \right) = \sum_{i=1}^{\infty} \dot{g}_i(t) u^i,$$

where the change in the order of the limits above is possible due to the analyticity of the solution and from the fact that power series which converges also converges uniformly. Since $\varphi_2(t, u)$ is the solution of the second equation in 2.2.3, we have

$$\frac{\partial}{\partial t} \varphi_2(t, u) = P_\alpha(\varphi_2(t, u)) = \sum_{i=1}^{\infty} \alpha_i \left(\sum_{j=1}^{\infty} g_j(t) u^j \right)^i.$$

From these two equalities and the uniqueness of the Taylor series we obtain the following system

$$E_g = \begin{cases} \dot{g}_1(t) & = & \alpha_1 g_1(t) \\ \dot{g}_2(t) & = & \alpha_1 g_2(t) + \alpha_2 g_1^2(t) \\ \dot{g}_3(t) & = & \alpha_1 g_3(t) + 2\alpha_2 g_1(t) g_2(t) + \alpha_3 g_1^3(t) \\ & \vdots & \\ \dot{g}_k(t) & = & \alpha_1 g_k(t) + P_k(\alpha_2, \ldots, \alpha_k; g_1, \ldots, g_{k-1}), \end{cases} \tag{2.2.5}$$

where, for each $k > 1$, P_k is a polynomial in $\alpha_2, \ldots, \alpha_k, g_1, \ldots, g_{k-1}$ with positive rational coefficients.

Therefore, $g_1(t) = e^{\alpha_1 t} g_1(0)$. Furthermore, from $P_\alpha(u) = P_\alpha(\varphi_2(0, u))$ we obtain $g_1(0) = 1$ and $g_i(0) = 0$ for $i \geq 2$. □

We want to study the structure of the functions g_i and the convergence of the series $\sum_{i=1}^{\infty} g_i(t)u^i$ as a function of t. In order to do this, consider the following equation, which will be referred as hyperbolic equation:

$$\dot{U} = \frac{1}{2}U + M\sum_{i=1}^{\infty} U^{i+1}. \tag{2.2.6}$$

Then, the following statement holds:

Lemma 2.3. *Let* $U(t,u) = \sum_{i=1}^{\infty} G_i(t)u^i$ *be the power series expansion of the trajectories of* (2.2.6)*. Then, for each* $i \geq 1$ *and* $t \geq 0$*,* $|g_i(t)| \leq G_i(t)$ *for any* $\alpha \in \mathcal{A}$*.*

Proof. The proof of this lemma can be found in [35] (Lemma 18, page 95). Notice that $\dot{U} = P_\alpha(U)$ with $\alpha = \left(\frac{1}{2}, M, M, \dots\right)$, so the functions $G_i(t)$ must verify

$$E_G = \begin{cases} \dot{G}_1(t) &= \frac{1}{2}G_1(t) \\ \dot{G}_2(t) &= \frac{1}{2}G_2(t) + MG_1^2(t) \\ \dot{G}_3(t) &= \frac{1}{2}G_3(t) + 2MG_1(t)G_2(t) + MG_1^3(t) \\ &\vdots \\ \dot{G}_k(t) &= \frac{1}{2}G_k(t) + P_k(M,\dots,M;G_1,\dots,G_{k-1}), \end{cases} \tag{2.2.7}$$

where, for each $k > 1$, P_k is a polynomial as in system 2.2.5.

Then, $G_1(t) = e^{\frac{t}{2}} > 0$ and $G_i(0) = 0$, for $i \geq 0$. So, $\dot{G}_2(t) = 1/2G_2(t) + Me^t$ and we obtain

$$G_2(t) = e^{\frac{t}{2}} \int_0^t MG_1^2(s)e^{-\frac{s}{2}}\mathrm{ds} \geq 0.$$

More generally, we obtain

$$G_k(t) = e^{\frac{t}{2}} \int_0^t P_k(M,\dots,M;G_1(s),\dots,G_{k-1}(s))e^{-\frac{s}{2}}\mathrm{ds} \geq 0.$$

Notice that $G_k(t) > 0$ if $t > 0$.

Now, we are going to prove the result by induction. For $\alpha \in \mathcal{A}$, $\alpha_1 < 1/2$ so

$$|g_1(t)| = e^{\alpha_1 t} \leq e^{\frac{t}{2}} = G_1(t), \quad \text{for all } t \geq 0.$$

Suppose, by induction hypothesis, that $|g_k(t)| \leq G_k(t)$ for each $1 \leq k \leq i-1$ and $t \geq 0$. We have,

$$\dot{g}_i(t) = \alpha_1 g_i(t) + P_i(\alpha_2,\dots,\alpha_i;g_1,\dots,g_{i-1})$$

and

$$\dot{G}_i(t) = \frac{1}{2}G_i(t) + P_i(M,\dots,M;G_1,\dots,G_{i-1}),$$

where P_i are polynomials with positive coefficients, then

$$\begin{aligned} |P_i(\alpha_2,\dots,\alpha_i;g_1,\dots,g_{i-1})| &\leq P_i(|\alpha_2|,\dots,|\alpha_i|;|g_1|,\dots,|g_{i-1}|) \\ &\leq |P_k(M,\dots,M;G_1,\dots,G_{k-1})|. \end{aligned}$$

Now, we know that $g_1(0) = 1$, $G_1(0) = 1$ and $g_k(0) = G_k(0) = 0$ for $k \geq 2$, hence

$$\dot{G}_k(0) = P_k(M,\dots M;G_1(0),\dots,G_{k-1}(0)) = MG_1^k(0) = M, \quad \text{for all } k \geq 2,$$

and

$$|\dot{g}_i(0)| \leq |\alpha_i g_1(0)^i| = |\alpha_i| < M, \quad \text{for all } k \geq 2.$$

Then, $|\dot{g}_k(0)| < \dot{G}_k(0)$, for all $k \geq 2$. By continuity, there exists $\delta > 0$ small enough such that $|\dot{g}_i(t)| < G_i(0)$ for all $t \in [0, \delta)$. It means that G_i increases faster than $|g_i|$, so $|g_i(t)| < G_i(t)$ for all $t \in (0, \delta)$. Now, suppose by contradiction that there exists $t_0 > 0$ such that $|\dot{g}_i(t_0)| \geq \dot{G}_i(t_0)$ and $|\dot{g}_i(t)| < \dot{G}_i(t)$ for all $t \in [0, t_0)$. Then, from continuity we must have $|g_i(t_0)| < G_i(t_0)$. Induction hypothesis gives us $|g_k(t_0) \leq G_k(t_0)$ for all $1 \leq k \leq i-1$, then for $t = t_0$

$$\dot{g}_i(t_0) = \alpha_1 g_i(t_0) + P_i(\alpha_2, \ldots, \alpha_i; g_1(t_0), \ldots, g_{i-1}(t_0))$$

and

$$\dot{G}_i(t_0) = \frac{1}{2} G_i(t_0) + P_i(M, \ldots, M; G_1(t_0), \ldots, G_{i-1}(t_0)).$$

So,

$$\begin{aligned} |\dot{g}_i(t_0)| &= |\alpha_1||g_i(t_0)| + P_i(|\alpha_2|, \ldots, |\alpha_i|; |g_1(t_0)|, \ldots, |g_{i-1}(t_0)|) \\ &< \frac{1}{2} G_i(t_0) + P_i(M, \ldots, M; G_i(t_0), \ldots, G_{i-1}(t_0)) = \dot{G}_i(t_0), \end{aligned}$$

so we have a contradiction with our assumption on t_0. It implies that $|\dot{g}_i(t)| < \dot{G}_i(t)$ for all $t \geq 0$. Therefore, $|g_i(t)| \leq G_i(t)$ for all $t \geq 0$. Finally, we have the result proved by induction. □

Next, we have more estimations to prove.

Lemma 2.4. *There exist constants $C, C_0 > 0$ such that*

$$|g_i(t)| \leq C_0 (C e^{t/2})^i \quad \textit{for any } i \geq 1, t \geq 0 \quad \textit{and any } \alpha \in \mathcal{A}.$$

Proof. The proof of this lemma can be found in[35] (Lemma 19, page 97). From Lemma 2.3 it is enough to prove that $G_i(t) \leq C_0 (C e^{\frac{t}{2}})^i$ for some positive constants C_0 and C and for all $i \geq 1$, $t \geq 0$ and $\alpha \in \mathcal{A}$. Since $U(t, u) = \sum_{i \geq 1} G_i(t) u^i$ is a trajectory of the hyperbolic equation $\dot{u} = P(u)$ or, equivalently, of the vector field $X = P(u) \dfrac{\partial}{\partial u}$ with $P(u) = \frac{1}{2} u + M \sum_{i \geq 1} u^{i+1}$.

The Linearization Theorem of Poincaré (see [10], page 71) ensures that there exists an analytic diffeomorphism $\xi(u) = u + O(u)$, converging for $|u| \leq K_1$, $K_1 > 0$ a fixed constant, such that

$$\xi_* \left(P(u) \frac{\partial}{\partial u} \right) = \frac{1}{2} u \frac{\partial}{\partial u}.$$

Moreover, this diffeomorphism sends the flow $U(t, u)$ of $P(u) \dfrac{\partial}{\partial u}$ into the flow $U_0(t, u) = u e^{\frac{t}{2}}$ of $\frac{1}{2} u \frac{\partial}{\partial u}$. It means that $U_0(t, \xi(u)) = \xi(U(t, u))$ for $|u|$, $|U(t, u)| \leq K_1$.

Now, we want to show that there exist constants $0 < b < B$ such that $b|u| \leq |\xi(u)| \leq B|u|$, for $|u| \leq K_1$. In fact, ξ is a diffeomorphism then it satisfies the Lipschitz condition, i.e., there exists $B > 0$ such that $|\xi(u)| \leq B|u|$, for all $|u| \leq K_1$. Also, ξ^{-1} is a diffeomorphism, so there exists $k_1 > 0$ such that $|\xi^{-1}(v)| \leq k_1 |v|$, for all $|v| \leq \max_{|u| \leq K_1} |\xi(u)|$. Then, for $v = \xi(u)$, $|u| \leq K_1$, we have $|u| = |\xi^{-1}(v)| \leq k_1 |v| = k_1 |\xi(u)|$. So, taking $b = 1/k_1$ we achieve the wanted inequalities.

Notice that, $0 < \frac{b}{B}e^{\frac{-t}{2}} < 1$ for all $t \geq 0$, so by defining $R(t) = \frac{b}{B}K_1 e^{-\frac{t}{2}}$ we have $R(t) \leq K_1$ for all $t \geq 0$. We now can restrict $|u| \leq R(t)$, then $|\xi(u)| \leq B|u| \leq BR(t) = bK_1 e^{-\frac{t}{2}}$. It implies that $|U_0(t, \xi(u))| = |\xi(u)|e^{\frac{t}{2}} \leq bK_1$. Since $U(t,u) = \xi^{-1}(U_0(t, \xi(u)))$ we have

$$|U(t,u)| = |\xi^{-1}(U_0(t,\xi(u)))| \leq \frac{1}{b}|U_0(t,\xi(u))| \leq \frac{1}{b}bK_1 = K_1.$$

By definition, $U(t,u) = \sum_{i\geq 1} G_i(t)u^i$ so, $\dfrac{\partial^i}{\partial u^i}U(t,u)\mid_{u=0} = i!\, G_i(t)$ and by applying Cauchy's inequalities (see Appendix A) for the coefficients $G_i(t)$ we find, for all $i \geq 1$ and $t \geq 0$,

$$|G_i(t)| \leq \frac{1}{R^i(t)} \sup\{|U(t,u)|; |u| = R(t)\} \leq \frac{K_1}{R^i(t)}. \tag{2.2.8}$$

So, for all $i \geq 1$ and $t \geq 0$,

$$|g_i(t)| \leq G_i(t) \leq \frac{K_1}{R^i(t)} = K_1 \left[\left(\frac{B}{b}K^{-1}\right)e^{\frac{t}{2}}\right]^i.$$

Therefore, the result is achieved by choosing $C_0 = K_1$ and $C = \frac{B}{b}K_1^{-1}$. ☐

Lemma 2.5. *For each $k \geq 1$, there exists a constant $C_k > 0$ such that*

$$\left|\frac{d^k g_i}{dt^k}(t)\right| \leq C_k(Ce^{t/2})^i \quad \textit{for any } i \geq 1, t \geq 0, \alpha \in \mathcal{A}, \tag{2.2.9}$$

where C is the same constant as in Lemma 2.4.

Proof. This result is proved in [35] (Lemma 20, page 98). Notice that the series in u of $\dfrac{\partial}{\partial t}\varphi_2$ has the same radius of convergence as $\varphi(t,u)$, it happens because $\dfrac{\partial}{\partial t}\varphi_2(t,u) = P_\alpha(\varphi_2(t,u))$. Similarly, we obtain, by induction on k, that $\dfrac{\partial^k}{\partial t^k}\varphi_2(t,u)$ has the same radius of convergence for all $k \geq 1$.

By means of the Cauchy's inequality we obtain

$$\left|\frac{d^k}{dt^k}g_i(t)\right| \leq \frac{1}{R^i(t)} \sup\left\{\left|\frac{\partial^k}{\partial t^k}\varphi_2(t,u)\right|; |u| = R(t)\right\}.$$

Now, for $C = \frac{B}{b}K_1^{-1}$ and some $C_k > 0$ we have $\left|\dfrac{d^k}{dt^k}g_i(t)\right| \leq C_k(Ce^{t/2})^i$. ☐

In order to obtain a more precise form for the functions $g_i(t)$ we introduce the function

$$\Omega(\alpha_1, t) = \begin{cases} \dfrac{e^{\alpha_1 t} - 1}{\alpha_1} & \text{for } \alpha_1 \neq 0 \\ t & \text{for } \alpha_1 = 0 \end{cases}.$$

Thus, we have the following

Proposition 2.3. *For each $k \geq 1$, $g_k(t) = e^{\alpha_1 t}Q_k(t)$, where Q_k is a polynomial of degree $\leq k-1$ in Ω. The coefficients of Q_k are polynomials in $\alpha_1, \ldots, \alpha_k$. More precisely,*

$$Q_k = \alpha_k + \bar{Q}_k(\alpha_1, \ldots, \alpha_k, \Omega), \tag{2.2.10}$$

where $\bar{Q}_k$ is a polynomial of degree $\leq k-1$ in Ω with coefficients in $\mathcal{I}(\alpha_1, \ldots, \alpha_{k-1}) \cap \mathcal{I}(\alpha_1, \ldots, \alpha_k)^2$
$\subset \mathbb{Q}[\alpha_1, \ldots, \alpha_k]$.

Proof. This result is proved in [35] (Proposition 10, page 99). Since $\varphi_2(t,u) = \sum_{i\geq 1} g_i(t)u^i$ is solution of $\dot{u} = P_\alpha(u)$ with $\varphi(0,u) = u$, we have from system E_g in equation 2.2.5 that $\dot{g}_k(t) = \alpha_1 g_k(t) + P_k(\alpha_2,\ldots,\alpha_k; g_1(t),\ldots,g_{k-1}(t))$ where P_k is polynomial of degree $\leq k$ in g_1, $g_1,\ldots,g_{k-1}$ and coefficients polynomial linear in α_2, $\alpha_3,\ldots,\alpha_k$. Observe that the monomials in P_k are of the form $g_1^{l_1} g_2^{l_2} \cdots g_{k-1}^{l_{k-1}}$ with $\sum\limits_{j=1}^{k-1} j \cdot l_j = k$ and $2 \leq \sum\limits_{j=1}^{k-1} l_j \leq k$.

Also, from E_g system we obtain $g_1(t) = e^{\alpha_1 t}$, $g_2(t) = \alpha_2 e^{\alpha_1 t}\left(\frac{e^{\alpha_1 t}-1}{\alpha_1}\right) = \alpha_2 e^{\alpha_1 t}\Omega$ and, more generally,

$$g_k(t) = e^{\alpha_1 t}\int_0^t e^{-\alpha_1 s} P_k(\alpha_2,\ldots,\alpha_k; g_1(s),\ldots,g_{k-1}(s))\text{ds}.$$

Observe that, $e^{\alpha_1 t} = 1+\alpha_1\Omega$. We are going to use induction to prove that $g_k(t) = e^{\alpha_1 t}Q_k(t)$ with Q_k polynomial of degree $\leq k$ in Ω and coefficients polynomial in $\alpha_1,\ldots,\alpha_k$. As seen above, the result is true for $k=1$ and $k=2$, so suppose that it true for all $j \leq k-1$. Consider the equation

$$\dot{g}_k = \alpha_1 g_k + P_k(\alpha_2,\ldots,\alpha_k; g_1,\ldots,g_{k-1}),$$

P_k has monomials of the form $g_1^{l_1} g_2^{l_2} \cdots g_{k-1}^{l_{k-1}}$, then by induction hypothesis, $g_j = e^{\alpha_i t}Q_j(t)$ for $1 \leq j \leq k-1$. Hence, for $i,j \leq k-1$ we have $g_i \cdot g_j = e^{2\alpha_1 t}Q_i(t)Q_j(t)$. Since $\sum\limits_{j=1}^{k-1} l_j \geq 2$ we have

$$g_1^{l_1} g_2^{l_2} \cdots g_{k-1}^{l_{k-1}} = e^{2\alpha_1 t}\bar{X}_k(t),$$

where $\bar{X}_k$ is a polynomial of degree $\leq k-2$ in $g_1,\ldots,g_{k-1}$. Now, since $e^{\alpha_1 t} = 1+\alpha_1\Omega$ we have that $g_1^{l_1} g_2^{l_2} \cdots g_{k-1}^{l_{k-1}}$ is polynomial of degree $\leq \sum\limits_{j=1}^{k-1} j \cdot l_j = k$ in Ω. Therefore, $P_k(\alpha_2,\ldots,\alpha_k; g_1,\ldots,g_{k-1}) = e^{2\alpha_1 t}X_k(\Omega)$ where X_k is polynomial of degree $\leq k-2$ in Ω. So,

$$\begin{aligned}
g_k(t) &= e^{\alpha_1 t}\int_0^t e^{-\alpha_1 s}e^{2\alpha_1 s}X_k(\Omega(s))\text{ds} \\
&= e^{\alpha_1 t}\int_0^t e^{\alpha_1 s}X_k(\Omega(s))\text{ds} \\
&= e^{\alpha_1 t}\int_0^t (1+\alpha_1\Omega(s))X_k(\Omega(s))\text{ds},
\end{aligned}$$

therefore, $g_k(t) = e^{\alpha_1 t}Q_k(t)$ with Q_k polynomial of degree $\leq k-1$ in Ω and coefficients polynomial in $\alpha_1,\ldots,\alpha_k$.

Observe that $P_k(\alpha_2,\ldots,\alpha_k; g_1,\ldots,g_{k-1}) = \alpha_k g_1^k + \tilde{P}_k(\alpha_2,\ldots,\alpha_{k-1}; g_1,\ldots,g_{k-1})$, for $k \geq 2$, where $\tilde{P}_k$ is linear in $\alpha_2,\ldots,\alpha_{k-1}$ and each monomial in $\tilde{P}_k$ contains at least one of g_i with $i \geq 2$. Moreover, the coefficients of g_i are in $\mathcal{I}(\alpha_1,\ldots,\alpha_i)$ and P_k contains, at least, one term multiplied by $\alpha_i\alpha_j g_i g_j$, so the coefficients of $\tilde{P}_k$ are in $\mathcal{I}(\alpha_2,\ldots,\alpha_{k-1})^2 \cap \mathcal{I}(\alpha_1,\ldots,\alpha_{k-1})$. Now,

$$\begin{aligned}
Q_k &= \int_0^t e^{-\alpha_1 s}P_k(\alpha_2,\ldots,\alpha_k; g_1,\ldots,g_{k-1})\text{ds} \\
&= \int_0^t e^{-\alpha_1 s}(\alpha_k g_1^k + \tilde{P}_k(\alpha_2,\ldots,\alpha_{k-1}; g_1,\ldots,g_{k-1}))\text{ds} \\
&= \int_0^t \alpha_k e^{-\alpha_1 s}e^{k\alpha_1 s}\text{ds} + \int_0^t e^{-\alpha_1 t}\tilde{P}_k(\alpha_2,\ldots,\alpha_k; g_1,\ldots,g_{k-1})\text{ds}.
\end{aligned}$$

The first term in this sum is

$$I_1 = \frac{\alpha_k}{\alpha_1(k-1)}\left(e^{(k-1)\alpha_1 t} - 1\right) = \frac{\alpha_k}{\alpha_1(k-1)}\left((1+\alpha_1\Omega)^{(k-1)} - 1\right) = \alpha_k\Omega + \frac{\alpha_1\alpha_k}{k-1}S(\Omega),$$

where $S(\Omega)$ is a polynomial of degree ≥ 2 and ≤ -2 in Ω. The second term in the expression of Q_k is

$$I_2 = \int_0^t e^{-\alpha_1 s}\tilde{P}_k(s)\mathrm{ds},$$

since $\tilde{P}_k$ has at least one term in g_i we have $\tilde{P}_k(s) = e^{\alpha_1 s}\bar{P}_k(s)$ where $\bar{P}_k$ has degree $\leq k-2$ in Ω. So, I_2 is polynomial in Ω with coefficients in $\mathcal{I}(\alpha_1,\ldots,\alpha_{k-1}) \cap \mathcal{I}(\alpha_2,\ldots,\alpha_{k-1})^2$. Therefore, $Q_k = I_1 + I_2 = \alpha_k\Omega + \bar{Q}_k$ where $\bar{Q}_k$ is a polynomial in Ω with coefficients in $\mathcal{I}(\alpha_1,\ldots,\alpha_{k-1}) \cap \mathcal{I}(\alpha_2,\ldots,\alpha_k)^2 \subset \mathbb{Q}[\alpha_1,\ldots,\alpha_k]$. □

Now we are ready to go back to the transition maps. The results above enable us to study the transition map D_i, $i = 1, 3$. We know that, in both cases, the time spent to go from σ to τ is equal to $t(x) = -\ln x$. Therefore, we have

- $u_{|\sigma_1} = x^p 1^q = x^p$ and $u_{|\tau_1} = (e^{t(x)}x)^p D_1(x)^q = D_1(x)^q$. Also, $u_{|\tau_1} = u(t(x), u_{|\sigma_1})$, i.e.,

  $$u_{|\tau_1} = u\left(-\ln x, x^p\right) = \sum_{i\geq 1} g_i\left(-\ln x\right)x^{pi},$$

  if this series converges for $t = -\ln x$. Therefore, we have

  $$D_1(x)^q \;\; = \sum_{i\geq 1} g_i\left(-\ln x\right)x^{pi}, \qquad (2.2.11)$$

  for $x > 0$ and $D_1(0) = 0$.

- $u_{|\sigma_3} = x^p x^q = x^{p+q}$ and $u_{|\tau_3} = (e^{t(x)}x)^p D_3(x)^q = D_3(x)^q$. Also, $u_{|\tau_3} = u(t(x), u_{|\sigma_3})$, i.e.,

  $$u_{|\tau_3} = u\left(-\ln x, x^{p+q}\right) = \sum_{i\geq 1} g_i\left(-\ln x\right)x^{(p+q)i},$$

  if this series converges for $t = -\ln x$. In this case, we have

  $$D_3(x)^q \;\; = \sum_{i\geq 1} g_i\left(-\ln x\right)x^{(p+q)i}, \qquad (2.2.12)$$

  for $x > 0$ and $D_3(0) = 0$.

Notice that D_i, $i = 1, 3$, is well defined for $x \in [0, \epsilon]$, where $\epsilon > 0$ is a real number, and it is analytic in (x, α), for $x \neq 0$, $\alpha \in \mathcal{A}$. In fact, for $x \neq 0$ and each $t \leq 0$ the convergence of the series in the expression of D_i^q is due to Lemma 2.4 and from this lemma we obtain that the convergence radius of the series $\sum_{i\geq 1} g_i\left(t\right)u^i$ is, for each $t \geq 0$, greater than or equal to $\frac{e^{-\frac{t}{2}}}{C}$. It implies that, for x small enough, the series converges for each $t < -2\ln x$. In particular, the series converges when $t = -\ln x$, so the expressions in (2.2.11) and (2.2.12), for D_1^q and D_3^q, hold. Furthermore, the convergence is normal on the interval $[0, \epsilon]$.

Lemma 2.6. *The series $\sum_{i\geq 1} g_i\left(-\ln x\right)x^{ai}$ is normally convergent for $x \in (0, \epsilon]$ for some $\epsilon > 0$, where $a \geq 1$ is a fixed constant.*

Proof. By definition, this series is normally convergent in a set S if the series $\sum_{i\geq 1} \sup_S |g_i(-\ln x)\, x^{ai}|$ converges. It follows from Lemma 2.4 that, for all $t \geq 0$,

$$|g_i(-\ln x)\, x^{ai}| = |g_i(-\ln x)\,||x|^{ai} \leq C_0|Ce^{\frac{-\ln x}{2}}|^i|x|^{ai} = C_0|Cx^{a-\frac{1}{2}}|^i.$$

So, by fixing a positive constant $0 < b < 1$ we can choose $\epsilon > 0$ small enough such that $|Cx^{a-\frac{1}{2}}| < b < 1$ for all $0 \leq x \leq \epsilon$. It implies that, $\sup_{x\in[0,\epsilon]} |g_i(-\ln x)\, x^{ai}| \leq C_0 b^i$. Since $0 < b < 1$, it follows that $\sum_{i\geq 1} \sup_{x\in[0,\epsilon]} |g_i(-\ln x)\, x^{ai}|$ is convergent and we have the result. $\square$

Remark 2.5. *From the proof above, we have that* $\lim_{t\to 0^+} g_i(-\ln x) = 0$*. Since* $D_j(0) = 0$, $j = 1, 3$*, we can assume the series in Lemma 2.6 is normally convergent in* $[0, \epsilon]$*.*

Before stating the next result, the following definition is needed.

Definition 2.5. *A function* $f : \mathbb{R}^n \to \mathbb{R}$ *of class* $\mathcal{C}^k$ *is said to be* $\mathcal{C}^k-$*flat in the coordinate* j*, at* $x_j = x_j^0$*, if* $\dfrac{\partial f}{\partial x_j}(x_0) = \cdots = \dfrac{\partial^k f}{\partial x_j^k}(x_0) = 0$*, where* $x_0 = (x_1, \ldots, x_{j-1}, x_j^0, x_{j+1}, \ldots, x_n)$.

From now on, we consider $a \in \{p, p+q\}$ and we show that $D_a(x)$, satisfying $D_a^q(x) = \sum_{i\geq 1} g_i(-\ln x)x^{ai}$, is quasi-regular. By doing that, we obtain the quasi-regularity of D_1 and D_3.

Proposition 2.4. *For any* $k \in \mathbb{N}$ *there exists* $K(k) \in \mathbb{N}$ *such that*

$$D_a^q(x) = \sum_{i=1}^{K(k)} g_i(-\ln x)x^{ai} + \psi_k(x, \alpha), \tag{2.2.13}$$

where $\psi_k(x, \alpha)$ *is a function of class* $\mathcal{C}^{ka}$ *in* (x, α) *(with* $\alpha \in \mathcal{A}$*) and* $ka-$*flat at* $x = 0$*.*

Proof. From the expression of $D_a^q(x)$, given $k > 0$ we want to find $K(k)$ such that

$$(D_a^q)^K(x) = \sum_{j=K(k)+1}^{\infty} g_j(-\ln x)x^{aj}$$

is a $\mathcal{C}^{ka}-$flat function. It is clear that $(D_\alpha^q)^K(0) = 0$, for all K. This is equivalent to showing that $\sum_{j\geq K(k)+1} \dfrac{d^s}{dx^s}\left[g_j(-\ln x)x^{aj}\right]$ converges and it is equal to zero at $x = 0$ for all $s \leq ka$.

Suppose that, for all $s \leq ka$,

$$\left|\frac{d^s}{dx^s}\left[g_j(-\ln x)x^{aj}\right]\right| \leq \beta_j(x),$$

where $\sum_{j\geq K(k)+1} \beta_j(x) < \infty$ and $\beta_j(0) = 0$. In this case,

$$\sum_{j\geq K(k)+1} \frac{d^s}{dx^s}\left[g_j(-\ln x)x^{aj}\right] \leq \sum_{j\geq K(k)+1} \left|\frac{d^s}{dx^s}\left[g_j(-\ln x)x^{aj}\right]\right| \leq \sum_{j\geq K(k)+1} \beta_j(x) < \infty,$$

moreover, $\beta(0) = 0$ implies that $\left|\dfrac{d^s}{dx^s}[g_j(-\ln x)x^{aj}]\right|$ converges to 0 when $x \to 0^+$. Then, the result we want is proved. Now, we will show that our assumption holds. Notice that,

$$
\begin{aligned}
\frac{d}{dx}\left[g_j(-\ln x)x^{aj}\right] &= \left(-\frac{dg_j}{dx}(-\ln x) + ajg_j(-\ln x)\right)x^{aj-1} \\
\frac{d^2}{dx^2}\left[g_j(-\ln x)x^{aj}\right] &= \left(\frac{d^2}{dx^2}g_j(-\ln x) - (2aj-1)\frac{d}{dx}g_j(-\ln x) + aj(aj-1)g_j(-\ln x)\right)x^{aj-2} \\
&\ \ \vdots \\
\frac{d^s}{dx^s}\left[g_j(-\ln x)x^{aj}\right] &= \Bigg((-1)^s\frac{d^s}{dx^s}g_j(-\ln x) + \sum_{l=1}^{s-1}F_l(aj)\frac{d^{s-l}}{dx^{s-l}}g_j(-\ln x) \\
&\qquad + \frac{aj!}{(aj-s)!}g_j(-\ln x)\Bigg)x^{aj-s},
\end{aligned}
$$

where F_l are polynomial of degree $\leq l$ in aj for $1 \leq l \leq s-1$. So, by defining $F_0(aj) = (-1)^s$ and $F_s(aj) = \dfrac{aj!}{(aj-s)!}$, we can write

$$
\frac{d^s}{dx^s}\left[g_j(-\ln x)x^{aj}\right] = \left(\sum_{l=0}^{s}F_l(aj)\frac{d^{s-l}}{dx^{s-l}}g_j(-\ln x)\right)x^{aj-s},
$$

where F_l is polynomial of degree $\leq l$ in aj for $0 \leq l \leq s$. For each $0 \leq l \leq s$ write

$$
F_l(y) = \mu_{l0} + \mu_{l1}y + \cdots + \mu_{ll}y^l,
$$

where $\mu_{li} \in \mathbb{R}$ for all $0 \leq i \leq l$ and $0 \leq l \leq s$. So, for $y \geq 1$, we have $y^s \geq y^l$ for $0 \leq l \leq s$ and

$$
|F_l(y)| \leq \sum_{i=0}^{l}|\mu_{li}|y^i \leq \sum_{i=0}^{l}|\mu_{li}|y^s.
$$

Define $\bar{M}_l = \sum\limits_{i=0}^{l}|\mu_{li}|$, then $|F_l(y)| \leq \bar{M}_l y^s$ for all $0 \leq l \leq s$. Therefore, by means of Lemmas 2.4 and 2.5, for $x \geq 0$, we obtain

$$
\begin{aligned}
\left|\frac{d^s}{dx^s}\left[g_j(-\ln x)x^{ai}\right]\right| &\leq \sum_{l=0}^{s}|F_l(aj)|\left|\frac{d^{s-l}}{dx^{s-l}}g_j(-\ln x)\right|x^{aj-s} \\
&\leq \sum_{l=0}^{s}\bar{M}_l(aj)^sC_{s-l}(Cx^{-\frac{1}{2}})^jx^{aj-s} \\
&\leq (aj)^s(Cx^{a-\frac{1}{2}})^jx^{-s}\sum_{l=0}^{s}\bar{M}_lC_{s-l} \\
&\leq M_s(aj)^s(Cx^{a-\frac{1}{2}})^jx^{-s},
\end{aligned}
$$

where $M_s = \sum\limits_{l=0}^{s}\bar{M}_lC_{s-l}$. Now, for each $j \geq j_0$ (j_0 to be defined below) and $x \geq 0$, define $\beta_j(x) = M_s(aj)^s(Cx^{a-\frac{1}{2}})^jx^{-s}$. Observe that,

$$
\frac{\beta_{j+1}(x)}{\beta_j(x)} = \left(\frac{j+1}{j}\right)^s Cx^{a-\frac{1}{2}},
$$

so $\lim_{j\to\infty} \frac{\beta_{j+1}(x)}{\beta_j(x)} = Cx^{a-\frac{1}{2}}$. Since $a \geq 1$, for $\delta > 0$ small enough, we have $Cx^{a-\frac{1}{2}} < 1$ for all $x \in [0, \delta)$. Now, D'Alembert criterion for convergence ensures that $\sum_{j\geq j_0} \beta_j(x)$ converges for each $x \in [0, \delta)$. Moreover, $\beta_j(0) = 0$ for all $j \geq j_0$. We want to find a j_0 such that $(a - \frac{1}{2})j_0 - s$ (it is necessary to make β_j well defined for $j \geq j_0$) and, for each $k \in \mathbb{N}$, the found result holds for all $s \leq ka$. So it is enough to choose $j_0 > 2a(k+1) - 1$. Now, the proof is completed by choosing $K(k) = 2a(k+1)$. □

Let $\omega(x, \alpha_1)$ be the function defined by

$$\omega(x, \alpha_1) = \begin{cases} \dfrac{x^{-\alpha_1} - 1}{\alpha_1} & \text{if } \alpha_1 \neq 0, \\ -\ln x & \text{if } \alpha_1 = 0. \end{cases}$$

With this definition we have $\omega(x, \alpha_1) = \Omega(\alpha_1, -\ln x)$, where $\Omega(\alpha_1, t)$ was previously defined. Also, for each $k > 0$, $x^k\omega \to -x^k \ln x$, as $\alpha_1 \to 0$ and the convergence is uniform for $x \in [0, \epsilon]$, where $\epsilon > 0$ is fixed.

Let i, j be natural numbers such that $0 \leq j \leq i$ and consider monomials of the form $x^i\omega^j$. These monomial functions form a totally ordered set with the following order relation:

$$x^i\omega^j \prec x^{i'}\omega^{j'} \iff i' > i \text{ or } i = i' \text{ and } j > j'.$$

Thus, we have $1 \prec x\omega \prec x \prec x^2\omega^2 \prec x^2\omega \prec x^2 \prec \cdots$. Now we will look to a $(x, \omega)-$expansion, of order k, of the transition map D_a.

Theorem 2.4. *The transition map D_a defined by* (2.2.13)*, has the following $(x, \omega)-$ expansion for order kp*

$$\begin{aligned} D_a^q(x) &= x^a + \alpha_1[x^a\omega + \cdots] + \alpha_2[x^a\omega + \cdots] + \cdots \\ &\quad + \alpha_K[x^{Ka} + \cdots] + \psi_k(x, \alpha), \end{aligned} \tag{2.2.14}$$

for any $k \in \mathbb{N}$. The index $K = K(k)$ is defined in Proposition 2.4, and each term between brackets is a finite combination of monomials $x^i\omega^j$ with the following convention:

a) the notation $x^i\omega^j + \cdots$ means that after the sign + we find a finite combination of $x^{i'}\omega^{j'}$ of strictly greater order;

b) the coefficients of the unwritten monomials after signs + are polynomial functions of $\alpha_1, \ldots, \alpha_K$, which are zero if $\alpha = 0$.

The remainder ψ_k is a $\mathcal{C}^{ka}-$function in (x, α), which is $ka-$flat at $x = 0$.

Proof. From Proposition 2.3 $g_k(t) = e^{\alpha_1 t}Q_k(t)$. For $t = -\ln x$ it becomes

$$g_k(-\ln x) = x^{-\alpha_1}Q_k(-\ln x) = x^{-\alpha_1}(\alpha_k\omega + \bar{Q}_k(\alpha_1, \ldots, \alpha_k; \omega)), \tag{2.2.15}$$

where $\bar{Q}_k$ is polynomial of degree $\leq k - 1$ in ω with coefficients in $\mathcal{I}(\alpha_1, \ldots, \alpha_{k-1}) \cap \mathcal{I}(\alpha_2, \ldots, \alpha_k)^2$. Hence, the general term $g_k(-\ln x)x^{ak}$ in $D_a^q(x)$ is given by

$$g_k(-\ln x)x^{ak} = x^{ak-\alpha_1}(\alpha_k\omega + \bar{Q}_k(\alpha_1, \ldots, \alpha_k, \omega)). \tag{2.2.16}$$

Notice that $x^{-\alpha_1} = \alpha_1\omega + 1$ then, rewriting the equation (2.2.16), we obtain, for $k \geq 2$,

$$\begin{aligned} g_k(-\ln x)x^{ak} &= x^{ak}(\alpha_1\omega + 1)(\alpha_k\omega + \bar{Q}_k(\alpha_1, \ldots, \alpha_k, \omega)) \\ &= \alpha_1\alpha_k x^{ak}\omega^2 + \alpha_k\omega x^{ak} + (1 + \alpha_1\omega)x^{ak}\bar{Q}_k, \end{aligned} \tag{2.2.17}$$

and, since $Q_1 \equiv 1$, $g_1(-\ln x)x^a = x^{ak-\alpha_1} = x^{ak}(\alpha_1\omega + 1) = x^{ak} + \alpha_1 x^p\omega$. Proposition 2.4 gives

$$\begin{aligned} D_a^q(x) &= \textstyle\sum_{i=1}^K g_i(-\ln x)x^{ak} + \psi_k \\ &= \alpha_1 x^a\omega + x^a + \alpha_1\alpha_2 x^{2a}\omega^2 + \alpha_2 x^{2a}\omega + (1+\alpha_1\omega)x^{2a}\bar{Q}_2 + \alpha_1\alpha_3 x^{3a}\omega^2 \\ &\quad +\alpha_3 x^{3a}\omega + (1+\alpha_1\omega)x^{3a}\bar{Q}_3 + \cdots + \psi_k, \end{aligned} \tag{2.2.18}$$

where $+\cdots$ is the expansion of $g_s(-\ln x)x^{sa}$ for $4 \le s \le K$. Now, we rearrange the sum above in the following way: first, take all the terms whose coefficients are divisible by α_1 (if $\alpha_1 = 0$ start from the second step). Next, all the remainders (they will not be divisible by α_1) divisible by α_2 and so on until α_K. From the characterization of $\bar{Q}_s(\alpha_1, \ldots, \alpha_s, \omega)$ we can write it as

$$\bar{Q}_s(\alpha_1, \ldots, \alpha_s, \omega) = b_0(\alpha_1, \ldots, \alpha_s) + b_1(\alpha_1, \ldots, \alpha_s)\omega + \cdots + b_{s-1}(\alpha_1, \ldots, \alpha_s)\omega^{s-1},$$

where $b_i(\alpha_1, \ldots, \alpha_s) \in \mathcal{I}(\alpha_1, \ldots, \alpha_{s-1}) \cap \mathcal{I}(\alpha_2, \ldots, \alpha_s)^2$, for $i = 0, \ldots, s-1$. If $\bar{Q}_s \equiv 0$ we have nothing to do. When $\bar{Q}_s \not\equiv 0$ then, b_i is non-zero for some $i = 0, \ldots, s-1$, and this term is divisible by some α_j, with $1 \le j \le s$. Thus, each non-zero term of $\bar{Q}_s$ is divisible by some α_j, $1 \le j \le s$. In this way, we obtain

$$\begin{aligned} D_a^q(x) = x^a \quad &+ \quad \alpha_1\left[x^a\omega + \alpha_2 x^{2a}\omega^2 + x^{2a}\omega\bar{Q}_2 + \alpha_3 x^{3a}\omega^2 + \cdots\right] \\ &+ \quad \alpha_2\left[x^{2a}\omega + \Theta_2\right] + \alpha_3\left[x^{3a}\omega + \Theta_3\right] \\ &\;\;\vdots \\ &+ \quad \alpha_K x^{Ka}\omega + \psi_k(x, \alpha), \end{aligned}$$

where Θ_j' are composed by the terms in $x^{(j+1)a}\bar{Q}_{j+1}, \ldots, x^{Ka}\bar{Q}_K$ divisible by α_j but not divisible by any α_i with $i = 1, \ldots, j-1$. It is clear that, each term after $x^{sa}\omega$ in the bracket related to α_s is of order greater than $x^{sa}\omega$ and has coefficients in $\mathcal{I}(\alpha_1, \ldots, \alpha_K)$. The Proposition 2.4 gives that the remainder ψ_k is a $ka-$flat function at $x = 0$, thus finishing the proof. □

Now, returning to the vector field X, Theorem 1.1 establishes a $\mathcal{C}^k-$equivalence between X and $X^{N(k)}$. This equivalence defines diffeomorphisms, Φ and Ψ, on $\mathbb{R}$, in a neighborhood of $0 = \Phi(0) = \Psi(0)$, such that if D_a is the transition map for $X^{N(k)}$, then,

$$D_a(x) = \Psi \circ D_a \circ \Phi(x). \tag{2.2.19}$$

Theorem 2.5. *Transition maps at saddle-regular, saddle-saddle or real saddle points are quasi-regular.*

Proof. The expansion of order k in (ω, x), given by Theorem 2.4, depends only on $\alpha_1, \ldots, \alpha_N$ because all $\alpha_i(\lambda) \equiv 0$ for $i \ge N+1$, then

$$D_a^q(x) = x^a + \alpha_1[x^a\omega + \cdots] + \cdots + \alpha_N x^{Na}\omega + \psi_k(x, \lambda). \tag{2.2.20}$$

The resonant coefficients α_i are independent of k, but it does not happen for the expansions in the brackets. To study the transition map for the vector field X, we have $\alpha_1 \equiv 0$ and, thus, $\omega = -\ln x$. Hence, $D(x) = \Psi \circ D_a \circ \Phi(x)$ is the transition map of X, with

$$D_a^q(x) = x^a + \alpha_2 x^{2a}(-\ln(x)) + \cdots + \alpha_N x^{Na}(-\ln x) + \psi_k(x),$$

where ψ_k is of class $\mathcal{C}^{ka}$ and $ka-$flat at $x=0$. So, for a fixed k, we have

$$D_a(x) = x^{\frac{a}{q}}\left(1+\alpha_2 x^a(-\ln(x))+\cdots+\alpha_N x^{(N-1)a}(-\ln x)+\overline{\psi}_k(x)\right)^{1/q}.$$

Consider the Taylor series of $y \mapsto (1+y)^\nu$ and $y \mapsto \ln(1+y)$ around $y=0$, then

$$(1+y)^\nu = 1+a_1y+a_2y^2+\cdots = 1+\sum_{i=1}^{\infty} a_i y^i$$

with $a_i = a_i(\nu)$ for all $i \geq 1$, and

$$\ln(1+y) = b_1y+b_2y^2+\cdots = \sum_{i=1}^{\infty} b_i y^i.$$

Since Φ and Ψ are diffeomorphisms with $\Phi(0)=0=\Psi(0)$, we can write them as

$$\Phi(y) = \sum_{i=1}^{\infty}\beta_i y^i \quad \text{and} \quad \Psi(y)=\sum_{i=1}^{\infty}\eta_i y^i,$$

with $\beta_1\eta_1 \neq 0$. Now, expanding D and ordering the terms, we obtain that, for any $k \in \mathbb{N}$, there exists a sequence of coefficients λ_i, with $\lambda_1 = \frac{a}{q} < \lambda_2 < \cdots < \lambda_{N(k)}$, and a sequence of polynomials, $P_1 \equiv A$ (positive constant), P_2,. . . ,$P_{N(k)}$, such that

$$D(x) = \Psi \circ D_a \circ \Phi(x) = \sum_{i=1}^{N(k)} x^{\lambda_i} P_i(\ln x)+\bar{\psi}_k(x), \tag{2.2.21}$$

where $\bar{\psi}_k$ is of class $\mathcal{C}^k$ and $k-$flat. The coefficients λ_i and the polynomials P_i are independent of k, so they are well defined. This means that taking $k' > k$, the sequence for k is the sequence for k' truncated at order $N(k)$.

Therefore, taking k arbitrarily large, we have a well defined infinite formal series $\hat{D}(x) = \sum\limits_{i=1}^{\infty} x^{\lambda_i}P_i(x)$ which is asymptotic to $D(x)$ in the following sense:

For any $s \in \mathbb{N}-\{0\}$, $\left|D(x) - \sum\limits_{i=1}^{s} x^{\lambda_i}P_i(\ln x)\right| = O(x^{\lambda_s})$ where $\lambda_1 = r < \lambda_2 < \cdots < \lambda_s < \cdots$ is an infinite sequence of positive coefficients tending to $+\infty$, and $P_1, P_2, \ldots, P_s, \ldots$ is an infinite sequence of polynomials with $P_1 \equiv A$ (positive constant). Hence, the transition map D is quasi-regular as required. □

2.3 A Cycle Having Limit Cycles Accumulating onto It

Consider the planar system of ordinary differential equations given in polar coordinates ($x=\rho\cos\theta$, $y=\rho\sin\theta$)

$$\begin{cases} \dot{\rho} &= f(\rho), \\ \dot{\theta} &= -1, \end{cases} \tag{2.3.1}$$

where,

$$f(\rho) = \begin{cases} \rho(\rho^2-1)^3\sin\left(\dfrac{1}{\rho^2-1}\right) & \text{if} \quad \rho \neq 1, \\ 0 & \text{if} \quad \rho = 1. \end{cases}$$

Observe that $f(\rho)$ is derivable and $f'(\rho)$ is continuous at $\rho = 1$, so the system (2.3.1) is of class $\mathcal{C}^1$ in the polar plane. Observe that by increasing the power of the term $\rho^2 - 1$ in the expression of $f(\rho)$ (for $\rho \neq 1$), the class of differentiability of f at $\rho = 1$ will also increase.

Notice that, $\rho \equiv 1$ corresponds to a closed periodic orbit of the system (2.3.1). Moreover, when $\rho \to 1^{\pm}$ there exist monotone sequences $\rho_k^{\pm} = \sqrt{1 \pm \frac{1}{k\pi}}$, $k \in \mathbb{N}$, such that

- $\rho_k^- < 1$, for all $k \in \mathbb{N}$, is an increasing sequence;
- $\rho_k^+ > 1$, for all $k \in \mathbb{N}$, is an decreasing sequence;
- $\rho_k^{\pm} \to 1$ when $k \to \infty$ and $f(\rho_k^{\pm}) = 0$ for all $k \in \mathbb{N}$.

Moreover, $f(\rho) = 0$ if and only if $\rho = 0, \rho^{\pm}$ or 1. Therefore, for each $k \in \mathbb{N}$, $\rho_k^{\pm}$ corresponds to a limit cycle of the system 2.3.1, more precisely, ρ_{2k+1}^- and ρ_{2k}^+ correspond to attractor limit cycles while ρ_{2k+1}^+ and ρ_{2k}^- correspond to repellor limit cycles. Thus, the cycle $\rho = 1$ has two sequences of limit cycles accumulating onto it, see Figure 2.5. This fact does not contradict the Dulac's problem because the system is not analytic.

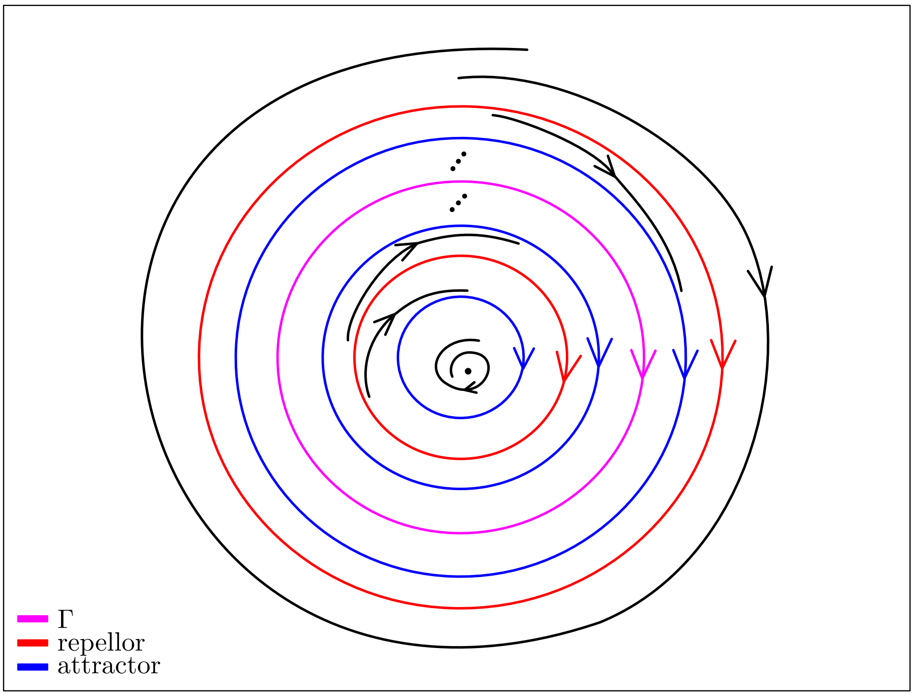

Figure 2.5: Phase portrait of the system (2.3.1).

2.4 Generalization

As seen in the previous section, the proof of the quasi-regularity of a transition map does not have a drastic change if we consider hyperbolic polycycles of discontinuous vector fields. In this section we consider a weaker hypothesis by allowing fold-regular, fold-fold, and fold-saddle singularities in the polycycle, see Figure 2.6. The construction and analysis of the first return maps are done in the exactly same way as in the hyperbolic polycycles. Therefore, as we have seen above, it is enough to prove that the transition

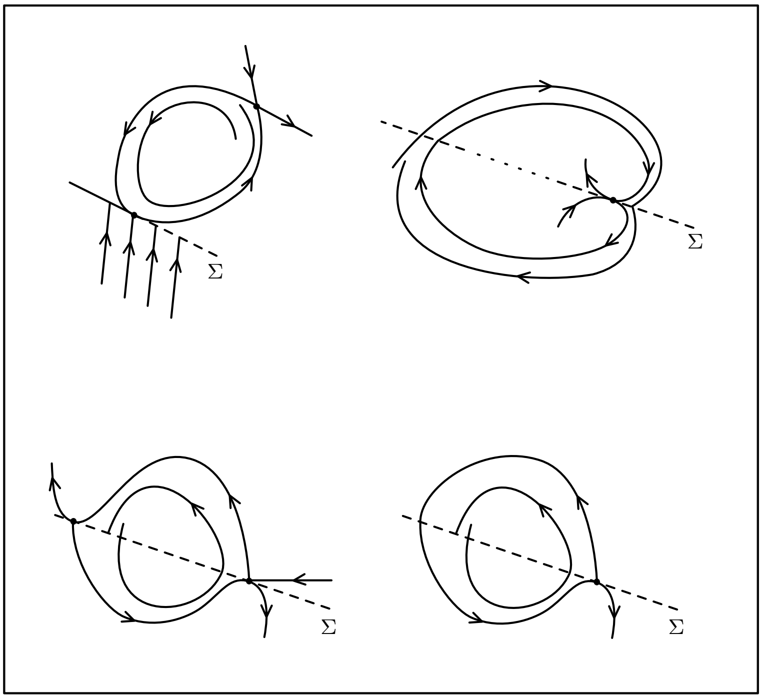

Figure 2.6: Examples of polycycles having singularities of the types saddle-fold, fold-regular and fold-folds.

map, in a neighborhood a fold-regular or a fold-fold singularity, is quasi-analytic. The first step in this direction is to prove that transition maps at fold points are quasi-regular.

In order to study the quasi-regularity of the transition maps at fold points, we consider Σ^+ and Σ^- as being manifolds with boundary Σ which is locally, around the origin, given by $\Sigma = h^{-1}(0)$ where $h(x,y) = y$. Assume that X is a vector field defined in Σ^+ and Y is a vector field defined in Σ^-. In addition, assume the origin is a visible fold point for X and Y, which are analytic vector fields. Then, as seen in Chapter 1, from [40], there exists a neighborhood of the origin such that X and Y are separately conjugated (by means of $C^\infty-$diffeomorphisms) to

$$X_{ab}(x,y) = \begin{pmatrix} a \\ bx \end{pmatrix} \text{ and } Y_{cd}(x,y) = \begin{pmatrix} c \\ dx \end{pmatrix}, \tag{2.4.1}$$

respectively, with $ab > 0$ and $cd < 0$. Now, we define the transversal sections to study the transition maps:

- For X_{ab}, consider $\varepsilon > 0$ and define
  - if $a > 0$: $\sigma \subset \Sigma$ and $\tau = \left\{(x,\varepsilon) \in \mathbb{R}^2; \sqrt{\frac{2a}{b}\varepsilon} - \delta < x < \sqrt{\frac{2a}{b}\varepsilon} + \delta\right\}$, for some $\delta > 0$. Moreover, if D^+_{ab} is the transition map at 0, in order to obtain $D^+_{ab}(0) = 0$, we consider another parametrization of τ, i.e., we consider the change $x \mapsto x - \sqrt{\frac{2a}{b}\varepsilon}$. Thus, by calculating the flow of X_{ab} we can also calculate the transition map from σ to τ, which is $D^+_{ab}(x) = \sqrt{x^2 + \frac{2a}{b}\varepsilon} - \sqrt{\frac{2a}{b}\varepsilon}$ for $x \geq 0$ with $(x,0) \in \sigma$;
  - if $a < 0$: interchange the definition of σ and τ in the case $a > 0$. In this case, we need a new parametrization of σ and we obtain that by doing $x \mapsto x - \sqrt{\frac{2a}{b}\varepsilon}$

the transition map from σ to τ is $D^-_{ab}(x) = \sqrt{\left(x - \sqrt{\frac{2a}{b}\varepsilon}\right)^2 - \frac{2a}{b}\varepsilon}$ for $x \geq 0$ with $\left(x + \sqrt{\frac{2a}{b}\varepsilon}, \varepsilon\right) \in \sigma$.

- For Y_{cd}, consider $\varepsilon > 0$ and define
  - if $c > 0$: $\sigma \subset \Sigma$ and $\tau = \left\{(x, -\varepsilon) \in \mathbb{R}^2; \sqrt{-\frac{2c}{d}\varepsilon} - \delta < x < \sqrt{-\frac{2c}{d}\varepsilon} + \delta\right\}$, for some $\delta > 0$. Analogously to the calculations for X_{ab} with $a > 0$ we obtain $D^+_{cd}(x) = \sqrt{x^2 + \frac{2c}{d}\varepsilon} - \sqrt{\frac{2c}{d}\varepsilon}$ for $x \geq 0$ with $(x, 0) \in \sigma$;
  - if $c < 0$: interchange the definition of σ and τ in the case $c > 0$. Analogously to the calculations for X_{ab} with $a < 0$ we obtain $D^-_{cd}(x) = \sqrt{\left(x - \sqrt{\frac{2c}{d}\varepsilon}\right)^2 - \frac{2c}{d}\varepsilon}$ for $x \geq 0$ with $\left(x + \sqrt{\frac{2c}{d}\varepsilon}, -\varepsilon\right) \in \sigma$.

See Figure 2.7 for a geometric illustration of the transition maps. Observe that, the case $a > 0$ (resp. $a < 0$) is analogous to the case $c > 0$ (resp. $c < 0$). Thus, it is enough to prove that $D^{\pm}_{ab}$ is quasi-regular.

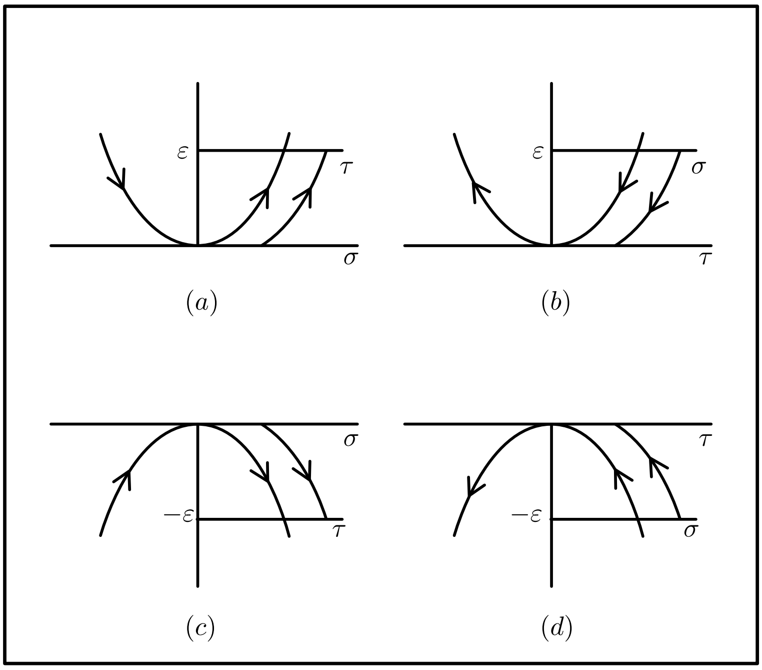

Figure 2.7: Transition maps through fold points: (a) X_{ab} with $a > 0$; (b) X_{ab} with $a < 0$; (c) X_{cd} with $c > 0$; (d) X_{cd} with $c < 0$.

Theorem 2.6. *Transition maps, near fold-regular points of C^∞ vector fields, are quasi-regular homeomorphisms.*

Proof. Let the origin be a fold point of the C^∞ vector field X defined in Σ^+. As we have observed above, the case where the origin is a fold point for the vector field defined in Σ^- is analogous, then it is not considered.

Since X is $C^\infty-$conjugated to X_{ab} with $\Sigma = \{(x, 0) \in \mathbb{R}^2\}$, there exist $C^\infty-$ diffeomorphisms, φ and ψ, defined around the origin, with $\varphi(0) = \psi(0) = 0$ such that the transition map D, of X, satisfies

- $D(x) = \varphi \circ D^+_{ab} \circ \psi(x)$ if $a > 0$;

- $D(x) = \varphi \circ D^{-}_{ab} \circ \psi(x)$ if $a < 0$.

Since the set $\mathcal{D}$, of all quasi-regular homeomorphisms, is a group which contains the set Diff_0 of all diffeomorphisms fixing the origin, it is enough to show that $D^{\pm}_{ab}$ are quasi-regular homeomorphisms.

Observe that, for $a > 0$, D^{+}_{ab} can be $C^{\infty}-$extended to an open neighborhood of the origin. Then, we can calculate the infinity formal Taylor series

$$\sum_{i=1}^{\infty} \frac{1}{i!} \frac{d^i}{dx^i} D^{+}_{ab}(0) x^i,$$

where, for each $n \in \mathbb{N}$, we have $\left| D^{+}_{ab}(x) - \sum_{i=1}^{n} \frac{1}{i!} \frac{d}{dx} D^{+}_{ab}(0) x^i \right| = O(x^n)$. Moreover, $\frac{d}{dx} D^{+}_{ab}(0) = 0$ and $\frac{d^2}{dx^2} D^{+}_{ab}(0) = \left(\frac{2a}{b}\varepsilon\right)^{-\frac{1}{2}} > 0$. Therefore, we obtain an asymptotic Dulac series by doing $\lambda_i = i + 1$ and $P_i(u) \equiv \frac{1}{(i+1)!} \frac{d^{i+1}}{dx^{i+1}} D^{+}_{ab}(0)$ for all $i \geq 2$, and so

$$\hat{D}^{+}_{ab}(x) = \sum_{i=1}^{\infty} x^{\lambda_i} P_i(\ln x),$$

with $P_1 \equiv \left(\frac{2a}{b}\varepsilon\right)^{-\frac{1}{2}} > 0$. Hence, D^{+}_{ab} is a quasi-regular homeomorphism. Now, observe that $D^{+}_{ab} \circ D^{-}_{ab}(x) \equiv x$, so $D^{+}_{ab} \circ D^{-}_{ab} = \mathrm{Id}$. Since the set $\mathcal{D}$ is a group and D^{+}_{ab} and Id are quasi-regular, we get that D^{-}_{ab} is a quasi-regular homeomorphism. Therefore, we have proved the result. □

Now, we proceed to prove the quasi-analyticity of the transition regular map at fold-regular points.

Theorem 2.7. *Transition maps at fold-regular points are quasi-analytic.*

Proof. It is enough to show that D^{+}_{ab} is quasi-analytic. From Theorem 2.7 it is enough to show the second part of Definition 2.4.

Let m be any positive real number and consider $g(x) = D^{+}_{ab}(e^{-x}) = \sqrt{e^{-2x} + \frac{2a}{b}\varepsilon} - \sqrt{\frac{2a}{b}\varepsilon}$. Since e^{-x} is a decreasing map, we have that $e^{-x} < e^{-m}$ for all $x > b$, moreover $e^{-x} \to 0$ as $x \to +\infty$. Therefore, e^{-2x} is bounded in the set $\{x \in \mathbb{R}; x > m\}$. Now, define the following complex extension of g,

$$G : \begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \mapsto & \sqrt{e^{-2z} + \frac{2a}{b}\varepsilon} - \sqrt{\frac{2a}{b}\varepsilon} \end{array}.$$

As $a, b, \varepsilon > 0$, G is a composition of holomorphic maps, G is thus a holomorphic map. Observe that, for $z = u + iv \in \mathbb{C}$, $|e^{-2z}| = |e^{-2u-2vi}| = |e^{-2u}(\cos(2v) - i\sin(2v))| \leq e^{-2u}$. Moreover, $\Delta_m = \{z = u + iv; u > m(1+v^2)^{1/4}\} \subset \{z = u + iv; u > m\}$. Therefore, G is bounded in the set $\Delta_m = \{z = u + iv; u > m(1+v^2)^{1/4}\}$ and it concludes the proof. □

Now, a straight consequence of this result and from the fact $\mathcal{D}$ is a group.

Corollary 2.1. *Transition maps, near fold-fold points of piecewise C^{∞} vector fields, are quasi-regular homeomorphisms and quasi-analytic maps.*

Remark 2.6. *From Theorem 2.7 and Corollary 2.1 we conclude the result obtained for hyperbolic polycycles does not change if fold points are allowed instead of saddle points. More specifically, polycycles for which all singular points are hyperbolic saddles (with invariant manifolds transversal to Σ if the saddle is on the switching manifold) and/or fold points cannot have limit cycles accumulating onto it.*

Chapter 3

Degenerate Cycle Through a Visible Fold-Regular Point

In this chapter we present a study of bifurcations of a degenerate cycles through visible fold-regular points. This kind of cycle has already been mentioned in literature, [9, 24, 31]. It is included in this work for completeness, since it appears in bifurcations of the degenerate cycles through hyperbolic saddle-regular points that will be studied in the next chapter. Moreover, we give a more complete study of the first return map and models realizing these cycles.

3.1 Generic Conditions

Consider a discontinuous vector field $Z = (X, Y) \in \Omega^r$. We say that Z satisfies the condition

- $FC(1)$: if Z has a visible fold-regular point $F_Z \in \Sigma$, being a fold point for X and regular for Y;
- $FC(2)$: if the unstable separatrix of the fold point, $W^u_+(X, F_Z)$, intersects Σ transversally at a point $Q_Z \neq F_Z$;
- $FC(3)$: if Y is transversal to Σ at Q_Z and the positive trajectory of Y through Q_Z meets Σ transversally at F_Z;

Notice that, if Z satisfies $FC(1)-FC(3)$ then Z has a degenerate cycle γ_Z passing through a visible fold-regular point. Moreover, in this case, Z satisfies:

- $FC(4)$: there exists a first return map defined in a half open neighborhood of F_Z contained in:
  (a) the bounded region in $\mathbb{R}^2$ delimited by γ_Z;
  (b) the unbounded region in $\mathbb{R}^2$ delimited by γ_Z.

Definition 3.1. *Given $Z = (X, Y) \in \Omega^r$ satisfying $FC(1) - FC(3)$ with cycle γ_Z. Then γ_Z is of type*

- *DFC_1: if Z satisfies $FC(4) - (a)$, see Figure 3.1–(a);*
- *DFC_2: if Z satisfies $FC(4) - (b)$, see Figure 3.1–(b).*

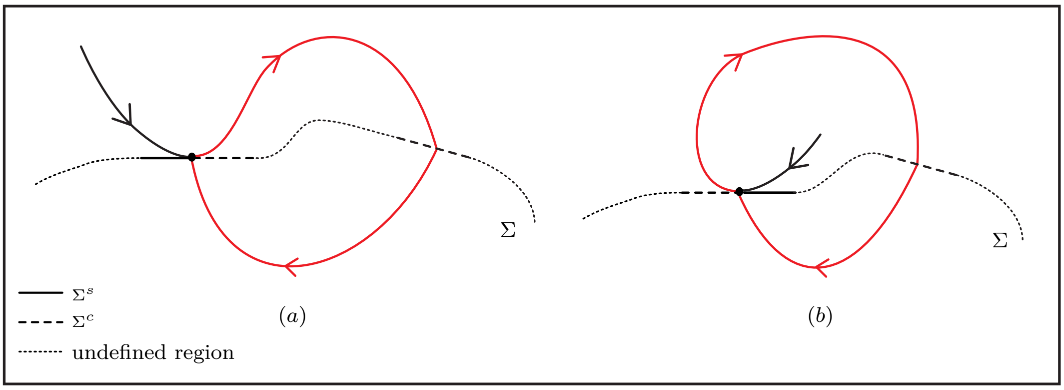

Figure 3.1: A degenerate cycle through a visible fold-regular point: (a) type DFC_1 and (b) type DFC_2.

3.1.1 Local Stability of a Fold-Regular Point

The first step towards the analysis of the unfolding of the degenerate cycles DFC_1 and DFC_2 is to perform a local study the fold-regular point.

Proposition 3.1. *Let $Z_0 = (X_0, Y_0) \in \Omega^r$ be a discontinuous vector field satisfying the condition $FC(1)$. Then, there exist neighborhoods $\mathcal{V}_{Z_0} \subset \Omega^r$ of Z_0 and $V_0 \subset \Sigma$ of the fold-regular point such that any $Z \in \mathcal{V}_{Z_0}$ has a visible fold-regular point in V_0.*

Proof. This result is proved in [15] (Theorem 3.5). Without loss of generality, we can suppose that Σ is locally determined by $h(x, y) = y$. Consider $X_0 = (X_{01}, X_{02})$ and $Y_0 = (Y_{01}, Y_{02})$. By hypothesis Y_0 is transversal to Σ at 0, so $Y_0 h(0) = Y_{02}(0) \neq 0$. Observe that the map $(Y, p) \in \chi^r \times \Sigma \mapsto Yh(p)$ is continuous, so there exist neighborhoods $U_{Y_0} \subset \chi^r$ of Y_0 and $U_0 \subset \Sigma$ of 0 such that $Yh(p) \neq 0$ and $\mathrm{Sgn} Yh(p) = \mathrm{Sgn} Y_0 h(0)$ for all $(Y, p) \in U_{Y_0} \times U_0$. Also, X_0 has a fold point at 0 so $X_0 h(0) = X_{02}(0) = 0$ and $X_0^2 h(0) = X_{01}(0) \dfrac{\partial}{\partial x} X_{02}(0) \neq 0$. Define,

$$\begin{array}{rccc} \Gamma : & \Omega^r \times \Sigma & \longrightarrow & \mathbb{R} \\ & ((X, Y), x) & \mapsto & Xh \cdot Yh(x) \end{array}.$$

It is clear that Γ is Fréchet differentiable at $((X_0, Y_0), 0)$ and we have $\Gamma((X_0, Y_0), 0) = 0$ and $\dfrac{\partial}{\partial x}\Gamma(X_0, Y_0, 0) = Y_{02}(0)\dfrac{\partial}{\partial x}X_{02}(0) \neq 0$. It follows from the Implicit Function Theorem for Banach spaces that there exist neighborhoods $\mathcal{V}_{X_0} \subset \chi^r$ of X_0, $\mathcal{V}_{Y_0} \subset \chi^r$ of Y_0, and $V_0 \subset \Sigma$ of 0 and a unique C^r-map $s : \mathcal{V}_{Z_0} = \mathcal{V}_{X_0} \times \mathcal{V}_{Y_0} \to V_0$ such that $\Gamma((X, Y), s(X, Y)) = 0$ and, for all $(Z, x) \in \mathcal{V}_{Z_0} \times V_0$, $\Gamma(Z, x) = 0$ if, and only if, $x = s(Z)$. Notice that we can take $\mathcal{V}_{Y_0} \subset U_{Y_0}$ and $V_0 \subset U_0$. Furthermore, given that $X_0^2 h(0) \neq 0$, by continuity $\mathcal{V}_{X_0}$ and V_0 can be determined in such a way that, for all $X \in \mathcal{V}_{X_0}$, $X^2 h(s(X)) \neq 0$ and $\mathrm{Sgn} X^2 h(s(X)) = \mathrm{Sgn} X_0^2 h(0)$. Therefore, if $Z \in \mathcal{V}_{Z_0}$ then $s(Z) \in V_0$ is a fold-regular point of Z of the same type as the fold-regular point of Z_0. □

The last result implies that Z_0 satisfying $FC(1)$ is structurally stable in Ω^r, see Proposition 3.4 in [15].

3.1.2 First Return Map

Consider $Z_0 = (X_0, Y_0) \in \Omega^r$ satisfying $FC(1) - FC(4)$, having the origin as the fold-regular point, and degenerate cycle γ_0. Once we are concerned with the study in a neighborhood of γ_0 there is no problem in assuming that Σ coincides, around the origin, with the $x-$axis and $h(x,y)=y$. Under these assumptions, by writing $X_0=(X_{01},X_{02})$ we have $X_0h(0)=X_{02}(0)=0$. As 0 is not a singular point of X_0 assume, without loss of generality, that $X_{01}(0)>0$. The case $X_{01}(0)<0$ is treated analogously.

From Proposition 3.1 and continuous dependence, we obtain a neighborhood $\mathcal{V}_{Z_0}\subset\Omega^r$ of Z_0 for which there is a well defined first return map defined in a half open interval contained in Σ. In fact, condition $FC(2)$ guarantees that, for some positive time, $W^u_+(X_0,0)$ intersects Σ transversely at Q_0. Then, reducing $\mathcal{V}_{Z_0}$ if necessary, by continuity we have that given $Z=(X,Y)\in\mathcal{V}_{Z_0}$ the unstable separatrix $W^u_+(X,F_Z)$ through the fold-regular point $F_Z=(a_Z,0)\in V_0$ intersects Σ transversely at a point $Q_Z\in\Sigma$, for some positive time and $Q_Z\approx Q_0$. It implies that, for each $x\geq a_Z$ small enough, there exists $0<t(x)<\infty$ satisfying $\varphi_X(t(x),(x,0))\in I^1_Z$, where $I^1_Z\subset\Sigma$ is an open neighborhood of Q_0 in Σ and $t(x)$ is of class C^r. Moreover, $\varphi_X(t(a_Z),F_Z)=Q_Z$ and the map $Z\in\mathcal{V}_{Z_0}\mapsto Q_Z$ is C^r. Similarly, condition $FC(3)$ ensures that, reducing I^1_Z if necessary, given $Q\in I^1_Z$ there exists $0<t(Q)<\infty$ such that $\varphi_Y(t(Q),Q)\in I^0_Z$ where $I^0_Z\subset\Sigma$ is an open neighborhood of the origin. Define $R_Z=\varphi_Y(t(Q_Z),Q_Z)\in I^0_Z$, then the map $Q_Z\in I^1_Z\mapsto R_Z\in I^0_Z$ is also of class C^r.

In order to describe the unfolding of the cycle γ_0 of Z_0 it is necessary to study the first return map, near F_Z, of all $Z\in\mathcal{V}_{Z_0}$. Both cases, $FC(4)-(a)$ and $FC(4)-(b)$, are analyzed with the same approach. The return map of Z_0 is defined in a half-open interval $[0,\delta_0)\subset\Sigma\cap V_0$ for some $\delta_0>0$.

Proposition 3.2. *Consider $Z\in\mathcal{V}_{Z_0}$ and let $F_Z=(a_Z,0)$ be the fold-regular point of Z in $\Sigma\cap V_0$. Then, the first return map of Z near F_Z can be written as*

$$\pi_Z(x)=b_Z+\kappa_2(x-a_Z)^2+O_3(x-a_Z), \tag{3.1.1}$$

where $x\in[a_Z,\delta_Z)\subset\Sigma\cap V_0$ and $\kappa_2>0$.

Proof. According to what we have seen above, the first return map for Z is defined in a half-open interval $[a_Z,\delta_Z)\subset\Sigma\cap V_0$ for some $\delta_Z>0$, where a_Z corresponds to the fold point of X. Let $\tau_\varepsilon\subset\Sigma^+$ be a transversal section to the flow of Z at $P_\varepsilon\in W^u_+(X,F_Z)$. For $\varepsilon>0$ small enough, assume $\tau_\varepsilon\subset\{(x,\varepsilon);x\in\mathbb{R}\}$ and $P_\varepsilon=(p_\varepsilon,\varepsilon)$. We identify τ_ε with an interval of $\mathbb{R}$ under projection on the first coordinate (the projection map on the $i-th$ coordinate is denoted by π_i, $i=1,2$). We also identify I^0_Z with an open interval of $\mathbb{R}$ through projection on the first coordinate. So, for all $x\in[a_Z,\delta_Z)\subset I^0_Z$ there exists $0<t_1(x)<\infty$ satisfying $\pi_2\circ\varphi_X(t_1(x),(x,0))=\varepsilon$. So, the following transition map is well defined

$$\begin{array}{rccc}\rho_1: & [a_Z,\delta_Z) & \longrightarrow & \tau_\varepsilon \\ & x & \mapsto & \pi_1\circ\varphi_X(t_1(x),(x,0))\end{array}.$$

From Proposition 1.2 (originally proved in [33]), for $\delta_Z>0$ small enough, we have:

$$\rho_1(x)=p_\varepsilon+\alpha_2(x-a_Z)^2+O((x-a_Z)^3), \tag{3.1.2}$$

with $\alpha_2>0$, Figure 3.2. The flow of X defines a diffeomorphism from τ_ε to I^1_Z:

$$\begin{array}{rccc}\rho_2: & \tau_\varepsilon & \longrightarrow & I^1_Z \\ & u & \mapsto & \varphi_X(t_2(u),u)\end{array},$$

where, for all $u\in\tau_\varepsilon$, $0<t_2(u)<\infty$ is the time spent by the trajectory of X through $P=(u,\varepsilon)$ to reach I^1_Z. Identify I^1_Z with an open interval $\tilde{I}\subset\mathbb{R}$ through a parametrization

$\tilde{u} \in \tilde{I} \mapsto \nu(\tilde{u}) \in I_Z^1$ satisfying $\nu(q_Z) = Q_Z$. Then, it is clear that ρ_2 is a decreasing diffeomorphism with $\frac{d}{du}\rho_2(p_\varepsilon) = \beta_1 < 0$. Thus, we obtain:

$$\rho_2(y) = q_Z + \beta_1(y - p_\varepsilon) + O((y - p_\varepsilon)^2), \tag{3.1.3}$$

see Figure 3.2. Finally, the flow of Y defines a diffeomorphism from I_Z^1 to I_Z^0:

$$\begin{array}{rccc} \rho_3 : & I_Z^1 & \longrightarrow & I_Z^0 \\ & v & \mapsto & \pi_1 \circ \varphi_Y(t_3(v), v) \end{array},$$

where $0 < t_3(v) < \infty$ is the time spent by the trajectory of Y through $Q = \nu(v)$ to reach I_Z^0. ρ_3 is a decreasing diffeomorphism, then $\frac{d}{dv}\rho_3(q_Z) = \eta_1 < 0$ and

$$\rho_3(v) = b_Z + \eta_1(v - q_Z) + O((v - q_Z)^2), \tag{3.1.4}$$

see Figure 3.2. Now, the first return map of Z can be written as:

$$\begin{array}{rccc} \pi_Z : & [a_Z, \delta_Z) & \longrightarrow & I_Z^0 \\ & x & \mapsto & \rho_3 \circ \rho_2 \circ \rho_1(x) \end{array}. \tag{3.1.5}$$

From equations 3.1.2, 3.1.3 and 3.1.4, for $x \in [a_Z, \delta_Z)$ we obtain

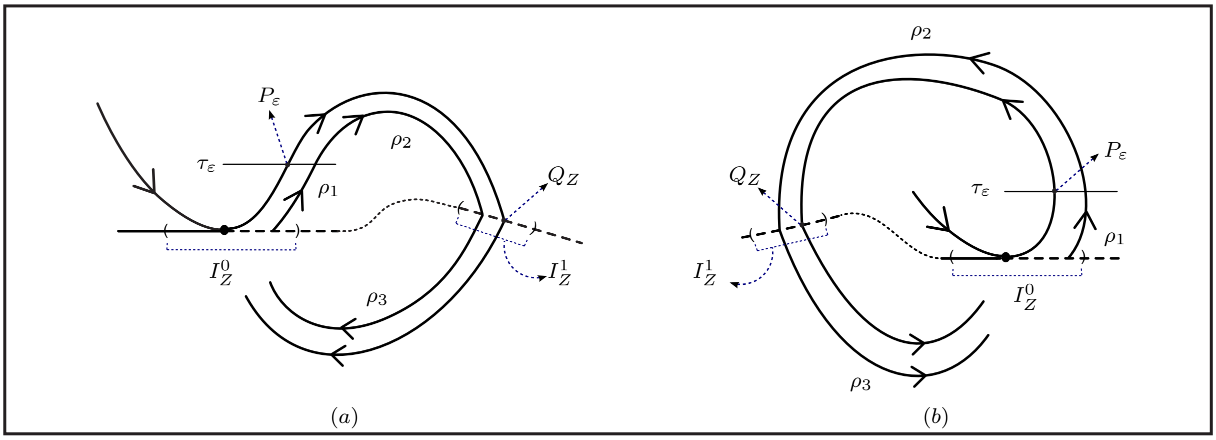

Figure 3.2: Construction of the first return map for: (a) type DFC_1 and (b) type DFC_2.

$$\pi_Z(x) = b_Z + \eta_1\beta_1\alpha_2(x - a_Z)^2 + O_3(x - a_Z). \tag{3.1.6}$$

The result is achieved by writing $\kappa_2 = \eta_1\beta_1\alpha_2$. □

3.2 Main Results and Bifurcation Diagrams

Notice that the unique non-stable feature of the cycles DFC_1 and DFC_2 is the fact the fold point coincides with a fixed point for the first return map. Then, from the previous analysis the unfolding of these cycles is completely determined by $b_Z - a_Z$ and we have the following result.

Theorem 3.1. *Consider $Z \in \mathcal{V}_{Z_0}$, let $F_Z = (a_Z, 0)$ be the fold point of Z in V_0, π_Z be the first return map of Z given in Proposition 3.2 and $\beta = \pi_Z(a_Z) - a_Z$. Then, Z has only one cycle γ_Z passing through V_0 and* *(i) if* $\beta > 0$ *then* γ_Z *is an attractor crossing limit cycle;*

(ii) if $\beta = 0$ *then* γ_Z *is a degenerate cycle through the fold-regular point* F_Z *which is an attractor for orbits passing through points in* $(a_Z, \delta_0) \times \{0\} \subset \Sigma$*;*

(iii) if $\beta < 0$ *then* γ_Z *is a sliding cycle through* F_Z*.*

Moreover, the bifurcation diagrams are illustrated in Figures 3.4 and 3.5.

Proof. Since the first return maps are analyzed separately for each $Z \in \mathcal{V}_{Z_0}$, there is no loss of generality in assuming $a_Z = 0$. The graph of π_Z, supposing $a_Z = 0$, is illustrated in Figure 3.3. Item (i) and (ii) follow from the facts that the graph of π_Z intersects the

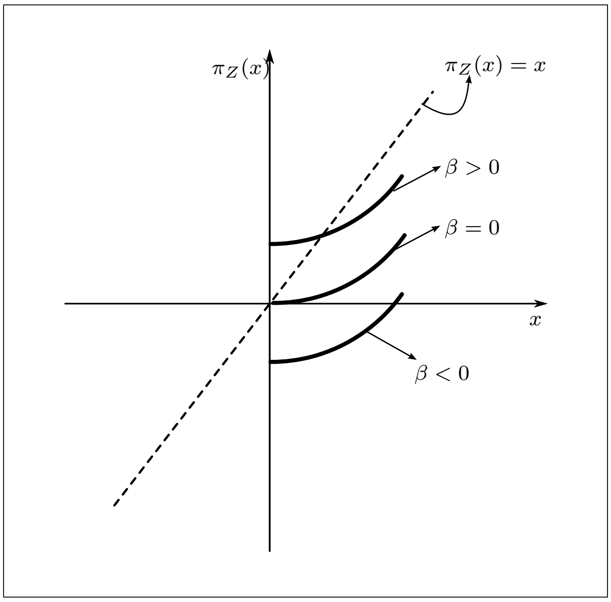

Figure 3.3: Graphs of π_Z for $a_Z = 0$.

graph of the identity map, providing a fixed point of π_Z in (a_Z, δ_Z). Item (iii) follows from the fact that F_Z is an attractor for the sliding vector field and $(\pi_Z(a_Z), 0) \in \Sigma^s$. □

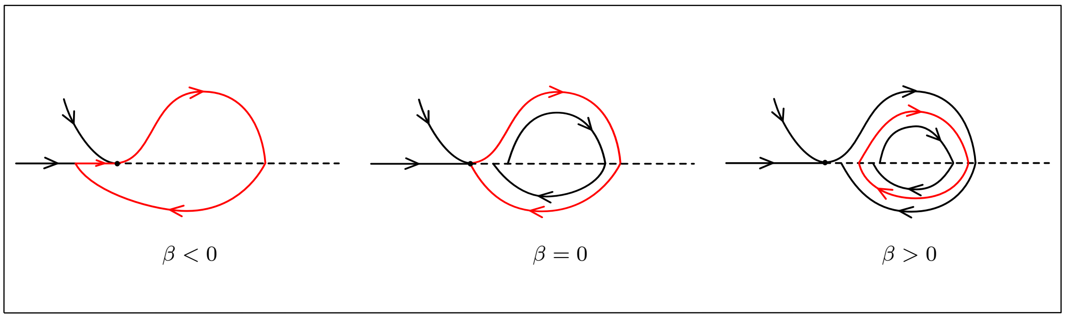

Figure 3.4: Bifurcation diagram for Z_0 of type DFC_1.

Notice that, although the result and the approach are the same for both cases, DFC_1 and DFC_2, they are topologically distinct.

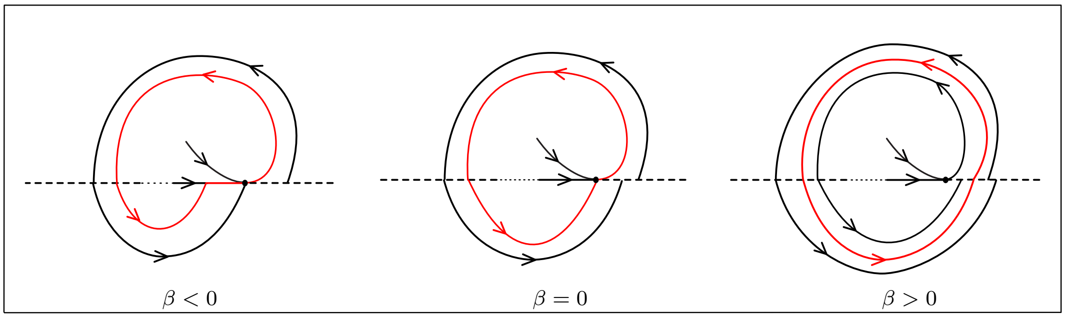

Figure 3.5: Bifurcation diagram for Z_0 of type DFC_2.

3.3 Examples

In this section we present models realizing the cycles DFC_1 and DFC_2. Due to the simplicity of the system we perform a more detailed analysis for the case DFC_1 to illustrate how rich can be a system presenting this kind of cycle.

Observe that cycles of type DFC_1 appear in bifurcations of a invisible fold-cusp singularity, see [9]. Moreover, cycles of type DFC_2 appear in bifurcations of a focus-fold singularity.

3.3.1 Cycle of Type DFC_1

Consider $Z_\beta = (X, Y_\beta)$ where X and Y_β are given by

$$X = \begin{pmatrix} 1 \\ x(4-3x) \end{pmatrix} \quad \text{and} \quad Y_\beta = \begin{pmatrix} -1 \\ 1+\beta-x \end{pmatrix}, \tag{3.3.1}$$

where $\beta \in \mathbb{R}$ is small enough for our purposes and $\Sigma = h^{-1}(0)$ with $h(x,y) = y$. Z_β satisfies the following properties:

1. both, X and Y_β, have no singular points;
2. X has a visible fold point at $F_0 = (0,0) \in \Sigma$ and an invisible fold point at $F_1 = (4/3, 0) \in \Sigma$;
3. Y_β has an invisible fold point at $F_\beta = (1+\beta, 0) \in \Sigma$;
4. the trajectory of X passing through F_0 meets Σ transversely, for a positive time, at $P_0 = (2,0)$;
5. the trajectory of Y_β passing through P_0 meets Σ again, for a positive time, at $(2\beta, 0)$;
6. once we are interested in $|\beta|$ sufficiently small we can assume that $0 < 1+\beta < \frac{4}{3}$ (i.e. $-1 < \beta < 1/3$);

Therefore, for $\beta = 0$ we have a degenerate cycle of type DFC_1 passing through the visible fold-regular point F_0. The first return map, π_β, of Z_β on the half interval $[0, 1+\beta)$ is given by

$$\begin{array}{rccc} \pi_\beta : & [0, 1+\beta) & \longrightarrow & [2\beta, \delta_\beta) \\ & x & \mapsto & 1 + 2\beta + \dfrac{x}{2} - \dfrac{1}{2}\sqrt{4 + 4x - 3x^2} \end{array}$$

where $\delta_\beta = \pi_\beta(1+\beta)$.

We are interested in the fixed points of π_β, i.e., in the solutions of $\pi_\beta(x) = x$ with $x \in [0, 1+\beta)$. This equation has two solutions $x_\pm = 1 + \beta \pm \sqrt{1 - 2\beta - 3\beta^2}$. For $0 < \beta < 1/3$ we have $x_+ > 1 + \beta$ and $0 \leq x_- < 1 + \beta$ then, only x_- corresponds to an acceptable fixed point. Moreover, $|\pi'_\beta(x_-)| < 1$ so x_- corresponds to an attractor limit cycle of Z_β. For $-1 < \beta < 0$ there exists no limit cycle.

In order to complete the study of the family Z_β it is necessary to analyze the sliding vector field. The sliding and escaping regions are

$$\Sigma^s = \{(x, 0) \in \Sigma; x < 0\} \quad \text{and} \quad \Sigma^e = \{(x, 0) \in \Sigma; 1 + \beta < x < 4/3\},$$

the sliding vector field is given by

$$Z^s_\beta(x) = \frac{1 + \beta + 3x - 3x^2}{1 + \beta - 5x + 3x^2}. \tag{3.3.2}$$

The equation $Z^s_\beta(x) = 0$ has two solutions, $z_\pm = \frac{1}{2} \pm \frac{1}{6}\sqrt{21 + 12\beta}$, satisfying:

- $z_- < 0$ and $Z'_\beta(z_-) > 0$ for $\beta > -1$. Then, $z_- \in \Sigma^s$ is a pseudo-saddle for Z_β;
- $z_+ \in (1+\beta, 4/3)$ and $Z'_\beta(z_+) > 0$ for $-7/4 < \beta < 1/3$. Then, $z_+ \in \Sigma^e$ is an unstable pseudo-node for Z_β;
- $z_- = 0$ for $\beta = -1$ then, $P_- = (z_-, 0)$ coincides with the visible fold point of X and it is a repellor for Z^s_β;
- $z_+ = 4/3$ for $\beta = 1/3$ then, $P_+ = (z_+, 0)$ coincides with the invisible fold point of X and it is a repellor for Z^s_β.

Observe that, when $\beta < 0$, the points F_0 and P_- can be connected with F_β, P_+ or F_1. In fact, for $\beta = 0$ we have a homoclinic-like connection of F_0, for $\beta = \beta_0 = \frac{7}{24} - \frac{1}{24}\sqrt{97}$ we have an heteroclinic connection between F_0 and P_-. It follows from continuity that for $\beta \in (\beta_0, 0)$ we have $\pi_\beta(F_0) \in (z_-, 0)$ and for $-1 < \beta < \beta_0$ we have $\pi_\beta(F_0) < z_-$. There exist lots of possible heteroclinic connections for this system, since our intention is just to show how rich this system can be we restrict our study for $\beta_0 < \beta < 1/3$.

Under some restrictions on β, the negative trajectory of Z_β through P_- intersects Σ in a point with first coordinate greater than 2, so there exists no connections between this point and F_β, P_+, F_1 or F_0. Therefore, we just need to analyze the connections of F_0. By using the software *Mathematica* and continuous dependence criteria we obtain

(1) for $\beta = \beta_1 = -1/3$ a trajectory of Y_β connects F_1 and F_0 without intersecting Σ in any other point;

(2) for $\beta = \beta_3 = -1/7$ the trajectory of Z_β through F_β meets F_0 after crossing Σ transversely once;

(3) items (1) and (2) imply in the existence of $\beta_2 \in (\beta_1, \beta_3)$ for which the trajectory of Z_β through P_+ meets F_0 after crossing Σ once;

(4) for $\beta = \beta_5 = (-3 + \sqrt{6})/9 \in (\beta_3, 0)$ the trajectory of Z_β through F_1 meets F_0 after crossing Σ transversely twice;

(5) items (2) and (4) imply in the existence of $\beta_4 \in (\beta_3, \beta_5)$ for which the trajectory of Z_β through P_+ meets F_0 after crossing Σ twice.

In this way, we obtain a strictly increasing sequence, $\beta_0 < \beta_1 < \beta_2 < \cdots < \beta_n < \cdots < 0$, such that

- for $\beta = \beta_{4i+1}$ the trajectory of Z_β through F_1 meets F_0 after crossing Σ transversely i times;
- for $\beta = \beta_{4i+2}$ the trajectory of Z_β through P_+ meets F_0 after crossing Σ transversely $2i+1$ times;
- for $\beta = \beta_{4i+3}$ the trajectory of Z_β through F_β meets F_0 after crossing Σ transversely $i+1$ times;
- for $\beta = \beta_{4i}$ the trajectory of Z_β through P_+ meets F_0 after crossing Σ transversely $i+1$ times;

Now we are able to sketch the bifurcation diagram of Z_β, see Figure 3.6.

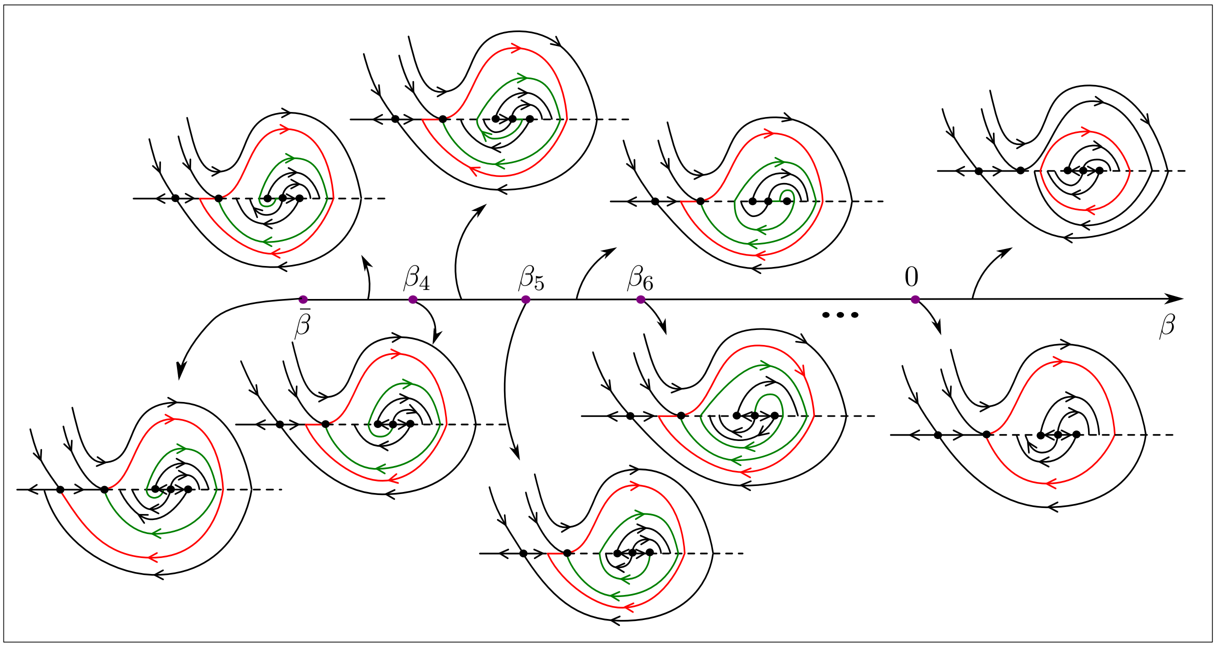

Figure 3.6: Bifurcation diagram of Z_β.

3.3.2 Cycle of Type DFC_2

In order to simplify the calculations consider $\Sigma = h^{-1}(0)$ with $h(x,y) = y - c$. Also, consider $Z_\beta = (X, Y_\beta)$ where X and Y_β are given by

$$X = \begin{pmatrix} x/4 - 2y \\ 2x + y/4 \end{pmatrix} \quad \text{and} \quad Y_\beta = \begin{pmatrix} 1 \\ x - \beta \end{pmatrix}, \tag{3.3.3}$$

with $\beta \in \mathbb{R}$. Observe that Z_β satisfies the following properties:

- X has a real unstable focus at the origin. For $c \neq 0$ X has a fold point at $F_0 = (-c/8, c)$ which is visible if $c < 0$ and invisible if $c > 0$;
- Y_β has an invisible fold point at $F_\beta = (\beta, c)$.

Notice that Z_β has a focus-fold point at the origin when $c = \beta = 0$. We are interested in showing that this vector field presents a cycle of the type DFC_2, it is just possible for $c < 0$. Since the focus at the origin is unstable and the orientation is counter-clockwise, the positive trajectory of X through F_0 crosses Σ transversely at a point $P_0 = (p_0, c)$ with $p_0 < -c/8$. On the other hand, the trajectories of Y_β intersect Σ symmetrically with relation to F_β. Therefore, for $\beta = -\frac{c+8p_0}{16}$ the vector field Z_β realizes a degenerate cycle of type DFC_2.

By means of numeric calculations we can plot the trajectories of Z_β and detect values of c and β for which Z_β belongs to each region of the bifurcation diagram presented in Figure 3.5.

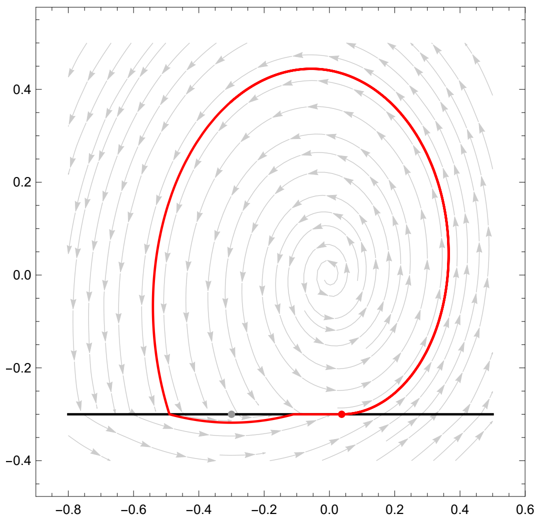

Figure 3.7: Trajectory of Z_β through F_0 (red) for $c = -0.3$ and $\beta = -0.3$. The gray point is F_β.

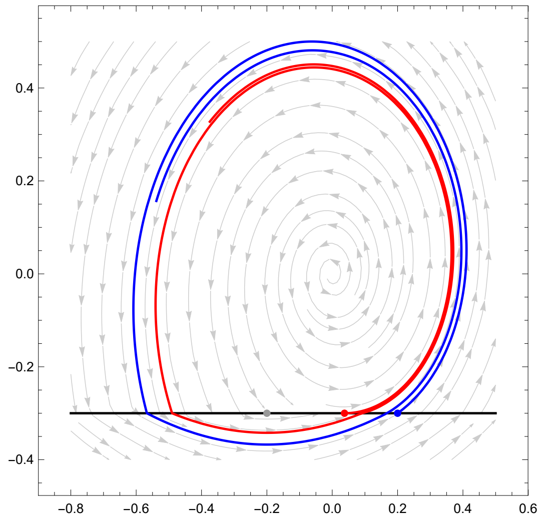

Figure 3.8: Trajectories through F_0 (red) and $P = (1/5, -0.3)$ (blue) for $c = -0.3$ and $\beta = -0.2$. The gray point is F_β, the red point is F_0, and the blue point is P.

(i) Z_β has a sliding cycle through F_0 for $c = -0.3$ and $\beta = -0.3$, see Figure 3.7;

(ii) Z_β has a stable crossing limit cycle for $c = -0.3$ and $\beta = -0.2$. The existence of the limit cycle is guaranteed by the fact that the positive trajectory of Z_β through F_0 intersects Σ, at the second time, at a point with first coordinate bigger than $-c/8$, while the trajectory through $P = (1/5, -0.3)$ intersects Σ, at the second time, at a point with first coordinate smaller than $1/5$, see Figure 3.8;

(iii) it follows from items (i) and (ii) that, for $c = -0.3$ and $\beta \in (-0.3, -0.2)$, Z_β has a degenerate cycle of type DFC_2.

Chapter 4

Degenerate Cycle Through a Saddle-Regular Point

In this chapter we present a study on a degenerate cycle passing through a hyperbolic saddle-regular point. Cycles through saddle points have been studied in the literature, see [18, 31, 41, 42], but all these works present either a real saddle with invariant manifolds crossing the switching manifold Σ transversely twice or a boundary saddle with invariant manifolds crossing Σ only at the saddle point. Our interest lies on the study of a boundary saddle with one of the invariant manifolds crossing Σ transversely at another point, different from the saddle. A systematical approach of the study of the bifurcations of this kind of cycle is presented here. The first step in this analysis is the study of local bifurcations of a saddle-regular point and then an analysis of the first return map defined near the cycle. Finally, we present all bifurcation diagrams for a family of vector fields $Z_{\alpha,\beta} = (X_{\alpha,\beta}, Y_{\alpha,\beta})$, such that $Z_{0,0}$ has a degenerate cycle of this type and the hyperbolicity ratios of the saddle point of $X_{\alpha,\beta}$ is an irrational number. Moreover, some models realizing the cycle and having saddle with hyperbolicity ratio as a rational number are studied.

4.1 Generic Conditions

We start by establishing the necessary generic conditions to obtain a degenerate cycle with lower possible codimension. Consider a discontinuous vector field $Z_0 = (X_0, Y_0) \in \Omega^r$ satisfying the following conditions

- $BS(1)$: X_0 has a hyperbolic saddle at $S_{X_0} \in \Sigma$ and the invariant manifolds of X_0 at the saddle point S_{X_0}, $W^u(X_0, S_{X_0})$ and $W^s(X_0, S_{X_0})$, are transversal to Σ at S_{X_0};
- $BS(2)$: Y_0 is transversal to Σ, $W^u(X_0, S_{X_0})$, and $W^s(X_0, S_{X_0})$ at S_{X_0};
- $BS(3)$: the sliding vector field near the saddle point has S_{X_0} as a non-degenerate singularity, i.e., by taking x as a local chart on Σ at S_{X_0}, $Z_s(x) = \mu x + O(x^2)$ with $\mu \neq 0$;
- $BSC(1)$: the unstable manifold of the saddle which lies in Σ^+, $W^u_+(X_0, S_{X_0})$, is transversal to Σ at $P_{X_0} \neq S_{X_0}$;

- $BSC(2)$: Y_0 is transversal to Σ at P_{X_0} and there exists $t_0 > 0$ such that $\varphi_{Y_0}(t_0, P_{X_0}) = S_{X_0}$, where $\varphi_{Y_0}(t_0, P_{X_0})$ denotes the trajectory of Y_0 through P_0. Moreover, $\varphi_{Y_0}(t, P_{X_0}) \in \Sigma^-$ for all $0 < t < t_0$.

Remark 4.1. *Conditions $BS(i)$, $i = 1, 2, 3$, concern about the behavior of a saddle-regular point and they are necessary to guarantee a codimension-1 bifurcation for this singularity, see [15] and [23]. Conditions $BSC(i)$, $i = 1, 2$, guarantee the existence of a degenerate cycle and, jointly with conditions $BS(i)$, $i = 1, 2, 3$, they avoid higher degeneracy for the cycle.*

Notice that, under the conditions above, the saddle-regular point is on the boundary of a sewing region and a escaping or sliding region, i.e, $S_{X_0} \in \partial\Sigma^e \cup \partial\Sigma^c$ or $S_{X_0} \in \partial\Sigma^s \cup \partial\Sigma^c$. We can also obtain such kind of cycle by changing the conditions $BSC(1)$ and $BSC(2)$ by

- $BSC(1')$: the stable manifold of the saddle which lies in Σ^+, $W^s_+(X_0, S_{X_0})$, is transversal to Σ at $P_{X_0} \neq S_{X_0}$;

- $BSC(2')$: there exists $t_0 < 0$ such that $\varphi_{Y_0}(t_0, P_{X_0}) = S_{X_0}$ and Y_0 is transversal to Σ at S_{X_0}. Moreover, $\varphi_{Y_0}(t, P_{X_0}) \in \Sigma^-$ for all $t_0 < t < 0$.

However, notice that, under conditions $BSC(1)$ and $BSC(2)$, results for $BSC(1')$ and $BSC(2')$ are obtained just by changing Z by $-Z$, i.e., by changing the orientation of all orbits. Variations of these cases are similarly obtained. For this reason, we just consider cases where $BSC(1)$ and $BSC(2)$ hold. There are two different topological types of cycles satisfying $BS(1) - BS(3)$ and $BSC(1) - BSC(3)$. In fact, the two possibilities are: the stable manifold $W^s_+(X_0, S_{X_0})$ is contained in the unbounded region (resp. bounded region) delimited by $W^u_+(X_0, S_{X_0})$ and the curve in Σ with extreme points in P_{X_0} and S_{X_0}, see Figure 4.1-(a) (resp. Figure 4.1-(b)).

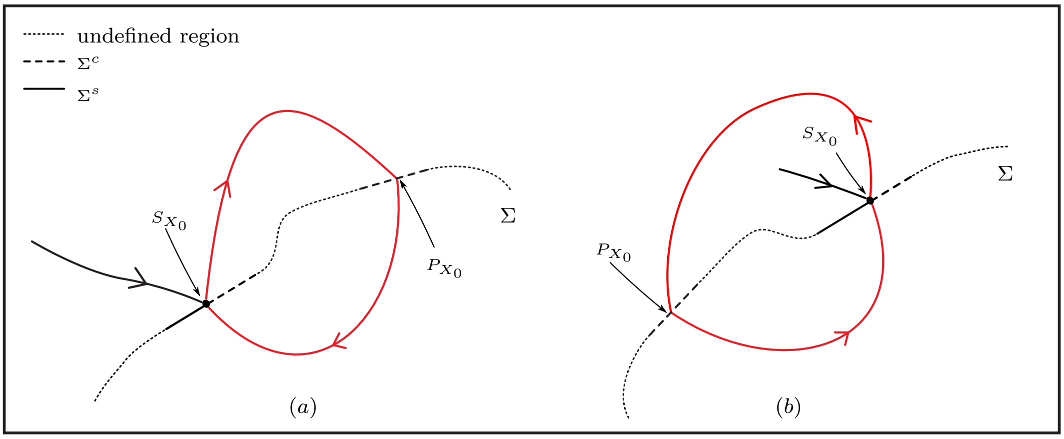

Figure 4.1: A degenerate cycle passing through a saddle-regular point: (a) $W^s_+(X_0, S_{X_0})$ contained in the unbounded region and (b) $W^s_+(X_0, S_{X_0})$ contained in the bounded region.

Despite the fact that both cases shown in Figure 4.1 are topologically distinct, the approach is conducted in the same manner as in case (a) and the proofs of the results in case (b) is straightforward. Therefore, we focus on case (a) and, hereinafter, whenever we refer to a degenerate cycle through a saddle point, we refer to a cycle of case (a) in Figure 4.1. Now, we proceed to study how this kind of cycle can be unfolded. We start by looking at the unfolding of the local saddle-regular bifurcation, then we perform a study on the first return map defined near the cycle. Finally, we state the main results and bifurcation diagrams.

4.1.1 Bifurcations of a Saddle-Regular Point

We are interested in the simplest case, i.e., the codimension 1 case studied in [15, 22, 23]. So, consider a discontinuous vector field $Z_0 = (X_0, Y_0) \in \Omega^r$ satisfying conditions $BS(1) - BS(3)$ and $BSC(3)$. In order to study local bifurcations of Z_0 near S_{X_0}, the following result, which is proved in [38] (Lemma 8.1, page 79), is an important tool. It describes bifurcations of a hyperbolic saddle point on the boundary Σ of the manifold with boundary $\overline{\Sigma^+} = \Sigma \cup \Sigma^+$.

Lemma 4.1. *Let $p \in \Sigma$ be a hyperbolic saddle point of $X_0|_{\overline{\Sigma^+}}$, where $X_0 \in \chi^r$. Then, there exist neighborhoods B_0 of p in $\mathbb{R}^2$ and $\mathcal{V}_0$ of X_0 in χ^r, and a C^r-map $\beta : \mathcal{V}_0 \to \mathbb{R}$, such that:*

(a) $\beta(X) = 0$ if and only if X has a unique equilibrium $p_X \in \Sigma \cap B_0$ which is a hyperbolic saddle point;

(b) if $\beta(X) > 0$, X has a unique equilibrium $p_X \in B_0 \cap int(\Sigma^+)$ which is a hyperbolic saddle point;

(c) if $\beta(X) < 0$, X has no equilibria in $B_0 \cap \overline{\Sigma^+}$.

Since Y_0 is transversal to Σ at S_{X_0}, there exist neighborhoods B_1, of S_{X_0} in Σ, and $\mathcal{V}_1$, of Y_0 in χ^r, such that for any $Y \in \mathcal{V}_1$ and $p \in B_1$, Y is transversal to Σ at p. So, reducing B_0 given in Lemma 4.1 and B_1 if necessary, we consider $B_0 \cap \Sigma = B_1$ and the neighborhood $\mathcal{V}_{Z_0} = \mathcal{V}_0 \times \mathcal{V}_1$ of Z_0 in Ω^r.

Notice that, if $\beta(X) \neq 0$, for $X \in \mathcal{V}_0$, there exists a fold point of X in $\Sigma \cap B_0$. The fold point is located between the points where the invariant manifolds of the saddle cross Σ. Moreover, the map $s : \mathcal{V}_0 \to \Sigma$ that associates each $X \in \mathcal{V}_0$ to a tangent point $F_X \in \Sigma$ is of class $\mathcal{C}^r$, F_X is visible fold point if $\beta(X) < 0$, F_X is a hyperbolic saddle point if $\beta(X) = 0$, F_X is an invisible point if $\beta(X) > 0$.

Given $Z = (X, Y) \in \mathcal{V}_{Z_0}$, we associate two curves, T_X and PE_Z, defined as following:

(i) T_X is the curve given implicitly by the equation $Xh(p) = 0$, i.e., T_X is formed by the point where X is parallel to Σ. Therefore, the intersection of T_X with Σ gives the fold point F_X;

(ii) PE_Z is the curve given implicitly by the equation $X(p) = \lambda(p)Y(p)$, where $\lambda(p) \in \mathbb{R}$, i.e., PE_Z is composed by those points where X and Y are parallel. So, when the intersection of PE_Z with Σ is in $\Sigma^s \cup \Sigma^e$, this intersection gives the position of the pseudo-equilibrium point.

Observe that the maps $X \mapsto T_X$ and $Z \mapsto PE_Z$ are of class C^r. It follows from conditions $BS(1) - BS(3)$ that the curves T_{X_0}, PE_{Z_0}, $W^u(X_0, S_{X_0})$ and $W^s(X_0, S_{X_0})$ have empty intersection in B_0 up to the saddle point X. In fact, all these curves contain the singular point S_{X_0}. Condition $BS(2)$ guarantees that PE_{Z_0}, $W^u(X_0, S_{X_0})$ and $W^s(X_0, S_{X_0})$ do not coincide in B_0 up to the saddle point, see [15, 23]. Condition $BS(2)$ also ensures that T_{X_0} and PE_{Z_0} are different in B_0 except at S_{X_0}, otherwise Y_0 would be tangent to Σ in B_1.

The continuous dependence of the curves T_{X_0}, PE_{Z_0}, $W^u(X_0, S_{X_0})$ and $W^s(X_0, S_{X_0})$ on the vector field ensures that $\mathcal{V}_{Z_0}$ can be reduced in order not change the relative position of these curves for all $Z \in \mathcal{V}_{Z_0}$. Moreover, curve T_X is located between $W^s_+(X_0, S_{X_0})$ and $W^u_+(X_0, S_{X_0})$, see Figure 4.2.

Now, we fix $S_{X_0} \in \partial\Sigma^s \cup \partial\Sigma^c$, the case where $S_{X_0} \in \partial\Sigma^e \cup \partial\Sigma^c$ can be easily obtained from the first one. Thus, there are three different cases to consider depending on the position of PE_{Z_0} in relation to T_X, $W^s_+(X_0, S_{X_0})$, and $W^u_+(X_0, S_{X_0})$, see 4.2. These cases are named in [22] as BS_1, BS_2 and BS_3, we keep this notation.

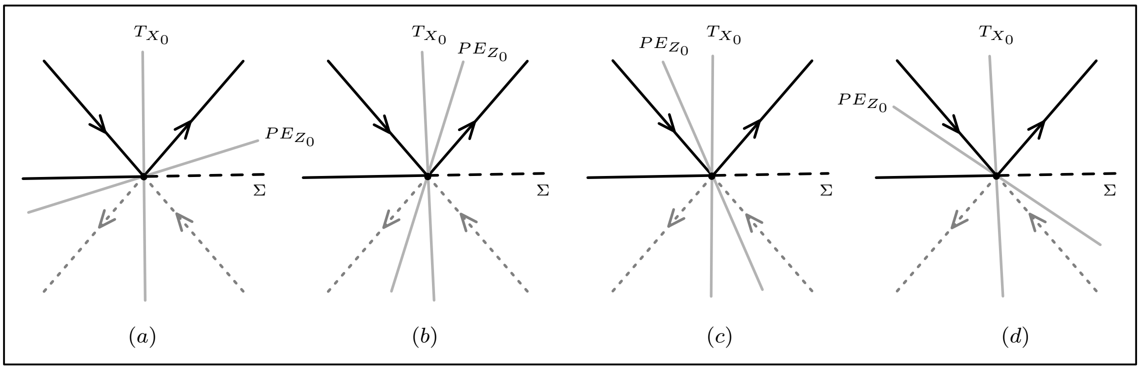

Figure 4.2: Relative position of the curves T_{X_0}, PE_{Z_0}, $W^s_+(X_0, S_{X_0})$, and $W^u_+(X_0, S_{X_0})$. (a) corresponds to case BS_1, (b) corresponds to case BS_2 and $(c) - (d)$ both correspond to case BS_3.

In order to understand the difference between BS_1, BS_2 and BS_3, for $Z = (X, Y) \in \mathcal{V}_{Z_0}$ let $\beta = \beta(X)$ be given by Lemma 4.1. Then, S_X is a real saddle if $\beta > 0$, a boundary saddle if $\beta = 0$, or a virtual saddle if $\beta < 0$.

- Case BS_1: this case happens when $W^u_+(S_{X_0}, X_0)$ is between T_{X_0} and PE_{Z_0} in Σ^+, see Figure 4.2$-(a)$. If the saddle is virtual ($\beta < 0$), the saddle-regular point turns into a visible fold-regular point and there is no pseudo-equilibrium. The fold-regular point is an attractor for the sliding vector field. Also, when the saddle is real ($\beta > 0$), an invisible fold-regular point merges and there exists an attractor pseudo-node. Moreover, the point in Σ^s where the unstable manifold of the saddle crosses Σ is located between the pseudo-equilibrium and the fold-regular point, see Figure 4.3.

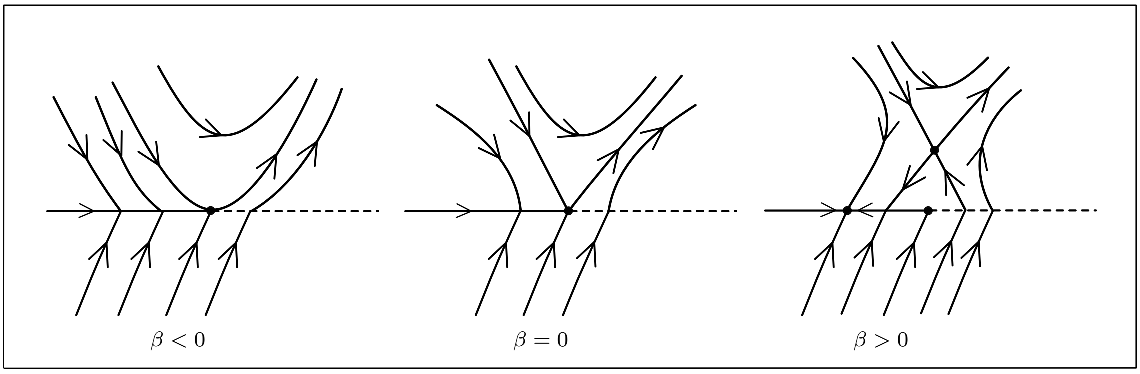

Figure 4.3: Bifurcation of a saddle-regular point: case BS_1.

2. Case BS_2: this case happens when PE_{Z_0} is between T_{X_0} and $W^u_+(S_{X_0}, X_0)$ in Σ^+, see Figure 4.2$-(b)$. There is a visible fold-regular point and there is no pseudo-equilibrium when the saddle is virtual ($\beta < 0$). The fold-regular point is an attractor

for the sliding vector field. When the saddle is real ($\beta > 0$), an invisible fold-regular point and an attractor pseudo-node coexist. Moreover, the pseudo-equilibrium is located between the point (in Σ^s) where the unstable manifold of the saddle meets Σ and the fold-regular point. See Figure 4.4.

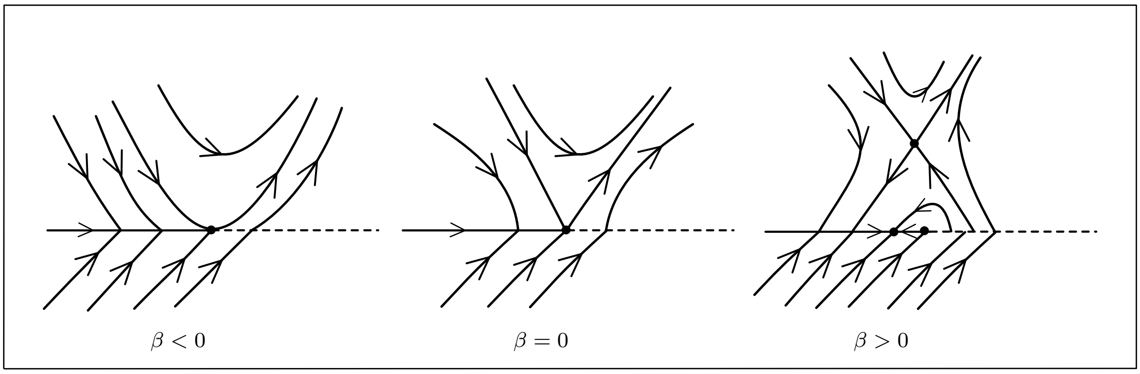

Figure 4.4: Bifurcation of a saddle-regular point: case BS_2.

3. Case BS_3: this case happens when T_{X_0} is between $W^u_+(S_{X_0}, X_0)$ and PE_{Z_0} in Σ^+, see Figures 4.2$(c)-(d)$. When the saddle is virtual ($\beta < 0$), a visible fold-point coexists with a pseudo-saddle. There exists no pseudo-equilibrium when the saddle is real ($\beta > 0$). In this case, the saddle-regular point turns into an invisible fold-regular point which is a repellor for the sliding vector field. See Figure 4.5.

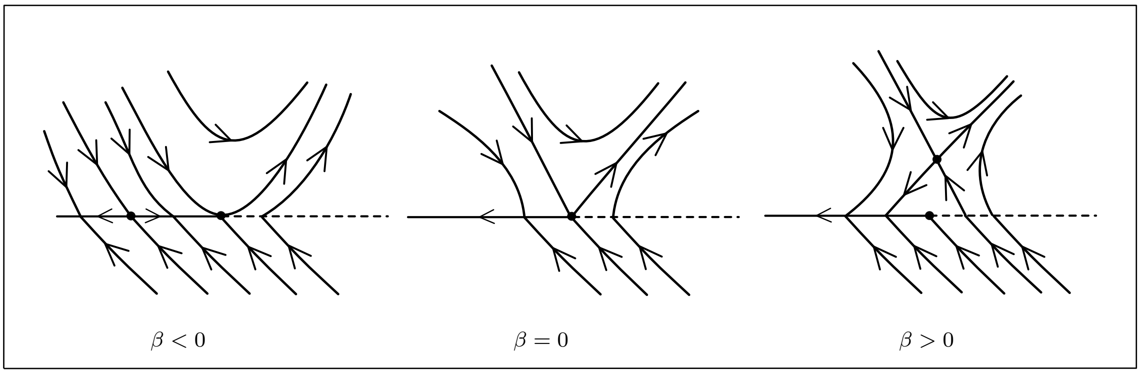

Figure 4.5: Bifurcation of a saddle-regular point: case BS_3.

4.1.2 First Return Map

Firstly, let Γ_0 be the degenerate cycle of Z_0. We show that $\mathcal{V}_{Z_0}$ can be reduced in such a way that, for each $Z \in \mathcal{V}_{Z_0}$, a first return map is defined in a half-open interval, near the cycle Γ_0.

Lemma 4.2. *Consider the discontinuous vector field Z_0 satisfying the conditions $BS(1)-BS(3)$ and $BSC(1) - BSC(2)$. In addition, suppose S_{X_0} is located at the origin. Then, there exists a neighborhood $\mathcal{V}_{Z_0}$ of Z_0 in Ω^r satisfying that, for all $Z \in \mathcal{V}_{Z_0}$, there exists a well defined first return map in a half-open interval $[a_Z, a_Z + \delta_Z)$ with $\delta_Z > 0$ and $a_Z \approx 0$.*

Proof. We have determined above a neighborhood $\mathcal{V}_{Z_0} = \mathcal{V}_0 \times \mathcal{V}_1$ where, for all $Z = (X, Y) \in \mathcal{V}_{Z_0}$, Lemma 4.1 holds for X and transversal conditions holds for Y in $B_1 = B_0 \cap \Sigma$. In order to reach the result, this neighborhood is reduced if it is necessary.

Given that the orbit of Y_0 connecting P_{X_0} and $S_{X_0} = 0$ (by hypothesis) is transversal to Σ in these two points there exist neighborhoods $\mathcal{V}_3$ of Y_0 in χ^r and I_0 of P_{X_0} in Σ such that, for any $Y \in \mathcal{V}_3$ and for any $p \in I_0$, the trajectory of Y passing through p is transversal to Σ at this point and meets Σ, again transversely, in a neighborhood of 0 in Σ that we can suppose to be B_1. Moreover, the segment of this trajectory is contained in Σ^-.

The vector field X_0 has a hyperbolic saddle point at the origin such that the eigenspaces of $DX_0(0)$ are transverse to Σ in 0. From Lemma 4.1 we have neighborhoods $\mathcal{V}_0$ of X_0 in χ^r and B_0 of 0 in $\mathbb{R}^2$ such that any $X \in \mathcal{V}_0$ has a hyperbolic saddle $S_X \in B_0$ as the only singularity of X in a neighborhood B_0 of 0 and the eigenspaces of $DX(S_X)$ are transverse to Σ at the origin. Therefore, if $\beta(X) \neq 0$, the invariant manifolds of X at S_X are transversal to Σ a neighborhood of 0. Moreover, since $W^u_+(X_0, 0)$ is transversal to Σ at P_{X_0}, there exist a neighborhood $\mathcal{V}_2$ of X_0 in χ^r and a neighborhood $I_1 \subset I_0$ of P_{X_0} in Σ such that, for any $X \in \mathcal{V}_2$, $W^u_+(X, S_X)$ also meets Σ transversely at a point in I_1.

When the saddle of $X \in \mathcal{V}_0 \cap \mathcal{V}_2$ is not in Σ, there are at least three different points in which the invariant manifolds $W^{u,s}(X, S_X)$ meet Σ. Denote these points by P^i_X, $i = 1, 2, 3$, where P^1_X, $P^2_X \in B_1$ and $P^3_X \in I_1$. Moreover, assume $P^1_X, P^3_X \in W^u(S_X, X) \cap \Sigma$ and $P^2_X \in W^s(S_X, X) \cap \Sigma$. We keep this notation if the saddle of $X \in \mathcal{V}_0 \cap \mathcal{V}_2$ is in Σ, the difference is $P^1_X = P^2_X = S_X$. Since we have assumed $S_{X_0} \in \partial\Sigma^s \cup \partial\Sigma^c$ (the other case is analogous), by taking x as a local chart for Σ near 0 (with $x < 0$ corresponding to sliding region and $x > 0$ corresponding to crossing region) and by denoting $P^i_X = x_i$, $i = 1, 2$, we have $x_1 \leq x_2$. As seen before, if $S_X \notin \Sigma$ then there exists a tangency point $F_X \in B_1$, also $S_X = F_X$ when the saddle is on the boundary. Considering the chart taken for Σ, denote $F_X = x_f$, then $x_1 \leq x_f \leq x_2$, see Figure 4.6. Observe that, $\lim_{Z \to Z_0} x_i = 0$, for $i = 1, 2, f$.

Now, redefine $\mathcal{V}_{Z_0} = (\mathcal{V}_0 \cap \mathcal{V}_2) \times (\mathcal{V}_1 \cap \mathcal{V}_3)$, and for $Z \in \mathcal{V}_{\bar{Z}}$ let a_Z be defined by:

- if $\beta(X) < 0$, then $a_Z = x_f$. The trajectory through the fold point crosses Σ in I_1 (flow of X) and, after that, it crosses Σ in B_1 (flow of Y);
- if $\beta(X) = 0$, then $a_Z = x_1 = x_2$. The unstable manifold of S_X crosses Σ in I_1 (flow of X) and, after that, it crosses Σ in B_1;
- if $\beta(X) > 0$, then $a_Z = x_2$. The stable manifold of the saddle crosses Σ at P^2_X unstable manifold of S_X crosses Σ in I_1 (flow of X) and, after that, it crosses Σ in B_1.

Therefore, by continuity, for $x > a_Z \in B_1$ the trajectory of Z through this point will return to B_1 after a positive finite time. Thus, by choosing $\delta_Z > 0$ small enough such that $[a_Z, a_Z + \delta_Z) \subset B_1$ we reach the result. □

The next step is the presentation of a more precise form for this return map. It is done as follows.

Proposition 4.1. *Consider the discontinuous vector field Z_0 satisfying the conditions $BS(1) - BS(3)$ and $BSC(1) - BSC(2)$ with $S_{X_0} = (0, 0)$. Then, by considering the notation in Lemma 4.2, the first return map of Z can be written as*

$$\begin{array}{rccc} \pi_Z : & [a_Z, a_Z + \delta_Z) & \longrightarrow & J_Y \\ & x & \mapsto & \rho_3 \circ \rho_2 \circ \rho_1(x) \end{array}, \tag{4.1.1}$$

where ρ_2 and ρ_3 are orientation reversing diffeomorphisms and ρ_1 is a transition map near a saddle or a fold point.

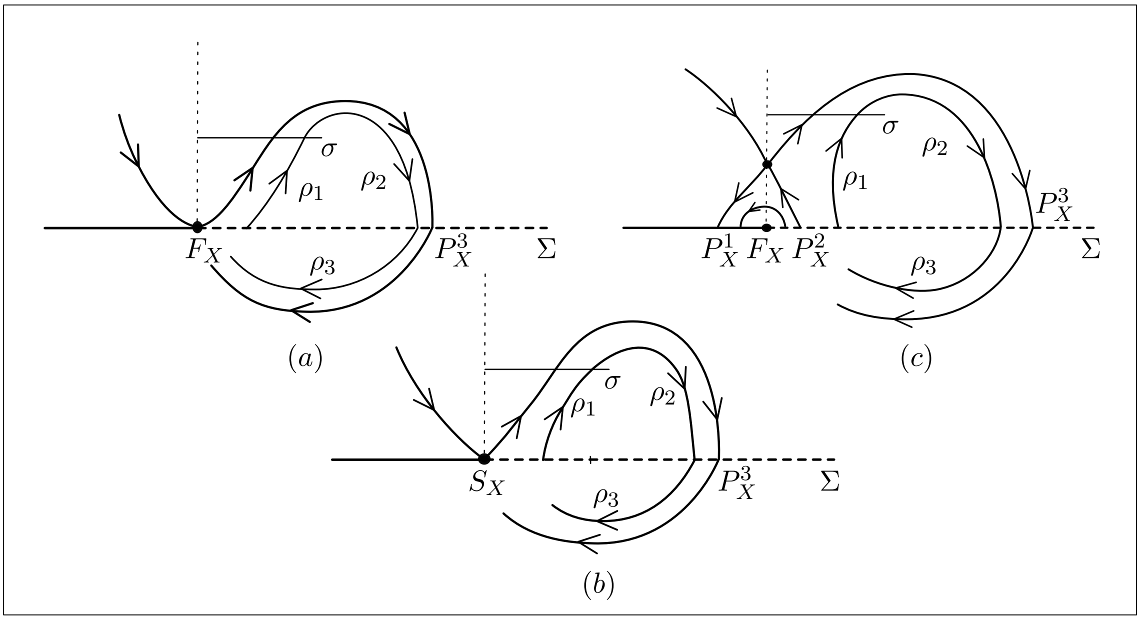

Figure 4.6: Illustration of the first return map with transversal section τ: (a) $\beta < 0$, (b) $\beta = 0$ and (c) $\beta > 0$.

Proof. As seen in Lemma 4.2 there exist neighborhoods $\mathcal{V}_{Z_0}$ of Z_0 in Ω^r and B_0 of $0 \in \Sigma$ such that, for all $Z \in \mathcal{V}_{Z_0}$ there exists $\delta_Z > 0$ for which π_Z is well defined in $[a_Z, a_Z + \delta_Z)$. Now, in order to study the structure of this first return map for $Z \in \mathcal{V}_{Z_0}$, we proceed to analyze it precisely near the saddle point.

First of all, without loss of generality, suppose that Σ is transversal to the axis y at the origin. Let σ denote a transversal section to the flow of X through the point $(0, \varepsilon)$, for some sufficiently small $\varepsilon > 0$. From Lemma 4.1 there are three options for the position of S_X in relation to Σ, in each possibility there is a different form to analyze the first return map. As above, consider $\beta = \beta(Z)$ then:

- if $\beta < 0$, S_X is virtual then, the first return map is limited by the visible fold point, F_X. So, we have a transition map, ρ_1, from Σ to σ near a fold point. See Figure 4.6$-(a)$;

- if $\beta = 0$, $S_X \in \Sigma$. Then, we have a transition map, ρ_1, from Σ to σ near a boundary saddle point. See Figure 4.6$-(b)$;

- if $\beta > 0$, S_X is real, then the limit point of the first return map is the point P_X^2 where the stable manifold $W_+^s(X, S_X)$ intersects Σ near 0. In this case, there is a transition map, ρ_1, from Σ to σ, near a real saddle point. See Figure 4.6$-(c)$.

After crossing σ, the orbits will cross Σ near P_X^3. Since σ is a transversal section, the transition from σ to Σ is performed by means of a diffeomorphism ρ_2. The flow of Y makes the transition from Σ (near P_X^3) to Σ (near 0), then the transversal conditions satisfied by Y give another diffeomorphism, ρ_3, performing this transition. By analyzing the flow, we obtain that ρ_2 and ρ_3 are orientation reversing diffeomorphisms. Thus, we have written $\pi_Z(x) = \rho_3 \circ \rho_2 \circ \rho_1(x)$, for $x \in [a_Z, a_Z + \delta_Z)$. □

Since ρ_2 and ρ_3 given in Proposition 4.1 are diffeomorphisms, the difficult part to understand in the expression of π_Z is to analyze the structure of ρ_1. For this reason, we will focus on understanding the structure of the transition map ρ_1.

As seen in Chapter 1, for each $Z = (X, Y) \in \mathcal{V}_{Z_0}$, we can assume that, around S_X, X is C^k-conjugated to the normal form given in Theorem 1.2. By performing a change of coordinates $x \mapsto y$ and $y \mapsto x$ in these normal forms we can consider axis y being the local unstable manifold and axis x being the local stable manifold of S_X. Moreover, we assume that $\Sigma = h_k^{-1}(0)$, where $h_k(x, y) = y - x + k$. Observe that Σ intersects the axis at the points $(0, -k)$ and $(k, 0)$, it means that, if the saddle is real, $k > 0$, if it is on the boundary, $k = 0$. and if it is virtual, $k < 0$.

Moreover, as seen in Chapter 2, we can assume $\varepsilon = 1$ in the definition of σ and $\sigma \subset \{(x, 1) \in \mathbb{R}^2; x \in \mathbb{R}\}$. Since we high differentiability classes are needed, we assume $Z_0 = (X_0, Y_0) \in \Omega^\infty$ and $\mathcal{V}_{Z_0} \subset \Omega^\infty$.

Hereinafter, r will denote the hyperbolicity ratio of X, i.e., $r = -\lambda_2/\lambda_1$ where $\lambda_2 < 0 < \lambda_1$ are the eigenvalues of $DX(S_X)$. Also, denote by $\varphi_{\tilde{X}}(t, x, y) = (\varphi_1(t, x, y), \varphi_2(t, x, y))^T$ the flow of $\tilde{X}$ given in the normal forms below.

(i) Suppose $r \notin \mathbb{Q}$. Then,

$$\tilde{X}(x, y) = -rx\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{4.1.2}$$

In this case, the transition time from Σ to σ is easily calculated and it is given by $t_1(x) = -\ln{(x - k)}$ for each $(x, x - k) \in \Sigma$. Therefore, the transition map $\tilde{\rho}$, from Σ to σ, is given by $\tilde{\rho}(x) = \varphi_2(t_1(x), x, x - k) = e^{-rt_1(x)}x = x(x - k)^r = k(x - k)^r + (x - k)^{r+1}$.

(ii) Suppose $r = \dfrac{p}{q} \in \mathbb{Q}$, $p, q \in \mathbb{N}$ without common factors. Then,

$$\tilde{X}(x, y) = y\frac{\partial}{\partial y} + \left(-r + \frac{1}{q}\sum_{i=1}^{N} \alpha_{i+1}(y^p x^q)^i\right) x\frac{\partial}{\partial x}. \tag{4.1.3}$$

In this case, the transition time from Σ to σ is also $t_1(x) = -\ln{(x - k)}$ for each $(x, x - k) \in \Sigma$. Thus, the transition map $\tilde{\rho}$, from Σ to σ, is given by $\tilde{\rho}(x) = \varphi_2(t_1(x), x, x - k)$ which we cannot to calculate explicitly.

For now, we focus on the case $r \notin \mathbb{Q}$.

Transition map for $r \notin \mathbb{Q}$

As seen in Chapter 1, there exist diffeomorphisms ϕ and ψ, defined in a neighborhood of the origin, such that $\rho = \phi \circ \tilde{\rho} \circ \psi$ and $\psi(0) = \phi(0) = 0$. More specifically, $\phi = \psi^{-1}$. We can also assume $\tilde{a} = \psi(a_Z)$ is equivalent to a_Z for $\tilde{X}$. i.e., the transition map $\tilde{\rho}$ of $\tilde{X}$ is defined in $[\tilde{a}, \tilde{a} + \tilde{\delta})$. With the notations established above we have:

- if $k \geq 0$, then $\tilde{a} = k$. This means $P_2 = (k, 0)$ for $k > 0$ and $S_X = (0, 0)$ for $k = 0$. See Figures 4.7-(a) and 4.7-(b);
- if $k < 0$, then $k < \tilde{a} = \dfrac{k}{1 + r} < 0$. This means that $F_X = \left(\dfrac{k}{1 + r}, \dfrac{-kr}{1 + r}\right)$. See Figure 4.7-$(c)$.

Lemma 4.3. *Let r be the hyperbolicity radius of the saddle of $\tilde{X}$, $r \notin \mathbb{Q}$, and suppose $\tilde{X}$ is given by equation 4.1.2. Define $s = [r]$, i.e., s is the highest integer smaller than r. Then, the transition map $\tilde{\rho}$ of $\tilde{X}$ satisfies:*

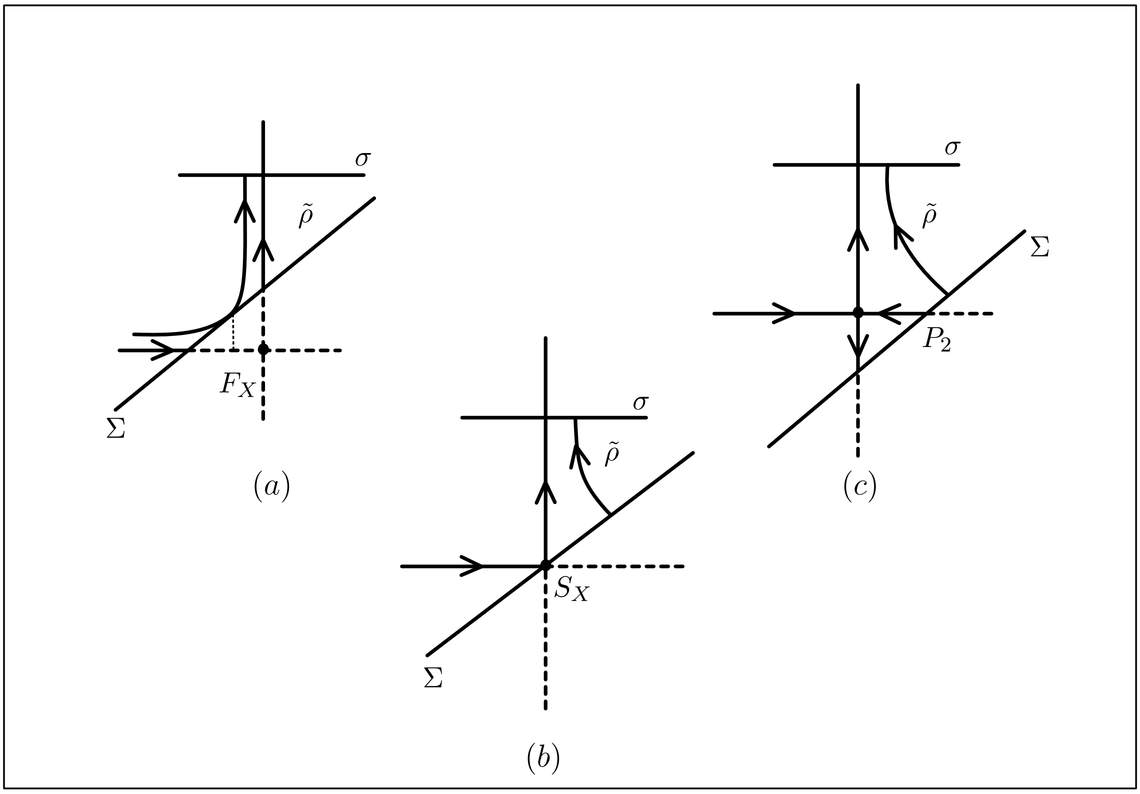

Figure 4.7: Illustration of the transition map $\tilde{\rho}$, from Σ to σ: (a) $k < 0$, (b) $k = 0$ and (c) $k > 0$.

(i) $\dfrac{d}{dx}\tilde{\rho}(\tilde{a}) = 0$ *and* $\dfrac{d^2}{dx^2}\tilde{\rho}(\tilde{a}) > 0$ *if* $k < 0$*;*

(ii) $\lim\limits_{x\to\tilde{a}^+} \dfrac{d}{dx}\tilde{\rho}(x) = \cdots = \lim\limits_{x\to\tilde{a}^+} \dfrac{d^{s+1}}{dx^{s+1}}\tilde{\rho}(x) = 0$ *and* $\lim\limits_{x\to\tilde{a}^+} \dfrac{d^{s+2}}{dx^{s+2}}\tilde{\rho}(x) = +\infty > 0$ *if* $k = 0$ *and* $r > 1$*;*

(iii) $\lim\limits_{x\to\tilde{a}^+} \dfrac{d}{dx}\tilde{\rho}(x) = +\infty$ *if* $k \leq 0$ *and* $r < 1$*;*

(iv) $\lim\limits_{x\to\tilde{a}^+} \dfrac{d}{dx}\tilde{\rho}(x) = \cdots = \lim\limits_{x\to\tilde{a}^+} \dfrac{d^{s}}{dx^{s}}\tilde{\rho}(x) = 0$ *and* $\lim\limits_{x\to\tilde{a}^+} \dfrac{d^{s+1}}{dx^{s+1}}\tilde{\rho}(x) = +\infty$ *if* $k > 0$ *and* $r > 1$.

Proof. Remember that $\tilde{\rho}(x) = \varphi_2(t_1(x), x, x-k) = k(x-k)^r + (x-k)^{r+1}$. To prove (i), notice that $r > 0$ and $\tilde{a} = k/(1+r) > k$ for $k < 0$. It is easy to see that $\tilde{\rho}$ can be smoothly extended to an open neighborhood of F_X in Σ. Moreover, we have

$$\frac{d}{dx}\tilde{\rho}(\tilde{a}) = 0 \quad \text{and} \quad \frac{d^2}{dx^2}\tilde{\rho}(\tilde{a}) = -kr(2+r)\left(-\frac{kr}{1+r}\right)^{r-2} > 0.$$

To prove $(ii)-(iv)$ it is enough to observe that, for $1 \leq i \leq s$, $r-i > 0$ and $r-(s+1) < 0$, also

$$\frac{d^i}{dx^i}\tilde{\rho}(x) = kr(r-1)\cdots(r-i+1)(x-k)^{r-i} + (r+1)r\cdots(r-i+2)(x-k)^{r+1-i}.$$

For $k \leq 0$ we have $\tilde{a} = k$. If $r > 1$, $r - i > 0$ for all $1 \leq i \leq s$ and $r - (s+1) < 0$. For $r < 1$ we have $r - i < 0$. Therefore, taking the limits, we have proved the result. □

Proposition 4.2. *Consider $Z = (X, Y) \in \mathcal{V}_{Z_0}$, let $r \notin \mathbb{Q}$ be the hyperbolicity ratio of S_X and $s = [r]$. Let $\beta = \beta(X)$ be defined in Lemma 4.1 and π_Z defined in Proposition 4.1. Then,*

(i) $\frac{d}{dx}\pi_Z(a_Z) = 0$ *and* $\frac{d^2}{dx^2}\pi_Z(a_Z) > 0$ *if* $\beta < 0$;

(ii) $\lim_{x\to a_Z^+} \frac{d}{dx}\pi_Z(x) = \cdots = \lim_{x\to a_Z^+} \frac{d^{s+1}}{dx^{s+1}}\pi_Z(x) = 0$ *and* $\lim_{x\to a_Z^+} \frac{d^{s+2}}{dx^{s+2}}\pi_Z(x) = +\infty > 0$ *if* $\beta = 0$ *and* $r > 1$;

(iii) $\lim_{x\to a_Z^+} \frac{d}{dx}\pi_Z(x) = +\infty$ *if* $\beta \le 0$ *and* $r < 1$;

(iv) $\lim_{x\to a_Z^+} \frac{d}{dx}\pi_Z(x) = \cdots = \lim_{x\to a_Z^+} \frac{d^{s}}{dx^{s}}\pi_Z(x) = 0$ *and* $\lim_{x\to a_Z^+} \frac{d^{s+1}}{dx^{s+1}}\pi_Z(x) = +\infty$ *if* $\beta > 0$ *and* $r > 1$.

Proof. From Proposition 4.1 we know $\pi_Z = \rho_3 \circ \rho_2 \circ \rho_1$. Observe that ρ_2 and ρ_3 are orientation reversing diffeomorphisms of class C^∞. Also, from Theorem 1.2, we obtain $\rho_1 = \phi \circ \tilde{\rho} \circ \psi$ where ϕ and ψ are orientation preserving diffeomorphisms of class $\mathcal{C}^l$, $l > s + 2$. Define $\Phi = \rho_3 \circ \rho_2 \circ \phi$, then $\pi_Z = \Phi \circ \tilde{\rho} \circ \psi$, where Φ and ψ are orientation preserving diffeomorphisms of class $\mathcal{C}^l$, $l > s + 2$. By definition, $\tilde{a} = \psi(a_Z)$. Let I_0 be a neighborhood of a_Z where $\pi_Z = \Phi \circ \tilde{\rho} \circ \psi(x)$ is well defined for $x \in I_0$. Then, the derivatives of order $1 \le i \le s+2$ of Φ and ψ are limited in I_0. Moreover, $\tilde{\rho}$ is differentiable in $(\tilde{a}, \tilde{a} + \tilde{\delta})$ for $\tilde{\delta} > 0$ sufficiently small.

Now,

$$\frac{d}{dx}\pi_Z(x) = \frac{d}{dx}\Phi(\tilde{\rho}(\psi(x)))\frac{d}{dx}\tilde{\rho}(\psi(x))\frac{d}{dx}\psi(x).$$

Thus, for each $1 < i \le s + 2$, by using the Chain and Product Rules for derivatives, we have

$$\begin{aligned}\frac{d^i}{dx^i}\pi_Z(x) =& \frac{d^i}{dx^i}\Phi(\tilde{\rho}(\psi(x)))\left[\frac{d}{dx}\tilde{\rho}(\psi(x))\frac{d}{dx}\psi(x)\right]^i + \frac{d}{dx}\Phi(\tilde{\rho}(\psi(x)))\frac{d^i}{dx^i}\tilde{\rho}(\psi(x))\left[\frac{d}{dx}\psi(x)\right]^i \\ &+ \frac{d}{dx}\Phi(\tilde{\rho}(\psi(x)))\frac{d}{dx}\tilde{\rho}(\psi(x))\frac{d^i}{dx^i}\psi(x) + S(x),\end{aligned}$$

where $S(x)$ is a sum of terms of composed by products of derivatives of order $< i$ of Φ, $\tilde{\rho}$, and ψ, at $\tilde{\rho}\psi(x)$, $\psi(x)$, and x, respectively. From Lemma 4.3 we obtain

$$\begin{aligned}\lim_{x\to a_Z^+}\frac{d^i}{dx^i}\pi_Z(x) &= \lim_{x\to a_Z^+}\frac{d}{dx}\Phi(\tilde{\rho}(\psi(x)))\frac{d^i}{dx^i}\tilde{\rho}(\psi(x))\left[\frac{d}{dx}\psi(x)\right]^i \\ &= \frac{d}{dx}\Phi(\tilde{\rho}(\psi(a_Z)))\left[\lim_{x\to a_Z^+}\frac{d^i}{dx^i}\tilde{\rho}(\psi(x))\right]\left[\frac{d}{dx}\psi(a_Z)\right]^i.\end{aligned}$$

Therefore, the result follows directly from the fact that $\psi(x) \to \tilde{a}^+$ when $x \to a_Z^+$ and from Lemma 4.2. □

Proposition 4.3. *Consider $Z = (X, Y) \in \mathcal{V}_{Z_0}$ and suppose that $r \notin \mathbb{Q}$ is the hyperbolicity ratio of S_X. Then, the following statements hold:*

(i) If $\pi_Z(a_Z) > a_Z$ and either $r > 1$ or $r < 1$ and $\beta(Z) \leq 0$, then there exists an attractor fixed point of π_Z, $x_0 \in (a_Z, \delta_Z)$, which corresponds to an attractor limit cycle of Z;

(ii) If $\pi_Z(a_Z) = a_Z$ and either $r > 1$ or $r < 1$ and $\beta(Z) \leq 0$ then, a_Z is an attractor for π_Z. So, there exists an attractor degenerate cycle for Z through a fold-regular point if $\beta(Z) < 0$, a saddle-regular point if $\beta(Z) = 0$, or a real saddle if $\beta(Z) > 0$;

(iii) If $\pi_Z(a_Z) < a_Z$, $r < 1$, and $\beta(Z) > 0$, then there exists a repellor fixed point of π_Z, $x_0 \in (a_Z, \delta_Z)$, which corresponds to a repellor limit cycle of Z.

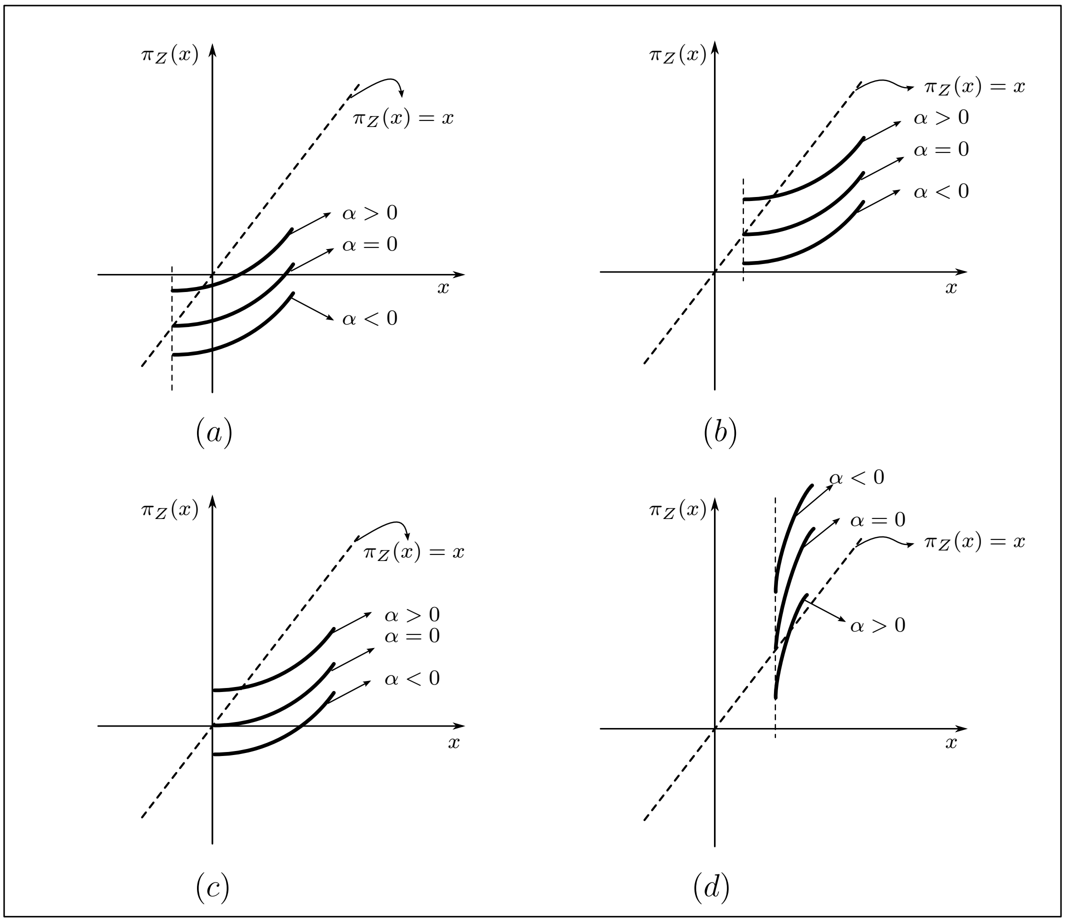

Figure 4.8: Illustration of the graph of the first return map $\pi_Z(x)$ for $x \in [a_Z, a_Z + \delta_Z)$ where $\alpha = \pi_Z(a_Z) - a_Z$. (a) $\beta(Z) < 0$, (b) $\beta(Z) > 0$ and $r > 1$, (c) $\beta(Z) = 0$, and (d) $\beta(Z) > 0$ and $r < 1$.

Proof. It follows from the definition of $\tilde{\rho}(x)$ that it is increasing in $[\tilde{a}, \tilde{a}+\tilde{\delta})$. Also, from the proof of Proposition 4.2 we have $\pi_Z = \Phi \circ \tilde{\rho} \circ \psi$ where Φ and ψ are orientation preserving diffeomorphisms, or, in other words, increasing maps since the first derivative of these maps are positive.

Also from Proposition 4.2, if $r > 1$, the tangent vector of the graph of $\pi_Z(x)$ tends to be horizontal when $x \to a_Z^+$. Analogously, if $r < 1$, the tangent vector of the graph of $\pi_Z(x)$ tends to be vertical when $x \to a_Z^+$. See Figure 4.8.

So, if $\pi_Z(a_Z) = a_Z$ and $r > 1$, reducing δ_Z if necessary, a_Z is the unique fixed point of π_Z in $[a_Z, a_Z + \delta_Z)$ and $\pi_Z(x) < x$ for all $x \in (a_Z, a_Z + \delta_Z)$. Therefore a_Z is an attractor fixed point of π_Z which corresponds to a degenerate cycle Z. The definition of a_Z gives the different types of cycles as listed in item (ii). Analogously, if $\pi_Z(a_Z) = a_Z$, $r < 1$ and $\beta(Z) > 0$, a_Z is the unique fixed point of p_Z in $[a_Z, a_Z + \delta_Z)$ and $\pi_Z(x) > x$ for all $x \in (a_Z, a_Z + \delta_Z)$. Then, a_Z is a repellor for π_Z which corresponds to a repellor degenerate cycle of Z through a real saddle point. It proves item (ii).

If $\pi_Z(a_Z) > a_Z$ and either $r > 1$ or $r < 1$ and $\beta(Z) \leq 0$, the analysis is similar to the case in item (ii), the only difference is that the fixed point will change. Indeed, since in these cases the tangent vector of the graph of π_Z tends to be horizontal at a_Z, for Z sufficiently near Z_0 (reducing $\mathcal{V}_{Z_0}$ if necessary), the graph of π_Z intersects the graph of the identity map at a point $x_0 \in (a_Z, a_Z + \delta_Z)$. Also, $\pi_Z(x) > x$ for all $x \in [a_Z, x_0)$ and $\pi_Z(x) < x$ for all $x \in (x_0, a_Z + \delta_Z)$. It proves item (i).

Finally, to prove item (iii) it is enough to observe that, since the tangent vector of the graph of π_Z tends to be vertical at a_Z, we obtain that the graph of π_Z will cross the graph of the identity map at a point $x_0 \in (a_Z, a_Z + \delta_Z)$. Also, $\pi_Z(x) < x$ for all $x \in [a_Z, x_0)$ and $\pi_Z(x) > x$ for all $x \in (x_0, a_Z + \delta_Z)$. It completes the proof. □

4.2 Main Results and Bifurcation Diagrams

Let $\mathcal{V}_{Z_0}$ be a neighborhood of Z_0 where all results found above hold. As seen before, we have some continuous correspondences, where Σ is identified with $\mathbb{R}$,

- for $Z = (X, Y) \in \mathcal{V}_{Z_0}$ it is associated the hyperbolic saddle S_X of X,

$$S: \begin{array}{ccc} \mathcal{V}(Z_0) & \rightarrow & \mathbb{R}^2 \\ (X,Y) & \mapsto & S_X \end{array}.$$

- for $Z = (X, Y) \in \mathcal{V}_{Z_0}$ it is associated a map that determines if the saddle of X is real, on the boundary, or virtual. This map was determined for X seen as a vector field in a Manifold with boundary in Lemma 4.1, now we extend this map to a $\mathcal{V}_{Z_0}$ and we will again call it β,

$$\beta: \begin{array}{ccc} \mathcal{V}(Z_0) & \rightarrow & \mathbb{R} \\ (X,Y) & \mapsto & \beta(X) \end{array}.$$

- for $Z = (X, Y) \in \mathcal{V}_{Z_0}$ it is associated the tangency point F_X of X with Σ,

$$F: \begin{array}{ccc} \mathcal{V}(Z_0) & \rightarrow & \Sigma \simeq \mathbb{R} \\ (X,Y) & \mapsto & F_X \end{array}.$$

  Observe that F_X is an invisible fold-regular point if $\beta(Z) > 0$, a visible fold-regular point if $\beta(Z) < 0$, and a saddle-regular point if $\beta(Z) = 0$.

- for $Z = (X, Y) \in \mathcal{V}_{Z_0}$ it is associated the points in Σ where the invariant manifolds of X at S_X cross Σ. The relative order in Σ of the points P_i, for $i = 1, 2, 3$, where established above.

$$P_i: \begin{array}{ccc} \mathcal{V}(Z_0) & \rightarrow & \Sigma \simeq \mathbb{R} \\ (X,Y) & \mapsto & P_X^i \end{array}$$

- for $Z=(X,Y)\in\mathcal{V}_{Z_0}$ it is associated the point $\Sigma\cap PE_Z$, where PE_Z is the curve where X is parallel to Y,

$$\begin{array}{rccc} P_E: & \mathcal{V}(Z_0) & \rightarrow & \Sigma\simeq\mathbb{R} \\ & Z & \mapsto & \Sigma\cap PE_Z \end{array}.$$

Observe that $P_E(Z)$ is a pseudo-equilibrium point of Z when $P_E(Z)\in\Sigma^s\cup\Sigma^e$.

- finally, for $Z=(X,Y)\in\mathcal{V}_{Z_0}$, it is associated a first return map π_Z. Thus, we can associate to Z a number that defines the existence of a degenerate cycle,

$$\begin{array}{rccc} \alpha: & \mathcal{V}(Z_0) & \rightarrow & \mathbb{R} \\ & Z & \mapsto & \pi_Z(a_Z)-a_Z \end{array}.$$

Observe that, despite the fact that a_Z was defined in terms of $\mathrm{Sgn}(\beta)$, the quantity $\alpha(Z)$ is an intrinsic feature of Z, it means that this dependence is just used in order to clarify the exposition.

Notice that $\alpha=\alpha(Z)$ and $\beta=\beta(Z)$ are bifurcation parameters from which we can obtain all the bifurcations of the degenerate cycle, Γ_0, of Z_0 in V_{Z_0}. Consider Z_0 satisfying $BS(1)-BS(3)$ and $BSC(1)-BSC(2)$ with hyperbolicity ratio of X_0 being irrational, from the study performed above it becomes clear that, to study these bifurcations, we have six cases to analyze:

- DSC_{11}: Z_0 has a saddle-regular point of type BS_1 and the hyperbolicity ratio of X_0 is greater than one;
- DSC_{12}: Z_0 has a saddle-regular point of type BS_1 and the hyperbolicity ratio of X_0 is smaller than one;
- DSC_{21}: Z_0 has a saddle-regular point of type BS_2 and the hyperbolicity ratio of X_0 is greater than one;
- DSC_{22}: Z_0 has a saddle-regular point of type BS_2 and the hyperbolicity ratio of X_0 is smaller than one;
- DSC_{31}: Z_0 has a saddle-regular point of type BS_3 and the hyperbolicity ratio of X_0 is greater than one;
- DSC_{32}: Z_0 has a saddle-regular point of type BS_3 and the hyperbolicity ratio of X_0 is smaller than one.

Define $\mathcal{V}_0=\{Z=(X,Y)\in\mathcal{V}_{Z_0};S_X$ has irrational hyperbolicity ratio$\}$. From now on we will consider vector fields in $\mathcal{V}_0$ and describe the bifurcations of Z_0 em $\mathcal{V}_0$. In order to do that, consider a family of discontinuous vector fields $Z_{\alpha,\beta}$ of vector fields in $\mathcal{V}_0$, $(\alpha,\beta)\in\mathcal{B}_0\subset\mathbb{R}^2$, such that $\mathcal{B}_0$ is an open neighborhood of $0\in\mathbb{R}^2$, $Z_{0,0}=Z_0$ and α, β are the bifurcation parameters discussed above. All the cases $DSC(1)-DSC(6)$ have at least four bifurcation curves, in the $(\alpha,\beta)-$plane, given implicitly as functions of α and β:

- let γ_{P_E} be the curve implicitly defined by $\pi(a_{Z_{\alpha,\beta}})-P_E(Z_{\alpha,\beta})=0$, i.e., $\gamma_{P_E}=\{(\alpha,\beta)\in\mathcal{B}_0;\pi(a_{Z_{\alpha,\beta}})-P_E(Z_{\alpha,\beta})=0\}$. Then γ_{P_E} is the curve where there exists a connection between the pseudo equilibrium and the point corresponding to $a_{Z_{\alpha,\beta}}$. Thus, this curve lies either on the half plane $\beta>0$ or in the half plane $\beta<0$;

- define $\gamma_F = \{(\alpha,\beta) \in \mathcal{B}_0; \pi(a_{Z_{\alpha,\beta}}) - F(Z_{\alpha,\beta}) = 0\}$. Then, γ_F is the curve providing a connection between the fold point $F_{X_{\alpha,\beta}}$ and the point corresponding to $a_{Z_{\alpha,\beta}}$. So, for $\beta \leq 0$ this curve coincides with the axis α;
- for $i = 1, 2$ the curve $\gamma_{P_i} = \{(\alpha,\beta) \in \mathcal{B}_0; \beta \geq 0$ and $\pi(a_{Z_{\alpha,\beta}}) - P_i(Z_{\alpha,\beta}) = 0\}$ provides a connection between $P_i(Z_{\alpha,\beta})$ and the saddle point ($a_{Z_{\alpha,\beta}}$ corresponds to the saddle point if $\beta \geq 0$). For $i = 1$ this connection i pseudo-homoclinic and it is homoclinic for $i = 2$. Moreover, γ_{P_2} coincides with the half axis $\{(0,\beta); \beta \geq 0\}$.

These curves will be illustrated later and the relative position between them depends on the case being analyzed. In some cases extra bifurcation curves will merge and they will be defined when it is necessary. In order to obtain an order relation between the curves, we use the abuse of notation $\gamma_j(\alpha,\beta) = \pi(a_{Z_{\alpha,\beta}}) - j(Z_{\alpha,\beta}) \in \mathbb{R}$, for $j = P_E,\ F, P_1$.

Now, we are ready to describe the bifurcation diagrams for the family $Z_{\alpha,\beta}$. We observe that all results follows almost directly from the analysis and constructions performed previously.

The first three theorems concern about cycles of types DSC_{11} and DSC_{12}.

Theorem A. *Suppose that Z_0 is in the case DSC_{11} and $\beta > 0$. Then, for the family $Z_{\alpha,\beta} = (X_{\alpha,\beta}, Y_{\alpha,\beta})$ defined above, bifurcation curves, γ_{P_E}, γ_{P_1}, and γ_{P_F}, merge from the origin, there exists an attractor pseudo-node, and the following statements hold:*

(a) if $(\alpha,\beta) \in R_7^1$, where $R_7^1 = \{(\alpha,\beta); 0 < \beta < \gamma_{P_E}(\alpha,\beta)\}$, then there exists a sliding polycycle passing through $S_{X_{\alpha,\beta}}$ and $P_E(Z_{\alpha,\beta})$, which contains two segments of sliding orbits;

(b) if $(\alpha,\beta) \in \gamma_{P_E}$, then there exists a sliding polycycle passing through $S_{X_{\alpha,\beta}}$ and $P_E(Z_{\alpha,\beta})$, which contains just one segment of sliding orbit;

(c) if $(\alpha,\beta) \in R_6^1 = \{(\alpha,\beta); \gamma_{P_E}(\alpha,\beta) < \beta < \gamma_{P_1}(\alpha,\beta)\}$, then there exists a sliding pseudo-cycle passing through $S_{X_{\alpha,\beta}}$;

(d) if $(\alpha,\beta) \in \gamma_{P_1}$, then there exists a pseudo-cycle passing through $S_{X_{\alpha,\beta}}$;

(e) if $(\alpha,\beta) \in R_5^1 = \{(\alpha,\beta); \gamma_{P_1}(\alpha,\beta) < \beta < \gamma_F(\alpha,\beta)\}$, then there exists a sliding pseudo-cycle passing through $S_{X_{\alpha,\beta}}$;

(f) if $(\alpha,\beta) \in \gamma_F$, then there exists a sliding pseudo-polycycle passing through $S_{X_{\alpha,\beta}}$ and $F(Z_{\alpha,\beta})$;

(g) if $(\alpha,\beta) \in R_4^1 = \{(\alpha,\beta); \gamma_F(\alpha,\beta) < \beta$ and $\alpha < 0\}$, then there exists a sliding pseudo-cycle passing through $S_{X_{\alpha,\beta}}$;

(h) if $\alpha = 0$ and $\beta > 0$, then there exists an attractor degenerate cycle passing through $S_{X_{\alpha,\beta}}$;

(i) if $(\alpha,\beta) \in R_3^1 = \{(\alpha,\beta); \beta > 0$ and $\alpha > 0\}$, then there exists an attractor limit cycle through Σ^c.

The bifurcation diagram is illustrated in Figure 4.9.

Proof. For $\beta > 0$ we have a real saddle $S_{X_{\alpha,\beta}}$. Since Z_0 is in the case DSC_{11}, the saddle-regular point of Z_0 is in the case BS_1, so there exists an attractor pseudo-node, $P_E(Z_{\alpha,\beta})$, satisfying $P_E(Z_{\alpha,\beta}) < P_1(Z_{\alpha,\beta})$ in Σ. Also, $P_1(Z_{\alpha,\beta}) < F(Z_{\alpha,\beta}) < P_2(Z_{\alpha,\beta})$ and, for $i = 1, 2, E$, $\lim_{(\alpha,\beta)\to(0,0)} F(Z_{\alpha,\beta}) = \lim_{(\alpha,\beta)\to(0,0)} P_i(Z_{\alpha,\beta}) = 0^-$. So, the curves γ_{F_Z}, γ_{P_1}, and γ_F merge from the origin and lie on $\{(\alpha,\beta); \alpha < 0 \text{ and } \beta > 0\}$. The existence of the limit cycle follows from Proposition 4.3. Now, the result is achieved by analyzing the dynamic of the system for the possible values of $\pi_{Z_{\alpha,\beta}}(a_{Z_{\alpha,\beta}})$. □

Theorem B. *Suppose that Z_0 is in the case DSC_{12} and $\beta > 0$. Then, for the family $Z_{\alpha,\beta} = (X_{\alpha,\beta}, Y_{\alpha,\beta})$ defined above, bifurcation curves, γ_{P_E}, γ_{P_1}, and γ_{P_F}, merge from the origin, there exists an attractor pseudo-node, and the following statements hold:*

(a) *if $(\alpha,\beta) \in R_7^1$, where $R_7^1 = \{(\alpha,\beta); 0 < \beta < \gamma_{P_E}(\alpha,\beta)\}$, then a repellor limit cycle through Σ^c coexists with a sliding polycycle passing through $S_{X_{\alpha,\beta}}$ and $P_E(Z_{\alpha,\beta})$, which contains two segments of sliding orbits;*

(b) *if $(\alpha,\beta) \in \gamma_{P_E}$, then a repellor limit cycle through Σ^c coexists with a sliding polycycle passing through $S_{X_{\alpha,\beta}}$ and $P_E(Z_{\alpha,\beta})$, which contains only one segment of sliding orbit;*

(c) *if $(\alpha,\beta) \in R_6^1 = \{(\alpha,\beta); \gamma_{P_E}(\alpha,\beta) < \beta < \gamma_{P_1}(\alpha,\beta)\}$, then a repellor limit cycle through Σ^c coexists with a sliding pseudo-cycle passing through $S_{X_{\alpha,\beta}}$;*

(d) *if $(\alpha,\beta) \in \gamma_{P_1}$, then a repellor limit cycle through Σ^c coexists with a pseudo-cycle passing through $S_{X_{\alpha,\beta}}$;*

(e) *if $(\alpha,\beta) \in R_5^1 = \{(\alpha,\beta); \gamma_{P_1}(\alpha,\beta) < \beta < \gamma_F(\alpha,\beta)\}$, then a repellor limit cycle through Σ^c coexists with a sliding pseudo-cycle passing through $S_{X_{\alpha,\beta}}$;*

(f) *if $(\alpha,\beta) \in \gamma_F$, then a repellor limit cycle through Σ^c coexists with a sliding pseudo-polycycle passing through $S_{X_{\alpha,\beta}}$ and $F(Z_{\alpha,\beta})$;*

(g) *if $(\alpha,\beta) \in R_4^1 = \{(\alpha,\beta); \gamma_F(\alpha,\beta) < \beta \text{ and } \alpha < 0\}$, then a repellor limit cycle through Σ^c coexists with a sliding pseudo-cycle passing through $S_{X_{\alpha,\beta}}$;*

(h) *if $\alpha = 0$ and $\beta > 0$, then there exists a repellor degenerate cycle passing through $S_{X_{\alpha,\beta}}$;*

(i) *if $(\alpha,\beta) \in R_3^1 = \{(\alpha,\beta); \beta > 0 \text{ and } \alpha > 0\}$, there exists no cycle.*

The bifurcation diagram is illustrated in Figure 4.10.

Proof. The proof is identical to the proof of Theorem A up to the fact that, as seen in Proposition 4.3, limit cycles appear for $\alpha < 0$ and the degenerate cycle for $\alpha = 0$ is a repellor. □

Theorem C. *Suppose that Z_0 is in the case DSC_{11} or DSC_{12} and $\beta \leq 0$. Then, the family $Z_{\alpha,\beta} = (X_{\alpha,\beta}, Y_{\alpha,\beta})$ defined above satisfies: there exists no pseudo-equilibrium, $S_{X_{\alpha,\beta}}$ is as attractor for the sliding vector field, and:*

(a) *if $\beta = 0$ and $\alpha < 0$, then there exists a sliding cycle passing through the $S_{X_{\alpha,\beta}}$;*

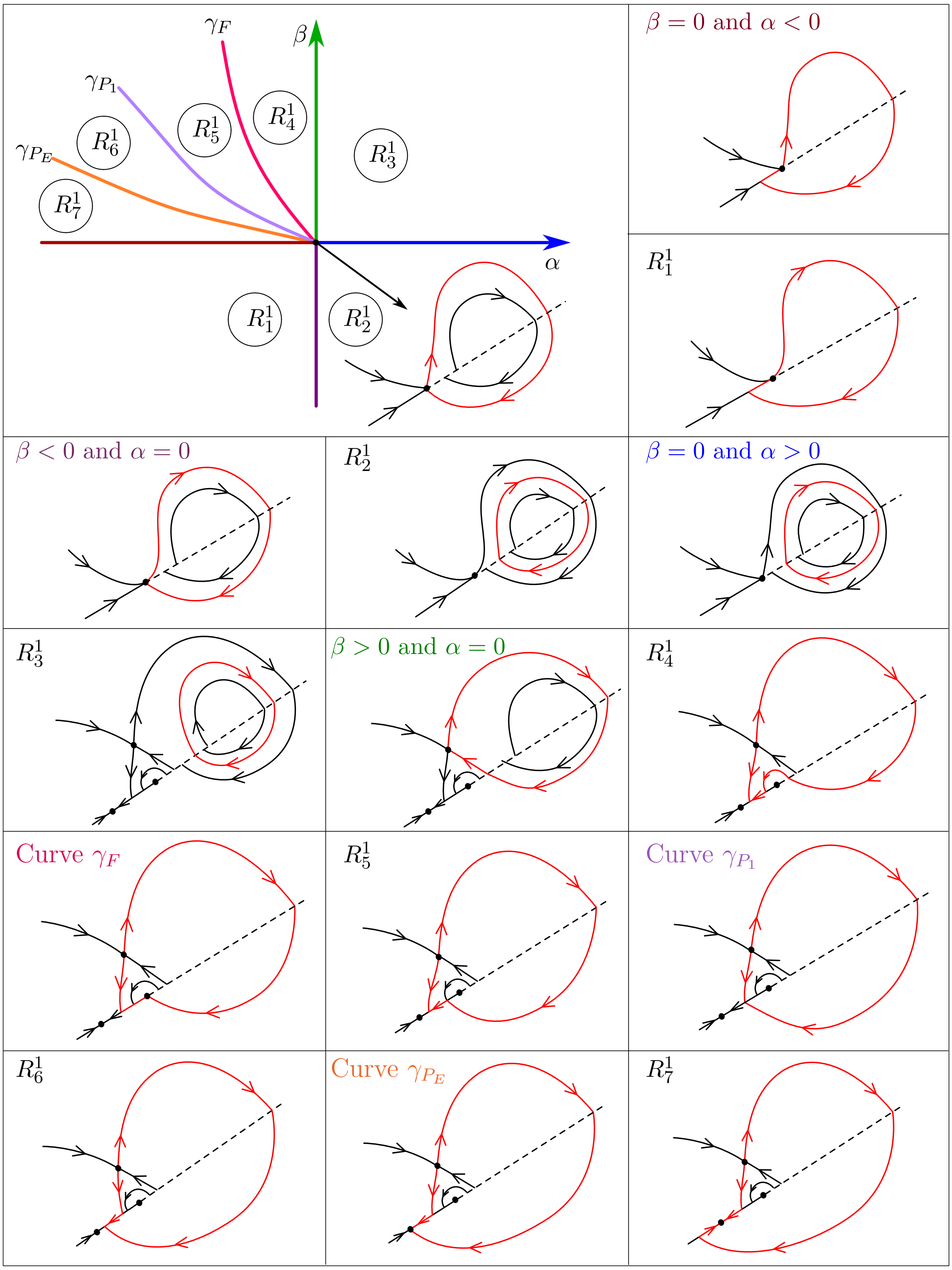

Figure 4.9: Bifurcation diagram of $Z_{\alpha,\beta}$: case DSC_{11}.

(b) if $\alpha = 0 = \beta$, then there exists an attractor cycle passing through $S_{X_{0,0}}$;

(c) if $\beta = 0$ and $\alpha > 0$, then there exists an attractor limit cycle through Σ^c;

(d) if $(\alpha, \beta) \in R_1^1 = \{(\alpha, \beta); \beta < 0$ and $\alpha < 0\}$, then there exists a sliding cycle passing through the fold-regular point $F(Z_{\alpha,\beta})$;

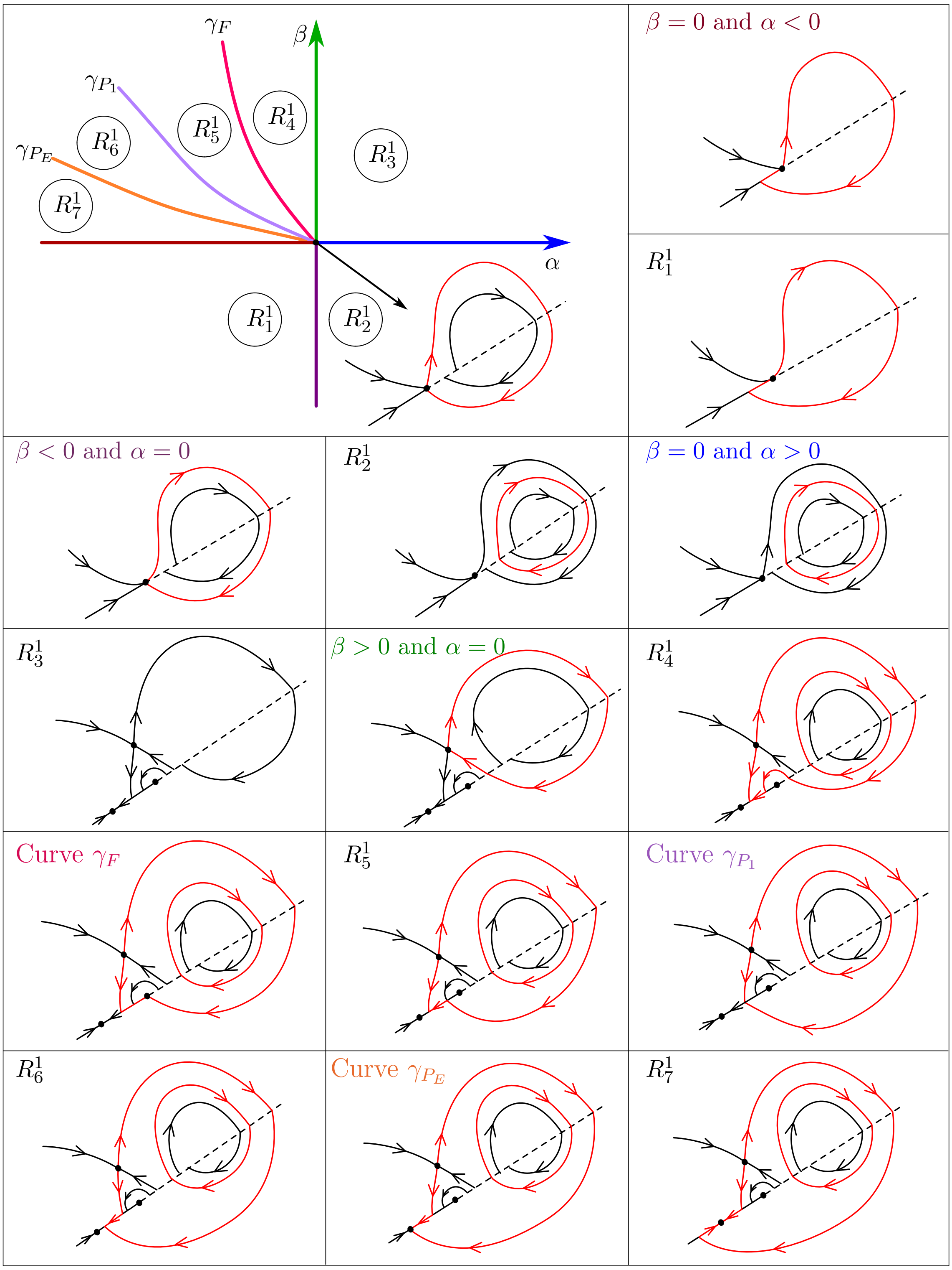

Figure 4.10: Bifurcation diagram of $Z_{\alpha,\beta}$: case DSC_{12}.

(e) *if $\alpha = 0$ and $\beta < 0$, then there exists a degenerate cycle passing through the fold-regular point $F(Z_{\alpha,\beta})$;*

(f) *if $(\alpha, \beta) \in R_2^1 = \{(\alpha, \beta); \beta < 0$ and $\alpha > 0\}$, then there exists an attractor limit cycle through Σ^c.*

The bifurcation diagrams for DSC_{11} and DSC_{12} are illustrated in Figures 4.9 and 4.10, *respectively.*

Proof. If $\beta = 0$ then $F(Z_{\alpha,\beta}) = S_{X_{\alpha,\beta}} \in \Sigma$. Items b and c follow directly from Proposition 4.3. If $\alpha < 0$ then the unstable manifold of the saddle in Σ^+ intersects Σ, after that it follows the flow of $Y_{\alpha,0}$ and it intersects the sliding region. Since $F(Z_{\alpha,\beta})$ is an attractor for the sliding vector field (see the bifurcation of a saddle-regular point of type BS_1), then there exists a sliding cycle through the saddle-regular point. If $\beta < 0$, then the saddle is virtual and there is no pseudo-equilibrium, the only distinguished singularity is the fold-regular point. Now, the result is achieved similarly to the proof of Theorem B. □

Now, the theorems concerning with cycles of type DSC_{21} and DSC_{22} are presented.

Theorem D. *Suppose that Z_0 is in the case DSC_{21}. Then, for the family $Z_{\alpha,\beta} = (X_{\alpha,\beta}, Y_{\alpha,\beta})$ defined above, bifurcation curves, γ_{P_E}, $\tilde{\gamma}_{P_E}$, γ_{P_1}, and γ_{P_F}, merge from the origin and the following statements hold:*

1. *for $\beta \leq 0$: identical to the cases given in Theorem C.*

2. *for $\beta > 0$: there exists an attractor pseudo-node and*

   (a) if $(\alpha, \beta) \in R_8^2 = \{(\alpha, \beta); 0 < \beta < \gamma_{P_1}(\alpha, \beta)\}$, then there exists a sliding pseudo-cycle passing through $S_{X_{\alpha,\beta}}$;

   (b) if $(\alpha, \beta) \in \gamma_{P_1}$ then there exists a pseudo-cycle passing through $S_{X_{\alpha,\beta}}$;

   (c) if $(\alpha, \beta) \in R_7^2 = \{(\alpha, \beta); \gamma_{P_1}(\alpha, \beta) < \beta < \gamma_{P_E}(\alpha, \beta)\}$, then there exists a sliding pseudo-cycle passing through $S_{X_{\alpha,\beta}}$;

   (d) if $(\alpha, \beta) \in \gamma_{P_E}$, then there exists a sliding polycycle passing through $S_{X_{\alpha,\beta}}$ and $P_E(Z_{\alpha,\beta})$, which contains only one segment of sliding orbits;

   (e) if $(\alpha, \beta) \in R_6^2 = \{(\alpha, \beta); \gamma_{P_E}(\alpha, \beta) < \beta < \gamma_F(\alpha, \beta)\}$, then there exists a sliding polycycle passing through $S_{X_{\alpha,\beta}}$ and $P_E(Z_{\alpha,\beta})$, which contains two segments of sliding orbits;

   (f) if $(\alpha, \beta) \in \gamma_F$, then there exists a sliding polycycle passing through $S_{X_{\alpha,\beta}}$, $P_E(Z_{\alpha,\beta})$ and $F(Z_{\alpha,\beta})$;

   (g) if $(\alpha, \beta) \in R_5^2 = \{(\alpha, \beta); \gamma_F(\alpha, \beta) < \beta < \tilde{\gamma}_{P_E}(\alpha, \beta)\}$, then there exists a sliding pseudo-cycle passing through $S_{X_{\alpha,\beta}}$;

   (h) if $(\alpha, \beta) \in \tilde{\gamma}_{P_E}$ then there exists a sliding polycycle passing through $S_{X_{\alpha,\beta}}$ and $Q_{Z_{\alpha,\beta}}$, which contains only one sliding segment;

   (i) if $(\alpha, \beta) \in R_4^2 = \{(\alpha, \beta); \beta > \tilde{\gamma}_{P_E}(\alpha, \beta)$ and $\alpha < 0\}$, then there exists a sliding polycycle passing through $S_{X_{\alpha,\beta}}$ and $P_E(Z_{\alpha,\beta})$, which contains two sliding segments;

   (j) if $\alpha = 0$ and $\beta > 0$, then there exists an attractor degenerate cycle through $S_{X_{\alpha,\beta}}$;

   (k) if $(\alpha, \beta) \in R_3^2 = \{(\alpha, \beta); \beta > 0$ and $\alpha > 0\}$, then there exists an attractor limit cycle passing through the crossing region near $P_{X_{\alpha,\beta}}$.

The bifurcation diagram is illustrated in Figure 4.11.

Proof. Since Z_0 is in the case DSC_{21} the saddle-regular point is in case BS_2, the position of the curves γ_{P_E} and γ_{P_1} changes in relation to cases DSC_{11} and DSC_{12}. For this reason, a new bifurcation curve, $\tilde{\gamma}_{P_E}$, merges from the origin in the half plane where $\beta > 0$. In fact, since $\alpha < 0$ and $\gamma_F < \beta < 0$, then $F(Z_{\alpha,\beta}) < \pi_{Z_{\alpha,\beta}}(a_{Z_{\alpha,\beta}}) < a_{Z_{\alpha,\beta}}$, it means that the trajectory which contains the unstable manifold of the saddle cross the sewing region twice before reach the sliding region from Σ^+. Therefore, by continuity, there exist values of α and β so that this trajectory reaches Σ^s at the pseudo-equilibrium point. This new curve provides a connection between $S_{X_{\alpha,\beta}}$ and $P_E(Z_{\alpha,\beta})$. The rest of the proof is similar to the proof of Theorem A. □

Theorem E. *Suppose that Z_0 is in the case DSC_{22}. Then, for the family $Z_{\alpha,\beta} = (X_{\alpha,\beta}, Y_{\alpha,\beta})$ defined above, bifurcation curves, γ_{P_E}, $\tilde{\gamma}_{P_E}$, γ_{P_1}, and γ_{P_F}, merge from the origin and the following statements hold:*

1. *for $\beta \leq 0$: identical to the cases given in Theorem C.*
2. *for $\beta > 0$: there exists a pseudo-node which is an attractor for the sliding vector field and*

   (a) if $(\alpha, \beta) \in R_8^2 = \{(\alpha,\beta); 0 < \beta < \gamma_{P_1}(\alpha,\beta)\}$, then a repellor limit cycle through Σ^c coexists with a sliding pseudo-cycle passing through $S_{X_{\alpha,\beta}}$;

   (b) if $(\alpha, \beta) \in \gamma_{P_1}$ then a repellor limit cycle through Σ^c coexists with a pseudo-cycle passing through $S_{X_{\alpha,\beta}}$;

   (c) if $(\alpha, \beta) \in R_7^2 = \{(\alpha,\beta); \gamma_{P_1}(\alpha,\beta) < \beta < \gamma_{P_E}(\alpha,\beta)\}$, then a repellor limit cycle through Σ^c coexists with a sliding pseudo-cycle passing through $S_{X_{\alpha,\beta}}$;

   (d) if $(\alpha, \beta) \in \gamma_{P_E}$, then a repellor limit cycle through Σ^c coexists with a sliding polycycle passing through $S_{X_{\alpha,\beta}}$ and $P_E(Z_{\alpha,\beta})$, which contains only one segment of sliding orbits;

   (e) if $(\alpha, \beta) \in R_6^2 = \{(\alpha,\beta); \gamma_{P_E}(\alpha,\beta) < \beta < \gamma_F(\alpha,\beta)\}$, then a repellor limit cycle through Σ^c coexists with a sliding polycycle passing through $S_{X_{\alpha,\beta}}$ and $P_E(Z_{\alpha,\beta})$, which contains two segments of sliding orbits;

   (f) if $(\alpha, \beta) \in \gamma_F$, then a repellor limit cycle through Σ^c coexists with a sliding polycycle passing through $S_{X_{\alpha,\beta}}$, $P_E(Z_{\alpha,\beta})$ and $F(Z_{\alpha,\beta})$;

   (g) if $(\alpha, \beta) \in R_5^2 = \{(\alpha,\beta); \gamma_F(\alpha,\beta) < \beta < \tilde{\gamma}_{P_E}(\alpha,\beta)\}$, then a repellor limit cycle through Σ^c coexists with a sliding pseudo-cycle passing through $S_{X_{\alpha,\beta}}$;

   (h) if $(\alpha, \beta) \in \tilde{\gamma}_{P_E}$, a repellor limit cycle through Σ^c coexists with a sliding polycycle passing through $S_{X_{\alpha,\beta}}$ and $Q_{Z_{\alpha,\beta}}$, which contains only one sliding segment;

   (i) if $(\alpha, \beta) \in R_4^2 = \{(\alpha,\beta); \beta > \tilde{\gamma}_{P_E}(\alpha,\beta)$ and $\alpha < 0\}$, then a repellor limit cycle through Σ^c coexists with a sliding polycycle passing through $S_{X_{\alpha,\beta}}$ and $P_E(Z_{\alpha,\beta})$, which contains two sliding segments;

   (j) if $\alpha = 0$ and $\beta > 0$, then there exists a repellor degenerate cycle through $S_{X_{\alpha,\beta}}$;

   (k) if $(\alpha, \beta) \in R_3^2 = \{(\alpha,\beta); \beta > 0$ and $\alpha > 0\}$, then there exist no cycles.

The bifurcation diagram is illustrated in Figure 4.12.

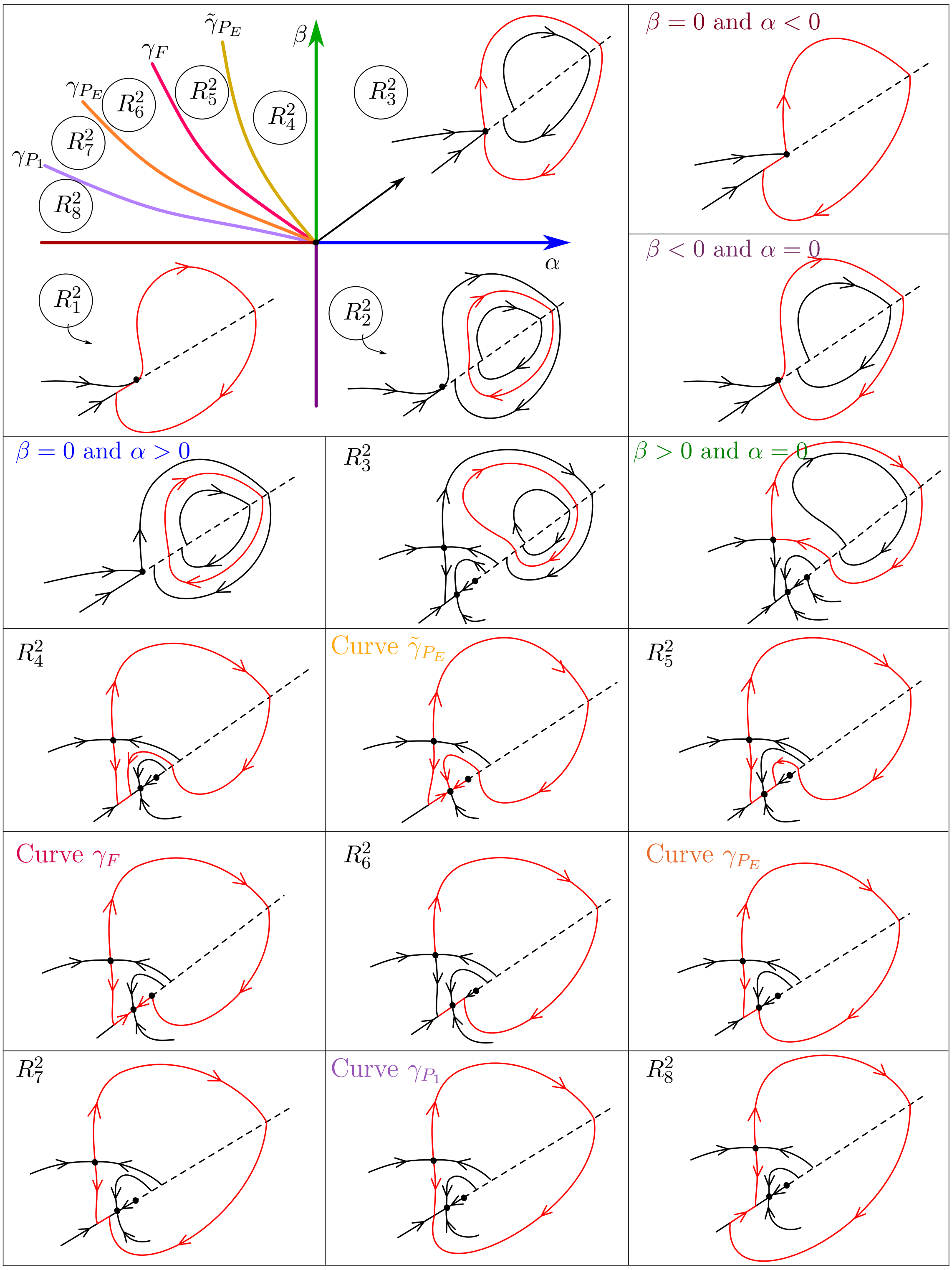

Figure 4.11: Bifurcation diagram of $Z_{\alpha,\beta}$: case DSC_{21}.

Proof. The only difference from the proof of Theorem A is the position of the limit cycles given in Proposition 4.3. □

Finally, the last cases, DSC_{31} and DSC_{32}, are described in the sequence.

Theorem F. *Suppose that Z_0 is in the case DSC_{31} and $\beta > 0$. Then, for the family $Z_{\alpha,\beta} = (X_{\alpha,\beta}, Y_{\alpha,\beta})$ defined above, bifurcation curves, γ_{P_1} and γ_{P_F} merge from the origin,*

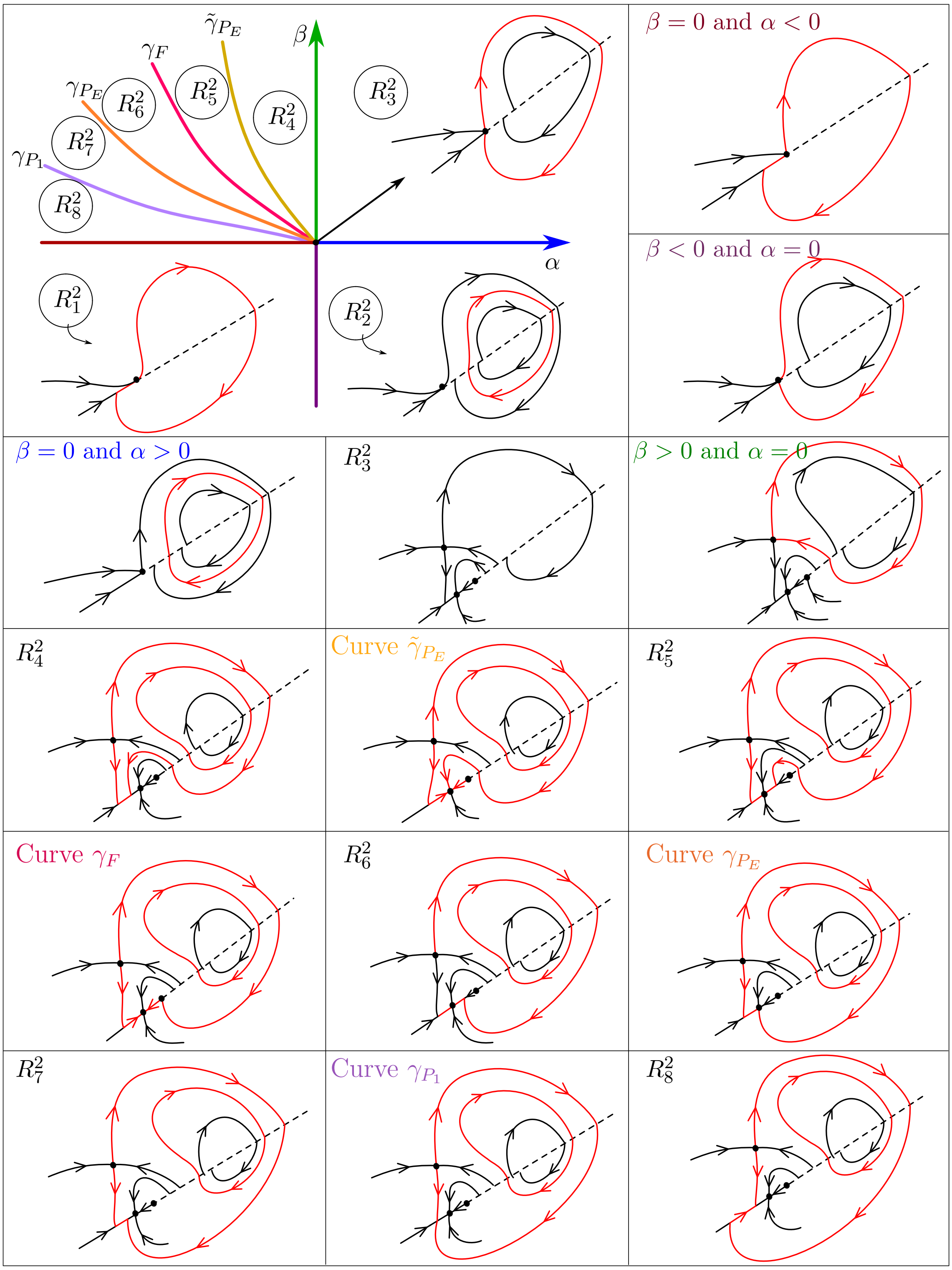

Figure 4.12: Bifurcation diagram of $Z_{\alpha,\beta}$: case DSC_{22}.

there exists no pseudo-equilibrium, $S_{X_{\alpha,\beta}}$ is a repellor for the sliding vector field, and the following statements hold:

(a) *if $(\alpha, \beta) \in R_7^1$, where $R_7^3 = \{(\alpha, \beta); 0 < \beta < \gamma_{P_1}(\alpha, \beta)\}$, then there exists a sliding pseudo-cycle through $S_{X_{\alpha,\beta}}$;*

(b) *if $(\alpha, \beta) \in \gamma_{P_1}$, then there exists a pseudo-cycle passing through $S_{X_{\alpha,\beta}}$;*

(c) *if* $(\alpha,\beta) \in R_6^3 = \{(\alpha,\beta); \gamma_{P_1}(\alpha,\beta) < \beta < \gamma_F(\alpha,\beta)\}$*, then there exists a sliding pseudo-cycle through* $S_{X_{\alpha,\beta}}$*;*

(d) *if* $(\alpha,\beta) \in \gamma_F$*, then there exists a sliding pseudo-polycycle through* $S_{X_{\alpha,\beta}}$ *and* $F(Z_{\alpha,\beta})$*;*

(e) *if* $(\alpha,\beta) \in R_5^3 = \{(\alpha,\beta); \gamma_F(\alpha,\beta) < \beta$ *and* $\alpha < 0\}$*, then there exists a sliding pseudo-cycle through* $S_{X_{\alpha,\beta}}$*;*

(f) *if* $\alpha = 0$ *and* $\beta > 0$*, then there exists an attractor degenerate cycle through* $S_{X_{\alpha,\beta}}$*;*

(g) *if* $(\alpha,\beta) \in R_4^3 = \{(\alpha,\beta); \beta > 0$ *and* $\alpha > 0\}$*, then there exists a limit cycle passing through* Σ^s*.*

The bifurcation diagram is illustrated in Figure 4.13.

Proof. Since Z_0 is in case DSC_{31}, the saddle-regular point is of type BS_3, i.e., the pseudo-equilibrium points appear when the saddle is virtual ($\beta < 0$). Thus, there is no pseudo-equilibrium for $\beta > 0$. The position of the limit cycles given in Proposition 4.3 implies that they happen for $\alpha > 0$. The rest of the proof is similar to the proof of Theorem A. □

Theorem G. *Suppose that* Z_0 *is in the case* DSC_{32} *and* $\beta > 0$*. Then, for the family* $Z_{\alpha,\beta} = (X_{\alpha,\beta}, Y_{\alpha,\beta})$ *defined above, bifurcation curves,* γ_{P_1} *and* γ_{P_F} *merge from the origin, there exists no pseudo-equilibrium,* $S_{X_{\alpha,\beta}}$ *is a repellor for the sliding vector field, and the following statements hold:*

(a) *if* $(\alpha,\beta) \in R_7^1$*, where* $R_7^3 = \{(\alpha,\beta); 0 < \beta < \gamma_{P_1}(\alpha,\beta)\}$*, then a repellor limit cycle through* Σ^c *coexists with a sliding pseudo-cycle through* $S_{X_{\alpha,\beta}}$*;*

(b) *if* $(\alpha,\beta) \in \gamma_{P_1}$*, then a repellor limit cycle through* Σ^c *coexists with a pseudo-cycle passing through* $S_{X_{\alpha,\beta}}$*;*

(c) *if* $(\alpha,\beta) \in R_6^3 = \{(\alpha,\beta); \gamma_{P_1}(\alpha,\beta) < \beta < \gamma_F(\alpha,\beta)\}$*, then a repellor limit cycle through* Σ^c *coexists with a sliding pseudo-cycle through* $S_{X_{\alpha,\beta}}$*;*

(d) *if* $(\alpha,\beta) \in \gamma_F$*, then a repellor limit cycle through* Σ^c *coexists with a sliding pseudo-polycycle through* $S_{X_{\alpha,\beta}}$ *and* $F(Z_{\alpha,\beta})$*;*

(e) *if* $(\alpha,\beta) \in R_5^3 = \{(\alpha,\beta); \gamma_F(\alpha,\beta) < \beta$ *and* $\alpha < 0\}$*, then a repellor limit cycle through* Σ^c *coexists with a sliding pseudo-cycle through* $S_{X_{\alpha,\beta}}$*;*

(f) *if* $\alpha = 0$ *and* $\beta > 0$*, then there exists a repellor degenerate cycle through* $S_{X_{\alpha,\beta}}$*;*

(g) *if* $(\alpha,\beta) \in R_4^3 = \{(\alpha,\beta); \beta > 0$ *and* $\alpha > 0\}$*, then there exists no cycles passing through* Σ^s*.*

The bifurcation diagram is illustrated in Figure 4.14.

Proof. The position of the limit cycles given in Proposition 4.3 implies that they happen for $\alpha < 0$. The rest of the proof is similar to the proof of Theorem F. □

Theorem H. *Suppose that* Z_0 *is in the case* DSC_{31} *or* DSC_{32} *and* $\beta \leq 0$*. Then, the family* $Z_{\alpha,\beta} = (X_{\alpha,\beta}, Y_{\alpha,\beta})$ *defined above satisfies: there exists no pseudo-equilibrium,* $S_{X_{\alpha,\beta}}$ *is an attractor for the sliding vector field, and:*

(a) if $\beta = 0$ *and* $\alpha < 0$*, then there exists a sliding cycle passing through the* $S_{X_{\alpha,\beta}}$*;*

(b) if $\alpha = 0 = \beta$*, then there exists an attractor degenerate cycle passing through* $S_{X_{0,0}}$*;*

(c) if $\beta = 0$ *and* $\alpha > 0$*, then there exists an attractor limit cycle through* Σ^c*;*

(d) if $(\alpha, \beta) \in R_1^3 = \{(\alpha, \beta); \gamma_{P_E}(\alpha, \beta) < \beta < 0\}$*, then there exists a sliding cycle passing through the fold-regular point* $F(Z_{\alpha,\beta})$*;*

(e) if $(\alpha, \beta) \in \gamma_{P_E}$*, there exists a polycycle passing through* $F(Z_{\alpha,\beta})$ *and* $P_E(Z_{\alpha,\beta})$*;*

(f) if $(\alpha, \beta) \in R_2^3 = \{(\alpha, \beta); \alpha < 0$ *and* $\beta < \gamma_{P_E}(\alpha, \beta)\}$*, then there exists a sliding cycle passing through* $P_{X_{\alpha,\beta}}$*;*

(g) if $\alpha = 0$ *and* $\beta < 0$*, then an attractor degenerate cycle through* $F(Z_{\alpha,\beta})$*;*

(h) if $(\alpha, \beta) \in R_3^3 = \{(\alpha, \beta); \beta < 0$ *and* $\alpha > 0\}$*, then there exists an attractor limit cycle through* Σ^+*.*

The bifurcation diagram for DSC_{31} *and* DSC_{32} *are illustrated in Figures 4.13 and 4.14, respectively.*

Proof. The position of the limit cycles given in Proposition 4.3 implies that they happen for $\alpha > 0$. The rest of the proof is similar to the proof of Theorem A. □

4.3 Illustrations with Hyperbolicity Ratio in $\mathbb{Q}$

In this section we present some models realizing degenerate cycles through hyperbolic saddles where the hyperbolicity ratio is a rational number. Moreover, we use these examples to illustrate how rich a system having a degenerate cycle through a saddle point can be. Firstly, we study a class of vector fields having hyperbolicity ratio equal to 1. After that, an example, also with hyperbolicity ratio equal to 1, is studied carefully taking in account the whole system showing how rich the dynamic of this model is. Finally, a model having hyperbolicity ratio r is studied numerically and the main objective of it is to illustrate the existence of degenerate cycles, limit cycles and to study the graph of the first return map when the hyperbolicity ratio is a rational number.

4.3.1 Class of Vector Fields with Hyperbolicity Ratio $= 1$

Let $\mathcal{A}$ be the set of all discontinuous vector fields $Z = (X, Y) \in \Omega^r$ where X has a hyperbolic saddle S_X and, in a neighborhood of S_X, X is $\mathcal{C}^2-$conjugated to a vector field $W(\mathbf{x}) = A\mathbf{x} + \mathbf{c}$, where

$$A = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} -a\tilde{y} \\ -b\tilde{x} \end{pmatrix} \tag{4.3.1}$$

with $a, b > 0$, $S_X = (\tilde{x}, \tilde{y})$, and $\Sigma = h^{-1}(0)$ where $h(x, y) = y$. Moreover, this conjugacy preserves the discontinuity set Σ. Observe that the eigenvalues of A in equation 4.3.1 are $\pm\sqrt{ab}$, therefore, for all $Z \in \mathcal{A}$, the hyperbolicity ratio of any S_X is equal to 1. Now, by considering vector fields in $\mathcal{V}_{Z_0} \cap \mathcal{A}$ we perform an analysis of the first return map. In order to do that, we proceed similarly to the analysis performed for the case $r \notin \mathbb{Q}$.

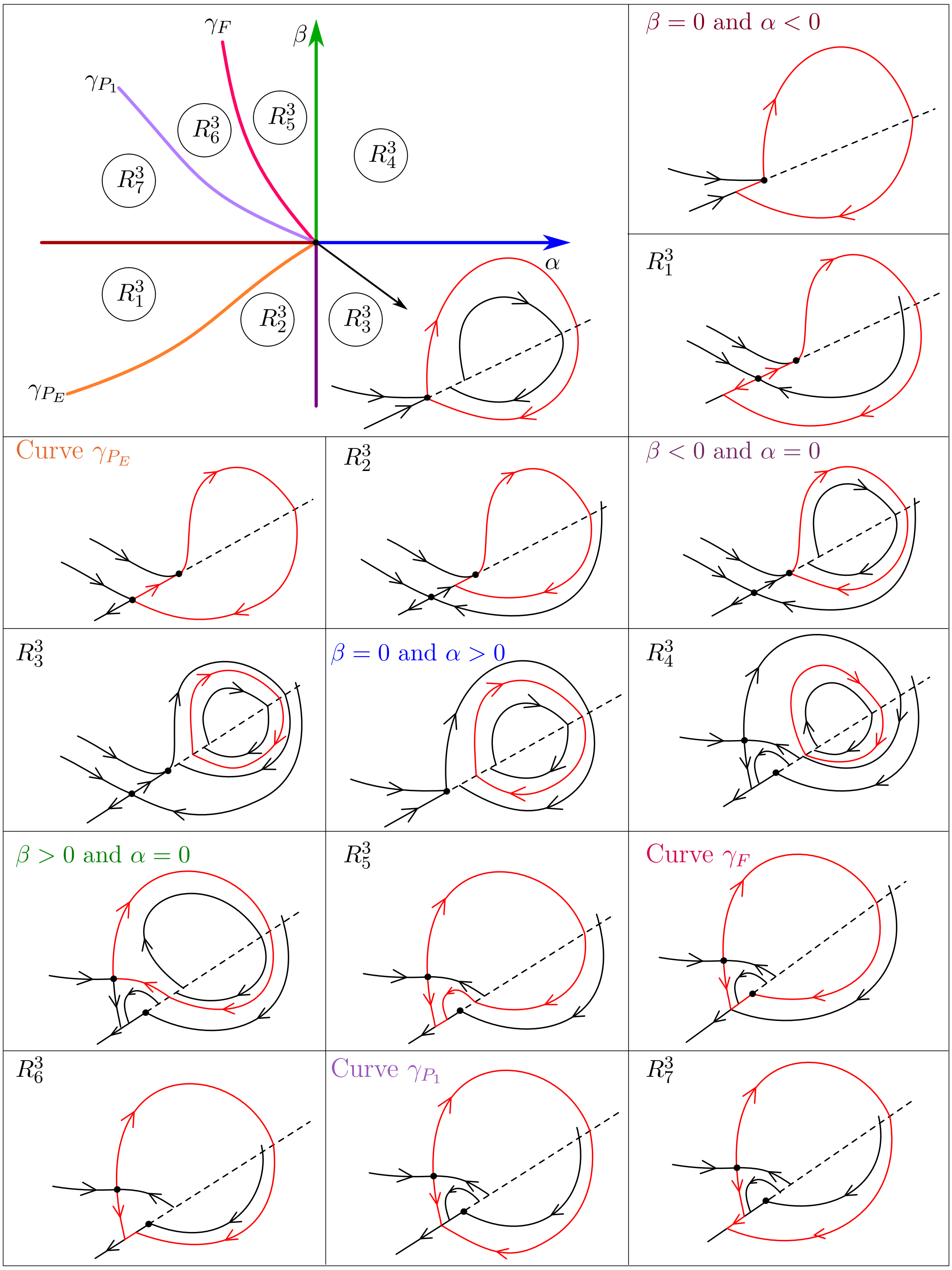

Figure 4.13: Bifurcation diagram of $Z_{\alpha,\beta}$: case DSC_{31}.

The main difference is that we consider Σ fixed and the saddle point variable. Then, for each $Z \in \mathcal{V}_{Z_0} \cap \mathcal{A}$, there is no loss of generality in assuming $a_Z = 0$ (it is possible up to translation maps) and the image of a_Z by conjugacy is also 0 (it is possible, because the conjugacy preserves Σ).

Consider $\tau \subset \Sigma^+$ a transversal section to the flow of X such that, if $S_X \in \Sigma^+$, then τ is above S_X. In this case, the transition map of X near S_X is $\rho_1 : [0, \delta_Z) \to \sigma$ with $\rho_1(x)$

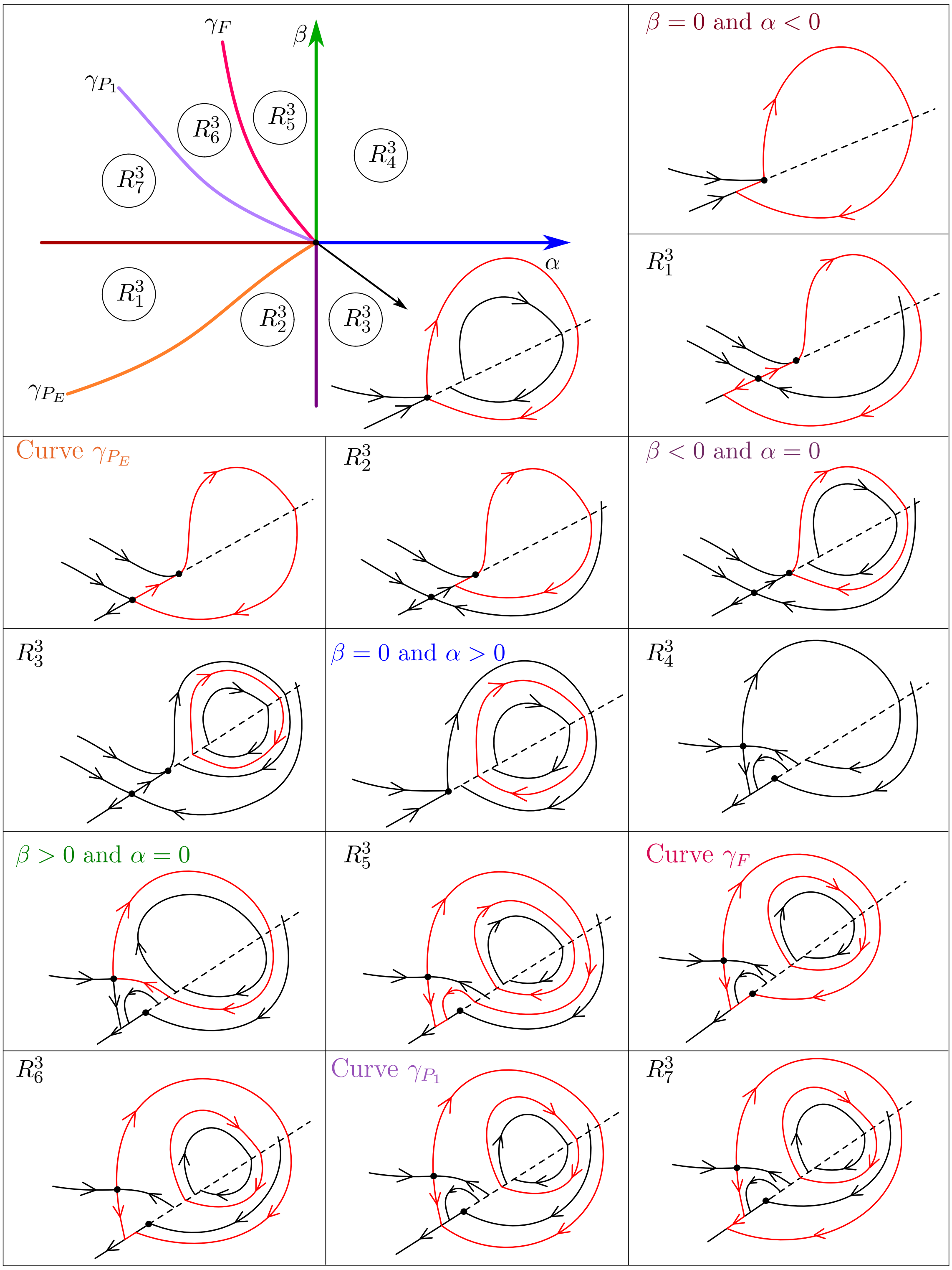

Figure 4.14: Bifurcation diagram of $Z_{\alpha,\beta}$: case DSC_{32}.

being the projection on the first coordinate of the point where the trajectory of X passing through $(x, 0) \in [0, \delta_Z) \times \{0\}$ meets τ at the first (positive) time. Let $\tau_W \subset \{(x, \varepsilon) \in \mathbb{R}^2\}$ be the transversal section to the flow of W for some $\varepsilon > 0$ is such way that this section is contained in the neighborhood where W and X are conjugated. Without loss of generality, we assume that τ is the image by conjugacy of σ_W.

Lemma 4.4. *The transition map* $\rho_W : [0, \delta) \to \sigma_W$ *can be differentially extended to an* *open neighborhood of* 0.

Proof. The trajectories of W lie in the level curves of the function $G(x,y) = bx^2 - ay^2 + 2c_1x - 2c_2y$, where $c_1 = -b\tilde{x}$ and $c_2 = -a\tilde{y}$. Assume that ρ_W is defined for $0 \le x < \delta$. Now, for each $(x,0) \in \Sigma$ with $0 \le x < \delta$ we obtain

$$\rho_W(x) = -\frac{c_1}{b} + \sqrt{x^2 + \frac{2c_1}{b}x + \frac{a}{b}\varepsilon^2 + \frac{2c_2}{b}\varepsilon + \frac{c_1^2}{b^2}}. \tag{4.3.2}$$

Observe that the expression inside the square root which does not depends on x is a polynomial of degree 2 in ε, $Q(\varepsilon) = \frac{a}{b}\varepsilon^2 + \frac{2c_2}{b}\varepsilon + \frac{c_1^2}{b^2}$. We analyze this polynomial in three different cases:

(i) if $\tilde{y} < 0$ we have $c_2 > 0$. As $P_X = (0,0)$ we must have $c_1 = 0$ and since $a, b > 0$ it follows that $Q(\varepsilon) > 0$ for $\varepsilon > 0$;

(ii) if $\tilde{y} = 0$ then $S_X = (0,0)$, consequently $c_1 = c_2 = 0$ and $Q(\varepsilon) > 0$ for $\varepsilon > 0$;

(iii) if $\tilde{y} > 0$, then $c_2 < 0$. Imposing the condition that the stable manifold of S_X intersects Σ in $(0,0)$ we obtain $c_1 = -c_2\sqrt{b/a} > 0$. It implies that Q has only one root, which is $-c_2/a = \tilde{y}$ and $Q(\varepsilon) > 0$ for all $\varepsilon > 0$ and $\varepsilon \ne \tilde{y}$. However, for our proposes, $\varepsilon > \tilde{y}$. Thus, $Q(\varepsilon) > 0$ for $\varepsilon > \tilde{y}$.

Therefore, choosing $\varepsilon > \max\{0, \tilde{y}\}$ we obtain $Q(\varepsilon) > 0$ and we can find $\delta_W > 0$ such that the expression inside the square root in (4.3.2) is positive for $x \in (-\delta_W, \delta_W)$. It follows that ρ_W can be differentially extended to an open neighborhood of 0. □

Since W is $\mathcal{C}^2-$conjugated to X in a neighborhood of S_X there exists a diffeomorphism of class $\mathcal{C}^2-$, ψ, defined in a neighborhood of S_X, such that $\psi(S_W) = S_X$ and

$$\rho_1(x) = \psi \circ \rho_W \circ \psi^{-1}(x). \tag{4.3.3}$$

It follows from this expression that ρ_1 can be, at least, $\mathcal{C}^2-$extended to an open neighborhood of 0 and it allows us to calculate the Taylor series, until order 2, of π_Z at $x = 0$.

Proposition 4.4. *Consider $Z = (X,Y) \in \mathcal{A} \cap \mathcal{V}_{Z_0}$ and let $S_X = (\tilde{x}, \tilde{y})$ be the saddle point associated with X. Suppose that, in the notation above, the first return map is defined in a half-open interval $[0, \delta_Z)$, for some $\delta_Z > 0$. Then the first return map of Z can be written as*

$$\pi_Z(x) = \alpha + k_1(\beta)x + k_2(\beta)x^2 + \ h.o.t., \tag{4.3.4}$$

where $0 < |k_1(\beta)| < 1$ if $\beta > 0$ and $k_1(\beta) = 0$, $k_2(\beta) > 0$ if $\beta \le 0$.

Proof. We have already shown the existence of the first return map $\pi_Z = \rho_3 \circ \rho_2 \circ \rho_1$ (see 4.1) and, as seen above, it can be, at least, $\mathcal{C}^2$ extended to an open interval $(-\delta_Z, \delta_Z)$. Thus, it is possible to calculate its Taylor's series until order 2 around $x = 0$, which is given by

$$\pi_Z(x) = \pi_Z(0) + \frac{d\pi_Z}{dx}(0)x + \frac{1}{2!}\frac{d^2\pi_Z}{dx^2}(0)x^2 + \mathrm{O}(x^3).$$

Observe that ρ_2 and ρ_3 are orientation reversing diffeomorphisms while ψ is an orientation preserving diffeomorphism. Also, $\rho_1 = \psi \circ \rho_W \circ \psi^{-1}$, $\pi_Z = \rho_3 \circ \rho_2 \circ \rho_1$,

$$\frac{d\rho_W}{dx}(0) = \frac{c_1}{\sqrt{ab\varepsilon^2 + 2bc_2\varepsilon + c_1^2}} \quad \text{and} \quad \frac{d^2\rho_W}{dx^2}(0) = \frac{b^2\varepsilon(a\varepsilon + 2c_2)}{(ab\varepsilon^2 + 2bc_2\varepsilon + c_1^2)^{3/2}}.$$

When $\tilde{y} \leq 0$ we assume $c_1 = 0$. It follows that $\frac{d\rho_W}{dx}(0) = 0$ and consequently $\frac{d\rho_1}{dx}(0) = 0$. Also, $c_2 \geq 0$ then $\frac{d^2\rho_W}{dx^2}(0) > 0$ and $\frac{d^2\rho_1}{dx^2}(0) > 0$. Therefore, $\frac{d\pi_Z}{dx}(0) = 0$ and $\frac{d^2\pi_Z}{dx\check{s}}(0) > 0$.

If $\tilde{y} > 0$ then $c_1 = -c_2\sqrt{b/a} > 0$. For each fixed $\varepsilon > \tilde{y}$, ρ_W depends continuously on $\tilde{y}$, consequently π_Z also depends continuously on $\tilde{y}$. Since $\tilde{y}$ is sufficiently small we conclude that

$$0 < \frac{d\pi_Z}{dx}(0) < 1$$

Finally, defining $\alpha = \pi_Z(0)$, $\beta = \tilde{y}$, $k_1(\beta) = \frac{d\pi_Z}{dx}(0)$ and $k_2(\beta) = \frac{1}{2!}\frac{d^2\pi_Z}{dx^2}(0)$ we achieve the result. □

It follows from this result that for $\bar{Z}$ we have $\pi_{\bar{Z}}(0) = 0$, $\frac{d\pi_{\bar{Z}}}{dx}(0) = 0$ and $\frac{d^2\pi_{\bar{Z}}}{dx^2}(0) > 0$. Let Γ be the degenerate cycle of $\bar{Z}$. Since the first return map is just defined in a half-open interval there exists a neighborhood of Γ such that the orbits of $\bar{Z}$ in this neighborhood, through points where the first return map is defined, have Γ as $\omega-$limit set. Another consequence of this proposition, which has a similar prove as the one of Proposition 4.3, is the following:

Corollary 4.1. *Under the hypotheses in the Proposition 4.4, if $Z \in \mathcal{V}_{Z_0} \cap \mathcal{A}$ is such that $\alpha > 0$ then, for some $x > 0$ sufficiently small, Z has a stable limit cycle passing through $(x, 0) \in \Sigma$. Moreover, if Z is such that $\alpha = 0$, then there exists a degenerate cycle Γ having a neighborhood for which the orbits of Z in this neighborhood passing through points where the first return map is defined have Γ as $\omega-$limit set.*

Therefore, for $Z \in \mathcal{V}_{Z_0} \cap \mathcal{A}$ the first return π_Z has the graph given in Figure 4.6. Moreover, depending on the structure of the local saddle-regular point, the bifurcation diagram in this case is given in Figure 4.9, or 4.11, or 4.13.

4.3.2 Model with Hyperbolicity Ratio $= 1$

Now, the objective is to illustrate how complex and rich a system having a degenerate cycle through a saddle-regular can be. All systems in the considered family are piecewise Hamiltonian, thus the hyperbolicity ratio of any saddle point is equal to 1. Moreover, the cycle is, topologically, of type DSC_{31}.

Consider the $2-$parameter family of discontinuous vector fields $Z_{\alpha,\beta} = (X_\alpha, Y_\beta)$ with

$$X_\alpha = \begin{pmatrix} -y + \alpha \\ x^2 - x \end{pmatrix} \quad \text{and} \quad Y_\beta = \begin{pmatrix} 1 \\ x - \dfrac{3}{4} - \beta \end{pmatrix}.$$

In order to analyze the vector field $Z_{\alpha,\beta}$ we first analyze, separately, the vector fields X_α and Y_β.

Analysis of X_α

The vector field X_α has two singular points, one of them is a hyperbolic saddle $S_\alpha = (0, \alpha)$, the other one is a center $C_\alpha = (1, \alpha)$. This is a Hamiltonian vector field with Hamiltonian function given by

$$H_\alpha(x, y) = -\frac{1}{2}y^2 + \alpha y - \frac{1}{3}x^3 + \frac{1}{2}x^2.$$

For $\alpha = 0$, $H_0^{-1}(0)$ meets Σ at the origin and at the points $P = \left(\frac{3}{2}, 0\right)$ and $Q = \left(-\frac{1}{2}, 0\right)$. There exists an homoclinic orbit through the origin which intersects the Σ at $P = (3/2, 0)$.

Notice that, when α varies, we just have a translation, on axis y, of the phase portrait for $\alpha = 0$. The level curves of H_α meet Σ at most in three distinct points. In fact, each $(x_0, 0) \in \Sigma$ belongs to the level $h_0 = H_\alpha(x_0, 0)$ of H_α, then to find the intersections of this level with Σ it is necessary to solve, in x, the equation $H_\alpha(x, 0) = h_0$. It is equivalent to find the roots of a third degree polynomial. Solving this problem we obtain the roots x_0 and $x_\pm$, where

$$x_\pm = \frac{3}{4} - \frac{x_0}{2} \pm \frac{\sqrt{9 + 12x_0 - 12x_0^2}}{4}.$$

Observe that, $x_\pm \in \mathbb{R}$ provided that $-\frac{1}{2} \leq x_0 \leq \frac{3}{2}$. Moreover,

- if $-\frac{1}{2} \leq x_0 \leq 0$, then $0 \leq x_-(x_0) \leq 1$ and $1 \leq x_+(x_0) \leq \frac{3}{2}$;
- if $0 \leq x_0 \leq \frac{3}{2}$, then $-\frac{1}{2} \leq x_-(x_0) \leq 0$ and $0 \leq x_+(x_0) \leq \frac{3}{2}$.

When $\alpha < 0$ the singular points of X_α become virtual singularities for $Z_{\alpha,\beta}$ but two tangency points merge in Σ in the same place where there are singularities for $\alpha = 0$. For this reason, the analysis on the existence of limit cycles for $\alpha < 0$ is analogous to the analysis for $\alpha = 0$. So, we can restrict our investigation considering $\alpha \geq 0$.

Consider the level $h_\alpha = H_\alpha(0, \alpha)$ of S_α, a point $(x, 0) \in \Sigma$ belongs to this level if and only if it is a root of $R_\alpha(x) = H_\alpha(x, 0) - h_\alpha$. The map R_α polynomial of degree 3 in x, denote the roots of R_α by $x_i = x_i(\alpha)$, $i = 1, 2, 3$, satisfying $x_1 \leq x_2 \leq x_3$ when all these roots are real. The biggest and smallest values of α for which x_1, x_2, $x_3 \in \mathbb{R}$ are $\alpha = \pm\sqrt{3}/3$ and so $x_1 = -1/2$, $x_2 = x_3 = 1$ and $h_\alpha = H_\alpha(1, 0)$.

Since we are interested in small values of α we assume $|\alpha| \leq \sqrt{3}/3$. By analyzing R_α we obtain that $x_1(\alpha)$ and $x_2(\alpha)$ are increasing maps while $x_3(\alpha)$ is a decreasing map.

Given $P_0 = (x_0, 0) \in \Sigma$ with $x_2 \leq x_0 \leq 1$ there exists a unique $P_1 = (\rho_1(x_0), 0) \in \Sigma$ that belongs to the same level P_0 and satisfies $1 \leq \rho_1(x_0) \leq x_3$. Therefore, the trajectory of X_α through P_0 meets Σ also in P_1. Therefore, we can consider the map $\rho_1 : [x_2, 1] \longrightarrow [1, x_3]$ given by

$$\rho_1(x) = x_+(x) = \frac{3}{4} - \frac{x}{2} + \frac{\sqrt{9 + 12x - 12x^2}}{4}.$$

Analysis of Y_β

The vector field Y_β does not have singular points but it has a fold point $F_\beta = (3/4 + \beta, 0) \in \Sigma$. Given $P_0 = (x_0, 0) \in \Sigma$, the trajectory of Y_β through P_0 meets again at $P_2 = (\rho_2(x_0), 0)$ and, we consider the map

$$\begin{array}{rccc} \rho_2 : & \left(-\infty, \frac{3}{4} + \beta\right] & \longrightarrow & \left[\frac{3}{4} + \beta, +\infty\right) \\ & x & \mapsto & \rho_2(x) = \frac{3}{2} - x + 2\beta \end{array}.$$

Observe that $\rho_2(0) = \frac{3}{2} + 2\beta$ and, if $\beta = 0$, then $\rho_2(0) = \frac{3}{2} = \rho_1(0)$. Therefore, $Z_{0,0}$ has a degenerate cycle through the hyperbolic saddle S_0 at the origin, see Figure 4.15.

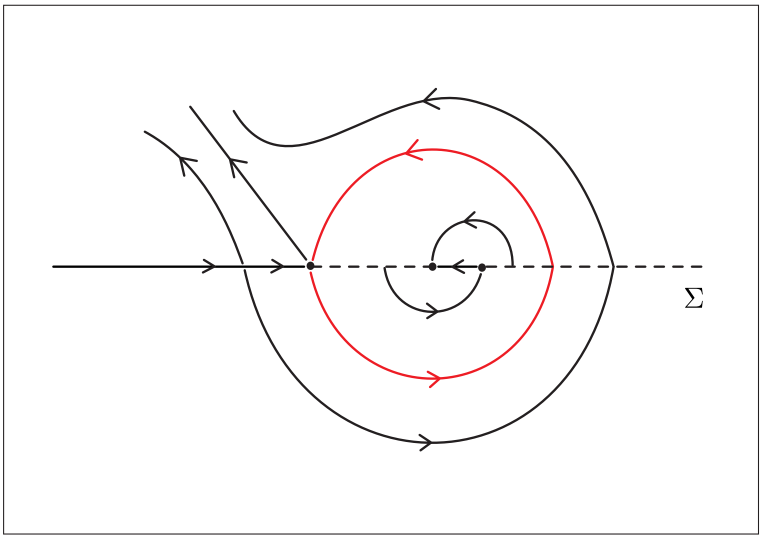

Figure 4.15: Phase portrait of $Z_{0,0}$.

Analysis of $Z_{\alpha,\beta}$

In order to analyze the behavior of $Z_{\alpha,\beta}$, it is necessary to determine the sewing, sliding and escaping regions. Due to simplicity, we consider $-3/4 \leq \beta \leq 1/4$ and, since $X_\alpha f \cdot Y_\beta f = x(x-1)\left(x - \dfrac{3}{4} - \beta\right)$ we obtain $\Sigma^c = \{(x,0); 0 < x < \beta + 3/4 \text{ or } x > 1\}$, $\Sigma^s = \{(x,0); \beta + 3/4 < x < 1\}$ and $\Sigma^e = \{(x,0); x < 0\}$.

The second step in this analysis is to study the sliding vector field, which is given by

$$Z^s_{\alpha,\beta} = \frac{1}{Y_\beta f - X_\alpha f}\left(Y_\beta f.X_\alpha - X_\alpha f.Y_\beta\right) = \frac{\alpha(4x-3-4\beta) - 4x(x-1)}{4x-3-4\beta-4x(x-1)}.$$

Observe that, by assuming $|\beta| < 1/4$ the denominator, $d(x,\beta)$, in the expression of $Z^s_{\alpha,\beta}$ is nonzero and

- $d(\beta,x) < 0$ if $x < 0$, i.e., if $x \in \Sigma^e$;
- $d(\beta,x) > 0$ if $3/4 + \beta < x < 1$, i.e., if $x \in \Sigma^s$.

The numerator in the expression of $Z^s_{\alpha,\beta}$ is $n(\alpha,\beta,x) = \alpha(4x-3-4\beta) - 4x(x-1)$, we have $n(\alpha,\beta,x) = 0$ if and only if $x = s_\pm = (1 + \alpha \pm \sqrt{\alpha^2 - \alpha(1+4\beta)+1})/2$. Observe that, $s_\pm \in \mathbb{R}$ when $|\beta| < 1/4$ and

- $n(\alpha,\beta,x) < 0$ if $x < s_-$ or $x > s_+$;
- $n(\alpha,\beta,x) = 0$ if $x = s_\pm$;
- $n(\alpha,\beta,x) > 0$ if $s_- < x < s_+$.

Now, we need to determine if $s_\pm \in \Sigma^s \cup \Sigma^e$. Observe that, $s_+ \in \Sigma^c$ for $|\beta| < 1/4$. Since $|\beta| < 1/4$, $s_- = 0 \Leftrightarrow \alpha = 0$ and $s_- < 0 \Leftrightarrow -1 < \alpha < 0$, in both cases s_- is an attractor for $Z^s_{\alpha,\beta}$. Otherwise, if $\alpha > 0$, $n(\alpha,\beta,x) < 0$ and, consequently, $Z_{\alpha,\beta}(x) > 0$ for $x < 0$ and $Z_{\alpha,\beta}(x) < 0$ for $3/4 + \beta < x < 1$.

The third step is to investigate the existence of limit cycles through points in Σ^c with $x_2 < x < 3/4 + \beta$. Define,

$$F_{\alpha,\beta}: \begin{array}{ccc} \left[x_2, \frac{3}{4} + \beta\right] & \longrightarrow & \mathbb{R} \\ x & \mapsto & (\rho_2 - \rho_1)(x) \end{array}. \tag{4.3.5}$$

Notice that, $F_{\alpha,\beta}(z) = 0$ iff $z = z_{\pm}$, where

$$z_{\pm} = \frac{3}{4} + \beta \pm \frac{\sqrt{9 - 24\beta - 48\beta^2}}{4}.$$

It is clear that $z_+ \geq 3/4 + \beta$, thus it is not in the domain of $F_{\alpha,\beta}$. If $\alpha = 0$ and $0 \leq \beta \leq 1/4$ we have $x_2 = 0$ and $0 < z_- < 3/4 + \beta$. Under these conditions, z_- is an increasing function in the variable β so, $F_{0,\beta}(x) < 0$ if $0 < x < z_-$ and $F_{0,\beta}(x) > 0$ if $z_- < x < 3/4 + \beta$. Therefore, varying β continuously in $(0, 1/4)$, the degenerate cycle for $\beta = 0$ becomes a limit cycle, with decreasing radius, degenerating in a point for $\beta = 1/4$. For $\alpha = \beta = 0$ we obtain $F_{0,0}(x) < 0$ for $0 < x < 3/4$, so the degenerate cycle through the origin is repellor, see figure 4.15. If $\alpha > 0$ it is not possible to find algebraically the expression of x_2, however it is known that, $x_2 = x_2(\alpha)$ and the equation $x_2(\alpha) = z_-(\beta)$ defines, implicitly a curve $\beta = \gamma_{x_2}(\alpha)$, in the $(\alpha, \beta)-$plane, for which $x_2 = z_-$ if $\beta = \gamma_{x_2}(\alpha)$, $x_2 < z_- < 3/4 + \beta$ if $\beta < \gamma_{x_2}(\alpha)$ and $x_2 > z_-$ if $\beta < \gamma_{x_2}(\alpha)$.

Finally, the last step in this analysis is to study connections between the singularities in Σ. In order to simplify the notation consider $b = \beta + 3/4$, notice that if $-3/4 \leq \beta \leq 1/4$ then $0 \leq b \leq 1$.

Since the trajectories of Y_β cross Σ in points having the fold $F_b = (b, 0)$ as middle point, we obtain some curves, $b = b(\alpha) = 3/4 + \beta(\alpha)$, which provide direct connections through the flow of Y_β:

- $b_1 = 0$: $F_b = (0, 0)$;
- $b_2 = \dfrac{x_1 + x_2}{2}$: connection between $P_1 = (x_1, 0)$ and $P_2 = (x_2, 0)$;
- $b_3 = \dfrac{x_1 + 1}{2}$: connection between P_1 and C_α;
- $b_4 = \dfrac{x_1 + x_3}{2}$: connection between P_1 and $P_3 = (x_3, 0)$;
- $b_5 = \dfrac{x_2}{2}$: connection between 0 and P_2;
- $b_6 = \dfrac{1}{2}$: connection between 0 and C_α;
- $b_7 = \dfrac{x_3}{2}$: connection between 0 and P_3;
- $b_8 = \dfrac{x_2 + 1}{2}$: connection between P_2 and C_α;
- $b_9 = \dfrac{x_2 + x_3}{2}$: connection between P_2 and P_3;
- $b_{10} = x_2$: $F_b = P_2$;

- $b_{11} = 1$: $F_b = (1,0)$.

By means of numeric calculations, considering $0 < \alpha < \sqrt{30}/24$, we obtain:

$$-\frac{1}{2} < x_1 < 0 < b_2 < b_5 < x_2 < b_3 < b_6 < b_8 < b_4 < b_7 < b_9 < 1 < x_3 \leq \frac{3}{2}.$$

Since $|\beta| < 1/4$ implies that $1/2 < b < 1$ and $b_8(0) = 1/2$ it is enough to consider $b_8 \leq b \leq 1$. However, just for a clearer exposition, we consider smaller values for b. Numeric calculations give some results:

- **Connections of $P_1 = (x_1, 0)$:**

  Observe that in the curve defined by b_3, we have a connection between P_1 and $(1,0)$ also, in the curve defined by b_4 we have a connection between P_1 and P_3. Therefore, for $b_3 \leq b \leq b_4$ there exists a sequence of disjoint curves (up to $\alpha = 0$) $b = \gamma_j^1(\alpha)$, converging to b_4 when $j \to \infty$, with $\gamma_0^1 = b_3$ and satisfying:

  - For $\gamma_{2j-2}^1 < b < \gamma_{2j-1}^1$, $j = 1, 2, \ldots$, the trajectory through P_1 crosses Σ $2j-1$ times before reaching the sliding region;
  - For $b = \gamma_{2j-1}^1$, $j = 1, 2, \ldots$, then the trajectory through P_1 intersects Σ $2j-1$ times before performing a connection with F_β;
  - For $\gamma_{2j-1}^1 < b < \gamma_{2j}^1$, $j = 1, 2, \ldots$, the trajectory through P_1 crosses Σ $2j$ times before reaching the sliding region;
  - For $b = \gamma_{2j}^1$, $j = 1, 2, \ldots$, then the trajectory through P_1 intersects Σ $2j$ times before performing a connection with $(1,0)$. See Figure 4.16.

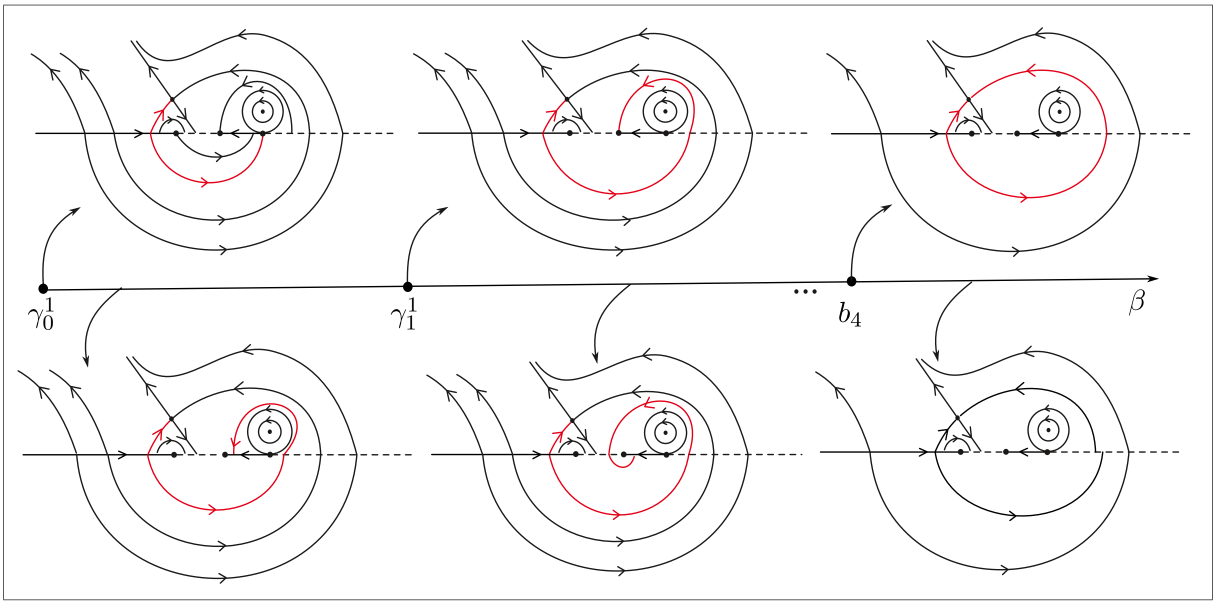

Figure 4.16: Illustration of the phase portraits of $Z_{\alpha,\beta}$ with values in the sequence γ_j^1.

- **Connections of $0 = (0,0)$:**

  Observe that in the curve defined by b_6, we have a connection between the origin and $(1,0)$. Moreover, in the curve defined by b_7 we have a connection between the origin and P_3. Therefore, for $b_6 \leq b \leq b_7$ there exists a sequence of disjoint curves (up to $\alpha = 0$) $b = \gamma_j^0(\alpha)$, converging to b_7 when $j \to \infty$, with $\gamma_0^0 = b_6$ and satisfying:

- For $\gamma^0_{2j-2} < b < \gamma^0_{2j-1}$, $j = 1, 2, \ldots$, the trajectory through 0 crosses Σ $2j-1$ times before reaching the sliding region;
- For $b = \gamma^0_{2j-1}$, $j = 1, 2, \ldots$, the trajectory through 0 intersects Σ $2j-1$ times before performing a connection with F_β;
- For $\gamma^0_{2j-1} < b < \gamma^0_{2j}$, $j = 1, 2, \ldots$, the trajectory through 0 crosses Σ $2j$ times before reaching the sliding region;
- For $b = \gamma^0_{2j}$, $j = 1, 2, \ldots$, the trajectory through 0 intersects Σ $2j$ times before performing a connection with $(1, 0)$. See Figure 4.17.

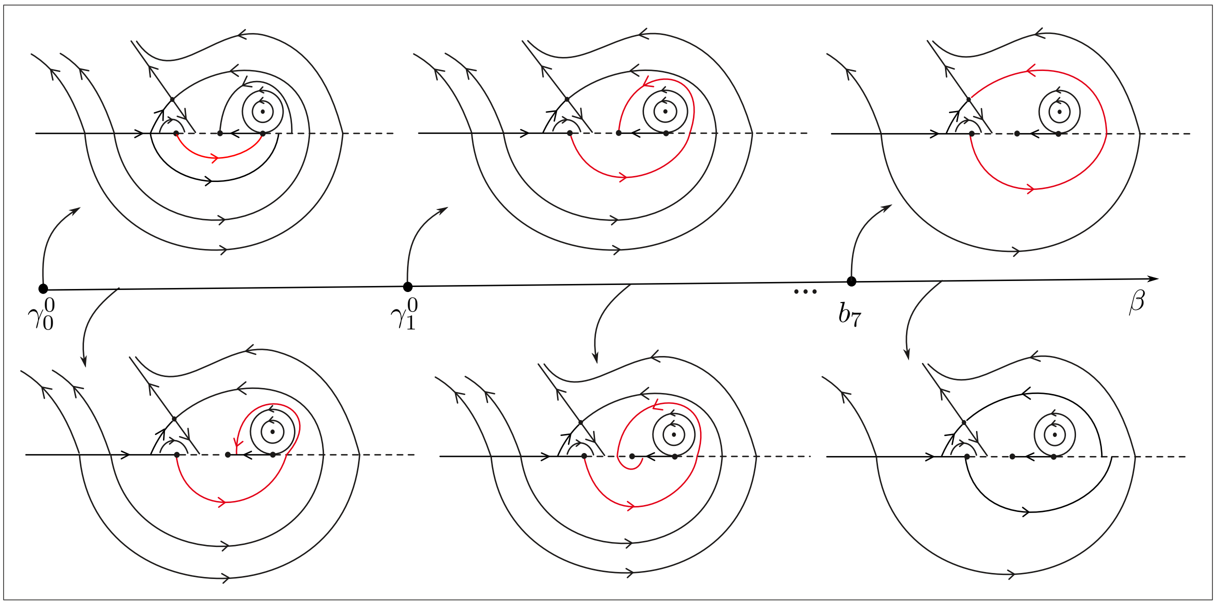

Figure 4.17: Illustration of the phase portraits of $Z_{\alpha,\beta}$ with values in the sequence γ^0_j.

- **Connections of P_2:** Observe that, in the curve defined by b_8, we have a connection between P_2 and $(1, 0)$. Moreover, in the curve defined by b_9 we have a connection between P_2 and P_3, this last connection provides a saddle connection. Therefore, for $b_8 \leq b \leq b_9$ there exists a sequence of disjoint curves (up to $\alpha = 0$) $b = \gamma^2_j(\alpha)$, converging to b_9 when $j \to \infty$, with $\gamma^2_0 = b_8$ and satisfying:
  - For $\gamma^2_{2j-2} < b < \gamma^2_{2j-1}$, $j = 1, 2, \ldots$, the trajectory through P_2 crosses Σ $2j-1$ times before reaching the sliding region;
  - For $b = \gamma^2_{2j-1}$, $j = 1, 2, \ldots$, the trajectory through P_2 intersects Σ $2j-1$ times before performing a connection with F_β;
  - For $\gamma^2_{2j-1} < b < \gamma^2_{2j}$, $j = 1, 2, \ldots$, the trajectory through P_2 crosses Σ $2j$ times before reaching the sliding region;
  - For $b = \gamma^2_{2j}$, $j = 1, 2, \ldots$, the trajectory through P_2 intersects Σ $2j$ times before performing a connection with $(1, 0)$. See Figure 4.18.

In addition to these cases we have:

- If $\gamma^1_j < b < \gamma^0_j$, $j = 0, 1, 2, \ldots$, then, after it crosses Σ $j+1$ times, the trajectory through P_1 can meet Σ^s at the same point the trajectory through 0 meets Σ^s after it crosses Σ j times, see Figure 4.19. It provides a new sequence η^{10}_j satisfying $\gamma^1_j < \eta^{10}_j < \gamma^0_j$ where this connection happens.

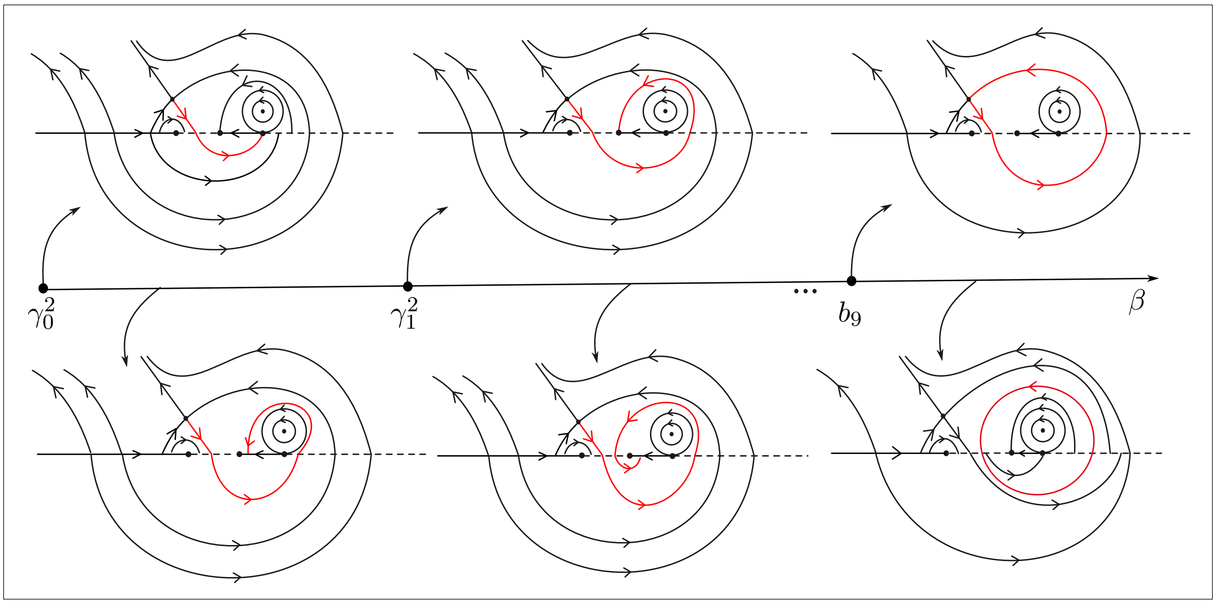

Figure 4.18: Illustration of the phase portraits of $Z_{\alpha,\beta}$ with values in the sequence γ_j^2.

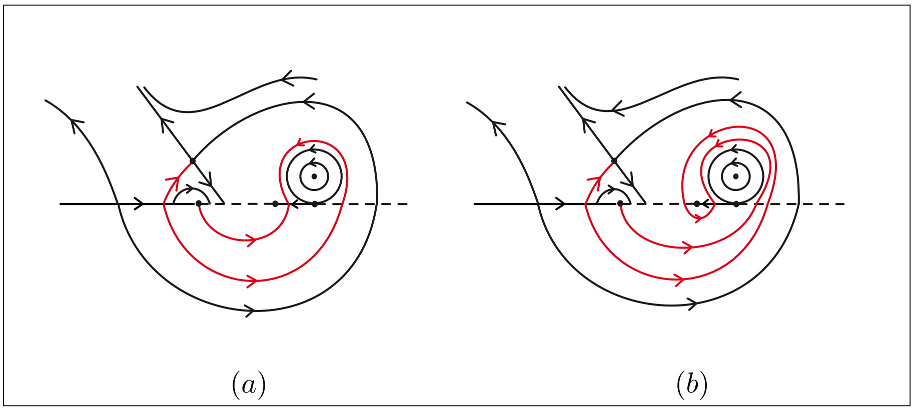

Figure 4.19: (a) in η_0^{10} and (b) in at η_1^{10}.

- There exists a sequence η_j^{12}, satisfying $\eta_j^{10} < \eta_j^{12} < \gamma_j^0$, such that, for $b = \eta_j^{12}$, after it crosses Σ $j+1$ times, the trajectory through P_1 meets Σ^s at the same point the trajectory through P_2 meets Σ^s after crossing Σ j times, see Figure 4.20.

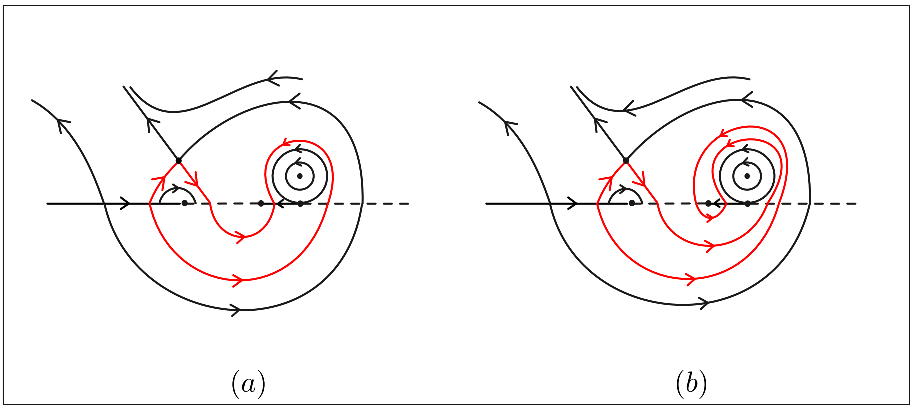

Figure 4.20: (a) in η_0^{12} and (b) in at η_1^{12}.

- There exists a sequence η_j^{02}, satisfying $\gamma_j^0 < \eta_j^{02} < \gamma_j^2$, providing a connection between the origin and P_2 in a similar way we have seen above, see Figure 4.21.

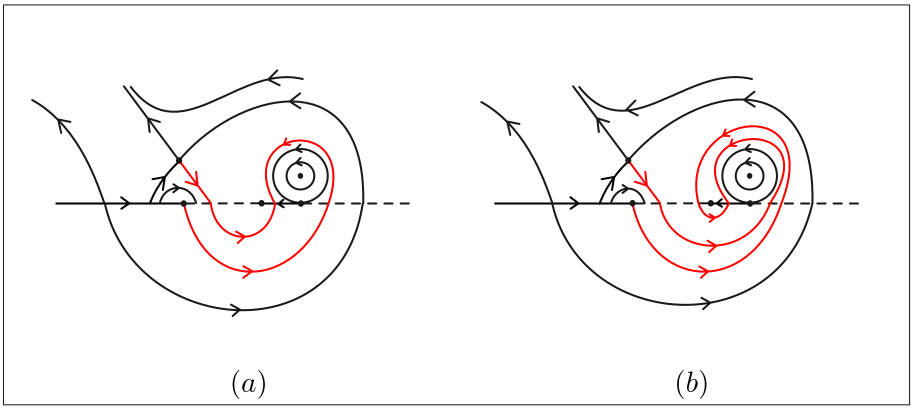

Figure 4.21: (a) in η_0^{02} and (b) in at η_1^{02}.

For $b > b_9$ there exists a repellor limit cycle through the sewing region. In fact, by means of numeric calculations it is possible to show that $z_- > x_2$, see Figure 4.18.

Moreover, $x_i(0) = 0$ for $i = 1, 2$, $x_3(0) = 3/2$, and $\gamma_j^i \to b_j^0$ for $i = 1, 2, 3$, where b_j^0 is a sequence of points tending to 0 when $j \to \infty$. Also, $\lim\limits_{\alpha \to \sqrt{3}/3} x_1(\alpha) = -\frac{1}{2}$ and $\lim\limits_{\alpha \to \sqrt{3}/3} x_i(\alpha) = 1$ for $i = 2, 3$, so

- $b_3(\sqrt{3}/3) = b_4(\sqrt{3}/3) = -1/2;$
- $b_6(\sqrt{3}/3) = b_7(\sqrt{3}/3) - 1/4;$
- $b_8(\sqrt{3}/3) = b_9(\sqrt{3}/3) = 1/4;$
- $b_3(0) = b_6(0) = b_8(0) = -1/4;$
- $b_4(0) = b_7(0) = b_9(0) = 0.$

So, for $\alpha = 0$ we find some other sequences which are limits of the existent sequences for $\alpha > 0$. Observe that $b_1 = b_2 = b_5 = b_{10} = 0$, $b_3 = b_6 = b_8 = 1/2$, and $b_4 = b_7 = b_9 = 3/4$. In this case:

- For $b = \dfrac{1}{2}$:

  There exists a heteroclinic connection between the origin and the point $(1, 0)$, which is a center for X_0, see Figure 4.22.

- For $\dfrac{1}{2} < b < \dfrac{3}{4}$:

  There exists a sequence b_j^0, with $b_0^0 = 1/2$, $1/2 < b_j^0 < 3/4$ and tending to $3/4$ when $j \to \infty$, satisfying

  - if $b_{2j-2}^0 < b < b_{2j-1}^0$, $j = 1, 2, \ldots$, the trajectory through the origin crosses Σ $2j - 1$ times before reaching the sliding region;
  - if $b = b_{2j-1}^0, j = 1, 2, \ldots$, the trajectory through the origin crosses Σ $2j - 1$ times and it connects with F_β;

- if $b^0_{2j-1} < b < b^{2j}$, $j = 1, 2, \ldots$, the trajectory through the origin crosses Σ $2j$ times before reaching the sliding region;
- if $b = b^0_{2j}$ the trajectory through the origin crosses Σ $2j$ times before connecting wit $(1, 0)$. See Figure 4.22.

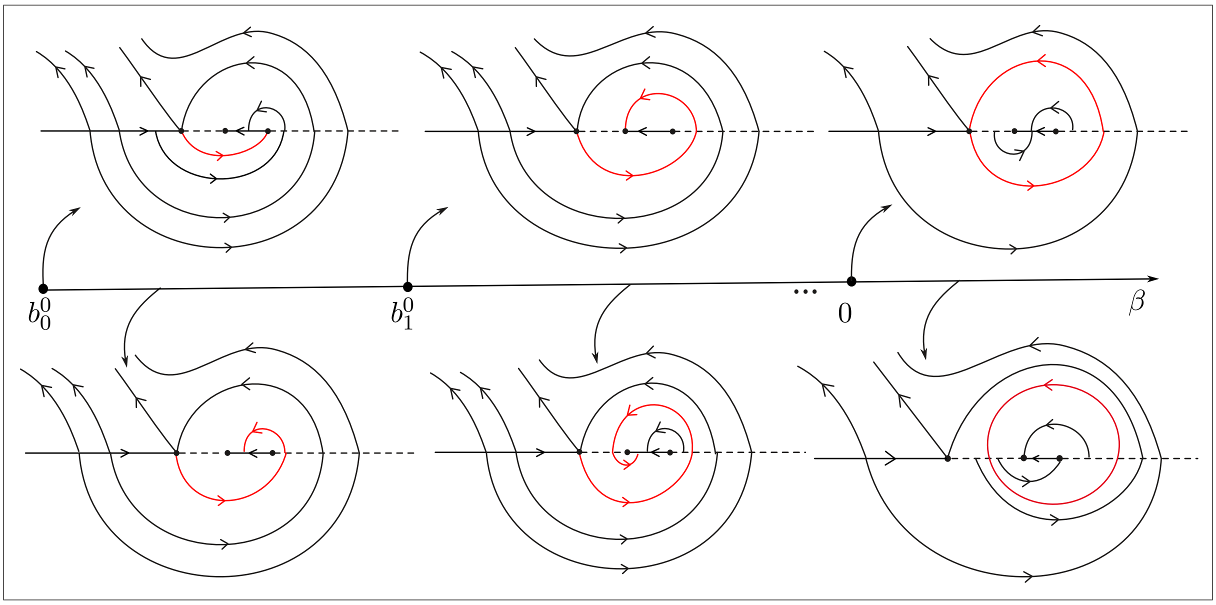

Figure 4.22: Illustration of the phase portraits of $Z_{0,\beta}$ with values in the sequence b^0_j.

- For $b = \dfrac{3}{4}$:

  There exists a homoclinic connection through the origin, i.e., there exists a degenerate cycle through the saddle-regular point at the origin, see Figure 4.15 or Figure 4.22.

- For $\dfrac{3}{4} < b < 1$:

  As seen above, there exists a repellor limit cycle through the sewing region, see 4.22.

Now, it remains to analyze the case when $\alpha < 0$. In this case we have a similar structure as in case $\alpha = 0$ but for the existence of the pseudo equilibrium point $Q = (s_-, 0)$. Thus, the same sequences for $\alpha = 0$ exist in this case, see Figure 4.23. Analogously to previous analysis, there exists a sequence ξ_j, providing a connection of the pseudo equilibrium Q and the points F_β and $(1, 0)$, see Figure 4.24.

In addition to these connections, it is also possible to connect Q and the origin by means of a pseudo connection at the sliding region, and this last sequenceis named ζ_j, see Figure 4.25.

Under these analysis we are ready to sketch the bifurcation curves, see Figure 4.26. Observe that, for any neighborhood V_0 of the $(\alpha, \beta)-$plane origin, there exists $n \in \mathbb{N}$ for which all curves defined by the sequences with index $\geq n$ have nonempty intersection with V_0. By changing the parameters, the hyperbolicity ratio does not change, it is always equal to one. We conclude that, despite the fact that the hyperbolicity ratio is not a irrational number, the studied family has a bifurcation diagram of type DSC_{31} given in Figure 4.13.

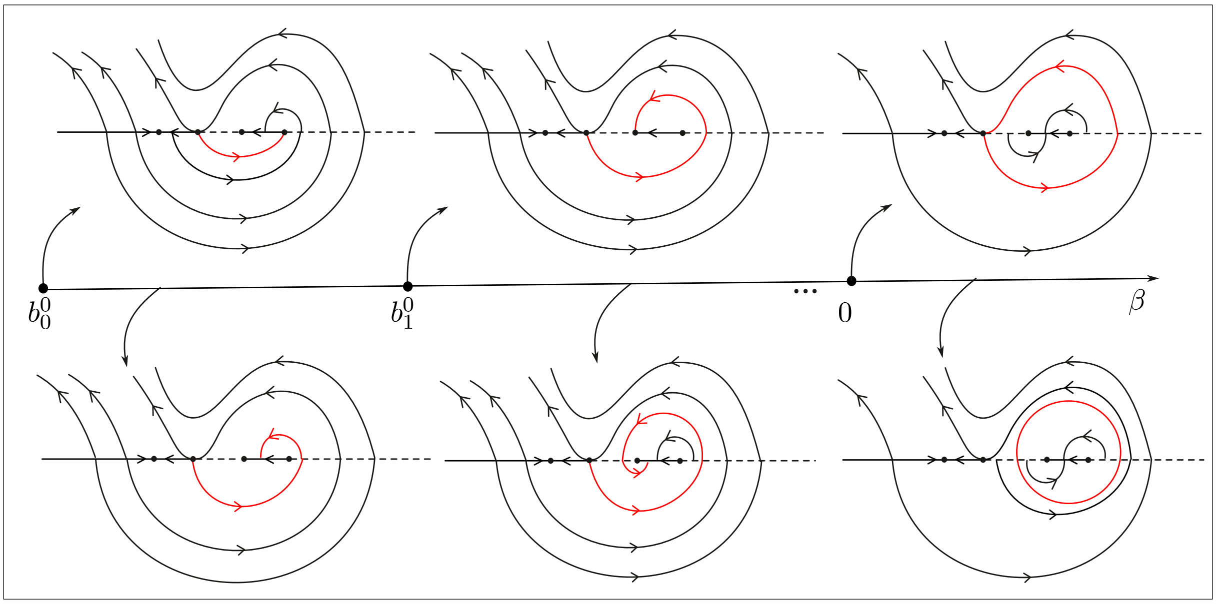

Figure 4.23: Illustration of the phase portraits of $Z_{\alpha,\beta}$ with values in the sequence b_j^0.

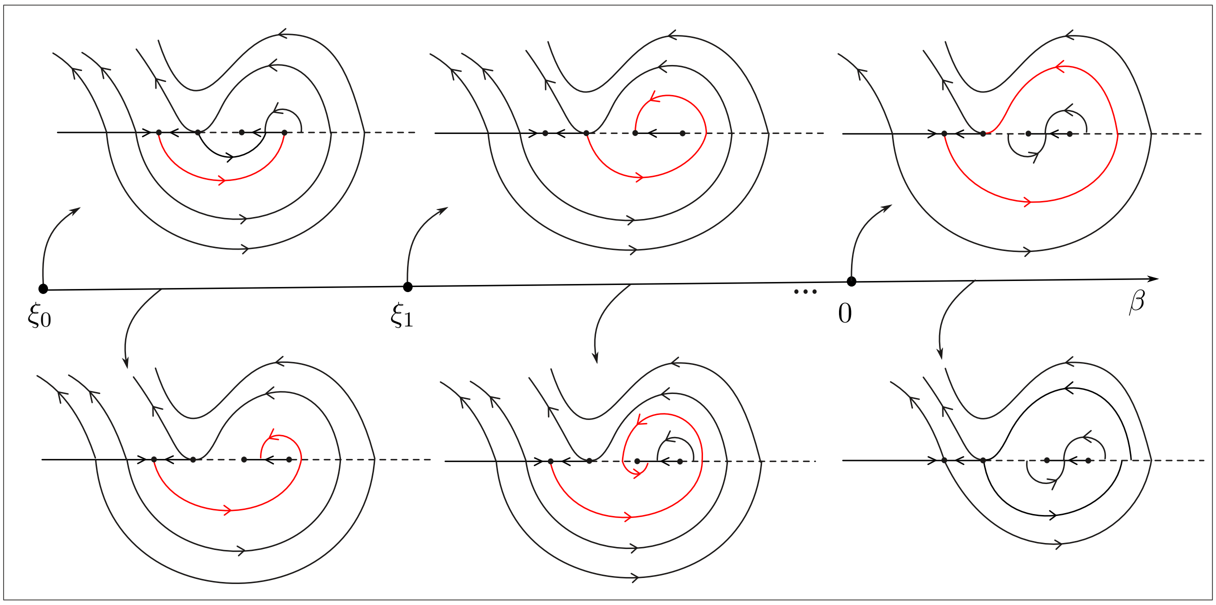

Figure 4.24: Illustration of the phase portraits of $Z_{\alpha,\beta}$ with values in the sequence ξ_j.

4.3.3 Model with Hyperbolicity Ratio in $\mathbb{Q}$

Now, we consider a model with hyperbolicity ratio $r > 0$ and we study how the system evolve for some values of $r \in \mathbb{Q}$. Consider $Z_a = (X_a, Y_a)$ with $a = (r, k, d, m)$, $\Sigma = h^{-1}(0)$, $h(x, y) = y + x/4 - m$, and

$$X_a = \begin{pmatrix} x \\ ry - x^3 - kx \end{pmatrix} \quad \text{and} \quad Y_a = \begin{pmatrix} -1 \\ -x + d \end{pmatrix}. \tag{4.3.6}$$

Now we consider the notation defined in Section 4.1.2. Observe that:

- for $r > 0$, $S_{X_a} = S = (0, 0)$ is a hyperbolic saddle of X_a with hyperbolicity ratio r;
- $W^s(S, X_a) = \{(x, y) \in \mathbb{R}; x = 0\}$ and $W^s(S, X) = \left\{(x, y) \in \mathbb{R}^2; y = -\frac{x^3}{r+3} - \frac{kx}{r+1}\right\}$;

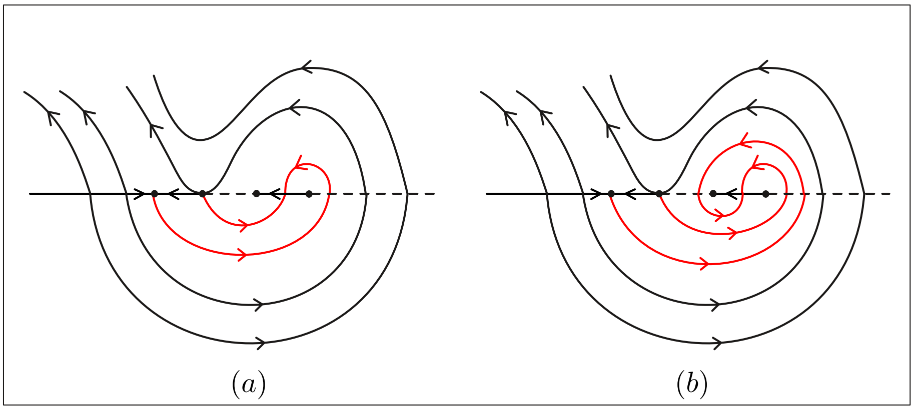

Figure 4.25: (a) in ζ_0^{02} and (b) in at ζ_1^{02}.

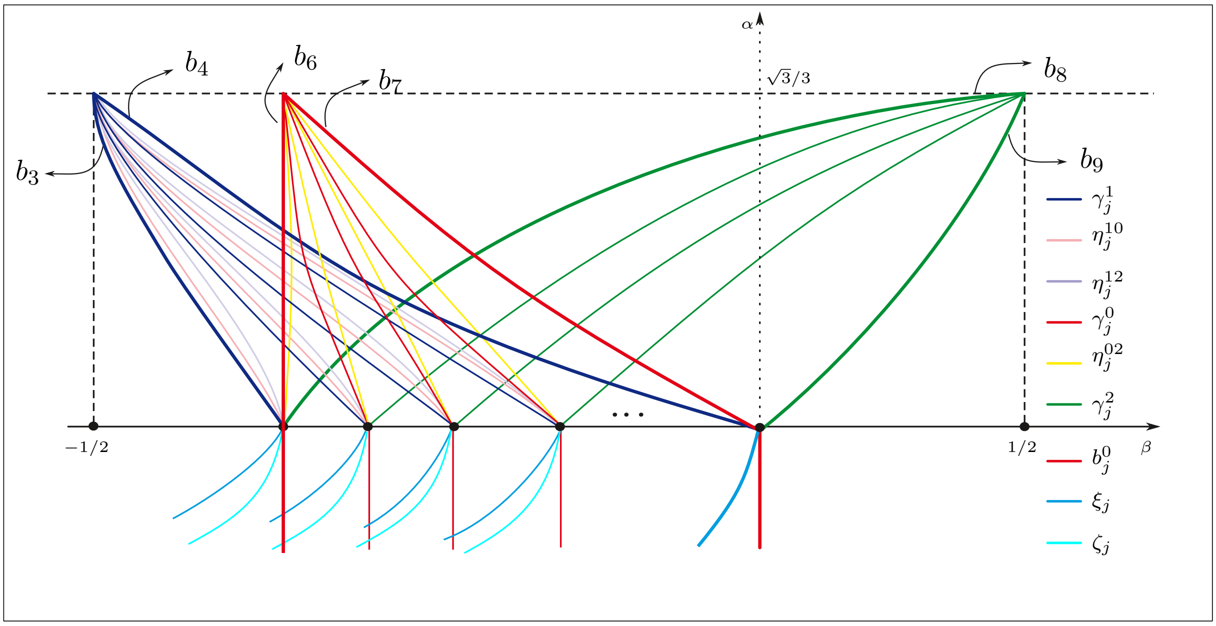

Figure 4.26: Sketch of the bifurcation diagram of $Z_{\alpha,\beta}$ with $|\beta|< 1/2$ and $|\alpha|< \sqrt{3}/3$.

- $W^s(S, X_a) \cap \Sigma = P_2$ and $W^u(S, X_a) \cap \Sigma = \{P_4, P_1, P_3\}$, where $P_j = P_j(r, k, m) = (x_j, -x_j/4 + m)$, $j = 1, 3, 4$, $P_2 = (0, m)$ and $x_4 < x_1 < x_3$. If $m > 0$ then $x_1 > 0$, $x_1 < 0$ for $m < 0$, and $x_1 = 0$ if $m = 0$. Also, for $m = 0$, $x_4 = -\sqrt{\frac{(r+1-4k)(r+3)}{4(r+1)}}$ and $x_3 = \sqrt{\frac{(r+1-4k)(r+3)}{4(r+1)}}$. Then, for $k < 0$, $W^u(S, X_a)$ crosses Σ in three different points if $m \approx 0$;
- $F_{Z_a} = (x_F, -x_F/4 + m)$ is the fold point of X_a near S for $m \neq 0$. F_{Z_a} is visible if $m > 0$ and invisible if $m < 0$;
- $F_d = \left(d - \frac{1}{4}, -\frac{d}{4} + \frac{1}{16} - m\right)$ is the unique fold point of Y_a, which is invisible;
- given $p_0 = (x_0, -x_0/4 + m) \in \Sigma$, the trajectory of Y_a through p_0 meets Σ again at the point $(2d - 1/2 - x_0, (x_0 - 2d)/4 + 1/8 + m)$. In this case, consider $\rho_3(x_0) = 2d - 1/2 - x_0$;
- for $k < 0$, $m = 0$, and $d > 0$ S is a saddle-regular point of type BS_3;

- by considering Σ parametrized by the first coordinate $(x \mapsto (x, -x/4+m)$), the sliding vector field is $Z^s(x) = -\dfrac{4x^3+4x^2+(4k-4d-r)+4mr}{4x^3-(5-4k+r)x+4mr+4d-1}$.

Observe that, for any two points p_1 and p_2 in Σ, parameter d in such a way that they are in the same trajectory of Y_a. Phase portraits of X_a and Y_a for $m=0$, $r>0$, and $k<0$ are given in Figure 4.27.

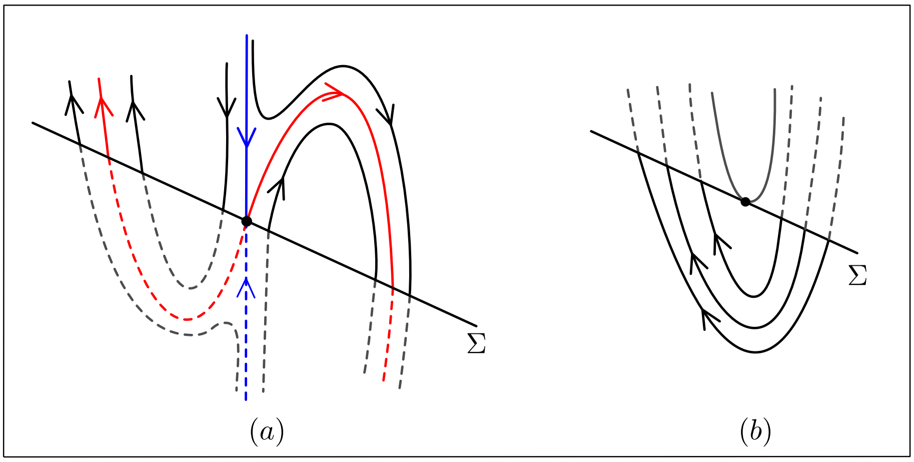

Figure 4.27: Phase portraits of (a) X_a and (b) Y_a with $m=0$, $r>0$, and $k<0$.

In what follows, assume $r>0$, $k<0$, and $m \approx 0$ sufficiently small such that $\sharp W^u(S, X_a) \cap \Sigma = 3$. Fix $r>0$ and $k, 0$, for each m, by varying d, we obtain all configurations given by the curves γ_F, γ_{P_1}, γ_{P_2}, and γ_{P_E} defined in Section 4.1.2 depending on the signal of m. Therefore, by changing m and d we want to show that Z_a realizes bifurcation diagram DSC_{31} if $r>1$ or DSC_{32} if $r<1$. In order to do that it is enough to show the existence of the limit cycles given in that bifurcation diagrams and that the pseudo-equilibrium appear when $m>0$, i.e., when the saddle is virtual. Since the algebraic accounts are not feasible, we illustrate it numerically.

Firstly, in Figures 4.28, 4.29, 4.30, and 4.31 we illustrate the position of the pseudo-equilibrium $P_E(Z_a)$ in relation to $F(Z_a)$, the invisible fold point of X_a near the origin, when d or m varies, for some values of r. Notice that, $P_E(Z_a) < F(Z_a)$ for $m>0$ it implies that $P_E(Z_a) \in \Sigma^s$, i.e., the pseudo-equilibrium exists when the saddle is virtual. Analogously, $P_E(Z_a) > F(Z_a)$ for $m>0$ it implies that $P_E(Z_a) \in \Sigma^c$, i.e., there exists no pseudo-equilibrium near the origin when the saddle is real.

Now, graphs of the first return map for some values of the parameters were done numerically, all graphs are given with the variable x having start point at the correspondent a_Z for which the first return is defined. See Figures 4.32,4.34, and 4.36 for some values of $r \in \mathbb{Q}$ satisfying $0<r<1$ and see Figures 4.33, 4.35, and 4.37 for some values of $r \in \mathbb{Q}$ with $r>1$. By observing that, for each m fixed, the first return map varies continuously as the other parameters vary, we conclude the results found in Section 4.1.2 remains true for this model. Therefore, cycles appear in the same way as seen for $r \notin \mathbb{Q}$. Moreover, from the analysis of the pseudo-equilibrium, we have that the bifurcations of the degenerate cycles obey bifurcation diagrams given in Figure 4.13 if $r>1$ and Figure 4.14 if $r<1$. As a complement to this analysis, we present some illustrations of the behavior of Z_a, for some values of parameters, with existence of limit cycles.

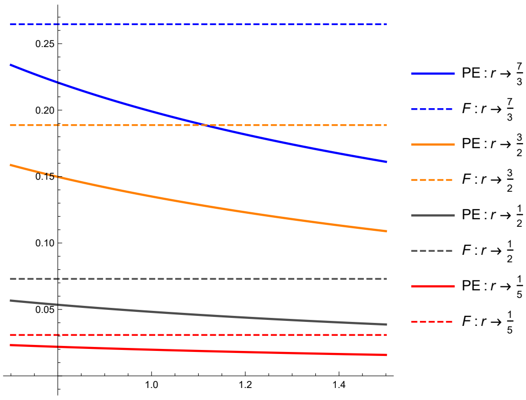

Figure 4.28: Graphs of P_E and F, considering $d \in (0.7, 1.5)$ variable, $m = 0.2$, $k = -1$.

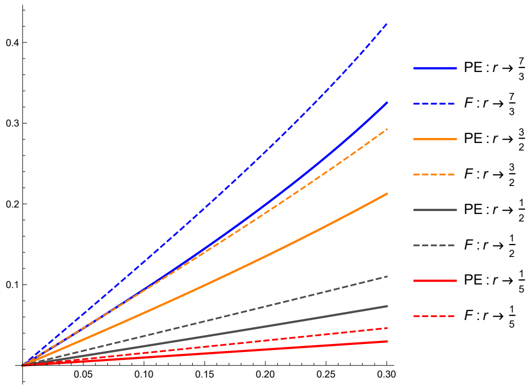

Figure 4.29: Graphs of P_E and F, considering $m \in (0, 0.3)$ variable, $d = 1$, $k = -1$.

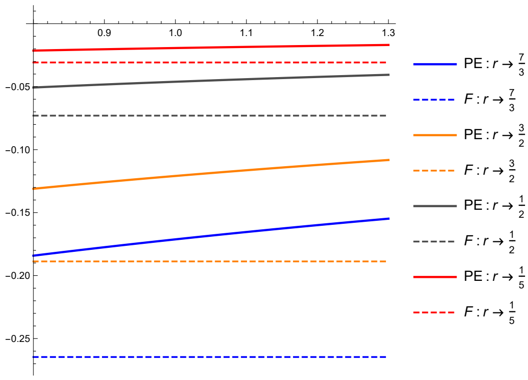

Figure 4.30: Graphs of P_E and F, considering $d \in (0.8, 1.3)$ variable, $m = -0.2$, $k = -1$.

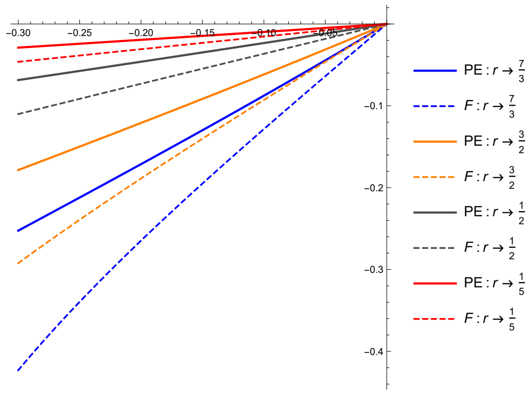

Figure 4.31: Graphs of P_E and F, considering $m \in (-0.3, 0)$ variable, $d = 1$, $k = -1$.

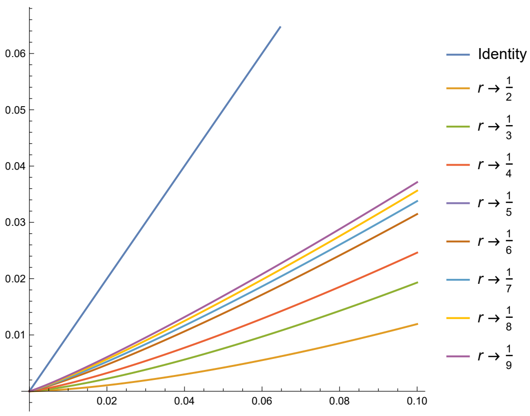

Figure 4.32: First return map: $m = 0$, $k = -1$, and $r < 1$.

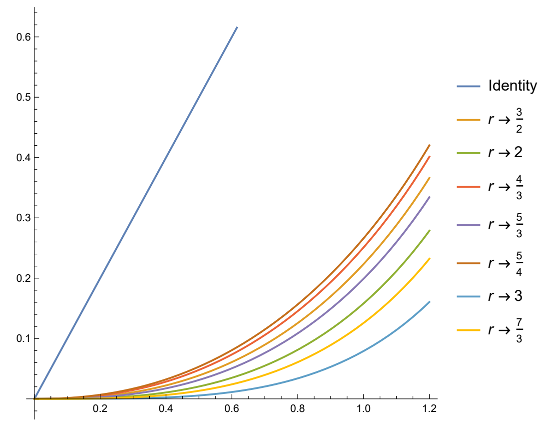

Figure 4.33: First return map: $m = 0$, $k = -1$, and $r > 1$.

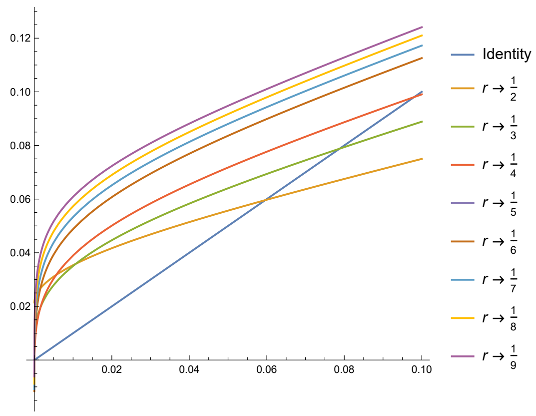

Figure 4.34: First return map: $m = -0.5$, $k = -1$, $d = 1.27$, and $r < 1$.

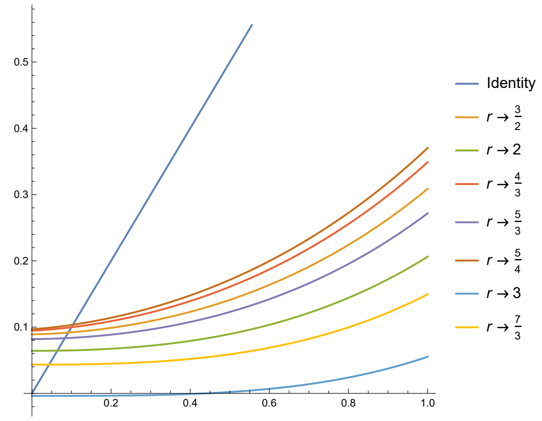

Figure 4.35: First return map: $m = -0.5$, $k = -1$, $d = 1.3$, and $r > 1$.

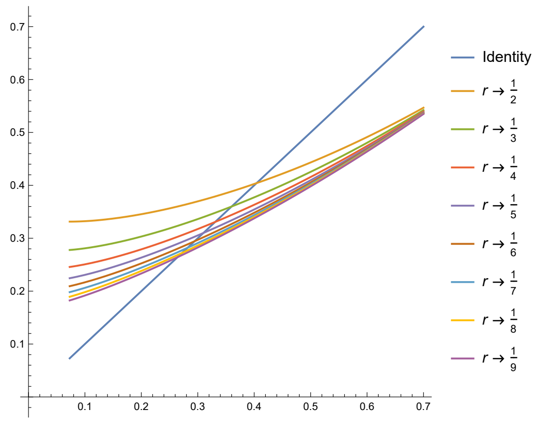

Figure 4.36: First return map: $m = 0.2$, $k = -1$, $d = 1.26$, and of $r < 1$.

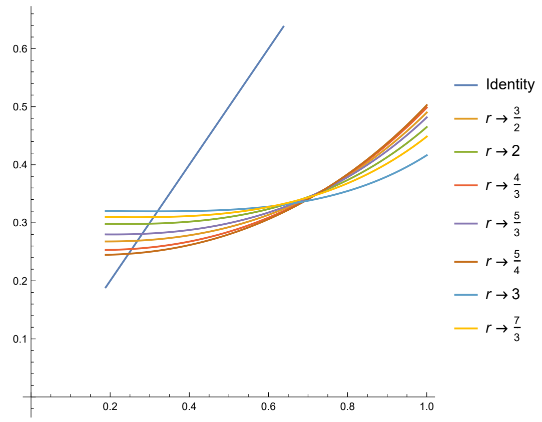

Figure 4.37: First return map: $m = 0.2$, $k = -1$, $d = 1.15$, and $r > 1$.

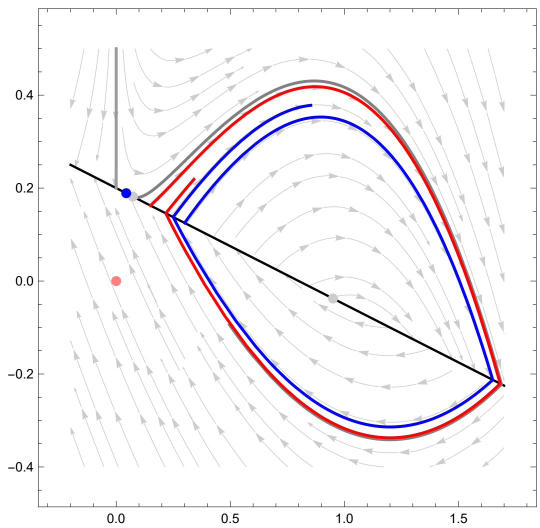

Figure 4.38: Trajectories through $p_1 = 0.15 \in \Sigma$ (red) and $p_2 = 0.3 \in \Sigma$ (blue), with $m = 0.2$, $k = -1$, $d = 1.2$, and $r = 1/2$. The blue point is the pseudo-equilibrium.

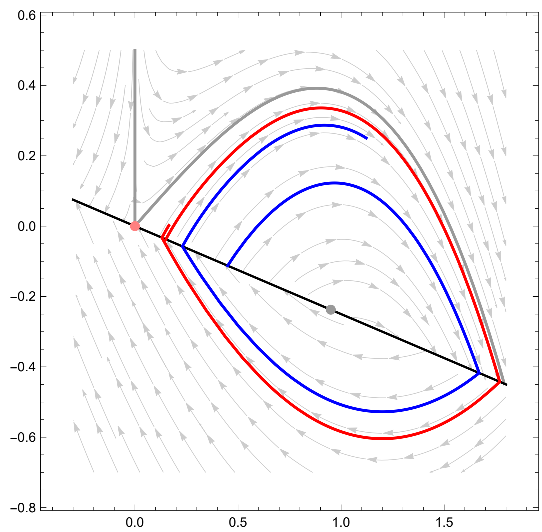

Figure 4.39: Trajectory through $p_1 = 0.15 \in \Sigma$ (red) and $p_2 = 0.45 \in \Sigma$ (blue), with $m = 0$, $k = -1$, $d = 1.2$, and $r = 1/2$.

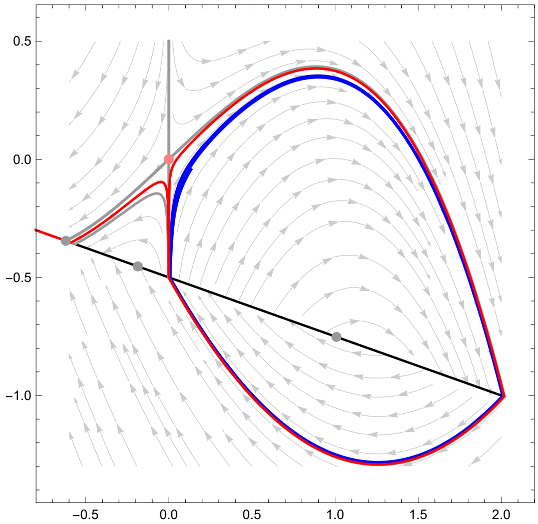

Figure 4.40: Trajectories through $p_1 = 0.0002 \in \Sigma$ (red) and $p_2 = 0.005 \in \Sigma$ (blue), with $m = -0.5$, $k = -1$, $d = 1.258$, and $r = 1/2$.

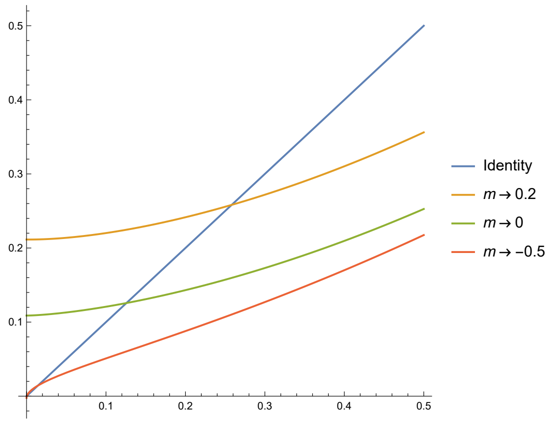

Figure 4.41: First return maps: $r = 1/2$, $k = -1$, $d = 1.2$ for $m = 0.2$, $d = 1.2$ for $m = 0$, and $d = 1.258$ for $m = -0.5$

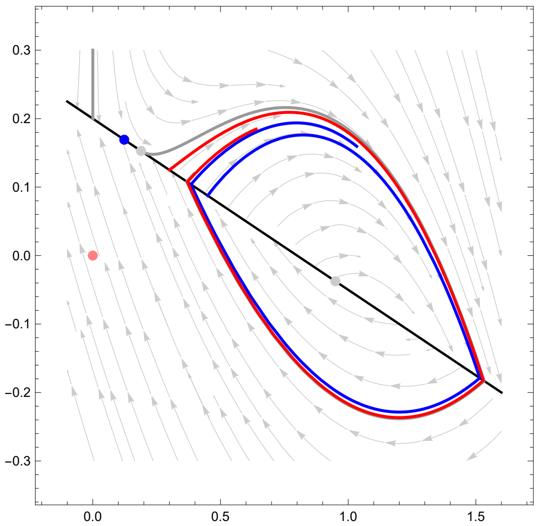

Figure 4.42: Trajectories through $p_1 = 0.3 \in \Sigma$ (red) and $p_2 = 0.45 \in \Sigma$ (blue), with $m = 0.2$, $k = -1$, $d = 1.2$, and $r = 3/2$. The blue point is the pseudo-equilibrium.

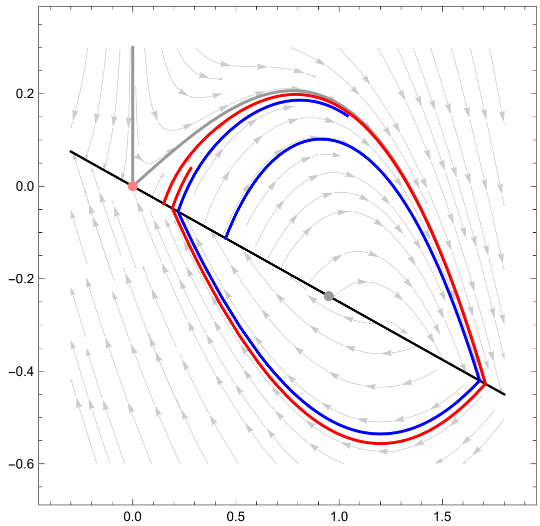

Figure 4.43: Trajectories through $p = 0.15 \in \Sigma$ (red) and $p_2 = 0.45 \in \Sigma$ (blue), with $m = 0$, $k = -1$, $d = 1.2$, and $r = 3/2$.

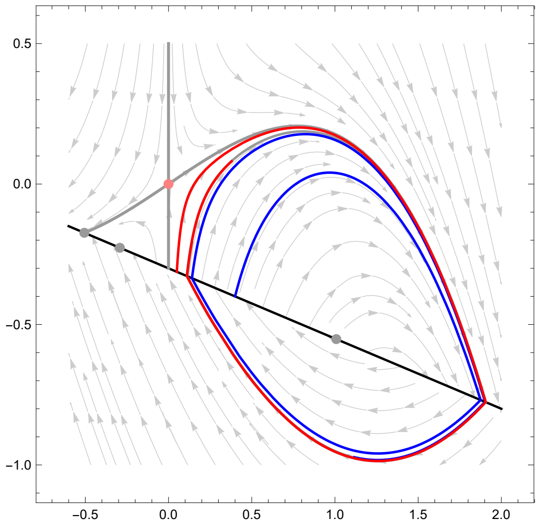

Figure 4.44: Trajectories through $p_1 = 0.4 \in \Sigma$ (red) and $p_2 = 0.05 \in \Sigma$ (blue), with $m = -0.5$, $k = -1$, $d = 1.258$, and $r = 3/2$.

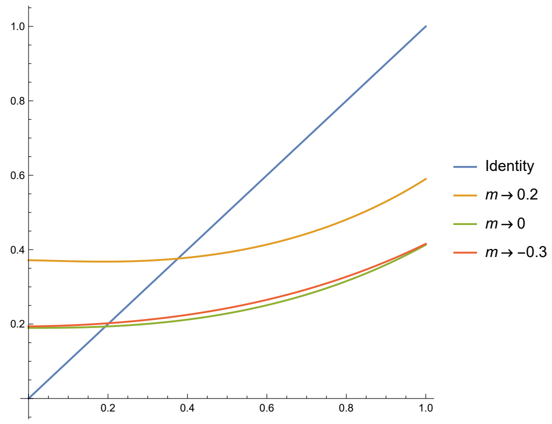

Figure 4.45: First return maps: $r = 3/2$, $k = -1$, $d = 1.2$ for $m = 0.2$, $d = 1.2$ for $m = 0$, and $d = 1.258$ for $m = -0.5$

Chapter 5

Applications

As important as exposing a theory is showing how useful and feasible it is. There are plenty of phenomena that are modeled by systems of ordinary differential equations, for instance see [2], [5] and [28]. Amid this variety of models, two "classes" will be presented here. The first concerns about a simple and particular example of oscillator, the pendulum. The second is included in the class of the integrable systems, the Hamiltonian ones.

The main objective of the present chapter is to illustrate that degenerate cycles through saddle-regular points are present in common models of applications. The study presented here is, in its majority, numeric and all calculations were performed by using the software *Mathematica*. The basic theory and construction of the models are not in the scope of this thesis, we just present the models without precise, physical, biological, economics, etc., interpretations. Moreover, the objects of our study focus on systems with the form:

$$Z(x,y) = \begin{pmatrix} g_1(x,y) \\ g_2(x,y) \end{pmatrix} + \begin{pmatrix} \alpha\phi(x,y) \\ (1-\alpha)\phi(x,y) \end{pmatrix}, \tag{5.0.1}$$

where $g_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$, are smooth functions, $\alpha = 0$ or $\alpha = 1$, and $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a piecewise smooth function:

$$\phi(p) = \begin{cases} \phi_1(p) & p \in \Sigma^+, \\ \phi_2(p) & p \in \Sigma^-. \end{cases}$$

In other words, we work with planar discontinuous vector fields $Z = (X, Y)$, where X and Y coincide up to one coordinate. Moreover, in each model, a different discontinuity manifold is considered, the choice of these manifolds is purely technical.

5.1 Pendulum Model

Pendulum is one of the simplest types of oscillators, many models and more details can be found in [2] and [28].

Consider the model for a simple pendulum with damping given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ a_1 y - \sin(x) \end{pmatrix} = X_{a_1}(x,y), \tag{5.1.1}$$

where x is the angle with the vertical axis, $y = \dot{x}$ is the angular speed and a_1 is a negative constant.

Suppose that a simple pendulum with damping is governed by the following law: if $\dot{x} > a_3 - a_4(x+\pi)$ it is given by system (5.1.1) and if $\dot{x} < a_3 - a_4(x+\pi)$ it is given by system (5.1.1) with an extra driving force, $a_2\left(x + \frac{\pi}{2}\right)\dfrac{\partial}{\partial y}$, being applied on it. This phenomenon can be modeled as a discontinuous system associated to the discontinuous vector field $Z_a = (X_{a_1}, Y_{a_1,a_2})$ with $a = (a_1, a_2, a_3, a_4)$, $\Sigma = \{(x, y) \in \mathbb{R}^2; y + a_4(x + \pi) = a_3\}$, and

$$Y_{a_1,a_2}(x, y) = \left(\begin{array}{c} y \\ a_1 y - \sin(x) + a_2\left(x + \dfrac{\pi}{2}\right) \end{array} \right).$$

Now, $S_X = (-\pi, 0)$ is a saddle point of X_{a_1}, with hyperbolicity ratio $r(a_1) = -\frac{a_1 - \sqrt{a_1^2+4}}{a_1 + \sqrt{a_1^2+4}}$. S_X is a real saddle when $a_3 < 0$, a boundary saddle when $a_3 = 0$, and a virtual saddle when $a_3 > 0$. For $a_3 \approx 0$ there exists a tangency point, in Σ, near S_X which is called P_{Z_a} and the first coordinate of this point will be denoted by p_a. Notice that P_{Z_a} coincides with S_X when $a_3 = 0$ and it is a fold point if $a_3 \neq 0$. A direct verification gives that, for $x \approx p_a$, there exists a crossing region when $x > p_a$ and there is a sliding region when $x < p_a$.

In what follows, we are going to check if this model realizes the degenerate cycle we are interested in. Let x_0 be a point in $\Sigma^+ \cup \Sigma^c$, sufficiently near the saddle point, and consider $\pi_a(x_0)$ as the first coordinate of the point were the trajectory passing through x_0 meets Σ for the second time. By using numeric calculations we analyze some trajectories, see Table 5.1.

$a = (a_1, a_2, a_3, a_4)$	x_0	$\pi_a(x_0)$
$(-0.1, -0.77, 0, 0.1)$	$(-2.8, -0.1(\pi - 2.8))$	$-4.33775\ldots$
$(-0.1, -0.77, 0, 0.1)$	$(-\pi, 0.5)$	$-4.54177\ldots$
$(-0.2, -0.77, 0, 0.1)$	$(-2.8, -0.1(\pi - 2.8))$	$-2.93979\ldots$
$(-0.2, -0.77, 0, 0.1)$	$(-\pi, 0.5)$	$-3.02473\ldots$

Table 5.1: Data for some values of the parameters of Z_a.

The trajectories with values given in Table 5.1 are illustrated in Figure 5.1. The first two values imply that the unstable manifold of the saddle in Σ^+ meets Σ in the sliding region, after it passes through Σ^c. The last two values imply that the unstable manifold of S_X in Σ^+ meets Σ in the crossing region twice. It follows from continuity that Z_a presents a degenerate cycle through a saddle-regular point for $a_2 = -0.77$, $a_3 = 0$, $a_4 = 0.1$ and some $a_1 \in (-0.2, -0.1)$.

Moreover, this system represents the case DSC_{11} given in Chapter 4. Now, we will show this system realizes each one of the regions in the bifurcation diagram in Figure 4.9. In Table 5.1 we have values of the parameter for which the vector field lies on the negative and positive parts of axes α of that bifurcation diagram. Just for simplicity, let q_a be the first coordinate of the point Q_{Z_a} (near P_{Z_a}) which vanishes the sliding vector field, i.e., Q_{Z_a} is the pseudo-equilibrium point when it is in the sliding region.

- **Region R_1^1**

  Consider the following values of parameters and initial conditions: $a = (-0.1, -0.77, 0.1, 0.1)$, $x_{01} = (-\pi, 05) \in \Sigma$, and $x_{02} = (-2.8, 0.1 - 0.1(\pi - 2.8)) \in \Sigma^+$. For

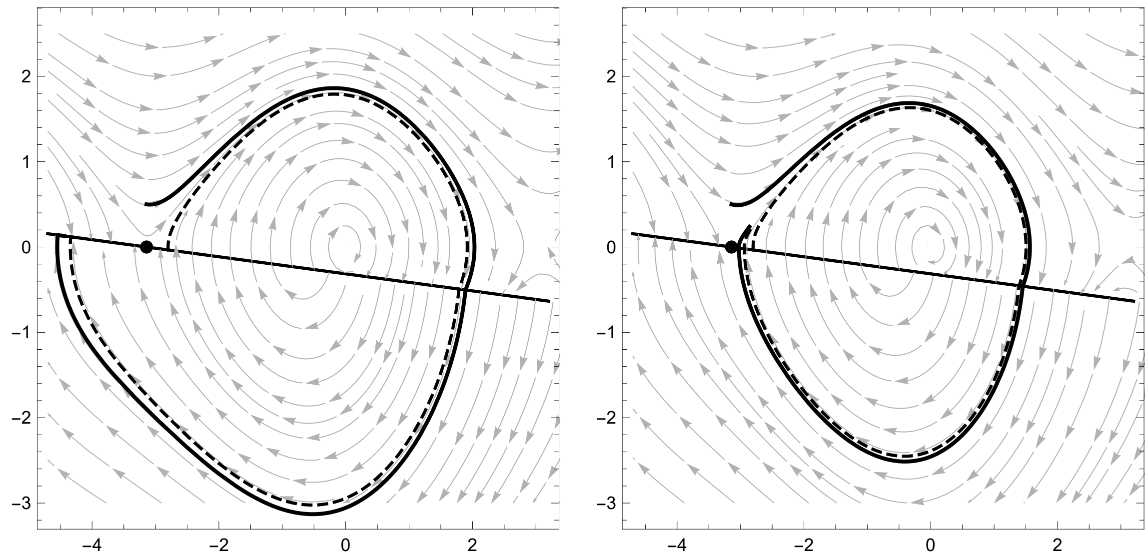

Figure 5.1: Trajectories of Z_a with data in Table 5.1. Left hand side corresponds to $a = (-0.1, -0.77, 0, 0.1)$ and right hand side corresponds to $a = (-0.2, -0.77, 0, 0.1)$.

these values $p_a = -3.14159\ldots$, $q_a = -2.14159\ldots$, $\pi_a(x_{01}) = -4.51446\ldots$, and $\pi_a(x_{02}) = -4.37873\ldots$. Observe that $q_a > p_a$, it means that there is no pseudo-equilibrium point. The correspondent trajectories are shown in Figure 5.2. By continuity, the trajectory passing through S_X intersects Σ in the sliding region.

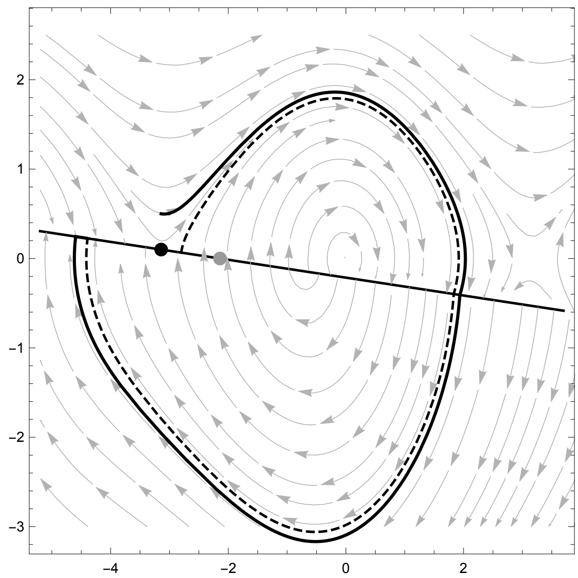

Figure 5.2: Trajectories of Z_a in R_1^1: $a = (-0.1, -0.77, 0.1, 0.1)$. The trajectory with solid line is correspondent to x_{01} and dashed one is correspondent to x_{02}. P_{Z_a} is the black point and Q_{Z_a} is the gray point.

- **Region** R_2^1

  Considering the following values of parameters and initial conditions: $a = (-0.2, -0.77, 0.1, 0.1)$, $x_{01} = (-\pi, 0.5)) \in \Sigma^+$ and $x_{02} = (-2.5, 0.1 - 0.1(\pi - 2.5)) \in \Sigma$, we obtain $p_a = -3.13169\ldots$, $q_a = -2.14159\ldots$, $\pi_a(x_{01}) = -3.06627\ldots$ and $\pi_a(x_{02}) =$

$-2.90533\ldots$. Since $q_a > p_a$ there is no pseudo-equilibrium point. These trajectories are shown in Figure 5.3. By continuity, the trajectory trough P_{Z_a} (which is a fold point) intersects Σ twice in the crossing region twice. Thus, these trajectories do not cross the sliding region near P_{Z_a}. Moreover, $\pi_a((-3.1, 0.1 - 0.1(\pi - 3.1))) = -3.00766\cdots > -3.1$ and $\pi_a((-2.9, 0.1 - 0.1(\pi - 2.9))) = -2.9955\cdots < -2.9$, since the first return map is continuous, must exists an attractor limit cycle through a point $(x_c, 0.1 - 0.1(\pi + x_c))$ for some $x_c \in (-3.1, -2.9)$. See the graph of the first return map in Figure 5.4.

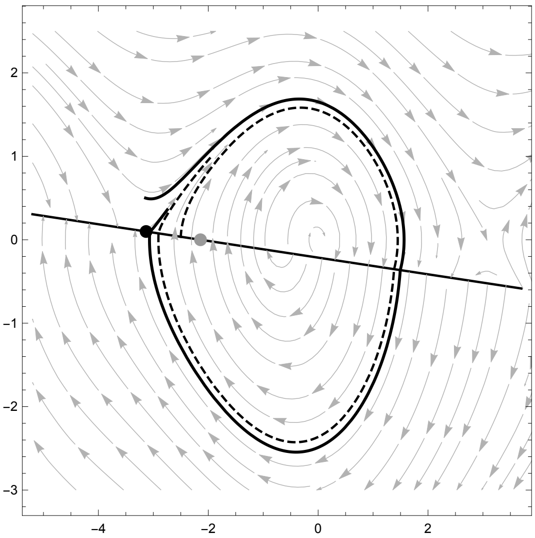

Figure 5.3: Trajectories of Z_a in R_2^1: $a = (-0.2, -0.77, 0.1, 0.1)$. The trajectory with solid line is correspondent to x_{01} and dashed one is correspondent to x_{02}. P_{Z_a} is the black point and Q_{Z_a} is the gray point.

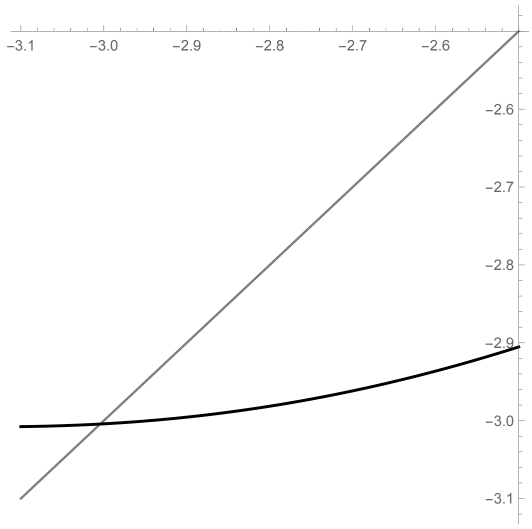

Figure 5.4: First return map in R_2: $a = (-0.2, -0.77, 0.1, 0.1)$. The origin is located in $(-2.5, -2.5)$.

- **Curve** $\alpha^+ = \{(\alpha, 0); \alpha > 0\}$

  When $a_3 = 0$ the saddle point is on the boundary. For the values of parameters and initial conditions $a = (-0.2, -0.77, 0, 0.1)$, $x_{01} = (-\pi, 0.5)$, and $x_{02} = (-2.8, -0.1(\pi - 2.8))$ we obtain $p_a = q_a = -3.14159\ldots$, $\pi_a(x_{01}) = -3.02473\ldots$, and $\pi_a(x_{02}) = -2.93979\ldots$. These trajectories are illustrated in Figure 5.5. Therefore, the unstable manifold of S_X in Σ^+ intersects Σ^c, at the second time, in a neighborhood of S_X. Moreover, $\pi_a((-3.1, -0.1(\pi - 3.1))) = -2.96489\cdots > -3.1$ and $\pi_a((-2.9, -0.1(\pi - 2.9))) = -2.95331\cdots < -2.9$. Since the first return map is continuous, it implies in the existence of an attractor limit cycle through $(x_c, -0.1(\pi + x_c))$ for some $x_c \in (-3.1, -2.9)$. See the graph of the first return map in figure 5.6.

- **Region** R_3^1

  Consider the values of parameters and initial conditions: $a{=}(-0.2, -0.77, -0.1, 0.1)$, $x_{01} = (-\pi, 0.6) \in \Sigma^+$, and $x_{02} = (-2.9, -0.1 - 0.1(\pi - 2.9)) \in \Sigma^+$. We obtain $p_a = -3.15149\ldots$, $q_a = -4.14159\ldots$, $\pi_a(x_{01}) = -2.99339\ldots$, and $\pi_a(x_{02}) = -2.89616\cdots < -2.9$. These trajectories are shown in Figure 5.7. Hence, the

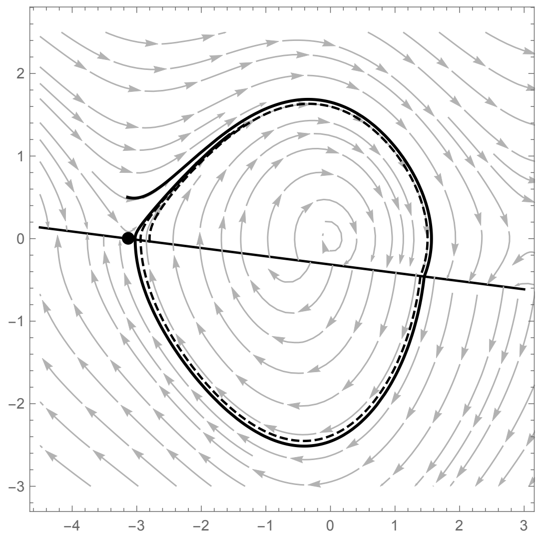

Figure 5.5: Trajectories of Z_a in α^+: $a = (-0.2, -0.77, 0, 0.1)$. The trajectory with solid line is correspondent to x_{01} and dashed one is correspondent to x_{02}. P_{Z_a} is the black point and Q_{Z_a} is the gray point.

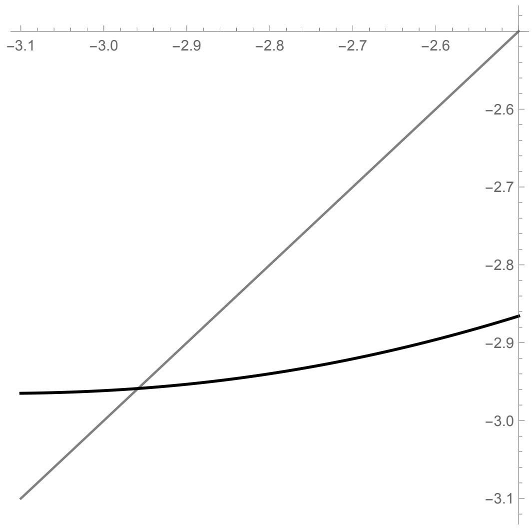

Figure 5.6: First return map in α^+: $a = (-0.2, -0.77, 0, 0.1)$. The origin of this axes is located at $(-2.5, -2.5)$.

unstable manifold in Σ^+ that intersects Σ at the crossing region must intersect Σ^c again near P_{Z_a}. Moreover,$\pi_a((-3.1, -0.1(\pi - 3.1))) = -3.31943\cdots > -3.1$ then there exists an attractor limit cycle through $(x_c, -0.1 - 0.1(\pi + x_c))$ for some $x_c \in (-3.1, -2.9)$. See the graph of the first return map in Figure 5.8.

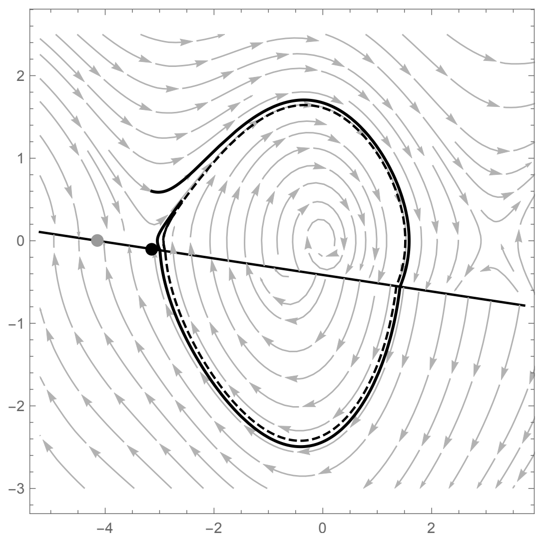

Figure 5.7: Trajectories of Z_a in R_3^1: $a = (-0.2, -0.77, -0.1, 0.1)$. The trajectory with solid line is correspondent to x_{01} and dashed one is correspondent to x_{02}. P_{Z_a} is the black point and Q_{Z_a} is the gray point.

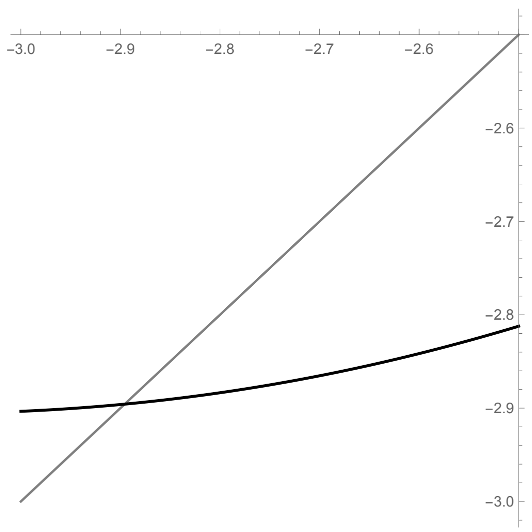

Figure 5.8: First return map in R_3^1: $a = (-0.2, -0.77, -0.1, 0.1)$. The origin of this axes is located in $(-2.5, -2.5)$.

- **Region** R_4^1

  Consider the values of parameters and initial conditions $a = (-0.185, -0.77, -0.2, 0.1)$, $x_{01} = (-\pi, 0.5) \in \Sigma^+$, and $x_{02} = (-2.8, -0.2 - 0.1(\pi - 2.8)) \in \Sigma$ we obtain $p_a = -3.15845\ldots$, $q_a = -5.14159\ldots$, $\pi_a(x_{01}) = -3.33481\ldots$, and $\pi_a(x_{02}) = -2.9545\ldots$. These trajectories are shown in Figure 5.9. In this way, the unstable manifold in Σ^+, which intersects Σ^c, reach the sliding region near P_{Z_a} after crossing Σ at twice.

- **Regions** $R_5^1 \cup \gamma_{x_1} \cup R_6^1$

  Consider the values of parameters and initial conditions $a = (-0.15, -0.77, -0.1, 0.1)$, $x_{01} = (-\pi, 0.5) \in \Sigma^+$, and $x_{02} = (-2.7, -0.1 - 0.1(\pi - 2.7)) \in \Sigma$ we obtain the values $p_a = -3.14657\ldots$, $q_a = -4.14159\ldots$, $\pi_a(x_{01}) = -3.57493\ldots$ and $\pi_a(x_{02}) = -3.41217\ldots$. These trajectories are shown in figure 5.10. So, the unstable manifold in Σ^+, which crosses Σ^c, intersects Σ^s at a point between the pseudo-equilibrium and the fold point.

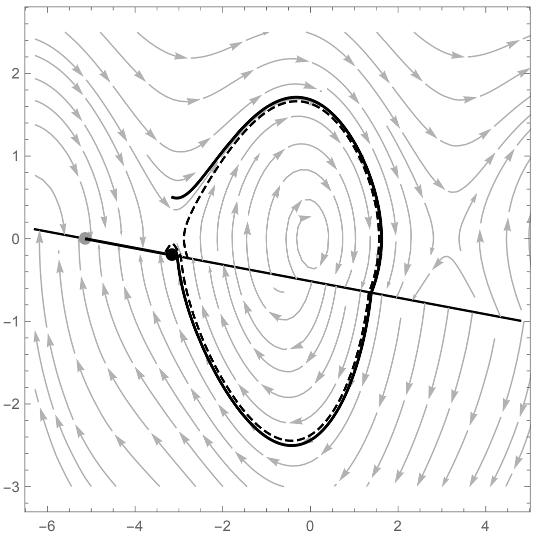

Figure 5.9: Trajectories of Z_a in R_4^1: $a = (-0.185, -0.77, -0.2, 0.1)$. The trajectory with solid line is correspondent to x_{01} and dashed one is correspondent to x_{02}. P_{Z_a} is the black point and Q_{Z_a} is the gray point.

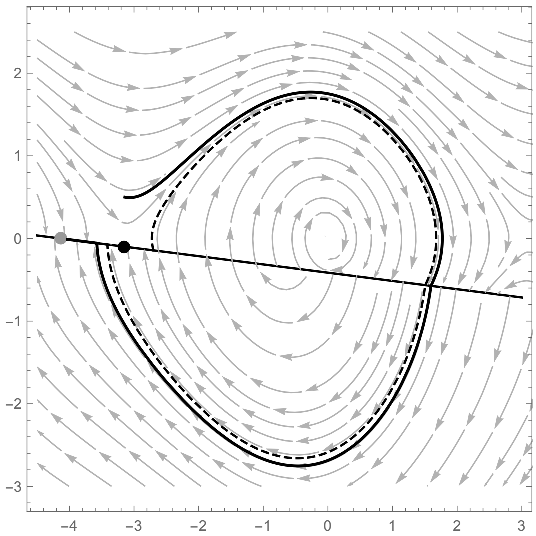

Figure 5.10: Trajectories of Z_a in $R_5^1 \cup \gamma_{x_1} \cup R_6^1$: $a = (-0.15, -0.77, -0.1, 0.1)$. The trajectory with solid line is correspondent to x_{01} and dashed one is correspondent to x_{02}. P_{Z_a} is the black point and Q_{Z_a} is the gray point.

- **Region** R_7^1

  Consider the values of parameters and initial conditions $a = (-0.1, -0.77, -0.1, 0.1)$, $x_{01} = (-2.9, -0.1 - 0.1(\pi - 2.9)) \in \Sigma^+$, and $x_{02} = (-2.9, -0.1 - 0.1(\pi - 2.9)) \in \Sigma^+$. So, $p_a = -3.14159\ldots$, $q_a = -4.14159\ldots$, $\pi_a(x_{01}) = -4.46432\ldots$, and $\pi_a(x_{02}) = -4.30114\ldots$. These trajectories are shown in figure 5.11. The branch of the unstable manifold in Σ^+, which crosses Σ transversely in Σ^c, intersects Σ^s at a point P_7 such that the pseudo-equilibrium is located between P_7 and P_{Z_a}.

- **Curve** $\alpha^- = \{(\alpha, 0); \alpha < 0\}$

Considering the values of parameters and initial conditions $a = (-0.1, -0.77, 0, 0.1)$, $x_{01} = (-\pi, 0.5) \in \Sigma^+$, and $x_{02} = (-2.8, -0.1(\pi - 2.8)) \in \Sigma$ we obtain $p_a = q_a = -3.14159\ldots$, $\pi_a(x_{01}) = -4.54177\ldots$ and $\pi_a(x_{02}) = -4.33775\ldots$. These trajectories are shown in Figure 5.12. So, the unstable manifold in Σ^+ crosses Σ^c once before it reaches Σ^s.

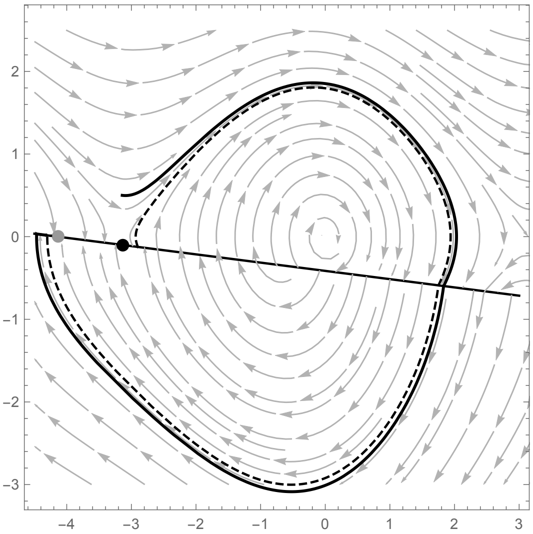

Figure 5.11: Trajectories of Z_a in R_7^1: $a = (-0.1, -0.77, -0.1, 0.1)$. The trajectory with solid line is correspondent to x_{01} and dashed one is correspondent to x_{02}. P_{Z_a} is the black point and Q_{Z_a} is the gray point.

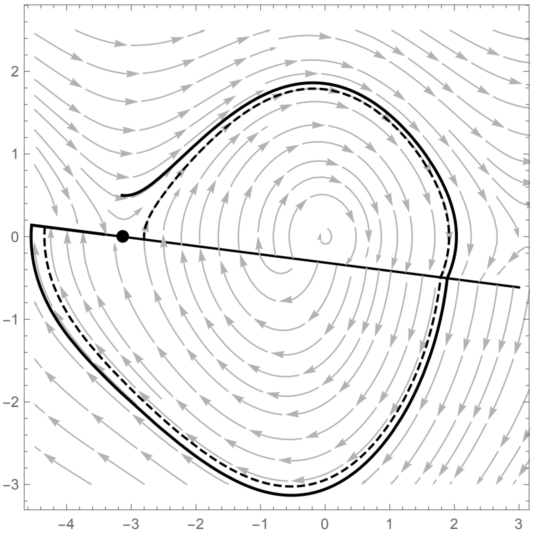

Figure 5.12: Trajectories of Z_a in α^-: $a = (-0.1, -0.77, 0, 0.1)$.The trajectory with solid line is correspondent to x_{01} and dashed one is correspondent to x_{02}. P_{Z_a} is the black point and Q_{Z_a} is the gray point.

5.2 Piecewise Hamiltonian Model

A Hamiltonian system in $\mathbb{R}^2$ is a system of the form

$$\begin{cases} \dot{x} &= \dfrac{\partial}{\partial y} H(x,y) \\ \dot{y} &= -\dfrac{\partial}{\partial x} H(x,y), \end{cases} \tag{5.2.1}$$

where $H : \mathbb{R}^2 \to \mathbb{R}$ is a function of class $\mathcal{C}^2$. H is called Hamiltonian function associated to the Hamiltonian system 5.2.1. Moreover, H is a first integral for the system, i.e., the trajectories of (5.2.1) lie in the level curves of H. This kind of system is widely studied in literature, the basic theory on this theme can be found in any introductory book on dynamical system, some specific references are [25, 41, 42].

Consider the discontinuous vector field

$$Z_{a,b,c}(x,y) = \begin{pmatrix} \phi_{a,b,c}(x,y) + 1 - y^2 \\ x \end{pmatrix}, \tag{5.2.2}$$

where

$$\phi_{a,b,c}(x,y) = \begin{cases} a - y & \text{if} \quad x \leq b(y+1)+c \\ 0 & \text{if} \quad x \geq b(y+1)+c \end{cases}.$$

Thus, $Z_{a,b,c} = (X_a, Y)$ where $\Sigma = h^{-1}(0)$, $h(x,y) = b(y+1) - x + c$, and

$$X_a(x,y) = \begin{pmatrix} 1 + a - y - y^2 \\ x \end{pmatrix} \quad \text{and} \quad Y(x,y) = \begin{pmatrix} 1 - y^2 \\ x \end{pmatrix}.$$

Observe that $Z_{a,b,c}$ is a piecewise Hamiltonian system, with associated Hamiltonian functions given by

$$H_{X_a}(x,y) = -\frac{y^3}{3} - \frac{y^2}{2} - \frac{x^2}{2} + (1+a)y \quad \text{and} \quad H_Y(x,y) = -\frac{y^3}{3} - \frac{x^2}{2} + y.$$

The vector field Y satisfies:

(i) it has two singular points $P_\pm = (0, \pm 1)^T$, where P_- is a saddle and P_+ is a center;

(ii) $P_- \in \Sigma$ if, and only if, $c = 0$, $P_- \in \Sigma^+$ iff $c > 0$, and $P_- \in \Sigma_-$ iff $c < 0$;

(iii) P_- has associated eigenvalues $\pm\sqrt{2}$ and $E^s = [(\sqrt{2}, 1)^T]$ and $E^u = [(\sqrt{2}, -1)^T]$ are the stable and unstable eigenspaces, respectively.

The vector field X_a satisfies:

(i) if $a > -5/4 = -1.25$ it has two singular points, $Q_\pm = (0, (-1 \pm \sqrt{5+4a})/2)^T$, where Q_+ is a center and Q_- is a saddle point;

(ii) if $-1 < a < 1$, then, ordering by the second coordinate, $Q_- < P_- < Q_+ < P_+$. Also $X_a(0,-1) = (1+a, 0)^T$.

We are going to show that, for $c = 0$, the system (5.2.2) admits a degenerate cycle of type DSC_{11} studied in Chapter 4. Consider $|a| < 1$, $0 < b < \sqrt{2}$, and c small enough in order to have defined a first return map in Σ. Let V_0 be a neighborhood of the saddle P_-, if for $p \in V_0$ the trajectory through p crosses Σ twice, define $p \in V_0 \mapsto \pi(p) \in \Sigma$ the map that associates to p the second coordinate of the point in Σ where this second intersection happens. Now, we have some numeric results.

(I) **Realizing the degenerate cycle**

The unstable manifold of saddle in Σ^-, W^u_-, is located between the trajectories through $(0.05, -0.9)$
$\in \Sigma$ and $(0.3, -1) \in \Sigma^-$ for all positive time. Having the values in Table 5.2, we obtain that W^u_- crosses Σ, at the second time, "above" the saddle point for $a = 0.5 = b$ and $c = 0$. Moreover, W^u_- crosses Σ, at the second time,"below" the saddle point for $a = 0.25$, $b = 0.5$, and $c = 0$. So, by continuity, W^u_- crosses Σ at P_- for some $a \in (0.25; 0.5)$.

(II) **Existence of limit cycles**

For $a = b = 0.5$ and $c = 0$ (boundary saddle), consider the points $p_0 = (0.1, -0.8)$ and $p_1 = (0.3, -0.4)$ in Σ. With values in Table 5.3 we obtain that the trajectory through p_0 returns to Σ "below" p_0 while the trajectory through p_1 returns to Σ "above" p_1. Therefore, by continuity, π has an attractor fixed point $\bar{p} = (0.5(\bar{y}+1), \bar{y}) \in \Sigma$ with

(a,b,c)	$p=(x,y)$	$\pi(p)$
$(0.5,0.5,0)$	$(0.05,-0.9)\in\Sigma$	$-0.681677\ldots$
$(0.5,0.5,0)$	$(0.3,-1)\in\Sigma^-$	$-0.730304\ldots$
$(0.25,0.5,0)$	$(0.05,-0.9)\in\Sigma$	$-1.12575\ldots$
$(0.25,0.5,0)$	$(0.3,-1)\in\Sigma^-$	$-1.20327\ldots$

Table 5.2: Data for some values of the parameters of $Z_{a,b,0}$.

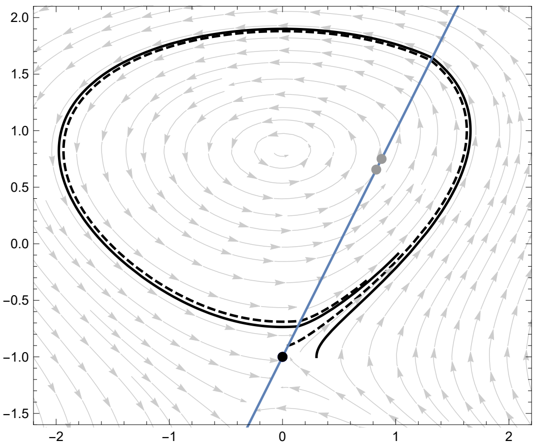

Figure 5.13: Trajectories of $Z_{a,b,c}$ with $a=b=0.5$ and $c=0$. The black point corresponds to the saddle P_- and the gray points are tangency points.

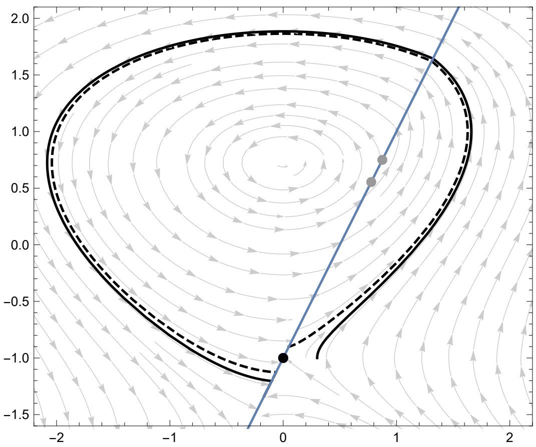

Figure 5.14: Trajectories of $Z_{a,b,c}$ with $a=0.25$, $b=0.5$, and $c=0$. The origin is located at $(-0.4,-0.4)$.

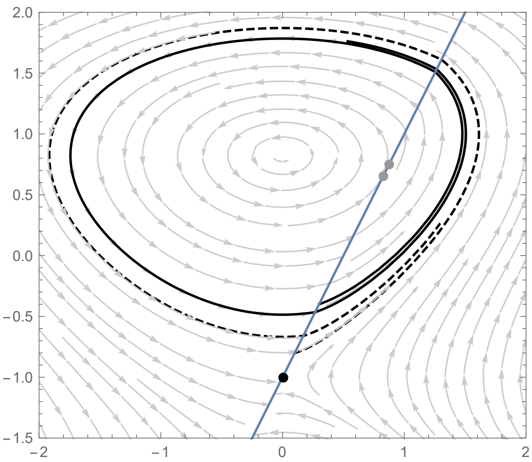

Figure 5.15: Trajectories of $Z_{a,b,c}$ with $a=b=0.5$ and $c=0$. The black point corresponds to the saddle P_- and the gray points are tangency points.

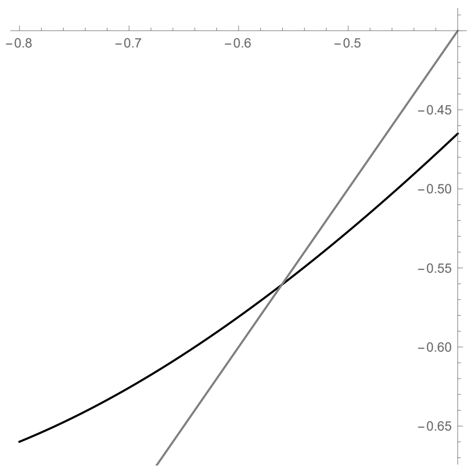

Figure 5.16: Graph of the first return map of $Z_{a,b,c}$ with $a=b=0.5$ and $c=0$. The origin is located at $(-0.4,-0.4)$.

$\bar{y}\in(-0.8,-0.4)$. This attractor fixed point corresponds to an attractor limit cycle of $Z_{a,b,c}$, see Figures 5.15 and 5.16.

The saddle point P_- is virtual if $c>0$, for $a=b=0.5$ and $c=0.2$ consider the points $p_0=(0.3,-0.8)$ and $p_1=(0.6,-0.2)$ in Σ. With the values in Table 5.4 we obtain that the trajectory through p_0 returns to Σ "below" p_0 while the trajectory through p_1 returns

(a,b,c)	$p=(x,y)$	$\pi(p)$
$(0.5,0.5,0)$	$p_0=(0.1,-0.8)\in\Sigma$	$-0.65989\ldots$
$(0.25,0.5,0)$	$p_1=(0.3,-0.4)\in\Sigma$	$-0.46503\ldots$

Table 5.3: Data for some values of the parameters of $Z_{a,b,0}$.

to Σ "above" p_1. Thus, by continuity, π has a fixed point $\bar{p}=(0.5(\bar{y}+1)+0.3,\bar{y})\in\Sigma$ with $\bar{y}\in(-0.8;-0.2)$, see Figures 5.17 and 5.18. Finally, P_- is a real saddle point

(a,b,c)	$p=(x,y)$	$\pi(p)$
$(0.5,0.5,0.2)$	$p_0=(0.3,-0.8)\in\Sigma$	$-0.574247\ldots$
$(0.5,0.5,0.2)$	$p_1=(0.6,-0.2)\in\Sigma$	$-0.268171\ldots$

Table 5.4: Data for some values of the parameters of $Z_{a,b,c}$.

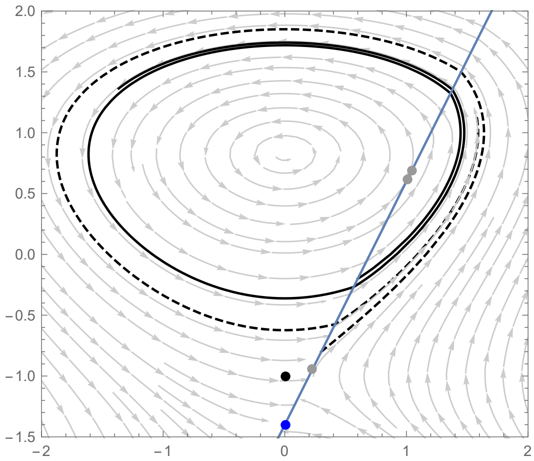

Figure 5.17: Trajectories of $Z_{a,b,c}$ with $a=b=0.5$ and $c=0.2$. The black point corresponds to the saddle P_- and the gray points are tangency points.

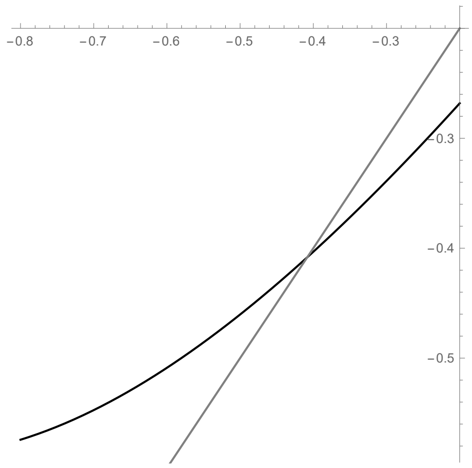

Figure 5.18: Graph of the first return map of $Z_{a,b,c}$ with $a=b=0.5$, and $c=0.2$. The origin is located at $(-0.2,-0.2)$.

if $c<0$, then for $a=b=0.5$ and $c=-0.2$, consider the points $p_0=(-0.1,-0.8)$ and $p_1=(0.1,-0.4)$ in Σ. With the values in Table 5.5 we obtain that the trajectory through p_0 returns to Σ "below" p_0 while the trajectory through p_1 returns to Σ "above" p_1. Therefore, by continuity, π has a fixed point $\bar{p}=(0.5(\bar{y}+1)-0.2,\bar{y})\in\Sigma$ with $\bar{y}\in(-0.8;-0.4)$, see Figures 5.19 and 5.20.

By taking a parametrization of Σ on the second coordinates, the sliding vector field is $Z^s_{b,c}(y)=by+b+c$. Provided that $P_0=(0,-(b+c)/b)\in\Sigma^e\cup\Sigma^s$, P_0 is the unique pseudo-equilibrium point of $Z_{a,b,c}$, which is a repellor one. Moreover, $P_0\in\Sigma^e\cup\Sigma^s$ when the saddle is virtual and the hyperbolicity ratio of the saddle point is equal to one. Despite this last fact, it is possible to see that this family is in the case DSC_{31} of the analysis in Chapter 4.

(a,b,c)	$p=(x,y)$	$\pi(p)$
$(0.5, 0.5, -0.2)$	$p_0 = (-0.1, -0.8) \in \Sigma$	$-0.737292\ldots$
$(0.5, 0.5, -0.2)$	$p_1 = (0.1, -0.4) \in \Sigma$	$-0.520851\ldots$

Table 5.5: Data for some values of the parameters of $Z_{a,b,c}$.

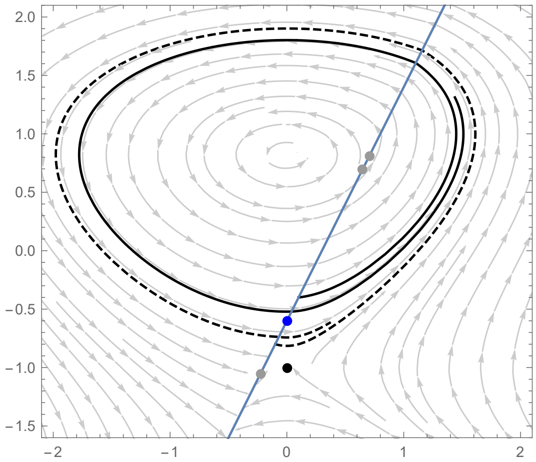

Figure 5.19: Trajectories of $Z_{a,b,c}$ with $a = b = 0.5$ and $c = -0.2$. The black point corresponds to the saddle P_- and the gray points are tangency points.

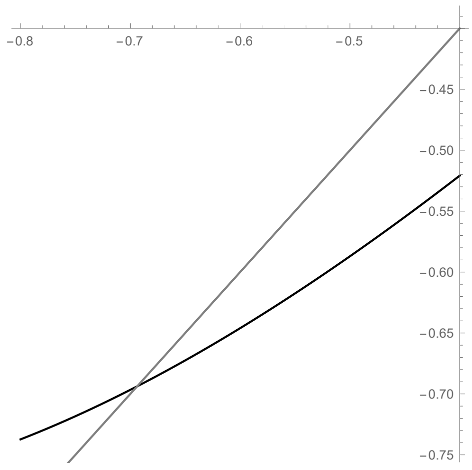

Figure 5.20: Graph of the first return map of $Z_{a,b,c}$ with $a = b = 0.5$ and $c = -0.2$. The origin is located at $(-0.4, -0.4)$.

Chapter 6

Future Work

In this chapter, we state some open questions remaining from the studies performed in this thesis as well as some related problems.

6.1 On the Dulac's Problem

The techniques used in Chapter 2 were used to study higher codimension of homoclinic connections for smooth vector fields, [35]. So, try to extend these results doing the necessary adaptations is the next step in this part of the work. Moreover, another direction is to give a step back and try to extend some results concerning with the finiteness of limit cycles in piecewise polynomial vector fields.

6.2 On Degenerate Cycles

We present two different approaches, to extend the results obtained for more complex cycles having boundary saddle points or to study a different class of typical cycles.

6.2.1 Cycles Through Hyperbolic Boundary Saddles

Based on the studies already performed some directions are the following:

- to study the first return map for saddles having hyperbolicity ratio in $\mathbb{Q}$ and to obtain the complete bifurcation diagrams for the cycle proposed in Chapter 4;
- to study the bifurcations of a degenerate cycle through a saddle-regular point by breaking some of the generic conditions imposed in Chapter 4. For instance, to admit tangency between the invariant manifolds of the saddle and the switching manifold or to allow non-hyperbolic pseudo-equilibrium points;
- to perform similar analysis for cycles having more than one singularity, i.e., having two saddle-regular points (Figure 6.1$-(a)$), a saddle-saddle point (Figure 6.1$-(b)$), a saddle-regular and a fold-regular point (Figure 6.1$-(c)$), or even a saddle-fold point (Figure 6.1$-(d)$).

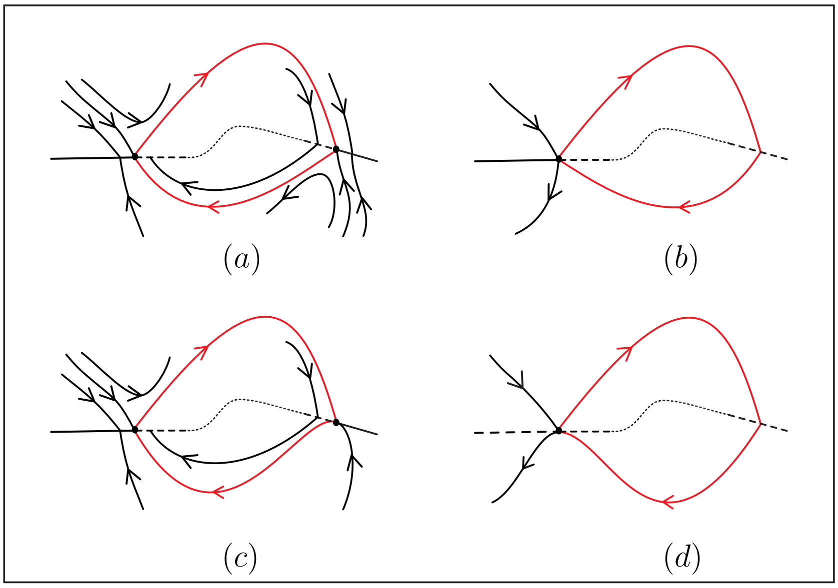

Figure 6.1: Cycle through: (a) two saddle-regular points, (b) a saddle-saddle point, (c) a saddle-regular point and a fold-regular point, and (d) a saddle-fold point.

6.2.2 Other Cycles

A different class of degenerate cycle is obtained by considering cycles through visible-invisible fold-fold points. Let $Z_0 = (X_0, Y_0) \in \Omega^r$ be a non-smooth vector field satisfying:

- $C(1)$: X_0 has a hyperbolic limit cycle γ_0 tangent to Σ at p_0 which is a visible fold point;
- $C(2)$: Y_0 has an invisible fold point at $p_0 \in \Sigma$;
- $C(3)$: $X_0(p_0)$ and $Y_0(p_0)$ point to the same direction.

Observe that $C(3)$ implies in the existence of a neighborhood V_0 of p_0 in Σ such that $V_0 - \{p_0\} \subset \Sigma^c$. In this way, we have two possibilities for Z_0 which are determined by the stability of γ_0, i.e., if γ_0 is an attractor, Figure 6.2, or if γ_0 is a repellor, Figure 6.3. This kind of cycle has already appeared in literature, [13, 24], but a proper bifurcation diagram need to be presented. Thus, we propose to study bifurcation diagrams of this kind of cycle.

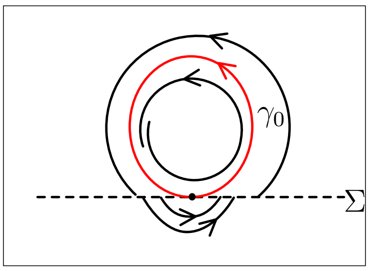

Figure 6.2: Attractor cycle.

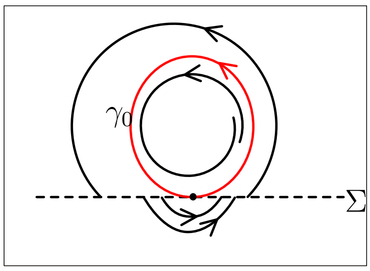

Figure 6.3: Repellor cycle.

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Appendix A

Cauchy's Inequality

The only objective of this appendix is to prove the Cauchy inequality used in Chapter 2. For a more complete study on this topic, see [1]. A key tool here is the well known Chauchy's integral formulas, i.e., if f is analytic inside and on the boundary C, of a simply connected region $R \subset \mathbb{C}$, and a is any point inside C, then:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz,$$

where C is traveled in the positive direction and $n = 0, 1, 2, \ldots$. Another important fact for line integrals, is

$$\left| \oint_C g(z) dz \right| \leq NL,$$

where $N = \sup\{|g(z)|; z \in \mathbb{C}\}$, $L =$ length of C and g is integrable along C. Now, the required result can be proved.

Lemma A.1. *If $f : U \subset \mathbb{C} \to \mathbb{C}$ is analytic inside $\overline{B(a,r)} \subset U$, then*

$$\left| f^{(n)}(a) \right| \leq \frac{M \cdot n!}{r^n}, \; n = 0, 1, 2, \ldots,$$

where $B(a,r)$ is the open ball with center a and radius $r > 0$ and $M > 0$ is a constant such that $|f(z)| \leq M$ for all $z \in \partial\overline{B(a,r)}$.

Proof. For $R = \overline{B(a,r)}$ and $C = \partial\overline{B(a,r)}$, by applying the Cauchy's integral formula we obtain

$$|f^{(n)}(a)| = \left| \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz \right| = \frac{n!}{2\pi} \left| \oint_C \frac{f(z)}{(z-a)^{n+1}} dz \right|.$$

Since $C = \{z \in \mathbb{C}; |z - a| = r\}$, the length of C is equal to $2\pi r$ and, for any $z \in C$,

$$\left| \frac{f(z)}{(z-a)^{n+1}} \right| \leq \frac{1}{r^{n+1}} \sup\{|f(z)|; z \in C\} = \frac{M}{r^{n+1}} \;\Rightarrow\; \left| \oint_C \frac{f(z)}{(z-a)^{n+1}} \right| \leq \frac{2\pi M}{r^n}.$$

Therefore, for $n = 0, 1, 2, \ldots$, we obtain $|f^{(n)}(a)| \leq \dfrac{M \cdot n!}{r^n}$. □