Douglas Duarte Novaes

Regularization and minimal sets for non–smooth dynamical systems

Regularização e conjuntos minimais para sistemas dinâmicos não suaves

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Coorientador: Jaume Llibre Saló

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Abstract

The problems discussed in this thesis focuses mainly in the theory of non-smooth differential system. Several topics of this subject are treated. The main results may be resumed as following. First, the hypotheses of the classical averaging theorems are relaxed to compute periodic solutions of non-smooth differential systems. Second, regarding planar piecewise linear differential system with two zones it is shown that oscillating the line of discontinuity several configurations of limit cycles can be obtained. In addition it is proved that for a given $n \in \mathbb{N}$ there exists a planar piecewise linear differential system with two zones having exactly n limit cycles. Moreover, using the Chebyshev theory, it is established sharp upper bounds for the maximum number of limit cycles that some classes of planar piecewise linear differential systems with two zones can have when the set of discontinuity is a straight line. Third, the concept of sliding Shilnikov orbit is introduced in the context of Filippov systems, then the Shilnikov problem is considered for this case. Finally, the recent extensions of the Filippov's conventions for solutions of discontinuous differential systems is studied and some results concerning its regularization are established. Moreover the pinching of continuous systems is studied in the context of these new conventions.

Keywords: Discontinuous vector fields, Filippov's systems, Averaging method, Singular perturbation.

Resumo

Os problemas discutidos nesta tese concentram-se principalmente na teoria dos sistemas dinâmicos não diferenciáveis, da qual vários tópicos são abordados. Os resultados principais podem ser resumidos da seguinte forma. Primeiramente, relaxa-se as hipóteses dos teoremas clássicos da teoria "averaging" para o cálculo de soluções periódicas de sistemas dinâmicos não diferenciáveis. Em segundo lugar, com relação a sistemas dinâmicos planares lineares por partes com duas zonas, mostra-se que ao oscilar a linha de descontinuidade obtém-se diferentes configurações de ciclos limite. Em particular, prova-se que para um dado $n \in \mathbb{N}$ existe um sistema dinâmico planar linear por partes com duas zonas tendo exatamente n ciclos limite. Além disso, usando a teoria de Chebyshev, fica estabelecido limites superiores ótimos para o número máximo de ciclos limites que algumas classes de sistemas dinâmicos planares lineares por partes com duas zonas podem ter quando o conjunto de descontinuidade é uma linha reta. Em terceiro lugar, introduz-se, no contexto de sistemas de Filippov, o conceito de órbita de Shilnikov deslizante e, em seguida, considera-se o problema Shilnikov para este caso. Por fim, estuda-se as recentes extensões das convenções de Filippov para soluções de sistemas dinâmicos descontínuos, obtendo-se resultados referentes a regularização e "pinching" no contexto destas novas convenções.

Palavras-chave: Campos vetoriais descontínuos, Sistemas de Filippov, Método averaging (Equações diferenciais), Perturbação singular (Matemática).

Contents

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Introduction

The main results provided in this thesis are essentially about non–smooth differential systems (both continuous and discontinuous) and can be split in three main parts: the first one, composed by chapters 1 and 2, deals with the averaging theory to compute periodic solutions of non–smooth differential systems; the second one, composed by chapters 3, 4, and 5, deals with piecewise linear differential systems; and the third one, composed only by chapter 6, deals with extensions of the Filippov's conventions for solutions of discontinuous differential systems and their regularizations. Apart of the chapters mentioned above, Chapter 7 is dedicated to present some possible directions for further investigations. In what follows a short introduction of the main results of each one of these parts is given.

Averaging theory for non–smooth differential systems

The main results of chapter 1 (Theorems A and B) are based on the papers [74, 73] and deal with nonlinear differential systems of the form

$$x'(t) = \sum_{i=0}^{k} \varepsilon^i F_i(t,x) + \varepsilon^{k+1} R(t,x,\varepsilon),$$

where $F_i : \mathbb{R}\times D \to \mathbb{R}^n$ for $i = 0, 1, \cdots, k$, and $R : \mathbb{R}\times D\times(-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are continuous functions, and T–periodic in the first variable, being D an open subset of $\mathbb{R}^n$, and ε a small parameter. For such differential systems, which there is no need to be of class $\mathcal{C}^1$, under suitable assumptions we extend the averaging theory for computing their periodic solutions.

The main results of sections 1.8.1 and 1.8.2 of chapter 1 (Theorems C and D) are based on the papers [53] and [78], respectively, and study the existence of periodic solutions of discontinuous differential systems in two different situations. First it is studied the existence of periodic solutions of discontinuous piecewise differential systems of the form

$$r' = \begin{cases} F^+(\theta, r, \varepsilon) & \text{if} \quad 0 \le \theta \le \alpha, \\ \\ F^-(\theta, r, \varepsilon) & \text{if} \quad \alpha \le \theta \le 2\pi, \end{cases}$$

where $F^{\pm}(\theta, r, \varepsilon) = \sum_{i=1}^{k} \varepsilon^i F_i^{\pm}(\theta, r) + \varepsilon^{k+1} R^{\pm}(\theta, r, \varepsilon)$ with $\theta \in \mathbb{S}^1$ and $r \in I$, where I is an open interval of $\mathbb{R}^+$. Second it is studied the existence of periodic solutions of discontinuous piecewise

differential systems of the form

$$\mathbf{x}' = \begin{cases} F^+(\theta, \mathbf{x}, \varepsilon) & \text{if} \quad 0 \le \theta \le \phi, \\ \\ F^-(\theta, \mathbf{x}, \varepsilon) & \text{if} \quad \phi \le \theta \le T, \end{cases}$$

where $F^{\pm}(\theta, \mathbf{x}, \varepsilon) = F_0^{\pm}(\theta, \mathbf{x}) + \varepsilon F_1^{\pm}(\theta, \mathbf{x}) + \varepsilon^2 F_2^{\pm}(\theta, \mathbf{x}) + \varepsilon^3 R^{\pm}(\theta, \mathbf{x}, \varepsilon)$ with $\theta \in \mathbb{S}^1$ and $\mathbf{x} \in D$, where D is an open bounded subset of $\mathbb{R}^n$. As the main hypothesis it is assumed that there exists a manifold $\mathcal{Z}$ embedded in D such that the solutions of the unperturbed system $\mathbf{x}' = F_0(\theta, \mathbf{x})$ starting in $\mathcal{Z}$ are all T–periodic functions.

The main results of chapter 2 (Theorems E, F, G, and H) are based on the papers [71, 72]. Motivated by problems coming from different areas of the applied science it is studied the periodic solutions of the following differential system

$$x'(t) = F_0(t, x) + \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 R(t, x, \varepsilon), \quad (t, x) \in \mathbb{S}^1 \times D$$

when F_0, F_1, F_2 and R are discontinuous piecewise functions, ε is a small parameter, and D is an open bounded subset of $\mathbb{R}^d$. The averaging theory is one of the best tools to attack this problem for continuous systems. Nevertheless until the works [77, 71, 72] such technique, to the best of our knowledge, was not known for studying the existence of periodic solutions of discontinuous differential system. Chapter 2 studies the above problem in two distinguished cases, namely $F_0 = 0$ and $F_0 \neq 0$.

When $F_0 = 0$, following [71], the averaging theory of first and second order is developed to study the periodic solutions of discontinuous piecewise differential systems in arbitrary dimension and with an arbitrary number of systems.

When $F_0 \neq 0$, following [72], the averaging theory of first order is developed provided that the manifold $\mathcal{Z}$ of all periodic solutions of the unperturbed system $x' = F_0(t, x)$ has dimension smaller or equal than d. In this case the theory is also developed in arbitrary dimension and with an arbitrary number of systems.

Piecewise linear differential systems

The main result of chapter 3 (Theorem I) is based on the paper [91]. Recently Braga and Mello [12] conjectured that for a given $n \in \mathbb{N}$ there is a piecewise linear system with two zones in the plane with exactly n limit cycles. In this chapter, it is proved a result from which the conjecture is an immediate consequence. Several explicit examples are given where location and stability of limit cycles are provided.

The main results of chapter 4 (Theorems J, K and L) are based on the paper [76]. They deal with the question of the determinacy of the maximum number of limit cycles for some classes of planar discontinuous piecewise linear differential systems defined in two half–planes separated by a straight line Σ. The problem is restricted to the non–sliding limit cycles case (see Figure 1), i.e.

limit cycles that do not contain any sliding segment. Among all cases treated, here it is proved that the maximum number of limit cycles is at most 2 if one of the two linear differential systems of the discontinuous piecewise linear differential system has a focus in Σ, a center, or a weak saddle. The theory of Chebyshev systems (see Appendix B) is used for establishing sharp upper bounds for the number of limit cycles. Some normal forms are also provided for these systems.

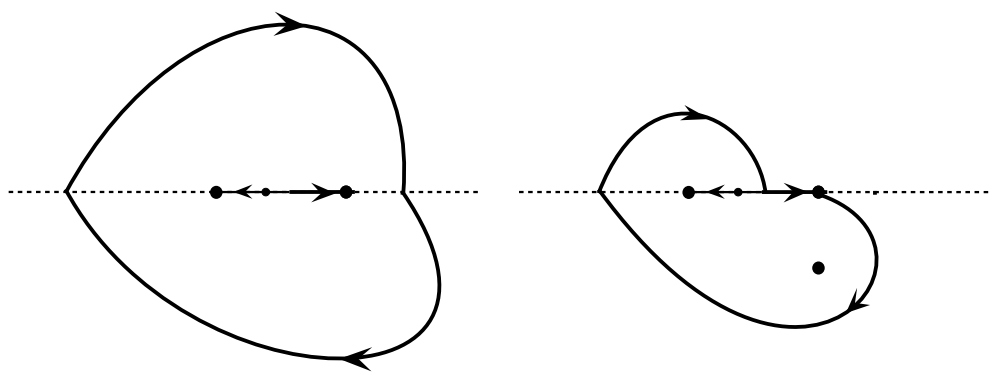

Figure 1: Left: non–sliding limit cycle, i.e. limit cycle that does not contain sliding segments. Right: sliding limit cycle, i.e limit cycle that contains a sliding segment.

The main results of chapter 5 (Theorems M, N, and 5.4.1) are based on the paper [92]. In this chapter, it is introduced the concept of a sliding Shilnikov orbit for 3D Filippov systems (see Figure 2). Versions of the Shilnikov's Theorems are provided for those systems. Specifically, it is shown that arbitrarily close to a sliding Shilnikov orbit there exist countable infinitely many sliding periodic orbits, and for a particular system having this kind of connection we investigate the existence of continuous systems close to it having an ordinary Shilnikov homoclinic orbit. It is also proved that, in general, a sliding Shilnikov orbit is a co-dimension 1 phenomenon. Furthermore a family $Z_{\alpha,\beta}$ of piecewise linear vector fields is provided as a prototype of systems having a sliding Shilnikov orbit.

Regularization of discontinuous differential systems

The main results of chapter 6 (Theorems P, Q, R, S, T, and U) are based on the paper [90]. This chapter studies the equivalence between differentiable and non-differentiable dynamics in $\mathbb{R}^n$. Filippov's theory of discontinuous differential equations allows us to find flow solutions of dynamical systems whose vector fields undergo switches at thresholds in phase space. The canonical *convex combination* at the discontinuity is only the linear part of a *nonlinear combination* that more fully explores Filippov's most general problem: the differential inclusion [34]. Here it is shown how recent works relating discontinuous systems to singular limits of continuous (or *regularized*)

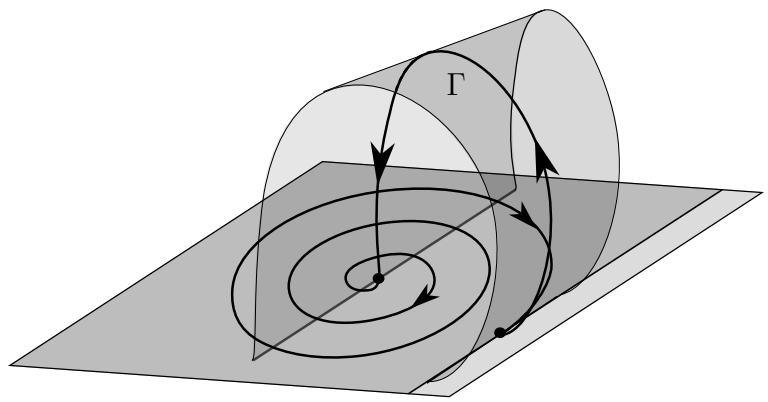

Figure 2: Sliding Shilnikov orbit Γ.

systems extends to nonlinear combinations. It is proved that if sliding occurs in a discontinuous systems, there exists a differentiable slow-fast system with equivalent slow invariant dynamics. It is also established the corresponding result for the *pinching* method (see Figures 3), a converse to regularization which approximates a smooth system by a discontinuous one.

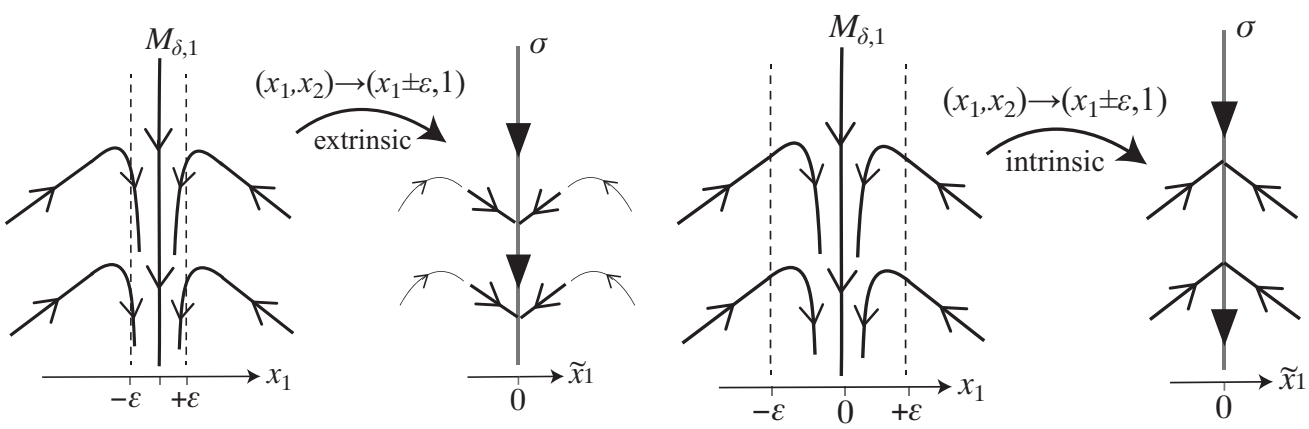

Figure 3: Two ways to pinch a continuous vector field. Left: Extrinsic pinching. Right: Intrinsic pinching

Chapter 1

Higher order averaging theory for finding periodic solutions via Brouwer degree

The main results of this chapter (Theorems A and B) are based on the papers [74, 73]. The main results of sections 1.8.1 and 1.8.2 (Theorems C and D) are based on the papers [53] and [78], respectively.

1.1 Introduction to averaging theory

The method of averaging is a classical and matured tool that allows to study the dynamics of the nonlinear *differential systems* under periodic forcing. The method of averaging has a long history that starts with the classical works of Lagrange and Laplace, who provided an intuitive justification of the method. The first formalization of this theory was done in 1928 by Fatou [32]. Important practical and theoretical contributions to the averaging theory were made in the 1930's by Bogoliubov and Krylov [9], and in 1945 by Bogoliubov [8]. In 2004, Buica and Llibre [19] extended the averaging theory for studying periodic orbits to *continuous differential systems* using the Brouwer degree. Recently a version of averaging theory for studying periodic orbits of *discontinuous differential systems* has been provided by Llibre, Novaes and Teixeira in [77]. We refer to the book of SV, Verhulst and Murdock [99] for a general introduction to this subject.

All these previous works develop the averaging theory usually up to first order in a small parameter ε, and at most up to third order. In a recent work of Giné, Grau and Llibre [40] the averaging theory for computing periodic solutions was developed to an arbitrary order in ε for analytical differential equations of one variable. An example that qualitative new phenomena can be found only when considering higher order analysis is the following. Consider arbitrary

polynomial perturbations

$$\begin{aligned} x' &= -y + \sum_{j\geq 1} \varepsilon^j f_j(x,y), \\ y' &= x + \sum_{j\geq 1} \varepsilon^j g_j(x,y), \end{aligned} \tag{1.1.1}$$

of the harmonic oscillator, where ε is a small parameter. In this differential system the polynomials f_j and g_j are of degree n in the variables x and y and the system is analytic in the variables x, y and ε. Then in [40] (see also Iliev [51]) it is proved that system (1.1.1) for $\varepsilon \neq 0$ sufficiently small has no more than $[s(n-1)/2]$ periodic solutions bifurcating from the periodic solutions of the linear center $\dot{x} = -y$, $\dot{y} = x$, using the averaging theory up to order s, and this bound can be reached. Here $[x]$ denotes the integer part function of the real number x. So, to take into account higher order averaging theory can improve qualitatively and quantitatively the results on the periodic solutions.

The goal of this chapter is to extend the averaging theory for computing periodic solutions to an arbitrary order in ε for continuous differential equations in n variables. Thus, the main theorem stated in this chapter extends the results of Buica and Llibre [19] to an arbitrary order in a small parameter ε and to an arbitrary number of variables.

1.2 Averaging theory at any order

Here we are interested in studying the existence of periodic orbits of general differential systems expressed by

$$x'(t) = \sum_{i=0}^{k} \varepsilon^i F_i(t,x) + \varepsilon^{k+1} R(t,x,\varepsilon), \tag{1.2.1}$$

where $F_i : \mathbb{R} \times D \to \mathbb{R}^n$ for $i = 1, 2, \cdots, k$, and $R : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are continuous functions, and T–periodic in the first variable, being D an open subset of $\mathbb{R}^n$.

In order to state our main results we introduce some notation. Let L be a positive integer, let $x = (x_1, \ldots, x_n) \in D$, $t \in \mathbb{R}$ and $y_j = (y_{j1}, \ldots, y_{jn}) \in \mathbb{R}^n$ for $j = 1, \ldots, L$. Given $F : \mathbb{R} \times D \to \mathbb{R}^n$ a sufficiently smooth function, for each $(t,x) \in \mathbb{R} \times D$ we denote by $\partial^L F(t,x)$ a symmetric L–multilinear map which is applied to a "product" of L vectors of $\mathbb{R}^n$, which we denote as $\bigodot_{j=1}^{L} y_j \in \mathbb{R}^{nL}$. The definition of this L–multilinear map is

$$\partial^L F(t,x) \bigodot_{j=1}^{L} y_j = \sum_{i_1,\ldots,i_L=1}^{n} \frac{\partial^L F(t,x)}{\partial x_{i_1} \cdots \partial x_{i_L}} y_{1i_1} \cdots y_{Li_L}. \tag{1.2.2}$$

We define ∂^0 as the identity functional. Given a positive integer b and a vector $y \in \mathbb{R}^n$ we also denote $y^b = \bigodot_{i=1}^{b} y \in \mathbb{R}^{nb}$.

Remark 1.2.1. The L–multilinear map defined in (1.2.2) is the Lth Fréchet derivative of the function $F(t,x)$ with respect to the variable x. Indeed, fixed $t \in \mathbb{R}$, if we consider the function $F_t : D \to \mathbb{R}^n$ such that $F_t(x) = F(t,x)$, then $\partial^L F(t,x) = F_t^{(L)}(x)$.

Example 1.2.1. To illustrate the above notation (1.2.2) we consider a smooth function $F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$. So for $x = (x_1, x_2)$ and $y^1 = (y_1^1, y_2^1)$ we have

$$\partial F(t,x)y^1 = \frac{\partial F}{\partial x_1}(t,x)y_1^1 + \frac{\partial F}{\partial x_2}(t,x)y_2^1.$$

Now, for $y^1 = (y_1^1, y_2^1)$ and $y^2 = (y_1^2, y_2^2)$ we have

$$\begin{aligned}\partial^2 F(t,x)(y^1,y^2) = \ & \frac{\partial^2 F(t,x)}{\partial x_1 \partial x_1} y_1^1 y_1^2 + \frac{\partial^2 F(t,x)}{\partial x_1 \partial x_2} y_1^1 y_2^2 \\ & + \frac{\partial^2 F(t,x)}{\partial x_2 \partial x_1} y_2^1 y_1^2 + \frac{\partial^2 F(t,x)}{\partial x_2 \partial x_2} y_2^1 y_2^2.\end{aligned}$$

Observe that for each $(t,x) \in \mathbb{R} \times D$, $\partial F(t,x)$ is a linear map in $\mathbb{R}^2$ and $\partial^2 F(t,x)$ is a bilinear map in $\mathbb{R}^2 \times \mathbb{R}^2$.

Let $\varphi(\cdot, z) : [0, t_z] \to \mathbb{R}^n$ be the solution of the unperturbed system, $x'(t) = F_0(t,x)$ such that $\varphi(0,z) = z$.

For $i = 1, 2, \ldots, k$, we define the *Averaged Function* $f_i : D \to \mathbb{R}^n$ of order i as

$$f_i(z) = \frac{y_i(T,z)}{i!}, \tag{1.2.3}$$

where $y_i : \mathbb{R} \times D \to \mathbb{R}^n$, for $i = 1, 2, \ldots, k-1$, are defined recurrently by the following integral equation

$$y_i(t,z) = i! \int_0^t \left(F_i\left(s, \varphi(s,z)\right) + \sum_{l=1}^{i} \sum_{S_l} \frac{1}{b_1!\ b_2!\ 2!^{b_2} \cdots b_l!\ l!^{b_l}} \partial^L F_{i-l}\left(s, \varphi(s,z)\right) \bigodot_{j=1}^{l} y_j(s,z)^{b_j} \right) ds, \tag{1.2.4}$$

where S_l is the set of all l-tuples of non–negative integers $(b_1, b_2, \cdots, b_l)$ satisfying $b_1 + 2b_2 + \cdots + lb_l = l$, and $L = b_1 + b_2 + \cdots + b_l$.

In section 1.5 we compute the sets S_l for $l = 1, 2, 3, 4, 5$. Furthermore, we make explicit the functions $f_k(z)$ up to $k = 5$ when $F_0 = 0$, and up to $k = 4$ when $F_0 \neq 0$.

Related to the averaging functions (1.2.3) there exist two cases of (1.2.1), essentially different, that must be treated separately. Namely, when $F_0 = 0$ and when $F_0 \neq 0$. It can be seen in the following remarks.

Remark 1.2.2. If $F_0 = 0$, then $\varphi(t,z) = z$ for each $t \in \mathbb{R}$. So

$$y_1(t,z) = \int_0^t F_1(t,z)ds, \quad \text{and} \quad f_1(t,z) = \int_0^T F_1(t,z)dt$$

as usual in averaging theory (see for instance [19]).

Remark 1.2.3. If $F_0 \neq 0$, then

$$y_1(t,z) = \int_0^t F_1\left(s, \varphi(s,z)\right) + \partial F_0\left(s, \varphi(s,z)\right) y_1(s,z) ds. \tag{1.2.5}$$

The integral equation (1.2.5) is equivalent to the following *Cauchy Problem*

$$\dot{u}(t) = F_1\left(t, \varphi(t,z)\right) + \partial F_0\left(t, \varphi(t,z)\right) u \quad \text{and} \quad u(0) = 0,$$

that is $y_1(t,z) = u(t)$. If we denote by $Y(t,z)$ the fundamental matrix of the system

$$u'(t) = \partial F_0(t, \varphi(t,z))\, u \tag{1.2.6}$$

such that $Y(0,z) = Id$ is the identity matrix, so

$$y_1(t,z) = Y(t,z) \int_0^t Y(s,z)^{-1} F_1(s, \varphi(s,z)) ds \tag{1.2.7}$$

and

$$f_1(z) = Y(T,z) \int_0^T Y(t,z)^{-1} F_1(t, \varphi(t,z)) dt.$$

Moreover, each $y_i(t,z)$ is obtained similarly from a Cauchy problem. The formulae are given explicitly in section 1.5. Later on, under hypotheses of Theorem B, it will follow that $Y(T,z) = Id$ because $Y(t,z) = D_2\varphi(t,z)$.

In the following, we state our main results: Theorem A when $F_0 = 0$, and Theorem B when $F_0 \neq 0$. The Brouwer degree d_B, which is defined in the Appendix A, is used.

Theorem A. Suppose that $F_0 = 0$. In addition, for the functions of (1.2.1), we assume the following conditions.

(i) For each $t \in \mathbb{R}$, $F_i(t,\cdot) \in \mathcal{C}^{k-i}$ for $i = 1, 2, \cdots, k$; $\partial^{k-i} F_i$ is locally Lipschitz in the second variable for $i = 1, 2, \cdots, k$; and R is a continuous function locally Lipschitz in the second variable.

(ii) Assume that $f_i = 0$ for $i = 1, 2, \ldots, r-1$ and $f_r \neq 0$ with $r \in \{1, 2, \ldots, k\}$ (here we are taking $f_0 = 0$). Moreover, suppose that for some $a \in D$ with $f_r(a) = 0$, there exists a bounded neighborhood $V \subset D$ of a such that $f_r(z) \neq 0$ for all $z \in \overline{V} \setminus \{a\}$, and that $d_B\left(f_r(z), V, 0\right) \neq 0$.

Then, for $|\varepsilon| > 0$ sufficiently small, there exists a T–periodic solution $x(\cdot, \varepsilon)$ of (1.2.1) such that $x(0,\varepsilon) \to a$ when $\varepsilon \to 0$.

Theorem B. Suppose that $F_0 \neq 0$. In addition, for the functions of (1.2.1), we assume the following conditions.

(j) There exists an open subset W of D such that for any $z \in \overline{W}$, $\varphi(t,z)$ is T–periodic in the variable t.

(jj) For each $t \in \mathbb{R}$, $F_i(t,\cdot) \in \mathcal{C}^{k-i}$ for $i = 0, 1, 2, \cdots, k$; $\partial^{k-i} F_i$ is locally Lipschitz in the second variable for $i = 0, 1, 2, \cdots, k$; and R is a continuous function locally Lipschitz in the second variable.

(jjj) Assume that $f_i = 0$ for $i = 1, 2, \ldots, r-1$ and $f_r \neq 0$ with $r \in \{1, 2, \ldots, k\}$. Moreover, suppose that for some $a \in W$ with $f_r(a) = 0$, there exists a bounded neighborhood $V \subset W$ of a such that $f_r(z) \neq 0$ for all $z \in \overline{V} \setminus \{a\}$, and that $d_B\left(f_r(z), V, 0\right) \neq 0$.

Then, for $|\varepsilon| > 0$ sufficiently small, there exists a T–periodic solution $x(\cdot, \varepsilon)$ of (1.2.1) such that $x(0, \varepsilon) \to a$ when $\varepsilon \to 0$.

Theorems A and B are proved in section 1.4.

Remark 1.2.4. When f_i for $i = 1, 2, \ldots, k$ (defined in (1.2.3)) are $\mathcal{C}^1$ functions the hypotheses (ii) and (jjj) become:

(k) Assume that $f_i = 0$ for $i = 1, 2 \ldots, r-1$ and $f_r \neq 0$ with $r \in \{1, 2, \ldots, k\}$. Moreover, suppose that for some $a \in W$ with $f_r(a) = 0$ we have that $f_r'(a) \neq 0$.

In this case, instead Brouwer degree theory, the Implicit Function Theorem could be used to prove Theorems A and B.

We emphasize that our main contribution to the advanced averaging theory is based on Theorems A and B. In fact, we provide conditions on the regularity of the functions, weaker than those given in [40].

1.3 Examples of applications

1.3.1 Application of Theorem A

Consider the following $n+2$–dimensional differential system

$$\begin{aligned} x'(t) &= \; y + \varepsilon\, F(x, y, \mathbf{z}) + \varepsilon^2\, R_F(x, y, \mathbf{z}, \varepsilon), \\ y'(t) &= \; -x + \varepsilon\, G(x, y, \mathbf{z}) + \varepsilon^2\, R_G(x, y, \mathbf{z}, \varepsilon), \\ z_i'(t) &= \; \varepsilon\, H^i(x, y, \mathbf{z}) + \varepsilon^2\, R_H^i(x, y, \mathbf{z}, \varepsilon), \end{aligned} \tag{1.3.1}$$

where $F, G, H^i : D \to \mathbb{R}$ and $R_F, R_G, R_H^i : D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ for $i = 1, 2, \ldots, n$ are $\mathcal{C}^1$ functions, $D \subset \mathbb{R}^{n+2}$ is an open subset, and $\mathbf{z} = (z_1, z_2, \cdots, z_n) \in \mathbb{R}^n$.

System (1.3.1), when $\varepsilon = 0$, is a linear oscillator having its phase portrait foliated by cylinders which are themselves foliated by periodic orbits. Theorem A allows the study of the persistence of periodic solutions when $\varepsilon \neq 0$.

Let $\mathcal{F}(x, y, \mathbf{z}) = (\mathcal{F}_0(x, y, \mathbf{z}), \mathcal{F}_1(x, y, \mathbf{z}), \cdots, \mathcal{F}_n(x, y, \mathbf{z}))$ be the function defined by

$$\begin{aligned} \mathcal{F}_0(r, \mathbf{z}) &= \int_0^{2\pi} \Big(F(r\cos\theta, r\sin\theta, \mathbf{z})\cos\theta + G(r\cos\theta, r\sin\theta, \mathbf{z})\sin\theta\Big) d\theta, \quad \text{and} \\ \mathcal{F}_i(r, \mathbf{z}) &= \int_0^{2\pi} H^i(r\cos\theta, r\sin\theta, \mathbf{z})\cos\theta d\theta, \end{aligned} \tag{1.3.2}$$

for $i = 1, 2, \ldots, n$.

Proposition 1.3.1. For each zero $(r^*, \mathbf{z}^*)$ of the system $\mathcal{F}(r, \mathbf{z}) = 0$ such that $|J\mathcal{F}(r^*, \mathbf{z}^*)| \neq 0$ there exists a periodic solution $\varphi(t, \varepsilon)$ of (1.3.1) and a point (x^*, y^*) with $|(x^*, y^*)| = r^*$ such that $\varphi(0, \varepsilon) \to (x^*, y^*, \mathbf{z}^*)$ when $\varepsilon \to 0$.

Here $|J\mathcal{F}(r^*, \mathbf{z}^*)|$ denotes the determinant of the Jacobian matrix of $\mathcal{F}$ evaluated at $(r^*, \mathbf{z}^*)$.

A proof of Proposition 1.3.1 is given in section 1.6.

1.3.2 Application of Theorem B

Consider the following non–autonomous perturbation of a linear oscillator.

$$x''(t) = -x + \varepsilon\,(a_1(t)x + b_1(t)y) + \varepsilon^2\left(a_2(t)x^2 + b_2(t)y^2\right) + \varepsilon^3 r(t, x, y, \varepsilon), \tag{1.3.3}$$

where $a_i, b_i : \mathbb{R} \to \mathbb{R}$ for $i = 1, 2$ are $\mathcal{C}^1$ functions, 2π–periodic in the variable t, and $D \subset \mathbb{R}^2$ is an open subset.

Let $\Gamma_i(t) = (a_i(t), b_i(t))$ and $\mathcal{A}_j(t) = \left(\sin^{j-1}\theta\cos^{2-j}\theta, (-1)^j\sin^{2-j}\theta\cos^{j-1}\theta\right)$ for $i, j = 1, 2$. We define the following functions

$$\begin{aligned} A_{ij}(t) = \;& (-1)^i\cos t\int_0^t \sin^{2-i}\theta\cos^{i-1}\theta\,\Gamma_1(\theta)\cdot\mathcal{A}_j(t)d\theta \\ & + \sin t\int_0^t \sin^{i-1}\theta\cos^{2-i}\theta\,\Gamma_1(\theta)\cdot\mathcal{A}_j(t)d\theta, \end{aligned} \tag{1.3.4}$$

for $i, j = 1, 2$. Here the dot denotes the inner product, that is $(u_1, u_2)\cdot(v_1, v_2) = u_1v_1 + u_2v_2$.

Now let $\mathcal{B}_j(t) = (A_{1j}(t), A_{2j}(t))$ for $j = 1, 2$ and $\mathbb{C}_{jk}(t) = \left(\cos^j t\sin^k t, (-1)^j\cos^k t\sin^j t\right)$. Similarly to (1.3.4), we define the following functions

$$\begin{aligned} B_{ij}(t) = \;& 2(-1)^i\cos t\int_0^t \sin^{2-i}\theta\cos^{i-1}\theta\,\Gamma_1(\theta)\cdot\mathcal{B}_j(t)d\theta \\ & + 2\sin t\int_0^t \sin^{i-1}\theta\cos^{2-i}\theta\,\Gamma_1(\theta)\cdot\mathcal{B}_j(t)d\theta, \end{aligned} \tag{1.3.5}$$

for $i, j = 1, 2$, and

$$\begin{aligned} C_{ijk}(t) = \;& 2^{jk+1}(-1)^i\cos t\int_0^t \sin^{2-i}\theta\cos^{i-1}\theta\,\Gamma_2(\theta)\cdot\mathbb{C}_{jk}(t)d\theta \\ & + 2^{jk+1}\sin t\int_0^t \sin^{i-1}\theta\cos^{2-i}\theta\,\Gamma_2(\theta)\cdot\mathbb{C}_{jk}(t)d\theta, \end{aligned} \tag{1.3.6}$$

for $i = 1, 2$, and $k, j = 0, 1, 2$.

Proposition 1.3.2. Denote $A(t) = (A_{ij}(t))_{ij}$, $B_{ij} = B_{ij}(2\pi)$ for $i, j = 1, 2$, and $C_{ijk} = C_{ijk}(2\pi)$ for $i = 1, 2$ and $j, k = 0, 1, 2$. Let $\mathcal{G}(x, y) = (\mathcal{G}_1(x, y), \mathcal{G}_2(x, y))$ be the function defined by

$$\begin{aligned} \mathcal{G}_1(x, y) &= B_{11}x + B_{12}y + C_{120}x^2 + C_{111}xy + C_{102}y^2, \quad \text{and} \\ \mathcal{G}_2(x, y) &= B_{21}x + B_{22}y + C_{220}x^2 + C_{211}xy + C_{202}y^2. \end{aligned} \tag{1.3.7}$$

So

(a) If $A(2\pi) \neq 0$ and $\det(A(2\pi)) \neq 0$, then for $|\varepsilon| > 0$ sufficiently small there exists a periodic solution $\varphi(t,\varepsilon)$ of (1.3.3) such that $\varphi(t,\varepsilon) \to 0$ (constant 0 solution) when $\varepsilon \to 0$.

(b) If $A(2\pi) = 0$, then for each zero (x^*, y^*) of the system $\mathcal{G}(x,y) = 0$ such that $|J\mathcal{G}(x^*,y^*)| \neq 0$ there exists, for $|\varepsilon| > 0$ sufficiently small, a periodic solution $\varphi(t,\varepsilon)$ of (1.3.3) such that $\varphi(0,\varepsilon) \to (x^*, y^*)$ when $\varepsilon \to 0$.

Here $|J\mathcal{G}(x^*,y^*)|$ denotes the determinant of the Jacobian matrix of $\mathcal{G}$ evaluated at (x^*, y^*).
A proof of Proposition 1.3.2 is given in section 1.6.
Now assume that

$$a_1(t) = 11\sin t - 16\sin^3 t, \quad b_1(t) = 5\cos t - 4\cos^4 t, \quad a_2(t) = \sin^2 t, \quad b_2(t) = \cos^2 t,$$

and $r(t,x,y,\varepsilon) = a_3(t)x^3 + b_3(t)y^3 + \varepsilon\tilde{r}(t,x,y,\varepsilon)$.

Proposition 1.3.3. Let $\mathcal{H}(x,y) = (\mathcal{H}_1(x,y), \mathcal{H}_2(x,y))$ be the function defined as

$$\begin{aligned}
\mathcal{H}_1(x,y) = \; & \frac{3\pi}{4}x^2 - \frac{63\pi}{4}y^2 - 6x^3\int_0^{2\pi}\Big(a_3(\theta)\sin\theta\cos^3\theta - b_3(\theta)\sin^4\theta\Big) \\
& -6y^3\int_0^{2\pi}\Big(a_3(\theta)\sin^4\theta + b_3(\theta)\sin\theta\cos^3\theta\Big) \\
& -18x^2y\int_0^{2\pi}\sin^2\theta\cos\theta\,(a_3(\theta)\cos\theta + b_3(\theta)\sin\theta) \\
& -18xy^2\int_0^{2\pi}\sin^2\theta\cos\theta\,(a_3(\theta)\sin\theta - b_3(\theta)\cos\theta)\,, \\
\mathcal{H}_2(x,y) = \; & \frac{149\pi}{48}x^2 - \frac{53\pi}{10}xy + 6x^3\int_0^{2\pi}\cos\theta\Big(a_3(\theta)\cos^3\theta - b_3(\theta)\sin^3\Big) \\
& +6y^3\int_0^{2\pi}\cos\theta\Big(a_3(\theta)\sin^3\theta + b_3(\theta)\cos^3\theta\Big) \\
& +18x^2y\int_0^{2\pi}\sin\theta\cos^2\theta\,(a_3(\theta)\cos\theta + b_3(\theta)\sin\theta) \\
& +18xy^2\int_0^{2\pi}\sin\theta\cos^2\theta\,(a_3(\theta)\sin\theta - b_3(\theta)\cos\theta)\,.
\end{aligned}$$

So for each zero (x^*, y^*) of the system $\mathcal{H}(x,y) = 0$ such that $|J\mathcal{H}(x^*,y^*)| \neq 0$ there exists a periodic solution $\varphi(t,\varepsilon)$ of (1.3.3) such that $\varphi(0,\varepsilon) \to (x^*, y^*)$ when $\varepsilon \to 0$.

Again $|J\mathcal{H}(x^*,y^*)|$ denotes the determinant of the Jacobian matrix of $\mathcal{H}$ evaluated at (x^*, y^*).
A proof of Proposition 1.3.3 is given in section 1.6.

1.4 Proofs of main results

Let $g : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ be a function defined on a small interval $(-\varepsilon_0, \varepsilon_0)$. We say that $g(\varepsilon) = \mathcal{O}(\varepsilon^\ell)$ for some positive integer ℓ if there exists constants $\varepsilon_1 > 0$ and $M > 0$ such that

$|g(\varepsilon)| \leq M|\varepsilon^\ell|$ for $-\varepsilon_1 < \varepsilon < \varepsilon_1$. Here $|\cdot|$ denotes the usual norm in the Euclidean space $\mathbb{R}^n$ for $n \geq 1$. The symbol $\mathcal{O}$ is one of the *Landau's symbols* (see for instance [99]).

To prove Theorems A and B we need the following lemma.

Lemma 1.4.1 (Fundamental Lemma)**.** Let $x(\cdot, z, \varepsilon) : [0, t_z) \to \mathbb{R}^n$ be the solution of (1.2.1) with $x(0, z, \varepsilon) = z$, and assume the hypothesis (jj) of Theorem B. If $t_z > T$, then

$$x(t, z, \varepsilon) = \varphi(t, z) + \sum_{i=1}^{k} \varepsilon^i \frac{y_i(t, z)}{i!} + \varepsilon^{k+1}\mathcal{O}(1),$$

where $y_i(t, z)$ for $i = 1, 2, \ldots, k$ are defined in (1.2.4).

Proof. By continuity of the solution $x(t, z, \varepsilon)$ and by compactness of the set $[0, T] \times \overline{V} \times [-\varepsilon_1, \varepsilon_1]$, there exits a compact subset K of D such that $x(t, z, \varepsilon) \in K$ for all $t \in [0, T]$, $z \in \overline{V}$ and $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$. Now, by the continuity of the function R, $|R(s, x(s, z, \varepsilon), \varepsilon)| \leq \max\{|R(t, x, \varepsilon)|, (t, x, \varepsilon) \in [0, T] \times K \times [-\varepsilon_1, \varepsilon_1]\} = N$. Then

$$\left| \int_0^t R(s, x(s, z, \varepsilon), \varepsilon) ds \right| \leq \int_0^T |R(s, x(s, z, \varepsilon), \varepsilon)| \, ds = TN,$$

which implies that

$$\int_0^t R(s, x(s, z, \varepsilon), \varepsilon) ds = \mathcal{O}(1). \tag{1.4.1}$$

Related to the functions $x(t, z, \varepsilon)$ and $\varphi(t, z)$ we have the followings equalities

$$\begin{aligned} x(t, z, \varepsilon) &= z + \sum_{i=0}^{k} \varepsilon^i \int_0^t F_i(s, x(s, z, \varepsilon)) ds + \mathcal{O}(\varepsilon^{k+1}), \quad \text{and} \\ \varphi(t, z) &= z + \int_0^t F_0(s, \varphi(s, z)) ds. \end{aligned} \tag{1.4.2}$$

Moreover $x(t, z, \varepsilon) = \varphi(t, z) + \mathcal{O}(\varepsilon)$. Indeed, F_0 is locally Lipschitz in the second variable, so from the compactness of the set $[0, T] \times \overline{V} \times [-\varepsilon_0, \varepsilon_0]$ and from (1.4.2) it follows

$$\begin{aligned} |x(t, z, \varepsilon) - \varphi(t, z)| \leq & \int_0^t |F_0(s, x(s, z, \varepsilon)) - F_0(s, \varphi(s, z))| ds + |\varepsilon| \int_0^t |F_1(s, x(s, z, \varepsilon))| ds \\ & + \mathcal{O}(\varepsilon^2) \\ \leq & \ |\varepsilon| M + \int_0^t L_0 |x(s, z, \varepsilon) - \varphi(s, z)| ds < |\varepsilon| M e^{TL_0}. \end{aligned}$$

Here L_0 is the Lipschitz constant of F_0 on the compact K. The first and second inequality was obtained similarly to (1.4.1). The last inequality is a consequence of Gronwall Lemma (see, for example, Lemma 1.3.1 of [99]).

In order to prove the present lemma we need the following claim.

Claim 1.4.1. For some positive integer m let $G : \mathbb{R} \times D \to \mathbb{R}^n$ be a $\mathcal{C}^m$ function. Then

$$\begin{aligned} G(t, x(t,z,\varepsilon)) = & \int_0^1 \lambda_1^{m-1} \int_0^1 \lambda_2^{m-2} \cdots \int_0^1 \lambda_{m-1} \int_0^1 \Big[\partial^m G(t, \ell_m \circ \ell_{m-1} \circ \cdots \circ \ell_1(x(t,z,\varepsilon))) \\ & -\partial^m G(t, \varphi(t,z))\Big] d\lambda_m d\lambda_{m-1} \cdots d\lambda_1 \cdot (x(t,z,\varepsilon) - \varphi(t,z))^m \\ & + \sum_{L=0}^{m} \partial^L G(t, \varphi(t,z)) \frac{(x(t,z,\varepsilon) - \varphi(t,z))^L}{L!}, \end{aligned}$$

where $\ell_i(v) = \lambda_i v + (1-\lambda_i)\varphi(t,z)$ for $v \in \mathbb{R}^n$.

We shall prove this claim using the principle of finite induction on m.

For $m = 1$, $G \in \mathcal{C}^1$. Let $\mathcal{L}_1(\lambda_1) = G(t, \ell_1(x(t,z,\varepsilon)))$. So

$$\begin{aligned} G(t, x(t,z,\varepsilon)) = & \; G(t, \varphi(t,z)) + \mathcal{L}_1(1) - \mathcal{L}_1(0) = G(t, \varphi(t,z)) + \int_0^1 \mathcal{L}_1'(\lambda_1) d\lambda_1 \\ = & \; G(t, \varphi(t,z)) + \int_0^1 \partial G(t, \ell_1(x(t,z,\varepsilon))) d\lambda_1 \cdot (x(t,z,\varepsilon) - \varphi(t,z)) \\ = & \; \int_0^1 \Big[\partial G(t, \ell_1(x(t,z,\varepsilon))) - \partial G(t, \varphi(t,z))\Big] d\lambda_1 \cdot (x(t,z,\varepsilon) - \varphi(t,z)) \\ & + G(t, \varphi(t,z)) + \partial G(t, \varphi(t,z))(x(t,z,\varepsilon) - \varphi(t,z)). \end{aligned}$$

Given an integer $\overline{k} > 1$ we assume as the *inductive hypothesis* $(I1)$ that the claim is true for $m = \overline{k} - 1$.

Now for $m = \overline{k}$, $G \in \mathcal{C}^{\overline{k}} \subset \mathcal{C}^{\overline{k}-1}$. So from inductive hypothesis $(I1)$,

$$\begin{aligned} G(t, x(t,z,\varepsilon)) = & \int_0^1 \lambda_1^{\overline{k}-2} \int_0^1 \lambda_2^{\overline{k}-3} \cdots \int_0^1 \lambda_{\overline{k}-2} \int_0^1 \Big[\partial^{\overline{k}-1} G(t, \ell_{\overline{k}-1} \circ \ell_{\overline{k}-2} \circ \cdots \\ & \circ \ell_1(x(t,z,\varepsilon))) - \partial^{\overline{k}-1} G(t, \varphi(t,z))\Big] d\lambda_{\overline{k}-1} d\lambda_{\overline{k}-2} \cdots d\lambda_1 \\ & \cdot (x(t,z,\varepsilon) - \varphi(t,z))^{\overline{k}-1} + \sum_{L=0}^{\overline{k}-1} \partial^L G(t, \varphi(t,z)) \frac{(x(t,z,\varepsilon) - \varphi(t,z))^L}{L!}. \end{aligned} \tag{1.4.3}$$

Let $\mathcal{L}(\lambda_{\overline{k}}) = \partial^{\overline{k}-1} G(t, \ell_{\overline{k}} \circ \ell_{\overline{k}-1} \circ \cdots \circ \ell_1(x(t,z,\varepsilon)))$. So

$$\begin{aligned} \int_0^1 \mathcal{L}'(\lambda_{\overline{k}}) d\lambda_{\overline{k}} = & \; \mathcal{L}(1) - \mathcal{L}(0) \\ = & \; \partial^{\overline{k}-1} G(t, \ell_{\overline{k}-1} \circ \ell_{\overline{k}-2} \circ \cdots \circ \ell_1(x(t,z,\varepsilon))) - \partial^m G(t, \varphi(t,z)). \end{aligned}$$

The derivative of $\mathcal{L}(\lambda_{\overline{k}})$ can be easily obtained as

$$\mathcal{L}'(\lambda_{\overline{k}}) = \lambda_{\overline{k}-1} \lambda_{\overline{k}-2} \cdots \lambda_1 \partial^{\overline{k}} G(t, \ell_{\overline{k}} \circ \ell_{\overline{k}-1} \circ \cdots \circ \ell_1(x(t,z,\varepsilon)))(x(t,z,\varepsilon) - \varphi(t,z)).$$

So

$$\begin{aligned}\int_0^1 \mathcal{L}'(\lambda_{\overline{k}})d\lambda_{\overline{k}} = \;& \lambda_{\overline{k}-1}\lambda_{\overline{k}-2}\cdots\lambda_1\int_0^1 \Big[\partial^{\overline{k}}G(t,\ell_{\overline{k}}\circ\ell_{\overline{k}-1}\circ\cdots\circ\ \ell_1(x(t,z,\varepsilon)))\\ & -\partial^{\overline{k}}G(t,\varphi(t,z))\Big]d\lambda_{\overline{k}}\cdot(x(t,z,\varepsilon)-\varphi(t,z))\\ & +\lambda_{\overline{k}-1}\lambda_{\overline{k}-2}\cdots\lambda_1\partial^{\overline{k}}G(t,\varphi(t,z))(x(t,z,\varepsilon)-\varphi(t,z)).\end{aligned} \tag{1.4.4}$$

Hence, from (1.4.3) and (1.4.4) we conclude that

$$\begin{aligned}G(t,x(t,z,\varepsilon)) = \;& \int_0^1 \lambda_1^{\overline{k}-1}\int_0^1 \lambda_2^{\overline{k}-2}\cdots\int_0^1 \lambda_{\overline{k}-1}\int_0^1 \Big[\partial^{\overline{k}}G(t,\ell_{\overline{k}}\circ\ell_{\overline{k}-1}\circ\cdots\circ\ell_1(x(t,z,\varepsilon)))\\ & -\partial^{\overline{k}}G(t,\varphi(t,z))\Big]d\lambda_{\overline{k}}d\lambda_{\overline{k}-1}\cdots d\lambda_1\cdot(x(t,z,\varepsilon)-\varphi(t,z))^{\overline{k}}\\ & +\sum_{L=0}^{\overline{k}}\partial^L G(t,\varphi(t,z))\frac{(x(t,z,\varepsilon)-\varphi(t,z))^L}{L!}.\end{aligned}$$

This completes the proof of the claim.

Given a non–negative integer m, we note that for a $\mathcal{C}^m$ function G such that $\partial^m G$ is locally Lipschitz in the second variable, the claim implies the following equality

$$G(t,x(t,z,\varepsilon)) = \sum_{L=0}^{m}\partial^L G(t,\varphi(t,z))\frac{(x(t,z,\varepsilon)-\varphi(t,z))^L}{L!} + \mathcal{O}(\varepsilon^{m+1}). \tag{1.4.5}$$

Indeed, for $m=0$ G is a continuous function locally Lipschitz in the second variable, so

$$|G(t,x(t,z,\varepsilon)) - G(t,\varphi(t,z))| \leq \;\; L_G|x(t,z,\varepsilon)-\varphi(t,z)| < |\varepsilon|L_G M e^{TL_0}.$$

Here L_G is the Lipschitz constant of the function G on the compact K. Thus

$$G(t,x(t,z,\varepsilon)) = G(t,\varphi(t,z)) + \mathcal{O}(\varepsilon).$$

Moreover for $m \geq 1$ the claim implies (1.4.5) in an similar way to (1.4.1).

Again we shall use the principle of finite induction, now on k, to prove the present lemma.

For $k=1$, $F_0 \in \mathbb{C}^1$ and the functions ∂F_0 and F_1 are locally Lipschitz in the second variable. Thus from (1.4.5), taking $G = F_0$ and $G = F_1$, we obtain

$$\begin{aligned}& F_0(t,x(t,z,\varepsilon)) = F_0(t,\varphi(t,z)) + \partial F_0(t,\varphi(t,z))(x(t,z,\varepsilon)-\varphi(t,z)) + \mathcal{O}(\varepsilon^2) \quad \text{and}\\ & F_1(t,x(t,z,\varepsilon)) = F_1(t,\varphi(t,z)) + \mathcal{O}(\varepsilon),\end{aligned} \tag{1.4.6}$$

respectively. From (1.4.2) and (1.4.6) we compute

$$\frac{d}{dt}\left(x(t,z,\varepsilon)-\varphi(t,z)\right) = \partial F_0(t,\varphi(t,z))\left(x(t,z,\varepsilon)-\varphi(t,z)\right) + \varepsilon F_1(t,\varphi(t,z)) + \mathcal{O}(\varepsilon^2).$$

Solving the linear differential equation (1.4.6) with respect to $x(t,z,\varepsilon)-\varphi(t,z)$ for the initial condition $x(0,z,\varepsilon)-\varphi(0,z,\varepsilon)=0$ and comparing the solution with (1.2.7) we conclude that

$$x(t,z,\varepsilon)=\varphi(t,z)+\varepsilon y_1(t,z)+\mathcal{O}(\varepsilon^2).$$

Given an integer $\overline{k}$ we assume as the *inductive hypothesis* $(I2)$ that the lemma is true for $k=\overline{k}-1$.

Now for $k=\overline{k}$, $F_i=\mathbb{C}^{\overline{k}-i}$ for $i=0,1,\ldots,\overline{k}$ and $\partial^{\overline{k}-i}F_i$ is locally Lipschitz in the second variable for $i=0,1,\ldots,\overline{k}$. So from (1.4.5)

$$F_i(t,x(t,z,\varepsilon))=\sum_{L=0}^{\overline{k}-i}\partial^L F_i(t,\varphi(t,z))\frac{(x(t,z,\varepsilon)-\varphi(t,z))^L}{L!}+\mathcal{O}(\varepsilon^{\overline{k}-i+1}), \tag{1.4.7}$$

for $i=0,1,\ldots,\overline{k}$.

Applying the inductive hypothesis $(I2)$ in (1.4.7) we get

$$F_i(t,x(t,z,\varepsilon))=F_1(t,\varphi(t,z))+\sum_{L=1}^{\overline{k}-i}\partial^L F_i(t,\varphi(t,z))\left(\sum_{i=1}^{\overline{k}-i-L+1}\varepsilon^i\frac{y_i(t,z)}{i!}\right)^L+\mathcal{O}(\varepsilon^{\overline{k}-i+1}) \tag{1.4.8}$$

for $i=1,2,\ldots,\overline{k}$. Now using the *Multinomial Theorem* (see for instance [47], p. 186) in (1.4.8) we obtain

$$\begin{aligned}F_i(t,x(t,z,\varepsilon))=\;& F_i\left(t,\varphi(t,z)\right)\\ &+\sum_{L=1}^{\overline{k}-i}\sum_{l=L}^{\overline{k}-i}\sum_{S_{l,L}^{\overline{k}-1}}\frac{\varepsilon^l}{b_1!\;b_2!\;2!^{b_2}\cdots b_{\overline{k}-1}!\,(\overline{k}-1)!^{b_{\overline{k}-1}}}\partial^L F_i\left(t,\varphi(t,z)\right)\bigodot_{j=1}^{\overline{k}-1}y_j(t,z)^{b_j}\\ &+\mathcal{O}(\varepsilon^{\overline{k}-i+1}),\end{aligned}$$

for $i=1,2,\ldots,\overline{k}$. Here $S_{l,L}^n$ is the set of all n–tuples of non–negative integers $(b_1,b_2,\ldots,b_n)$ satisfying $b_1+2b_2+\cdots+nb_n=l$ and $b_1+b_2+\cdots+b_n=L$. We note that if $n>l$ then $b_{l+1}=b_{l+2}=\cdots=b_n=0$. Hence

$$\begin{aligned}F_i(t,x(t,z,\varepsilon))=\;& F_i\left(t,\varphi(t,z)\right)\\ &+\sum_{L=1}^{\overline{k}-i}\sum_{l=L}^{\overline{k}-i}\sum_{S_{l,L}^{l}}\frac{\varepsilon^l}{b_1!\;b_2!\;2!^{b_2}\cdots b_l!\;l!^{b_l}}\partial^L F_i\left(t,\varphi(t,z)\right)\bigodot_{j=1}^{l}y_j(t,z)^{b_j}\\ &+\mathcal{O}(\varepsilon^{\overline{k}-i+1}).\end{aligned} \tag{1.4.9}$$

for $i=1,2,\ldots,\overline{k}$, because $\overline{k}-i\geq l$

Finally, doing a change of indexes in (1.4.9) and observing that $\cup_{L=1}^{l} S_{l,L}^{l} = S_l$, we may write

$$\begin{aligned} F_i(t, x(t,z,\varepsilon)) = \; & F_i\left(t, \varphi(t,z)\right) \\ & + \sum_{l=1}^{\overline{k}-i} \varepsilon^l \sum_{S_l} \frac{1}{b_1!\; b_2!\, 2!^{b_2} \cdots b_l!\, l!^{b_l}} \partial^L F_i\left(t, \varphi(t,z)\right) \bigodot_{j=1}^{l} y_j(t,z)^{b_j} \\ & + \mathcal{O}(\varepsilon^{\overline{k}-i+1}), \end{aligned} \tag{1.4.10}$$

for $i = 1, 2, \ldots, \overline{k}$.

Following the above steps we also obtain

$$\begin{aligned} F_0(t, x(t,z,\varepsilon)) = \; & F_0\left(t, \varphi(t,z)\right) + \partial F_0(t, \varphi(t,z))(x(t,z,\varepsilon) - \varphi(t,z)) \\ & + \sum_{i=1}^{\overline{k}} \varepsilon^i \Big[\sum_{S_i} \frac{1}{b_1!\; b_2!\, 2!^{b_2} \cdots b_i!\, i!^{b_r}} \partial^L F_0\left(t, \varphi(t,z)\right) \bigodot_{j=1}^{i} y_j(t,z)^{b_j} \\ & - \partial F_0(t, \varphi(t,z)) \frac{y_i(t,z)}{i!} \Big] + \mathcal{O}(\varepsilon^{\overline{k}+1}). \end{aligned} \tag{1.4.11}$$

Now from (1.4.2) we compute

$$\frac{d}{dt}\left(x(t,z,\varepsilon) - \varphi(t,z)\right) = F_0(t, x(t,z,\varepsilon)) - F_0(t, \varphi(t,z)) + \sum_{i=1}^{\overline{k}} \varepsilon^i F_i(t, x(t,z,\varepsilon)) + \mathcal{O}(\varepsilon^{\overline{k}+1}). \tag{1.4.12}$$

Proceeding with a change of index we obtain from (1.4.10) that

$$\begin{aligned} \sum_{i=1}^{\overline{k}} \varepsilon^i F_i(t, x(t,z,\varepsilon)) = \; & \sum_{i=1}^{\overline{k}} \varepsilon^i \sum_{l=0}^{i-1} \sum_{S_l} \frac{1}{b_1!\; b_2!\, 2!^{b_2} \cdots b_l!\, l!^{b_l}} \partial^L F_{i-l}\left(t, \varphi(t,z)\right) \\ & \bigodot_{j=1}^{l} y_j(t,z)^{b_j} + \mathcal{O}(\varepsilon^{\overline{k}+1}). \end{aligned} \tag{1.4.13}$$

Substituting (1.4.11) and (1.4.13) in (1.4.12) we conclude that

$$\begin{aligned} \frac{d}{dt}\left(x(t,z,\varepsilon) - \varphi(t,z)\right) = \; & \partial F_0(t, \varphi(t,z))\left(x(t,z,\varepsilon) - \varphi(t,z)\right) \\ & + \sum_{i=1}^{\overline{k}} \varepsilon^i \Bigg[\sum_{l=0}^{i} \sum_{S_l} \frac{1}{b_1!\; b_2!\, 2!^{b_2} \cdots b_l!\, l!^{b_l}} \partial^L F_{i-l}\left(t, \varphi(t,z)\right) \\ & \bigodot_{j=1}^{l} y_j(s,z)^{b_j} - \partial F_0(t, \varphi(t,z)) \frac{y_i(t,z)}{i!} \Bigg] + \mathcal{O}(\varepsilon^{\overline{k}+1}). \end{aligned} \tag{1.4.14}$$

Solving the linear differential equation (1.4.14) with respect to $x(t,z,\varepsilon) - \varphi(t,z)$ for the initial condition $x(0,z,\varepsilon) - \varphi(0,z) = 0$ we obtain

$$x(t,z,\varepsilon) = \varphi(t,z) + \sum_{i=1}^{\overline{k}} \varepsilon^i \frac{Y_i(t,z)}{i!} + \mathcal{O}(\varepsilon^{\overline{k}+1}),$$

where

$$Y_i(t,z) = \; Y(t,z)\int_0^t Y(s,z)^{-1}\Bigg[\sum_{l=0}^{i}\sum_{S_l}\frac{i!}{b_1!\;b_2!\;2!^{b_2}\cdots b_l!\;l!^{b_l}}\partial^L F_{i-l}\left(s,\varphi(s,z)\right)\bigodot_{j=1}^{l} y_j(s,z)^{b_j}$$
$$-\partial F_0(s,\varphi(s,z))y_i(s,z)\Bigg]ds.$$

The function $Y(t,z)$ was defined in (1.2.6). Hence

$$\frac{d}{dt}Y_i(t,z) = \; \partial F_0(t,\varphi(t,z))Y_i(t,z) + \sum_{l=0}^{i}\sum_{S_l}\frac{i!}{b_1!\;b_2!\;2!^{b_2}\cdots b_l!\;l!^{b_l}}\partial^L F_{i-l}\left(t,\varphi(t,z)\right)\bigodot_{j=1}^{l} y_j(t,z)^{b_j}$$
$$-\partial F_0(t,\varphi(t,z))y_i(t,z)ds.$$

Computing the derivative of the function $y_i(t,z)$ we conclude that the functions $y_i(t,z)$ and $Y_i(t,z)$ are defined by the same differential equation. Since $Y_i(0,z) = y_i(0,z) = 0$ it follows that $Y_r(t,z) \equiv y_r(t,z)$ for every $i = 1,2,\ldots,\overline{k}$. So we have concluded the induction, which completes the proof of the lemma. □

1.4.1 Proof of Theorem A

In few words the proof of Theorem A is an application of the Brouwer degree (see Appendix A) to the approximated solution given by Lemma 1.4.1.

Proof of Theorem A. Let $x(\cdot,z,\varepsilon)$ be a solution of (1.2.1) such that $x(0,z,\varepsilon) = z$. For each $z \in \overline{V}$, there exists $\varepsilon_1 > 0$ such that if $\varepsilon \in [-\varepsilon_1,\varepsilon_1]$ then $x(\cdot,z,\varepsilon)$ is defined in $[0,T]$. Indeed, by the *Existence and Uniqueness Theorem* of solutions (see, for example, Theorem 1.2.4 of [99]), $x(\cdot,z,\varepsilon)$ is defined for all $0 \le t \le \inf\left(T, d/M(\varepsilon)\right)$, where

$$M(\varepsilon) \ge \left|\sum_{i=1}^{k}\varepsilon^i F_i(t,x) + \varepsilon^{k+1}R(t,x,\varepsilon)\right|$$

for all $t \in [0,T]$, for each x with $|x-z| < d$ and for every $z \in \overline{V}$. When ε is sufficiently small we can take $d/M(\varepsilon)$ sufficiently large in order that $\inf\left(T, d/M(\varepsilon)\right) = T$ for all $z \in \overline{V}$.

We denote

$$\varepsilon f(z,\varepsilon) = x(T,z,\varepsilon) - z.$$

From Lemma 1.4.1 and equation (1.4.1) we have that

$$f(z,\varepsilon) = f_1(z) + \varepsilon f_2(z) + \varepsilon^2 f_3(z) + \cdots + \varepsilon^{k-1}f_k(z) + \varepsilon^k\mathcal{O}(1),$$

where the function f_i is the one defined in (1.2.3) for $i = 1,2,\cdots,k$. From the assumption (ii) of the theorem we have that

$$f(z,\varepsilon) = \varepsilon^{r-1}f_r(z) + \cdots + \varepsilon^{k-1}f_k(z) + \varepsilon^k\mathcal{O}(1),$$

Clearly $x(\cdot, z, \varepsilon)$ is a T–periodic solution if and only if $f(z, \varepsilon) = 0$, because $x(t, z, \varepsilon)$ is defined for all $t \in [0, T]$.

From the Brouwer degree theory (see Lemma A.0.1 of Appendix A) and hypothesis (ii) we have for $|\varepsilon| > 0$ sufficiently small that

$$d_B\left(f_r(z), V, 0\right) = d_B\left(f(z, \varepsilon), V, 0\right) \neq 0.$$

Hence, by item (i) of Proposition A.0.1 (see Appendix A), $0 \in f(V, \varepsilon)$ for $|\varepsilon| > 0$ sufficiently small, that is there exists $a_\varepsilon \in V$ such that $f(a_\varepsilon, \varepsilon) = 0$.

Therefore, for $|\varepsilon| > 0$ sufficiently small, $x(t, a_\varepsilon, \varepsilon)$ is a periodic solution of (1.2.1). Clearly we can choose a_ε such that $a_\varepsilon \to a$ when $\varepsilon \to 0$, because $f(z, \varepsilon) \neq 0$ in $V \setminus \{a\}$. This completes the proof of the theorem. □

1.4.2 Proof of Theorem B

For proving Theorem B we also need the following lemma.

Lemma 1.4.2. Let $w(\cdot, z, \varepsilon) : [0, \check{t}_z] \to \mathbb{R}^n$ be the solution of the system

$$w'(t) = \sum_{i=1}^{k} \varepsilon^i \left(\left[D_2\varphi(t, w)\right]^{-1} F_i(t, \varphi(t, w)) \right) + \varepsilon^{k+1} \left[D_2\varphi(t, w)\right]^{-1} R(t, \varphi(t, w), \varepsilon), \tag{1.4.15}$$

such that $w(0, z, \varepsilon) = z$. Then $\psi(\cdot, z, \varepsilon) : [0, \tilde{t}_z] \to \mathbb{R}^n$ defined as $\psi(t, z, \varepsilon) = \varphi\left(t, w(t, z, \varepsilon)\right)$ is the solution of (1.2.1) such that $\psi(0, z, \varepsilon) = z$.

Proof. Given $z \in D$, let $M(t) = D_2\varphi(t, z)$. The result about differentiable dependence on initial conditions implies that the function $M(t)$ is given as the fundamental matrix of the differential equation $u' = \partial F_0(t, \varphi(t, z))u$. So the matrix $M(t)$ is invertible for each $t \in [0, T]$. From here, the proof follows immediately from the derivative of $\psi(t, \xi, \varepsilon)$ with respect to t. □

Proof of Theorem B. Let $x(\cdot, z, \varepsilon)$ be a solution of (1.2.1) such that $x(0, z, \varepsilon) = z$. For each $z \in \overline{V}$, there exists $\varepsilon_1 > 0$ such that if $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$ then $x(\cdot, z, \varepsilon)$ is defined in $[0, T]$. Indeed, from Lemma 1.4.2, $x(t, z, \varepsilon) = \varphi\left(t, w(t, z, \varepsilon)\right)$ for each $z \in \overline{V}$, where $w(\cdot, z, \varepsilon)$ is the solution of (1.4.15). Moreover for $|\varepsilon_1| > 0$ sufficiently small, $w(t, z, \varepsilon) \in W$ for each $(t, z, \varepsilon) \in [0, T] \times \overline{V} \times [-\varepsilon_1, \varepsilon_1]$. Repeating the argument of the proof of Theorem A we can show that $\check{t}_z = T$ for every $z \in \overline{V}$. Since $\varphi(\cdot, z)$ is defined in [0,T] for every $z \in W$, it follows that $\tilde{t}_z = T$, that is $x(\cdot, z, \varepsilon)$ is also defined in $[0, T]$.

From here the proof follows similarly of Theorem A. □

1.5 Computing formulae

In this section we illustrate how to compute the formulae of Theorems A and B for some $k \in \mathbb{N}$. In 3.1 we compute the formulae when $F_0 = 0$ for Theorem A up to $k = 5$. In 3.2 we compute the formulae when $F_0 \neq 0$ for Theorem B up to $k = 4$.

First of all from (1.2.4) we should determine the sets S_l for $l = 1, 2, 3, 4, 5$.

$$S_1 = \{1\},$$

$$S_2 = \{(0,1), (2,0)\},$$

$$S_3 = \{(0,0,1), (1,1,0), (3,0,0)\},$$

$$S_4 = \{(0,0,0,1), (1,0,1,0), (2,1,0,0), (0,2,0,0), (4,0,0,0)\}.$$

To compute S_l is conveniently to exhibit a table of possibilities with the value b_i in the column i. We starts it from the last column.

Clearly the last column can be only filled by 0 and 1, because $5b_5 > 5$ for $b_5 > 1$. The same happens with the fourth and the third column, because $3b_3$, $4b_4 > 5$, for b_3, $b_4 > 1$. Taking $b_5 = 1$, the unique possibility is $b_1 = b_2 = b_3 = b_4 = 0$, thus any other solution satisfies $b_5 = 0$. Taking $b_5 = 0$ and $b_4 = 1$, the unique possibility is $b_1 = 1$ and $b_2 = b_3 = 0$, thus any other solution must have $b_4 = b_5 = 0$. Finally, taking $b_5 = b_4 = 0$ and $b_3 = 1$, we have two possibilities either $b_1 = 2$ and $b_2 = 0$, or $b_1 = 0$ and $b_2 = 1$. Thus any other solution satisfies $b_3 = b_4 = b_5 = 0$.

Now we observe that the second column can be only filled by 0, 1 or 2, since $2b_2 > 5$ for $b_2 > 2$; and taking $b_3 = b_4 = b_5 = 0$ and $b_2 = 1$ the unique possibility is $b_1 = 3$. Taking $b_3 = b_4 = b_5 = 0$ and $b_2 = 2$ the unique possibility is $b_1 = 1$, thus any other solution satisfies $b_2 = b_3 = b_4 = b_5 = 0$. Finally, taking $b_2 = b_3 = b_4 = b_5 = 0$ the unique possibility is $b_1 = 5$. Therefore the complete table of solutions is

$S_5 =$

b_1	b_2	b_3	b_4	b_5
0	0	0	0	1
1	0	0	1	0
0	1	1	0	0
2	0	1	0	0
3	1	0	0	0
1	2	0	0	0
5	0	0	0	0

Now we can use the (1.2.4) and (1.2.3) to compute the expressions of y_i and f_i.

1.5.1 Fifth order averaging theorem (assuming a vanishing F_0)

From (1.2.4) we obtain the functions $y_i(t,z)$ for $k = 1,2,3,4,5$.

$$
\begin{aligned}
y_1(t,z) = & \int_0^t F_1(s,z)ds, \\
y_2(t,z) = & \int_0^t \Big(2F_2(s,z) + 2\partial F_1(s,z)y_1(s,z)\Big)ds, \\
y_3(t,z) = & \int_0^t \Big(6F_3(s,z) + 6\partial F_2(s,z)y_1(t,z) \\
& +3\partial^2 F_1(s,z)y_1(s,z)^2 + 3\partial F_1(s,z)\, y_2(s,z)\Big)ds, \\
y_4(t,z) = & \int_0^t \Big(24F_4(s,z) + 24\partial F_3(s,z)y_1(s,z) \\
& +12\partial^2 F_2(s,z)y_1(s,z)^2 + 12\partial F_2(s,z)y_2(s,z) \\
& +12\partial^2 F_1(s,z)y_1(s,z) \odot y_2(s,z) \\
& +4\partial^3 F_1(s,z)y_1(s,z)^3 + 4\partial F_1(s,z)y_3(s,z)\Big)ds,
\end{aligned}
$$

$$
\begin{aligned}
y_5(t,z) = & \int_0^t \Big(120F_5(s,z) + 120\partial F_4(s,z)y_1(s,z) \\
& +60\partial^2 F_3(s,z)y_1(s,z)^2 + 60\partial F_3(s,z)y_2(s,z) + 60\partial^2 F_2(s,z)y_1(s,z) \odot y_2(s,z) \\
& +20\partial^3 F_2(s,z)y_1(s,z)^3 + 20\partial F_2(s,z)y_3(s,z) + 20\partial^2 F_1(s,z)y_1(s,z) \odot y_3(s,z) \\
& +15\partial^2 F_1(s,z)y_2(s,z)^2 + 30\partial^3 F_1(s,z)y_1(s,z)^2 \odot y_2(s,z) \\
& +5\partial^4 F_1(s,z)y_1(s,z)^4 + 5\partial F_1(s,z)y_4(s,z)\Big)ds.
\end{aligned}
$$

So from (1.2.3) we have that

$$
\begin{aligned}
f_0(z) = &\ 0,\\
f_1(z) = &\ \int_0^T F_1(t,z)dt,\\
f_2(z) = &\ \int_0^T \Big(F_2(t,z)ds + \partial F_1(t,z)y_1(t,z)\Big)dt,\\
f_3(z) = &\ \int_0^T \Big(F_3(t,z) + \partial F_2(t,z)y_1(t,z)\\
&+\frac{1}{2}\partial^2 F_1(t,z)y_1(t,z)^2 + \frac{1}{2}\partial F_1(t,z)y_2(t,z)\Big)dt,\\
f_4(z) = &\ \int_0^T \Big(F_4(t,z) + \partial F_3(t,z)y_1(t,z)\\
&+\frac{1}{2}\partial^2 F_2(t,z)y_1(t,z)^2 + \frac{1}{2}\partial F_2(t,z)y_2(t,z)\\
&+\frac{1}{2}\partial^2 F_1(t,z)y_1(t,z)\odot y_2(t,z)dt\\
&+\frac{1}{6}\partial^3 F_1(t,z)y_1(t,z)^3 + \frac{1}{6}\partial F_1(t,z)y_3(t,z)\Big)dt,
\end{aligned}
$$

$$
\begin{aligned}
f_5(z) = &\ \int_0^T \Big(F_5(t,z) + \partial F_4(t,z)y_1(t,z)\\
&+\frac{1}{2}\partial^2 F_3(t,z)y_1(t,z)^2 + \frac{1}{2}\partial F_3(t,z)y_2(t,z) + \frac{1}{2}\partial^2 F_2(t,z)y_1(t,z)\odot y_2(t,z)\\
&+\frac{1}{6}\partial^3 F_2(t,z)y_1(t,z)^3 + \frac{1}{6}\partial F_2(t,z)y_3(t,z) + \frac{1}{6}\partial^2 F_1(t,z)y_1(t,z)\odot y_3(t,z)\\
&+\frac{1}{8}\partial^2 F_1(t,z)y_2(t,z)^2 + \frac{1}{4}\partial^3 F_1(t,z)y_1(t,z)^2\odot y_2(t,z)\\
&+\frac{1}{24}\partial^4 F_1(t,z)y_1(t,z)^4 + \frac{1}{24}\partial F_1(t,z)y_4(t,z)\Big)dt.
\end{aligned}
$$

1.5.2 Fourth order averaging theorem (assuming a nonvanishing F_0)

First of all, a Cauchy problem, or equivalently an integral equation (see Remark 1.2.3), must be solved to compute the expressions $y_i(t,z)$ for $i = 1, 2, \ldots, k$. We give the integral equations and its solutions for $k = 1, 2, 3, 4$.

Let $Y(t,z)$ be the function defined in (1.2.6). Hence, from (1.2.4) and (1.2.3) we obtain the functions $y_1(t,z)$ and $f_1(z)$:

$$
y_1(t,z) = \int_0^t \Big(F_1(s,\varphi(s,z)) + \partial F_0(s,\varphi(s,z))y_1(s,z)\Big)ds,
$$

so

$$y_1(t,z) = Y(t,z)\int_0^t Y(s,z)^{-1}F_1(s,\varphi(s,z))ds,$$

and

$$f_1(z) = \int_0^T Y(t,z)^{-1}F_1(t,\varphi(t,z))dt.$$

Similarly, the functions $y_2(t,z)$ and $f_2(z)$ are given by

$$\begin{aligned} y_2(t,z) = & \int_0^t \Big(2F_2(s,\varphi(s,z)) + 2\partial F_1(s,\varphi(s,z))y_1(s,z) \\ & +\partial^2 F_0(s,\varphi(s,z))y_1(s,z)^2 + \partial F_0(s,\varphi(s,z))\, y_2(s,z)\Big)dt, \end{aligned}$$

so

$$\begin{aligned} y_2(t,z) = & \; Y(t,z)\int_0^t Y(s,z)^{-1}\Big(2F_2(s,\varphi(s,z)) + 2\partial F_1(s,\varphi(s,z))y_1(s,z) \\ & +\partial^2 F_0(s,\varphi(s,z))y_1(s,z)^2\Big)ds, \end{aligned}$$

and

$$\begin{aligned} f_2(z) = & \int_0^T Y(t,z)^{-1}\Big(F_2(t,\varphi(t,z)) + \partial F_1(t,\varphi(t,z))y_1(t,z) \\ & +\frac{1}{2}\partial^2 F_0(t,\varphi(t,z))y_1(t,z)^2\Big)dt, \end{aligned}$$

The functions $y_3(t,z)$ and $f_3(z)$ are given by

$$\begin{aligned} y_3(t,z) = & \int_0^t \Big(6F_3(s,\varphi(s,z)) + 6\partial F_2(s,\varphi(s,z))y_1(s,z) \\ & +3\partial^2 F_1(s,\varphi(s,z))y_1(s,z)^2 + 3\partial F_1(s,\varphi(s,z))\, y_2(s,z) \\ & +3\partial^2 F_0(s,\varphi(s,z))y_1(s,z)\odot y_2(s,z) \\ & +\partial^3 F_0(s,\varphi(s,z))y_1(s,z)^3 + \partial F_0(s,\varphi(s,z))y_3(s,z)\Big)ds, \end{aligned}$$

so

$$\begin{aligned} y_3(t,z) = & \; Y(t,z)\int_0^t Y(s,z)^{-1}\Big(6F_3(s,\varphi(s,z)) + 6\partial F_2(s,\varphi(s,z))y_1(s,z) \\ & +3\partial^2 F_1(s,\varphi(s,z))y_1(s,z)^2 + 3\partial F_1(s,\varphi(s,z))\, y_2(s,z) \\ & +3\partial^2 F_0(s,\varphi(s,z))y_1(s,z)\odot y_2(s,z) \\ & +\partial^3 F_0(s,\varphi(s,z))y_1(s,z)^3\Big)ds, \end{aligned}$$

and

$$
\begin{aligned}
f_3(z) = \quad & \int_0^T Y(t,z)^{-1}\Big(F_3(t,\varphi(t,z)) + \partial F_2(t,\varphi(t,z))y_1(t,z) \\
& + \frac{1}{2}\partial^2 F_1(t,\varphi(t,z))y_1(t,z)^2 + \frac{1}{2}\partial F_1(t,\varphi(t,z))\, y_2(t,z) \\
& + \frac{1}{2}\partial^2 F_0(t,\varphi(t,z))y_1(t,z) \odot y_2(t,z) \\
& + \frac{1}{6}\partial^3 F_0(t,\varphi(t,z))y_1(t,z)^3\Big)ds,
\end{aligned}
$$

Finally, the functions $y_4(t,z)$ and $f_4(z)$ are given by

$$
\begin{aligned}
y_4(t,z) = \quad & \int_0^t \Big(24F_4(s,\varphi(s,z)) + 24\partial F_3(s,\varphi(s,z))y_1(s,z) \\
& +12\partial^2 F_2(s,\varphi(s,z))y_1(s,z)^2 + 12\partial F_2(s,\varphi(s,z))y_2(s,z) \\
& +12\partial^2 F_1(s,\varphi(s,z))y_1(s,z) \odot y_2(s,z) \\
& +4\partial^3 F_1(s,\varphi(s,z))y_1(s,z)^3 + 4\partial F_1(s,\varphi(s,z))y_3(s,z) \\
& +4\partial^2 F_0(s,\varphi(s,z))y_1(s,z) \odot y_3(s,z) \\
& +3\partial^2 F_0(s,\varphi(s,z))y_2(s,z)^2 ds + 6\partial^3 F_0(s,\varphi(s,z))y_1(s,z)^2 \odot y_2(s,z) \\
& +\partial^4 F_0(s,\varphi(s,z))y_1(s,z)^4 + \partial F_0(s,\varphi(s,z))y_4(s,z)\Big)ds.
\end{aligned}
$$

so

$$
\begin{aligned}
y_4(t,z) = \quad & Y(t,z)\int_0^t Y(s,z)^{-1}\Big(24F_4(s,\varphi(s,z)) + 24\partial F_3(s,\varphi(s,z))y_1(s,z) \\
& +12\partial^2 F_2(s,\varphi(s,z))y_1(s,z)^2 + 12\partial F_2(s,\varphi(s,z))y_2(s,z) \\
& +12\partial^2 F_1(s,\varphi(s,z))y_1(s,z) \odot y_2(s,z) \\
& +4\partial^3 F_1(s,\varphi(s,z))y_1(s,z)^3 + 4\partial F_1(s,\varphi(s,z))y_3(s,z) \\
& +4\partial^2 F_0(s,\varphi(s,z))y_1(s,z) \odot y_3(s,z) \\
& +3\partial^2 F_0(s,\varphi(s,z))y_2(s,z)^2 ds + 6\partial^3 F_0(s,\varphi(s,z))y_1(s,z)^2 \odot y_2(s,z) \\
& +\partial^4 F_0(s,\varphi(s,z))y_1(s,z)^4\Big)ds.
\end{aligned}
$$

and

$$
\begin{aligned}
f_4(z) = \quad & \int_0^T Y(t,z)^{-1}\Big(F_4(t,\varphi(t,z)) + \partial F_3(t,\varphi(t,z))y_1(t,z) \\
& + \frac{1}{2}\partial^2 F_2(t,\varphi(t,z))y_1(t,z)^2 + \frac{1}{2}\partial F_2(t,\varphi(t,z))y_2(t,z) \\
& + \frac{1}{2}\partial^2 F_1(t,\varphi(t,z))y_1(t,z) \odot y_2(t,z) \\
& + \frac{1}{6}\partial^3 F_1(t,\varphi(t,z))y_1(t,z)^3 + \frac{1}{6}\partial F_1(t,\varphi(t,z))y_3(t,z) \\
& + \frac{1}{6}\partial^2 F_0(t,\varphi(t,z))y_1(t,z) \odot y_3(t,z) \\
& + \frac{1}{8}\partial^2 F_0(t,\varphi(t,z))y_2(t,z)^2 ds + \frac{1}{4}\partial^3 F_0(t,\varphi(t,z))y_1(t,z)^2 \odot y_2(t,z) \\
& + \frac{1}{24}\partial^4 F_0(t,\varphi(t,z))y_1(t,z)^4\Big)ds.
\end{aligned}
$$

1.6 Proofs of examples

For proving Propositions 1.3.1, 1.3.2 and 1.3.3 we use the formulae obtained in section 1.5.

Proof of Proposition 1.3.1. Applying the change of variables $(x, y, \mathbf{z}) = (r\cos\theta, r\sin\theta, \mathbf{z})$, system (1.3.1) becomes

$$
\begin{aligned}
r'(t) = \quad & \varepsilon\Big(F(r\cos\theta, r\sin\theta, \mathbf{z})\cos\theta + G(r\cos\theta, r\sin\theta, \mathbf{z})\sin\theta\Big) + \mathcal{O}(\varepsilon^2), \\
\theta'(t) = \quad & \varepsilon\,\frac{1}{r}\Big(G(r\cos\theta, r\sin\theta, \mathbf{z})\cos\theta - F(r\cos\theta, r\sin\theta, \mathbf{z})\sin\theta\Big) + \mathcal{O}(\varepsilon^2), \\
z_i'(t) = \quad & \varepsilon\, H^i(r\cos\theta, r\sin\theta, \mathbf{z}) + \mathcal{O}(\varepsilon^2).
\end{aligned}
\tag{1.6.1}
$$

Now rescaling the time by $t = \theta$, the system (1.6.1) is reduced to

$$
\begin{aligned}
r'(\theta) = \quad & -\varepsilon\,(F(r\cos\theta, r\sin\theta, \mathbf{z})\cos\theta + G(r\cos\theta, r\sin\theta, \mathbf{z})\sin\theta) + \mathcal{O}(\varepsilon^2), \\
z_i'(\theta) = \quad & -\varepsilon\, H^i(r\cos\theta, r\sin\theta, \mathbf{z}) + \mathcal{O}(\varepsilon^2).
\end{aligned}
\tag{1.6.2}
$$

Computing the function f_1 (defined in (1.2.3)) for the system (1.6.2) we have that

$$
\begin{aligned}
f_1(r, \mathbf{z}) = \quad & \Big(-\int_0^{2\pi}\Big(F(r\cos\theta, r\sin\theta, \mathbf{z})\cos\theta + G(r\cos\theta, r\sin\theta, \mathbf{z})\sin\theta\Big)d\theta, \\
& -\int_0^{2\pi} H^1(r\cos\theta, r\sin\theta, \mathbf{z}))d\theta, \\
& \vdots \\
& -\int_0^{2\pi} H^n(r\cos\theta, r\sin\theta, \mathbf{z}))d\theta\Big).
\end{aligned}
$$

We observe that the system $f_1(r, \mathbf{z}) = 0$ is equivalent to the system $\mathcal{F}(r, \mathbf{z}) = 0$ with $\mathcal{F}$ defined in (1.3.2). So applying Theorem A and observing Remark 1.2.4 the result follows. □

Proof of Proposition 1.3.2. Computing the function y_1 and y_2 (defined in (1.2.3)) for the system (1.3.3) we have that

$$y_1(t, x, y) = \Big(A_{11}(t)x + A_{12}(t)y\,,\, A_{21}(t)x + A_{22}(t)y\Big),$$

and

$$\begin{aligned} y_2(t, x, y) = \ & \Big(B_{11}(t)x + B_{12}(t)y + C_{120}(t)x^2 + C_{111}(t)xy + C_{102}(t)y^2\,, \\ & B_{21}(t)x + B_{22}(t)y + C_{220}(t)x^2 + C_{211}(t)xy + C_{202}(t)y^2\Big), \end{aligned}$$

where $A_{ij}(t)$, $B_{ij}(t)$ for $i, j = 1, 2$, and C_{ijk} for $i = 1, 2$ and $j = 0, 1, 2$, are defined respectively in (1.3.4), (1.3.5), and (1.3.6).

If $A(t) = (A_{ij}(t))_{ij}$, then

$$f_1(x, y) = y_1(2\pi, x, y) = A(2\pi)\begin{pmatrix} x \\ y \end{pmatrix}.$$

So, for $\det(A(2\pi)) \neq 0$ we have that $(x, y) = (0, 0)$ is the unique solution of the linear system $f_1(x, y) = 0$. Applying Theorem B and observing Remark 1.2.4 the proof of item (a) of theorem follows.

Now, if $A(2\pi) = 0$ then $f_1 = 0$. So to find the periodic solutions of (1.3.3) using Theorem B we have to study the system $f_2(x, y) = 0$, where

$$\begin{aligned} f_2(x, y) = \frac{y_2(2\pi, x, y)}{2} = \ & \frac{1}{2}\Big(B_{11}x + B_{12}y + C_{120}x^2 + C_{111}xy + C_{102}y^2\,, \\ & B_{21}x + B_{22}y + C_{220}x^2 + C_{211}xy + C_{202}y^2\Big). \end{aligned}$$

Since the system $f_2(x, y) = 0$ is equivalent to the system $\mathcal{G}(x, y) = 0$, with $\mathcal{G}$ defined in (1.3.7), the proof of item (b) of theorem follows. □

Proof of Proposition 1.3.2. Analogously to the proof of item (b) of Proposition 1.3.2, since the hypotheses implies $f_1 = f_2 = 0$. The result follows immediately by computing the function $f_3(x, y)$, applying Theorem B and observing Remark 1.2.4. □

1.7 Simpler proof of the fundamental lemma

In this section, using the *Faá di Bruno's Formula* instead the finite induction, we present an alternative proof of Lemma 1.4.1 assuming that (1.2.1) is a $\mathbb{C}^k$ system.

We recall the Faá di Bruno's Formula (see [58]) about the lth derivative of a composite function.

Faá di Bruno's Formula *If g and f are functions with a sufficient number of derivatives, then*

$$\frac{d^l}{dt^l} g(f(t)) = \sum_{S_l} \frac{l!}{b_1!\ b_2!\, 2!^{b_2} \cdots b_l!\, l!^{b_l}} g^{(L)}(f(t)) \bigodot_{j=1}^{l} f^{(j)}(t)^{b_j},$$

where S_l is the set of all l–tuples of non–negative integers $(b_1, b_2, \cdots, b_l)$ which are solutions of the equation $b_1 + 2b_2 + \cdots + lb_l = l$ and $L = b_1 + b_2 + \cdots + b_l$.

The result about differentiable dependence on parameters implies that the map $\varepsilon \mapsto x(t, z, \varepsilon)$ is k times differentiable. So we can use the Faá di Bruno's Formula to prove Lemma 1.4.1 as following.

Since $x(t, z, 0) = \varphi(t, z)$, the Taylor expansion of $F_i(t, x(t, z, \varepsilon))$ around $\varepsilon = 0$, for $i = 0, 1, \ldots, k-1$, is given by

$$F_i(t, x(t, z, \varepsilon)) = F_i\left(t, \varphi(t, z)\right) + \sum_{l=1}^{k-i} \frac{\varepsilon^l}{l!} \left(\frac{\partial^l}{\partial \varepsilon^l} F_i(t, x(t, z, \varepsilon)) \right) \Bigg|_{\varepsilon=0} + \varepsilon^{k-i+1} \mathcal{O}(1). \tag{1.7.1}$$

The Faá di Bruno's formula allows to compute the l–derivatives of $F_i(t, x(t, z, \varepsilon))$ in ε, for $i = 0, 1, \ldots, k-1$:

$$\frac{\partial^l}{\partial \varepsilon^l} F_i(t, x(t, z, \varepsilon)) \Bigg|_{\varepsilon=0} = \sum_{S_l} \frac{l!}{b_1!\ b_2!\, 2!^{b_2} \cdots b_l!\, l!^{b_l}} \left(\partial^L F_i(t, x(t, z, \varepsilon)) \right) \Bigg|_{\varepsilon=0} \bigodot_{j=1}^{l} y_j(t, z)^{b_j}. \tag{1.7.2}$$

Here S_l is the set of all l-tuples of non–negative integers $(b_1, b_2, \cdots, b_l)$ which are solutions of the equation $b_1 + 2b_2 + \cdots + lb_l = l$, $L = b_1 + b_2 + \cdots + b_l$, and

$$y_j(t, z) = \left(\frac{\partial^j}{\partial \varepsilon^j} x(t, z, \varepsilon) \right) \Bigg|_{\varepsilon=0}. \tag{1.7.3}$$

Substituting (1.7.2) in (1.7.1) the Taylor expansion at $\varepsilon = 0$ of $F_i(s, x(t, z, \varepsilon))$ becomes

$$\begin{aligned} F_i(s, x(s, z, \varepsilon)) = \ & F_i\left(s, \varphi(s, z)\right) \\ & + \sum_{l=1}^{k-i} \sum_{S_l} \frac{\varepsilon^l}{b_1!\ b_2!\, 2!^{b_2} \cdots b_l!\, l!^{b_l}} \partial^L F_i\left(s, \varphi(s, z)\right) \bigodot_{j=1}^{l} y_j(s, z)^{b_j} \\ & + \varepsilon^{k-i+1} \mathcal{O}(1), \end{aligned} \tag{1.7.4}$$

for $i = 0, 1, \ldots, k-1$. Moreover, for $i = k$ we have that

$$F_k(s, x(s, z, \varepsilon)) = F_k\left(s, \varphi(s, z)\right) + \varepsilon \mathcal{O}(1). \tag{1.7.5}$$

Now, from (1.4.2), (1.7.4), (1.7.5), and (1.4.1), the following equation holds

$$x(t, z, \varepsilon) = z + \int_0^t Q(s, z, \varepsilon) ds + \sum_{i=0}^{k} \varepsilon^i \int_0^t F_i(s, \varphi(s, z)) ds + \varepsilon^{k+1} \mathcal{O}(1), \tag{1.7.6}$$

where

$$Q(s,z,\varepsilon)=\sum_{i=0}^{k-1}\sum_{l=1}^{k-i}\varepsilon^{l+i}\sum_{S_l}\int_0^t\frac{1}{b_1!\ b_2!\,2!^{b_2}\cdots b_l!\,l!^{b_l}}\partial^L F_i(s,\varphi(s,z))\bigodot_{j=1}^{l}y_j(s,z)^{b_j}ds.$$

We may write

$$\begin{aligned}Q(s,z,\varepsilon)=&\ \sum_{l=1}^{k}\sum_{i=l}^{k}\varepsilon^{i}\sum_{S_l}\int_0^t\frac{1}{b_1!\ b_2!\,2!^{b_2}\cdots b_l!\,l!^{b_l}}\partial^L F_{i-l}(s,\varphi(s,z))\bigodot_{j=1}^{l}y_j(s,z)^{b_j}ds\\ =&\ \sum_{i=1}^{k}\varepsilon^{i}\sum_{l=1}^{i}\sum_{S_l}\int_0^t\frac{1}{b_1!\ b_2!\,2!^{b_2}\cdots b_l!\,l!^{b_l}}\partial^L F_{i-l}(s,\varphi(s,z))\bigodot_{j=1}^{l}y_j(s,z)^{b_j}ds.\end{aligned}\tag{1.7.7}$$

Finally, from (1.7.6) and (1.7.7), we get

$$\begin{aligned}&x(t,z,\varepsilon)=z+\int_0^t F_0(t,\varphi(s,z))ds\\ &+\sum_{i=1}^{k}\varepsilon^{i}\left(\int_0^t F_i(s,\varphi(s,z))+\sum_{l=1}^{i}\sum_{S_l}\frac{1}{b_1!\ b_2!\,2!^{b_2}\cdots b_l!\,l!^{b_l}}\partial^L F_{i-l}(s,\varphi(s,z))\bigodot_{j=1}^{l}y_j(s,z)^{b_j}\,ds\right)\\ &+\varepsilon^{k+1}\int_0^t R(s,x(s,z,\varepsilon),\varepsilon)ds+\varepsilon^{k+1}\mathcal{O}(1).\end{aligned}$$

Now, using this last expression of $x(t,z,\varepsilon)$ we conclude that functions $y_i(t,z)$ defined in (1.7.3), for $i=1,2,\ldots,k-1$, can be computed recurrently from the following integral equation

$$\begin{aligned}y_i(t,z)&=\left(\frac{\partial^i x}{\partial\varepsilon^i}(t,z,\varepsilon)\right)\Bigg|_{\varepsilon=0}\\ &=i!\int_0^t\left(F_i(s,\varphi(s,z))+\sum_{l=1}^{i}\sum_{S_l}\frac{1}{b_1!\ b_2!\,2!^{b_2}\cdots b_l!\,l!^{b_l}}\partial^L F_{i-l}(s,\varphi(s,z))\bigodot_{j=1}^{l}y_j(s,z)^{b_j}\right)ds.\end{aligned}$$

Since

$$\varphi(t,z)=z+\int_0^t F_0(t,\varphi(s,z))ds,$$

we obtain

$$x(t,z,\varepsilon)=\varphi(t,z)+\sum_{i=1}^{k}\varepsilon^{i}\frac{y_i(t,z)}{i!}+\varepsilon^{k+1}\mathcal{O}(1),$$

which completes the proof of Lemma 1.4.1.

1.8 Applications in discontinuous dynamical systems

1.8.1 Application 1

In this section we provide the bifurcation function at any order for computing the periodic solutions of discontinuous piecewise differential system of the form

$$r' = \begin{cases} F^+(\theta, r, \varepsilon) & \text{if} \quad 0 \leq \alpha, \\ \\ F^-(\theta, r, \varepsilon) & \text{if} \quad \alpha \leq 2\pi, \end{cases} \tag{1.8.1}$$

where

$$F^{\pm}(\theta, r, \varepsilon) = \sum_{i=1}^{k} \varepsilon^i F_i^{\pm}(\theta, r) + \varepsilon^{k+1} R^{\pm}(\theta, r, \varepsilon).$$

The set of discontinuity of system (1.8.1) is $\Sigma = \{\theta = 0\} \cup \{\theta = \alpha\}$ if $0 < \alpha < 2\pi$. Here $F_i^{\pm} : \mathbb{S}^1 \times D \to \mathbb{R}$ for $i = 0, 1, \ldots, n$, and $R^{\pm} : \mathbb{S}^1 \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ are $\mathcal{C}^{k+1}$ functions, where D is an open and bounded interval of $(0, \infty)$, and $\mathbb{S}^1 \equiv \mathbb{R}/(2\pi)$.

For $i = 1, 2, \ldots, k$, we define the *averaged function* $f_i : D \to \mathbb{R}$ of order i as

$$f_i(\rho) = \frac{y_i^+(\alpha, \rho) - y_i^-(\alpha - 2\pi, \rho)}{i!}, \tag{1.8.2}$$

where $y_i^{\pm} : \mathbb{S}^1 \times D \to \mathbb{R}$, for $i = 1, 2, \ldots, k-1$, are defined recurrently as

$$\begin{aligned} y_i^{\pm}(\theta, \rho) = \; & i! \int_0^{\theta} \Bigg(F_i^{\pm}(\phi, \rho) + \sum_{l=1}^{i} \sum_{S_l} \frac{1}{b_1!\; b_2!\, 2!^{b_2} \cdots b_l!\, l!^{b_l}} \cdot \\ & \partial^L F_{i-l}^{\pm}(\phi, \rho) \prod_{j=1}^{l} y_j^{\pm}(\phi, \rho)^{b_j} \Bigg) d\phi, \end{aligned} \tag{1.8.3}$$

where S_l is the set of all l-tuples of non–negative integers $(b_1, b_2, \cdots, b_l)$ satisfying $b_1 + 2b_2 + \cdots + lb_l = l$, and $L = b_1 + b_2 + \cdots + b_l$.

Our main result on the periodic solutions of (1.8.1) is the following.

Theorem C. Assume that, for some $\ell \in \{1, 2, \ldots, k\}$, $f_i = 0$ for $i = 1, 2, \ldots, \ell - 1$ and $f_\ell \neq 0$. If there exists $\rho^* \in D$ such that $f_\ell(\rho^*) = 0$ and $f'_\ell(\rho^*) \neq 0$, then for $|\varepsilon| > 0$ sufficiently small there exists a 2π–periodic solution $r(\theta, \varepsilon)$ of (1.8.1) such that $r(0, \varepsilon) \to \rho^*$ when $\varepsilon \to 0$.

The proof of Theorem C is based on the following lemma.

Lemma 1.8.1. Let $r^{\pm}(\cdot, \rho, \varepsilon) : [0, \theta_\rho) \to \mathbb{R}^k$ be the solution of $r' = F^{\pm}(\theta, r, \varepsilon)$ with $r^{\pm}(0, \rho, \varepsilon) = \rho$. If $\theta_\rho > T$, then

$$r^{\pm}(\theta, \rho, \varepsilon) = \rho + \sum_{i=1}^{k} \varepsilon^i \frac{y_i^{\pm}(\theta, \rho)}{i!} + \mathcal{O}_{k+1}(\varepsilon),$$

where $y_i^{\pm}(t, z)$ for $i = 1, 2, \ldots, k$ are defined in (1.8.3).

Proof. The proof of this lemma is a direct consequence of the Fundamental Lemma 1.4.1. □

Now we prove Theorem C.

Proof of Theorem C. First of all we have to show that there exists ε_0 sufficiently small such that for each $\rho \in \overline{D}$ and for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ the solutions $r^{\pm}(\theta, \rho, \varepsilon)$ are defined for every $\theta \in [0, T]$. Indeed, by the *Existence and Uniqueness Theorem* of solutions (see, for example, Theorem 1.2.4 of [99]), $r^{\pm}(\theta, \rho, \varepsilon)$ is defined for all $0 \leq \theta \leq \inf\left(T, d/M^{\pm}(\varepsilon)\right)$, for each x with $|r - \rho| < d$ and for every $\rho \in \overline{D}$, where

$$M^{\pm}(\varepsilon) \geq \left| \sum_{i=1}^{k} \varepsilon^i F_i^{\pm}(\theta, \rho) + \varepsilon^{k+1} R^{\pm}(\theta, \rho, \varepsilon) \right|.$$

Clearly ε can be taken sufficiently small in order that $\inf\left(T, d/M^{\pm}(\varepsilon)\right) = T$ for all $\rho \in \overline{D}$. Moreover, since the vector fields $F^{\pm}(\theta, r, \varepsilon)$ are T–periodic, the solutions $r^{\pm}(\theta, \rho, \varepsilon)$ can be extended for $\theta \in \mathbb{R}$.

We denote

$$f(\rho, \varepsilon) = r^{+}(\alpha, \rho, \varepsilon) - r^{-}(\alpha - T, \rho, \varepsilon).$$

It is easy to see that system (1.8.1) for $\varepsilon = \bar{\varepsilon} \in (-\varepsilon_0, \varepsilon_0)$ has a periodic solution passing through $\bar{\rho} \in D$ if and only if $f(\bar{\rho}, \bar{\varepsilon}) = 0$.

From Lemma 1.8.1 we have that

$$\begin{aligned} f(\rho, \varepsilon) = & \sum_{i=1}^{k} \varepsilon^i \frac{y_i(\theta, \rho) - y_i(\theta, \rho)}{i!} + \mathcal{O}_{k+1}(\varepsilon) \\ = & \sum_{i=1}^{k} \varepsilon^i f_i(\rho) + \mathcal{O}_{k+1}(\varepsilon) \end{aligned}$$

where the function f_i is the one defined in (1.8.2) for $i = 1, 2, \cdots, k$. From hypothesis

$$f(\rho, \varepsilon) = \varepsilon^r f_r(\rho) + \cdots + \varepsilon^k f_k(\rho) + \mathcal{O}_{k+1}(\varepsilon).$$

Since $f_r(\rho^*) = 0$ and $f_r'(\rho^*) \neq 0$, the *implicit function theorem* applied to the function $\mathcal{F}(\rho, \varepsilon) = f(\rho, \varepsilon)/\varepsilon^r$ guarantees the existence of a differentiable function $\rho(\varepsilon)$ such that $\rho(0) = \rho^*$ and $f(\rho(\varepsilon), \varepsilon) = \varepsilon^r \mathcal{F}(\rho(\varepsilon\ ), \varepsilon) = 0$ for every $|\varepsilon| \neq 0$ sufficiently small. Then the proof of the theorem follows. □

1.8.2 Application 2

In this section we provide the bifurcation function up to order 2 for computing the periodic solutions of discontinuous piecewise differential system of the form

$$\mathbf{x}' = \begin{cases} F^{+}(\theta, \mathbf{x}, \varepsilon) & \text{if} \quad 0 \leq \theta \leq \phi, \\ F^{-}(\theta, \mathbf{x}, \varepsilon) & \text{if} \quad \phi \leq \theta \leq T, \end{cases} \tag{1.8.4}$$

where

$$F^{\pm}(\theta,\mathbf{x},\varepsilon)=F_0^{\pm}(\theta,\mathbf{x})+\varepsilon F_1^{\pm}(\theta,\mathbf{x})+\varepsilon^2F_2^{\pm}(\theta,\mathbf{x})+\varepsilon^3R^{\pm}(\theta,\mathbf{x},\varepsilon).$$

The set of discontinuity of system (1.8.4) is given by $\Sigma=\{\theta=0\}\cup\{\theta=\phi\}$. Here $F_i^{\pm}:\mathbb{S}^1\times D\to\mathbb{R}^{d+1}$ for $i=0,1,2$, and $R^{\pm}:\mathbb{S}^1\times D\times(-\varepsilon_0,\varepsilon_0)\to\mathbb{R}^{d+1}$ are $\mathbb{C}^3$ functions, where D is an open bounded subset of $\mathbb{R}^{d+1}$, and $\mathbb{S}^1\equiv\mathbb{R}/T$ for a positive real number T.

For $\mathbf{z}\in D$ let $\varphi^{\pm}(\theta,\mathbf{z})$ be the solutions of the systems

$$\mathbf{x}'=F_0^{\pm}(\theta,\mathbf{x}), \tag{1.8.5}$$

such that $\varphi^{\pm}(0,\mathbf{z})=\mathbf{z}$, respectively, and let $\varphi(\theta,\mathbf{z})$ be the solution of the unperturbed system $\mathbf{x}'=F_0(\theta,\mathbf{x})$, such that $\varphi(0,\mathbf{z})=\mathbf{z}$. Clearly

$$\varphi(\theta,\mathbf{z})=\begin{cases}\varphi^{+}(\theta,\mathbf{z}) & \text{if} \quad 0\le\theta\le\phi,\\ \\ \varphi^{-}(\theta,\mathbf{z}) & \text{if} \quad \phi\le\theta\le T,\end{cases}$$

As the main hypothesis of this section we shall assume that there exists a manifold $\mathcal{Z}$ embedded in D such that the solutions starting in $\mathcal{Z}$ are all T–periodic functions. Formally for $p=d+1$ and $q\le p$ let $\sigma:\overline{V}\to\mathbb{R}^{p-q}$ be a $\mathbb{C}^3$ function with V an open and bounded subset of $\mathbb{R}^q$, and let $\mathcal{Z}=\{\mathbf{z}_\nu=(\nu,\sigma(\nu)):\nu\in\overline{V}\}$. We suppose that

(H) $\mathcal{Z}\subset D$ and for each $\mathbf{z}_\nu$ the unique solution $\varphi(\theta,\mathbf{z}_\nu)$ such that $\varphi(0,\mathbf{z}_\nu)=\mathbf{z}_\nu$ is T–periodic.

Now for $\mathbf{z}\in D$ we consider the linearization of the systems (1.8.5) along the solution $\varphi^{\pm}(\theta,\mathbf{z})$, that is

$$Y'=D_{\mathbf{x}}F_0^{\pm}(\theta,\varphi^{\pm}(\theta,\mathbf{z}))\,Y. \tag{1.8.6}$$

Let $Y^{\pm}(\theta,\mathbf{z})$ be the fundamental matrices of the differential system (1.8.6).

Let $\xi:\mathbb{R}^q\times\mathbb{R}^{p-q}\to\mathbb{R}^q$ and $\xi^{\perp}:\mathbb{R}^q\times\mathbb{R}^{p-q}\to\mathbb{R}^{p-q}$ be the projections onto the first q coordinates and onto the last $p-q$ coordinates, respectively. For a point $\mathbf{z}\in$ D we also consider $\mathrm{z}=(u,v)\in\mathbb{R}^p\times\mathbb{R}^{p-q}$. Thus we define the averaged functions $f_1,f_2:\overline{V}\to\mathbb{R}^q$ as

$$\begin{aligned} f_1(\nu) &= \xi g_1(\mathbf{z}_\nu),\\ f_2(\nu) &= 2\xi g_2(\mathbf{z}_\nu)+2\frac{\partial\xi g_1}{\partial v}(\mathbf{z}_\nu)\gamma(\nu)\\ &\quad+\frac{\partial^2\xi g_0}{\partial v^2}(\mathbf{z}_\nu)\gamma(\nu)^2, \end{aligned} \tag{1.8.7}$$

where

$$\gamma(\nu)=-\Delta_\nu^{-1}\xi^{\perp}g_1(\mathbf{z}_\nu), \tag{1.8.8}$$

and

$$g_i(\mathbf{z})=y_i^{+}(\phi,\mathbf{z})-y_i^{-}(\phi-T,\mathbf{z}), \tag{1.8.9}$$

being

$$
\begin{aligned}
y_0^{\pm}(\theta,\mathbf{z}) = &\ \varphi^{\pm}(\theta,\mathbf{z}),\\
y_1^{\pm}(\theta,\mathbf{z}) = &\ Y^{\pm}(\theta,\mathbf{z})\int_0^{\theta} Y^{\pm}(s,\mathbf{z})^{-1}F_1^{\pm}(s,\varphi^{\pm}(s,\mathbf{z}))ds,\\
y_2^{\pm}(\theta,\mathbf{z}) = &\ Y^{\pm}(\theta,\mathbf{z})\int_0^{\theta} Y^{\pm}(s,\mathbf{z})^{-1}\Big(2F_2^{\pm}(s,\varphi^{\pm}(s,\mathbf{z}))\\
&+2\frac{\partial F_1^{\pm}}{\partial \mathbf{x}}(s,\varphi(s,\mathbf{z}))y_1^{\pm}(s,\mathbf{z}) + \frac{\partial^2 F_0^{\pm}}{\partial \mathbf{x}^2}(s,\varphi^{\pm}(s,\mathbf{z}))y_1^{\pm}(s,\mathbf{z})^2\Big)ds.
\end{aligned}
\tag{1.8.10}
$$

Our main result on the periodic solutions of the DPDS (1.8.4) is the following.

Theorem D. In addition to the hypothesis (H) we assume that for any $\nu \in \overline{V}$ the matrix $Y^+(\phi,\nu) - Y^-(\phi - T,\nu)$ has in the upper right corner the null $q \times (p-q)$ matrix, and in the lower right corner has the $(p-q)\times(p-q)$ matrix Δ_ν with $\det(\Delta_\nu) \neq 0$. So the following statement hold.

(a) If there exists $\nu^* \in V$ such that $f_1(\nu^*) = 0$ and $\det(f_1'(\nu^*)) \neq 0$, then for $|\varepsilon| > 0$ sufficiently small there exists a T–periodic solution $\mathbf{x}(\theta,\varepsilon)$ of system (1.8.4) such that $\mathbf{x}(0,\varepsilon) \to \mathbf{z}_{\nu^*}$ as $\varepsilon \to 0$.

(b) Assume that $f_1 = 0$. If there exists $\nu^* \in V$ such that $f_2(\nu^*) = 0$ and $\det(f_2'(\nu^*)) \neq 0$, then for $|\varepsilon| > 0$ sufficiently small there exists a T–periodic solution $\mathbf{x}(\theta,\varepsilon)$ of system (1.8.4) such that $\mathbf{x}(0,\varepsilon) \to \mathbf{z}_{\nu^*}$ as $\varepsilon \to 0$.

The following result is an immediate consequence of Theorem D.

Corollary 1.8.1. Assume that $q = p$, in this case $\mathcal{Z} = \overline{V} \subset D$ is a compact bounded p–dimensional manifold. Then the statements (a) and (b) of Theorem D hold by taking $f_1 = g_1$ and $f_2 = 2g_2$, and without any assumption about the matrix Δ_ν.

The proof of Theorem D is based on the next lemma which is a particular case of the *Lyapunov–Schmidt reduction* for finite dimensional function (see for instance [27]).

Lemma 1.8.2. Assuming $q \leq p$ are positive integers, let D and V be open bounded subsets of $\mathbb{R}^p$ nd $\mathbb{R}^q$, respectively. Let $g : D \times (-\varepsilon_0,\varepsilon_0) \to \mathbb{R}^p$ and $\sigma : \overline{V} \to \mathbb{R}^{p-q}$ be $\mathbb{C}^3$ functions such that $g(\mathbf{z},\varepsilon) = g_0(\mathbf{z}) + \varepsilon g_1(\mathbf{z}) + \varepsilon^2 g_2(\mathbf{z}) + \mathcal{O}(\varepsilon^3)$ and $\mathcal{Z} = \{\mathbf{z}_\nu = (\nu,\sigma(\nu)) : \nu \in \overline{V}\} \subset D$. We denote by Γ_ν the upper right corner $q \times (p-q)$ matrix of $D\,g_0(\mathbf{z}_\nu)$, and by Δ_ν the lower right corner $(p-q)\times(p-q)$ matrix of $D\,g_0(\mathbf{z}_\nu)$. Assume that for each $\mathbf{z}_\nu \in \mathcal{Z}$, $\det(\Delta_\nu) \neq 0$ and $g_0(\mathbf{z}_\nu) = 0$. We consider the functions $f_1, f_2 : \overline{V} \to \mathbb{R}^q$ defined in (1.8.7). Then the following statements hold.

(a) If there exists $\nu^* \in V$ with $f_1(\nu^*) = 0$ and $\det(D\,f_1(\nu^*)) \neq 0$, then there exists ν_ε such that $g(\mathbf{z}_{\nu_\varepsilon},\varepsilon) = 0$ and $\mathbf{z}_{\nu_\varepsilon} \to \mathbf{z}_{\nu^*}$ when $\varepsilon \to 0$.

(*b*) Assume that $f_1 = 0$. If there exists $\nu^* \in V$ with $f_2(\nu^*) = 0$ and $\det(D\, f_2(\nu^*)) \neq 0$, then there exists ν_ε such that $g(\mathbf{z}_{\nu_\varepsilon}, \varepsilon) = 0$ and $\mathbf{z}_{\nu_\varepsilon} \to \mathbf{z}_{\nu^*}$ when $\varepsilon \to 0$.

The proof of this lemma can be found in [17, 72].

Note that in Lemma 1.8.2 the functions g_i for $i = 0, 1, 2$ which appears in the expression of (1.8.7) and (1.8.8) are the ones of the function $g(z, \varepsilon) = g_0(z) + \varepsilon g_1(z) + \varepsilon^2 g_2(z) + \mathcal{O}(\varepsilon^3)$, instead of the functions which appear in (1.8.9).

Now we prove Theorem D

Proof of Theorem D. Let $\psi(\theta, \mathbf{z}, \varepsilon)$ be the solution of system (1.8.4) such that $\psi(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. Similarly let $\psi^\pm(\theta, \mathbf{z}, \varepsilon)$ be the solutions of the systems $\mathbf{x}' = F^\pm(\theta, \mathbf{x}, \varepsilon)$ such that $\psi^\pm(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. So

$$\psi(\theta, \mathbf{z}, \varepsilon) = \begin{cases} \psi^+(\theta, \mathbf{z}, \varepsilon) & \text{if} \quad 0 \leq \theta \leq \phi, \\ \\ \psi^-(\theta, \mathbf{z}, \varepsilon) & \text{if} \quad \phi \leq \theta \leq T, \end{cases}$$

Since the vector field (1.8.4) is T–periodic it may also read

$$\psi(\theta, \mathbf{z}, \varepsilon) = \begin{cases} \psi^+(\theta, \mathbf{z}, \varepsilon) & \text{if} \quad 0 \leq \theta \leq \phi, \\ \\ \psi^-(\theta, \mathbf{z}, \varepsilon) & \text{if} \quad \phi - T \leq \theta \leq 2\pi. \end{cases}$$

Now we consider the function $g(\mathbf{z}, \varepsilon) = \psi^+(\phi, \mathbf{z}, \varepsilon) - \psi^-(\phi - \theta, \mathbf{z}, \varepsilon)$. It is easy to see that the solution $\psi(\theta, \mathbf{z}, \varepsilon)$ is T–periodic in θ if and only if $g(\mathbf{z}, \varepsilon) = 0$. So from hypothesis (H) we have that $g(\mathbf{z}_\nu) = 0$ for every $\mathbf{z}_\nu \in \mathcal{Z}$.

Applying Lemma 1.4.1 to the functions $\psi^\pm(\theta, \mathbf{z}, \varepsilon)$ we obtain

$$\psi^\pm(\theta, \mathbf{z}, \varepsilon) = y_0^\pm(\theta, \mathbf{z}) + \varepsilon y_1^\pm(\theta, \mathbf{z}) + \varepsilon^2 \frac{y_2^\pm(\theta, \mathbf{z})}{2} + \mathcal{O}(\varepsilon^2),$$

where $y_i(\theta, \mathbf{z})$ is given in (1.8.10). Therefore $g(\mathbf{z}, \varepsilon) = g_0(\mathbf{z}) + \varepsilon g_1(\mathbf{z}) + \varepsilon^2 g_2(\mathbf{z}) + \mathcal{O}(\varepsilon^2)$, where $g_i(\mathbf{z}) = y_i^+(\phi, \mathbf{z}) - g_i^-(\phi - T, \mathbf{z})$ for $i = 0, 1, 2$. Moreover

$$Dg_0(\mathbf{z}) = \frac{\partial \varphi^+}{\partial \mathbf{z}}(\phi, \mathbf{z}) - \frac{\partial \varphi^-}{\partial \mathbf{z}}(\phi - T, \mathbf{z}) = Y^+(\phi, \mathbf{z}) - Y^-(\phi - T, \mathbf{z}).$$

So from hypothesis (H) we have that the matrix $Dg_0(\mathbf{z})$ has in the upper right corner the null $q \times (d-q)$ matrix, and in the lower right corner has the $(p-q) \times (p-q)$ matrix Δ_ν with $\det(\Delta_\nu) \neq 0$.

The proof of this theorem concludes by applying Lemma 1.8.2 for the function $g(\mathbf{z}, \varepsilon)$ defined above. $\square$

Chapter 2

On the continuation of periodic solutions in discontinuous dynamical systems

The main results of this chapter (Theorems E, F, G, and H) are based on the papers [71, 72].

2.1 Introduction to the non–smooth averaging theory

One of the main problem in the qualitative theory of differential systems is the study of their periodic solutions. A good tool to study the periodic solutions is the averaging theory, see for instance the books of Sanders, Verhulst, and Murdock [99] and Verhulst [113]. We point out that the method of averaging is a classical and matured tool that provides a useful means to study the behaviour of nonlinear smooth dynamical systems. The method of averaging has a long history that starts with the classical works of Lagrange and Laplace who provided an intuitive justification of the process. The first formalization of this procedure was given by Fatou in 1928 [32]. Very important practical and theoretical contributions in the averaging theory were made by Krylov and Bogoliubov [9] in the 1930s and Bogoliubov [8] in 1945.

On the other hand the study of the discontinuous differential systems has it importance and motivation lying in some fields of the applied sciences. Indeed, in these last years a big interest has appeared for studying such systems. This interest has been stimulated by discontinuous phenomena in control systems [5], impact and friction mechanics [13], nonlinear oscillations [2, 88], economics [44, 54], and biology [6, 62], and it has become certainly one of the common frontiers between Mathematics, Physics and Engineering. For more details see Teixeira [106]. A recent review appears in [112].

Despite to the importance of the discontinuous differential systems mentioned above, there still exist only a few analytical techniques to study the invariant sets of discontinuous differential systems. In [77] the averaging theory has been extended for the following class of discontinuous differential systems

$$x'(t) = \begin{cases} \varepsilon F_1(t,x) + \varepsilon^2 R_1(t,x,\varepsilon) & \text{if} \quad h(t,x) > 0, \\ \varepsilon F_2(t,x) + \varepsilon^2 R_2(t,x,\varepsilon) & \text{if} \quad h(t,x) < 0. \end{cases} \tag{2.1.1}$$

where F_1, F_2, R_1, R_2 and h are continuous functions, locally Lipschitz in the variable x, T–periodic in the variable t, and h is a $\mathcal{C}^1$ function having 0 as a regular value. The results stated in [77] have been extensively used, see for instance the works [69, 70, 89, 68, 84].

In this chapter we focus on the development and improvement of the averaging theory for studying periodic solutions of a much bigger class of discontinuous differential systems than (2.1.1). Regarding the averaging theory for finding periodic solutions there are essentially three main theorems. In what follows we describe these theorems.

The first one is concerning about the study of the periodic solutions of the periodic differential systems of the form

$$x' = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \cdots + \varepsilon^m F_m(t,x) + \varepsilon^{m+1} R(t,x,\varepsilon),$$

with $x \in \mathbb{R}^d$. For continuous differential systems, even for the non–differentiable ones, this theory is already completely developed (see for instance [113, 99, 19, 40, 74, 73]), and for discontinuous differential systems this theory is develop up to order 2 in ε (see [77, 71]).

The other two theorems go back to the works of Malkin [86] and Roseau [98]. They studied the periodic solutions of the periodic differential systems of the form

$$x' = F_0(t,x) + \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \cdots + \varepsilon^m F_m(t,x) + \varepsilon^{m+1} R(t,x,\varepsilon),$$

with $x \in \mathbb{R}^d$, distinguishing when the manifold $\mathcal{Z}$ of all periodic solutions of the unperturbed system $x' = F_0(t,x)$ has dimension d or smaller than d. These theories are well developed for continuous differential systems (see for instance [16, 17, 18, 97, 40, 74, 73]). Nevertheless there is no theory for studying such problems in discontinuous differential systems. Thus our main objective in this chapter is to develop these last theorems for a big class of discontinuous differential systems.

Here, assuming $F_0 \neq 0$ (resp. $F_0 \neq 0$), we develop the averaging theory of first order (resp. first and second order) for studying the periodic solutions of discontinuous piecewise differential systems in arbitrary dimension and with an arbitrary number of systems (pieces). We generalize the results established in [19, 77] considering minimal conditions of differentiability. Furthermore, we use this theory to study perturbed linear systems.

2.2 Preliminaries: Discontinuous dynamical systems

In what follows we define the necessary elements for the statements of our main results.

Let D be an open subset of $\mathbb{R}^d$ and $\mathbb{S}^1 = \mathbb{R}/T$ for some period $T > 0$. We consider a finite set of ODE's

$$x'(t) = f^n(t,x), \quad (t,x) \in I \times D \quad \text{for} \quad n = 1, 2, \ldots, M, \tag{2.2.1}$$

where $f^n : \mathbb{S}^1 \times D \to \mathbb{R}^d$ is a continuous function. Here the prime denotes derivative with respect to the time t. Let (S_n) be a finite sequence of open disjoints subset of $\mathbb{S}^1 \times D$ for $n = 1, 2, \ldots, M$. We suppose that the boundaries of each S_n are piecewise $\mathcal{C}^k$ embedded hypersurfaces with $k \geq 1$. Furthermore the union of all boundaries, denoted by Σ, and all S_n together cover $\mathbb{S}^1 \times D$. So we

define a *M–Discontinuous Piecewise Differential System* (*M–DPDS*) as

$$x'(t) = f(t,x) = \begin{cases} f^1(t,x), & (t,x) \in \overline{S}_1, \\ f^2(t,x), & (t,x) \in \overline{S}_2, \\ & \vdots \\ f^M(t,x), & (t,x) \in \overline{S}_M, \end{cases} \tag{2.2.2}$$

where $\overline{S}_k$ denotes the closure of S_k in D. When the context is clear we shall refer to the systems of kind (2.2.2) only by $DPDS$. Later on in this chapter it will be assumed that the functions f^n are Lipschitz in the second variable for $n = 1, 2, \ldots, M$. However the theory described in the following is developed without these assumptions.

Let A be a subset of $\mathbb{S}^1 \times D$ and let $\chi_A(t,x)$ be the *characteristic function* defined as

$$\chi_A(t,x) = \begin{cases} 1 & \text{if} \quad (t,x) \in A, \\ 0 & \text{if} \quad (t,x) \notin A. \end{cases}$$

So system (2.2.2) can be written as

$$x'(t) = f(t,x) = \sum_{n=1}^{M} \chi_{\overline{S}_n}(t,x) f^n(t,x), \quad (t,x) \in \mathbb{S}^1 \times D. \tag{2.2.3}$$

We stress that systems (2.2.2) and (2.2.3) does not coincides in Σ. Indeed system (2.2.2) is multivalued in Σ whereas system (2.2.3) is single valued in Σ. Nevertheless the Filippov's convention for the solutions of these systems (see [34]) passing through a point $(t,x) \in \Sigma$ does not depend on the value $f(t,x)$. So the solutions of systems (2.2.2) and (2.2.3), in the sense of Filippov, are the same.

We say that a point $p \in \Sigma$ is a *generic point of discontinuity* if there exists a neighborhood $G_p \subset \mathbb{S}^1 \times D$ of p such that $\mathcal{S}_p = G_p \cap \Sigma$ is a $\mathcal{C}^k$ embedded hypersurface in $\mathbb{S}^1 \times D$ with $k \geq 1$. In this case we can always assume that $\mathcal{S}_p$ splits $G_p \backslash \mathcal{S}_p$ in two disconnected regions, namely G_p^+ and G_p^-, and that the vector fields $f_p^+ = f|_{G_p^+}$ and $f_p^- = f|_{G_p^-}$ are continuous. We define $l(p)$ as the segment connecting the vectors $f_p^+(p)$ and $f_p^-(p)$ when they have the same origin p

Let $\mathcal{S} \subset \Sigma$ be an embedded hypersurface in $\mathbb{S}^1 \times D$ and $T_p\mathcal{S}$ denotes the tangent space of $\mathcal{S}$ at the point p. The set $\Sigma^c(\mathcal{S}) = \{p \in \mathcal{S} : l(p) \cap T_p\mathcal{S} = \emptyset\}$ is called the *crossing region* of the hypersurface $\mathcal{S}$. This definition only makes sense when the linear space $T_p\mathcal{S}$ is based at the origin of the vectors $F_p^+(p)$ and $F_p^-(p)$. Moreover when the hypersurface $\mathcal{S} \subset \Sigma$ is given by $\mathcal{S} = h^{-1}(0)$ for some $\mathcal{C}^1$ function $h : \mathbb{S}^1 \times D \to \mathbb{R}$ having 0 as a regular value, the crossing region of $\mathcal{S}$ writes

$$\Sigma^c(\mathcal{S}) = \Big\{ p \in \mathcal{S} : \langle \nabla h(p), (1, F^+(p)) \rangle \langle \nabla h(p), (1, F^-(p)) \rangle > 0 \Big\} \tag{2.2.4}$$

Globally we define the *crossing region* Σ^c as the generic points of discontinuity p such that $p \in \Sigma^c(\mathcal{S}_p)$. Later on this chapter for a point $p \in \Sigma^c$ we shall denote $T_p\Sigma = T_p\mathcal{S}_p$.

For a point $q \in S_n$ we denote by $\varphi_{F^n}(t,q)$ the solution of system (2.2.1) such that $\varphi_{F^n}(0,q) = q$. Now for a point $p \in \Sigma^c$ such that $l(p) \subset G_p^+$ and taking the origin of time at p, the trajectory through p, given by the Filippov's convention, is defined as $\varphi_F(t,p) = \varphi_{F_p^-}(t,p)$ for $t \in I_p \cap \{t < 0\}$, and $\varphi_F(t,p) = \varphi_{F_p^+}(t,p)$ for $t \in I_p \cap \{t > 0\}$. Here I_p is an open interval having the 0 in its interior. For the case $l(p) \subset G_p^-$ the definition is the same reversing the time.

The following proposition gives a condition for the existence and uniqueness of solutions of system (2.2.3).

Proposition 2.2.1. For every point $p \in \Sigma^c$ there is a solution passing either from G_p^- into G_p^+, or from G_p^+ into G_p^-, and uniqueness in not violated.

For a proof of Proposition 2.2.1 see Corollary 1 of section 10 of chapter 2 of [34].

Assuming that the functions $f^n(t,x)$ are Lipschitz in the variable x for $n = 1, 2, \ldots, N$, Proposition 2.2.1 implies the uniqueness of the solutions which reach the set of discontinuity only at points of Σ^c.

2.3 Case 1: Averaging of first and second order for a vanishing F_0

We consider the following DPDS.

$$x'(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 R(t,x,\varepsilon), \tag{2.3.1}$$

with

$$F_i(t,x) = \sum_{j=1}^{M} \chi_{\overline{S}_j}(t,x) F_i^j(t,x), \quad \text{for } i = 1,2, \text{ and}$$

$$R(t,x,\varepsilon) = \sum_{j=1}^{M} \chi_{\overline{S_j}}(t,x) R^j(t,x),$$

where $F_i^j : \mathbb{S}^1 \times D \to \mathbb{R}^d$, $R^j : \mathbb{S}^1 \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^d$ for $i = 1,2$ and $j = 1, 2, \ldots, M$ are continuous functions, T–periodic in the variable t and D is an open subset of $\mathbb{R}^d$. For $i = 1,2$ we denote

$$D_x F_i(t,z) = \sum_{j=1}^{M} \chi_{\overline{S}_j}(t,z) D_x F_i^j(t,z). \tag{2.3.2}$$

In order to state our main results we define the averaged functions $f_1, f_2 : D \to \mathbb{R}^d$ as

$$f_1(z) = \int_0^T F_1(t,z)dt, \quad \text{and} \tag{2.3.3}$$

$$f_2(z) = \int_0^T \Big(D_x F_1(t,z) y_1(t,z) + F_2(t,z) \Big) dt, \tag{2.3.4}$$

where

$$y_1(t,z) = \int_0^t F_1(s,z)ds.$$

Moreover we state the next condition which is common for our main results.

(HC) There exists an open bounded set $C \subset D$ such that for each $z \in \overline{C}$ the curve $\{(t,z) : t \in \mathbb{S}^1\}$ reaches transversely the set Σ and only at generic points of discontinuity.

The principal consequence of assumption (HC) is the following:

Proposition 2.3.1. The assumption (HC) implies that, for $|\varepsilon| \neq 0$ sufficiently small, every solution of (2.3.1) starting in $\overline{C}$ reaches the set of discontinuity Σ only at its crossing region.

Proposition 2.3.1 is proved in subsection 2.7.1.

Our main results on the periodic orbits of DPDS (2.3.1) are given in the next two theorems. Their proofs use the Brouwer degree theory for finite dimensional spaces (see Appendix A for a definition of the Brouwer degree $d_B(f, V, 0)$).

Theorem E (First order averaging)**.** In addition to the crossing hypothesis (HC) assume the following conditions.

$(Ha1)$ For $i = 1, 2$ and $j = 1, 2, \ldots, M$, the continuous functions F_i^j and R_i^j are locally Lipschitz with respect to x, and T–periodic with respect to the time t. Furthermore, for $j = 1, 2, \ldots, M$, the boundaries of S_j are piecewise $\mathcal{C}^k$ embedded hypersurfaces with $k \geq 1$.

$(Ha2)$ For $a^* \in C$ with $f_1(a^*) = 0$, there exist a neighborhood $U \subset C$ of a^* such that $f_1(z) \neq 0$ for all $z \in \overline{U} \backslash \{a^*\}$ and $d_B(f_1, U, 0) \neq 0$.

Then for $|\varepsilon| \neq 0$ sufficiently small, there exists a T–periodic solution $x(t, \varepsilon)$ of system (2.3.1) such that $x(0, \varepsilon) \to a^*$ as $\varepsilon \to 0$.

Theorem E is proved in subsection 2.7.1.

Theorem F (Second order averaging)**.** Suppose that $f_1(z) \equiv 0$. In addition to the crossing hypothesis (HC) assume the following conditions.

$(Hb1)$ For $j = 1, 2, \ldots, M$, the functions $F_1^j(t, \cdot)$ are of class $\mathcal{C}^1$ for all $t \in \mathbb{R}$; for $j = 1, 2, \ldots, M$, the functions $D_x F_1^j$, F_2^j and R are locally Lipschitz with respect to x. Furthermore, for $j = 1, 2, \ldots, M$, the boundaries of S_j are piecewise $\mathcal{C}^k$ embedded hypersurfaces with $k \geq 1$.

$(Hb2)$ If $(t, z) \in \Sigma$ then $(0, y_1(t, z)) \in T_{(t,z)}\Sigma$.

$(Hb3)$ For $a^* \in C$ with $f_2(a^*) = 0$, there exist a neighborhood $U \subset C$ of a^* such that $f_2(z) \neq 0$ for all $z \in \overline{U} \backslash \{a^*\}$ and $d_B(f_2, U, 0) \neq 0$.

Then for $|\varepsilon| \neq 0$ sufficiently small, there exists a T–periodic solution $x(t, \varepsilon)$ of system (2.3.1) such that $x(0, \varepsilon) \to a^*$ as $\varepsilon \to 0$.

Theorem F is also proved in subsection 2.7.1.

We remark that when f_1 (resp. f_2) is a $\mathcal{C}^1$ function the assumption "there exists $a^* \in V$ such that $f_1(a^*) = 0$ (resp. $f_2(a^*) = 0$) and $\det(f_1'(a^*)) \neq 0$ (resp. $\det(f_2'(a^*)) \neq 0$)" is a sufficient condition to guarantees the validity of the hypothesis $(Ha2)$ (resp. $(Hb3)$).

2.4 Remark on discontinuous perturbation of planar linear centers

In this subsection we show how to use the Theorems E and F for studying the linear centers perturbed by DPDS having the set of discontinuity composed by rays passing through the origin of coordinates. In other words we shall show that the hypothesis of crossing (HC) and the hypothesis $(Hb2)$ of Theorem F always hold for such systems after a change of variables and a time–rescaling.

Let M be a positive integer greater than 1, let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_M) \in \mathbb{T}^M$ (M–Torus) be a M–tuple of angles such that $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_M < 2\pi$ and let $\mathcal{X} = (X_1, X_2, \cdots, X_M)$ be a M–tuple of locally Lipschitz vector fields defined on an open neighborhood $D \subset \mathbb{R}^2$ of the origin.

We define the set of discontinuity $\Sigma = \bigcup_{i=1}^{M} L_i$, where L_i for $i = 1, 2, \ldots, M$, is the intersection between the ray starting at the origin and passing through the point $(\cos\alpha_i, \sin\alpha_i)$ with the set D. We note that the set Σ splits the set $D\backslash\Sigma \subset \mathbb{R}^2$ in M disjoint open sectors. We denote the sector delimited by L_i and L_{i+1} by C_i for $i = 1, 2, \ldots, M$.

Now let $Z_{\mathcal{X},\alpha}(x, y)$ be the DPDS defined in D as

$$Z_{\mathcal{X},\alpha}(x, y) = X_i(x, y) \quad \text{if} \quad (x, y) \in C_i.$$

Let $\mathcal{X}$ and $\mathcal{Y}$ be two M–tuples of vector fields. We shall study the following DPDS.

$$(\dot{x}\,,\,\dot{y}) = (y\,,\,-x) + \varepsilon Z_{\mathcal{X},\alpha}(x, y) + \varepsilon^2 Z_{\mathcal{Y},\alpha}(x, y). \tag{2.4.1}$$

Here the dot denotes derivative with respect to the time t.

Using the polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, system (2.4.1) becomes equivalent to

$$\left(\dot{\theta}\,,\,\dot{r}\right) = (-1, 0) + \varepsilon A(\theta, r) + \varepsilon^2 B(\theta, r), \tag{2.4.2}$$

where A and B are DPDS with the set of discontinuity $\widetilde{\Sigma}$ being the union of the rays $\{(\alpha_i, r) : r > 0\}$ for $i = 1, 2, \ldots, M$. Moreover $A(\theta, r) = A_i(\theta, r)$ and $B(\theta, r) = B_i(\theta, r)$ if $\alpha_i \leq \theta \leq \alpha_{i+1}$ for $i = 1, 2, \ldots, M$, where $\alpha_{M+1} = \alpha_1$, and

$$\begin{aligned} A_i(\theta, r) = \; & \Bigg(\frac{1}{r}\left(X_i^2\left(r\cos\theta, r\sin\theta\right)\cos\theta - X_i^1\left(r\cos\theta, r\sin\theta\right)\sin\theta\right), \\ & X_i^1\left(r\cos\theta, r\sin\theta\right)\cos\theta + X_i^2\left(r\cos\theta, r\sin\theta\right)\sin\theta\Bigg) \end{aligned}$$

and

$$\begin{aligned} B_i(\theta, r) = \; & \Bigg(\frac{1}{r}\left(Y_i^2\left(r\cos\theta, r\sin\theta\right)\cos\theta - Y_i^1\left(r\cos\theta, r\sin\theta\right)\sin\theta\right), \\ & Y_i^1\left(r\cos\theta, r\sin\theta\right)\cos\theta + Y_i^2\left(r\cos\theta, r\sin\theta\right)\sin\theta\Bigg). \end{aligned}$$

Here $X_i = (X_i^1, X_i^2)$ and $Y_i = (Y_i^1, Y_i^2)$ for $i = 1, 2, \ldots, M$.

Taking θ as the new time system (2.4.2) writes

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = \frac{\varepsilon A_i^2(\theta, r) + \varepsilon^2 B_i^2(\theta, r)}{-1 + \varepsilon A_i^1(\theta, r) + \varepsilon^2 B_i^1(\theta, r)},$$

for $\alpha_i \leq \theta \leq \alpha_{i+1}$. Here $A_i = (A_i^1, A_i^2)$ and $B_i = (B_i^1, B_i^2)$ for $i = 1, 2, \ldots, M$. So system (2.4.2) and consequently system (2.4.1) become equivalent to

$$r' = \mathcal{R}(\theta, r, \varepsilon), \tag{2.4.3}$$

where, for $i = 1, 2, \ldots, M$, $\mathcal{R}(\theta, r, \varepsilon) = \mathcal{R}_i(\theta, r, \varepsilon)$ if $\alpha_i \leq \theta \leq \alpha_{i+1}$, and

$$\begin{aligned}
\mathcal{R}_i(\theta, r, \varepsilon) = \; & -\varepsilon \left(X_i^1 \left(r\cos\theta, r\sin\theta\right)\cos\theta + X_i^2 \left(r\cos\theta, r\sin\theta\right)\sin\theta \right) \\
& -\varepsilon^2 \Big(\frac{1}{r} \left(X_i^2 \left(r\cos\theta, r\sin\theta\right)\cos\theta - X_i^1 \left(r\cos\theta, r\sin\theta\right)\sin\theta \right) \\
& \cdot \left(X_i^1 \left(r\cos\theta, r\sin\theta\right)\cos\theta + X_i^2 \left(r\cos\theta, r\sin\theta\right)\sin\theta \right) \\
& + \left(Y_i^1 \left(r\cos\theta, r\sin\theta\right)\cos\theta + Y_i^2 \left(r\cos\theta, r\sin\theta\right)\sin\theta \right) \Big) + \mathcal{O}(\varepsilon^3).
\end{aligned}$$

Now the prime denotes derivative with respect to the time θ.

Proposition 2.4.1. The hypotheses (HC) and $(Hb2)$ hold for system (2.4.3).

Proof. The assumption (HC) holds because the set of discontinuity of system (2.4.3) is the union of the rays $\widetilde{\Sigma}_i = \{(\alpha_i, r) : r > 0\}$ for $i = 1, 2, \ldots, M$. Let $h_i(\theta, r) = \theta - \alpha_i$, so $\widetilde{\Sigma}_i = h_i^{-1}(0)$. Hence $(s, y_1(\alpha_i, r)) \in T_{(\alpha_i, r)}\widetilde{\Sigma}$ if and only if $0 = \langle (1, 0), (s, y_1(\alpha_i, r)) \rangle = \langle \nabla h_i(\alpha_i, r), (s, y_1(\alpha_i, r)) \rangle = s$. Therefore $(0, y_1(\theta, r)) \in T_{(\theta, r)}\widetilde{\Sigma}$ for every $(\theta, r) \in \widetilde{\Sigma}$. $\square$

2.5 Case 2: Averaging of first order for a nonvanishing F_0

Let D be an open subset of $\mathbb{R}^d$ and for $n = 1, 2, \ldots, N$ let $F_0^n : \mathbb{S}^1 \times D \to \mathbb{R}^d$ be a $\mathbb{C}^m$ function with $m \geq 1$, and $F_1^n : \mathbb{S}^1 \times D \to \mathbb{R}^d$, and $R^n : \mathbb{S}^1 \times D \times [0, 1] \to \mathbb{R}^d$ be continuous functions which are Lipschitz in the second variable. Later on in this chapter we shall assume more conditions under these functions.

Now taking

$$\begin{aligned}
F_i(t, x) &= \sum_{n=1}^{N} \chi_{\overline{S_n}}(t, x) F_i^n(t, x), \quad \text{for } i = 0, 1, \text{ and} \\
R(t, x, \varepsilon) &= \sum_{n=1}^{N} \chi_{\overline{S_n}}(t, x) R^n(t, x),
\end{aligned}$$

we consider the following DPDS,

$$x'(t) = F_0(t, x) + \varepsilon F_1(t, x) + \varepsilon^2 R(t, x, \varepsilon). \tag{2.5.1}$$

The parameter ε is assumed to be small. We recall that Σ denotes the union of the boundaries of S_n for $n = 1, 2, \ldots, N$.

A first approach to deal with the periodic solutions of system (2.5.1) would use the regularizarion technique (see [103]) mimetizing the procedure of [77]. Nevertheless this approach does not apply directly in our problem because it demands more information about the set of discontinuity than we have, for instance in [77] it is assume that Σ is a regular manifold.

In order to present our main results we have to introduce more definitions and notation.

For $z \in D$ and $\varepsilon > 0$ sufficiently small we denote by $x(\cdot, z, \varepsilon) : [0, t_{(z,\varepsilon)}) \to \mathbb{R}^d$ the solution of system (2.5.1) such that $x(0, z, \varepsilon) = z$. Given a subset B of D we define $\widetilde{B}^\varepsilon = \overline{\{(t, x(t, z, \varepsilon)) : z \in B, t \in [0, t_{(z,\varepsilon)})\}}$.

We denote by Σ_0 the set of points $x \in D$ such that the function $F(0, x)$ is discontinuous, clearly $\{0\} \times \Sigma_0 \subset \Sigma$.

One of the main hypothesis of this chapter is that the unperturbed system

$$x'(t) = F_0(t, x), \tag{2.5.2}$$

has a manifold $\mathcal{Z}$ embedded in $D \backslash \partial\Sigma_0$ such that the solutions starting in $\mathcal{Z}$ are all T–periodic functions and reach the set of discontinuity Σ only at its crossing region Σ^c. Here $\partial\Sigma_0$ denotes the boundary of Σ_0 with respect to topology of D. Precisely,

(H) let $\mathcal{Z} = \{z_\alpha = (\alpha, \beta_0(\alpha)) : \alpha \in \overline{V}\}$, where V is an open bounded subset of $\mathbb{R}^k$, and $\beta_0 : \overline{V} \to \mathbb{R}^{d-k}$ is a $\mathbb{C}^m$ function with $m \geq 1$. We shall assume that $\mathcal{Z} \subset D$, $\mathcal{Z} \cap \partial\Sigma_0 = \emptyset$, $\widetilde{\mathcal{Z}}^0 \cap \Sigma \subset \Sigma^c$ and for each $z_\alpha \in \mathcal{Z}$ the unique solution $x_\alpha(t) = x(t, z_\alpha, 0)$ is T–periodic.

Remark 2.5.1. Suppose that the solution $x_\alpha(t)$ reaches the set Σ^c κ_α times. The assumption $\mathcal{Z} \cap \partial\Sigma_0 = \emptyset$ in hypothesis (H) implies that for each $z_\alpha \in \mathcal{Z}$ there exists a small neighborhood $U_\alpha \subset D$ of z_α such that for $\varepsilon > 0$ sufficiently small every solution of the perturbed system (2.5.1) starting in U_α reach the crossing region of the set of discontinuity Σ^c also κ_α times. This fact will be well justified in the proofs of Lemmas 2.7.5 and 2.7.6 in subsection 2.7.2

For $z \in D$ we take the following discontinuous piecewise linear differential system

$$y' = D_x F_0(t, x(t, z, 0))\, y, \tag{2.5.3}$$

which can be seen as the linearization of the unperturbed system (2.5.2) along the solution $x(t, z, 0)$. We note that for each $z \in D$ the matrix–valued function $Q(t) = D_x F_0(t, x(t, z, 0))$ is piecewise $\mathbb{C}^m$ differentiable with $m \geq 1$, so we can consider a fundamental matrix $Y(t, z)$ of the differential system (2.5.3).

$$y_1(t, z) = Y(t, z) \int_0^t Y(s, z)^{-1} F_1(s, x(s, z, 0)) ds. \tag{2.5.4}$$

Now for $z_\alpha \in \mathcal{Z}$ we denote $Y_\alpha(t) = Y(t, z_\alpha)$. Let $\pi : \mathbb{R}^k \times \mathbb{R}^{d-k} \to \mathbb{R}^k$ and $\pi^\perp : \mathbb{R}^k \times \mathbb{R}^{d-k} \to \mathbb{R}^{d-k}$ be the projections onto the first k coordinates and onto the last $d - k$ coordinates, respectively. Thus we define the averaged function $f_1 : \overline{V} \to \mathbb{R}^k$ as

$$f_1(\alpha) = \pi y_1(T, z_\alpha).$$

In what follows $\mathrm{dis}(x, A)$ denotes the Hausdorff distance function between a point $x \in D$ and a set $A \subset D$, and as usual the function $d_B(f_1, W, 0)$ denotes the Brouwer degree (see for instance [15] for details on the Brouwer degree). Our main result on the periodic solutions of DPDS (2.5.1) is the following.

Theorem G. In addition to the hypothesis (H) we assume that

$(H1)$ for $n = 1, 2, \ldots, N$, the functions F_0^n and β_0 are of class $\mathcal{C}^1$; the continuous functions $D_x F_0^n$, F_1^n and R are locally Lipschitz with respect to x; and the boundary of S_n are piecewise $\mathcal{C}^1$ embedded hypersurface in $\mathbb{R} \times D$;

$(H2)$ there exists a fundamental matrix solution $Y(t, z)$ of (2.5.3) such that, for every $\alpha \in \overline{V}$, the matrix $Y_\alpha(T)Y_\alpha(0)^{-1} - Id$ has in the upper right corner the null $k \times (d-k)$ matrix, and in the lower right corner has the $(d-k) \times (d-k)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$;

$(H3)$ for an open subset U of D such that $\mathcal{Z} \subset U$ we have that $(0, y_1(s, z)) \in T_{(s,x(s,z,0))}\Sigma$ whenever $(s, x(s, z, 0)) \in \Sigma^c$ for $(s, z) \in \mathbb{S}^1 \times U$;

$(H4)$ there exists W open subset of V such that $f_1(\alpha) \neq 0$ for $\alpha \in \partial W$ and $d_B(f_1, W, 0) \neq 0$.

Then for $\varepsilon > 0$ sufficiently small, there exists a T–periodic solution $\varphi(t, \varepsilon)$ of system (2.5.1) such that $\mathrm{dis}(\varphi(0, \varepsilon), \mathcal{Z}) \to 0$ as $\varepsilon \to 0$.

Theorem G is proved in subsection 2.7.2.

Remark 2.5.2. When f_1 is a $\mathbb{C}^1$ function the assumption

(h4) there exists $a \in V$ such that $f_1(a) = 0$ and $\det(f_1'(a)) \neq 0$,

is a sufficient condition to guarantees the validity of the hypothesis $(H4)$.

Theorem H. We suppose that the hypotheses (H), $(H2)$ and $(H3)$ of Theorem G hold. If we assume that

$(h1)$ for $n = 1, 2, \ldots, N$, F_0^n, $D_x F_0^n$, F_1^n, R^n, and β_0 are $\mathcal{C}^2$ functions and the boundary of S_n are piecewise $\mathcal{C}^2$ embedded hypersurface in $\mathbb{R} \times D$,

then $f_1(\alpha)$ is a $\mathcal{C}^1$ function for every $\alpha \in \overline{V}$. Moreover, if we assume in addition that hypothesis $(h4)$ holds, then for $\varepsilon > 0$ sufficiently small, there exists a T–periodic solution $\varphi(t, \varepsilon)$ of system (2.5.1) such that $\varphi(0, \varepsilon) \to z_a$ as $\varepsilon \to 0$.

2.6 Examples of applications

2.6.1 Example 1: Application of Theorem E

In the following example we solve a problem of type (2.4.1).

Consider $\alpha = (0, \pi/2, \pi, 3\pi/2, 2\pi) \in \mathbb{T}^4$. Thus $L_1 = \{(x, 0) : x > 0\}$, $L_2 = \{(0, y) : y > 0\}$, $L_3 = \{(x, 0) : x < 0\}$, and $L_4 = \{(0, y) : y < 0\}$. Then for $i = 1, ..., 4$ we have that C_i is the first, second, third and fourth quadrants, respectively.

In this example we study the maximum number of limit cycles given by the averaging theory of first and second order for DPDS, which can bifurcate from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$, perturbed inside the following class of linear DPDS:

$$\dot{X} = Y_i(x, y) \quad \text{if} \quad (x, y) \in C_i, \ i = 1, ..., 4 \tag{2.6.1}$$

where

$$Y_i(x, y) = \begin{pmatrix} y + \varepsilon P_i^1(x, y) + \varepsilon^2 P_i^2(x, y) \\ \\ -x + \varepsilon Q_i^1(x, y) + \varepsilon^2 Q_i^2(x, y) \end{pmatrix},$$

with $P_i^1(x, y) = a_{0i} + a_{1i}x + a_{2i}y$, $P_i^2(x, y) = c_{0i} + c_{1i}x + c_{2i}y$, $Q_i^1(x, y) = b_{0i} + b_{1i}x + b_{2i}y$, $Q_i^2(x, y) = d_{0i} + d_{1i}x + d_{2i}y$ and $|\varepsilon| \neq 0$ is a small parameter.

Let $\mathcal{A}$ denote the set of the following two conditions

$$4a_{01} - 4(a_{02} + a_{03} - a_{04} - b_{01} - b_{02} + b_{03} + b_{04}) = 0 \quad \text{and}$$

$$2a_{21} - 2(a_{22} - a_{23} + a_{24} - b_{11} + b_{12} - b_{13} + b_{14}) +$$

$$(a_{11} + a_{12} + a_{13} + a_{14} + b_{21} + b_{22} + b_{23} + b_{24})\pi = 0.$$

Our results on the limit cycles of system (2.6.1) are stated in the next two propositions.

Proposition 2.6.1. For $|\varepsilon| \neq 0$ sufficiently small and using Theorem E system (2.6.1) has at most 1 limit cycle for any chosen of parameters for which the conditions of $\mathcal{A}$ do not hold. Moreover we can find parameters a_{ij}, b_{ij}, c_{ij}, and d_{ij} such that system (2.6.1) has exactly 0 or 1 limit cycle.

Proposition 2.6.2. For $|\varepsilon| \neq 0$ sufficiently small and using Theorem F system (2.6.1) has at most 4 limit cycles for any chosen of parameters for which the two conditions of $\mathcal{A}$ holds. Moreover we can find parameters a_{ij}, b_{ij}, c_{ij}, and d_{ij} such that system (2.6.1) has exactly 0, 1, 2, 3 or 4 limit cycles.

Proposition 2.6.1 and 2.6.2 are proved in section 2.8.

2.6.2 Example 2: Application of Theorem F

In the following example we solve a problem which is not of type (2.4.1).

Let $h(x,y) = y - x^2$. The set $\Sigma = h^{-1}(0)$ is a regular manifold which splits the set $\mathbb{R}^2 \backslash \Sigma$ in two disjoint open regions. We consider the following system

$$\begin{pmatrix} \dot{x} \\ \\ \dot{y} \end{pmatrix} = \begin{cases} x + \varepsilon P^1(x,y) + \varepsilon^2 P^2(x,y), & \\ & \text{if} \quad h(x,y) \geq 0, \\ -y + \varepsilon Q^1(x,y) + \varepsilon^2 Q^2(x,y), & \\ \\ x + \varepsilon R^1(x,y) + \varepsilon^2 R^2(x,y,) & \\ & \text{if} \quad h(x,y) \leq 0, \\ -y + \varepsilon S^1(x,y) + \varepsilon^2 S^2(x,y), & \end{cases} \tag{2.6.2}$$

where

$$P^i = p^i_{00} + p^i_{10}x + p^i_{01}y + p^i_{20}x^2 + p^i_{11}xy + p^i_{02}y^2,$$

$$Q^i = q^i_{00} + q^i_{10}x + q^i_{01}y + q^i_{20}x^2 + q^i_{11}xy + q^i_{02}y^2,$$

$$R^i = r^i_{00} + r^i_{10}x + r^i_{01}y + r^i_{20}x^2 + r^i_{11}xy + r^i_{02}y^2,$$

$$S^i = s^i_{00} + s^i_{10}x + s^i_{01}y + s^i_{20}x^2 + s^i_{11}xy + s^i_{02}y^2,$$

for $i = 1, 2$.

Let $\mathcal{B}$ denote the set of conditions

$$p^1_{00} = p^1_{10} = q^1_{00} = q^1_{01} = q^1_{02} = s^1_{00} = s^1_{02} = 0,$$

$$q^1_{10} = -p^1_{01} - 2p^1_{20}, \quad q^1_{11} = -p^1_{02} - 2p^1_{20},$$

$$q^1_{20} = -p^1_{11}, \quad s^1_{01} = -r^1_{10}, \quad \text{and}$$

$$s^1_{20} = 3r^1_{10} - r^1_{11}.$$

Our results on the limit cycles of system (2.6.2) are given in the next two propositions.

Proposition 2.6.3. For $|\varepsilon| \neq 0$ sufficiently small and using Theorem E system (2.6.2) has at most 4 limit cycles for any chosen of parameters for which the conditions of $\mathcal{B}$ do not hold. Moreover we can find parameters p^1_{ij}, q^1_{ij}, r^1_{ij}, and s^1_{ij} such that system (2.6.2) has exactly 0, 1, 2, 3 or 4 limit cycles.

Proposition 2.6.4. For $|\varepsilon| \neq 0$ sufficiently small and using Theorem F system (2.6.2) has at most 6 limit cycles for any chosen of parameters for which the conditions of $\mathcal{B}$ hold. Moreover we can find parameters p^1_{01}, p^1_{20}, p^1_{11}, p^1_{02}, s^1_{10}, r^1_{ij} p^2_{ij}, q^2_{ij}, r^2_{ij}, and s^2_{ij} such that system (2.6.2) has exactly 0, 1, 2, 3, 4, 5 or 6 limit cycles.

Proposition 2.6.3 and 2.6.4 are proved in subsections 2.8.1 and 2.8.2, respectively.

2.6.3 Example 3: Applications of Theorems G and H

In what follows we provide an application of Theorems G and H. We study the existence of limit cycles which bifurcate from the periodic solutions of the linear differential system $(\dot{u}, \dot{v}, \dot{w}) = (-v, u, w)$ perturbed inside the class of all discontinuous piecewise linear differential systems with two zones separated by the plane $\Sigma = \{v = 0\} \subset \mathbb{R}^3$, i.e.

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{cases} \begin{pmatrix} -v + \varepsilon(a_1^+ + b_1^+ u + c_1^+ v + d_1^+ w) \\ u + \varepsilon(a_2^+ + b_2^+ u + c_2^+ v + d_2^+ w) \\ w + \varepsilon(a_3^+ + b_3^+ u + c_3^+ v + d_3^+ w) \end{pmatrix} & \text{if} \quad v > 0, \\ \begin{pmatrix} -v + \varepsilon(a_1^- + b_1^- u + c_1^- v + d_1^- w) \\ u + \varepsilon(a_2^- + b_2^- u + c_2^- v + d_2^- w) \\ w + \varepsilon(a_3^- + b_3^- u + c_3^- v + d_3^- w) \end{pmatrix} & \text{if} \quad v < 0. \end{cases} \tag{2.6.3}$$

Our result on the existence of a limit cycle of system (2.6.3) is the following.

Proposition 2.6.5. If $(a_2^- - a_2^+)(b_1^- + b_1^+ + c_2^- + c_2^-) > 0$, then for $|\varepsilon| > 0$ sufficiently small there exists a periodic solution $(u(t,\varepsilon), v(t,\varepsilon), w(t,\varepsilon))$ of system (2.6.3) such that $w(0,\varepsilon) \to 0$ when $\varepsilon \to 0$. Moreover, we can find $(u^*, v^*) \in \mathbb{R}^2$ such that

$$||(u^*, v^*)|| = \frac{4(a_2^- - a_2^+)}{\pi(b_1^- + b_1^+ + c_2^- + c_2^+)},$$

and $(u(0,\varepsilon), v(0,\varepsilon)) \to (u^*, v^*)$ when $\varepsilon \to 0$.

Proposition 2.6.5 is proved in subsection 2.8.3.

2.7 Proofs of main results

2.7.1 Proofs of main results of section 2.3

We start this section proving Proposition 2.3.1. Then we state some preliminary lemmas needed to prove our main results. After that, the remainder of this section consists of the proof of Theorems E and F. As usual μ denotes the *Lebesgue Measure*.

Proof of Proposition 2.3.1. For a fixed $z \in \overline{C}$ let $(t^i, z) \in \Sigma$ be a generic point of discontinuity. So there exists a neighborhood $G_{(t^i,z)}$ of (t^i, z) such that $\mathcal{S}_{(t^i,z)} = G_{(t^i,z)} \cap \Sigma$ is a $\mathcal{C}^k$ embedded hypersurface of $\mathbb{S}^1 \times \mathbb{R}^d$ with $k \geq 1$. It is well known that $\mathcal{S}_{(t^i,z)}$ can be locally described as the inverse image of a regular value of a $\mathcal{C}^k$ function, that is, there exists a $\mathcal{C}^k$ function $h_i : G_{(t^i,z)} \to \mathbb{R}$ such

that $\widetilde{G}_{(t^i,z)} \cap \mathcal{S}_{(t^i,z)} = h_i^{-1}(0) \cap \Sigma$. Here $\widetilde{G}_{(t^i,z)}$ is an open subset such that $(t^i, z) \in \widetilde{G}_{(t^i,z)} \subseteq G_{(t^i,z)}$. For $(t, z) \in \widetilde{G}_{(t^i,z)}$ system (2.3.1) becomes

$$x' = \begin{cases} f^+_{(t^i,z)}(t,x,\varepsilon) = \varepsilon F_1^{j_{i+1}}(t,x) + \varepsilon^2 F_2^{j_{i+1}}(t,x) + \varepsilon^3 R^{j_{i+1}}(t,x,\varepsilon) & \text{if} \quad h_i(t,x) > 0, \\ f^-_{(t^i,z)}(t,x,\varepsilon) = \varepsilon F_1^{j_i}(t,x) + \varepsilon^2 F_2^{j_i}(t,x) + \varepsilon^3 R^{j_i}(t,x,\varepsilon) & \text{if} \quad h_i(t,x) < 0. \end{cases}$$

From hypothesis (HC) we know that $(\partial/\partial t)h_i(t^i, z)^2 > 0$. Hence

$$\left\langle \nabla h_i(t^i,z), f^+_{(t^i,z)}(t,x,\varepsilon) \right\rangle \left\langle \nabla h_i(t^i,z), f^-_{(t^i,z)}(t,x,\varepsilon) \right\rangle = \left(\frac{\partial h_i}{\partial t}(t^i,z) \right)^2 + \mathcal{O}(\varepsilon),$$

which is positive for $|\varepsilon| \neq 0$ sufficiently small. So from (2.2.4) we conclude this proof. □

Lemma 2.7.1. The averaged functions (2.3.3) and (2.3.4) are continuous in C.

Proof. Let $z_0 \in C$ and let V be a neighborhood of z_0 with a compact closure contained in C. For $z \in V$ we define the sets $A_z^i(t) = \{s \in [0,t] : (s,z) \in S_i\}$, and $A_z^0(t) = \{s \in [0,t] : (s,z) \in \Sigma\}$. From hypothesis (HC) we have that $\mu\left(A_z^0(t)\right) = 0$ for every $t \in [0,T]$ and $\mathbf{z} \in \overline{C}$. So

$$\begin{aligned} \Delta(t,z,z_0) &= |y_1(t,z_0) - y_1(t,z)| \\ &= \left| \sum_{j=1}^M \int_{A^j_{z_0}(t)} F_1^j(s,z_0)ds - \sum_{j=1}^M \int_{A^j_z(t)} F_1^j(s,z)ds \right| \\ &\leq \sum_{j=1}^M \left| \int_{A^j_{z_0}(t)} F_1^j(s,z_0)ds - \int_{A^j_z(t)} F_1^j(s,z)ds \right| \\ &\leq \sum_{j=1}^M \int_{A^j_{z_0}(t)\cap A^j_z(t)} \left| F_1^j(s,z_0) - F_1^j(s,z) \right| ds + \sum_{j=1}^M \left| \int_{A^j_{z_0}(t)\setminus A^j_z(t)} F_1^j(s,z_0) - \int_{A^j_z(t)\setminus A^j_{z_0}(t)} F_1^j(s,z)ds \right| \\ &\leq MTL|z_0 - z| + \sum_{j=1}^M \left| \int_{A^j_{z_0}(t)\setminus A^j_z(t)} F_1^j(s,z_0)ds - \int_{A^j_z(t)\setminus A^j_{z_0}(t)} F_1^j(s,z)ds \right| \\ &\leq MTL|z_0 - z| + \sum_{j=1}^M L_{1,j} \left(\mu\left(A^j_{z_0}(t)\setminus A^j_z(t)\right) + \mu\left(A^j_z(t)\setminus A^j_{z_0}(t)\right) \right), \end{aligned} \tag{2.7.1}$$

where L is maximum of the Lipschitz constants of the functions F_i^j for $j = 1, 2, \ldots, M$, and $L_{1,j} = \max\{F_1^j(s,z) : (s,z) \in [0,T] \times \overline{V}\}$ for $j = 1, 2, \ldots, M$. We observe that $\mu\left(A^j_{z_0}(t)\setminus A^j_z(t)\right) \to 0$ and $\mu\left(A^j_z(t)\setminus A^j_{z_0}(t)\right) \to 0$, as $z \to z_0$ for every $t \in [0,T]$. Thus $\Delta(t,z,z_0) \to 0$, as $z \to z_0$ for every $t \in [0,T]$. So the function $y_1(t,z)$ is continuous in C for each $t \in [0,T]$. Since $f_1(z) = y_1(T,z)$, we conclude that the averaged function f_1 is continuous in C.

Repeating the computations (2.7.1), now for $\int_0^t F_2(s,z)ds$, we get that this function is continuous for $z \in C$. So to prove the continuity of the function f_2 it is sufficient to estimate the difference

$$D(z_0,z) = \left| \int_0^T \left(D_x F_1(t,z_0) y_1(t,z_0) - D_x F_1(t,z) y_1(t,z) \right) dt \right|,$$

for $z \in V$. Thus

$$\begin{aligned} D(z_0, z) \leq & \int_0^T |D_x F_1(t, z_0) - D_x F_1(t, z)|\, |y_1(t, z_0)| dt + \int_0^T |D_x F_1(t, z)|\, |y_1(t, z_0) - y_1(t, z)| dt \\ \leq & \; TY \int_0^T |D_x F_1(t, z_0) - D_x F_1(t, z)|\, dt + TL' \int_0^T |y_1(t, z_0) - y_1(t, z)| dt, \end{aligned}$$

where $Y = \max\{|y_1(s, z)|\colon\ (s, z) \in [0, T] \times \overline{V}\}$ and $L' = \max_{j=1}^{M}\{|D_x F_1^j(s, z)|\colon\ (s, z) \in [0, T] \times \overline{V}\}$. The function $y_1(t, z)$ is continuous in z. Hence repeating the computations (2.7.1), now for $D_x F_1(t, z)$, we conclude that $D(z_0, z) \to 0$ when $z \to z_0$, which implies the continuity of the averaged function f_2 in C. □

Let $g : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^d$ be a function defined on a small interval $(-\varepsilon_0, \varepsilon_0)$. We say that

1. $g(\varepsilon) = \mathcal{O}(\varepsilon^\ell)$ for some positive integer ℓ if there exists constants $\varepsilon_1 > 0$ and $k > 0$ such that $|g(\varepsilon)| \leq k|\varepsilon^\ell|$ for $-\varepsilon_1 < \varepsilon < \varepsilon_1$.

2. $g(\varepsilon) = o(\varepsilon^\ell)$ for some positive integer ℓ if

$$\lim_{\varepsilon \to 0} \frac{|g(\varepsilon)|}{\varepsilon^\ell} = 0.$$

Here $|\cdot|$ denotes the usual norm in the Euclidean space $\mathbb{R}^n$ for $n \geq 1$. The symbols $\mathcal{O}$ and o are called the *Landau's symbols* (see for instance [99]).

Lemma 2.7.2. Let $x(\cdot, z, \varepsilon) : [0, t_z) \to \mathbb{R}^n$ be the solution of system (2.2.3) with $x(0, z, \varepsilon) = z$. Then we have the following statements.

(a) Under the hypotheses of Theorem E $t_z \geq T$ and the equality $x(t, z, \varepsilon) = z + \varepsilon y_1(t, z) + \mathcal{O}(\varepsilon^2)$ holds.

(b) Under the hypotheses of Theorem F $t_z \geq T$ and the equality $x(t, z, \varepsilon) = z + \varepsilon y_1(t, z) + \varepsilon^2 \int_0^t (D_x F_1(s, z)\ y_1(s, z) + F_2(s, z)) ds + \varepsilon\, o(\varepsilon)$ holds. Furthermore if for $j = 1, 2, \ldots, M$ the boundaries of S_j are piecewise $\mathcal{C}^k$ embedded hypersurfaces with $k \geq 2$ then we have that $x(t, z, \varepsilon) = z + \varepsilon y_1(t, z) + \varepsilon^2 \int_0^t (D_x F_1(s, z) y_1(s, z) + F_2(s, z)) ds + \mathcal{O}(\varepsilon^3)$.

Proof. For each $z \in C$ the function $t \in [0, t_z) \mapsto x(t, z, \varepsilon)$ is continuous and piecewise differentiable. From hypothesis (HC), for $|\varepsilon| \neq 0$ sufficiently small, we can assume that

$$x(t, z, \varepsilon) = \begin{cases} x_1(t, z, \varepsilon) & \text{if} \quad 0 = \quad t_\varepsilon^0 \leq t \leq t_\varepsilon^1, \\ x_2(t, z, \varepsilon) & \text{if} \qquad\quad t_\varepsilon^1 \leq t \leq t_\varepsilon^2, \\ \quad\vdots & \\ x_i(t, z, \varepsilon) & \text{if} \qquad\quad t_\varepsilon^{i-1} \leq t \leq t_\varepsilon^i, \\ \quad\vdots & \\ x_\kappa(t, z, \varepsilon) & \text{if} \qquad\quad t_\varepsilon^{\kappa-1} \leq t \leq t_\varepsilon^\kappa = t_z \leq T, \end{cases}$$

for which we have the following recurrence

$$x_1(0,z,\varepsilon)=z \quad \text{and} \quad x_i(t_\varepsilon^{i-1},z,\varepsilon)=x_{i-1}(t_\varepsilon^{i-1},z,\varepsilon), \tag{2.7.2}$$

for $i=2,\dots,\kappa$. Moreover each function $x_i(t,z,\varepsilon)$ satisfies the DPDS (2.3.1), that is, there exists a subsequence (j_i) for $i=1,\dots,\kappa$ such that

$$\frac{\partial}{\partial t}x_i(t,z,\varepsilon)=\varepsilon F_1^{j_i}(t,x_i(t,z,\varepsilon))+\varepsilon^2F_2^{j_i}(t,x_i(t,z,\varepsilon))+\varepsilon^3R^{j_i}(t,x_i(t,z,\varepsilon),\varepsilon), \tag{2.7.3}$$

In other words, for $i=1,\dots,\kappa$, the function $x_i(t,z,\varepsilon)$ is the solution of the *Cauchy Problem* defined by the differential equation (2.7.3) together with the initial condition (2.7.2).

We note that there exists $|\varepsilon_0|\neq 0$ sufficiently small such that, for each $z\in\overline{C}$, the solution $x_i(t,z,\varepsilon)$ of (2.7.3) is defined in $[0,T]$ for every $\varepsilon\in[-\varepsilon_0,\varepsilon_0]$ and $i=1,2,\dots,\kappa$. Indeed, using the *Existence and Uniqueness Theorem* of solutions (see, for instance, Theorem 1.2.4 of [99]) we have that, for each $z\in C$, $x_i(t,z,\varepsilon)$ is defined for all $0\le t\le \inf\left(T,d/M_i(\varepsilon)\right)$, where

$$M_i(\varepsilon)\ge\left|\varepsilon F_1^{j_i}(t,x_i(t,z,\varepsilon))+\varepsilon^2F_2^{j_i}(t,x_i(t,z,\varepsilon))+\varepsilon^3R^{j_i}(t,x_i(t,z,\varepsilon),\varepsilon)\right|$$

for all $t\in[0,T]$, for each x with $|x-z|\le d$ and for every $z\in\overline{C}$. When ε is sufficiently small we can take $d/M_i(\varepsilon)$ sufficiently large in order that $\inf\left(T,d/M_i(\varepsilon)\right)=T$ for all $z\in\overline{C}$. So for any $z\in\overline{C}$ we have that the solution $x(t,z,\varepsilon)$ of system (2.2.3) is also defined for every $t\in[0,T]$.

From the continuity of the solution $x(t,z,\varepsilon)$ and by compactness of the set $[0,T]\times\overline{C}\times[-\varepsilon_0,\varepsilon_0]$, there exits a compact subset K of D such that $x(t,z,\varepsilon)\in K$ for all $t\in[0,T]$, $z\in\overline{C}$ and $\varepsilon\in[-\varepsilon_0,\varepsilon_0]$. Now, by the piecewise continuity of the function R, $|R(s,x(s,z,\varepsilon),\varepsilon)|\le\max\{|R(t,x,\varepsilon)|,(t,x,\varepsilon)\in[0,T]\times K\times[-\varepsilon_1,\varepsilon_1]\}=N$. Then

$$\left|\int_0^t R(s,x(s,z,\varepsilon),\varepsilon)ds\right|\le\int_0^T\left|R(s,x(s,z,\varepsilon),\varepsilon)\right|ds=TN,$$

which implies that

$$\int_0^t R(s,x(s,z,\varepsilon),\varepsilon)ds=\mathcal{O}(1).$$

Now for a given $t\in(0,T)$ there exists $\bar\kappa\in\{1,2,\dots,\kappa-1\}$ such that $t\in[t_\varepsilon^{\bar\kappa-1},t_\varepsilon^{\bar\kappa})$ and

$$\begin{aligned}x(t,z,\varepsilon)=&\ x_{\bar\kappa}(t,z,\varepsilon)\\ =&\ x_{\bar\kappa-1}(t_\varepsilon^{\bar\kappa-1},z,\varepsilon)+\varepsilon\int_{t_\varepsilon^{\bar\kappa-1}}^t F_1(s,x(s,z,\varepsilon))ds+\varepsilon^2\int_{t_\varepsilon^{\bar\kappa-1}}^t F_2(s,x(s,z,\varepsilon))ds+\mathcal{O}(\varepsilon^3).\end{aligned}$$

Since

$$x_{i+1}(t_\varepsilon^{i+1},z,\varepsilon)\ =x_i(t_\varepsilon^i,z,\varepsilon)+\varepsilon\int_{t_\varepsilon^i}^{t_\varepsilon^{i+1}}F_1(t,x(t,z,\varepsilon))dt+\varepsilon^2\int_{t_\varepsilon^i}^{t_\varepsilon^{i+1}}F_2(t,x(t,z,\varepsilon))dt+\mathcal{O}(\varepsilon^3),$$

proceeding by induction on i, we obtain

$$x(t,z,\varepsilon)=z+\varepsilon\int_0^t F_1(s,x(s,z,\varepsilon))ds+\varepsilon^2\int_0^t F_2(s,x(s,z,\varepsilon))ds+\mathcal{O}(\varepsilon^3). \tag{2.7.4}$$

Claim 2.7.1. Statement (a) of Lemma 2.7.2 holds.

For $i = 1, 2, \ldots, \kappa$ and for $t_\varepsilon^{i-1} \leq t \leq t_\varepsilon^i$, $x_i(t, z, \varepsilon) = x(t, z, \varepsilon)$. Since $F_1^{j_i}$ is Lipschitz for $i = 1, 2, \ldots, \kappa$ in the variable x, we have that

$$\begin{aligned}\left|F_1^{j_i}(t, x_i(t, z, \varepsilon)) - F_1^{j_i}(t, z)\right| = & \left|F_1^{j_i}(t, x(t, z, \varepsilon)) - F_1^{j_i}(t, z)\right| \\ \leq & L_{j_i}|x(t, z, \varepsilon) - z| = \mathcal{O}(\varepsilon),\end{aligned}$$

for all $t_\varepsilon^{i-1} \leq t < t_\varepsilon^i$, where L_{j_i} is the Lipschitz constant of the function $F_1^{j_i}$. It implies that

$$F_1^{j_i}(t, x_i(t, z, \varepsilon)) = F_1^{j_i}(t, z) + \mathcal{O}(\varepsilon), \tag{2.7.5}$$

for $t_\varepsilon^{i-1} \leq t < t_\varepsilon^i$ and for each $i = 1, 2, \ldots, \kappa$.

Let $t^i = \lim_{\varepsilon \to 0} t_\varepsilon^i$ for $i = 1, 2, \ldots, \kappa - 1$. Observing that, for $t^{i-1} \leq t < t^i$, $F_1^{j_i}(s, z) = F_1(s, z)$ and using (2.7.5) we compute

$$\begin{aligned}\int_0^t F_1(s, x(s, z, \varepsilon))ds &= \left(\sum_{i=1}^{\bar{\kappa}-1} \int_{t_\varepsilon^{i-1}}^{t_\varepsilon^i} F_1^{j_i}(s, x_i(s, z, \varepsilon))ds\right) + \int_{t_\varepsilon^{\bar{\kappa}-1}}^t F_1^{j_{\bar{\kappa}}}(s, x_{\bar{\kappa}}(s, z, \varepsilon))ds \\ &= \left(\sum_{i=1}^{\bar{\kappa}-1} \int_{t_\varepsilon^{i-1}}^{t_\varepsilon^i} F_1^{j_i}(s, z)ds\right) + \int_{t_\varepsilon^{\bar{\kappa}-1}}^t F_1^{j_{\bar{\kappa}}}(s, z)ds + \mathcal{O}(\varepsilon) \\ &= \left(\sum_{i=1}^{\bar{\kappa}-1} \int_{t^{i-1}}^{t^i} F_1^{j_i}(s, z)ds\right) + \int_{t^{\bar{\kappa}-1}}^t F_1^{j_{\bar{\kappa}}}(s, z)ds + E_1(\varepsilon) + \mathcal{O}(\varepsilon) \\ &= \left(\sum_{i=1}^{\bar{\kappa}-1} \int_{t^{i-1}}^{t^i} F_1(s, z)ds\right) + \int_{t^{\bar{\kappa}-1}}^t F_1(s, z)ds + E_1(\varepsilon) + \mathcal{O}(\varepsilon) \\ &= \int_0^t F_1(s, z)ds + E_1(\varepsilon) + \mathcal{O}(\varepsilon),\end{aligned} \tag{2.7.6}$$

where

$$E_1(\varepsilon) = \sum_{i=1}^{\bar{\kappa}-1} \left(\int_{t_\varepsilon^{i-1}}^{t^{i-1}} F_1^{j_i}(s, z)ds - \int_{t_\varepsilon^i}^{t^i} F_1^{j_i}(s, z)ds\right) + \int_{t_\varepsilon^{\bar{\kappa}-1}}^{t^{\bar{\kappa}-1}} F_1^{j_{\bar{\kappa}}}(s, z)ds.$$

It is easy to see that there exists a constant $\widetilde{E}$ such that

$$|E_1(\varepsilon)| \leq \widetilde{E} \sum_{i=0}^{\bar{\kappa}-1} |t^i - t_\varepsilon^i|.$$

We shall prove that $\tau^i : \varepsilon \mapsto t_\varepsilon^i$ is a $\mathcal{C}^k$ function with $k \geq 1$.

As in the proof of Proposition 2.3.1, for a generic point of discontinuity $(t^i, z) \in \Sigma$ with $z \in \overline{C}$, let $\widetilde{G}_{(t^i,z)}$ be a neighbourhood of (t^i, z) such that $\mathcal{S}_{(t^i,z)} = \widetilde{G}_{(t^i,z)} \cap \Sigma$ is a $\mathcal{C}^k$ embedded hypersurface of $\mathbb{S}^1 \times \mathbb{R}^d$ with $k \geq 1$, for which there exists a $\mathcal{C}^k$ function $h_i : \widetilde{G}_{(t^i,z)} \to \mathbb{R}$ such

that $\widetilde{G}_{(t^i,z)} \cap \mathcal{S}_{(t^i,z)} = h_i^{-1}(0) \cap \Sigma$. We define $H_i(t,\varepsilon) = h_i(t, x_i(t,z,\varepsilon))$. So $H_i(t^i,0) = 0$ and from hypothesis (HC)

$$\begin{aligned}
\frac{\partial}{\partial t} H_i(t^i,0) &= \frac{\partial}{\partial t} h_i(t, x_i(t,z,\varepsilon))\Big|_{(t^i,0)} \\
&= \frac{\partial}{\partial t} h_i(t^i, x_i(t^i,z,0)) + \frac{\partial}{\partial z} h_i(t^i, x_i(t^i,z,0)) \frac{\partial}{\partial t} x_i(t^i,z,0) \\
&= \frac{\partial}{\partial t} h_i(t^i, x_i(t^i,z,0)) \neq 0,
\end{aligned}$$

because (2.7.3) implies $(\partial/\partial t) x_i(t,z,0) = 0$. Hence from the Implicit Function Theorem, $\tau^i(\varepsilon)$ is a $\mathcal{C}^k$ function with $H(\tau^i(\varepsilon),\varepsilon) = 0$ for every $|\varepsilon| \neq 0$ sufficiently small and $\tau^i(0) = t^i$. So

$$\tau^i(\varepsilon) = t^i + (\tau^i)'(0)\varepsilon + o(\varepsilon) \tag{2.7.7}$$

for every $i = 1, 2, \ldots, \kappa - 1$, because $k \geq 1$. This implies that $E_1(\varepsilon) = \mathcal{O}(\varepsilon)$.

Going back to the equality (2.7.6) we have

$$\int_0^t F_1(s, x(s,z,\varepsilon)) = \int_0^t F_1(s,z)ds + \mathcal{O}(\varepsilon). \tag{2.7.8}$$

Hence from (2.7.4) and (2.7.8) we conclude that

$$x(t,z,\varepsilon) = z + \varepsilon \int_0^t F_1(s,z)ds + \mathcal{O}(\varepsilon^2).$$

Therefore the claim 1 is proved.

Claim 2.7.2. Statement (b) of Lemma 2.7.2 holds.

For $i = 1, 2, \ldots, \kappa$ and for $t_\varepsilon^{i-1} \leq t \leq t_\varepsilon^i$ we prove that

$$|F_1^{j_i}(t, x_i(t,z,\varepsilon)) - F_1^{j_i}(t,z) - \varepsilon D_x F_1^{j_i}(t,z) y_1(t,z)| = \mathcal{O}(\varepsilon^2). \tag{2.7.9}$$

To do this we define

$$G(\lambda) = F_1^{j_i}(t, \lambda x_i(t,z,\varepsilon) + (1-\lambda)z).$$

Computing the derivative in λ we get

$$G'(\lambda) = D_x F_1^{j_i}(t, \lambda x_i(t,z,\varepsilon) + (1-\lambda)z)(x_i(t,z,\varepsilon) - z).$$

So from the Fundamental Theorem of Calculus and observing that, for $t_\varepsilon^{i-1} \leq t \leq t_\varepsilon^i$, $x_i(t,z,\varepsilon) = x(t,z,\varepsilon)$ it follows that

$$G(1) - G(0) = \int_0^1 D_x F_1^{j_i}(t, \lambda x(t,z,\varepsilon) + (1-\lambda)z)(x(t,z,\varepsilon) - z)d\lambda.$$

Thus for $t_\varepsilon^{i-1} \le t \le t_\varepsilon^i$,

$$\frac{1}{\varepsilon^2}\left(F_1^{j_i}(t,x_i(t,z,\varepsilon)) - F_1^{j_i}(t,z) - \varepsilon D_x F_1^{j_i}(t,z)y_1(t,z)\right) =$$
$$\frac{1}{\varepsilon^2}\left(G(1) - G(0) - \varepsilon D_x F_1^{j_i}(t,z)y_1(t,z)\right) =$$
$$\frac{1}{\varepsilon}\int_0^1 D_x F_1^{j_i}(t,\lambda x(t,z,\varepsilon) + (1-\lambda)z)\frac{(x(t,z,\varepsilon)-z)}{\varepsilon}d\lambda - \frac{1}{\varepsilon}D_x F_1^{j_i}(t,z)y_1(t,z) =$$
$$\frac{1}{\varepsilon}\left(\int_0^1 D_x F_1^{j_i}(t,\lambda x(t,z,\varepsilon) + (1-\lambda)z)d\lambda\right)\int_0^t F_1(s,x(s,z,\varepsilon))ds -$$
$$\frac{1}{\varepsilon}D_x F_1^{j_i}(t,z)y_1(t,z) + \mathcal{O}(1) =$$
$$\frac{1}{\varepsilon}\left(\int_0^1 \left[D_x F_1^{j_i}(t,\lambda x(t,z,\varepsilon) + (1-\lambda)z) - D_x F_1^{j_i}(t,z)\right]d\lambda\right)\int_0^t F_1(s,x(s,z,\varepsilon))ds +$$
$$\frac{1}{\varepsilon}D_x F_1^{j_i}(t,z)\left[\int_0^t F_1(s,x(s,z,\varepsilon)) - F_1(s,z)ds\right] + \mathcal{O}(1).$$

Let $B = \max\{|F_1(s,x(s,z,\varepsilon))|\colon (t,z) \in [0,T]\times\overline{C}\}$. Observing that $D_x F_1^{j_i}$ is locally Lipschitz in the second variable, and (from (2.7.8)) that $\int_0^t F_1(s,x(s,z,\varepsilon)) - \int_0^t F_1(s,z) = \mathcal{O}(\varepsilon)$, it follows that

$$\left|\frac{1}{\varepsilon^2}\left(F_1^{j_i}(t,x_i(t,z,\varepsilon)) - F_1^{j_i}(t,z) - \varepsilon D_x F_1^{j_i}(t,z)y_1(t,z)\right)\right| \le$$
$$\frac{1}{\varepsilon}\int_0^1 \left|D_x F_1^{j_i}(t,\lambda x(t,z,\varepsilon) + (1-\lambda)z) - D_x F_1^{j_i}(t,z)\right|d\lambda \left|\int_0^t F_1(s,x(s,z,\varepsilon))ds\right| +$$
$$\frac{1}{\varepsilon}\left|D_x F_1^{j_i}(t,z)\right|\left|\int_0^t F_1(s,x(s,z,\varepsilon)) - F_1(s,z)ds\right| + \mathcal{O}(1) \le$$
$$TL_iB\frac{|x(t,z,\varepsilon)-z|}{\varepsilon} + \mathcal{O}(1) = \mathcal{O}(1),$$

where L_i is the Lipschitz constant of the function $D_x F_1^{j_i}$. Hence for $t_\varepsilon^{i-1} \le t \le t_\varepsilon^i$ and for every $i = 1,2,\ldots,\kappa$ the equality (2.7.9) holds, which implies that $F_1^{j_i}(t,x(t,z,\varepsilon)) = F_1^{j_i}(t,z) + \varepsilon D_x F_1^{j_i}(t,z)y_1(t,z) + \mathcal{O}(\varepsilon^2)$.

Observing that for $t^{i-1} \leq s < t^i$, $F_1^{j_i}(s,z) = F_1(s,z)$ we compute

$$\begin{aligned}
\int_0^t F_1(s,x(s,z,\varepsilon))ds = & \sum_{i=1}^{\bar{\kappa}-1}\left(\int_{t_\varepsilon^{i-1}}^{t_\varepsilon^i} F_1^{j_i}(s,x_i(s,z,\varepsilon))ds\right) + \int_{t_\varepsilon^{\bar{\kappa}-1}}^{t} F_1^{j_{\bar{\kappa}}}(s,x_{\bar{\kappa}}(s,z,\varepsilon))ds \\
= & \sum_{i=1}^{\bar{\kappa}-1}\left(\int_{t_\varepsilon^{i-1}}^{t_\varepsilon^i} [F_1^{j_i}(s,z) + \varepsilon D_x F_1^{j_i}(s,z)y_1(s,z)]ds\right) \\
& + \int_{t_\varepsilon^{\bar{\kappa}-1}}^{t} [F_1^{j_{\bar{\kappa}}}(s,z) + \varepsilon D_x F_1^{j_{\bar{\kappa}}}(s,z)y_1(s,z)]ds + \mathcal{O}(\varepsilon^2) \\
= & \sum_{i=1}^{\bar{\kappa}-1}\left(\int_{t^{i-1}}^{t^i} [F_1^{j_i}(s,z) + \varepsilon D_x F_1^{j_i}(s,z)y_1(s,z)]ds\right) \\
& + \int_{t^{\bar{\kappa}-1}}^{t} [F_1^{j_{\bar{\kappa}}}(s,z) + \varepsilon D_x F_1^{j_{\bar{\kappa}}}(s,z)y_1(s,z)]ds + E_2(\varepsilon) + \mathcal{O}(\varepsilon^2) \\
= & \sum_{i=1}^{\bar{\kappa}-1}\left(\int_{t^{i-1}}^{t^i} [F_1(s,z) + \varepsilon D_x F_1^{j_i}(s,z)y_1(s,z)]ds\right) \\
& + \int_{t^{\bar{\kappa}-1}}^{t} [F_1(s,z) + \varepsilon D_x F_1^{j_{\bar{\kappa}}}(s,z)y_1(s,z)]ds + E_2(\varepsilon) + \mathcal{O}(\varepsilon^2) \\
= & \int_0^t [F_1(s,z) + \varepsilon D_x F_1(s,z)y_1(s,z)]ds + E_2(\varepsilon) + \mathcal{O}(\varepsilon^2).
\end{aligned} \tag{2.7.10}$$

The last equality comes from (2.3.2). Here

$$\begin{aligned}
E_2(\varepsilon) = & \sum_{i=1}^{\bar{\kappa}-1}\left(\int_{t_\varepsilon^{i-1}}^{t^{i-1}} [F_1^{j_i}(s,z) + \varepsilon D_x F_1^{j_i}(s,z)y_1(s,z)]ds\right. \\
& \left. - \int_{t_\varepsilon^{i}}^{t^{i}} [F_1^{j_i}(s,z) + \varepsilon D_x F_1^{j_i}(s,z)y_1(s,z)]ds\right) \\
& + \int_{t_\varepsilon^{\bar{\kappa}-1}}^{t^{\bar{\kappa}-1}} [F_1^{j_{\bar{\kappa}}}(s,z) + \varepsilon D_x F_1^{j_{\bar{\kappa}}}(s,z)y_1(s,z)]ds.
\end{aligned}$$

It is easy to see that there exists a constant $\widehat{E}$ such that

$$|E_2(\varepsilon)| \leq \widehat{E} \sum_{i=0}^{\bar{\kappa}-1} |t^i - t_\varepsilon^i|. \tag{2.7.11}$$

From statement (a) the function $\varepsilon \mapsto x(\tau^i(\varepsilon), z, \varepsilon))$ is differentiable at $\varepsilon = 0$. Moreover $y_1(t,z) =$

$(\partial/\partial\varepsilon)x(t,z,0)$. Since for $|\varepsilon|\neq 0$ sufficiently small $h_i(\tau^i(\varepsilon), x(\tau^i(\varepsilon), z, \varepsilon)) = 0$, so

$$\begin{aligned}
0 = & \frac{\partial}{\partial\varepsilon} h(\tau^i(\varepsilon), x(\tau^i(\varepsilon), z, \varepsilon))\Big|_{\varepsilon=0} \\
= & \frac{\partial}{\partial t} h(t^i, z)(\tau^i)'(0) + \frac{\partial}{\partial z} h(t^i, z)\left(\frac{\partial}{\partial t} x(t^i, z, 0)(\tau^i)'(0) + \frac{\partial}{\partial\varepsilon} x(t^i, z, \varepsilon)\Big|_{\varepsilon=0}\right) \\
= & \left\langle \nabla h(t^i, z), \Big((\tau^i)'(0), y_1(t^i, z)\Big)\right\rangle,
\end{aligned}$$

So $((\tau^i)'(0), y_1(t^i, z)) \in T_{(t^i,z)}\Sigma$.

Now we shall prove that $(\tau^i)'(0) = 0$ and $E_2(\varepsilon) = o(\varepsilon)$. If $(\tau^i)'(0) \neq 0$, we get that $(s, y_1(t^i, z)) \in T_{(t^i,z)}\Sigma$ for every $s \in \mathbb{R}$, because from hypothesis $(Hb2)$, $(0, y_1(t^i, z)) \in T_{(t^i,z)}\Sigma$. Thus

$$0 = \left\langle \nabla h(t^i, z), \Big(s, y_1(t^i, z)\Big)\right\rangle = \frac{\partial}{\partial t} h(t^i, z)\, s + \frac{\partial}{\partial z} h(t^i, z) y_1(t^i, z),$$

for every $s \in \mathbb{R}$. Computing the derivative in s of the last equality it follows that $(\partial h/\partial t)(t^i, z) = 0$ contradicting then the hypothesis (HC). Hence we conclude that $(\tau^i)'(0) = 0$. Moreover from (2.7.11) and (2.7.7) we obtain that $E_2(\varepsilon) = o(\varepsilon)$.

Going back to the equality (2.7.10) we have

$$\int_0^t F_1(s, x(s, z, \varepsilon))ds = \int_0^t F_1(s, z)ds - \varepsilon \int_0^t D_x F_1(s, z) y_1(s, z)ds + o(\varepsilon). \tag{2.7.12}$$

Analogously to the proof of statement (a) and using that $E_2(\varepsilon) = o(\varepsilon) \subset \mathcal{O}(\varepsilon)$ we can show that

$$\int_0^t F_2(s, x(s, z, \varepsilon))ds = \int_0^t F_2(s, z)ds + \mathcal{O}(\varepsilon). \tag{2.7.13}$$

So from (2.7.4), (2.7.12) and (2.7.13) we get

$$x(t, z, \varepsilon) = z + \varepsilon \int_0^t F_1(s, z)ds - \varepsilon^2 \int_0^t [D_x F_1(s, z) y_1(s, z) + F_2(s, z)]ds + \varepsilon o(\varepsilon). \tag{2.7.14}$$

To conclude the proof of statement (b) we assume that for $j = 1, 2, \ldots, M$ the boundaries of S_j are piecewise $\mathcal{C}^k$ embedded hypersurfaces with $k \geq 2$. From (HC) and following the steps of the proof of Claim 1 we can find a $\mathcal{C}^k$ function $h_i : G_{(t^i,z)} \to \mathbb{R}$, now with $k \geq 2$, such that $\widetilde{G}_{(t^i,z)} \cap \mathcal{S}_{(t^i,z)} = h_i^{-1}(0) \cap \Sigma$. Again, $\widetilde{G}_{(t^i,z)}$ is an open subset such that $(t^i, z) \in \widetilde{G}_{(t^i,z)} \subseteq G_{(t^i,z)}$. Applying the Inverse Function Theorem we conclude that $\tau^i(\varepsilon)$ is a $\mathcal{C}^2$ function. So

$$\tau^i(\varepsilon) = t^i + (\tau^i)'(0)\varepsilon + \mathcal{O}(\varepsilon^2).$$

which implies that $E_2(\varepsilon) = \mathcal{O}(\varepsilon^2)$. From here, analogously to (2.7.14), we obtain that

$$x(t, z, \varepsilon) = z + \varepsilon \int_0^t F_1(s, z)ds - \varepsilon^2 \int_0^t [D_x F_1(s, z) y_1(s, z) + F_2(s, z)]ds + \mathcal{O}(\varepsilon^3).$$

It concludes this proof. □

Lemma 2.7.3. Let U be a bounded open set of $\mathbb{R}^n$ and let $f : \overline{U} \times [-\varepsilon_0, \varepsilon_0] \to \mathbb{R}^n$ be a continuous function. We assume that $f(x, 0) \neq 0$ for all $x \in \partial U$. Then for $|\varepsilon| \neq 0$ sufficiently small $d(f(x, \varepsilon), U, 0)$ is well defined and $d(f(x, \varepsilon), U, 0) = d(f(x, 0), U, 0)$ for $|\varepsilon| \neq 0$ sufficiently small.

Proof. For each $\varepsilon \in [-\varepsilon_0, \varepsilon_0] \backslash \{0\}$ we consider the continuous homotopy

$$f_t(x, \varepsilon) = f(x, 0) + t\,(f(x, \varepsilon) - f(x, 0))\,.$$

Suppose that there exist sequences $(\varepsilon_i) \subset [-\varepsilon_0, \varepsilon_0]$, $(x_i) \subset \partial V$ and $(t_i) \in [0, 1]$ with $\varepsilon_i \to 0$ when $\varepsilon \to \infty$ such that $f_{t_i}(x_i, \varepsilon_i) = 0$, that is $0 \in f_{t_i}(\partial V, \varepsilon_i)$. Since the sets ∂V and $[0, 1]$ are compacts, there exists convergent subsequences $(x_{i_j}) \subset \partial V$ and $(t_{i_j}) \in [0, 1]$, namely $x_{i_j} \to \bar{x} \in \partial V$ and $t_{i_j} \to \bar{t} \in [0, 1]$ when $j \to \infty$. So $t_{i_j} f(x_{i_j}, 0) - f(x_{i_j}, 0) = t_{i_j} f(x_{i_j}, \varepsilon_{i_j})$. Passing the limit we conclude that $f(\bar{x}, 0) = 0$, contradicting then the hypotheses. So it must exists $\widetilde{\varepsilon} \in [0, \varepsilon_0]$ such that $0 \notin f_t(\partial V, \varepsilon)$ for every $\varepsilon \in [-\widetilde{\varepsilon}, \widetilde{\varepsilon}]$. From statement (iii) of Theorem A.0.1 (see Appendix A) we conclude that $d(f(x, \varepsilon), V, 0) = d(f(x, 0), V, 0)$ for every $\varepsilon \in [-\widetilde{\varepsilon}, \widetilde{\varepsilon}]$. □

Proof of Theorem E. Let f be the function such that $\varepsilon f(z, \varepsilon) = x(T, z, \varepsilon) - z$. This function is well defined because, from statement (a) of Lemma 2.7.2, the solution $x(t, z, \varepsilon)$ is defined for all $t \in [0, T]$. Moreover f is continuous on C. Also from statement (a) of Lemma 2.7.2 we have that

$$f(z, \varepsilon) = f_1(z) + \mathcal{O}(\varepsilon),$$

where the function f_1 is the one defined in (2.3.3), which, from Lemma 2.7.1, is continuous. Clearly, $x(t, z, \varepsilon)$ is a T–periodic solution if and only if $f(z, \varepsilon) = 0$. However from Lemma 2.7.3 and hypothesis $(Ha2)$ we have, for $|\varepsilon| \neq 0$ sufficiently small, that

$$d_B\,(f_1(z), U, 0) = d_B\,(f(z, \varepsilon), U, 0) \neq 0.$$

Hence, by item (i) of Theorem A.0.1 (see the Appendix A), $0 \in f(U, \varepsilon)$ for $|\varepsilon| \neq 0$ sufficiently small, that is, there exists $a_\varepsilon \in U$ such that $f(a_\varepsilon, \varepsilon) = 0$. Therefore, for $|\varepsilon| \neq 0$ sufficiently small, $x(t, a_\varepsilon, \varepsilon)$ is a periodic solution of (2.2.3). We can choose a_ε such that $a_\varepsilon \to a^*$ when $\varepsilon \to 0$, because $f(z, \varepsilon) \neq 0$ in $U \setminus \{a^*\}$. It completes this proof. □

Proof of Theorem F. Let f be the function such that $\varepsilon^2 f(z, \varepsilon) = x(T, z, \varepsilon) - z$. From statement (b) of Lemma 2.7.2 we have that

$$f(z, \varepsilon) = f_2(z) + \frac{o(\varepsilon)}{\varepsilon},$$

where the function f_2 is the one defined in (2.3.4), which, from Lemma 2.7.1, is continuous. Since $o(\varepsilon)/\varepsilon \to 0$ when $\varepsilon \to 0$ the proof follows similarly to the proof of Theorem E. □

2.7.2 Proofs of main results of section 2.5

Before proving our main result we state some preliminary lemmas.

Given a function $\xi : [0,1] \to \mathbb{R}^d$ we say that $\xi(\varepsilon) = \mathcal{O}(\varepsilon^\ell)$ for some positive integer ℓ if there exists constants $\varepsilon_1 > 0$ and $k > 0$ such that $||\xi(\varepsilon)|| \leq k|\varepsilon^\ell|$ for $0 \leq \varepsilon \leq \varepsilon_1$, and that $\xi(\varepsilon) = o(\varepsilon^\ell)$ for some positive integer ℓ if

$$\lim_{\varepsilon \to 0} \frac{||\xi(\varepsilon)||}{\varepsilon^\ell} = 0.$$

Here $||\cdot||$ denotes the usual Euclidean norm of $\mathbb{R}^d$. The symbols $\mathcal{O}$ and o are called the *Landau's symbols* (see for instance [99]).

Lemma 2.7.4. Under the hypotheses (H), $(H1)$, and $(H3)$ of Theorem G there exist a neighbourhood C of $\mathcal{Z}$ with $\overline{C} \subset U \backslash \partial\Sigma_0$ and a small parameter $\varepsilon_0 > 0$ such that $t_{(z,\varepsilon)} > T$ and $x(t,z,\varepsilon) = x(t,z,0) + \varepsilon y_1(t,z) + o(\varepsilon)$ for every $z \in \overline{C}$, $\varepsilon \in [0,\varepsilon_0]$, and $t \in \mathbb{S}^1$.

Proof. We note that $\mathcal{Z}$ and $\partial\Sigma_0$ are compact subsets of D such that, from the hypothesis (H), $\mathcal{Z} \cap \partial\Sigma_0 = \emptyset$. So there exists an open subset A of D such that $\mathcal{Z} \subset A$ and $\overline{A} \cap \partial\Sigma_0 = \emptyset$.

Also from hypothesis (H) we have that for $\alpha \in \overline{V}$ the continuous function $x_\alpha(t)$ reaches the set Σ only at points of Σ^c. From the definition of the crossing region Σ^c these intersections are transversal. Since this function is T–periodic, we can find a finite sequence (t_α^i) for $i = 0, 1, \ldots, \kappa_\alpha$, with $t_\alpha^0 = 0$ and $t_\alpha^{\kappa_\alpha} = T$ such that

$$x_\alpha(t) = \begin{cases} x_\alpha^1(t) & \text{if } \; 0 = \; t_\alpha^0 \leq t \leq t_\alpha^1, \\ x_\alpha^2(t) & \text{if} \qquad t_\alpha^1 \leq t \leq t_\alpha^2, \\ \vdots & \\ x_\alpha^i(t) & \text{if} \qquad t_\alpha^{i-1} \leq t \leq t_\alpha^i, \\ \vdots & \\ x_\alpha^{\kappa_\alpha}(t) & \text{if} \qquad t_\alpha^{\kappa_\alpha - 1} \leq t \leq t_\alpha^{\kappa_\alpha} = T, \end{cases}$$

where each curve $x_\alpha^i(t)$, for $t \in [t_\alpha^{i-1}, t_\alpha^i]$, reaches the set Σ^c only at $t = t_\alpha^{i-1}$ and $t = t_\alpha^i$ for $i = 2, 3, \ldots, \kappa_\alpha - 1$; the curve x_α^1 reaches the set Σ^c only at $t = 0$ and $t = t_\alpha^1$ if $(0, z_\alpha) \in \Sigma$, and only at $t = t_\alpha^1$ if $(0, z_\alpha) \notin \Sigma$; and the curve $x_\alpha^{\kappa_\alpha}$ reaches the set Σ^c only at $t = t_\alpha^{\kappa_\alpha - 1}$ and $t = T$ if $(T, x(T, z_\alpha, 0)) \in \Sigma$, and only at $t = t_\alpha^{\kappa_\alpha - 1}$ if $(T, x(T, z_\alpha, 0)) \notin \Sigma$.

Since x_α^i for $i = 1, 2, \ldots, \kappa_\alpha$ are solutions of Lipschitz differential equations, the results of continuous dependence of the solutions on initial conditions and parameters ensure the existence of a small parameter ε_α and a small neighborhood $C_\alpha \subset A \cap U$ of z_α such that $\widetilde{C_\alpha^\varepsilon} \cap \Sigma \subset \Sigma^c$ for every $\varepsilon \in [0, \varepsilon_\alpha]$. The family $\{C_\alpha : \alpha \in \overline{V}\}$ is a cover of the compact set $\mathcal{Z}$. Therefore there exists a finite subcover $\{C_{\alpha_j} : j = 1, 2, \ldots, j_0\}$ of $\mathcal{Z}$. We fix then $\varepsilon_1 = \min\{\varepsilon_{\alpha_j} : j = 1, 2, \ldots, j_0\}$. Now taking $C = \cup_{j=1}^{j_0} C_{\alpha_j}$ it follows that $\widetilde{C^\varepsilon} \cap \Sigma \subset \Sigma^c$ for every $\varepsilon \in [0, \varepsilon_1]$. Moreover, we can take $\varepsilon_1 > 0$ and C smaller in order that the function $t \mapsto x(t, z, \varepsilon)$ is defined for all $(t, z, \varepsilon) \in \mathbb{S}^1 \times \overline{C} \times [0, \varepsilon_1]$. This is again a simple consequence of the continuous dependence on initial conditions and parameters.

Thus for $z \in \overline{C}$ and $\varepsilon \in [0, \varepsilon_1]$ the function $t \mapsto x(t, z, \varepsilon)$ is continuous and piecewise $\mathbb{C}^1$. So we

can find a finite sequence $(t^i(z,\varepsilon))$ for $i=0,1,\dots\kappa_z^\varepsilon$ with $t^1(z,\varepsilon)=0$ and $t^{\kappa_z^\varepsilon}(z,\varepsilon)=T$ such that

$$x(t,z,\varepsilon)=\begin{cases} x^1(t,z,\varepsilon) & \text{if } \ 0= \ t^0(z,\varepsilon)\le t\le t^1(z,\varepsilon),\\ x^2(t,z,\varepsilon) & \text{if } \qquad t^1(z,\varepsilon)\le t\le t^2(z,\varepsilon),\\ \quad\vdots & \\ x^i(t,z,\varepsilon) & \text{if } \qquad t^{i-1}(z,\varepsilon)\le t\le t^i(z,\varepsilon),\\ \quad\vdots & \\ x^{\kappa_z^\varepsilon}(t,z,\varepsilon) & \text{if } \qquad t^{\kappa_z^\varepsilon-1}(z,\varepsilon)\le t\le t^{\kappa_z^\varepsilon}(z,\varepsilon)=T, \end{cases} \tag{2.7.15}$$

for which we have the following recurrence

$$x^1(0,z,\varepsilon)=z \quad\text{and}\quad x^i(t^{i-1}(z,\varepsilon),z,\varepsilon)=x^{i-1}(t^{i-1}(z,\varepsilon),z,\varepsilon), \tag{2.7.16}$$

for $i=2,3,\dots,\kappa_z^\varepsilon$. Each $x^i(t,z,\varepsilon)$ for $t\in[t^{i-1}(z,\varepsilon),t^i(z,\varepsilon)]$ is called a *differentiable piece* of the solution $x(t,z,\varepsilon)$.

The crossing region Σ^c is an open subset of Σ, so for each $z\in\overline{C}$ we can find $0<\varepsilon_z\le\varepsilon_1$ and a neighbourhood $U_z\subset D$ of z such that the number κ_z^ε of intersections between the curve $t\mapsto x(t,z,\varepsilon)$ with the set Σ^c for $0\le t\le T$ is constant for $\varepsilon\in[0,\varepsilon_z]$. From compactness of $\overline{C}$ we can find $\varepsilon_2\le\varepsilon_1$ such that the function $\varepsilon\mapsto\kappa_z^\varepsilon$ is constant for $\varepsilon\in[0,\varepsilon_2]$, and the function $z\mapsto\kappa_z^\varepsilon$ is piecewise constant for $z\in\overline{C}$. So for $\varepsilon\in[0,\varepsilon_2]$ we can take $\kappa_z^\varepsilon=\kappa_z$.

Here again for every $z\in\overline{C}$ and $\varepsilon\in[0,\varepsilon_2]$ each curve $t\mapsto x^i(t,z,\varepsilon)$ reaches the set Σ^c only at $t=t^{i-1}(z,\varepsilon)$ and $t=t^i(z,\varepsilon)$ for $i=2,3,\dots,\kappa_z-1$; the curve $x^1(t,z,\varepsilon)$ reaches the set Σ^c only at $t=0$ and $t=t^1(z,\varepsilon)$ if $(0,z)\in\Sigma$, and only at $t=t^1(z,\varepsilon)$ if $(0,z)\notin\Sigma$; and the curve $x^{\kappa_z}(t,z,\varepsilon)$ reaches the set Σ^c only at $t=t^{\kappa_z-1}(z,\varepsilon)$ and $t=T$ if $(T,x(T,z,0))\in\Sigma$, and only at $t=t^{\kappa_z-1}(z,\varepsilon)$ if $(T,x(T,z,0))\notin\Sigma$.

The functions $t\mapsto x^i(t,z,\varepsilon)$ for $i=1,2,\dots,\kappa_z$ are $\mathbb{C}^1$ and satisfy the DPDS (2.5.1), so there exists a subsequence (n_i) for $i=1,\dots,\kappa_z$ with $n_i\in\{1,2,\dots,N\}$ such that

$$\frac{\partial}{\partial t}x^i(t,z,\varepsilon)=F_0^{n_i}(t,x^i(t,z,\varepsilon))+\varepsilon F_1^{n_i}(t,x^i(t,z,\varepsilon))+\varepsilon^2R^{n_i}(t,x^i(t,z,\varepsilon),\varepsilon). \tag{2.7.17}$$

Therefore the function $x^i(t,z,\varepsilon)$ is the solution of the *Cauchy Problem* defined by the differential system (2.7.17) together with the corresponding initial condition given in (2.7.16). Moreover $x^i(t,z_\alpha,0)=x_\alpha^i(t)$ and $t^i(z_\alpha,0)=t_\alpha^i$ for $i=1,2,\dots,\kappa_z$.

From the continuity of the function $x(t,z,\varepsilon)$ we can choose a compact subset K of D such that $x(t,z,\varepsilon)\in K$ for all $(t,z,\varepsilon)\in\mathbb{S}^1\times\overline{C}\times[0,\varepsilon_2]$. From the continuity of the functions F_i^n and R^n for $i=0,1$ and $n=1,2,\dots,N$ we have that these functions are bounded on the compact set $\mathbb{S}^1\times K\times[0,\varepsilon_2]$. So let M be an upper bound for all these functions, and let L be being the maximum Lipschitz constant of the functions F_i^n, DF_0^n, and R^n for $i=0,1$ and $n=1,2,\dots,N$ on the compact set $\mathbb{S}^1\times K\times[0,\varepsilon_2]$.

We compute

$$\left|\left|\int_0^t R(s,x(s,z,\varepsilon),\varepsilon)ds\right|\right|\le\int_0^T||R(s,x(s,z,\varepsilon),\varepsilon)||ds=TM,$$

which implies that $\int_0^t R(s, x(s,z,\varepsilon),\varepsilon)ds = \mathcal{O}(1)$ in the parameter ε.

For $z \in \overline{C}$ and $t \in (0,T)$ we can find $\overline{\kappa} \in \{1,2,\dots,\kappa_z - 1\}$ such that $t \in [t^{\overline{\kappa}-1}(z,\varepsilon), t^{\overline{\kappa}}(z,\varepsilon))$ and

$$\begin{aligned} x(t,z,\varepsilon) = \;& x^{\overline{\kappa}}(t,z,\varepsilon) \\ = \;& x^{\overline{\kappa}-1}(t^{\overline{\kappa}-1}(z,\varepsilon),z,\varepsilon) + \int_{t^{\overline{\kappa}-1}(z,\varepsilon)}^t F_0(s,x(s,z,\varepsilon))ds \\ & +\varepsilon \int_{t^{\overline{\kappa}-1}(z,\varepsilon)}^t F_1(s,x(s,z,\varepsilon))ds + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Since

$$\begin{aligned} x^i(t^i(z,\varepsilon),z,\varepsilon) = \;& x^{i-1}(t^{i-1}(z,\varepsilon),z,\varepsilon) + \int_{t^{i-1}(z,\varepsilon)}^{t^i(z,\varepsilon)} F_0(t,x(t,z,\varepsilon))dt \\ & +\varepsilon \int_{t^{i-1}(z,\varepsilon)}^{t^i(z,\varepsilon)} F_1(t,x(t,z,\varepsilon))dt + \mathcal{O}(\varepsilon^2), \end{aligned}$$

for $i = 1,2,\dots,\kappa_z$, we obtain, by induction on i, that

$$x(t,z,\varepsilon) = z + \int_0^t F_0(s,x(s,z,\varepsilon))ds + \varepsilon \int_0^t F_1(s,x(s,z,\varepsilon))ds + \mathcal{O}(\varepsilon^2). \tag{2.7.18}$$

Claim 2.7.3. There exists a small parameter $\varepsilon_0 > 0$ such that the function $t^i(z,\varepsilon)$, for $i = 0,1,2,\dots,\kappa_z$, is of class $\mathbb{C}^1$ for $(z,\varepsilon) \in \overline{C} \times [0,\varepsilon_0]$, and $(\partial t^i/\partial\varepsilon)(z,0) = 0$. Moreover, $y_1(t,z) = (\partial x^i/\partial\varepsilon)(t,z,0)$ for $t^{i-i}(z,0) \le t \le t^i(z,0)$ and $i = 1,2,\dots,\kappa_z$.

First of all we note that $t^1(z,\varepsilon) = 0$ and $t^{\kappa_z}(z,\varepsilon) = T$. So the first part of Claim 2.7.3 is clearly true for $i = 0$ and $i = \kappa_z$.

We have already concluded that for each $z \in \overline{C}$ the curve $t \mapsto x(t,z,0)$ reaches the discontinuity set only at points of Σ^c. Let $z^i = x^i(t^i(z,0),z,0)$ and $p_z^i = (t^i(z,0),z^i) \in \Sigma^c$, then $p_z^i \in \Sigma^c$ for every $i = 1,2,\dots,\kappa_z$ if $(0,x(T,z,0)) \in \Sigma$, and for every $i = 1,2,\dots,\overline{\kappa}_z - 1$ if $(0,x(T,z,0)) \notin \Sigma$. Particularly p_i is a generic point of Σ, so there exists a neighborhood $G_{p_z^i}$ of p_z^i such that $\mathcal{S}_{p_z^i} = G_{p_z^i} \cap \Sigma$ is a $\mathcal{C}^m$ embedded hypersurface of $\mathbb{S}^1 \times D$ with $m \ge 1$. It is well known that $\mathcal{S}_{p_z^i}$ can be locally described as the inverse image of a regular value of a $\mathcal{C}^m$ function. Thus there exists a small neighborhood $\breve{G}_{p_z^i}$ of p_z^i with $\breve{G}_{p_z^i} \subset G_{p_z^i}$ and a $\mathcal{C}^m$ function $h_i : \breve{G}_{p_z^i} \to \mathbb{R}$ such that $\breve{G}_{p_z^i} \cap \mathcal{S}_{p_z^i} = h_i^{-1}(0) \cap \Sigma$.

For $(t,x) \in \breve{G}_{p_z^i}$ system (2.5.1) can be written as the autonomous system

$$\begin{pmatrix} \tau' \\ x' \end{pmatrix} = \begin{cases} X(\tau,x,\varepsilon) & \text{if} \quad h_i(\tau,x) > 0, \\ Y(\tau,x,\varepsilon) & \text{if} \quad h_i(\tau,x) < 0, \end{cases}$$

where

$$X(\tau,x,\varepsilon)=\begin{pmatrix} 1 \\ \\ F_0^{n_{i+1}}(\tau,x)\varepsilon F_1^{n_{i+1}}(\tau,x)+\varepsilon^2 R^{n_{i+1}}(\tau,x,\varepsilon) \end{pmatrix},$$

$$Y(\tau,x,\varepsilon)=\begin{pmatrix} 1 \\ \\ F_0^{n_i}(\tau,x)+\varepsilon F_1^{n_i}(\tau,x)+\varepsilon^2 R^{n_i}(\tau,x,\varepsilon) \end{pmatrix}.$$

From the definition of crossing region we also have $Xh_i(p_z^i,0)Yh_i(p_z^i,0)>0$, therefore

$$\begin{aligned} 0\neq Yh_i(p_z^i,0) &= \left\langle \left(\frac{\partial h_i}{\partial t}(p_z^i),\frac{\partial h_i}{\partial x}(p_z^i)\right),\left(1,F_0^{n_i}(p_z^i)\right)\right\rangle \\ &= \frac{\partial h_i}{\partial t}(p_z^i)+\frac{\partial h_i}{\partial x}(p_z^i)F_0^{n_i}(p_z^i). \end{aligned} \tag{2.7.19}$$

Now defining $H_i(t,\zeta,\varepsilon)=h_i(t,x^i(t,\zeta,\varepsilon))$ we get $H_i(t^i(z,0),z,0)=0$, and

$$\begin{aligned} \frac{\partial H_i}{\partial t}(t^i(z,0),z,0) &= \frac{\partial}{\partial t}h_i(t,x^i(t,\zeta,\varepsilon))\Big|_{(t,\zeta,\varepsilon)=(t^i(z,0),z,0)} \\ &= \frac{\partial h_i}{\partial t}(t^i(z,0),x^i(t^i(z,0),z,0)) \\ &\quad +\frac{\partial h_i}{\partial x}(t^i(z,0),x^i(t^i(z,0),z,0))\frac{\partial x^i}{\partial t}(t^i(z,0),z,0) \\ &= \frac{\partial h_i}{\partial t}(p_z^i)+\frac{\partial h_i}{\partial x}(p_z^i)\frac{\partial x^i}{\partial t}(t^i(z,0),z,0) \\ &= \frac{\partial h_i}{\partial t}(p_z^i)+\frac{\partial h_i}{\partial x}(p_z^i)F_0^{n_i}(p_z^i) \\ &= Yh_i(p_z^i,0)\neq 0. \end{aligned}$$

The Implicit Function Theorem leads to the existence of a small neighborhood $V_z\subset D$ of z and a small parameter $\widehat{\varepsilon}_z>0$ such that $t^i(\zeta,\varepsilon)$ is the unique $\mathcal{C}^m$ function such that $H(t^i(\zeta,\varepsilon),\varepsilon)=0$ for every $\zeta\in V_z$ and $\varepsilon\in[0,\widehat{\varepsilon}_z]$. So

$$t^i(\zeta,\varepsilon)=t^i(\zeta,0)+\varepsilon\frac{\partial t^i}{\partial\varepsilon}(\zeta,0)+o(\varepsilon)$$

for every $i=1,2,\ldots,\overline{\kappa}_z-1$. Now, from the compactness of $\overline{C}$, there exists $\varepsilon_0>0$ such that the above conclusion is true for every $\varepsilon\in[0,\varepsilon_0]$.

Now we shall use finite induction to conclude the proof of Claim 2.7.3. We note that $h_i(t^i(z,\varepsilon), x^i(t^i(z,\varepsilon), z, \varepsilon$
0 for $\varepsilon \in [0, \varepsilon_0]$, so

$$\begin{aligned}
0 = & \frac{\partial}{\partial \varepsilon} h(t^i(z,\varepsilon), x^i(t^i(z,\varepsilon), z, \varepsilon))\Big|_{\varepsilon=0} \\
= & \frac{\partial h}{\partial t}(p_z^i)\frac{\partial t^i}{\partial \varepsilon}(z,0) + \frac{\partial h}{\partial z}(p_z^i)\left(\frac{\partial x^i}{\partial t}(t^i(z,0),z,0)\frac{\partial t^i}{\partial \varepsilon}(z,0)\right. \\
& \left. + \frac{\partial x^i}{\partial \varepsilon}(t^i(z,0),z,0)\right) \qquad (2.7.20) \\
= & \frac{\partial h}{\partial t}(p_z^i)\frac{\partial t^i}{\partial \varepsilon}(z,0) + \frac{\partial h}{\partial z}(p_z^i)\left(F_0^{n_i}(p_z^i)\frac{\partial t^i}{\partial \varepsilon}(z,0) + \frac{\partial x^i}{\partial \varepsilon}(t^i(z,0),z,0)\right) \\
= & \left\langle \nabla h(p_z^i), \Big(\frac{\partial t^i}{\partial \varepsilon}(z,0), F_0^{n_i}(p_z^i)\frac{\partial t^i}{\partial \varepsilon}(z,0) + \frac{\partial x^i}{\partial \varepsilon}(t^i(z,0),z,0)\Big)\right\rangle,
\end{aligned}$$

for $i = 1, 2, \ldots, \kappa_z$.

Taking $i = 1$, from (2.7.17) we obtain

$$\frac{d}{dt}\left(\frac{\partial x^1}{\partial \varepsilon}(t,z,0)\right) = DF_0^{n_1}(t, x^1(t,z,0))\left(\frac{\partial x^1}{\partial \varepsilon}(t,z,0)\right) + F_1^{n_1}(t, x^1(t,z,0)). \qquad (2.7.21)$$

So for $0 \le t \le t^1(z,0)$ the differential system (2.7.21) becomes

$$\frac{d}{dt}\left(\frac{\partial x^1}{\partial \varepsilon}(t,z,0)\right) = DF_0(t, x(t,z,0))\left(\frac{\partial x^1}{\partial \varepsilon}(t,z,0)\right) + F_1(t, x(t,z,0)). \qquad (2.7.22)$$

Since $\frac{\partial x^1}{\partial \varepsilon}(0,z,0) = 0$ the solution of the linear differential system (2.7.22) is

$$\frac{\partial x^1}{\partial \varepsilon}(t,z,0) = Y(t,z)\int_0^t Y(s,z)^{-1}F_1(x(s,z,0))ds = y_1(t,z), \qquad (2.7.23)$$

for $0 \le t \le t^1(z,0)$. Now from hypothesis $(H3)$ and from equality (2.7.20), for $i = 1$, we have that

$$\Big(\lambda\frac{\partial t^1}{\partial \varepsilon}(z,0), \lambda F_0^{n_1}(p_z^1)\frac{\partial t^1}{\partial \varepsilon}(z,0) + y_1(t^1(z,0),z)\Big) \in T_{p_z^1}\Sigma \qquad (2.7.24)$$

for every $\lambda \in [0,1]$. Thus

$$\begin{aligned}
0 = & \left\langle \nabla h(p_z^1), \Big(\lambda\frac{\partial t^1}{\partial \varepsilon}(z,0), \lambda F_0^{n_1}(p_z^1)\frac{\partial t^1}{\partial \varepsilon}(z,0) + y_1(t^1(z,0),z)\Big)\right\rangle \\
= & \lambda\left(\frac{\partial h}{\partial t}(p_z^1)\frac{\partial t^1}{\partial \varepsilon}(z,0) + \frac{\partial h}{\partial z}(p_z^1)F_0^{n_1}(p_z^1)\frac{\partial t^1}{\partial \varepsilon}(z,0)\right) + \frac{\partial h}{\partial z}(p_z^1)y_1(t^1(z,0),z) \qquad (2.7.25) \\
= & \lambda Y h_1(p_z^1,0)\frac{\partial t^1}{\partial \varepsilon}(z,0) + \frac{\partial h}{\partial z}(p_z^1)y_1(t^1(z,0),z),
\end{aligned}$$

for every $\lambda \in [0,1]$. Computing the derivative with respect to λ in (2.7.25) it follows that $Yh_1(p_z^1,0)\dfrac{\partial t^1}{\partial \varepsilon}(z,0)=0$. So from (2.7.19) we conclude

$$\frac{\partial t^1}{\partial \varepsilon}(z,0)=0. \tag{2.7.26}$$

Hence from (2.7.23) and (2.7.26) the claim is proved for $i=1$.

Given a positive integer $\ell>1$, we assume by induction hypothesis that Claim 2.7.3 is true for $i=\ell-1$. Taking $i=\ell$, the relation (2.7.17) implies

$$\frac{d}{dt}\left(\frac{\partial x^\ell}{\partial \varepsilon}(t,z,0)\right)=DF_0^{n_\ell}(t,x^\ell(t,z,0))\left(\frac{\partial x^\ell}{\partial \varepsilon}(t,z,0)\right)+F_1^{n_\ell}(t,x^\ell(t,z,0)). \tag{2.7.27}$$

So for $t^{\ell-1}(z,0)\le t\le t^\ell(z,0)$ the differential system (2.7.27) becomes

$$\frac{d}{dt}\left(\frac{\partial x^\ell}{\partial \varepsilon}(t,z,0)\right)=DF_0(t,x(t,z,0))\left(\frac{\partial x^\ell}{\partial \varepsilon}(t,z,0)\right)+F_1(t,x(t,z,0)). \tag{2.7.28}$$

From (2.7.16) we have that $x^\ell(t^{\ell-1}(z,\varepsilon),z,\varepsilon)=x^{\ell-1}(t^{\ell-1}(z,\varepsilon),z,\varepsilon)$ for every $\varepsilon\in[0,\varepsilon_0]$. Computing its derivative with respect to ε at $\varepsilon=0$ we obtain that

$$\begin{aligned}&\frac{\partial x^\ell}{\partial t}(t^{\ell-1}(z,0),z,0)\frac{\partial t^{\ell-1}}{\partial \varepsilon}(z,0)+\frac{\partial x^\ell}{\partial \varepsilon}(t^{\ell-1}(z,0),z,0)=\\&\frac{\partial x^{\ell-1}}{\partial t}(t^{\ell-1}(z,0),z,0)\frac{\partial t^{\ell-1}}{\partial \varepsilon}(z,0)+\frac{\partial x^{\ell-1}}{\partial \varepsilon}(t^{\ell-1}(z,0),z,0).\end{aligned}$$

So from induction hypothesis it follows that

$$\frac{\partial x^\ell}{\partial \varepsilon}(t^{\ell-1}(z,0),z,0)=\frac{\partial x^{\ell-1}}{\partial \varepsilon}(t^{\ell-1}(z,0),z,0)=y_1(t^{\ell-1}(z,0),z). \tag{2.7.29}$$

We note that (2.7.29) is the initial condition of the differential equation (2.7.28). Thus for $t^{\ell-1}(z,0)\le t\le t^\ell(z,0)$ regarding linear differential equation (2.7.28) we get that

$$\frac{\partial x^\ell}{\partial \varepsilon}(t,z,0)=\widehat{Y}(t,z)y_1(t^{\ell-1}(z,0),z)+\widehat{Y}(t,z)\int_{t^{\ell-1}(z,0)}^t\widehat{Y}(s,z)^{-1}F_1(x(s,z,0))ds, \tag{2.7.30}$$

where $\widehat{Y}(t,z)$ is the fundamental matrix of the linear differential system (2.5.3) such that $\widehat{Y}(t^{\ell-1}(z,0),z)$ is the identity matrix. Clearly $\widehat{Y}(t,z)=Y(t,z)Y(t^{\ell-1}\ (z,0),z)^{-1}$, where $Y(t,z)$ is fixed in (2.5.4). So substituting $\widehat{Y}(t,z)$ and (2.5.4) in (2.7.30) we get

$$\begin{aligned}\frac{\partial x^\ell}{\partial \varepsilon}(t,z,0)=&\ Y(t,z)\int_0^{t^{\ell-1}(z,0)}Y(s,z)^{-1}F_1(x(s,z,0))ds\\&+Y(t,z)\int_{t^{\ell-1}(z,0)}^t Y(s,z)^{-1}F_1(x(s,z,0))ds\\=&\ Y(t,z)\int_{t^{\ell-1}(z,0)}^t Y(s,z)^{-1}F_1(x(s,z,0))ds=y_1(t,z),\end{aligned}$$

for $t^{\ell-1}(z,0) \le t \le t^{\ell}(z,0)$. This claim follows by repeating the procedure of (2.7.24) and (2.7.25) for $i = \ell$ to obtain $\dfrac{\partial t^{\ell}}{\partial \varepsilon}(z,0) = 0$. So we have proved Claim 2.7.3.

Claim 2.7.4. The equality $x(t,z,\varepsilon) = x(t,z,0) + \mathcal{O}(\varepsilon)$ holds for every $z \in \overline{C}$ and $\varepsilon \in [0,\varepsilon_0]$.

For $t \in [t^{\overline{\kappa}-1}(z,\varepsilon), t^{\overline{\kappa}}(z,\varepsilon))$ we compute

$$\begin{aligned}
\int_0^t F_0(s,x(s,z,\varepsilon))ds = & \sum_{i=1}^{\overline{\kappa}-1}\left(\int_{t^{i-1}(z,\varepsilon)}^{t^i(z,\varepsilon)} F_0^{n_i}(s,x(s,z,\varepsilon))ds\right) \\
& + \int_{t^{\overline{\kappa}-1}(z,\varepsilon)}^{t} F_0^{n_{\overline{\kappa}}}(s,x(s,z,\varepsilon))ds \\
= & \sum_{i=1}^{\overline{\kappa}-1}\left(\int_{t^{i-1}(z,0)}^{t^i(z,0)} F_0^{n_i}(s,x(s,z,\varepsilon))ds\right) \\
& + \int_{t^{\overline{\kappa}-1}(z,0)}^{t} F_0^{n_{\overline{\kappa}}}(s,x(s,z,\varepsilon))ds + E_0(\varepsilon),
\end{aligned}$$

where

$$\begin{aligned}
E_0(\varepsilon) = & \sum_{i=1}^{\overline{\kappa}-1}\left(\int_{t^{i-1}(z,\varepsilon)}^{t^{i-1}(z,0)} F_0^{n_i}(s,x(s,z,\varepsilon))ds - \int_{t^{i}(z,\varepsilon)}^{t^{i}(z,0)} F_0^{n_i}(s,x(s,z,\varepsilon))ds\right) \\
& + \int_{t^{\overline{\kappa}-1}(z,\varepsilon)}^{t^{\overline{\kappa}-1}(z,0)} F_0^{n_{\overline{\kappa}}}(s,x(s,z,\varepsilon))ds.
\end{aligned}$$

The function $F_0^{n_i}(t,x)$ is bounded in the set $\mathbb{S}^1 \times K$, so

$$\begin{aligned}
\left|\left|\int_{t^i(z,\varepsilon)}^{t^i(z,0)} F_0^{n_i}(s,x(s,z,\varepsilon))ds\right|\right| & \le \int_{t^i(z,\varepsilon)}^{t^i(z,0)} ||F_0^{n_i}(s,x(s,z,\varepsilon))||\, ds \\
& \le M|t^i(z,0) - t^i(z,\varepsilon)|,
\end{aligned}$$

for $i = 0,1,2,\ldots,\overline{\kappa}$. Therefore there exists a constant $\overline{E}$ such that

$$||E_0(\varepsilon)|| \le \overline{E}\sum_{i=0}^{\overline{\kappa}-1}|t^i(z,0) - t^i(z,\varepsilon)|.$$

From Claim 2.7.3, $t^i(z,\varepsilon) = t^i(z,0) + o(\varepsilon)$, implying $E_0(\varepsilon) = o(\varepsilon)$, particularly $E_0(\varepsilon) = \mathcal{O}(\varepsilon)$. Thus

$$\begin{aligned}
\int_0^t F_0(s,x(s,z,\varepsilon))ds = & \sum_{i=1}^{\overline{\kappa}-1}\left(\int_{t^{i-1}(z,0)}^{t^i(z,0)} F_0^{n_i}(s,x(s,z,\varepsilon))ds\right) \\
& + \int_{t^{\overline{\kappa}-1}(z,0)}^{t} F_0^{n_{\overline{\kappa}}}(s,x(s,z,\varepsilon))ds + \mathcal{O}(\varepsilon).
\end{aligned} \tag{2.7.31}$$

Using that the functions $F_0^{n_i}$, $i = 1, 2, \ldots, \kappa_z$, are locally Lipschitz in the second variable we obtain, from (2.7.31), that

$$\begin{aligned}
& \left|\left|\int_0^t F_0(s, x(s,z,\varepsilon)) - F_0(s, x(s,z,0))ds\right|\right| \\
\leq & \sum_{i=1}^{\overline{\kappa}-1} \int_{t^{i-1}(z,0)}^{t^i(z,0)} ||F_0^{n_i}(s, x(s,z,\varepsilon)) - F_0^{n_i}(s, x(s,z,0))||ds \\
& + \int_{t^{\overline{\kappa}-1}(z,0)}^{t} ||F_0^{n_{\overline{\kappa}}}(s, x(s,z,\varepsilon)) - F_0^{n_{\overline{\kappa}}}(s, x(s,z,0))||ds + \mathcal{O}(\varepsilon) \\
\leq & L \sum_{i=1}^{\overline{\kappa}-1} \int_{t^{i-1}(z,0)}^{t^i(z,0)} ||x(s,z,\varepsilon) - x(s,z,0)||ds \\
& + L \int_{t^{\overline{\kappa}-1}(z,0)}^{t} ||x(s,z,\varepsilon) - x(s,z,0)||ds + \mathcal{O}(\varepsilon) \\
= & L \int_0^t ||x(s,z,\varepsilon) - x(s,z,0)||ds + \mathcal{O}(\varepsilon).
\end{aligned}$$

From (2.7.18) we obtain

$$\begin{aligned}
||x(t,z,\varepsilon) - x(t,z,0)|| \leq & \int_0^t ||F_0(s, x(s,z,\varepsilon)) - F_0(s, x(s,z,0))||ds \\
& + |\varepsilon| \int_0^t ||F_1(s, x(s,z,\varepsilon))||ds + \mathcal{O}(\varepsilon^2) \\
\leq & |\varepsilon| MT + L \int_0^t ||x(s,z,\varepsilon) - x(s,z,0)||ds \\
\leq & |\varepsilon| MTe^{TL}.
\end{aligned} \quad , \tag{2.7.32}$$

The last inequality is a consequence of Gronwall Lemma (see, for example, Lemma 1.3.1 of [99]).

The inequality (2.7.32) implies that $x(t,z,\varepsilon) = x(t,z,0) + \mathcal{O}(\varepsilon)$, which proves this claim.

Claim 2.7.5. The equality $x(t,z,\varepsilon) = x(t,z,0) + \varepsilon y_1(t,z) + o(\varepsilon)$ holds for every $z \in \overline{C}$ and $\varepsilon \in [0, \varepsilon_0]$.

In the proof of Lemma 1.4.1 it has been proved that

$$\begin{aligned}
F_0^{n_i}(t, x^i(t,z,\varepsilon)) = & F_0^{n_i}(t, x^i(t,z,0)) + D_x F_0^{n_i}(t, x^i(t,z,0)) \\
& \cdot (x^i(t,z,\varepsilon) - x^i(t,z,0)) + \mathcal{O}(\varepsilon^2),
\end{aligned} \tag{2.7.33}$$

$$F_1^{n_i}(t, x^i(t,z,\varepsilon)) = F_1^{n_i}(t, x^i(t,z,0)) + \mathcal{O}(\varepsilon),$$

for all $t^{i-1}(z,\varepsilon) \leq t \leq t^i(z,\varepsilon)$ and for every $i = 1, 2, \ldots, \kappa_z$. So we obtain that

$$\begin{aligned} F_0^{n_i}(t, x^i(t,z,\varepsilon)) = \quad & F_0^{n_i}(t, x^i(t,z,0)) + \varepsilon D_x F_0^{n_i}(t, x^i(t,z,0)) \\ & \cdot \frac{\partial x^i}{\partial \varepsilon}(t,z,0) + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{2.7.34}$$

for all $t^{i-1}(z,\varepsilon) \leq t \leq t^i(z,\varepsilon)$ and for every $i = 1, 2, \ldots, \kappa_z$. For the moment we cannot use Claim 2.7.3 to ensure that $\dfrac{\partial x^i}{\partial \varepsilon}(t,z,0) = y_1(t,z)$ because it is only true when $t^{i-1}(z,0) \leq t \leq t^i(z,0)$.

Given $z \in \overline{C}$ we have that, for every $t^{i-1}(z,\varepsilon) \leq t \leq t^i(z,\varepsilon)$, $x^i(t,z,\varepsilon) = x(t,z,\varepsilon)$ for $i = 1, 2, \ldots, \kappa_\alpha$. Moreover if $t^{i-1}(z,\varepsilon) \leq s < t^i(z,\varepsilon)$ and $\varepsilon \in [0, \varepsilon_0]$, then $F_j^{n_i}(s, x^i(s,z,\varepsilon)) = F_j(s, x(t,z,\varepsilon))$ for $j = 0, 1$ and for every $i = 1, 2, \ldots, \overline{\kappa}$. So from (2.7.33) we compute

$$\begin{aligned} & \int_0^t F_1(s, x(s,z,\varepsilon))ds = \\ & \left(\sum_{i=1}^{\overline{\kappa}-1} \int_{t^{i-1}(z,\varepsilon)}^{t^i(z,\varepsilon)} F_1^{n_i}(s, x^i(s,z,\varepsilon))ds \right) + \int_{t^{\overline{\kappa}-1}(z,\varepsilon)}^{t} F_1^{n_{\overline{\kappa}}}(s, x^{\overline{\kappa}}(s,z,\varepsilon))ds = \\ & \left(\sum_{i=1}^{\overline{\kappa}-1} \int_{t^{i-1}(z,\varepsilon)}^{t^i(z,\varepsilon)} F_1^{n_i}(s, x^i(s,z,0))ds \right) + \int_{t^{\overline{\kappa}-1}(z,\varepsilon)}^{t} F_1^{n_{\overline{\kappa}}}(s, x^{\overline{\kappa}}(s,z,0))ds + \mathcal{O}(\varepsilon) = \\ & \left(\sum_{i=1}^{\overline{\kappa}-1} \int_{t^{i-1}(z,0)}^{t^i(z,0)} F_1^{n_i}(s, x^i(s,z,0))ds \right) + \int_{t^{\overline{\kappa}-1}(z,0)}^{t} F_1^{n_{\overline{\kappa}}}(s, x^{\overline{\kappa}}(s,z,0))ds + E_1(\varepsilon) \\ & + \mathcal{O}(\varepsilon) = \\ & \left(\sum_{i=1}^{\overline{\kappa}-1} \int_{t^{i-1}(z,0)}^{t^i(z,0)} F_1(s, x(s,z,0))ds \right) + \int_{t^{\overline{\kappa}-1}(z,0)}^{t} F_1(s, x(s,z,0))ds + E_1(\varepsilon) + \mathcal{O}(\varepsilon) = \\ & \int_0^t F_1(s, x(s,z,0))ds + E_1(\varepsilon) + \mathcal{O}(\varepsilon), \end{aligned} \tag{2.7.35}$$

for $t^{\kappa-1}(z,\varepsilon) \leq t \leq t^\kappa(z,\varepsilon)$. Here

$$\begin{aligned} E_1(\varepsilon) = \quad & \sum_{i=1}^{\overline{\kappa}-1} \left(\int_{t^{i-1}(z,\varepsilon)}^{t^{i-1}(z,0)} F_1^{n_i}(s, x^i(s,z,0))ds - \int_{t^i(z,\varepsilon)}^{t^i(z,0)} F_1^{n_i}(s, x^i(s,z,0))ds \right) \\ & + \int_{t^{\overline{\kappa}-1}(z,\varepsilon)}^{t^{\overline{\kappa}-1}(z,0)} F_1^{n_{\overline{\kappa}}}(s, x^{\overline{\kappa}}(s,z,0))ds. \end{aligned}$$

Now, as in the case $E_0(\varepsilon)$ of the proof of Claim 2.7.4, it is easy to see that there exists a constant $\widetilde{E}$ such that

$$||E_1(\varepsilon)|| \leq \widetilde{E} \sum_{i=0}^{\overline{\kappa}-1} |t^i(z,0) - t^i(z,\varepsilon)|.$$

From Claim 2.7.3 we have that $E_1(\varepsilon) = o(\varepsilon)$, consequently $E_1(\varepsilon) = \mathcal{O}(\varepsilon)$. Going back to inequality (2.7.35) we obtain

$$\int_0^t F_1(s, x(s,z,\varepsilon))ds = \int_0^t F_1(s, x(s,z,0))ds + \mathcal{O}(\varepsilon). \tag{2.7.36}$$

Claim 2.7.3 also implies that $\dfrac{\partial x^i}{\partial \varepsilon}(t,z,0) = y_1(t,z)$ for $t^{i-1}(z,0) \le t \le t^i(z,0)$, so from (2.7.34) we compute

$$\begin{aligned}
&\int_0^t F_0(s, x(s,z,\varepsilon))ds = \\
&\sum_{i=1}^{\overline{\kappa}-1}\left(\int_{t^{i-1}(z,\varepsilon)}^{t^i(z,\varepsilon)} F_0^{n_i}(s, x^i(s,z,\varepsilon))ds\right) + \int_{t^{\overline{\kappa}-1}(z,\varepsilon)}^{t} F_0^{n_{\overline{\kappa}}}(s, x^{\overline{\kappa}}(s,z,\varepsilon))ds = \\
&\sum_{i=1}^{\overline{\kappa}-1}\left(\int_{t^{i-1}(z,\varepsilon)}^{t^i(z,\varepsilon)} [F_0^{n_i}(s, x^i(s,z,0)) + \varepsilon D_x F_0^{n_i}(s, x^i(s,z,0))\frac{\partial x^i}{\partial \varepsilon}(s,z,0)]ds\right) + \\
&\int_{t^{\overline{\kappa}-1}(z,\varepsilon)}^{t} [F_0^{n_{\overline{\kappa}}}(s, x^{\overline{\kappa}}(s,z,0)) + \varepsilon D_x F_0^{n_{\overline{\kappa}}}(s, x^{\overline{\kappa}}(s,z,0))\frac{\partial x^{\overline{\kappa}}}{\partial \varepsilon}(t,z,0)]ds + \mathcal{O}(\varepsilon^2) = \\
&\sum_{i=1}^{\overline{\kappa}-1}\left(\int_{t^{i-1}(z,0)}^{t^i(z,0)} [F_0^{n_i}(s, x^i(s,z,0)) + \varepsilon D_x F_0^{n_i}(s, x^i(s,z,0))\frac{\partial x^i}{\partial \varepsilon}(t,z,0)]ds\right) + \\
&\int_{t^{\overline{\kappa}-1}(z,0)}^{t} [F_0^{n_{\overline{\kappa}}}(s, x^{\overline{\kappa}}(s,z,0)) + \varepsilon D_x F_0^{n_{\overline{\kappa}}}(s, x^{\overline{\kappa}}(s,z,0))\frac{\partial x^{\overline{\kappa}}}{\partial \varepsilon}(t,z,0)]ds + E_2(\varepsilon) \\
&+\mathcal{O}(\varepsilon^2) = \\
&\sum_{i=1}^{\overline{\kappa}-1}\left(\int_{t^{i-1}(z,0)}^{t^i(z,0)} [F_0(s, x(s,z,0)) + \varepsilon D_x F_0^{n_i}(s, x(s,z,0))y_1(s,z)]ds\right) + \\
&\int_{t^{\overline{\kappa}-1}(z,0)}^{t} [F_0(s, x(s,z,0)) + \varepsilon D_x F_0^{n_{\overline{\kappa}}}(s, x(s,z,0))y_1(s,z)]ds + E_2(\varepsilon) + \mathcal{O}(\varepsilon^2)
\end{aligned} \tag{2.7.37}$$

The last equality comes from observing that $F_0^{n_i}(s, x^i(s,z,0)) = F_0(s, x(s,z,0))$ for every $s \in [t^{i-1}(z,0), t^i(z,0))$ and $i = 1, 2, \ldots, \overline{\kappa}$. From definition (2.3.2) the inequality (2.7.37) becomes

$$\begin{aligned}
\int_0^t F_0(s, x(s,z,\varepsilon))ds = &\int_0^t [F_0(s, x(s,z,0)) + \varepsilon D_x F_0(s, x(s,z,0)) \\
&\cdot y_1(s,z)]ds + E_2(\varepsilon) + \mathcal{O}(\varepsilon^2).
\end{aligned} \tag{2.7.38}$$

Here

$$
\begin{aligned}
E_2(\varepsilon) = & \sum_{i=1}^{\overline{\kappa}-1} \left(\int_{t^{i-1}(z,\varepsilon)}^{t^{i-1}(z,0)} [F_0^{n_i}(s, x^i(s,z,0)) + \varepsilon D_x F_0^{n_i}(s, x^i(s,z,0)) \frac{\partial x^i}{\partial \varepsilon}(t,z,0)] ds \right. \\
& \left. - \int_{t^i(z,\varepsilon)}^{t^i(z,0)} [F_0^{n_i}(s, x(s,z,0)) + \varepsilon D_x F_0^{n_i}(s, x^i(s,z,0)) \frac{\partial x^i}{\partial \varepsilon}(t,z,0)] ds \right) \\
& + \int_{t^{\overline{\kappa}-1}(z,\varepsilon)}^{t^{\overline{\kappa}-1}(z,0)} [F_0^{n_{\overline{\kappa}}}(s, x^i(s,z,0)) + \varepsilon D_x F_0^{n_{\overline{\kappa}}}(s, x^i(s,z,0)) \frac{\partial x^i}{\partial \varepsilon}(t,z,0)] ds.
\end{aligned}
$$

Again, it is easy to see that there exists a constant $\widehat{E}$ such that

$$
||E_2(\varepsilon)|| \leq \widehat{E} \sum_{i=0}^{\overline{\kappa}-1} |t^i(z,0) - t^i(z,\varepsilon)|.
$$

From Claim 2.7.3 it follows that $E_2(\varepsilon) = o(\varepsilon)$. Going back to inequality (2.7.38) we have

$$
\begin{aligned}
\int_0^t F_0(s, x(s,z,\varepsilon)) ds = & \int_0^t F_0(s, x(s,z,0)) ds \\
& + \varepsilon \int_0^t D_x F_0(s, x(s,z,0)) y_1(s,z) ds + o(\varepsilon).
\end{aligned} \tag{2.7.39}
$$

So from (2.7.18), (2.7.36), and (2.7.39) we conclude that

$$
\begin{aligned}
x(t,z,\varepsilon) = & \; z + \int_0^t F_0(s, x(s,z,0)) ds \\
& + \varepsilon \int_0^t [D_x F_0(s, x(s,z,0)) y_1(s,z) + F_1(s, x(s,z,0))] ds + o(\varepsilon) \\
= & \; x(s,z,0) + \varepsilon y_1(t,z) + o(\varepsilon).
\end{aligned}
$$

The last equality is a simple consequence of the computations made in Claim 2.7.3. Indeed from (2.7.28) and Claim 2.7.3 if $t^{\ell-1}(z,0) \leq t \leq t^{\ell}(z,0)$, then

$$
y_1(t,z) = y_1(t^{\ell-1}(z,0),0) + \int_{t^{\ell-1}(z,0)}^t [D_x F_0(s, x(s,z,0)) y_1(s,z) + F_1(s, x(s,z,0))] ds.
$$

From here, proceeding by induction on ℓ, we obtain that

$$
y_1(t,z) = \int_0^t [D_x F_0(s, x(s,z,0)) y_1(s,z) + F_1(s, x(s,z,0))] ds
$$

This completes the proof of Claim 2.7.5 and, consequently, the proof of this lemma. □

Lemma 2.7.5. Under the hypothesis of Theorem G the solution $x(t,z,0)$ of the unperturbed differential system (2.5.2) is of class $\mathbb{C}^1$ in the variable z for every $z \in \overline{C}$. Moreover $(\partial x/\partial z)(t,z,0) = Y(t,z)Y(0,z)^{-1}$. The set C is defined in the statement of Lemma 2.7.4 and Y is the fundamental matrix solution of (2.5.3).

Proof. Given $z \in \overline{C}$, the solution of system (2.5.1) (resp. of the uperturbed system (2.5.2)) starting at z is given by (2.7.15) (resp. by (2.7.15) taking $\varepsilon = 0$). From the proof of Lemma 2.7.4 we know that for each $z \in \overline{C}$ there exists a small neighborhood $U_z \subset D$ of z such that the solution $x(t,\zeta,0)$ can be written as (2.7.15) for every $\zeta \in U_z$ having the same number κ_z of differentiable pieces.

Let $\varphi_n(t,t_0,x_0)$ be the solution of the differential equation $x' = F_0^n(t,x)$ such that $\varphi_n(t_0,t_0,x_0) = x_0$. From the results of the differential dependence of the solutions we conclude that each of these functions are of class $\mathbb{C}^1$ in the variables (t,t_0,x_0). Indeed the function F_0^n is $\mathbb{C}^1$ for $i = 1,2,\ldots,\kappa_z$. From Claim 2.7.3 of the proof of Lemma 2.7.4 we know that the function $t^i(\zeta,\varepsilon)$, for $i = 1,2,\ldots,\kappa_z$, is of class $\mathbb{C}^1$ for every $\zeta \in U_z$ and $\varepsilon \in [0,\varepsilon_0]$.

From (2.7.16) we have that

$$\begin{aligned} &x^1(t,\zeta,0) = \varphi_{n_1}(t,0,\zeta) \quad \text{and} \\ &x^i(t,\zeta,0) = \varphi_{n_i}(t,t^{i-1}(\zeta,0),x^{i-1}(t^{i-1}(\zeta,0),\zeta,0)), \end{aligned} \tag{2.7.40}$$

for $\zeta \in U_z$ and for $i = 2,3,\ldots,\kappa_z$. So for $i = 1$ the function $(t,\zeta) \mapsto x^1(t,\zeta,0) = \varphi_{n_1}(t,0,\zeta)$ is $\mathbb{C}^1$. Moreover for $0 \le t \le t^1(\zeta,0)$ we have that $\dfrac{\partial x^1}{\partial z}(t,\zeta,0) = Y(t,\zeta)$. Indeed, it follows from (2.7.17) that

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\partial x^1}{\partial z}(t,\zeta,0)\right) &= D_x F_0^{n_1}(t,x^1(t,\zeta,0))\frac{\partial x^1}{\partial z}(t,\zeta,0) \\ &= D_x F_0(t,x(t,\zeta,0))\frac{\partial x^1}{\partial z}(t,\zeta,0), \end{aligned} \tag{2.7.41}$$

for $0 \le t \le t^1(\zeta,0)$. Solving the linear differential equation (2.7.41) we obtain that $\dfrac{\partial x^1}{\partial z}(t,\zeta,0)$ is a fundamental matrix solution of system (2.5.3), for $0 \le t \le t^1(\zeta,0)$ and $\zeta \in U_z$, which is the identity matrix for $t = 0$. So we conclude that $\dfrac{\partial x^1}{\partial z}(t,\zeta,0) = Y(t,\zeta)Y(0,z)^{-1}$ for $0 \le t \le t^1(\zeta,0)$ and $\zeta \in U_z$.

Now we assume by induction hypothesis that the function $\zeta \mapsto x^{\ell-1}(t,\zeta,0)$ is $\mathbb{C}^1$ for each $t \in \mathbb{S}^1$, and that the equality $\dfrac{\partial x^\ell}{\partial z}(t,\zeta,0) = Y(t,\zeta)Y(0,z)^{-1}$ holds for $t^{\ell-2}(\zeta,0) \le t \le t^{\ell-1}(\zeta,0)$.

From (2.7.40) we have that, for $i = \ell$, $x^\ell(t,\zeta,0) = \varphi_{n_\ell}(t,t^{\ell-1}(\zeta,0),x^{\ell-1}(t^{\ell-1}(\zeta,0),\zeta,0))$. So the the function $\zeta \mapsto x^\ell(t,\zeta,0)$ is $\mathbb{C}^1$ because from the induction hypothesis it is composition of $\mathbb{C}^1$

functions. From (2.7.17) and (2.7.40) we compute

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial x^\ell}{\partial z}(t,\zeta,0)\right) &= D_x F_0^{n_\ell}(t,x^\ell(t,\zeta,0))\frac{\partial x^\ell}{\partial z}(t,\zeta,0)\\ &= D_x F_0(t,x(t,\zeta,0))\frac{\partial x^\ell}{\partial z}(t,\zeta,0),\end{aligned}$$

for $t^{\ell-1}(\zeta,0)\le t\le t^\ell(\zeta,0)$. Solving the above linear differential equation we get

$$\begin{aligned}\frac{\partial x^\ell}{\partial z}(t,\zeta,0) = &\ Y(t,\zeta)Y(t^{\ell-1}(\zeta,0),\zeta)^{-1}\frac{\partial x^{\ell-1}}{\partial z}(t^\ell(\zeta,0),\zeta,0)\\ = &\ Y(t,\zeta)Y(0,\zeta)^{-1},\end{aligned}$$

for $t^{\ell-1}(\zeta,0)\le t\le t^\ell(\zeta,0)$ and $\zeta\in U_z$. The last equality comes from the induction hypothesis because

$$\frac{\partial x^\ell}{\partial z}(t^{\ell-1}(\zeta,0),\zeta,0)=\frac{\partial x^{\ell-1}}{\partial z}(t^{\ell-1}(\zeta,0),\zeta,0)=Y(t^{\ell-1}(\zeta,0),\zeta)Y(0,\zeta)^{-1}.$$

The above induction has proved that for every $z\in\overline{C}$, $x^i(t,z,0)$ is a $\mathbb{C}^1$ function in the second variable and $\frac{\partial x^i}{\partial z}(t,z,0)=Y(t,z)Y(0,z)^{-1}$, provided that $t^{i-1}\le t\le t^i$. The proof of this lemma follows by observing that for $z\in\overline{C}$ and $t\in\mathbb{S}^1$ there exists $\ell\in\{1,2,\dots,\kappa_z\}$ such that $t^{\ell-1}(z,0)\le t\le t^i(z,0)$, hence $x(t,z,0)=x^\ell(t,z,0)$. □

Lemma 2.7.6. Under the hypotheses of Theorem G there exists a small parameter $\overline{\varepsilon}\in[0,\varepsilon_0]$ such that for every $\varepsilon\in[0,\varepsilon_0]$ the function $z\mapsto x(T,z,\varepsilon)$ is locally Lipshchitz for $z\in\overline{C}$. The parameter ε_0 and the set C are defined in the statement of Lemma 2.7.4.

Proof. Given $z\in\overline{C}$, the solution of system (2.5.1) starting at z is given by (2.7.15). From the proof of Lemma 2.7.4 we know that for each $z\in\overline{C}$ there exists a small neighborhood $U_z\subset D$ of z such that the solution $x(t,\zeta,\varepsilon)$ can be written as (2.7.15) having the same number κ_z of differentiable pieces for every $\zeta\in U_z$ and $\varepsilon\in[0,\varepsilon_0]$.

Let $\psi_n(t,t_0,x_0,\varepsilon)$ be the solution of the differential equation

$$x'=F^n(t,x)=F_0^n(t,x)+\varepsilon F_1^n(t,x)+\varepsilon^2R^n(t,x,\varepsilon),$$

such that $\psi_n(t_0,t_0,x_0,\varepsilon)=x_0$. Clearly $\psi_n(t,t_0,\ x_0,0)=\varphi_n(t,t_0,x_0)$ which has been defined in Lemma 2.7.5. From the result of the continuous dependence of the solutions on the initial conditions we conclude that each of these functions are continuous in the variables (t,t_0,x_0). Indeed F^n is a continuous function which is Lipschitz in the second variable for $i=1,2,\dots,\kappa_z$. So using the Gronwall Lemma (see, for instance, [99]) we conclude that

$$||\psi_n(t,s_1,z_1,\varepsilon)-\psi_n(t,s_2,z_2,\varepsilon)||\le Me^{LT}|s_1-s_2|+e^{LT}||z_1-z_2||, \tag{2.7.42}$$

for each $t, s_1, s_2 \in \mathbb{S}^1$, $z_1, z_2 \in U_z$, and $\varepsilon \in [0, \varepsilon_0]$, where the constant L and M are defined in the proof of Lemma 2.7.4. Moreover,

$$\left|\left|\frac{\partial \psi_n}{\partial t}(t, s, z, \varepsilon)\right|\right| = ||F_n(t, s, \psi_n(t, s, z, \varepsilon), \varepsilon)|| \leq M,$$

therefore

$$\begin{aligned} ||\psi_n(t_1, s, z, \varepsilon) - \psi_n(t_2, s, z, \varepsilon)|| &\leq \max_{t \in \mathbb{S}^1} \left|\left|\frac{\partial \psi_n}{\partial t}(t, s, z, \varepsilon)\right|\right| \cdot |t_1 - t_2| \\ &\leq M|t_1 - t_2|. \end{aligned} \tag{2.7.43}$$

The relations (2.7.42) and (2.7.43) gives the following inequality

$$\begin{aligned} ||\psi_n(t_1, s_1, z_1, \varepsilon) - \psi_n(t_2, s_2, z_2, \varepsilon)|| \leq \; & M|t_1 - t_2| + Me^{LT}|s_1 - s_2| \\ & + e^{LT}||z_1 - z_2||, \end{aligned} \tag{2.7.44}$$

for $t_1, t_2, s_1, s_2 \in \mathbb{S}^1$, $z_1, z_2 \in U_z$, $\varepsilon \in [0, \varepsilon_0]$, and $n = 1, 2, \ldots, N$.

From (2.7.16) we obtain

$$\begin{aligned} & x^1(t, \zeta, \varepsilon) = \psi_{n_1}(t, 0, \zeta, \varepsilon) \quad \text{and} \\ & x^i(t, \zeta, 0) = \psi_{n_i}(t, t^{i-1}(\zeta, \varepsilon), x^{i-1}(t^{i-1}(\zeta, \varepsilon), \zeta, \varepsilon), \varepsilon), \end{aligned} \tag{2.7.45}$$

for $\zeta \in \mathcal{U}_z$ and for $i = 2, 3, \ldots, \kappa_z$. Thus, for $i = 1$, $x^1(t, \zeta, \varepsilon) = \varphi_{n_1}(t, 0, \zeta)$. So from (2.7.44) we have that

$$\begin{aligned} ||x^1(t_1, z_1, \varepsilon) - x^1(t_2, z_2, \varepsilon)|| = \; & ||\psi_{n_1}(t_1, 0, z_1, \varepsilon) - \psi_{n_1}(t_2, 0, z_2, \varepsilon)|| \\ \leq \; & e^{LT}||z_1 - z_2|| + M|t_1 - t_2|, \end{aligned}$$

for every $z_1, z_2 \in \mathcal{U}_z$, $0 \leq t_1 \leq t^1(z_1, \varepsilon)$, $0 \leq t_2 \leq t^1(z_2, \varepsilon)$, and $\varepsilon \in [0, \overline{\varepsilon}]$.

We assume by induction hypothesis that there exist constants $A_{\ell-1}$ and $B_{\ell-1}$ such that

$$||x^{\ell-1}(t_1, z_1, \varepsilon) - x^{\ell-1}(t_2, z_2, \varepsilon)|| \leq A_{\ell-1}|t_1 - t_2| + B_{\ell-1}||z_1 - z_2||,$$

for every $z_1, z_2 \in U_z$, $t^{\ell-2}(z_1, \varepsilon) \leq t_1 \leq t^{\ell-1}(z_1, \varepsilon)$, $t^{\ell-2}(z_2, \varepsilon) \leq t_2 \leq t^{\ell-1}(z_2, \varepsilon)$, and $\varepsilon \in [0, \varepsilon_0]$.

Now for $i = \ell$ the relation (2.7.45) implies that $x^\ell(t, \zeta, \varepsilon) = \psi_{n_\ell}(t, t^{\ell-1}(\zeta, \varepsilon), x^{\ell-1}(t^{\ell-1}(\zeta, \varepsilon), \zeta, \varepsilon), \varepsilon)$ for $\zeta \in U_z$, $t^{\ell-1}(\zeta, \varepsilon) \leq t \leq t^\ell(\zeta, \varepsilon)$ and $\varepsilon \in [0, \varepsilon_0]$. So from induction hypothesis we obtain

that

$$
\begin{aligned}
&||x^\ell(t_1,z_1,\varepsilon)-x^\ell(t_2,z_2,\varepsilon)||=||\psi_{n_\ell}(t_1,t^{\ell-1}(z_1,\varepsilon),x^{\ell-1}(t^{\ell-1}(z_1,\varepsilon),z_1,\,\varepsilon),\varepsilon)\\
&-\psi_{n_\ell}(t_2,t^{\ell-1}(z_2,\varepsilon),x^{\ell-1}(t^{\ell-1}(z_2,\varepsilon),z_2,\,\varepsilon),\varepsilon)||\leq\\
&M|t_1-t_2|+Me^{LT}|t^{\ell-1}(z_1,\varepsilon)-t^{\ell-1}(z_2,\varepsilon)|\\
&+e^{LT}||x^{\ell-1}(t^{\ell-1}(z_1,\varepsilon),z_1,\,\varepsilon)-x^{\ell-1}(t^{\ell-1}(z_2,\varepsilon),z_2,\,\varepsilon)||\leq\\
&M|t_1-t_2|+e^{LT}(M+A_{\ell-1})|t^{\ell-1}(z_1,\varepsilon)-t^{\ell-1}(z_2,\varepsilon)|+e^{LT}B_{\ell-1}||z_1-z_2||
\end{aligned}
\tag{2.7.46}
$$

for every $z_1,z_2\in U_z$, $t^{\ell-1}(z_1,\varepsilon)\leq t_1\leq t^\ell(z_1,\varepsilon)$, $t^{\ell-1}(z_2,\varepsilon)\leq t_2\leq t^\ell(z_2,\varepsilon)$, and $\varepsilon\in[0,\varepsilon_0]$.

From Claim 1 of the proof of Lemma 2.7.4 we have that $t^{\ell-1}(z,\varepsilon)$ is a $\mathbb{C}^1$ function, then there exists a constant $\delta>0$ such that $|t^{\ell-1}(z_1,\varepsilon)-t^{\ell-1}(z_2,\varepsilon)|\leq\delta||z_1-z_2||$ for every $\varepsilon\in[0,\varepsilon_0]$. Going back to the inequality (2.7.46) we get

$$||x^\ell(t_1,z_1,\varepsilon)-x^\ell(t_2,z_2,\varepsilon)||\leq A_\ell|t_1-t_2|+B_\ell||z_1-z_2||,$$

for every $z_1,z_2\in U_z$, $t^{\ell-1}(z_1,\varepsilon)\leq t_1\leq t^\ell(z_1,\varepsilon)$, $t^{\ell-1}(z_2,\varepsilon)\leq t_2\leq t^\ell(z_2,\varepsilon)$, and $\varepsilon\in[0,\varepsilon_0]$, where $A_\ell=Me^{LT}$ and $B_\ell=e^{LT}\left(\delta(M+A_{\ell-1})+B_{\ell-1}\right)$.

The proof of this lemma follows by noting that $x(T,z,\varepsilon)=x^{\kappa_z}(T,z,\varepsilon)$ which, from the above induction, is locally Lipschitz in the variable z. □

Lemma 2.7.7. Under the hypothesis of Theorem H the solution $x(t,z,\varepsilon)$ of the unperturbed differential system (2.5.2) is $\mathbb{C}^2$ in the variable z for every $z\in\overline{C}$. Moreover $(\partial x/\partial z)(t,z,0)=Y(t,z)Y(0,z)^{-1}$. The set C is defined in the statement of Lemma 2.7.4 and Y is the fundamental matrix solution of (2.5.3).

Proof. Assuming the hypothesis $(h1)$ instead of $(H1)$ we can prove, analogously to Claim 2.7.3 of the proof of Lemma 2.7.4, that for a given $z\in\overline{C}$ the functions $t^i(z,\varepsilon)$, $i=0,1,2,\cdots,\kappa_z$, are of class $\mathbb{C}^2$ for every ζ in a neighborhood $U_z\subset C$ of z and $\varepsilon\in[0,\varepsilon_0]$. The proof of this lemma follows analogous the proof of Lemma 2.7.5 but now considering the functions $\psi_n(t,t_0,x_0,\varepsilon)$ defined in Lemma 2.7.6. □

The next two lemmas are versions of the so called *Lyapunov–Schmidt reduction* for finite dimensional function (see for instance [24]). Their proofs can be found in [20] and [17, 18], respectively. The first lemma will be used for proving Theorem G, and the second one will be used for proving Theorem H.

Lemma 2.7.8. Let $P:\mathbb{R}^d\to\mathbb{R}^d$ be a $\mathbb{C}^1$ function, and let $Q:\mathbb{R}^d\times[0,\varepsilon_0]\to\mathbb{R}^d$ be a continuous functions which is locally Lipschitz in the first variable, and define $f:\mathbb{R}^d\times[0,\varepsilon_0]\to\mathbb{R}^d$ as $f(z,\varepsilon)=P(z)+\varepsilon Q(z,\varepsilon)$. We assume that there exists an open bounded subset $V\subset\mathbb{R}^k$ with $k\leq n$

and a $\mathcal{C}^1$ function $\beta_0 : \overline{V} \to \mathbb{R}^{d-k}$ such that P vanishes on the set $\mathcal{Z} = \{z_\alpha = (\alpha, \beta_0(\alpha)) : \alpha \in \overline{V}\}$ and that for any $\alpha \in \overline{V}$ the matrix $DP(z_\alpha)$ has in its upper right corner the null $k \times (d-k)$ matrix and in the lower corner the $(d-k) \times (d-k)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$. For any $\alpha \in \overline{V}$ we define $f_1(\alpha) = \pi Q(z_\alpha, 0)$. Thus if $f_1(\alpha) \neq 0$ for all $\alpha \in \partial V$ and $d_B(f_1, V, 0) \neq 0$, then there exists $\varepsilon_1 > 0$ sufficiently small such that for each $\varepsilon \in (0, \varepsilon_1]$ there exists at least one $z_\varepsilon \in \mathbb{R}^d$ with $F(z_\varepsilon, \varepsilon) = 0$ and $\mathrm{dis}(z_\varepsilon, \mathcal{Z}) \to 0$ as $\varepsilon \to 0$.

Lemma 2.7.9. Let $P : \mathbb{R}^d \to \mathbb{R}^d$ and $Q : \mathbb{R}^d \times [0, \varepsilon_0] \to \mathbb{R}^d$ be $\mathbb{C}^2$ functions, and define $f : \mathbb{R}^d \times [0, \varepsilon_0] \to \mathbb{R}^d$ as $f(z, \varepsilon) = P(z) + \varepsilon Q(z, \varepsilon)$. We assume that there exists an open bounded subset $V \subset \mathbb{R}^k$ with $k \leq n$ and a $\mathbb{C}^2$ function $\beta_0 : \overline{V} \to \mathbb{R}^{d-k}$ such that P vanishes on the set $\mathcal{Z} = \{z_\alpha = (\alpha, \beta_0(\alpha)) : \alpha \in \overline{V}\}$ and that for any $\alpha \in \overline{V}$ the matrix $DP(z_\alpha)$ has in its upper right corner the null $k \times (d-k)$ matrix and in the lower corner the $(d-k) \times (d-k)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$. For any $\alpha \in \overline{V}$ we define $f_1(\alpha) = \pi Q(z_\alpha, 0)$. Thus if there exists $a \in V$ with $f_1(a) \neq 0$ and $\det(f'(a)) \neq 0$, then there exists α_ε such that $f(z_{\alpha_\varepsilon}, \varepsilon) = 0$ and $z_{\alpha_\varepsilon} \to z_a$ as $\varepsilon \to 0$.

Now we prove our main results.

Proof of Theorem G. We consider the $\mathbb{C}^1$ function $f : \overline{C} \times [0, \varepsilon_0] \to \mathbb{R}^d$, given by

$$f(z, \varepsilon) = x(T, z, \varepsilon) - z. \tag{2.7.47}$$

Its differentiability comes from Lemma 2.7.5. Clearly system (2.5.1) for $\varepsilon = \overline{\varepsilon} \in [0, \varepsilon_0]$ has a periodic solution passing through $\overline{z} \in C$ if and only if $f(\overline{z}, \overline{\varepsilon}) = 0$.

From Lemma 2.7.4 we have that $x(t, z, \varepsilon) = x(t, z, 0) + \varepsilon y_1(t, z) + o(\varepsilon)$. Taking $P(z) = x(t, z, 0) - z$ and $Q(z, \varepsilon) = y_1(t, z) + o(\varepsilon)/\varepsilon$, thus $f(z, \varepsilon) = P(z) + \varepsilon Q(z, \varepsilon)$. Moreover from Lemma 2.7.5 $P(z)$ is a $\mathbb{C}^1$ function, and from Lemma 2.7.6 $Q(z, \varepsilon)$ is a continuous function which is locally Lipschitz in the first variable because $Q(z, \varepsilon) = (x(T, z, \varepsilon) - x(T, z, 0))/\varepsilon$.

In order to apply Lemma 2.7.8 to function (2.7.47) we compute

$$P(z_\alpha) = x(T, z_\alpha, 0) - z_\alpha = 0,$$

and

$$\begin{aligned} \frac{\partial P}{\partial z}(z_\alpha) = & \frac{\partial x}{\partial z}(T, z_\alpha, 0) - Id \\ = & Y_\alpha(T) Y_\alpha(0)^{-1} - Id. \end{aligned}$$

From hypothesis (H) the function P vanishes on the set $\mathcal{Z}$, and from hypothesis $(H2)$ the matrix $DP(z_\alpha)$, for each $\alpha \in \overline{V}$, has in its upper right corner the null $k \times (d-k)$ matrix and in the lower corner the $(d-k) \times (d-k)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$. Since $\pi Q(\alpha, \beta_0(\alpha)) = \pi y_1(T, z_\alpha) = f_1(\alpha)$, the proof follows by applying Lemma 2.7.8. □

Proof of Theorem H. The proof is analogous to the proof of Theorem G applying Lemma 2.7.7 instead of Lemmas 2.7.5 and 2.7.6, and applying Lemma 2.7.9 instead of Lemma 2.7.8. □

2.8 Studying examples

2.8.1 Proof of example 1

Proof of Proposition 2.6.1. The linear DPDS (2.6.1) in polar coordinates (r, θ) becomes

$$\begin{aligned}
\dot{r} = \; & \varepsilon\left(a_{0i}\cos\theta + a_{1i}r\cos^2\theta + b_{0i}\sin\theta + a_{2i}r\cos\theta\sin\theta + b_{1i}r\cos\theta\sin\theta + b_{2i}r\sin^2\theta\right) + \\
& \varepsilon^2\left(c_{0i}\cos\theta + c_{1i}r\cos^2\theta + c_{2i}r\cos\theta\sin\theta + d_{1i}r\cos\theta\sin\theta + d_{2i}r\sin^2\theta + d_{0i}\sin\theta\right), \\
\dot{\theta} = \; & -1 - \frac{\varepsilon}{r}\left(-b_{0i}\cos\theta - b_{1i}r\cos^2\theta + a_{0i}\sin\theta + a_{1i}r\cos\theta\sin\theta - b_{2i}r\cos\theta\sin\theta + a_{2i}r\sin^2\theta\right) - \\
& \frac{\varepsilon^2}{r}\left(-d_{0i}\cos\theta - d_{1i}r\cos^2\theta + c_{0i}\sin\theta + c_{1i}r\cos\theta\sin\theta - d_{2i}r\cos\theta\sin\theta + c_{2i}r\sin^2\theta\right),
\end{aligned}$$

with $i = 1$ if $0 < \theta < \pi/2$, $i = 2$ if $\pi/2 < \theta < \pi$, $i = 3$ if $\pi < \theta < 3\pi/2$, and $i = 4$ if $3\pi/2 < \theta < 2\pi$. Taking the angle θ as the new independent variable the DPDS (2.6.1) writes

$$\dot{r} = \varepsilon F_{1i} + \varepsilon^2 F_{2i} + \mathcal{O}(\varepsilon^3), \tag{2.8.1}$$

where

$$\begin{aligned}
F_{1i} = \; & -r(a_{0i}\cos\theta + a_{1i}r\cos^2\theta + b_{0i}\sin\theta + a_{2i}r\cos\theta\sin\theta + b_{1i}r\cos\theta\sin\theta + b_{2i}r\sin^2\theta), \\
F_{2i} = \; & \frac{1}{r}(-b_{1i}r\cos^2\theta - b_{0i}\cos\theta + a_{1i}r\sin\theta\cos\theta - b_{2i}r\sin\theta\cos\theta + a_{2i}r\sin^2\theta + a_{0i}\sin\theta) \\
& (a_{1i}r\cos^2\theta + a_{0i}\cos\theta + a_{2i}r\sin\theta\cos\theta + b_{1i}r\sin\theta\cos\theta + b_{2i}r\sin^2\theta + b_{0i}\sin\theta) \\
& -(c_{1i}r\cos^2\theta + c_{0i}\cos\theta + c_{2i}r\sin\theta\cos\theta + d_{1i}r\sin\theta\cos\theta + d_{2i}r\sin^2\theta + d_{0i}\sin\theta).
\end{aligned}$$

From Proposition 2.6.1 the assumptions of Theorem E hold for the DPDS (2.8.1). Computing the averaged function f_1 we obtain

$$\begin{aligned}
f_1(r) = \; & \frac{1}{4}r(-4a_{01} + 4(a_{02} + a_{03} - a_{04} - b_{01} - b_{02} + b_{03} + b_{04}) \\
& -(2a_{21} - 2(a_{22} - a_{23} + a_{24} - b_{11} + b_{12} - b_{13} + b_{14}) \\
& +(a_{11} + a_{12} + a_{13} + a_{14} + b_{21} + b_{22} + b_{23} + b_{24})\pi)r).
\end{aligned}$$

Clearly f_1 has at most 1 zero. Moreover we can choose coefficients a_{ij}, in such a way that f_1 has a simple positive zero. Hence this proposition is proved. □

Proof of Proposition 2.6.2. We choose coefficients a_{ij}, such that the conditions contained in $\mathcal{A}$ hold. Then $f_1(r) \equiv 0$. Again from Proposition 2.4.1 the assumptions of Theorem F hold for the DPDS (2.8.1). Using some algebraic manipulator as Mathematica or Maple we obtain

$$f_2(r) = k_0 + k_1 r + k_2 r^2 + k_3 r^3 + k_4 r^4, \tag{2.8.2}$$

where k_i, $i = 0, ..., 4$, depends on the coefficients a_{ij}, $i = 0, 1$, $j = 1, ..., 4$ and can be taken freely. The function (2.8.2) is a polynomial in the variable r of degree 4. So, clearly, it has at most 4 zeros. Moreover we can choose coefficients a_{ij}, $i = 0, 1$, such that (2.8.2) has 0, 1, 2, 3 or 4 simple zeros. So this proposition is proved. □

2.8.2 Proof of example 2

First of all we recall the *Descartes Theorem* about the number of zeros of a real polynomial (for a proof see for instance either the pages 82 and 83 of [7], or the appendix of [82]).

Descartes Theorem *Consider the real polynomial* $p(x) = a_{i_1}x^{i_1} + a_{i_2}x^{i_2} + \cdots + a_{i_r}x^{i_r}$ *with* $0 \le i_1 < i_2 < \cdots < i_r$ *and* $a_{i_j} \neq 0$ *real constants for* $j \in \{1, 2, \cdots, r\}$. *When* $a_{i_j}a_{i_{j+1}} < 0$, *we say that* a_{i_j} *and* $a_{i_{j+1}}$ *have a variation of sign. If the number of variations of signs is* m, *then* $p(x)$ *has at most* m *positive real roots.*

Now consider the functions

$$g_1(u) = 1,$$

$$g_2(u) = u^2,$$

$$g_3(u) = u^4,$$

$$g_4(u) = u\left(2 + u^2\right)\arccos\left(\frac{u}{\sqrt{2+u^2}}\right),$$

$$g_5^1(u) = u\left(2 + u^2\right),$$

$$g_5^2(u) = u\left(2 + u^2\right)\left(\pi - \arccos\left(\frac{u}{\sqrt{2+u^2}}\right)\right),$$

$$g_6(u) = \sqrt{2}u^6 - u\left(8 - 4u^4 - u^6\right)\left(\frac{\pi}{2} + \arcsin\left(\frac{u}{\sqrt{2+u^2}}\right)\right), \quad \text{and}$$

$$g_7(u) = -\sqrt{2}u^6 - \frac{3\pi u^3\left(2+u^2\right)^2}{2} - u\left(8 - 4u^4 - u^6\right)\arccos\left(\frac{u}{\sqrt{2+u^2}}\right).$$

We define the sets of functions $G^1 = \{g_1, g_2, g_3, g_4, g_5^1\}$ and $G^2 = \{g_1, g_2, g_3, g_4, g_5^2, g_6, g_7\}$.

Lemma 2.8.1. The sets of functions G^1 and G^2 are ECT-systems (see Appendix B) on the interval $(0, \infty)$.

Proof. To prove the statement we compute the Wronskians $W_1(u) = g_1(u)$, $W_2(u) = W(g_1, g_2)(u)$, $W_3(u) = W(g_1, g_2, g_3)(u)$, $W_4(u) = W(g_1, g_2, g_3, g_4)(u)$, $W_5^1(u) = W(g_1, g_2, g_3, g_4, g_5^1)(u)$, $W_5^2(u) =$

$W(g_1, g_2, g_3, g_4, g_5^2)(u)$, $W_6(u) = W(g_1, g_2, g_3, g_4, g_5^2, g_6)(u)$, and $W_6(u) = W(g_1, g_2, g_3, g_4, g_5^2, g_6, g_7)(u)$. So

$$\begin{aligned}
W_1(u) &= 1, \\
W_2(u) &= 2u, \\
W_3(u) &= 16u^3, \\
W_4(u) &= \frac{16u}{(2+u^2)^2}\left(P_1(u) + P_2(u)\arccos\left(\frac{u}{\sqrt{2+u^2}}\right)\right), \\
W_5^1(u) &= -\frac{6144\sqrt{2}u^3}{(2+u^2)^3}, \\
W_5^2(u) &= -\frac{6144\sqrt{2}\pi u^3}{(2+u^2)^3}, \\
W_6(u) &= \frac{-12\,288\pi u}{(2+u^2)^6}\left(Q_1(u) + Q_2(u)\arcsin\left(\frac{u}{\sqrt{2+u^2}}\right)\right), \quad \text{and} \\
W_7(u) &= -\frac{14\,495\,514\,624\pi^2 u^3\,(11+u^2)}{(2+u^2)^8},
\end{aligned}$$

where

$$\begin{aligned}
P_1(u) &= \sqrt{2}u\left(12 + 4u^2 + 3u^4\right), \\
P_2(u) &= 3\left(2 - u^2\right)\left(2 + u^2\right)^2, \\
Q_1(u) &= 144\sqrt{2}\pi - 288u - 336\sqrt{2}\pi u_7^2 20u^3 + 1\,656\sqrt{2}\pi u^4 + 14\,584u^5 + 5\,760\sqrt{2}\pi u^6 \\
&\quad +14\,700u^7 + 4\,305\sqrt{2}\pi u^8 + 3\,780u^9 + 945\sqrt{2}\pi u^{10}, \quad \text{and} \\
Q_2(u) &= 6\sqrt{2}\left(2 + u^2\right)^2\left(12 - 40u^2 + 175u^4 + 315u^6\right).
\end{aligned}$$

Clearly $W_1(u) \neq 0$, $W_2(u) \neq 0$, $W_3(u) \neq 0$, $W_5^1(u) \neq 0$, $W_5^2(u) \neq 0$ and $W_7(u) \neq 0$ for $u > 0$. To see that the function $W_4(u)$ does not vanish for any $u > 0$ we shall prove that

$$\widetilde{P}(u) = P_1(u) + P_2(u)\arccos\left(\frac{u}{\sqrt{2+u^2}}\right)$$

is an increasing function. Computing its derivative we have

$$P'(u) = 6u\left(\sqrt{2}u\left(2 + 3u^2\right) + \left(3u^4 + 4u^2 - 4\right)\arcsin\left(\frac{u}{\sqrt{2+u^2}}\right)\right).$$

It is easy to see that $(3u^4+4u^2-4)$ is increasing. So $\widetilde{P}'(u)$ is also a increasing function for $u>0$, because it is sums and products of increasing functions. Since $\widetilde{P}'(0)=0$ it follows that $\widetilde{P}'(u)>0$ for every $u>0$. This implies that $\widetilde{P}(u)$ is an increasing function for $u>0$. Again, since $\widetilde{P}(0)=0$ it follows that $\widetilde{P}(u)>0$ for every $u>0$. Thus $W_4(u)\neq 0$ for $u>0$.

To see that the function $W_6(u)$ does not vanish for any $u>0$ we shall prove that

$$\widetilde{Q}(u)=Q_1(u)+Q_2(u)\arcsin\left(\frac{u}{\sqrt{2+u^2}}\right)$$

is a positive function for $u>0$. From Descartes Theorem the polynomials Q_1 and Q_2 have at most 2 zeros, and 1 minimum or maximum. Numerically we find $u_1\approx 0.247$ and $u_2\approx 0.269$ as the minimums for Q_1 and Q_2 respectively. So $\widetilde{Q}(u)$ is an increasing function for $u>\max\{u_1,u_2\}$. Finally it is easy to see that $\widetilde{Q}(u)>0$ for $0<u\leq\max\{u_1,u_2\}$. Thus $W_6(u)\neq 0$ for $u>0$. Hence this lemma is proved. □

Proof of Proposition 2.6.3. Consider system (2.6.2). Proceeding with the change of variables $x=r\cos\theta$ and $y=r\sin\theta$, and taking θ as the new time, system (2.6.2) becomes equivalent

$$r'=\begin{cases} A(\theta,r) & \text{if} \quad r\sin^2\theta+\sin\theta-r>0,\\ B(\theta,r) & \text{if} \quad r\sin^2\theta+\sin\theta-r<0,\end{cases} \tag{2.8.3}$$

where

$$\begin{aligned} A(\theta,r)=\ & -p_{20}^1r^2\cos^3\theta-r\cos^2\theta\left(p_{10}^1+\left(p_{11}^1+q_{20}^1\right)r\sin\theta\right)\\ & -\cos\theta\left(p_{00}^1+r\sin\theta\left(p_{01}^1+q_{10}^1+\left(p_{02}^1+q_1^{11}\right)r\sin\theta\right)\right)\\ & -\sin\theta\left(q_{00}^1+r\sin\theta\left(q_{01}^1+q_{02}^1r\sin\theta\right)\right),\end{aligned}$$

$$\begin{aligned} B(\theta,r)=\ & -r_{20}^1r^2\cos^3\theta-r\cos^2\theta\left(r_{10}^1+\left(r_{11}^1+s_{20}^1\right)r\sin\theta\right)\\ & -\cos\theta\left(r_{00}^1+r\sin\theta\left(r_{01}^1+s_{10}^1+\left(r_{02}^1+s_1^{11}\right)r\sin\theta\right)\right)\\ & -\sin\theta\left(s_{00}^1+r\sin\theta\left(s_{01}^1+s_{02}^1r\sin\theta\right)\right).\end{aligned}$$

Clearly hypothesis $(Ha1)$ holds for system (2.8.3). Given

$$\theta_1(r)=\arcsin\left(\frac{u}{\sqrt{2+u^2}}\right)\quad\text{and}\quad\theta_2(r)=\pi-\arcsin\left(\frac{u}{\sqrt{2+u^2}}\right),$$

we have that for $r>0$, $r\sin^2\theta+\sin\theta-r>0$ if and only $0\leq\theta<\theta_1(r)$ and $\theta_2(r)<\theta\leq 2\pi$; and $r\sin^2\theta+\sin\theta-r<0$ if and only if $\theta_1(r)<\theta<\theta_2(r)$. Let $\widetilde{h}(\theta,r)=r\sin^2\theta+\sin\theta-r$, thus the set of

discontinuity of system (2.8.3) is given by $\widetilde{\Sigma} = \widetilde{h}^{-1}(0) = \{(\theta_1(r), r) : r > 0\} \cup \{(\theta_2(r), r) : r > 0\}$. Since

$$\left\langle \nabla \widetilde{h}(\theta_1(r), r), (1, A(\theta_1(r), r)) \right\rangle \left\langle \nabla \widetilde{h}(\theta_1(r), r), (1, B(\theta_1(r), r)) \right\rangle = \frac{(1+4r^2)\left(-1+\sqrt{1+4r^2}\right)}{2r^2},$$

$$\left\langle \nabla \widetilde{h}(\theta_2(r), r), (1, A(\theta_2(r), r)) \right\rangle \left\langle \nabla \widetilde{h}(\theta_2(r), r), (1, B(\theta_2(r), r)) \right\rangle = \frac{(1+4r^2)\left(-1+\sqrt{1+4r^2}\right)}{2r^2},$$

we conclude that $\widetilde{\Sigma}$ has only crossing regions. So hypothesis (HC) holds for system (2.8.3).

Taking $r = u\sqrt{2+u^2}/2$ and computing the averaged function f_1 we obtain

$$f_1(u) = k_1 g_1(u) + k_2 g_2(u) + k_3 g_3(u) + k_4 g_4(u) + k_5 g_5^1(u),$$

where

$$\begin{aligned}
k_1 &= 24\sqrt{2}\left(q_{00}^1 - s_{00}^1\right), \\
k_2 &= 2\sqrt{2}\left(-3p_{10}^1 + 2p_{11}^1 + 3q_{01}^1 + 4q_{02}^1 + 2q_{20}^1 + 3r_{10}^1 - 2r_{11}^1 - 3s_{01}^1 - 4s_{02}^1 - 2s_{20}^1\right), \\
k_3 &= 6\sqrt{2}\left(q_{02}^1 - s_{02}^1\right), \\
k_4 &= -6\left(p_{10}^1 + q_{01}^1 - r_{10}^1 - s_{01}^1\right), \\
k_5 &= -3\left(p_{10}^1 + q_{01}^1 - r_{10}^1 - s_{01}^1\right).
\end{aligned}$$

So from Lemma 2.8.1 and Theorem E the proof follows. □

Proof of Proposition 2.6.4. In order to apply Theorem F to system (2.8.3) we have to guarantee that $f_1(u) \equiv 0$. By the linearity of the set of functions G^1, $f_1(u) \equiv 0$ if and only if $k_i = 0$ for $i = 1, 2, \ldots, 5$. Thus assuming that $k_i = 0$ for $i = 1, 2, \ldots, 5$, it is easy to see, using some algebraic manipulator as Mathematica or Maple, that the statement $\langle \nabla h(\theta_1(r), r), (s, y_1(\theta_1(r), t)) \rangle = 0$ implies $s = 0$ holds if and only if the conditions $\mathcal{B}$ holds. So assuming conditions $\mathcal{B}$ the hypothesis $(Hb2)$ holds.

Taking $r = u\sqrt{2+u^2}/2$ and computing the averaged function f_1 we obtain

$$f_2(u) = k_1 g_1(u) + k_2 g_2(u) + k_3 g_3(u) + k_4 g_4(u) + k_5 g_5^2(u) + k_6 g_6(u) + k_6 g_6(u).$$

Hence from Lemma 2.8.1 and Theorem F the proof follows. □

2.8.3 Proof of example 3

To study system (2.6.3) it is conveniently to proceed with the change of variables $(u, v, w) = (r\cos\theta, r\sin\theta, z)$. Taking θ as the new time by doing $r' = \dot{r}/\dot{\theta}$ and $z' = \dot{z}/\dot{\theta}$ system (2.6.3) becomes

$$(r', z') = \begin{cases} (0, z) + \varepsilon\, G^{+}(\theta, r, z) + \mathcal{O}(\varepsilon^2) & \text{if} \quad 0 \le \theta \le \pi, \\ (0, z) + \varepsilon\, G^{-}(\theta, r, z) + \mathcal{O}(\varepsilon^2) & \text{if} \quad \pi \le \theta \le 2\pi, \end{cases} \tag{2.8.4}$$

where $G^{\pm} = \left(G_1^{\pm}, G_2^{\pm}\right)$, and

$$\begin{aligned} G_1^{\pm} = \;& b_1^{\pm} r \cos^2\theta + \left(a_1^{\pm} + d_1^{\pm} z + (b_2^{\pm} + c_1^{\pm}) r \sin\theta\right)\cos\theta \\ & + \left(a_2^{\pm} + d_2^{\pm} z + c_2^{\pm} r \sin\theta\right)\sin\theta, \\ G_2^{\pm} = \;& \frac{1}{r}\Big(r(a_3^{\pm} + d_3^{\pm} z) - b_2^{\pm} r z \cos^2\theta + (C_3^{\pm} r^2 + (a_1^{\pm} + d_1^{\pm} z) z + c_1^{\pm} r \sin\theta)\sin\theta \\ & (b_3^{\pm} r^2 - (a_2^{\pm} + d_2^{\pm}) z + (b_1^{\pm} - c_2^{\pm}) r z \sin\theta)\cos\theta\Big). \end{aligned}$$

Here the prime denotes the derivative with respect to θ.

For system (2.8.4) we have that $D = \{(r, z) : 0 < r < r_0,\, z \in \mathbb{R}\}$ and $T = 2\pi$ with r_0 arbitrarily large. We note that $\Sigma = \{(0, r) : r > 0\} \cup \{(\pi, r) : r > 0\} \cup \{(2\pi, r) : r > 0\}$, thus taking $h(\theta, r, z) = \theta(\theta - \pi)(\theta - 2\pi)$ it follows that $\Sigma = h^{-1}(0)$.

In what follows we prove Proposition 2.6.5.

Proof of Proposition 2.6.5. To prove this proposition we shall study the elements of hypothesis (H) of Theorem G. For $\varepsilon = 0$ the solution $x(\theta, r, z, 0)$ of system (2.8.4) such that $x(0, r, z, 0) = (r, z)$ is given by $x(\theta, r, z, 0) = (r, e^{\theta} z)$. Taking $V = \{r \in \mathbb{R} : r_1 < \alpha < r_2\}$ with $r_1 > 0$ arbitrarily small and $r_0 > r_2 > r_1$, and $\beta_0 = 0$ we have that the solution $x_\alpha(\theta) = (\alpha, 0)$ is constant for every $\alpha \in \overline{V}$, particularly 2π–periodic. In this case the compact manifold $\mathcal{Z}$ of periodic solution of the system (2.8.4) when $\varepsilon = 0$ is given by $\mathcal{Z} = \{(\alpha, 0) : r_1 \le \alpha \le r_2\}$, and $\Sigma_0 = D$ is an open bounded set. Since $\mathcal{Z} \subset \Sigma_0$ it follows that $\mathcal{Z} \cap \partial\Sigma_0 = \emptyset$. Moreover computing the crossing region of system (2.8.4) for $\varepsilon > 0$ sufficiently small we conclude that $\Sigma^c = \Sigma$, so we obtain that $\widetilde{\mathcal{Z}^0} \cap \Sigma \subset \Sigma^c$. Therefore hypothesis (H) hods for system (2.8.4).

Hypothesis $(H1)$ of Theorem G clearly holds for system (2.8.4). To check hypothesis $(H2)$ we take

$$Y(\theta, r, z) = \frac{\partial x}{\partial z}(\theta, r, z, 0) = \begin{pmatrix} 1 & 0 \\ 0 & e^{\theta} \end{pmatrix}$$

as the fundamental matrix solution of system (2.5.3) in the case of system (2.8.4). So

$$Y_\alpha(2\pi) Y_\alpha(0)^{-1} - Id = Y(2\pi, \alpha, 0) Y(0, \alpha, 0)^{-1} - Id = \begin{pmatrix} 0 & 0 \\ 0 & e^{2\pi} - 1 \end{pmatrix}.$$

Note that $\Delta_\alpha = e^{2\pi} - 1 \neq 0$ for every $\alpha \in \overline{V}$. Hence hypothesis $(H2)$ holds for system (2.8.4).

Now if $(\theta, r, z) \in \Sigma$, then $\theta \in \{0, \pi\}$. On the other hand $\nabla h(0, r, z) = (2\pi^2, 0, 0)$ and $\nabla h(\pi, r, z) = (-\pi^2, 0, 0)$ for every $(r, z) \in D$. So $\langle \nabla h(\theta, r, z), (0, v) \rangle = 0$ for every $\theta \in \{0, \pi\}$, $(r, z) \in D$, and $v \in \mathbb{R}^2$, which means that for any $v \in \mathbb{R}^2$ we have that $(0, v) \in T_{(\theta,r,z)}\Sigma$ for every $\theta \in \{0, \pi\}$ and $(r, z) \in D$. In short hypothesis $(H3)$ holds for system (2.8.4).

Using an algebraic manipulator as Mathematica or Maple we compute

$$f_1(\alpha) = \frac{\pi}{2}\left(b_1^+ + b_1^- + c_2^+ + c_2^-\right)\alpha + 2\left(a_2^+ - a_2^-\right).$$

From hypotheses $\left(b_1^+ + b_1^- + c_2^+ + c_2^-\right)\left(a_2^- - a_2^+\right) > 0$, thus

$$a = \frac{4\left(a_2^- - a_2^+\right)}{\pi\left(b_1^+ + b_1^- + c_2^+ + c_2^-\right)}$$

is a solutions of the equation $f_1(\alpha) = 0$ such that $f_1'(a) \neq 0$. From Remark 2.5.2 it is a sufficient condition to guarantee the existence of a small neighborhood $W \subset V$ of a such that $d_B(f_1, W, 0) \neq 0$. Since f_1 is linear, it is clear that $f_1(\alpha) \neq 0$ for every $\alpha \in \partial W$. Therefore hypothesis $(H4)$ of Theorem G holds for system (2.8.4).

Now the proof of the proposition follows directly by applying Theorems G and H. $\square$

Chapter 3

Limit cycles of planar piecewise linear systems with two zones

The main result of this chapter (Theorem I) is based on the paper [91].

3.1 Introduction to the Braga–Mello conjecture

The computation of upper bounds for the number of limit cycles in all possible configurations within the family of planar piecewise linear differential systems with two zones has been the subject of some recent papers. Assuming that the separation boundary is a straight line, Han and Zhang [43] conjectured in 2010 that for such planar piecewise linear systems there can be at most two limit cycles. However, Huan and Yang [48] promptly gave a negative answer to this conjecture by means of a numerical example with three limit cycles under a focus-focus configuration. Such counter-intuitive example led researchers to look for rigorous proofs of this fact, see [80] for a computer-assisted proof, and [37] for an analytical proof under a more general setting.

Recently, in [11] one can find a study showing that the three limit cycles of the Huan and Yang's example can be simultaneously obtained through a rather special bifurcation. Later, a general and analytical proof for the existence of three nested limit cycles in certain open regions of the parameter space in the focus-focus configuration was given in [37]. In [38] it is proved that one can have three limit cycles not only in the focus-focus case, as shown in [37], so that the lower bounds for the maximum number of limit cycles corresponding both to the focus-node and focus-saddle cases is three, one more than stated before, see [83]. This number three seems to be the maximum number of limit cycles that can be obtained through piecewise linear perturbation of a linear center, see [22]. In all the cases, the existence of a sliding set is crucial for existence of multiple limit cycles; otherwise, it can be shown the uniqueness of limit cycles, see [87].

When the boundary between the two linear zones is not a straight line any longer, it is possible to obtain more than three limit cycles. Thus, by resorting to the same example introduced in [48] and analyzed in [80], Braga and Mello studied in [12] some members of the following class of

discontinuous piecewise linear differential system with two zones

$$X' = \begin{cases} G^- X & \text{if} \quad H(X,p) < 0, \\ \\ G^+ X & \text{if} \quad H(X,p) \geq 0, \end{cases} \tag{3.1.1}$$

where the prime denotes derivative with respect to the independent variable t, p is a parameter vector, $X = (x, y)$, and

$$G^{\pm} = \begin{pmatrix} g_{11}^{\pm} & g_{12}^{\pm} \\ \\ g_{21}^{\pm} & g_{22}^{\pm} \end{pmatrix}$$

are matrices with real entries satisfying the following assumptions:

$(H1)$ $g_{12}^{\pm} < 0$,

$(H2)$ G^- has complex eigenvalues with negative real parts while G^+ has complex eigenvalues with positive real parts, and

$(H3)$ the function H is at least continuous.

After using some broken line as the boundary between linear zones, Braga and Mello in [12] put in evidence the important role of the separation boundary in the number of limit cycles. They obtained examples with different number of limit cycles, and accordingly stated the following conjecture in [12].

Braga–Mello Conjecture *Given $n \in \mathbb{N}$ there is a piecewise linear system with two zones in the plane with exactly n limit cycles.*

As a consequence of this conjecture we have that the number of limit cycles for the family of planar piecewise linear differential systems with two zones is unbounded.

3.2 Oscillating the line of discontinuity to create several limit cycles

In this section, we prove that the Braga–Mello Conjecture is true by showing how to perturb a rather simple vector field in order to get as many limit cycles as wanted. We provide several concrete examples. Furthermore, we show that the involved methodology allow us to locate the position of the limit cycles and determine their stability.

We start from the normal form given in [36] for systems of kind (3.1.1) and after selecting an appropriate value for $\gamma > 0$, we take

$$G^{\pm} = \begin{pmatrix} \pm 2\gamma & -1 \\ \\ \gamma^2 + 1 & 0 \end{pmatrix}, \tag{3.2.1}$$

and

$$H(X) = \begin{cases} x & \text{if} \quad y \leq 0, \\ \\ x - h(y) & \text{if} \quad y > 0, \end{cases} \tag{3.2.2}$$

where $h(y)$ is a $\mathcal{C}^1$ function such that $h(0) = 0$. We also assume that for $y > 0$ the following hypotheses:

$(H1')$ $|h(y)| < y/\gamma$,

$(H2')$ $h(y)(2\gamma - (1+\gamma^2)h'(y)) < y$, and

$(H3')$ $h(y)(2\gamma + (1+\gamma^2)h'(y)) > -y$.

As shown in Section 3.3, Hypothesis $H1'$ is just assumed to facilitate the computation of solutions, whilst Hypotheses $H2'$ and $H3'$ allow us to assure that the two linear vector fields can be concatenated across the discontinuity curve in the natural way, so avoiding the existence of sliding sets, see [64].

It should be noticed that when $h(y) \equiv 0$ the boundary separating the two linear zones is a straight line, namely the y-axis. In such a case, we have indeed a continuous vector field with a global nonlinear center at the origin, since from each side the origin is a focus and the expansion in the right part is perfectly balanced with the contraction in the left part. Furthermore, all the periodic orbits of the center are homothetic. See the left panel of Figure 3.2 and Proposition 4.2 of [36] for more details.

In the case of a non-vanishing function h, the discontinuity set for system (3.1.1) is given by

$$\Sigma = \{(h(y), y) : \, y > 0, h(y) \neq 0\},$$

so that the balance between expansion and contraction is lost; as it will be seen, some periodic orbits from the original center configuration can persist, becoming isolated and so leading to limit cycles.

Our main result is the following.

Theorem I. Assume $\gamma > 0$ and consider system (3.1.1)-(3.2.1) and the switching curve $H(X) = 0$ as in (3.2.2), where h satisfies Hypotheses $H1'$, $H2'$ and $H3'$. For a given positive real number y^* there exists a periodic solution of system (3.1.1) passing through $(h(y^*), y^*)$ if and only if $h(y^*) = 0$, in this case the periodic solution cut the y–axis at the points $(0, y^*)$ and $(0, -e^{-\gamma\pi}y^*)$. Moreover if $h'(y^*) < 0$ ($h'(y^*) > 0$) this periodic solution is a stable (unstable) limit cycle.

Theorem I is proved in Section 3.3.

The Braga–Mello Conjecture is a direct consequence of Theorem I as we can see in the following corollaries. See also the right panel of Figure 3.2.

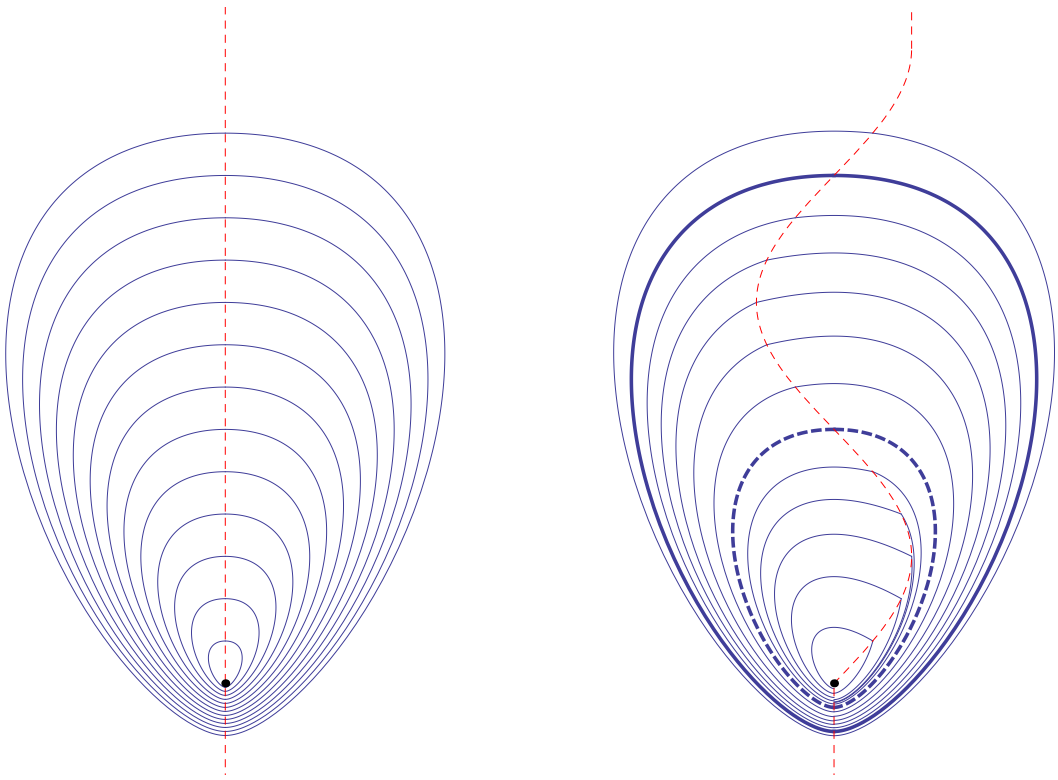

Figure 3.1: Left: The unperturbed piecewise linear center for $\gamma = 0.75$. Right: Here we consider the system of Corollary 3.2.1 for $n = 2$ and $\gamma = 0.75$. The continuous bold (dashed) closed curve surrounding the origin represent one stable (unstable) limit cycle, while the remaining orbits are not closed any longer. The discontinuity set is represented by the dashed line crossing the y–axis twice for $y > 0$.

Corollary 3.2.1. If $0 < \gamma < \sqrt{3/5}$ and

$$h(y) = \frac{2\gamma}{(\gamma^2+1)\pi} \begin{cases} \sin(\pi y), & 0 \le y \le (2n+1)/2, \\ (-1)^n, & y > (2n+1)/2, \end{cases}$$

then system (3.1.1) has exactly n limit cycles for any $n \in \mathbb{N}$. These limit cycles are nested and surround the origin, which is a stable singular point of focus type. The limit cycles cut the y–axis at the points $(0, k)$ and $(0, -k\, e^{-\gamma\pi})$ for $k = 1, \ldots, n$, being stable (unstable) for k even (odd).

Using Theorem I we can find some systems exhibiting exotic configurations of limit cycles. As an example we prove the following corollary, where we find an infinite sequence of limit cycles accumulating at the origin.

Corollary 3.2.2. Given $0 < \alpha < (-1+\sqrt{3})/2$, if $\gamma = 1$ and $h(y) = \alpha y^2 \sin(1/y)$ for $y > 0$ with $h(0) = 0$, then for each $k = 1, 2, \cdots$ there exists a limit cycle of system (3.1.1) cutting the y–axis

at the points $(0, 1/(k\pi))$ and $(0, -e^{-\pi}/(k\pi))$ being stable (unstable) for k even (odd). These limit cycles are nested and surround the origin, which is a stable singular point of focus type.

Corollaries 3.2.1 and 3.2.2 are proved in Section 3.3. Note that the function h is bounded and that its upper bound can be taken as small as desired. Thus, we do not need a big perturbation to obtain as much limit cycles as wanted.

The oscillating line used here to define the discontinuity set or switching curve seems to have the same effect for getting several limit cycles than the one achieved in piecewise linear Liénard systems with an oscillating continuous function, see [79] and [81], or with a discontinuous one, see [108]. It is difficult however to establish a relationship between the Liénard systems studied in the quoted papers (whose phase plane is split into many bands with different linear vector fields) and the discontinuous systems considered here with only two linear pieces; the study of the existence of (non-smooth) changes of variables relating these two contexts is beyond the scope of this chapter.

To finish, we emphasize that with a suitable choice of function h one can also get as much semi-stable limit cycles as you want. We call semi-stable limit cycles the isolated periodic orbits that are stable from the interior and unstable from the exterior or vice versa. Thus, we next state our last result.

Corollary 3.2.3. If $0 < \gamma < \sqrt{3/13}$ and

$$h(y) = \frac{2\gamma}{(\gamma^2+1)\pi} \begin{cases} 1 - \cos(\pi y), & 0 \le y \le 2n+1, \\ 2, & y > 2n+1, \end{cases}$$

then system (3.1.1) has exactly n limit cycles for any $n \in \mathbb{N}$. These limit cycles are nested and surround the origin, which is a stable singular point of focus type. The limit cycles cut the y–axis at the points $(0, 2k)$ and $(0, -2k\, e^{-\gamma\pi})$ for $k = 1, \ldots, n$, being all of semi-stable type.

Corollary 3.2.3 can be proved in a very similar way than Corollary 3.2.1; in fact, its proof (to be omitted for sake of brevity) is even easier since the function h is non-negative. In showing the semi-stable character of limit cycles, first two statements of Lemma 3.3.2 should be taken into account, see below.

3.3 Proofs of main results

The proof of Theorem I is made by constructing a displacement function for points of kind $(h(y), y)$ with $y > 0$. Since system (3.1.1) has a focus at the origin in the both sides, we obtain this displacement function by computing the difference between the position of the first return to the section $\{x = 0\,,\, y < 0\}$ in forward time and the position of the first return to the section $\{x = 0\,,\, y < 0\}$ in backward time considering the flow starting in $(h(y), y)$.

Proof of Theorem I. We start by computing

$$\langle \nabla H(h(y),y)\,,\, G^{\pm}(h(y),y)\rangle = (1,-h'(y))^T \begin{pmatrix} \pm 2\gamma h(y) - y \\ \\ (\gamma^2+1)h(y) \end{pmatrix},$$

so that, from Hypotheses $H2'$ and $H3'$, we get

$$\langle \nabla H(h(y),y)\,,\, G^{+}(h(y),y)\rangle = -y + h(y)\,(2\gamma - (1+\gamma^2)h'(y)) < 0, \quad \text{and}$$

$$\langle \nabla H(h(y),y)\,,\, G^{-}(h(y),y)\rangle = -y + h(y)\,(-2\gamma - (1+\gamma^2)h'(y)) < 0.$$

Therefore for $y > 0$ the flow of system (3.1.1) in all points $(h(y), y)$ crosses always Σ from the right to the left, all becoming crossing points, in the usual terminology of Filippov systems, see [64]. In other words, excepting at the origin, the two vector fields have with respect to Σ nontrivial normal components of the same sign. In short, all orbits cross the curve $H(x,y) = 0$ in an anti-clockwise sense with respect to the origin. Note that this property also guarantees that orbits only intersect the discontinuity curve once after completing a turn around the origin.

Let $\varphi^+(t,x,y) = \left(\varphi_1^+(t,x,y), \varphi_2^+(t,x,y)\right)$ be the solution of system (3.1.1) for $H(x,y) > 0$ such that $\varphi^+(0,x,y) = (x,y)$, and let $\varphi^-(t,x,y) = (\varphi_1^-(t,x,y), \varphi_2^-(t,\ x,y))$ be the solution of system (3.1.1) for $H(x,y) < 0$ such that $\varphi^-(0,x,y) = (x,y)$. Since system (3.1.1) is piecewise linear, these solutions can be easily computed as

$$\begin{aligned} \varphi_1^{\pm}(t,x,y) &= e^{\pm\gamma t}\left[(\pm\gamma x - y)\sin t + x\cos t\right], \\ \varphi_2^{\pm}(t,x,y) &= e^{\pm\gamma t}\left[(\gamma^2 x \mp \gamma y + x)\sin t + y\cos t\right]. \end{aligned} \tag{3.3.1}$$

Let $\pi/2 < t_L(y) < 3\pi/2$ be the smallest positive time such that $\varphi_1^-(t_L(y), h(y)\ , y) = 0$ and $\varphi_2^-(t_L(y), h(y), y) < 0$. We note that $t_L(y) > \pi$ if $h(y) > 0$, $t_L(y) = \pi$ if $h(y) = 0$, and $t_L(y) < \pi$ if $h(y) < 0$. Similarly let $-3\pi/2 < t_R(y) < -\pi/2$ be the biggest negative time such that $\varphi_1^+(t_R(y), h(y), y) = 0$ and $\varphi_2^+(t_R(y), h(y), y) < 0$. We note that $t_R(y) > -\pi$ if $h(y) > 0$, $t_R(y) = -\pi$ if $h(y) = 0$, and $t_R(y) < -\pi$ if $h(y) < 0$. From hypothesis $H1'$ we have $|h(y)| < y/\gamma$, so $y + \gamma h(y) > 0$ and $y - \gamma h(y) > 0$, therefore we can easily compute $t_L(y)$ and $t_R(y)$ as

$$\begin{aligned} t_L(y) &= \pi + \arctan\left(\frac{h(y)}{y+\gamma h(y)}\right), \\ t_R(y) &= -\pi + \arctan\left(\frac{h(y)}{y-\gamma h(y)}\right). \end{aligned}$$

Accordingly, we have

$$\cos(t_L(y)) = -\frac{y+\gamma h(y)}{\sqrt{(y+\gamma h(y))^2 + h(y)^2}}, \quad \sin(t_L(y)) = -\frac{h(y)}{\sqrt{(y+\gamma h(y))^2 + h(y)^2}},$$

and

$$\cos(t_R(y)) = -\frac{y-\gamma h(y)}{\sqrt{(y-\gamma h(y))^2+h(y)^2}}, \quad \sin(t_R(y)) = -\frac{h(y)}{\sqrt{(y-\gamma h(y))^2+h(y)^2}}.$$

Finally, substituting in (3.3.1) and after some standard manipulations, we obtain

$$\varphi_2^-(t_L(y),h(y),y) = -e^{-\gamma\pi-\gamma\arctan\left(\frac{h(y)}{y+\gamma h(y)}\right)}\sqrt{(y+\gamma h(y))^2+h(y)^2},$$

and

$$\varphi_2^+(t_R(y),h(y),y) = -e^{-\gamma\pi+\gamma\arctan\left(\frac{h(y)}{y-\gamma h(y)}\right)}\sqrt{(y-\gamma h(y))^2+h(y)^2}.$$

We construct now the displacement function as $f(y) = \varphi_2^-(t_L(y),h(y),y) - \varphi_2^+(t_R(y),h(y),y)$, thus

$$\begin{aligned} f(y) = \quad & e^{-\gamma\pi+\gamma\arctan\left(\frac{h(y)}{y-\gamma h(y)}\right)}\sqrt{(y-\gamma h(y))^2+h(y)^2} \\ & -e^{-\gamma\pi-\gamma\arctan\left(\frac{h(y)}{y+\gamma h(y)}\right)}\sqrt{(y+\gamma h(y))^2+h(y)^2}. \end{aligned}$$

If $y^* > 0$ is such that $h(y^*) = 0$ it is easy to see that $f(y^*) = 0$. Therefore there exists a periodic solution passing through $(h(y), y)$. The following auxiliary results, where we prove a little bit more than needed for Theorem I, can be easily shown under the previous hypotheses.

Lemma 3.3.1. Taking $y > 0$, if we have $h(y) > 0$ ($h(y) < 0$) then $f(y) > 0$ ($f(y) < 0$).

Proof. We show first that if $h(y) > 0$ then $f(y) > 0$. To see this, we consider for a fixed $y > 0$ the function

$$\begin{aligned} F_y(x) = \quad & e^{\gamma\arctan\left(\frac{x}{y-\gamma x}\right)}\sqrt{(y-\gamma x)^2+x^2} \\ & -e^{-\gamma\arctan\left(\frac{x}{y+\gamma x}\right)}\sqrt{(y+\gamma x)^2+x^2}. \end{aligned}$$

Clearly $f(y) = e^{-\gamma\pi}F_y(h(y))$. We note that $F_y(0) = 0$ and

$$F_y(x) = \delta_y(x) - \delta_y(-x),$$

where

$$\delta_y(x) = e^{\gamma\arctan\left(\frac{x}{y-\gamma x}\right)}\sqrt{(y-\gamma x)^2+x^2}.$$

Since we are dealing with a difference of two positive terms, for determining its sign we can work with the difference of squares, avoiding so to deal with square roots. Now the derivative of $\delta_y(x)^2 - \delta_y(-x)^2$ with respect to x simplifies to

$$2x\left(1+\gamma^2\right)e^{2\gamma\arctan\left(\frac{x}{y-\gamma x}\right)} - 2x\left(1+\gamma^2\right)e^{-2\gamma\arctan\left(\frac{x}{y+\gamma x}\right)},$$

which is obviously positive for all $0 < x < y/\gamma$. Then $F_y(x)$ is monotone increasing for the same range, and we can assure that $f(y) = e^{-\gamma\pi}F_y(h(y))$ is positive.

Since $F_y(-x) = -F_y(x)$, the case $h(y) < 0$ is a direct consequence of the above reasoning and the lemma follows. □

Lemma 3.3.2. Assume $y^* > 0$ such that $h(y^*) = 0$. The following statements hold.

(*i*) If there exists $\varepsilon > 0$ such that $h(y) < 0$ ($h(y) > 0$) for $y^* - \varepsilon < y < y^*$ then the periodic orbit passing for $(0, y^*)$ is stable (unstable) from the interior.

(*ii*) If there exists $\varepsilon > 0$ such that $h(y) > 0$ ($h(y) < 0$) for $y^* < y < y^* + \varepsilon$ then the periodic orbit passing for $(0, y^*)$ is stable (unstable) from the exterior.

(*iii*) If $h'(y^*) > 0$ ($h'(y^*) < 0$) then there exist a periodic solution passing for $(0, y^*)$ which is a stable (unstable) limit cycle.

Proof. The three statements follow from the standard properties of displacement function and Lemma 3.3.1.

In the case of statement (iii), an alternative proof can be obtained by direct computations of derivatives of the displacement function f at y^*. The expressions are quite long but simplify a lot after substituting $h(y^*) = 0$. One obtains $f'(y^*) = 0$, so that the limit cycles are non-hyperbolic and we need to resort to successive derivatives, getting $f''(y^*) = 0$ and

$$f'''(y^*) = \frac{8\gamma(1+\gamma^2)e^{-\gamma\pi}h'(y^*)^3}{y^2}.$$

We conclude again that the periodic solution is a stable limit cycle if $h'(y^*) > 0$, and an unstable limit cycle if $h'(y^*) < 0$. □

From Lemma 3.3.2, Theorem I is shown. □

It follows the proofs of corollaries.

Proof of Corollary 3.2.1. It is easy to see that

$$|h'(y)| \leq \frac{2\gamma}{\gamma^2+1},$$

and to fulfill Hypothesis $H1'$ we should need

$$\frac{2\gamma}{\gamma^2+1} < \frac{1}{\gamma},$$

which is true for all $0 < \gamma < 1$.

Furthermore, for $y \leq (2n+1)/2$ we have

$$h(y)(2\gamma \pm (1+\gamma^2)h'(y)) = \frac{4\gamma^2}{(\gamma^2+1)\pi}\sin(\pi y)(1 \pm \cos(\pi y)).$$

We note that for $0 < \gamma < \sqrt{3/5}$ the inequality $8\gamma^2/(\pi + \pi\gamma^2) < 1$ holds. Again using that $|\sin(y)| < y$ for $y > 0$ we obtain

$$h(y)(2\gamma - (1+\gamma^2)h'(y)) < \frac{8\gamma^2}{(1+\gamma^2)\pi}y < y, \quad \text{and}$$

$$h(y)(2\gamma + (1+\gamma^2)h'(y)) > -\frac{8\gamma^2}{(1+\gamma^2)\pi}y > -y.$$

So the Hypotheses $H2'$ and $H3'$ hold for $y \leq (2n+1)/2$.

Now for $y > (2n+1)/2 \geq 3/2$ we have $h'(y) = 0$, so that

$$h(y)(2\gamma \pm (1+\gamma^2)h'(y)) = \frac{4\gamma^2}{(\gamma^2+1)\pi}(-1)^n,$$

and the inequalities in Hypotheses $H2'$ and $H3'$ trivially hold.

Computing the zeros of the function h and applying Theorem I we conclude that system (3.1.1) has exactly n limit cycles for any given $n \in \mathbb{N}$ cuting the y-axis at the points $(0, k)$ and $(0, -k\, e^{-\gamma\pi})$ for $k = 1, \ldots, n$. □

Proof of Corollary 3.2.2. We note first that the hypothesis $0 < \alpha < (-1+\sqrt{3})/2$ implies that $2\alpha(1+\alpha) < 1$. Thus, we also have $\alpha < 1$, so that using the inequality $|\sin(1/y)| < 1/y$ for $y > 0$, it is easy to see that $|h(y)| < \alpha y < y$, and Hypothesis $H1'$ holds.

Since $\gamma = 1$, we have

$$h(y)(2\gamma \pm (1+\gamma^2)h'(y)) = 2h(y)(1 \pm h'(y)),$$

and then

$$2h(y)(1 \pm h'(y)) = 2\alpha y^2 \sin\left(\frac{1}{y}\right) \mp 2\alpha^2 y^2 \sin\left(\frac{1}{y}\right)\cos\left(\frac{1}{y}\right) \pm 4\alpha^2 y^3 \sin^2\left(\frac{1}{y}\right).$$

So using again the inequality $|\sin(1/y)| < 1/y$ for $y > 0$ we obtain

$$2h(y)(1 - h'(y)) < 2\alpha y + 2\alpha^2 y = 2\alpha(1+\alpha)y < y, \quad \text{and}$$

$$2h(y)(1 + h'(y)) > -2\alpha y - 2\alpha^2 y = -2\alpha(1+\alpha)y > -y.$$

Hence the inequalities in Hypotheses $H2'$ and $H3'$ hold for $y > 0$.

Computing the zeros of the function h and applying Theorem I, we conclude the proof of the corollary. □

Chapter 4

Maximum number of limit cycles for certain piecewise linear dynamical systems

The main results of this chapter (Theorems J, K and L) are based on the paper [76].

4.1 Introduction to the Lum–Chua's problem

Non–smooth dynamical systems emerge in a natural way modelling many real processes and phenomena, for instance, recently piecewise linear differential equations appeared as idealized models of cell activity, see [26, 107, 109]. Due to that, in these last years, the mathematical community became very interested in understanding the dynamics of these kind of system. In general, some of the main source of motivation to study non–smooth systems can be found in control theory [5], impact and friction mechanics [10, 13, 63], nonlinear oscillations [2, 88], economics [44, 54], and biology [6, 62]. See for more details the book [29] and the references therein. In this chapter we are interested in discontinuous piecewise linear differential systems. The study of this particular class of non–smooth dynamical systems has started with Andronov and coworkers [2].

We start with a historical fact. Lum and Chua [96] in 1990 conjectured that a continuous piecewise linear vector field in the plane with two zones separated by a straight line, which is the easiest example of this kind of system, has at most one limit cycle. This conjecture was proved by Freire et al [35] in 1998. Even this relatively easy case demanded a hard work to show the existence of at most one limit cycle.

In this chapter we address the problem of Lum and Chua extended to the class of discontinuous piecewise linear differential systems in the plane with two zones separated by a straight line. Here we deal with non–sliding limit cycle, which is a limit cycle that does not contain any sliding segment in Σ. This problem is very related to the Hilbert's 16th problem [52].

Limit cycles of discontinuous piecewise linear differential systems with two zones separated by a straight line have been studied recently by several authors, see among others [3, 22, 11, 39, 43, 48, 49, 50, 65, 75, 80, 83]. Nevertheless the problem of Lum and Chua remains open for this class of

differential equations. In this work we give a partial solution for this problem. We note that in [31] the authors proved that if one of the two linear systems has its singular point on the discontinuity straight line then the number of limit cycles of such a system is at most 4. Our results reduce this upper bound to 2 and, additionally, we prove that it is reached.

Our point of interest in the Lum and Chua problem is aligned with two directions which face serious technical difficulties. First, while solutions in each linear regions are easy to find, the times of passage along the regions are not simple to achieve. It means that matching solutions across regions is a very difficult task. Second, to control all possible configurations one must generally consider a large number of parameters.

It was conjectured in [43] that a planar piecewise linear differential systems with two zones separated by a straight line has at most 2 non–sliding limit cycles. A negative answer for this conjecture was provided in [48] via a numerical example having 3 non–sliding limit cycles. Analytical proofs for the existence of these 3 limit cycles were given in [80, 37]. Finaly in [38] it was studied general conditions to obtain 3 non–sliding limit cycles in planar piecewise linear differential systems with two zones separated by a straight line. Recently, perturbative techniques (see [77, 75]) were used together with newly developed tools on Chebyshev systems (see [93]) to obtain 3 limit cycles in such systems when they are near to non–smooth centers.

When a general curve of discontinuity is considered instead of a straight line, there is no upper bound for the maximum number of non–sliding limit cycles that a system of this family can have. It is a consequence of a conjecture stated by Braga and Mello in [12] and then proved by Novaes and Ponce in [91].

4.2 Bounds for the maximum number of limit cycles

In this section we deal with planar vector fields Z expressed as $\dot{z} = F(z) + \text{sign}(x)G(z)$, where $z = (x, y) \in \mathbb{R}^2$, and F and G are linear vector fields in $\mathbb{R}^2$ or, equivalently,

$$\dot{z} = \begin{cases} X(z) & \text{if} \quad x > 0, \\ \\ Y(z) & \text{if} \quad x < 0, \end{cases} \tag{4.2.1}$$

where $X(z) = F(z) + G(z)$ and $Y(z) = F(z) - G(z)$. The line $\Sigma = \{x = 0\}$ is called discontinuity set. Our main goal is to study the maximum number of non–sliding limit cycles that the discontinuous piecewise linear differential system (4.2.1) can have.

The systems $\dot{z} = X(z)$ and $\dot{z} = Y(z)$ are called *lateral linear differential systems* (or just *lateral systems*) and more specifically *right system* and *left system*, respectively.

A linear differential system is called *degenerate* if its determinant is zero, otherwise it is called *non–degenerate*. From now on in this chapter we only consider non–degenerate linear differential systems.

System (4.2.1) can be classified according to the singularities of the lateral linear differential systems. A non–degenerate linear differential system can have the following singularities: *saddle*

(S), *node* (N), *focus* (F), and *center* (C). Among the above classes of singularities we shall also distinguish the following ones: a *weak saddle*, i.e. a saddle such that the sum of its eigenvalues is zero (S^0); a diagonalizable node with distinct eigenvalues (N); *star node*, i.e. a diagonalizable node with equal eigenvalues (N^*); and an *improper node*, i.e. a non–diagonalizable node (iN). We say that the discontinuous differential system (4.2.1) is an *LR–system* with $L, R \in \{S, S^0, N, N^*, iN, F, C\}$, when the left system has a singularity of type L and the right system has a singularity of type R.

We define subclasses of LR–systems according to the position of the singularity of each lateral system. The right system can have a *virtual singularity* (R_v), i.e. a singularity $p = (p_x, p_y)$ with $p_x < 0$; a *boundary singularity* (R_b), i.e. a singularity $p = (p_x, p_y)$ with $p_x = 0$; or a *real singularity* (R_r) i.e. a singularity $p = (p_x, p_y)$ with $p_x > 0$. Accordingly the left system can have a *virtual singularity* (L_v), i.e. a singularity $p = (p_x, p_y)$ with $p_x > 0$; a *boundary singularity* (L_b), i.e. a singularity $p = (p_x, p_y)$ with $p_x = 0$; or a *real singularity* (L_r) i.e. a singularity $p = (p_x, p_y)$ with $p_x < 0$.

We denote by $\mathcal{N}(L, R)$ the maximum number of non–sliding limit cycles that an LR–system can have. Clearly $\mathcal{N}(L, R) = \mathcal{N}(R, L)$.

In this chapter we compute the exact value of $\mathcal{N}(L, R)$ always when one of the lateral systems is a saddle of kind S_v, S_b, S^0, a node of kind N_r, N_b, N^*, iN_r, iN_b, a focus of kind F_b, and a center C. Particularly we obtain that $\mathcal{N}(L, R) \leq 2$ in all the above cases.

It is easy to see that if one of the lateral linear differential systems is of type S_v, S_b, N_r, N_b, N^*, iN_r, or iN_b, then the first return map on the straight line $x = 0$ of system (4.2.1) is not defined. Consequently system (4.2.1) does not admit non–sliding limit cycles in all these cases. So $\mathcal{N}(R, L) = 0$ for the systems having one of these kind of equilibria.

It remains to study the cases when one of the lateral system is F_b, C or S_r^0. For these cases we shall prove the following theorems.

Theorem J. All numbers $\mathcal{N}(F_b, F_v)$, $\mathcal{N}(F_b, F_r)$, $\mathcal{N}(F_b, N_v)$, $\mathcal{N}(F_b, iN_v)$ and $\mathcal{N}(F_b, S_r)$ are equal to 2, and all numbers $\mathcal{N}(F_b, F_b)$, $\mathcal{N}(F_b, C)$ and $\mathcal{N}(F_b, S_r^0)$ are equal to 1.

Theorem K. The equality $\mathcal{N}(S_r^0, F_r) = 2$ holds, all numbers $\mathcal{N}(S_r^0, F_v)$, $\mathcal{N}(S_r^0, F_b)$, $\mathcal{N}(S_r^0, N_v)$, $\mathcal{N}(S_r^0, iN_v)$ and $\mathcal{N}(S_r^0, S_r)$ are equal to 1, and all numbers $\mathcal{N}(S_r^0, C)$ and $\mathcal{N}(S_r^0, S_r^0)$ are equal to 0.

We shall see that the next result can be obtained as an immediately corollary of the proofs of Theorems J and K.

Corollary 4.2.1. The equality $\mathcal{N}(C_b, F_r) = 2$ holds, all numbers $\mathcal{N}(C_b, F_v)$, $\mathcal{N}(C_b, F_b)$, $\mathcal{N}(C_b, N_v)$, $\mathcal{N}(C_b, iN_v)$ and $\mathcal{N}(C_b, S_r)$ are equal to 1, and all numbers $\mathcal{N}(C_b, C)$ and $\mathcal{N}(C_b, S_r^0)$ are equal to 0.

The equalities of Corollary 4.2.1 can be extended for all linear centers.

Theorem L. The equality $\mathcal{N}(C, F_r) = 2$ holds, all numbers $\mathcal{N}(C, F_v)$, $\mathcal{N}(C, F_b)$, $\mathcal{N}(C, N_v)$, $\mathcal{N}(C, iN_v)$ and $\mathcal{N}(C, S_r)$ are equal to 1, and all numbers $\mathcal{N}(C, C)$ and $\mathcal{N}(C, S_r^0)$ are equal to 0.

Theorems J, K, and L, and Corollary 4.2.1 are proved in Section 4.4.

Our results give sufficient conditions in order to guarantee that system (4.2.1) has at most 2, 1, or 0 limit cycles. We study the non–degenerate cases for which the expression of the time that

a trajectory starting in $p \in \Sigma$ remains in the region $x > 0$ (or $x < 0$) is known. The remaining cases are those ones whose this associated time is not explicitly determined for both regions.

The systems studied in [48, 80, 77, 37, 38], possessing 3 limit cycles, have in one side a real focus, and in the other side either a real focus or a linear system with trace distinct from zero. Thus they do not satisfy the conditions of our theorems.

4.3 Preliminary results

A linear change of variables in the plane preserving the vertical lines will be called a *vertical lines–preserving linear change of variables.*

Proposition 4.3.1. Let $M = (m_{ij})_{i,j}$ be a 2×2 matrix. If the linear differential system

$$(\dot{x}, \dot{y})^T = M(x, y)^T \tag{4.3.1}$$

is a

(a) S–system then after a vertical lines–preserving linear change of variables and a time–rescaling system (4.3.1) becomes $(\dot{x}, \dot{y})^T = M_1(x, y)^T$;

(b) N–system then after a vertical lines–preserving linear change of variables and a time–rescaling system (4.3.1) becomes $(\dot{x}, \dot{y})^T = M_2(x, y)^T$;

(c) F–system (C–system) then after a vertical lines–preserving linear change of variables and a time–rescaling system (4.3.1) becomes $(\dot{x}, \dot{y})^T = M_3(x, y)^T$ with $a \neq 0$ ($a = 0$);

(d) iN–system then after a vertical lines–preserving linear change of variables and a time–rescaling system (4.3.1) becomes $(\dot{x}, \dot{y})^T = M_4(x, y)^T$,

where

$$M_1 = \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix} \quad \text{with} \quad |a| < 1; \qquad M_2 = \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix} \quad \text{with} \quad |a| > 1;$$

$$M_3 = \begin{pmatrix} a & 1 \\ -1 & a \end{pmatrix} \quad \text{with} \quad a \in \mathbb{R}; \quad \text{and} \quad M_4 = \begin{pmatrix} \lambda & \lambda \\ 0 & \lambda \end{pmatrix} \quad \text{with} \quad \lambda = \pm 1.$$

Proof of Proposition 4.3.1. Let $S = (s_{ij})_{i,j}$ be a 2×2 matrix. The change of variables $(u, v)^T = S(x, y)^T$ is a vertical lines–preserving linear change of variables if and only if $s_{12} = 0$ and $s_{11} = 1$. Indeed, $S(x, y) = (s_{11}x + s_{12}y, s_{21}x + s_{22}y)$ and $s_{11}x + s_{12}y = x$ for every $x \in \mathbb{R}$ if and only if $s_{11} = 1$ and $s_{12} = 0$. So in what follows we fix $s_{12} = 0$ and $s_{11} = 1$.

Claim 4.3.1. The statement (a) holds.

Since we are assuming that we have a saddle at the origin and in the expression of its eigenvalues appears $\sqrt{4m_{12}m_{21}+(m_{11}-m_{22})^2}$, we must assume that $4m_{12}m_{21}+(m_{11}-m_{22})^2>0$. Taking

$$s_{21}=\frac{m_{11}-m_{22}}{\sqrt{4m_{12}m_{21}+(m_{11}-m_{22})^2}},\quad \text{and}\quad s_{22}=\frac{2m_{12}}{\sqrt{4m_{12}m_{21}+(m_{11}-m_{22})^2}},$$

it follows that

$$SMS^{-1}=\frac{1}{2}\begin{pmatrix} m_{11}+m_{22} & \sqrt{4m_{12}m_{21}+(m_{11}-m_{22})^2} \\ \sqrt{4m_{12}m_{21}+(m_{11}-m_{22})^2} & m_{11}+m_{22} \end{pmatrix}.$$

Then we can rescale the time by

$$\tau=\frac{1}{2}\sqrt{4m_{12}m_{21}+(m_{11}-m_{22})^2}\,t.$$

Denoting $a=(m_{11}+m_{22})/\sqrt{4m_{12}m_{21}+(m_{11}-m_{22})^2}$ system (4.3.1) becomes

$$\begin{pmatrix} x' \\ y' \end{pmatrix}=\begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

where now the prime denotes the derivative with respect to the new time variable τ. Computing the eigenvalues of the above system $\{-1+a,1+a\}$ we conclude that $|a|<1$, because this system is a saddle, i.e. the eigenvalues have different sign. Therefore we have proved statement (a).

Claim 4.3.2. The statement (b) holds.

The proof of statement (b) follows similarly to the proof of statement (a). Nevertheless we conclude that $|a|>1$, because in this case the system is a diagonalizable node, i.e. the eigenvalue have the same sign. Thus we have proved statement (b).

Claim 4.3.3. The statement (c) holds.

Taking

$$s_{21}=\frac{m_{11}-m_{22}}{\sqrt{-4m_{12}m_{21}-(m_{11}-m_{22})^2}},\quad \text{and}\quad s_{22}=\frac{2m_{12}}{\sqrt{-4m_{12}m_{21}-(m_{11}-m_{22})^2}},$$

it follows that

$$SMS^{-1}=\frac{1}{2}\begin{pmatrix} m_{11}+m_{22} & \sqrt{-4m_{12}m_{21}-(m_{11}-m_{22})^2} \\ -\sqrt{-4m_{12}m_{21}-(m_{11}-m_{22})^2} & m_{11}+m_{22} \end{pmatrix}.$$

From hypotheses this system is a focus thus $-4m_{12}m_{21}-(m_{11}-m_{22})^2>0$. So we can rescale the time by $\tau=\dfrac{1}{2}\sqrt{-4m_{12}m_{21}-(m_{11}-m_{22})^2}\,t$. Denoting $a=(m_{11}+m_{22})/\sqrt{-4m_{12}m_{21}-(m_{11}-m_{22})^2}$ system (4.3.1) becomes

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & 1 \\ -1 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

where now the prime denotes the derivative with respect to the new time variable τ. Computing the eigenvalues of the above system $\{-i+a, i+a\}$ we conclude that when $a\neq 0$ this system has a focus and a center when $a=0$. Hence statement (c) is proved.

Claim 4.3.4. The statement (d) holds.

One of the entries m_{12} or m_{21} are distinct of zero. Indeed, suppose that $m_{12}=0$, so $\{m_{11}, m_{22}\}$ are the eigenvalues of the matrix M. Since system (4.3.1) is a non–diagonalizable node we have that $m_{11}=m_{22}$ which implies that $m_{21}\neq 0$, in other way the matrix M would be diagonalizable. On the other hand, supposing that $m_{21}=0$ we obtain $m_{12}\neq 0$. From here we assume, without loss of generality, that $m_{12}\neq 0$.

We also have that $m_{11}+m_{22}\neq 0$, we prove this by reduction to the absurd. Suppose that $m_{11}+m_{22}=0$, then $\pm\sqrt{m_{11}^2+m_{12}m_{21}}$ are the eigenvalues of the matrix M. Since system (4.3.1) is a non–diagonalizable node we have that the matrix M has only one eigenvalue with multiplicity 2. This implies that the eigenvalues are zero, which is a contradiction with the fact that we are working with non–degenerate linear differential systems. In short we have proved that $m_{11}+m_{22}\neq 0$.

From the expression of the eigenvalues it is also easy to see that $4m_{12}m_{21}+(m_{11}-m_{22})^2=0$.

Taking

$$s_{21}=\frac{m_{11}-m_{22}}{2m_{12}} \quad \text{and} \quad s_{22}=\frac{2m_{12}}{m_{11}+m_{22}},$$

it follows that

$$SMS^{-1}=\frac{1}{2}\begin{pmatrix} m_{11}+m_{22} & m_{11}+m_{22} \\ 0 & m_{11}+m_{22} \end{pmatrix}.$$

So we can rescale the time by $\tau=\dfrac{1}{2}|m_{11}+m_{22}|\,t$ system (4.3.1) becomes

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \lambda & \lambda \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

where $\lambda=\pm 1$, and now the prime denotes the derivative with respect to the new time variable τ. This completes the proof of statement (c). $\square$

A limit cycle of our piecewise linear differential system (4.2.1) expends a time t_R in the region $x > 0$ and a time t_L in the region $x < 0$. As we shall see later on we know explicitly the time t_L, and we do not know explicitly the time t_R. The next lemma will help us to work with one of the intersection points of the limit cycle with the discontinuity straight line instead of the unknown time t_R.

Lemma 4.3.1. We consider the functions

$$\begin{aligned} &F(t) = e^{-at}\csc(t) - \cot(t), \quad G(t) = e^{-at}\mathrm{csch}(t) - \coth(t), \\ &H(t) = \frac{e^{-t}-1}{t}. \end{aligned} \tag{4.3.2}$$

The following statements hold.

(a) For every $a \in \mathbb{R}$, $F(t)$ is a monotonic increasing function in the interval $(-\pi, \pi)$ such that $F(t) < -a$ for $t \in (-\pi, 0)$, and $F(t) > -a$ for $t \in (0, \pi)$.

(a') For every $a > 0$ (resp. $a < 0$), $F(t)$ is a monotonic increasing function in the interval $(\pi, 2\pi)$ (resp. $(-2\pi, -\pi)$).

(b) For $|a| > 1$, $G(t)$ is a monotonic increasing function on $\mathbb{R}$ such that $G(t) > -a$ for $t > 0$; and for $|a| < 1$ $G(t)$ is a monotonic decreasing function on $\mathbb{R}$ such that $G(t) > -a$ for $t > 0$.

(c) $H(t)$ is a monotonic increasing function on $\mathbb{R}$ such that $H(t) \lesseqgtr -1$ for $t \lesseqgtr 0$.

Proof. To prove statement (a) we compute

$$F'(t) = \csc^2(t)\left(1 - e^{-at}(\cos(t) + a\sin(t))\right) = \csc^2(t)p(t),$$

where $p(t) = 1 - e^{-at}(\cos(t) + a\sin(t))$ and $p'(t) = (1+a^2)e^{-at}\sin(t)$. Clearly, $p'(t) > 0$ when $0 < t < \pi$, and $p'(t) < 0$ when $-\pi < t < 0$. So $p(t)$ is a decreasing function in the interval $(-\pi, 0)$ and it is an increasing function in the interval $(0, \pi)$. Since $p(0) = 0$ we conclude that $p(t) > 0$ for $t \in (-\pi, \pi) \setminus \{0\}$. Finally, $F'(0) = (1+a^2)/2 > 0$ so $F'(t) > 0$ for every $t \in (-\pi, \pi)$, which implies that F is monotonic increasing for $t \in (-\pi, \pi)$. The proof of statement (a) follows by noting that $\lim_{t\to 0} F(t) = -a$. The proof of statement (a') is completely analogous to the proof of statement (a).

To prove statement (b) we compute

$$\begin{aligned} G'(t) = &\ \mathrm{csch}(t)\left(\mathrm{csch}(t) - e^{-at}(a + \coth(t))\right) \\ = &\ \frac{e^{-at}\mathrm{csch}(t)}{(e^t-1)(e^t+1)}\left(a - 1 - e^{2t} - ae^{2t} + 2e^{t+at}\right) \\ = &\ \frac{e^{-at}\mathrm{csch}(t)}{(e^t-1)(e^t+1)}q(t), \end{aligned}$$

where $q(t) = a-1-e^{2t}-ae^{2t}+2e^{t+at}$ and $q'(t) = -2(1+a)e^t\,(e^t - e^{at})$. When $|a|> 1$, $q'(t) \gtrless 0$ for $t \gtrless 0$, because $e^t - e^{at} \gtrless 0$ for $t \lessgtr 0$ (resp. $t \gtrless 0$) when $a > 1$ (resp. $a < -1$). Since $q(0) = 0$ we conclude that $q(t) > 0$, consequently $G'(t) > 0$, for every $t \neq 0$. It implies, for $|a|> 1$, that G is a monotonic increasing function on $\mathbb{R}$ such that $G(t) \gtrless -a$ for $t \gtrless 0$, because $\lim_{t\to 0} G(t) = -a$. On the other hand, when $|a|< 1$, $q'(t) \gtrless 0$ for $t \lessgtr 0$, because in this case $e^t - e^{at} \gtrless 0$ for $t \gtrless 0$. Hence, for $|a|< 1$, we conclude that G is a monotonic decreasing function on $\mathbb{R}$ such that $G(t) < -a$ for every $t > 0$. It concludes the proof of statement (b).

To prove statement (c) we compute

$$H'(t) = \frac{e^{-t}}{t^2}\,(e^t - t - 1) = \frac{e^{-t}}{t^2} r(t),$$

where $r(t) = e^t - t - 1$ and $r'(t) = e^t - 1$. Since $r(0) = 0$ and $r'(t) \lessgtr 0$ for $t \lessgtr 0$ we conclude that $r(t) > 0$, consequently $H'(t) > 0$, for $t \neq 0$. It implies that H is an monotonic increasing function for $t > 0$. The proof of statement (c) follows by noting that $\lim_{t\to 0} H(t) = -1$. $\square$

Now consider the functions

$$\xi_1(t) = 1,$$

$$\xi_2^1(t) = \cot(t) - e^{at}\csc(t), \quad \xi_2^2(t) = \coth(t) - e^{at}\mathrm{csch}(t), \quad \xi_2^3 = \frac{1-e^t}{t}$$

$$\xi_3^1(t) = \cot(t) - e^{-at}\csc(t), \quad \xi_3^2(t) = \coth(t) - e^{-at}\mathrm{csch}(t), \quad \xi_3^3 = \frac{1-e^{-t}}{t}$$

$$\xi_2^4(t) = \frac{\xi_3^1(t) - \xi_2^1(t)}{2} = \csc(t)\sinh(at),$$

$$\xi_2^5(t) = \frac{\xi_3^2(t) - \xi_2^2(t)}{2} = \mathrm{csch}(t)\sinh(at),$$

$$\xi_2^6(t) = \frac{\xi_3^3(t) - \xi_2^3(t)}{2} = \frac{\sinh(t)}{t}.$$

We define the ordered sets of functions $\mathcal{F}^i = (\xi_1, \xi_2^i, \xi_3^i)$ and $\widetilde{\mathcal{F}}^i = (\xi_1, \xi_3^i, \xi_2^i)$ for $i = 1, 2, 3$, and $\mathcal{F}^i = (\xi_1, \xi_2^i)$ for $i = 4, 5, 6$.

The next two technical lemmas together with Definition B.0.1 and Propositions B.0.3 and B.0.4 will be used later on in the proofs of Theorems 1, 2 and 4 to establish sharp upper bounds for the maximum numbers of non–sliding limit cycles that system (4.2.1) can have.

Lemma 4.3.2. The following statements hold.

(a) The set of functions $\mathcal{F}^1$ is an ECT–system on the intervals $(0, \pi)$ and $(-\pi, 0)$ for every $a \neq 0$.

(a') The set of functions $\widetilde{\mathcal{F}}^1$ is an ECT–system on the interval $(\pi, 2\pi)$ (resp. $(-2\pi, -\pi)$) for every $a > 0$ (resp. $a < 0$).

(b) The set of functions $\mathcal{F}^2$ is an ECT–system on $\mathbb{R}^+$ for every $a \notin \{0, \pm 1\}$.

(c) The set of functions $\mathcal{F}^3$ is an ECT–system on $\mathbb{R}^+$.

(d) The set of functions $\mathcal{F}^4$ is an ECT–system on the intervals $(0, \pi)$ and $(-\pi, 0)$ for every $a \neq 0$.

(d') The set of functions $\mathcal{F}^4$ defined on the intervals $(\pi, 2\pi)$ (or $(-2\pi, -\pi)$) satisfies $Z(\mathcal{F}^4) = 2$ for every $a \neq 0$.

(e) The set of functions $\mathcal{F}^5$ is an ECT–system on $\mathbb{R}^+$ for every $a \notin \{0, \pm 1\}$.

(f) The set of functions $\mathcal{F}^6$ is an ECT–system on $\mathbb{R}^+$.

Proof. To prove the statements (a)–(f) we compute the Wronskians $W_1(t) = W(\xi_1)(t)$, $W_2^i(t) = W(\xi_1, \xi_2^i)(t)$ for $i = 1, 2, \ldots, 6$, and $W_3^i(t) = W(\xi_1, \xi_2^i, \xi_3^i)(t)$ for $i = 1, 2, 3$.

$$W_1(t) = 1,$$

$$W_2^1(t) = \csc(t)\left(e^{at}(\cot(t) - a) - \csc(t)\right),$$

$$W_3^1(t) = 2\left(1 + a^2\right)\csc^2(t)\left(a - \csc(t)\sinh(at)\right),$$

$$W_2^2(t) = \operatorname{csch}(t)\left(e^{at}(\coth(t) - a) - \operatorname{csch}(t)\right),$$

$$W_3^2(t) = 2\left(1 - a^2\right)\operatorname{csch}^2(t)\left(\operatorname{csch}(t)\sinh(at) - a\right),$$

$$W_2^3 = \frac{e^t(1-t) - 1}{t^2},$$

$$W_3^3 = \frac{2\left(t - \sinh(t)\right)}{t^3},$$

$$W_2^4(t) = \csc(t)\left(a\cosh(at) - \cot(t)\sinh(at)\right),$$

$$W_2^5(t) = \operatorname{csch}(t)\left(a\cosh(at) - \coth(t)\sinh(at)\right),$$

$$W_2^6(t) = \frac{t\cosh(t) - \sinh(t)}{t^2}.$$

From here, it is easy to see that for each $a \neq 0$ the Wronskians W_2^1, W_3^1 and W_2^4 do not vanish at any point of the intervals $(0, \pi)$ and $(-\pi, 0)$; for each $a \notin \{0, \pm 1\}$ the Wronskians W_2^2, W_3^2 and W_2^5 do not vanish at any point of $\mathbb{R}^+$; and the Wronskians W_2^3, W_3^3 and W_2^6 do not vanish at any point of $\mathbb{R}^+$. So statements $(a) - (f)$ are proved.

To see statement (a') we compute the Wronskians

$$\widetilde{W}_2^1(t) = W(\xi_1, \xi_3^1)(t) = \csc(t)\left(e^{-at}(\cot(t) - a) - \csc(t)\right),$$

$$\widetilde{W}_3^1(t) = W(\xi_1, \xi_3^1, \xi_3^1)(t) = -W_3^1(t).$$

Again it is easy to see that for each $a > 0$ (resp. $a < 0$) the Wronskian $\widetilde{W}_2^1$ does not vanish at any point of the interval $(\pi, 2\pi)$ and (resp. $(-2\pi, -\pi)$).

Finally, statement (d') follows by showing that the Wronskian $W_2^4(t)$ has exactly one zero in each one of the intervals $(\pi, 2\pi)$ and $(-2\pi, -\pi)$. Indeed

$$W_2^4(t) = \csc(t)\cosh(at)(a - \cot(t)\tanh(at)) = \csc(t)\mathrm{csch}(at)P_a(t).$$

Since $\csc(t)\cosh(at)$ is nonvanishing for every $a \in \mathbb{R}$, it is sufficient to study the zeros of $P_a(t)$ in order to study the zeros of $W_2^4(t)$. For $a > 0$

$$\lim_{t\uparrow 2\pi} P_a(t) = -\lim_{t\downarrow \pi} P_a(t) = \infty \quad \text{and} \quad \lim_{t\downarrow -2\pi} P_a(t) = -\lim_{t\uparrow -\pi} P_a(t) = \infty,$$

and for $a < 0$

$$\lim_{t\downarrow \pi} P_a(t) = -\lim_{t\uparrow 2\pi} P_a(t) = \infty \quad \text{and} \quad \lim_{t\uparrow -\pi} P_a(t) = -\lim_{t\downarrow -2\pi} P_a(t) = \infty.$$

So, for $a \neq 0$, there exist $\bar{t}_a \in (\pi, 2\pi)$ and $\underline{t}_a \in (-2\pi, -\pi)$ such that $P_a(\bar{t}_a) = P_a(\underline{t}_a) = 0$. Indeed function $P_a(t)$ is continuous on the intervals $(\pi, 2\pi)$ and $(-2\pi, -\pi)$. Computing $P_a'(t) = \csc^2(t)\tanh(at) - a\cot(t)\mathrm{sech}^2(at)$ we see that $P_a'(t) \neq 0$ for every $a \neq 0$ and $t \in (\pi, 2\pi) \cup (-2\pi, -\pi)$, which implies that $P_a(t)$ has at most one zero in each one of these intervals. This proof ends by applying Proposition B.0.4 for $n = \ell = 1$.

□

Lemma 4.3.2 was stated assuming $a \neq 0$. For $a = 0$ we define the sets of functions $\mathcal{G}^i = \{\xi_1, \xi_2^i\}$ for $i = 1, 2$ and we prove the next lemma.

Lemma 4.3.3. Then following statements hold.

- (a) The set of functions $\mathcal{G}^1$ is an ECT–system on the intervals $(0, \pi)$, $(-\pi, 0)$, $(\pi, 2\pi)$, and $(-2\pi, -\pi)$.
- (b) The set of functions $\mathcal{G}^2$ is an ECT–system on $\mathbb{R}^+$.

Proof. Assuming $a = 0$ and proceeding analogously to the proof of Lemma 4.3.2 we compute the Wronskians.

$$W_1(t) = 1,$$

$$W_2^1(t) = \csc(t)\cot(t) - \csc^2(t),$$

$$W_2^2(t) = \mathrm{csch}(t)\coth(t) - \mathrm{csch}^2(t).$$

From here, it is easy to see that the Wronskian W_2^1 does not vanish at any point of the interval $(0,\pi)$, $(-\pi,0)$, $(\pi,2\pi)$, and $(-2\pi,-\pi)$, and that the Wronskian W_2^2 does not vanish at any point of $\mathbb{R}^+$. $\square$

4.4 Proofs of main results

The proofs of Theorem J and Corollary 4.2.1 will be immediate consequences of Propositions 4.4.1–4.4.6; the proof of Theorem K will be an immediate consequence of Propositions 4.4.6–4.4.11; and the proof of Theorem L will be an immediate consequence of Propositions 4.4.12–4.4.15 and Corollary 4.2.1.

We note that some of the partial results contained in this section could be obtained using different approaches. Particularly, the results in [36] may lead to the Propositions 4.4.1, 4.4.2, and 4.4.3. For sake of completeness, we shall prove all propositions using the same technique.

Using Proposition 4.3.1 the matrix, which defines the right system X of (4.2.1) is transformed into one of the matrices of the statements (a)–(d), namely $A=(a_{ij})_{i,j}$. Of course the transformation is applied to the whole system (4.2.1), so the matrix which defines the left system Y is also transformed into a (general) matrix $B=(b_{ij})_{i,j}$. Then system (4.2.1), after this transformation, reads

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x+u_1 \\ y+u_2 \end{pmatrix} & \text{if} \quad x>0, \\ \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} x+v_1 \\ y+v_2 \end{pmatrix} & \text{if} \quad x<0. \end{cases} \tag{4.4.1}$$

The solution of (4.4.1) can be easily computed, because it is a piecewise linear differential system. So let $\varphi^+(t,x,y)=\big(\varphi_1^+(t,x,y),\varphi_2^+(t,x,y)\big)$ be the solution of (4.4.1) for $x>0$ such that $\varphi^+(0,x,y)=(x,y)$. Similarly, let $\varphi^-(t,x,y)=(\varphi_1^-(t,x,y),\ \varphi_2^-(t,x,y))$ be the solution of (4.4.1) for $x<0$ such that $\varphi^-(0,x,y)=(x,y)$.

In what follows, let $t^+(y)>0$ be the smallest positive time such that $\varphi_1^+(t^+(y),0,\ y)=0$, and let $t_+(y)<0$ be the biggest negative time such that $\varphi_1^+(t_+(y),0,y)=0$. Analogously, let $t^-(y)<0$ be the biggest negative time such that $\varphi_1^-(t^-(y),0,y)=0$, and $t_-(y)>0$ be the smallest positive time such that $\varphi_1^-(t_-(y),\ 0,y)=0$. Observe that the functions $t^+(y)$, $t_+(y)$, $t^-(y)$, and $t_-(y)$ are not necessarily always defined.

Assuming that $t^+(y)>0$ and $t^-(y)<0$ are defined then there exists a limit cycle passing through the point $(0,y)$ with $y\in J^*=\mathrm{Dom}(t^+)\cap\mathrm{Dom}(t^-)$ if and only if $\varphi_2^+(t^+(y),0,y)=\varphi_2^-(t^-(y),0,y)$. Thus, in this case, we must study the zeros y^* of the function

$$f(y)= \ \varphi_2^+(t^+(y),0,y)-\varphi_2^-(t^-(y),0,y), \tag{4.4.2}$$

on the domain J^*.

Equivalently, if $t_+(y) < 0$ and $t_-(y) > 0$ are defined then there exists a limit cycle passing through $(0, y)$ with $y \in J_* = \mathrm{Dom}(t_+) \cap \mathrm{Dom}(t_-)$ if and only if $\varphi_2^+(t_+(y), 0, y) = \varphi_2^-(t_-(y), 0, y)$. Thus, in this case, we must study the zeros y_* of the function

$$f(y) = \quad \varphi_2^+(t_+(y), 0, y) - \varphi_2^-(t_-(y), 0, y), \tag{4.4.3}$$

on the domain J_*.

Since the vectors fields X and Y are linear, then a limit cycle passing through a point (x_0, y_0) must contain points of kind $(0, y^*)$ and $(0, y_*)$ such that $y^* \in J^*$ and $y_* \in J_*$. Therefore detecting all the zeros of (4.4.2) or (4.4.3) we must detect all non–sliding limit cycles of (4.4.1).

Let $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$. We say that a point $(0, y)$ is an

(a) *invisible fold point for the right system* when

$$X_1(0, y) = 0 \quad \text{and} \quad \frac{\partial X_1}{\partial y}(0, y) X_2(0, y) < 0;$$

(c) *invisible fold point for the left system* when

$$Y_1(0, y) = 0 \quad \text{and} \quad \frac{\partial Y_1}{\partial y}(0, y) Y_2(0, y) > 0.$$

An affine (linear) change of variables in the plane preserving the straight line $x = 0$ will be called in what follows a Σ*–preserving affine (linear) change of variables*, and a Σ–preserving affine (linear) change of variables which also preserves the semiplane $x > 0$ will be called in what follows a Σ^+*–preserving affine (linear) change of variables.* Clearly a Σ^+–preserving affine (linear) change of variables also preserves the semiplane $x < 0$.

The case when the left system has a focus or a center on Σ will be studied in subsection 4.4.1, the case when the left system has a weak saddle will be studied in subsection 4.4.2, and the case when the left system has a virtual or real center will be studied in subsection 4.4.3.

4.4.1 Left system has a focus or a center on Σ

In this case $v_1 = 0$, $4b_{12}b_{21} + (b_{11} - b_{22})^2 \leq 0$ and the point $(0, -v_2)$ is a singularity of focus or center type.

Let $\Gamma = \sqrt{-4b_{12}b_{21} - (b_{11} - b_{22})^2}$. The function $t^-(y) < 0$ is defined for every $y > -v_2$, and we compute $t^-(y) = -2\pi/\Gamma$. Analogously the function $t_-(y) > 0$ is defined for every $y < -v_2$, and we compute $t_-(y) = 2\pi/\Gamma$.

In order to fix the clockwise orientation of the flow of system (4.4.1) we assume that $Y_1(0, 1 - v_2) = b_{12} > 0$.

Proposition 4.4.1. The equalities $\mathcal{N}(F_b, F_v) = 2$, $\mathcal{N}(F_b, C_v) = \mathcal{N}(C_b, F_v) = 1$ and $\mathcal{N}(C_b, C_v) = 0$ hold.

Proof. From Proposition 4.3.1(c) we can assume that $a_{11} = a_{22} = a$ with $a \in \mathbb{R}$, $a_{12} = -a_{21} = 1$, and by a Σ^+–preserving translation we can take $u_2 = 0$. Moreover $u_1 > 0$ because the right system has a focus or a center which is virtual for system (4.4.1).

It is easy to see that the point $(0, -a\,u_1) \in \Sigma$ is an invisible fold point for the right system. So the function $t^+(y) > 0$ is defined for every $y > -a\,u_1$ (see Figure 4.1). Moreover its image is the interval $(0, \pi)$. Indeed, given $y > -a\,u_1$ consider the line $\ell(y)$ passing through the focus point $(-u_1, 0)$ and $(0, y)$. The trajectory of the left system starting at $(0, y)$ returns to the line $\ell(y)$ at $t = \pi$, so it must return to Σ in a time $t < \pi$. Thus $t^+(y) \in (0, \pi)$ for every $y > -a\,u_1$.

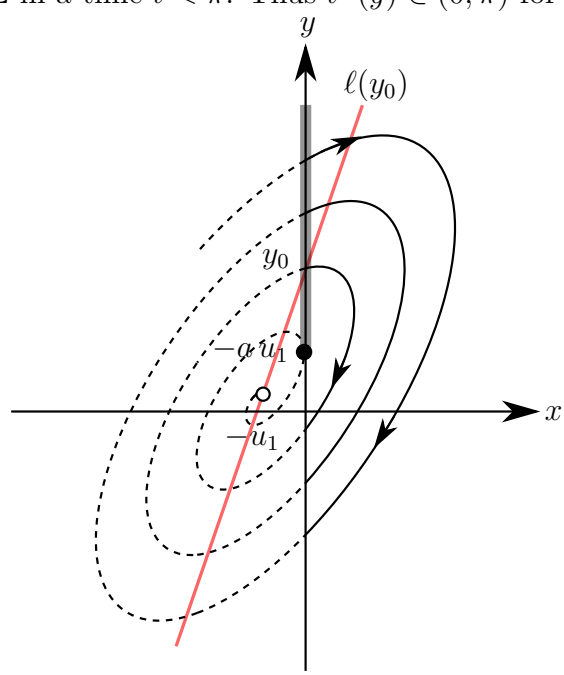

Figure 4.1: Virtual focus for the right system. The shaded line represents the domain of the definition of the function $t^+(y) > 0$.

We know that $\varphi_1^+(t^+(y), 0, y) = 0$ for every $y > -a\,u_1$, that is

$$-u_1 + e^{at^+(y)}\Big(u_1 \cos(t^+(y)) + y \sin(t^+(y))\Big) = 0.$$

Hence taking $y^+(t) = u_1 F(t)$ for $t \in (0, \pi)$ we have that $y^+\left(t^+(y)\right) = y$ for every $y > -a\,u_1$. The function F is defined in (4.3.2).

Now we claim that $t^+\left(y^+(t)\right) = t$ for every $t \in (0, \pi)$. Indeed, for $t_0 \in (0, \pi)$ let $y_0 = y^+(t_0)$. From Lemma 4.3.1(a) $y_0 > -a\,u_1$, so from the above comments we obtain that $y_0 = y^+\left(t^+(y_0)\right)$. Thus $y^+(t_0) = y^+\left(t^+(y_0)\right)$. Again from Lemma 4.3.1(a) $y^+(t) = u_1 F(t)$ is injective on the interval $(0, \pi)$, so $t_0 = t^+(y_0)$. Hence $t_0 = t^+(y_0) = t^+\left(y^+(t_0)\right)$. Since t_0 was arbitrarily chosen in $(0, \pi)$ we conclude that $t^+\left(y^+(t)\right) = t$ for every $t \in (0, \pi)$. Therefore the function $t^+ : (-a\,u_1, \infty) \to (0, \pi)$ is invertible with inverse equal to $y^+ : (0, \pi) \to (-a\,u_1, \infty)$.

Let $Y_M = \max\{-a\,u_1, -v_2\}$, so computing the zeros of the function (4.4.2) for $y > Y_M$ is equivalent to compute the zeros of the function $g_1(t) = f(y^+(t))$. Since

$$f(y) = v_2 + e^{-\dfrac{(b_{11}+b_{22})\pi}{\Gamma}}(v_2 + y) + e^{a\,t^+(y)}\Big(y\cos(t^+(y)) - u_1\sin(t^+(y))\Big),$$

taking $\delta = e^{-\frac{(b_{11}+b_{22})\pi}{\Gamma}}$ we obtain

$$\begin{aligned} g_1(t) = &\ v_2(1+\delta) + u_1\left(\cot(t) - e^{at}\csc(t)\right) - \delta u_1\left(\cot(t) - e^{-at}\csc(t)\right) \\ = &\ k_1\,\xi_1 + k_2\,\xi_2^1 + k_3\,\xi_3^1, \end{aligned} \tag{4.4.4}$$

for $t \in I \subset (0,\pi)$. Clearly $k_1 = v_2(1+\delta)$, $k_2 = u_1$, $k_3 = -\delta\,u_1$, and $I = t^+\left((Y_M,\infty)\right)$. Note that $I = (0,\pi)$ provided $v_2 \geq au_1$.

Applying Lemma 4.3.2(a) we obtain the inequality $\mathcal{N}(F_b, F_v) \leq 2$. The equality is not a direct consequence of Lemma 4.3.2(a) and Proposition B.0.4 because the parameters k_1 and k_2 are not free to be chosen among the real numbers. However choosing $a = -1$, $u_1 = 8$, $v_2 = -40/9$, and $b_{i,j}$ for $i,j = 1,2$ such that $\delta = 1/8$ we obtain $k_1 = -5$, $k_2 = 8$, and $k_3 = -1$. We claim that for these choice of parameters the function (4.4.4) has exactly 2 simple zeros in $(0,\pi)$.

To see the claim it is sufficient to prove the existence of two distinct zeros in $(0,\pi)$. Indeed, once proved their existence, Lemma 4.3.2(a) and Proposition B.0.4 imply, directly, that they are simple and that the function g_1 has no more zeros in $(0,\pi)$. This argumentation will be recurrently used, with less details, throughout the proofs in this section. Accordingly we compute $g_1(1/2) \approx 1.13 > 0$, $g_1(3/2) \approx -1.80 < 0$, and $g_1(5/2) \approx 4.89 > 0$. Thus, from continuity, there exist at least two zeros in the interval $(1/2, 5/2) \subset (0,\pi)$, which leads to the claim. We may also estimate $t_1 \approx 0.770$ and $t_2 \approx 2.203$. Hence for $y^+(t_1) \approx 16.572 > Y_M = 8$ and $y^+(t_2) \approx 95.667 > 8$ there exist two limit cycles of system (4.4.1) passing respectively through the points $(0, y^+(t_1))$ and $(0, y^+(t_2))$.

Now, the right system is a center if and only if $a = 0$, in this case $\xi_2^1(t) = \xi_3^1(t) = \cot(t) - \csc(t)$, so the function (4.4.4) becomes

$$g_1(t) = k_1\,\xi_1 + \bar{k}_2\,\xi_2^1,$$

where $\bar{k}_2 = k_2 + k_3$. Since now k_1 and $\bar{k}_2$ can be chosen freely, we conclude that $\mathcal{N}(F_b, C_v) = 1$ we obtain, from Lemma 4.3.3(a), that $\mathcal{N}(F_b, C_v) \leq 1$.

The left system has a center if and only if $b_{22} = -b_{11}$ and $b_{11}^2 + b_{12}b_{21} < 0$. In this case $\delta = 1$, $k_1 = 2v_2$, $k_3 = -k_2 = -u_1$, so the function (4.4.4) becomes

$$g_4(t) = k_1\,\xi_1 - 2k_2\,\xi_2^4.$$

Multiplying g_4 by a parameter, if needed, we see that k_1 and k_2 can be chosen freely. Hence applying Lemma 4.3.2(d) we conclude that $\mathcal{N}(C_b, F_v) = 1$.

Finally the lateral systems are centers if and only if $a = 0$, $b_{22} = -b_{11}$ and $b_{11}^2 + b_{12}b_{21} < 0$. In this case the function (4.4.4) becomes $g_1(t) = k_1$. So if $k_1 \neq 0$, that is $v_2 \neq 0$, then there is no

solutions for the equation $g_1(t) = 0$; if $k_1 = 0$, that is $v_2 = 0$, then $g_1 = 0$, which implies that all the solutions of system (4.4.1) passing through $(0, y)$ for $y > Y_M$ are periodic solutions, in other words there are no limit cycles. Hence we conclude that $\mathcal{N}(C_b, C_v) = 0$. □

Proposition 4.4.2. The equalities $\mathcal{N}(F_b, F_r) = \mathcal{N}(C_b, F_r) = 2$, $\mathcal{N}(F_b, C_r) = 1$ and $\mathcal{N}(C_b, C_r) = 0$ hold.

Proof. From Proposition 4.3.1(c) and by a Σ^+–preserving translation we can assume that $a_{11} = a_{22} = a$ with $a \in \mathbb{R}$, $a_{12} = -a_{21} = 1$, $u_2 = 0$, and $u_1 < 0$ because the right system has a focus which is real for system (4.4.1).

In the case that $a < 0$ it is easy to see that the focus $(-u_1, 0)$ is an attractor singularity and that the point $(0, -au_1) \in \Sigma$ is a visible fold point for the right system. So the function $t_+(y) < 0$ is defined for every $y < -au_1$. Moreover its image is the interval $(-\tau, -\pi)$, where $\tau = -t_+(-au_1)$ so $\pi < \tau < 2\pi$. Indeed given $y < -au_1$ consider the line $\ell(y)$ passing through the focus point $(-u_1, 0)$ and $(0, y)$. The trajectory of the left system starting at $(0, y)$ returns to the line $\ell(y)$ at $t = -\pi$, so it must return to Σ for $-2\pi < -\tau < t < -\pi$. Thus $t_+(y) \in (-\tau, -\pi)$ for every $y < -au_1$ (see Figure 4.2 left).

In the other case $a > 0$ the focus $(-u_1, 0)$ is a repulsive singularity. Considering now the function $t^+(y) > 0$ defined for every $y > -au_1$, the same analysis can be done (see Figure 4.2 right).

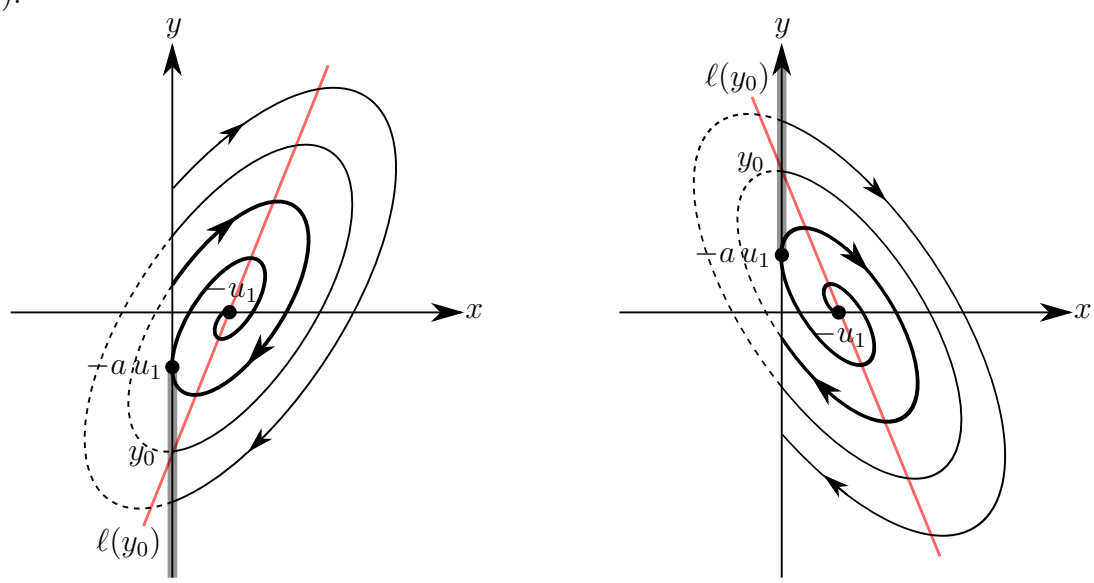

Figure 4.2: Left: Real focus for the right system when $a < 0$. In this case the shaded line represents the domain of the definition of the function $t_+(y) < 0$. Right: Real focus for the right system when $a > 0$. In this case the shaded line represents the domain of the definition of the function $t^+(y) > 0$.

From now on in this proof we assume, without loss of generality, that $a < 0$.

We know that $\varphi_1^+(t_+(y), 0, y) = 0$ for every $y < -au_1$, that is

$$-u_1 + e^{at_+(y)} \left(u_1 \cos(t_+(y)) + y \sin(t_+(y))\right) = 0.$$

Hence taking $y_+(t) = u_1 F(t)$ for $t \in (-\tau, -\pi)$ we have that $y_+(t_+(y)) = y$ for every $y < -au_1$.

Now we claim that $t_+(y_+(t)) = t$ for every $t \in (-\tau, -\pi)$. Indeed for $t_0 \in (-\tau, -\pi)$, let $y_0 = y_+(t_0)$. From Lemma 4.3.1(a') $y_+(t)$ is decreasing on the interval $(-\tau, -\pi) \subset (-2\pi, -\pi)$, and since $y_+(\tau) = -au_1$ it follows that $y_0 < -au_1$. So from the above comments we obtain that $y_0 = y_+(t_+(y_0))$. Thus $y_+(t_0) = y_+(t_+(y_0))$. Again from Lemma 4.3.1(a') $y_+(t) = u_1 F(t)$ is injective on the interval $(-\tau, -\pi) \subset (-2\pi, -\pi)$, so $t_0 = t_+(y_0)$. Hence $t_0 = t_+(y_0) = t_+(y_+(t_0))$. Since t_0 was arbitrarily chosen in $(-\tau, -\pi)$ we conclude that $t_+(y_+(t)) = t$ for every $t \in (-\tau, -\pi)$. Therefore the function $t_+ : (-\infty, -au_1) \to (-\tau, -\pi)$ is invertible with inverse equal to $y_+ : (-\tau, -\pi) \to (-\infty, -au_1)$.

Let $Y_m = \min\{-au_1, -v_2\}$, so computing the zeros of the function (4.4.3) for $y < Y_m$ is also equivalent to compute the zeros of the function (4.4.4) now for $t \in I \subset (-\tau, -\pi)$, where $I = t_+((-\infty, Y_m))$. Note that $I = (-\tau, -\pi)$ provided $v_2 \leq au_1$.

Applying Lemma 4.3.2(a') we conclude that $\mathcal{N}(F_b, F_r) \leq 2$. Now choosing $a = -3/4$, $u_1 = -1/10$, $v_2 = -3/22$, and $b_{i,j}$ for $i, j = 1, 2$ such that $\delta = 10$ we obtain $k_1 = -3/2$, $k_2 = -1/10$, and $k_3 = 1$ which implies, analogously to the proof of Proposition 4.4.1, that (4.4.4) has exactly 2 simple zeros, at first, in $(-2\pi, -\pi)$, namely $t_1 \approx -3.733$ and $t_2 \approx -3.250$. We wish to conclude that $t_1, t_2 \in (-\tau, -\pi)$ or, equivalently, $y^+(t_1), y^+(t_2) < Y_m$. Indeed, $y^+(t_1) \approx -0.160 < Y_m = -0.075$ and $y^+(t_2) \approx -0.999 < -0.075$. So there exist two limit cycles of system (4.4.1) passing respectively through the points $(0, y_+(t_1))$ and $(0, y_+(t_2))$.

The equalities $\mathcal{N}(F_b, C_r) = 1$ and $\mathcal{N}(C_b, C_r) = 0$ follows in a similar way to the proof of Proposition 4.4.1.

The inequality $\mathcal{N}(C_b, F_r) \leq 2$ also follows in a similar way to the proof of Proposition 4.4.1 but now applying Lemma 4.3.2(d') to the function $g_4(t) = 2v_2\,\xi_1 - 2u_1\,\xi_2^4$ for $t \in (-\tau, -\pi)$. To see the equality we take $a = u_1 = -1/10$, and $v_2 = -1/20$. It implies that g_4 has exactly 2 simple zeros, at first, in $(-2\pi, -\pi)$, namely $t_1 \approx -4.176$ and $t_2 \approx -4.796$. Again, $y^+(t_1) \approx -0.136 < Y_m = -0.01$ and $y^+(t_2) \approx -0.054 < -0.01$. So there exist two limit cycles of system (4.4.1) passing respectively through the points $(0, y_+(t_1))$ and $(0, y_+(t_2))$. It concludes the proof of this proposition. □

Proposition 4.4.3. The equalities $\mathcal{N}(F_b, F_b) = \mathcal{N}(F_b, C_b) = 1$ and $\mathcal{N}(C_b, C_b) = 0$ hold.

Proof. Here $u_1 = 0$, because the right system have its focus on the line Σ. From Proposition 4.3.1(c) and by a Σ^+–preserving translation we can assume that $a_{11} = a_{22} = a$ with $a \in \mathbb{R}$, $a_{12} = -a_{21} = 1$, and $u_2 = 0$.

The function $t^+(y) > 0$ is defined for every $y > 0$, because the point $(0, 0)$ is a focus for the right system. Moreover we compute $t^+(y) = \pi$.

Let $Y_M = \max\{0, -v_2\}$, so computing the zeros of the function (4.4.2) for $y > Y_M$ is equivalent to compute the zeros of the linear function

$$f_1(y) = k_1 + k_2 y, \tag{4.4.5}$$

for $y > Y_M$, where $k_1 = v_2(1 + \delta)$ and $k_2 = (\delta - e^{a\pi})$. Hence $\mathcal{N}(F_b, F_b) \leq 1$. Nevertheless we can choose coefficients such that $\bar{y} = \dfrac{(1+\delta)v_2}{e^{a\pi} - \delta} > Y_M$ is the unique zero of (4.4.5).

From here, the equalities $\mathcal{N}(F_b, C_b) = 1$ and $\mathcal{N}(C_b, C_b) = 0$ follows similarly to the proof of Proposition 4.4.1. It concludes the proof of this proposition. $\square$

Proposition 4.4.4. The equalities $\mathcal{N}(F_b, N_v) = 2$ and $\mathcal{N}(C_b, N_v) = 1$ hold.

Proof. From Proposition 4.3.1(b) and by a Σ^+–preserving translation, we can assume that $a_{11} = a_{22} = a$ with $|a| > 1$, $a_{12} = a_{21} = 1$, $u_2 = 0$, and $u_1 > 0$, because the right system is a diagonalizable node, which is virtual for system (4.4.1).

It is easy to see that the point $(0, -a\,u_1) \in \Sigma$ is an invisible fold point for the right system.

In the case $a < -1$ the node $(-u_1, 0)$ is an attractor singularity. The stable manifold and the strong stable manifold of the node intersect Σ at the points $(0, y^s)$ and $(0, y^{ss})$, respectively, where $y^s = u_1 < -a\,u_1$ and $y^{ss} = -u_1 < u_1$. So the function $t^+(y) > 0$ is defined for every $y > -au_1$ (see Figure 4.3 left).

In the other case $a > 1$ the node $(-u_1, 0)$ is an repulsive singularity. The stable manifold and the strong stable manifold of the node intersect Σ at the points $(0, y^s)$ and $(0, y^{ss})$, respectively, where $y^s = -u_1 > -a\,u_1$ and $y^{ss} = u_1 > -u_1$. So the function $t_+(y) < 0$ is defined for every $y < -au_1$ (see Figure 4.3 right).

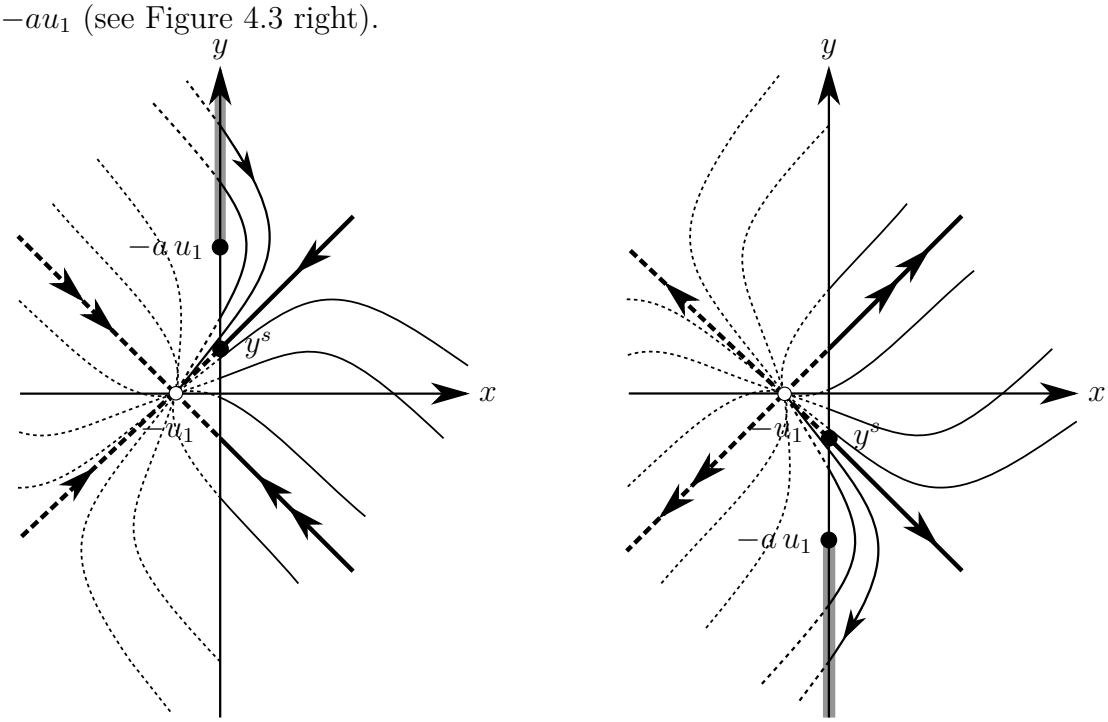

Figure 4.3: Left: Virtual diagonalizable node for the right system when $a < -1$. In this case the shaded line represents the domain of the definition of the function $t^+(y) > 0$. Right: Virtual diagonalizable node for the right system when $a > 1$. In this case the shaded line represents the domain of the definition of the function $t_+(y) < 0$

From now on in this proof we assume, without loss of generality, that $a < -1$.

We know that $\varphi_1^+(t^+(y), 0, y) = 0$ for every $y > -au_1$, that is

$$-u_1 + e^{at^+(y)} \Big(u_1 \cosh(t^+(y)) + y \sinh(t^+(y)) \Big) = 0.$$

Hence taking $y^+(t) = u_1 G(t)$ for $t \in \mathbb{R}^+$ we have that $y^+\left(t^+(y)\right) = y$ for every $y > -au_1$.

The image of the function t^+ is $\mathbb{R}^+$. Indeed, computing implicitly the derivative in the variable y of the identity $y^+\left(t^+(y)\right) = y$ we obtain

$$\frac{dt^+(y)}{dy} = P\left(t^+(y)\right), \quad \text{where} \quad P(\theta) = \frac{\sinh(\theta)}{u_1\left(\operatorname{csch}(\theta) - e^{-a\theta}\left(a + \coth(\theta)\right)\right)}.$$

It is easy to see that $P(\theta) > 0$ for every $\theta > 0$. So any solution $\theta(y)$ of the differential equation $\dot{\theta} = F(\theta)$ starting at $\theta = \overline{\theta} > 0$ and $y = \bar{y}$, i.e. $\theta(\bar{y}) = \overline{\theta}$, keeps itself positive for every $y > \bar{y}$, moreover this solution will be strictly increasing. Hence we conclude that $t^+(y) > 0$ is a positive strictly increasing function for $y > -au_1$.

We claim that $t^+\left(y^+(t)\right) = t$ for every $t > 0$. Indeed for $t_0 > 0$, let $y_0 = y^+(t_0)$. From Lemma 4.3.1(b) $y_0 > -au_1$, so from the above comments we obtain that $y_0 = y^+\left(t^+(y_0)\right)$. Thus $y^+(t_0) = y^+\left(t^+(y_0)\right)$. Again from Lemma 4.3.1(b) $y^+(t) = u_1 G(t)$ is injective on $\mathbb{R}^+$, so $t_0 = t^+(y_0)$. Hence $t_0 = t^+(y_0) = t^+\left(y^+(t_0)\right)$. Since $t_0 > 0$ was arbitrarily chosen we conclude that $t^+\left(y^+(t)\right) = t$ for every $t > 0$. Therefore the function $t^+ : (-au_1, \infty) \to \mathbb{R}^+$ is invertible with inverse equal to $y^+ : \mathbb{R}^+ \to (-au_1, \infty)$.

Computing the zeros of the function (4.4.2) for $y > Y_M = \max\{-au_1, -v_2\}$ is equivalent to compute the zeros of the function

$$g_2(t) = f(y^+(t)) = k_1\,\xi_1 + k_2\,\xi_2^2 + k_3\,\xi_3^2 \tag{4.4.6}$$

for $t \in I \subset \mathbb{R}^+$, where $k_1 = v_2(1+\delta)$, $k_2 = u_1$, $k_3 = -\delta u_1$, $\delta = e^{-\frac{(b_{11}+b_{22})\pi}{\Gamma}}$, and here $I = t^+\left((Y_M, \infty)\right)$. Note that $I = \mathbb{R}^+$ provided $v_2 \geq a\,u_1$.

Applying Lemma 4.3.2(b) we conclude that $\mathcal{N}(F_b, N_v) \leq 2$. Now choosing $a = -3/2$, $u_1 = 75$, $v_2 = -375/4$, and $b_{i,j}$ for $i, j = 1, 2$ such that $\delta = 1/15$ we obtain $k_1 = -100$, $k_2 = 75$, and $k_3 = -5$ which implies, analogously to the proof of Proposition 4.4.1, that (4.4.6) has 2 zeros in $\mathbb{R}^+$, namely $t_1 \approx 0.704$ and $t_2 \approx 2.069$. Hence for $y^+(t_1) \approx 158.781 > Y_M = 112.5$ and $y^+(t_2) \approx 351.490 > 112.5$ there exist two limit cycles of system (4.4.1) passing respectively through the points $(0, y^+(t_1))$ and $(0, y^+(t_2))$.

From here, the equality $\mathcal{N}(C_b, N_v) = 1$ follows similarly to the proof of Proposition 4.4.1 but now applying Lemma 4.3.2(e) to the function $g_2(t) = k_1\xi_1 - 2k_2\,\xi_2^5$. It completes the proof of this proposition. □

Proposition 4.4.5. The equalities $\mathcal{N}(F_b, iN_v) = 2$ and $\mathcal{N}(C_b, iN_v) = 1$ hold.

Proof. From Proposition 4.3.1(d) and by a Σ^+–preserving translation, we can assume that $a_{11} = a_{12} = a_{22} = \lambda$ with $\lambda = \pm 1$, $a_{21} = 0$, $u_2 = 0$, and $u_1 > 0$, because the right system is a non diagonalizable node, which is virtual for system (4.4.1).

It is easy to see that for $\lambda = \pm 1$ the point $(0, -u_1) \in \Sigma$ is a invisible fold point for the right system and that the invariant manifold of the node intersects Σ at the origin $(0, 0)$ (see Figure 4.4). In order to fix the clockwise orientation of the flow of system (4.4.1) we assume that $\lambda = 1$, otherwise the first return map would not be defined and there would not exist limit cycles. In this case the function $t_+(y) < 0$ is defined for every $y < -u_1$.

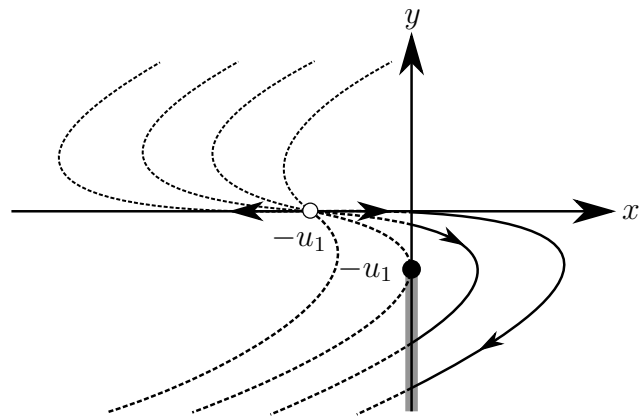

Figure 4.4: Virtual non–diagonalizable node for the right system when $\lambda = 1$. In this case the shaded line represents the domain of the definition of the function $t_+(y) < 0$.

We know that $\varphi_1^+(t_+(y), 0, y) = 0$ for every $y < -u_1$, that is

$$-u_1 + e^{t_+(y)}\,(u_1 + y\,t) = 0.$$

Hence taking $y_+(t) = u_1 H(t)$ for $t \in \mathbb{R}^-$ we have that $y_+\,(t_+(y)) = y$ for every $y < -u_1$.

The image of the function t_+ is $\mathbb{R}^-$. Indeed, computing implicitly the derivative in the variable y of the identity $y_+\,(t_+(y)) = y$ we obtain

$$\frac{dt_+(y)}{dy} = Q\,(t_+(y))\,, \quad \text{where} \quad Q(\theta) = \frac{e^\theta \theta^2}{u_1\,(e^\theta - \theta - 1)}.$$

So the function t_+ is the solution $\theta(y)$ of the above differential equation such that $\theta(-u_1) = 0$. It is easy to see that $Q(\theta) > 0$, moreover by continuity we have that $Q(0) = 2$. So it follows that the solution $\theta(y)$ is strictly increasing. Hence we conclude that $t_+(y) < 0$ is strictly increasing function such that $t_+(-u_1) = 0$, which implies that $t_+(y) < 0$ for $y < -u_1$.

Now we claim that $t_+\,(y_+(t)) = t$ for every $t < 0$. Indeed for $t_0 < 0$, let $y_0 = y_+(t_0)$. From Lemma 4.3.1(c) $y_0 < -u_1$, so from the above comments we obtain that $y_0 = y_+\,(t_+(y_0))$. Thus $y_+(t_0) = y_+\,(t_+(y_0))$. Again from Lemma 4.3.1(c) the function $y_+(t) = u_1 H(t)$ is injective, so $t_0 = t_+(y_0)$. Hence $t_0 = t_+(y_0) = t_+\,(y_+(t_0))$. Since $t_0 < 0$ was arbitrarily chosen we conclude that $t_+\,(y_+(t)) = t$ for every $t > 0$. Therefore the function $t_+ : (-\infty, -u_1) \to \mathbb{R}^-$ is invertible with inverse equal to $y_+ : \mathbb{R}^- \to (-\infty, -u_1)$.

Let $Y_m = \min\{-u_1, -v_2\}$, so computing the zeros of the function (4.4.3) for $y < Y_m$ is equivalent to compute the zeros of the function

$$g_3(t) = f(y_+(t)) = k_1\,\xi_1 + k_2\,\xi_2^3 + k_3\,\xi_3^3 \tag{4.4.7}$$

for $t \in I \subset \mathbb{R}^-$, where $k_1 = v_2(1+\delta)$, $k_2 = u_1$, $k_3 = -\delta u_1$, $\delta = e^{-\frac{(b_{11}+b_{22})\pi}{\Gamma}}$, and here $I = y_-\,((-\infty, Y_M))$. Note that $I = \mathbb{R}^-$ provided $v_2 \geq u_1$.

Applying Lemma 4.3.2(c) we conclude that $\mathcal{N}(F_b, iN_v) \leq 2$. Now choosing $u_1 = 149$, $v_2 = 298/3$, and $b_{i,j}$ for $i, j = 1, 2$ such that $\delta = 1/149$ we obtain $k_1 = 100$, $k_2 = 149$, and $k_3 = -1$

which implies, analogously to the proof of Proposition 4.4.1, that (4.4.7) has 2 zeros in $\mathbb{R}^-$, namely $t_1 \approx -6.146$ and $t_2 \approx -0.897$. Hence for $y^+(t_1) \approx -11295.600 < Y_m = -149$ and $y^+(t_2) \approx -241.197 < -149$ there exist two limit cycles of system (4.4.1) passing respectively through the points $(0, y^+(t_1))$ and $(0, y^+(t_2))$.

From here the equality $\mathcal{N}(C_b, iN_v) = 1$ follows similarly to the proof of Proposition 4.4.1 but now applying Lemma 4.3.2(f) to the function $g_3(t) = k_1\,\xi_1 - 2k_2\,\xi_2^6$. It concludes the proof of proposition. □

Proposition 4.4.6. The equalities $\mathcal{N}(F_b, S_r) = 2$, $\mathcal{N}(F_b, S_r^0) = \mathcal{N}(C_b, S_r) = 1$ and $\mathcal{N}(C_b, S_r^0) = 0$ hold.

Proof. From Proposition 4.3.1(a) and by a Σ^+–preserving translation, we can assume that $a_{11} = a_{22} = a$ with $|a| < 1$, $a_{12} = a_{21} = 1$, $u_2 = 0$, and $u_1 < 0$, because the right system is a saddle, which is real for system (4.4.1).

It is easy to see that the point $(0, -au_1) \in \Sigma$ is an invisible fold point for the right system and that the stable and unstable invariant manifolds of the saddle intersect Σ at the points $(0, y^s)$ and $(0, y^u)$, respectively, where $y^s = -u_1$ and $y_s = u_1$. So the function $t^+(y) > 0$ is defined for every $-au_1 < y < -u_1$.

We know that $\varphi_1^+(t^+(y), 0, y) = 0$ for every $-au_1 < y < -u_1$, that is

$$-u_1 + e^{at^+(y)}\left(u_1 \cosh(t^+(y)) + y \sinh(t^+(y))\right) = 0.$$

Hence taking $y^+(t) = u_1 G(t)$ for $t \in \mathbb{R}^+$ we have that $y^+\left(t^+(y)\right) = y$ for every $-au_1 < y < -u_1$.

In the proof of Proposition 4.4.4 we have seen that the function $t^+ : (-au_1, \infty) \to \mathbb{R}^+$ is invertible with inverse equal to $y^+ : \mathbb{R}^+ \to (-au_1, \infty)$. So its restriction to $-au_1 < y < -u_1$ is also invertible with inverse defined in $t^+\left(-au_1, -u_1\right)$.

Computing the zeros of the function (4.4.2) for $\max\{-au_1, -v_2\} = Y_M < y < -u_1$ is equivalent to compute the zeros of the function (4.4.6) for $t \in I \subset \mathbb{R}^+$, where $k_1 = v_2(1+\delta)$, $k_2 = u_1$, $k_3 = -\delta u_1$, $\delta = e^{-\frac{(b_{11}+b_{22})\pi}{\Gamma}}$, and $I = t^+\left((Y_M, -u_1)\right)$.

Applying Lemma 4.3.2(b) we conclude that $\mathcal{N}(F_b, S_r) \leq 2$. Now choosing $a = -1/2$, $u_1 = -100$, $v_2 = 1600/27$, and $b_{i,j}$ for $i, j = 1, 2$ such that $\delta = 7/20$ we obtain $k_1 = 80$, $k_2 = -100$, and $k_3 = 35$ which implies, analogously to the proof of Proposition 4.4.1, that (4.4.6) has 2 zeros in I_5, namely $t_1 \approx 0.689$ and $t_2 \approx 2.761$. Hence for $y^+(t_1) \approx -22.071 \in (Y_M, -u_1) = (-50, 100)$ and $y^+(t_2) \approx 50.318 \in (-50, 100)$ there exist two limit cycles of system (4.4.1) passing respectively through the points $(0, y^+(t_1))$ and $(0, y^+(t_2))$.

The right system has a saddle with trace equal 0 if and only if $a = 0$, in this case $\xi_2^2(t) = \xi_3^2(t) = \coth(t) - \operatorname{csch}(t)$. So the equality $\mathcal{N}(F_b, S_r^0) = 1$ follows applying lemma 4.3.3(b) to the function $g_2(t) = k_1\xi_1(t) + 2k_2\xi_2^2$. From here the equalities $\mathcal{N}(C_b, S_r) = 1$ and $\mathcal{N}(C_b, S_r^0) = 0$ follows similar to the proof of Proposition 4.4.1 but now by applying Lemma 4.3.2(e) to the function $g_2(t) = k_1\xi_1(t) - 2k_2\xi_2^5(t)$. It concludes the proof of this proposition. □

4.4.2 Left system has a weak saddle

In this case $b_{22} = -b_{11}$, $b_{11}^2 + b_{12}b_{21} > 0$ and $v_1 > 0$ and the point $(-v_1, -v_2)$ is a singularity of saddle type.

Let $\Gamma = \sqrt{b_{11}^2 + b_{12}b_{21}}$, let y^u be the y–coordinate of the intersection between the unstable manifold with Σ, and let y^s be the y–coordinate of the intersection between the stable manifold with Σ. We compute

$$y^u = -v_2 + \frac{v_1\,(\Gamma - b_{11})}{b_{12}} \quad \text{and} \quad y^s = -v_2 - \frac{v_1\,(\Gamma + b_{11})}{b_{12}}.$$

In order to fix the clockwise orientation of the flow of system (4.4.1) we assume that $y^s < y^s$, which is equivalent to assume that $b_{12} > 0$.

The left system has an invisible fold point $(0, \check{y})$ given by

$$\check{y} = -v_2 - \frac{b_{11}v_1}{b_{12}}.$$

For $y^s < y < y^y$ we define

$$t^*(y) = \frac{1}{\Gamma}\log\left(\frac{v_1\,(\Gamma - b_{11}) - b_{12}\,(v_2 + y)}{v_1\,(\Gamma + b_{11}) + b_{12}\,(v_2 + y)}\right).$$

So $t^-(y) = t^*(y) < 0$ for $\check{y} < y < y^u$ and $t_-(y) = t^*(y) > 0$ for $y^s < y < \check{y}$.

Proposition 4.4.7. The equalities $\mathcal{N}(S_r^0, F_v) = 1$ and $\mathcal{N}(S_r^0, C_v) = 0$ holds.

Proof. From Proposition 4.3.1(c) we can assume that $a_{11} = a_{22} = a$ with $a \in \mathbb{R}$, $a_{12} = -a_{21} = 1$, and by a Σ^+–preserving translation we can take $u_2 = 0$. Moreover $u_1 > 0$ because the right system has a focus which is virtual for system (4.4.1).

From the proof of Proposition 4.4.1 we know that the function $t^+ : (-a\,u_1, \infty) \to (0, \pi)$, such that $\varphi^+(t^+(y), 0, y) = 0$ for $y > -a\,u_1$, is invertible with inverse $y^+ : (0, \pi) \to (-a\,u_1, \infty)$ given by $y^+(t) = u_1 F(t)$.

Let $Y_M = \max\{-a\,u_1, \check{y}\}$, so computing the zeros of the function (4.4.2) for $Y_M < y < y^u$ is equivalent to compute the zeros of the function

$$g_4(t) = f(y^+(t)) = k_1\,\xi_1 + k_2\,\xi_2^4 \tag{4.4.8}$$

for $t \in I \subset (0, \pi)$, where $k_1 = 2(b_{11}v_1 + b_{12}v_2)/b_{12}$ and $k_2 = -2u_1$, and here $I = t^+\left((Y_M, y^u)\right)$. Multiplying the function g_4 by a parameter, if necessary, we see that k_1 and k_2 can be chosen freely. So applying Lemma 4.3.2(d) we conclude that $\mathcal{N}(S_r^0, F_v) = 1$.

The right system has a center if and only if $a = 0$. In this case $\xi_2^4 = 0$ and the function (4.4.8) becomes $g_4(t) = k_1$. So if $k_1 \neq 0$, that is $b_{11}v_1 \neq -b_{12}v_2$, then there are no solutions for the equation $g_4(t) = 0$; and if $k_1 = 0$, that is $b_{11}v_1 = -b_{12}v_2$, then $g_4 = 0$, that is system (4.4.1) is a center. Hence we conclude that $\mathcal{N}(S_r^0, C_v) = 0$. $\square$

Proposition 4.4.8. The equalities $\mathcal{N}(S_r^0, F_r) = 2$ and $\mathcal{N}(S_r^0, C_r) = 0$ hold.

Proof. From Proposition 4.3.1(c) we can assume that $a_{11} = a_{22} = a$ with $a \in \mathbb{R}$, $a_{12} = -a_{21} = 1$, and by a Σ^+–preserving translation we can take $u_2 = 0$. Moreover $u_1 < 0$ because the right system has a focus which is real for system (4.4.1).

From the proof of Proposition 4.4.2 we know that the function $t_+ : (-\infty, -au_1) \to (-\tau, -\pi)$ is invertible with inverse $y_+ : (-\tau, -\pi) \to (-\infty, -au_1)$ given by $y_+(t) = u_1 F(t)$. Here as we have done in the proof of Proposition 4.4.2 we are assuming, without loss of generality, that $a < 0$.

Let $Y_m = \min\{-au_1, \check{y}\}$, so computing the zeros of the function (4.4.3) for $y^s < y < Y_m$ is also equivalent to compute the zeros of the function (4.4.8) now for $t \in I \subset (-\tau, -\pi)$, where $I = t_+((y^s, Y_m))$.

Applying Lemma 4.3.2(d') we conclude that $\mathcal{N}(S_r^0, F_v) \leq 2$. Now choosing $b_{11} = b_{12} = b_{21} = v_1 = 1$, $a = -1/10$, $u_1 = -1/20$, $v_2 = -21/20$, we obtain $b_{11}^2 + b_{12}b_{21} = 2 > 0$, and $k_2 = -k_1 = 1/10$. It implies, analogously to the proof of Proposition 4.4.1, that (4.4.6) has 2 zeros in $(-2\pi, -\pi)$, namely $t_1 \approx -3.508$ and $t_2 \approx -5.646$. Hence for $y^+(t_1) \approx -0.048 \in (y^s, Y_m) \approx (-1.364, -0.005)$ and $y^+(t_2) \approx -0.05 \in (-1.364, -0.005)$ there exist two limit cycles of system (4.4.1) passing respectively through the points $(0, y^+(t_1))$ and $(0, y^+(t_2))$.

The equality $\mathcal{N}(S_r^0, C_r) = 0$ follows similarly to the proof of Proposition 4.4.7. It concludes the proof of this proposition. □

Proposition 4.4.9. The equality $\mathcal{N}(S_r^0, N_v) = 1$ holds.

Proof. From Proposition 4.3.1(b) and by a Σ^+–preserving translation, we can assume that $a_{11} = a_{22} = a$ with $|a| > 1$, $a_{12} = a_{21} = 1$, $u_2 = 0$, and $u_1 > 0$, because the right system has a diagonalizable node, which is virtual for system (4.4.1).

Following the proof of Proposition 4.4.4 the function $t^+ : (-au_1, \infty) \to \mathbb{R}^+$ is invertible with inverse $y^+ : \mathbb{R}^+ \to (-au_1, \infty)$ given by $y^+(t) = u_1 G(t)$. Here as we have done in the proof of Proposition 4.4.3 we are assuming, without loss of generality, that $a < 1$.

Let $Y_M = \max\{-a\,u_1, \check{y}\}$, so computing the zeros of the function (4.4.2) for $Y_M < y < y^u$ is equivalent to compute the zeros of the function

$$g_5(t) = f(y^+(t)) = k_1\,\xi_1 + k_2\,\xi_2^5 \tag{4.4.9}$$

for $t \in I \subset \mathbb{R}^+$, where $k_1 = 2(b_{11}v_1 + b_{12}v_2)/b_{12}$, $k_2 = -2u_1$, and $I = y^+((Y_M, y^u))$. Multiplying the function $g_5(t)$ by a parameter, if necessary, we see that the parameters k_1 and k_2 can be chosen freely. So applying Lemma 4.3.2(e) we conclude that $\mathcal{N}(S_r^0, N_v) = 1$. □

Proposition 4.4.10. The equality $\mathcal{N}(S_r^0, iN_v) = 1$ holds.

Proof. From Proposition 4.3.1(b) and by a Σ^+–preserving translation, we can assume that $a_{11} = a_{12} = a_{22} = \lambda$ with $\lambda = \pm 1$, $a_{21} = 0$, $u_2 = 0$, and $u_1 > 0$, because the right system has a non diagonalizable node, which is virtual for system (4.4.1).

Following the proof of Proposition 4.4.5 the function $t^+ : (-u_1, \infty) \to \mathbb{R}^+$ is invertible with inverse $y^+ : \mathbb{R}^+ \to (-u_1, \infty)$ given by $y^+(t) = u_1 H(t)$. Here as we have done in the proof of Proposition 4.4.5 we are assuming, without loss of generality, that $\lambda = 1$.

Let $Y_M = \max\{-u_1, \check{y}\}$, so computing the zeros of the function (4.4.2) for $Y_M < y < y^u$ is equivalent to compute the zeros of the function

$$g_6(t) = f(y^+(t)) = k_1\,\xi_1 + k_2\,\xi_2^6$$

for $t \in I \subset \mathbb{R}^+$, where $k_1 = 2(b_{11}v_1 + b_{12}v_2)/b_{12}$, $k_2 = -2u_1$, and $I = y^+\left((Y_M, y^u)\right)$. Multiplying the function $g_6(t)$ by a parameter, if necessary, we see that k_1 and k_2 can be chosen freely. So applying Lemma 4.3.2(f) we conclude that $\mathcal{N}(S_r^0, iN_v) = 1$. □

Proposition 4.4.11. The equalities $\mathcal{N}(S_r^0, S_r) = 1$ and $\mathcal{N}(S_r^0, S_r^0) = 0$ hold.

Proof. From Proposition 4.3.1(d) and by a Σ^+–preserving translation, we can assume that $a_{11} = a_{22} = a$ with $|a| < 1$, $a_{12} = a_{21} = 1$, $u_2 = 0$, and $u_1 < 0$, because the right system has a saddle, which is real for system (4.4.1).

Following the proof of Proposition 4.4.6 the function $t^+ : (-au_1, \infty) \to \mathbb{R}^+$ is invertible with inverse $y^+ : I \to (-au_1, u_1)$ given by $y^+(t) = u_1 G(t)$, where $I = t^+\left(-au_1, -u_1\right)$.

Let $Y_M = \max\{-a\,u_1, \check{y}\}$ and $Y_m = \min\{u_1, y^u\}$, so computing the zeros of the function (4.4.2) for $Y_M < y < Y_m$ is equivalent to compute the zeros of the function (4.4.9) for $t \in I \subset \mathbb{R}^+$, where $k_1 = 2(b_{11}v_1 + b_{12}v_2)/b_{12}$ and $k_2 = -2u_1$. Multiplying the function (4.4.9) by a parameter, if necessary, we see that k_1 and k_2 can be chosen freely. So applying Lemma 4.3.2(e) we conclude that $\mathcal{N}(S_r^0, S_r) = 1$.

The right system has a saddle with trace equal 0 if and only if $a = 0$. In this case $\xi_2^5 = 0$ and the function (4.4.9) becomes $g_5(t) = k_1$. So if $k_1 \neq 0$, that is $b_{11}v_1 \neq 0$, then there are no solutions for the equation $g_5(t) = 0$. If $k_1 = 0$, that is $b_{11}v_1 = 0$, then $g_5 = 0$, which implies that all the solutions of system (4.4.1) passing through $(0, y)$ for $Y_M < y < Y_m$ are periodic solutions, in other words there are no limit cycles. Hence we conclude that $\mathcal{N}(S_r^0, S_r^0) = 0$. □

4.4.3 Left system has a virtual or real center

In this case $v_1 \neq 0$, $b_{22} = -b_{11}$, $b_{11}^2 + b_{12}b_{21} < 0$ and the point $(-v_1, -v_2)$ is a singularity of center type.

The left system has a fold point $(0, \check{y})$ given by

$$\check{y} = -v_2 - \frac{b_{11}v_1}{b_{12}}.$$

which is visible if $v_1 > 0$, and invisible if $v_1 < 0$. In order to fix the clockwise orientation of the flow of system (4.4.1) we assume that $Y_1(-v_1, 1 - v_2) = b_{12} > 0$.

Let $\Gamma = 2\sqrt{-b_{11}^2 - b_{12}b_{21}}$. We define

$$t^*(y) = \frac{4}{\Gamma}\arctan\left(\frac{2\left(b_{11}v_1 + b_{12}(v_2 + y)\right)}{v_1\Gamma}\right).$$

If $v_1 < 0$, then $t^-(y) = t^*(y)$ for $y > \check{y}$ and $t_-(y) = t^*(y)$ for $y < \check{y}$. If $v_1 > 0$, then $t^-(y) = t^*(y) - 4\pi/\Gamma$ for $y > \check{y}$ and $t^-(y) = t^*(y) + 4\pi/\Gamma$ for $y < \check{y}$.

Proposition 4.4.12. The equalities $\mathcal{N}(C, F_v) = 1$, $\mathcal{N}(C, F_r) = 2$ and $\mathcal{N}(C, C_v) = \mathcal{N}(C, C_r) = 0$ hold.

Proof. In Corollary 4.2.1 these equalities have already been proved when the left system has a center in Σ. So we can take $v_1 \neq 0$.

To obtain $\mathcal{N}(C, F_v) = 1$ we follow the proof of Proposition 4.4.1 and then we compute the solutions of the function (4.4.2) for $y > Y_M = \max\{\check{y}, -a\, u_1\}$. To obtain $\mathcal{N}(C, F_r) = 2$ we follow the proof of Proposition 4.4.2 and then we compute the solutions of the function (4.4.3) for $y < Y_m = \min\{\check{y}, -a\, u_1\}$. In both cases the equations to be solved are equivalent to $k_1 + k_2\xi_2^4(t) = 0$, for $t \in (0, \pi)$ and $t \in (-\tau, -\pi)$, respectively. Here $k_1 = (b_{11}v_1 + b_{12}v_2)/b_{12}$ and $k_2 = -u_1$. So applying statements (d) and (d') of Lemma 4.3.2 we conclude that $\mathcal{N}(C, F_v) = 1$ and $\mathcal{N}(C, F_r) \leq 2$, respectively. Moreover, since $\mathcal{N}(C_b, F_r) = 2$, we actually have the equality $\mathcal{N}(C, F_r) = 2$. The equality $\mathcal{N}(C, C_v) = \mathcal{N}(C, C_r) = 0$ follows similarly to the proof of Proposition 4.4.7. It concludes the proof of this proposition. □

Proposition 4.4.13. The equalities $\mathcal{N}(C, F_b) = 1$ and $\mathcal{N}(C, C_b) = 0$ hold.

Proof. In Corollary 4.2.1 these equalities have already been proved when the left system has a center in Σ. So we can take $v_1 \neq 0$.

To obtain $\mathcal{N}(C, F_v) = 1$ we follow the proof of Proposition 4.4.3 and then we compute the solutions of the function (4.4.2) for $y > Y_M = \max\{\check{y}, 0\}$, which is equivalent to compute the zeros of the liner equation $k_1 + k_2 y = 0$. Here $k_1 = 2(b_{11}v_1 + b_{12}v_2)/b_{12}$ and $k_2 = (1 - e^{a\pi})$. The equalities $\mathcal{N}(C, F_b) = 1$ and $\mathcal{N}(C, C_b) = 0$ follows similarly to the proof of Proposition 4.4.1. It concludes the proof of this proposition. □

Proposition 4.4.14. The equalities $\mathcal{N}(C, N_v) = \mathcal{N}(C, S_r) = 1$ and $\mathcal{N}(C, S_r^0) = 0$ hold.

Proof. In Corollary 4.2.1 these equalities have already been proved when the left system has a center in Σ. So we can take $v_1 \neq 0$.

To prove the equality $\mathcal{N}(C, N_v) = 1$ we follow the proof of Proposition 4.4.4 and then we compute the solutions of the function (4.4.2) for $y > Y_M = \max\{\check{y}, -a\, u_1\}$. To prove the equality $\mathcal{N}(C, S_r) = 1$ we follow the proof of Proposition 4.4.6, then we compute the solutions of the function (4.4.2) for $Y_M < y < u_1$. In both cases the equations to be solved are equivalent to $k_1 + k_2\xi_2^5 = 0$, where $k_1 = 2(b_{11}v_1 + b_{12}v_2)/b_{12}$ and $k_2 = -2u_1$. From here, the proofs of the equalities $\mathcal{N}(C, N_v) = 1$ and $\mathcal{N}(C, S_r) = 1$ follows similarly to the proofs of the Propositions 4.4.9 and 4.4.11, respectively. The equality $\mathcal{N}(C, S_r^0) = 0$ follows similarly to the proof of Proposition 4.4.11. It concludes the proof of this proposition. □

Proposition 4.4.15. The equality $\mathcal{N}(C, iN_v) = 1$ holds.

Proof. In Corollary 4.2.1 this equality has already been proved when the left system has a center in Σ. So we can take $v_1 \neq 0$.

Following the proof of Proposition 4.4.5 we compute the solutions of the function (4.4.3) for $y < Y_m = \{\check{y}, -u_1\}$, which is equivalent to compute the zeros of the following equation $k_1 + k_2\xi_2^6 = 0$, where $k_1 = (b_{11}v_1 + b_{12}v_2)/b_{12}$ and $k_2 = -u_1$. So analogously to the proof of Proposition 4.4.10 we conclude that $\mathcal{N}(C, iN_v) = 1$. It concludes the proof of this proposition. □

Chapter 5

Shilnikov problem in Filippov dynamical systems

The main results of this chapter (Theorems M, N, and 5.4.1) are based on the paper [92].

5.1 Introduction to the Shilnikov problem

The study of discontinuous piecewise dynamical systems (DPDS) produces interesting and amazing mathematical challenges and plays an important part of so many applications in several branches of science (see, for instance, [106, 85, 112, 21] and the references therein). The present work focuses on the analysis of a typical phenomenon that occurs in this area which evidences a striking resemblance to Shilnikov homoclínic loop

Consider a smooth three dimensional vector field for which $p \in \mathbb{R}^3$ is a hyperbolic saddle–focus equilibrium admitting a two dimensional stable (resp. unstable) manifold and an one dimensional unstable (resp. stable) manifold. In the classical theory of dynamical systems a *Shilnikov homoclinic orbit* Γ of this vector field is a trajectory connecting p to itself, bi–asymptotically. Under suitable genericity conditions this connection is a codimension one scenario, and its unfolding depends on the *saddle quantity* $\sigma = \lambda^u + \mathrm{Re}(\lambda^s_{1,2})$ (resp. $\sigma = \lambda^s + \mathrm{Re}(\lambda^u_{1,2})$), where $\lambda^u > 0$ (resp. $\lambda^s > 0$) and $\lambda^s_{1,2} \in \mathbb{C}$ (resp. $\lambda^u_{1,2} \in \mathbb{C}$) are the eigenvalues of p, clealry $\mathrm{Re}(\lambda^s_{1,2}) < 0$ (resp. $\mathrm{Re}(\lambda^u_{1,2}) > 0$). In this case, when $\sigma > 0$ (resp. $\sigma < 0$) a chaotic behaviour occurs. We point out that chaotic behaviour is mostly understood as the existence of strange attractors. These attractors appear when the Shilnikov homoclinic orbit is unfolded (see, for instance, [95, 46]). It is also proved that there exists a compact hyperbolic invariant set $\mathcal{S}$ which contains countable infinitely many periodic orbits of saddle type in any sufficiently small neighbourhood of Γ(see, for instance, [100, 101, 110, 111, 102, 114, 1]).

In the theory of discontinuous piecewise dynamical systems the notion of solutions of a discontinuous differential equation is stated by the Filippov's convention (see [34]). In this context there exist some special points that must be distinguished and treated as typical singularities, one of those is a pseudo–equilibrium, which we shall introduce it formally later on this chapter. This kind of singularity gives rise to the definition of the sliding homoclinic orbit, that is a trajectory,

in the Filippov sense, connecting a pseudo–equilibrium to itself in an infinity time at least by one side (future or past). Particularly a *sliding Shilnikov orbit* is a sliding homoclinic orbit connecting a hyperbolic pseudo saddle–focus pseudo to it self.

A sliding Shilnikov orbit is an intrinsic phenomenon of DPDS. However, for each piecewise smooth system Z^0, we conjecture (see Conjecture 1) the existence of an one parameter family Z^δ of smooth systems approaching continuously to Z^0 ($\mathcal{C}^0 \times \mathcal{C}^0$ topology) such that, for each $\delta > 0$ small enough, Z^δ admits an ordinary Shilnikov homoclinic orbit with chaotic behaviour, that is $\lambda^s < 0$, $\mathrm{Re}(\lambda^u_{1,2}) > 0$, and $\sigma < 0$.

The main goal of this chapter is to produce versions of the Shilnikov's Theorems for systems having a sliding Shilnikov orbit, and also to track the above conjecture. This conjecture is formalized in Subsection 5.2 (see Conjecture 1), which also contains some basic notions and definitions. Our main results can be summarized as following. In Section 5.3, we prove that, in general, a sliding Shilnikov orbit is a co-dimension 1 phenomenon (see Theorem M), and that arbitrarily close to a sliding Shilnikov orbit there exist countable infinitely many sliding periodic orbits (see Theorem N). Furthermore, in Section 5.4, we provide a family $Z_{\alpha,\beta}$ of discontinuous piecewise linear vector fields as a prototype of systems having a sliding Shilnikov orbit (see Theorem 5.4.1). Finally, using techniques of regularization and singular perturbation, we illustrate, in Section 5.5, the Conjecture 1 for the the family $Z_{\alpha,\beta}$ (see Theorem O).

5.2 Sliding Shilnikov orbit

In this subsection the basic theory of non–smooth dynamical systems is given in order to define the sliding Shilnikov orbits and to state our main results.

Let U be an open bounded subset of $\mathbb{R}^3$. We denote by $\mathcal{C}^r(K, \mathbb{R}^3)$, $K = \overline{U}$, the set of all $\mathcal{C}^r$ vector fields $X : K \to \mathbb{R}^3$ endowed with the topology induced by the norm $||X||_r = \sup\{||D^i X(x)||: x \in K,\ i \in \{0, 1, \ldots, r\}\}$. Here D^r is the identity operator for $r = 0$, and the rth–derivative for $r > 0$. In order to keep the uniqueness property of the trajectories of vector fields in $\mathcal{C}^0(K, \mathbb{R}^3)$ we shall assume, additionally, that these vector fields are Lipschitz.

Given $h : K \to \mathbb{R}$ a differentiable function having 0 as a regular value we denote by $\Omega^r_h(K, \mathbb{R}^3)$ the space of piecewise vector fields

$$Z(x) = \begin{cases} X(x), & \text{if} \quad h(x) > 0, \\ \\ Y(x), & \text{if} \quad h(x) > 0, \end{cases} \tag{5.2.1}$$

with $X, Y \in \mathcal{C}^r(K, \mathbb{R}^3)$. As usual, system (5.2.1) is denoted by $Z = (X, Y)$ and the switching surface $h^{-1}(0)$ by Σ . So we are taking $\Omega^r_h(K, \mathbb{R}^3) = \mathcal{C}^r(K, \mathbb{R}^3) \times \mathcal{C}^r(K, \mathbb{R}^3)$ endowed with the product topology. When the context is clear we shall refer the sets $\Omega^r_h(K, \mathbb{R}^3)$ and $\mathcal{C}^r(K, \mathbb{R}^3)$ only by Ω^r and $\mathcal{C}^r$, respectively. It is worth to say that the space $\mathcal{C}^r$ can be identified as the diagonal of Ω^r, that is $X \approx (X, X)$ which is a $\mathcal{C}^r$ vector space in Ω^r.

The points on Σ where both vectors fields X and Y simultaneously point outward or inward from Σ define, respectively, the *escaping* Σ^e or *sliding* Σ^s regions, and the interior of its complement in Σ defines the *crossing region* Σ^c. The complementary of the union of those regions constitute by the *tangency* points between X or Y with Σ (see Figure 5.1).

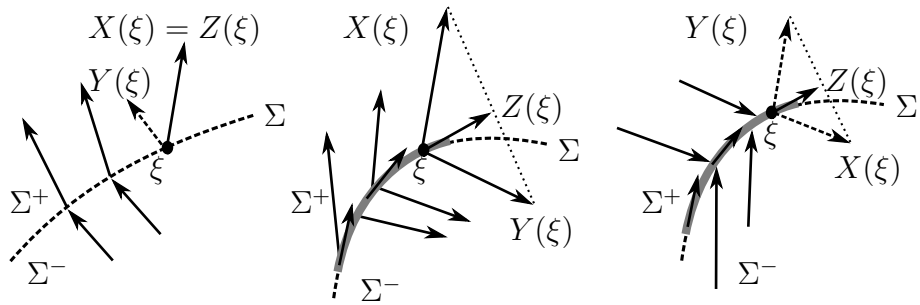

Figure 5.1: Definition of the vector field on Σ following Filippov's convention in the sewing, escaping, and sliding regions, respectively. This figure has been gotten from [22]

The points in Σ^c satisfy $Xh(\xi) \cdot Yh(\xi) > 0$, where Xh denote the derivative of the function h in the direction of the vector X, i.e. $Xh(\xi) = \langle \nabla h(\xi), X(\xi) \rangle$. The points in Σ^s (resp. Σ^e) satisfy $Xh(\xi) < 0$ and $Yh(\xi) > 0$ (resp. $Xh(\xi) > 0$ and $Yh(\xi) < 0$). Finally, the tangency points of X (resp. Y) satisfy $Xh(\xi) = 0$ (resp. $Yh(\xi) = 0$).

Now we define the *sliding vector field*

$$\widetilde{Z}(\xi) = \frac{Yh(\xi)X(\xi) - Xh(\xi)Y(\xi)}{Yh(\xi) - Xh(\xi)}. \tag{5.2.2}$$

The local trajectory $\varphi_Z(t,p)$ of the discontinuous piecewise differential system $\dot{x} = Z(x)$ passing through a point $p \in \mathbb{R}^3$ is given by the Filippov convention (see [32, 41]). Here $0 \in I_p \subset \mathbb{R}$ denotes the maximum interval of definition of $\varphi_Z(t,p)$, and φ_W denotes the flow of a vector field W. The Filippov convention is resumed as following:

(i) for $p \in \mathbb{R}^3$ such that $h(p) > 0$ (resp. $h(p) < 0$) and taking the origin of time at p, the trajectory is defined as $\varphi_Z(t,p) = \varphi_X(t,p)$ (resp. $\varphi_Z(t,p) = \varphi_Y(t,p)$) for $t \in I_p$.

(ii) for $p \in \Sigma^c$ such that $(Xh)(p), (Yh)(p) > 0$ and taking the origin of time at p, the trajectory is defined as $\varphi_Z(t,p) = \varphi_Y(t,p)$ for $t \in I_p \cap \{t < 0\}$ and $\varphi_Z(t,p) = \varphi_X(t,p)$ for $t \in I_p \cap \{t > 0\}$. For the case $(Xh)(p), (Yh)(p) < 0$ the definition is the same reversing time;

(iii) for $p \in \Sigma^s$ and taking the origin of time at p, the trajectory is defined as $\varphi_Z(t,p) = \varphi_{\widetilde{Z}}(t,p)$ for $t \in I_p \cap \{t \geq 0\}$ and $\varphi_Z(t,p)$ is either $\varphi_X(t,p)$ or $\varphi_Y(t,p)$ or $\varphi_{\widetilde{Z}}(t,p)$ for $t \in I_p \cap \{t \leq 0\}$. For the case $p \in \Sigma^e$ the definition is the same reversing time;

(iv) For $p \in \partial\Sigma^c \cup \partial\Sigma^s \cup \partial\Sigma^e$ such that the definitions of trajectories for points in Σ in both sides of p can be extended to p and coincide, the orbit through p is this limiting orbit. We will call these points *regular tangency* points.

(v) for any other point (*singular tangency* points) $\varphi_Z(t,p) = p$ for all $t \in \mathbb{R}$;

Remark 5.2.1. A tangency point $\xi \in \Sigma$ is called a *visible fold* of X (resp. Y) if $(X)^2h(\xi) > 0$ (resp. $(Y)^2h(\xi) < 0$). Analogously, reversing the inequalities, we define a *invisible fold*. Suppose that p is a visible fold of X such that $Yh(p) > 0$, then p is an example of a regular tangency point. In this case, taking the origin of time at p, the trajectory passing through p is defined as $\varphi_Z(t,p) = \varphi_1(t,p)$ for $t \in I_p \cap \{t \leq 0\}$ and $\varphi_Z(t,p) = \varphi_2(t,p)$ for $t \in I_p \cap \{t \geq 0\}$, where each φ_1, φ_2 is either φ_X or φ_Y or $\varphi_{\widetilde{Z}}$.

A *pseudo–equilibrium* is a critical point $\xi^* \in \Sigma^{s,e}$ of the sliding vector field, i.e. $\widetilde{Z}(\xi^*) = 0$. When ξ^* is a hyperbolic critical point of $\widetilde{Z}$, it is called a *hyperbolic pseudo–equilibrium*. Particularly if $\xi^* \in \Sigma^s$ (resp. $\xi^* \in \Sigma^e$) is an unstable (resp. stable) hyperbolic focus of $\widetilde{Z}$ then we call ξ^* a *hyperbolic saddle–focus pseudo–equilibrium* or just *hyperbolic pseudo saddle–focus*.

In order to study the orbits of the sliding vector field it is convenient to define the $(\mathcal{C}^r)$ normalized sliding vector field

$$\widehat{Z}(\xi) = (Yh(\xi) - Xh(\xi))\widetilde{Z}(\xi) = Yh(\xi)X(\xi) - Xh(\xi)Y(\xi), \tag{5.2.3}$$

which has the same phase portrait of $\widetilde{Z}$ reversing the direction of the flow in the escaping region. Indeed, system (5.2.3) is obtained from (5.2.2) through a time rescaling multiplying (5.2.2) by the function $Yh(\xi) - Xh(\xi)$ which is positive (resp. negative) for $\xi \in \Sigma^s$ (resp. $\xi \in \Sigma^e$).

Definition 5.2.1. Let $Z = (X,Y)$ be a piecewise continuous vector field having a hyperbolic pseudo saddle–focus $p \in \Sigma^s$ (resp. $p \in \Sigma^e$). We assume that there exists a tangential point $q \in \partial\Sigma^s$ (resp. $q \in \partial\Sigma^e$) which is a visible fold point of the vector field X such that

(j) the orbit passing through q following the sliding vector field $\widetilde{Z}$ converges to p backward in time (resp. forward in time);

(jj) the orbit starting at q and following the vector field X spends a time $t_0 > 0$ (resp. $t_0 < 0$) to reach p.

So through p and q a sliding loop Γ is easily characterized. We call Γ a *sliding Shilnikov orbit* (see Figures 5.2 for $\alpha = 0$, 5.3, and 5.5).

Remark 5.2.2. Given $Z = (X,Y) \in \Omega^r$ it is worth to say that if $p \in \partial\Sigma^{e,s}$ is a fold–regular point of Z, that is p is a fold of X (resp. of Y) such that $Yh(p) \neq 0$ (resp. $Xh(p) \neq 0$), then the sliding vector field $\widetilde{Z}$ given in (5.2.2) is transverse to $\partial\Sigma^{s,e}$ at p. A proof of this fact can be found in [104].

In the sequel we formalize the conjecture made in the introduction.

Conjecture 1. Assume that $Z^0 \in \Omega^r$ admits a sliding Shilnikov orbit Γ^0. So, for all $0 \leq s \leq r$, there exists an one parameter family $Z^\delta \in \mathcal{C}^s$ approaching continuously to Z^0 (Ω^0 topology), such that, for each $\delta > 0$ small enough, Z^δ admits an ordinary Shilnikov homoclinic orbit Γ^δ, bi–asymptotic to a saddle–focus p_δ, $\lambda^s < 0$, $\mathrm{Re}(\lambda^u_{1,2}) > 0$, and $\sigma < 0$. Here $\lambda^s \in \mathbb{R}$ and $\lambda^u_{1,2} \in \mathbb{C}$ are the eigenvalues of the singularity p_δ, and $\sigma = \lambda^s + \mathrm{Re}(\lambda^u_{1,2})$ is the saddle quantity.

5.3 Main results on sliding Shilnikov orbits

In the theory of ordinary differential equations a Shilnikov homoclinic orbit of a 3D vector field is a co–dimension 1 phenomenon in $\mathcal{C}^r$. Our first main result shows that the sliding Shilnikov is also a co-dimension 1 phenomenon in Ω^r.

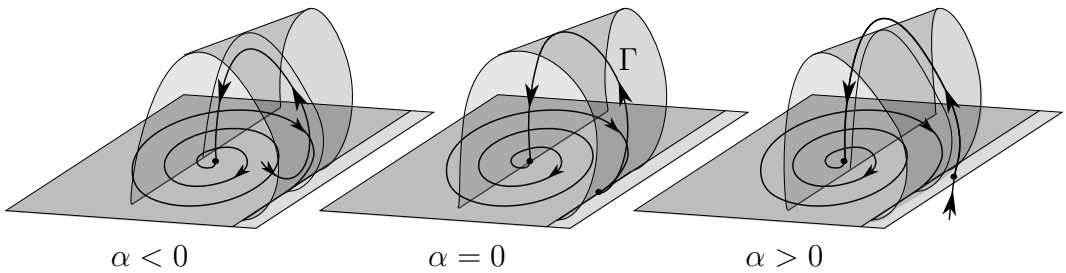

Figure 5.2: Unfolding $Z_\alpha = (X_\alpha, Y_\alpha)$ of a sliding Shilnikov orbit Γ in $Z_0 = (X_0, Y_0) \in \Omega^r$.

Theorem M. Assume that $Z_0 = (X_0, Y_0) \in \Omega^r$ (with $r \geq 1$) has a sliding Shilnikov orbit Γ_0 and let $W \subset \Omega^r$ be a small neighbourhood of Z_0. Then there exists a $\mathcal{C}^1$ function $g : W \to \mathbb{R}$ having 0 as a regular value such that $Z \in W$ has a sliding Shilnikov orbit Γ if and only if $g(Z) = 0$.

Proof. For simplicity we assume that $h(x, y, z) = z$, that is $\Sigma = \{z = 0\}$. Let $Z_0 = (X_0, Y_0) \in \Omega^r$ having a sliding Silnikov orbit Γ_0. We assume that Γ_0 is a sliding loop through a pseudo–equilibrium of a focus–saddle type $p_0 \in \Sigma^s$ and a tangential point q_0 which is a visible fold point for the vector field X_0. The case when $p_0 \in \Sigma^e$ would follow similarly.

Let $\gamma_0 = B_r(q_0) \cap \partial\Sigma^s$. Here $B_a(q_0) \subset \Sigma$ is the planar ball with center at q_0 and radius r. Of course γ_0 is a branch of the fold line contained in the boundary of the sliding region $\partial\Sigma^s$. We remark that in the sliding region the orbit of the sliding vector field is always transversal to the fold line. In addition, the orbits of $\widetilde{Z}_0$ through the points of γ_0 converge to p_0 in backward time. The forward saturation of γ_0 through the flow of X_0 meets Σ in a curve μ_0 in a finite time. Moreover $p_0 \in \mu_0$.

Let W be a small neighborhood of $Z_0 \in \Omega^r$. So associated to each $Z \in W$ we can define similar objects: p_Z, γ_Z and μ_Z. Clearly Z will have a sliding Shilnikov orbit if and only if $p_Z \in \mu_Z$.

We may assume that, in suitable local coordinate system (x, y) around $p_0 = (0, 0) \in \Sigma^s$, μ_0 is the graph of a function $y = r(x)$. So for $Z \in W$, μ_Z is given by $y = k_Z(x) = a_0 + a_1 x + \mathcal{O}_2(x)$ with a_0, a_1 small parameters.

Let $p_Z = (x_Z, y_Z)$ and define $g : W \to \mathbb{R}$ by $g(Z) = k_Z(x_Z) - y_Z$. Of course g is a $\mathcal{C}^1$ function and $g(Z_0) = 0$. We prove now that 0 is a regular value of g, that is the linear map $g'(Z_0) : \Omega^r \to \mathbb{R}$ is surjective.

First of all we note that, for $Z^* \in W$, $g(Z^*) = 0$ if and only if $p_{Z^*} \in \mu_{Z^*}$, equivalently, Z^* admits a sliding Shilnikov orbit. Since

$$g'(Z^*) \cdot V = \frac{d}{dv} g(Z(v))\Big|_{v=0} = \lim_{v \to 0} \frac{g(Z(v)) - g(Z^*)}{v}$$

for any curve $Z(v) \in \Omega^r$ such that $Z(0) = Z^*$ and $Z'(0) = V \in \Omega^r$, we can take $Z(v)$ in such a way that $p_{Z(v)} = (0,0)$ and $k_{Z(v)}(x) = v$ (constant). Hence $g(Z(v)) = v$ and $g'(Z^*) \cdot V = 1$, which implies that $g'(Z^*)$ is surjective for every $Z^* \in g^{-1}(0)$. It concludes the proof of this theorem. $\square$

Our second main result is a version of Shilnikov's theorem for sliding Shilnikov orbits.

Theorem N. Assume that $Z_0 = (X_0, Y_0) \in \Omega^r$ (with $r \geq 0$) has a sliding Shilnikov orbit Γ_0 and let $Z_\alpha = (X_\alpha, Y_\alpha) \in \Omega^r$ be an unfolding of Z_0 with respect to Γ_0. Then the following statements hold:

- (a) for $\alpha = 0$ every neighbourhood $G \subset \mathbb{R}^3$ of Γ_0 contains countable infinitely many sliding periodic orbits of Z_0;
- (b) for every $|\alpha| \neq 0$ sufficiently small there exists a neighbourhood $G_\alpha \subset \mathbb{R}^3$ of Γ_0 containing a finite number $N(\alpha) > 0$ of sliding periodic orbits of Z_α. Moreover $N(\alpha) \to \infty$ when $\alpha \to 0$;
- (c) for every neighbourhood $G \subset \mathbb{R}^3$ of Γ_0 there exists $|\alpha_0| \neq 0$ sufficiently small such that G contains a finite number $N_G(\alpha_0) > 0$ of sliding periodic orbits of Z_α. Moreover $N_G(\alpha) \to \infty$ when $\alpha \to 0$.

Proof. We assume that Γ_0 is a loop through $p_0 \in \Sigma^s$ and $q_0 \in \partial\Sigma^s$. The case $p_0 \in \Sigma^e$ and $q_0 \in \partial\Sigma^e$ would follow analogously.

To prove statement (a) let $\gamma_r = \overline{B_r(q_0) \cap \partial\Sigma^s}$ and let S_r be the backward saturation of γ_r through the flow of the sliding vector field $\widetilde{Z}$. The forward saturation of γ_r through the flow of X meets Σ in a curve μ_r in a finite time. So

$$S_r \cap \mu_r = \bigcup_{i=1}^{\infty} I_i,$$

where $I_i \cap I_j = \emptyset$ if $i \neq j$. The sequence of compact sets $(I_i)_{i=1}^{\infty}$ can be taken such that $I_i \to \{p_0\}$ (see Figure 5.3).

For each $i = 1, 2, \ldots$, we define J_i as the intersection between the backward saturation of I_i through the flow of X with the curve γ_r. Clearly $J_i \cap J_j = \emptyset$ if $i \neq j$ and $J_i \to \{q_0\}$.

For $\xi \in \Sigma^s$ and $z \in \mathbb{R}^3$ let $\varphi^s(t, \xi)$ and $\varphi^X(t, z)$ be the flows of the sliding vector field $\widetilde{Z}$ and X, respectively.

In what follows we define the applications $\psi_i : J_i \to J_i$. For $\xi \in J_i$ there exists $t_i^s(\xi) < 0$ such that $\xi_i(\xi) = \varphi^s(t_i^s(\xi), \xi) \in I_i$; and there exists $t_i^X(\xi) < 0$ such that $\varphi^X(t_i^X(\xi), \xi_i(\xi)) \in J_i$. So we take $\psi_i(\xi) = \varphi^X(t_i^X(\xi), \xi_i(\xi))$. Note that ψ_i is a composition of $\mathcal{C}^r$ function, being then itself a $\mathcal{C}^r$ function.

It is easy to see that each fixed point of ψ_i corresponds to a sliding periodic orbit of Z (see Figure 5.4). Now, for each $i = 1, 2, \ldots$, ψ_i is a continuous function from a compact interval J_i to itself. So applying the *Brouwer fixed–point Theorem* we obtain a sequence $(q_i)_{i=1}^{\infty}$ such that $q_i \in J_i$ and $\psi_i(q_i) = q_i$. Hence we conclude that there exists a sequence of sliding periodic orbits of Z passing through q_i. The proof of statement (a) follows just by observing that $q_i \to q_0$.

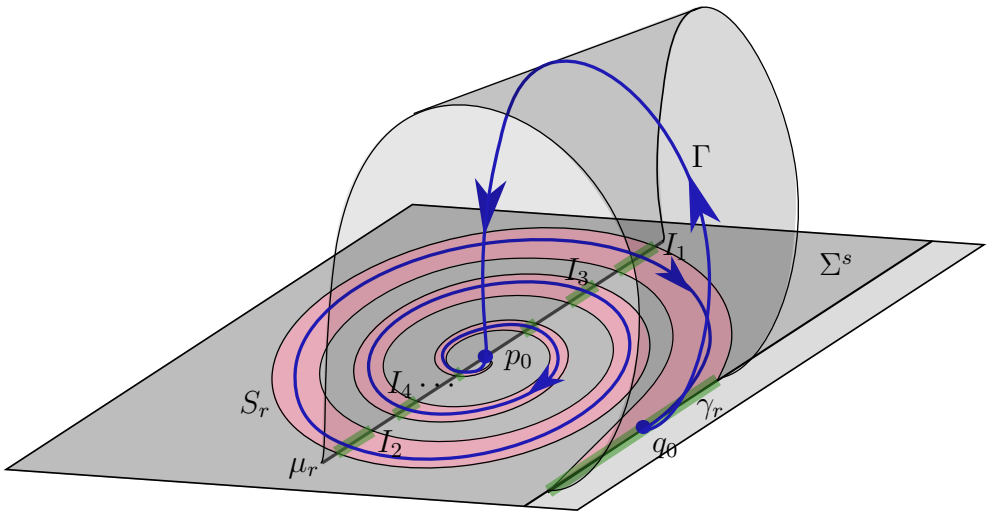

Figure 5.3: A schematic representation of a Shilnikov sliding orbit Γ.

In what follows we prove the statements (b) and (c). Firstly for $|\alpha| \neq 0$ sufficiently small we build elements γ_r^α, S_r^α and μ_r^α similarly to the elements γ_r, S_r and μ_r, respectively.

Since the new pseudo–equilibrium p_α is not in μ_r^α, the intersection $S_r^\alpha \cap \mu_r^\alpha$ has only a finite number $N(\alpha)$ of disjoint sets I_i. Furthermore the number of disjoint sets $N(\alpha)$ in this intersection goes to infinity when α goes to 0, and they converges to $\{p\}$ when $i \to \infty$. From here the proof of statement (b) follows analogously to the proof of statement (a).

For a fixed neighbourhood G of Γ_0 there exists $|\alpha| \neq 0$ sufficiently small such that $p_\alpha \in G \cap \Sigma$, because $p_\alpha \to 0$ when $\alpha \to 0$, so that $\mu_\alpha \subset G \cap \Sigma$. From here the proof of statement (c) follows analogously to the proof of statement (b). $\square$

5.4 A piecewise linear model

In this section we present a 2–parameter family of discontinuous piecewise linear dynamical system $Z_{\alpha,\beta}$ admitting a sliding Shilnikov orbit $\Gamma_{\alpha,\beta}$.

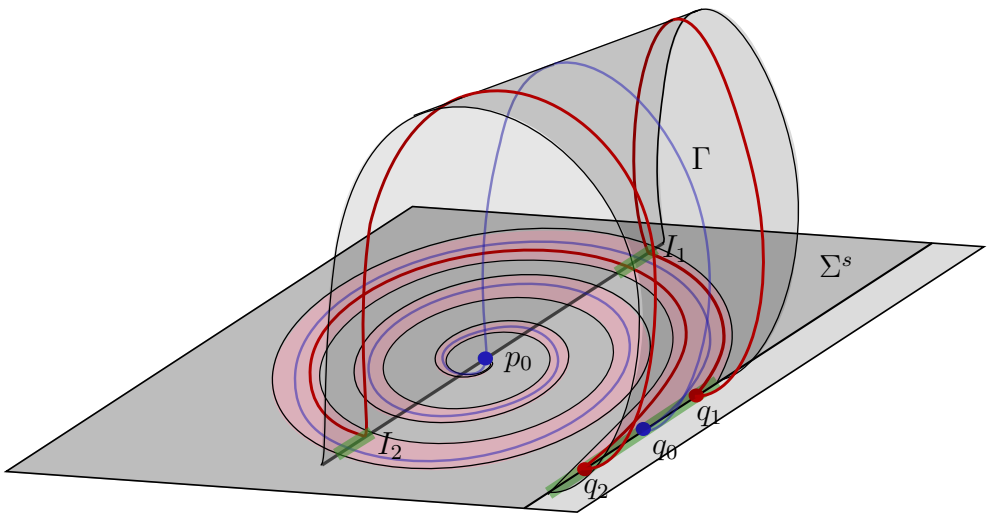

Figure 5.4: Periodic orbits close to a Shilnikov sliding orbit Γ.

For $\alpha > 0$ and $\beta > 0$, consider the following discontinuous piecewise linear vector field.

$$Z_{\alpha,\beta}(x,y,z) = \begin{cases} X_{\alpha,\beta}(x,y,z) = \begin{pmatrix} -\alpha \\ x-\beta \\ y - \dfrac{3\beta^2}{8\alpha} \end{pmatrix} & \text{if} \quad z > 0, \\ Y_{\alpha,\beta}(x,y,z) = \begin{pmatrix} \alpha \\ \dfrac{3\alpha}{\beta} y + \beta \\ \dfrac{3\beta^2}{8\alpha} \end{pmatrix} & \text{if} \quad z < 0. \end{cases} \tag{5.4.1}$$

The plane $\Sigma = \{z = 0\}$ is a switching manifold for system (5.4.1), which can be decomposed as $\Sigma = \overline{\Sigma^c} \cup \overline{\Sigma^s} \cup \overline{\Sigma^e}$ being

$$\Sigma^c = \left\{(x,y,0) : \, y > \frac{3\beta^2}{8\alpha}\right\}, \quad \Sigma^s = \left\{(x,y,0) : \, y < \frac{3\beta^2}{8\alpha}\right\} \quad \text{and} \quad \Sigma^e = \emptyset.$$

Thus $p = (0,0,0) \in \Sigma^s$. Moreover $c = (\beta, 3\beta^2/(8\alpha), 0)$ is a cuspid-regular singularity for system (5.4.1) (see Figure 5.5).

Proposition 5.4.1. For every positive real numbers α and β the following statements hold:

(a) the origin $p = (0,0,0)$ is a hyperbolic pseudo saddle–focus of system $Z_{\alpha,\beta}$ (5.4.1) in such way that its projection onto Σ is an unstable hyperbolic focus of the sliding vector field $\widetilde{Z}_{\alpha,\beta}$ (5.2.2) associated with (5.4.1);

(b) there exists a sliding Shilnikov orbit $\Gamma_{\alpha,\beta}$, connecting $p = (0,0,0)$ to itself, passing through the fold–regular point $q = (3\beta/2, 3\beta^2/(8\alpha))$ (see Figure 5.5).

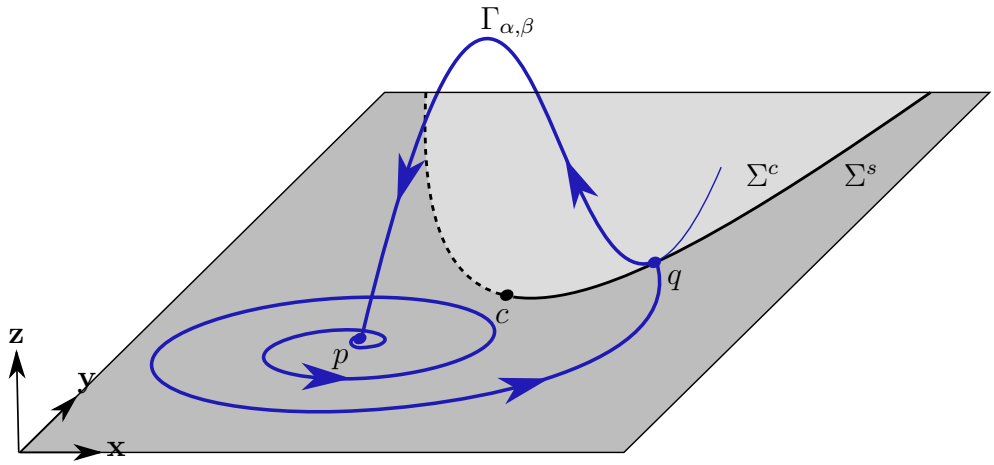

Figure 5.5: A representation of the Shilnikov sliding orbit of system (5.4.1). Here, in order to make easy the visualization, we have used the change of variables $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (x, y - x^2, z)$ to bend the y–axis.

Proof. We compute the sliding vector field and the normalized sliding vector field of (5.4.1) as

$$\begin{aligned} \widetilde{Z}_{\alpha,\beta}(x,y) &= \left(\frac{4\alpha^2 y}{4\alpha y - 3\beta^2}\,,\ \frac{3\beta^3 x + \alpha\beta^2 y - 24\alpha^2 y^2}{6\beta^3 - 8\alpha\beta y} \right) \quad \text{and} \\ \widehat{Z}_{\alpha,\beta}(x,y) &= \left(-\alpha y\,,\ \frac{3\beta^2}{8\alpha}x + \frac{\beta}{8}y - \frac{3\alpha}{\beta}y^2 \right), \end{aligned} \tag{5.4.2}$$

respectively. It is easy to see that $(0,0) \in \Sigma^s$ is a hyperbolic focus of $\widetilde{Z}_{\alpha,\beta}$. Indeed, their eigenvalues are given by

$$\lambda^{\pm} = \frac{\alpha}{12\beta} \pm i\frac{\sqrt{95}\alpha}{12\beta}. \tag{5.4.3}$$

It implies that the origin is a hyperbolic pseudo saddle–focus of vector field (5.4.1). Moreover, since $\mathrm{Re}(\lambda^{\pm}) > 0$ then $(0,0)$ is an unstable hyperbolic focus of the (normalized) sliding vector field (5.4.2).

After a change of variables and a time rescaling expressed by

$$(x,y) = \left(\frac{3\beta}{2}u, \frac{3\beta^2}{8\alpha}v \right), \quad t = -\frac{4}{\beta}\tau, \tag{5.4.4}$$

respectively, the normalized sliding vector field $\widehat{Z}_{\alpha,\beta}$ becomes

$$\overline{Z} = \left(v, -6u - \frac{1}{2}v + \frac{9}{2}v^2\right). \tag{5.4.5}$$

We note that the time rescaling (5.4.4) reverses the direction of the flow of (5.4.2). The fold line $\partial\Sigma^s$ is given now, in (u, v) coordinates, by $\ell = \{(u, 1) : u \in \mathbb{R}\}$.

We claim that the orbit of system (5.4.5) starting at the point $(1, 1) \in \ell$ is attracted to the focus equilibrium $(0, 0)$ without touching the line ℓ. Clearly, going back through the transformation (5.4.4), this claim implies that the orbit of system (5.4.2) starting at the point $q = (3\beta/2, 3\beta^2/(8\alpha)) \in \partial\Sigma^s$ is attracted, now backward in time, to the focus $(0, 0)$ without touching the fold line $\partial\Sigma^s$.

To prove the claim we shall construct a compact region $\mathcal{R}$ in the u, v–plane that is positively invariant through the flow of the vector field (5.4.5). To do that, let $m(y) = -13/108 + 9y^2/13 + 54y^3/169$, and take the curves

$$\mathcal{C}_1 = \{(u, 1) : m(1) \leq u \leq 1\},$$

$$\mathcal{C}_2 = \{(u, -2u + 3) : 1 \leq u \leq 3/2\},$$

$$\mathcal{C}_3 = \{(3/2, v) : -91/72 < v < 0\},$$

$$\mathcal{C}_4 = \{(u, -91/72) : m(-91/72) \leq u \leq 3/2\},$$

$$\mathcal{C}_5 = \{(m(v), v) : -91/71 \leq v \leq 1\}.$$

We define $\mathcal{R}$ as being the compact region delimited by the curves $\mathcal{C}_i$ for $i = 1, 2, \ldots, 5$ (see Figure 5.6). After some standard computations we conclude that $\mathcal{R}$ is positively invariant through the flow of (5.4.5). Furthermore, the vector field (5.4.5) has at most one limit cycle (see Theorem A of [25]), which is hyperbolic. So from the positive invariance of $\mathcal{R}$, from the stability of the equilibrium $(0, 0)$, and from the uniqueness of a possible limit cycle we conclude that, if this limit cycles exists, then it cannot be inside $\mathcal{R}$. Applying Poincaré–Bendixson theorem we conclude that the stable focus of (5.4.5) attracts the orbits, forward in time, of all points in $\mathcal{R}$ without touching the line ℓ. The claim follows by noting that $(1, 1) \in \mathcal{R}$.

On the other hand the vector field $X_{\alpha,\beta}$ is also linear. Thus its orbit starting at q is easily computed as

$$\varphi^+(t, q) = \left(-\alpha\, t + \frac{3\beta}{2}, \frac{(3\beta - 2\alpha\, t)(\beta + 2\alpha\, t)}{8\alpha}, \frac{(3\beta - 2\alpha\, t)t^2}{12}\right).$$

So for $t^+ = 3\beta/(2\alpha) > 0$ we have that $\varphi^+(t^+, q) = p$. It implies that there exists a sliding Shilnikov orbit $\Gamma_{\alpha,\beta}$ of $Z_{\alpha,\beta}$ connecting p to itself passing through q. $\square$

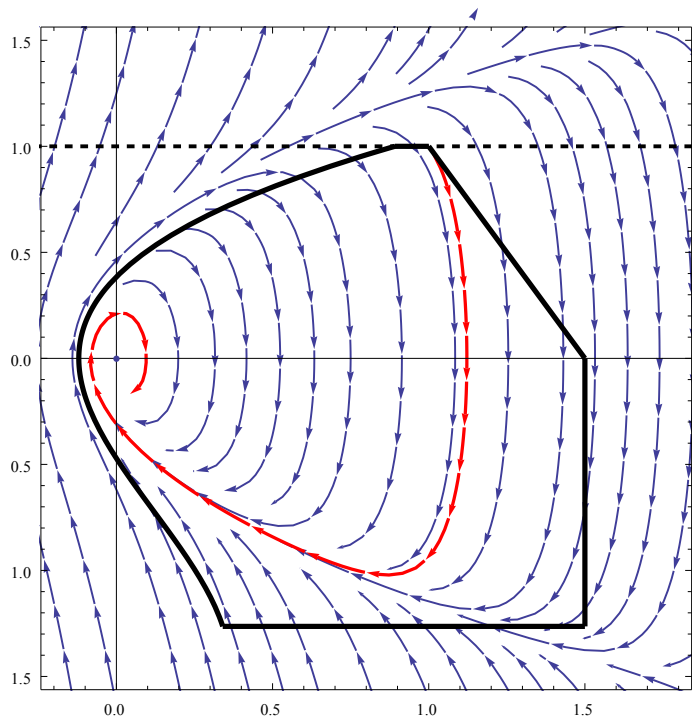

Figure 5.6: The dashed bold line represents the line ℓ. The continuous bold line delimits the compact region $\mathcal{R}$ which is positively invariant through the flow of (5.4.5). The red trajectory is the orbit starting at $(1,1)$ being attracted to the focus $(0,0)$.

5.5 Regularization

Shilnikov [100, 101] showed that any smooth 3–dimensional vector field possessing a hyperbolic saddle–focus $p \in \mathbb{R}^3$ with a 2–dimensional stable (resp. unstable) manifold and an 1–dimensional unstable (resp. stable) manifold admits a chaotic behaviour always when its saddle quantity $\sigma = \lambda^u + \mathrm{Re}(\lambda^s_{1,2})$ (resp. $\sigma = \lambda^s + \mathrm{Re}(\lambda^u_{1,2})$) is positive (negative). Tresser extended the Shilnikov's results for $\mathcal{C}^{1,1}$ vector fields [110] and for Lipschitz continuous piecewise $\mathcal{C}^{1,1}$ vector fields [111] when the Shilnikov homoclinic orbit is transversal to the sets of non–differentiability.

As an immediate consequence of the main result of this section we shall obtain that, for each positive real numbers α and β, every neighbourhood $\mathcal{U} \subset \Omega^0$ of the piecewise linear model $Z_{\alpha,\beta}(x,y,z)$, built in the previous section, contains a continuous piecewise quadratic vector field possessing a Shilnikov homoclinic orbit. Moreover this vector field presents a chaotic behaviour, and any neighbourhood of its Shilnikov homoclinic orbit contains infinitely many periodic orbits.

Theorem O. For each positive real numbers α and β, and for $\delta > 0$ small enough, there exists a family $W^\delta_{\alpha,\beta} \subset \Omega^0$ of continuous piecewise smooth vector fields δ–close to $Z_{\alpha,\beta}$ (Ω^0 topology) having the following properties for $\delta > 0$ small enough.

(a) The origin is a hyperbolic saddle–focus singularity of $W^\delta_{\alpha,\beta}$ admitting an 1–dimensional stable manifold $\mathcal{W}^s_\delta$ and a 2–dimensional unstable manifold $\mathcal{W}^u_\delta$;

(*b*) The stable and unstable manifolds intersect each other in a Shilnikov homoclinic orbit $\Gamma^{\delta}_{\alpha,\beta} = \mathcal{W}^s_\delta \cap \mathcal{W}^u_\delta$, which is δ–close to $\Gamma_{\alpha,\beta}$.

(*c*) The saddle quantity σ of the origin is negative. So any neighbourhood of $\Gamma^{\delta}_{\alpha,\beta}$ contains infinitely many periodic orbits for every $\delta > 0$ sufficiently small.

Before proving Theorem O we describe the *regularization process*, which is the main tool we shall use in its proof. Roughly speaking, a regularization of a discontinuous system $Z = (X, Y)$ is a one–parameter family Z^δ of continuous vector fields such that Z^δ converges (Ω^0 topology) to the discontinuous system when $\delta \to 0$. The regularized system Z^δ represents a class of continuous functions approximated by Z as $\delta \to 0$.

The Sotomayor-Teixeira method of regularization [103] takes

$$\begin{aligned} &Z^\delta(x,y) = W^\delta(x,y) = \frac{1+\phi_\delta(h(x,y))}{2}X(x,y) + \frac{1-\phi_\delta(h(x,y))}{2}Y(x,y), \quad \text{being} \\ &\phi_\delta(h) := \phi(h/\delta), \end{aligned} \tag{5.5.1}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous function which is $\mathcal{C}^1$ for $s \in (-1, 1)$ such that $\phi(s) = \text{sign}(s)$ for $|s| \geq 1$, and $\phi'(s) > 0$ for $s \in (-1, 1)$. We call ϕ a *monotonic transition function* and $Z^\delta(x)$ the *ϕ–regularization* of Z.

We point out that the Sotomayor-Teixeira regularization is not the unique method to regularize a vector field $Z = (X, Y)$. Indeed, let $F : K \times [0, \delta_0] \to \mathbb{R}^n$ be a continuous function such that $F(x, y, 0) = 0$, then $Z^\delta(x, y) = W^\delta(x, y) + F(x, y, \delta)$ is also a regularization of Z, where W^δ is given by (5.5.1).

Proof of Theorem O. Let $\phi : \mathbb{R} \to \mathbb{R}$ be the following monotonic transition function

$$\phi(u) = \begin{cases} 1 & \text{if} \quad u > 1, \\ u & \text{if} \quad -1 < u < 1, \\ -1 & \text{if} \quad u < -1. \end{cases}$$

and $h(x, y, z) = z$. We take $Z^\delta_{\alpha,\beta} = W^\delta_{\alpha,\beta} + \delta(0\,,\,0\,,\,Ax + By)$ where $W^\delta_{\alpha,\beta}$ is the ϕ–regularization of the vector field (5.4.1). Thus the differential system induced by $Z^\delta_{\alpha,\beta}$, for $-\delta \leq z \leq \delta$, is given by

$$\begin{aligned} \dot{x} &= -\frac{\alpha\, z}{\delta}, \\ \dot{y} &= \frac{(\beta x - 3\alpha\, y - 2\beta^2)z}{2\delta\beta} + \frac{\beta\, x + 3\alpha\, y}{2\beta}, \\ \dot{z} &= \frac{(4\alpha y - 3\beta^2)z}{8\delta\alpha} + \frac{y + (Ax + By)z}{2} + \delta\frac{(Ax + By)}{2}. \end{aligned} \tag{5.5.2}$$

We note that for $z \geq \delta$, $Z^{\delta}_{\alpha,\beta} = X^{\delta}_{\alpha,\beta} + \delta(0\,,\,0\,,\,A+Bx)$, and for $z \leq -\delta$, $Z^{\delta}_{\alpha,\beta} = Y^{\delta}_{\alpha,\beta} + \delta(0\,,\,0\,,\,A+Bx)$, which are linear vector fields.

In order to simplify the study, we take $z = \delta w$. Thus system (5.5.2) becomes

$$\begin{aligned} \dot{x} &= -\alpha\, w, \\ \dot{y} &= \frac{\beta\, x + 3\alpha\, y + (\beta x - 3\alpha\, y - 2\beta^2)w}{2\beta}, \\ \delta \dot{w} &= \frac{y}{2} + \frac{(4\alpha y - 3\beta^2)w}{8\alpha} + \delta\frac{(Ax+By)(w+1)}{2}, \end{aligned} \tag{5.5.3}$$

for $-1 \leq w \leq 1$. Here the dot denotes derivative with respect to the variable t.

Let (x_0, y_0, w_0) be a singularity of (5.5.3). Clearly $w_0 = 0$, and x_0, y_0 satisfy the equation

$$\underbrace{\begin{pmatrix} 1 & \dfrac{3\alpha}{\beta} \\ \delta A & 1+\delta B \end{pmatrix}}_{P} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 0.$$

Since $\beta > 0$ we have that $\beta \det(P) = \beta - \delta(3\alpha A - \beta B) > 0$ for $\delta \neq 0$ small enough. So the origin $(x_0, y_0, z_0) = (0,0,0)$ is the unique singularity of system (5.5.3) for every $\delta > 0$small enough. Moreover we can estimate their eigenvalues as

$$\lambda^s = -\frac{3\beta^2}{8\delta\alpha} + \frac{4\alpha}{3\beta} + \mathcal{O}(\delta) \quad \text{and} \quad \lambda^u_{1,2} = \frac{\alpha}{12\beta} \pm i\frac{\alpha\sqrt{95}}{12\beta} + \mathcal{O}(\delta).$$

In the above equalities the effects of the parameters A and B are contained in $\mathcal{O}(\delta)$. Note that, for $\delta = 0$, the eigenvalues $\lambda^u_{1,2}$ coincide with the eigenvalues $\lambda^{\pm}$ (see (5.4.3)) of the sliding vector field $\widetilde{Z}_{\alpha,\beta}$ given by (5.4.2). We conclude then that the origin is a hyperbolic saddle–focus singularity for every $\delta > 0$ sufficiently small, which has an 1–dimensional stable manifold $\mathcal{W}^s_\delta$ and a 2–dimensional unstable manifold $\mathcal{W}^u_\delta$. It concludes the proof of statement (a).

System (5.5.3), known as *slow system*, can be studied using singular perturbation methods. Doing $\delta = 0$ we obtain the *reduced problem*

$$\begin{aligned} \dot{x} &= -\alpha\, w, \\ \dot{y} &= \frac{\beta\, x + 3\alpha\, y + (\beta x - 3\alpha\, y - 2\beta^2)w}{2\beta}, \\ 0 &= \frac{y}{2} + \frac{(4\alpha y - 3\beta^2)w}{8\alpha}, \end{aligned}$$

which is a differential equation defined on a manifold. Solving the last equality for $-1 \leq w \leq 1$ this manifold writes

$$\mathcal{M}_0 = \left\{ (x, y, m_0(x, y)) : \, x \in \mathbb{R}, \, y \leq \frac{3\beta^2}{8\alpha}, \, m_0(x, y) = \frac{4\alpha y}{3\beta^2 - 4\alpha y} \right\}. \tag{5.5.4}$$

We note that $(0, 0, 0) \in \mathcal{M}_0$.

Now performing the time rescaling $t = \delta\tau$ we get the so called *fast system*

$$\begin{aligned} x' &= -\delta\alpha\, w, \\ y' &= \delta \frac{(\beta\, x + 3\alpha\, y + (\beta x - 3\alpha\, y - 2\beta^2)w)}{2\beta}, \\ w' &= \frac{y}{2} + \frac{(4\alpha y - 3\beta^2)w}{8\alpha} + \delta \frac{(Ax + By)(w + 1)}{2}. \end{aligned} \tag{5.5.5}$$

that we shall denote by $F_\delta(x, y, w)$. Here the prime denotes derivative with respect to the variable τ. We note that $\mathcal{M}_0$ is a manifold of critical points of F_0, that is system (5.5.5), for $\delta = 0$, namely

$$\begin{aligned} x' &= 0, \\ y' &= 0, \\ w' &= \frac{y}{2} + \frac{(4\alpha y - 3\beta^2)w}{8\alpha}. \end{aligned} \tag{5.5.6}$$

System (5.5.6) is known as the *layer problem.*

Using systems (5.5.5) and (5.5.6) it is straightforward to prove that the solution $\varphi(\tau, \delta) = (\varphi_1(\tau, \delta), \, \varphi_2(\tau, \delta), \varphi_3(\tau, \delta))$ of system (5.5.5) such that $f_3(0, \delta) = 1$ and $\lim_{t\to\infty} \varphi(\tau\ , \delta) = (0, 0, 0)$ can be estimated, for $\delta > 0$ small enough, as

$$\varphi_1(t, \delta) = \frac{8\delta\alpha^2}{3\beta^2} e^{-\dfrac{3\beta^2 t}{8\alpha}} + \mathcal{O}(\delta^2), \quad \varphi_2(t, \delta) = \frac{8\delta\alpha}{3\beta} e^{-\dfrac{3\beta^2 t}{8\alpha}} + \mathcal{O}(\delta^2), \quad \text{and}$$

$$\varphi_3(t, \delta) = e^{-\dfrac{3\beta^2 t}{8\alpha}} + \frac{4\delta\alpha}{9\beta^3} \left(4\alpha(8\alpha + 3\beta^2 t) e^{-\dfrac{3\beta^2 t}{8\alpha}} - 32 e^{-\dfrac{3\beta^2 t}{4\alpha}} \right) + \mathcal{O}(\delta^2).$$

So the stable manifold $\mathcal{W}^s_\delta$ intersects the plane $w = 1$ at the point

$$p_\delta = \varphi(0, \delta) = \left(\frac{\delta 8\alpha^2}{3\beta^2}, \, \frac{\delta 8\alpha}{3\beta}, \, 1 \right) + \mathcal{O}(\delta^2).$$

For $(x, y, w) \in \mathcal{M}_0$ we compute

$$DF_0(x, y, w) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \dfrac{-3\beta^2}{8\alpha y - 6\beta^2} & \dfrac{4\alpha y - 3\beta^2}{8\alpha} \end{pmatrix}.$$

Since $(4\alpha y - 3\beta^2)/(8\alpha) \neq 0$ for all the points of $\mathcal{M}_0$, it follows that the manifold $\mathcal{M}_0$ is a normally hyperbolic attracting manifold for F_0. So in any compact set of $\mathcal{M}_0$ we can apply the well known *first Fenichel theorem* (see, for instance, [33, 59, 60]), which ensures the existence of a normally hyperbolic attracting invariant manifold $\mathcal{M}_\delta$ for $\delta > 0$, small enough, of system (5.5.3), which is known as *slow manifold*. The slow manifold $\mathcal{M}_\delta$ is δ–close to $\mathcal{M}_0$, that is $\mathcal{M}_\delta = \{(x, y, m(x, y, \delta)) : m(x, y, \delta) = m_0(x, y) + \delta m_1(x, y)\}$, where m_0 is defined in (5.5.4). Considering that $(x(t), y(t), \delta\, m(x(t), y(t), \delta))$ is a solution of system (5.5.3) we compute

$$m_1(x, y) = \frac{12\alpha\beta^2(A\,x + B\,y)}{(4\alpha\,y - 3\beta^2)^2} - \frac{48\alpha^2\beta(3\beta^3\,x + \alpha\beta^2\,y - 24\alpha^2 y^2)}{(4\alpha\,y - 3\beta^2)^4}.$$

We claim that the slow manifold $\mathcal{M}_\delta$ contains the origin for $\delta > 0$ sufficiently small. Indeed, suppose that $(0, 0, 0) \notin \mathcal{M}_\delta$ so it is δ–close to $\mathcal{M}_\delta$ because $(0, 0, 0) \in \mathcal{M}_0$. Since $\mathcal{M}_\delta$ is an attracting invariant manifold for $\delta > 0$ sufficiently small, it must attract the origin which is contradiction because the origin is a singularity. Thus we conclude that $(0, 0, 0) \in \mathcal{M}_\delta$ for $\delta > 0$ sufficiently small. From similar reasons the slow manifold also contains the unstable manifold $\mathcal{W}^u_\delta$ of the singularity $(0, 0, 0)$ for $\delta > 0$ sufficiently small.

We can easily check that the slow manifold $\mathcal{M}_\delta$ intersects the plane $w = 1$ transversely along the curve $(x, \ell(x, \delta), 1)$, where

$$\ell(x, \delta) = \frac{3\beta^2}{8\alpha} + \delta\left(\frac{16\alpha\,x}{3\beta^2} - Ax - \frac{3B\beta^2}{8\alpha} - \frac{16\alpha}{3\beta}\right) + \mathcal{O}(\delta^2).$$

Now we consider the solution $(x(t, \delta), y(t, \delta), w(t, \delta))$ of system (5.5.3) starting at a point of the slow manifold $\mathcal{M}_\delta$. From its invariance property we know that $w(t) = m_0(x(t, \delta), y(t, \delta)) + \delta m_1(x(t, \delta), y(t, \delta)) + \mathcal{O}(\delta^2)$. Substituting this relation in the slow system (5.5.3) we obtain the following planar differential system

$$\begin{aligned} x' &= \frac{4\alpha^2 y}{4\alpha y - 3\beta^2} + \mathcal{O}(\delta), \\ y' &= \frac{3\beta^2 x + \alpha\beta^2 y - 24\alpha^2 y^2}{6\beta^3 - 8\alpha\beta y} + \mathcal{O}(\delta), \end{aligned}$$

which is topologically equivalent to the sliding vector field (5.4.2) for $y < 3\beta^2/(4\alpha)$ and $\delta > 0$ small enough.

Let $q_\delta = (3\beta/2, \ell(3\beta/2,\delta), m_\delta(q_\delta))$. From the proof of Proposition 5.4.1 we know that, for $\delta = 0$, the orbit starting at $q_0 = (3\beta/2, 3\beta^2/(8\alpha), 1) = (q, 1)$ is attracted, backward in time, to the focus $(0,0,0)$. So, for $\delta > 0$ sufficiently small, the orbit starting at

$$q_\delta = \left(\frac{3\beta}{2}\,,\, \frac{3\beta^2}{8\alpha} + \frac{\delta(64\alpha^2 - 36A\,\alpha\beta^2 - 9B\,\beta^3)}{24\alpha\beta}\,,\, 1\right) + \mathcal{O}(\delta^2)$$

is also attracted, backward in time, to the focus $(0,0,0)$.

Let $\overline{q}_\delta$ and $\overline{p}_\delta$ be the points q_δ and p_δ in the variables (x,y,z) (that is $z = \delta w$). The proof will follow by showing that for some branches A_δ and B_δ the flow of the linear system $X_{\alpha,\beta}$ connects the points $\overline{q}_\delta$ and $\overline{p}_\delta$ for $\delta > 0$ sufficiently small.

For $z \geq 1$ the vector field $Z^\delta_{\alpha,\beta}$ is equal to the linear vector field $X_{\alpha,\beta}(x,y,z) + \delta(0\,,\,0\,,\,A + Bx)$. Computing its solution $\psi(t,\delta) = (\psi_1(t,\delta), \psi_2(t,\delta), \psi_3(t,\delta))$ such that $\psi(0,\delta) = \overline{q}_\delta$ we obtain that

$$\begin{aligned}
\psi_1(t,\delta) &= \frac{3\beta}{2} - \alpha\, t + \mathcal{O}(\delta^2),\\
\psi_2(t,\delta) &= \frac{(3\beta - 2\alpha\, t)(\beta + 2\alpha\, t)}{8\alpha} + \frac{\delta(6a\alpha^2 - 36A\alpha\beta^2 - 9B\beta^3)}{24\alpha\beta} + \mathcal{O}(\delta^2),\\
\psi_3(t,\delta) &= \frac{(3\beta - 2\alpha\, t)t^2}{12} + \frac{\delta}{12}\left(12 + \frac{32\alpha\, t}{\beta} - (6A\alpha - 3B\beta)t^2 - 2B\alpha\, t^3\right) + \mathcal{O}(\delta^2),
\end{aligned}$$

Since the orbit $\psi(t,0)$ reaches transversally the plane $\Sigma = \{z = 0\}$ in a finite time $t_0 = 3\beta/(2\alpha)$, we can prove that the orbit $\psi(t,\delta)$, for $\delta > 0$ small enough, will also reach transversally the plane $z = \delta$ in a finite time t_δ. Moreover we can estimate $t_\delta = 3\beta/(2\alpha) + \delta(32\alpha - 9A\beta^2)/(3\beta^2) + \mathcal{O}(\delta^2)$.

Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ and $\pi^\perp : \mathbb{R}^3 \to \mathbb{R}$ be the projections onto the two first coordinates and onto the last coordinate, respectively. Define $\mathcal{F}(A,B,\delta) = (\psi(t_\delta,\delta) - \overline{p}_\delta)/\delta$. It is easy to see that, for every $\delta > 0$ sufficiently small, $\pi^\perp\mathcal{F}(A,B,\delta) = 0$ and

$$\pi\mathcal{F}(A,B,\delta) = \left(3A\alpha - \frac{40\alpha^2}{3\beta^2} + \mathcal{O}(\delta)\,,\, -\frac{32\alpha}{3\beta} + \frac{3A\beta}{2} - \frac{3B\beta^2}{8\alpha} + \mathcal{O}(\delta)\right).$$

We note that $\mathcal{F}(A_0,B_0,\delta_0) = 0$, for some A_0, B_0, and $\delta_0 > 0$, if and only if the vector field (5.5.2) (for $A = A_0$, $B = B_0$, and $\delta = \delta_0$) admits an orbit connecting the points $\overline{q}_{\delta_0}$ and $\overline{p}_{\delta_0}$, that is an sliding Shilnikov orbit. Since for $A^* = 40\alpha/(9\beta^2)$ and $B^* = -32\alpha^2/(3\beta^3)$, we have that $\pi\mathcal{F}(A^*,B^*,0) = 0$ and $\det(\pi D\mathcal{F}(A^*,B^*,0)) = -9\beta^2/8 \neq 0$, then, using the *implicit function Theorem*, we conclude that, for $\delta > 0$ sufficiently small, there exist two branches A_δ and B_δ such that $\pi\mathcal{F}(A_\delta,B_\delta,\delta) = 0$, and $A_\delta \to A^*$ and $B_\delta \to B^*$ when $\delta \to 0$. It concludes the proof of statement (b).

Finally, we compute the saddle quantity as $\sigma = -3\beta^2/(8\delta\,\alpha) + 17\alpha/(12\beta) + \mathcal{O}(\delta)$ which negative for $\delta > 0$ small enough. The proof of statement (c) follows by applying the classical results for Shilnikov homoclinic orbits [110, 111].

□

Chapter 6

Regularization of hidden dynamics in piecewise smooth flows

The main results of this chapter (Theorems P, Q, R, S, T, and U) are based on the paper [90].

6.1 Introduction

Consider an ordinary differential equation in $x \in \mathbb{R}^n$ with a discontinuous righthand side,

$$\dot{x} = \begin{cases} f^+(x) & \text{if} \quad h(x) > 0, \\ \\ f^-(x) & \text{if} \quad h(x) < 0, \end{cases} \tag{6.1.1}$$

where f^+ and f^- are smooth vector fields, and h is a differentiable scalar function whose gradient ∇h is well-defined and non-vanishing everywhere. Throughout this chapter we consider an open region $x \in D$ in which (6.1.1) holds. The set $\Sigma = \{x \in \mathcal{D} : \ h(x) = 0\}$ is called the *switching manifold*, and the regions either side of it are denoted as $\mathcal{R}^{\pm} = \{x \in \mathcal{D} : \ h(x) \gtrless 0\}$.

The term 'hidden dynamics' refers to what happens on Σ, specifically to behaviours governed by terms that disappear in $\mathcal{R}^{\pm}$ (hence they are 'hidden' in (6.1.1)), and which go beyond Filippov's standard theory [34]. The theory of Filippov relies heavily on two alternatives for extending (6.1.1) across $h = 0$. The first is a differential inclusion

$$\dot{x} \in \mathcal{F}(x) \qquad \text{s.t.} \quad f^+(x), f^-(x) \in \mathcal{F}(x) \tag{6.1.2}$$

which is very general because $\mathcal{F}$ is any set that contains $f^{\pm}$ ($\mathcal{F}$ is usually assumed to be convex to provide certain restrictions on sequences of solutions [34], but this does not prevent $\mathcal{F}$ being arbitrarily large). The second alternative is a smaller set, the convex hull of f^+ and f^-,

$$\dot{x} = Z(x;\lambda) := \frac{1+\lambda}{2} f^+(x) + \frac{1-\lambda}{2} f^-(x), \quad \lambda \in \begin{cases} \text{sign}(h(x)) & \text{if} \quad h(x) \neq 0 \ , \\ \\ [-1,+1] & \text{if} \quad h(x) = 0 \ , \end{cases} \tag{6.1.3}$$

which is very restrictive in the sense that it selects only values of (6.1.2) that are linear combinations of $f^{\pm}$. Examples of the set $\mathcal{F}$ and hull $\{Z(x;\lambda) : \lambda \in [-1,+1]\}$ will be illustrated in Example 6.1.1 below, along with a third alternative that unties them.

We will refer to the transition as h changes sign in (6.1.3) as *linear switching* (implying linear dependence with respect to λ). In Filippov's theory, one seeks values of $\dot{x}$ in the sets (6.1.2) or (6.1.3) that result in continuous (though typically non-differentiable) flows at Σ. In many situations of interest, the flow obtained from (6.1.3) is unique (making possible, for example, substantial classifications of singularities and bifurcations for such systems [34, 105, 29]).

The problem highlighted in [56] was that between the set-valued flow of (6.1.2) and the piecewise-smooth flow of (6.1.3), a vast expanse of non-equivalent but no less valid dynamical systems can be considered. All that is lacking is a way to express them explicitly. This is provided quite simply by permitting nonlinear dependence on the transition parameter λ, in the form

$$\dot{x} = f(x;\lambda) := \frac{1+\lambda}{2} f^{+}(x) + \frac{1-\lambda}{2} f^{-}(x) + G(x;\lambda), \tag{6.1.4}$$

where

$$h(x)G(x;\lambda) = 0\ , \qquad \lambda \in \begin{cases} \operatorname{sign}(h(x)) & \text{if} \quad h(x) \neq 0\ , \\ [-1,+1] & \text{if} \quad h(x) = 0\ , \end{cases} \tag{6.1.5}$$

with G some continuous vector field that is nonlinear in λ. An example of the set generated by $\{f(x;\lambda) : \lambda \in [-1,+1]\}$ is given in Example 6.1.1 below. We shall refer to (6.1.4) as the nonlinear combination, and the transition it undergoes as h changes sign as *nonlinear switching*. (Moreover the term 'nonlinear' throughout this chapter will refer to nonlinear dependence on λ via the function G).

Example 6.1.1. Consider in coordinates $x = (x_1, x_2)$ the piecewise constant system (6.1.1) with vector fields $f^{+} = (1,1)$, $f^{-} = (1,-2)$, and $G(\lambda) = (\lambda^2 - 1)(2,0)$, with $h(x) = x_1$. In Figure 6.1 we illustrate a convex set $\mathcal{F}$ satisfying (6.1.2), the linear combination $Z(x;\lambda)$ defined in (6.1.3), and the nonlinear combination from (6.1.4), represented by the shaded region, dashed line, and dotted curve, respectively. By choosing different forms of G (subject to $hG = 0$) we can choose different curves $\{f(x;\lambda) : \lambda \in [-1,+1]\}$ which explore different subsets of $\mathcal{F}$.

Although Filippov (followed by many authors since) favoured (6.1.3), it is worthwhile exploring the more general form (6.1.4), not least because in [56, 57] it was shown to provide new ways of modeling real mechanical phenomena (namely static friction, the phenomenon that the force of dry-friction during sticking can exceed that during motion, not captured by applying Filippov's method to the basic discontinuous Coulomb friction law), and in [42, 55] it is shown that similar nonlinearities become inescapable when multiple switches are involved (specifically it is shown that multiple switches create the possibility of multiple sliding solutions, which must be resolved by some kind of regularization or blow up of the discontinuity). It is therefore important obtain greater insight into the discontinuous dynamical systems represented by (6.1.4), one of the first concerns being typically their persistence within larger classes of systems. To this end it has been

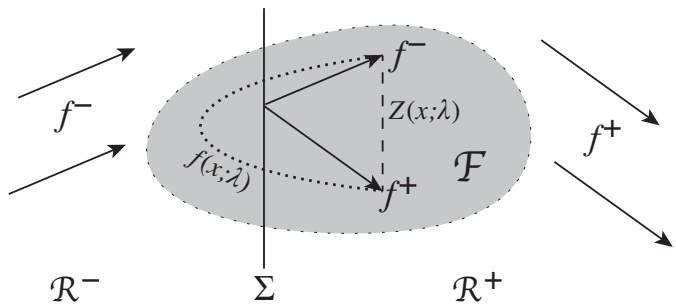

Figure 6.1: The vector field f switches between f^+ and f^- in regions $\mathcal{R}^+$ and $\mathcal{R}^-$. At the boundary Σ Filippov considered either a general convex set $\mathcal{F}$ containing $f^\pm$ (shaded area), or a convex hull $Z(x;\lambda)$ of $f^\pm$ (dashed line). The nonlinear combination $f(x;\lambda)$ allows us to explore $\mathcal{F}$ more explicitly (dotted curve), by choosing a different G we obtain a different curve of values $f(x;\lambda) \subset \mathcal{F}$.

shown that the dynamics of (6.1.3) persists when the discontinuity is *regularized* (i.e. smoothed) [67] and, as we will show here, the same is equally true of the nonlinear combination (6.1.4).

The behaviours associated with adding G in (6.1.4) have been referred to as *hidden dynamics*, because the first condition in (6.1.5) means that G vanishes for $h \neq 0$, i.e. everywhere except at the discontinuity itself. The function G may, for example, be any finite vector field multiplied by a scalar term like $\lambda(\lambda^2 - 1)$, $\sin(\lambda^2 - 1)$, or $\lambda^{2r} - 1$ for any natural number r.

In this chapter we will consider how the nonlinear combinations (6.1.4) relate to singular limits of continuous systems via both regularization [103], and a converse to regularization known as pinching [14, 28]. Much of our analysis will concern the closeness of dynamics on Σ in the discontinuous system (6.1.4) to invariant dynamics near Σ in a topologically equivalent smooth system.

We set up the problem in Section 6.2, then prove results regarding regularization and pinching in Sections 6.3-6.4.

6.2 Preliminaries: crossing or sliding in the nonlinear system

The first step in studying (6.1.4) is to define more precisely what happens on Σ, our main interest being what happens when $G(x;\lambda)$ is allowed not to vanish there. We denote the interval of values taken by λ as $\mathcal{I} := [-1, +1]$.

Henceforth the symbol p will always denote a point inside Σ, and where specific coordinates are useful we will sometimes let $h(x) = x_1$ and write $p = (0, \mathbf{y})$.

For any $p \in \Sigma$ we define the scalar function

$$K(p;\lambda) := f(p;\lambda) \cdot \nabla h(p) \ , \tag{6.2.1}$$

which is a multiple of the normal component of f to Σ. This vanishes on the set

$$S(p) := \{\lambda^* \in \mathcal{I} : \ K(p;\lambda^*) = 0\}, \tag{6.2.2}$$

which may or may not have solutions for $\lambda^* \in \mathcal{I}$. Places where there exist solutions to (6.2.2) define regions where the vector field f lies tangent to Σ for one or more values of $\lambda^* \in \mathcal{I}$, allowing the flow of (6.1.4) to *slide* along Σ, and we call the set of all such points $p \in \Sigma$ the nonlinear *sliding region* Σ^{ns}, given by

$$\Sigma^{ns} := \{p \in \Sigma : \ S(p) \neq \emptyset\}.$$

The complement to this on Σ is the set where (6.2.2) has no solutions, so f is transverse to Σ for all $\lambda \in \mathcal{I}$, defining the nonlinear *crossing region* Σ^{nc},

$$\Sigma^{nc} := \{p \in \Sigma : \ S(p) = \emptyset\} \ ,$$

such that $\Sigma = \overline{\Sigma^{ns}} \cup \overline{\Sigma^{nc}}$, ($\overline{\Sigma^{ns}}$ and $\overline{\Sigma^{nc}}$ denoting the closures of Σ^{ns} and Σ^{nc}).

The implication is that for $p \in \Sigma^{nc}$ the vector field $f(p;\lambda)$ pushes the flow transversally across Σ between $\mathcal{R}^+$ and $\mathcal{R}^-$, while for $p \in \Sigma^{ns}$ the flow is able to slide along Σ. Substituting the solution λ^* of (6.2.2) into (6.1.4), the system that defines these nonlinear *sliding modes* is given by

$$\dot{p} = f^{ns}(p) := f(p;\lambda^*(p)) \ , \qquad \lambda^*(p) \in S(p) \ , \tag{6.2.3}$$

with f^{ns} defining the nonlinear *sliding vector field*. Typically there may exist a set of such functions λ_i^*, $i = 1, 2, ...$, defining branches of solutions of $K(p;\lambda^*) = 0$ in (6.2.2), each on a subset $\sigma_i \subset \Sigma^{ns}$, such that the union of all σ_i's covers Σ^{ns}, and $\lambda_i^* : \sigma_i \subset \Sigma^{ns} \mapsto \mathcal{I}$. We then have a set of sliding modes specified by a set of equations defined by (6.2.3) on different branches $p \in \sigma_i$.

If we fix $G \equiv 0$ everywhere then the nonlinear crossing region Σ^{nc} is exactly the crossing region defined by the Filippov's convention for the system (6.1.3), and the nonlinear sliding region Σ^{ns} is the union of the sliding region, defined by the Filippov's convention, with the tangential points. We therefore call the *linear crossing region* Σ^c and *linear sliding region* Σ^s (obtained directly by solving the above conditions neglecting G). The linear system (i.e. without G) can only have one (linear) sliding mode, on Σ^s, while the full system (G nonzero on Σ) may have multiple (nonlinear) sliding modes as defined by (6.2.3) with (6.1.4). It is easily shown (see [56]) that $\Sigma^s \subseteq \Sigma^{ns}$ and $\Sigma^{nc} \subseteq \Sigma^c$.

6.3 Regularization

Let us first show that regularizations of the linear combination (6.1.3) or of the nonlinear combination (6.1.4) can be related by a simple substitution.

Let C^r denote the class of r-times differentiable functions. We shall denote by

$\psi : \mathbb{R} \to \mathbb{R}$ a continuous function which is C^1 for $s \in (-1, 1)$

such that $\psi(s) = \text{sign}(s)$ for $|s| \geq 1$.

We call ψ a *transition function.*

$\phi : \mathbb{R} \to \mathbb{R}$ a continuous function which is C^1 for $s \in (-1, 1)$

such that $\phi(s) = \text{sign}(s)$ for $|s| \geq 1$, and $\phi'(s) > 0$ for $s \in (-1, 1)$.

We call ϕ a *monotonic transition function.*

We also let

$$\phi_\delta(h) := \phi(h/\delta) \qquad \text{and} \qquad \psi_\delta(h) := \psi(h/\delta) \ .$$

A regularization of a discontinuous system (6.1.3) or (6.1.4) is a one–parameter family $Z_\delta \in C^r$ for $r \geq 0$ such that f_δ converges to the discontinuous system when $\delta \to 0$. The intention is that this represents a class of continuous functions approximated by (6.1.1) as $\delta \to 0$, the importance of (6.1.4) is that it will show this class to be larger than those derived from (6.1.3). The Sotomayor-Teixeira method of regularization, see e.g. [103], replaces λ in (6.1.3) by a monotonic transition function ϕ, to consider

$$\dot{x} = \frac{1 + \phi_\delta(h(x))}{2} f^+(x) + \frac{1 - \phi_\delta(h(x))}{2} f^-(x).$$

We refer to this as a *linear*-regularization (or ϕ–regularization in other references). It is shown in [4, 23, 66, 67] that this defines a system with slow invariant dynamics topologically equivalent to Filippov's (linear) sliding dynamics. One may ask what happens if we consider instead (6.1.3) with a non-monotonic transition function ψ. When modeling a physical system, for example, there is no clear reason to exclude such possibilities, and we shall see below how they fit with established theory for discontinuous differential equations.

We will show that the (non-monotonic) ψ regularization of Filippov's linear combination (6.1.3),

$$\dot{x} = Z_\delta(x) := \frac{1 + \psi_\delta(h(x))}{2} f^+(x) + \frac{1 - \psi_\delta(h(x))}{2} f^-(x) \tag{6.3.1}$$

is equivalent to the (monotonic) ϕ regularization of a nonlinear combination (6.1.4), given by $f_\delta(x) = f(x; \phi(h(x)/\delta))$, i.e.

$$\dot{x} = f_\delta(x) := \frac{1 + \phi_\delta(h(x))}{2} f^+(x) + \frac{1 - \phi_\delta(h(x))}{2} f^-(x) + G(x; \phi_\delta(h(x))) \ . \tag{6.3.2}$$

Theorem P. If ϕ is a monotonic transition function and ψ is a non–monotonic transition function, then there exists a unique function $G(x; \lambda)$ satisfying (6.1.5) such that the ψ–regularization of (6.1.3) is a ϕ–regularization of (6.1.4).

Proof. Let $\lambda = \phi(s)$, the function ϕ is monotonic in the interval $\mathcal{I}$ and therefore has an inverse $s = \phi^{-1}(\lambda)$, so we can express ψ in terms of λ via a function $\Psi(\lambda) = \psi\left(\phi^{-1}(\lambda)\right)$. The ψ–regularization of (6.1.3) as given by (6.3.1) can thus be re-arranged to

$$\dot{x} = \frac{1+\lambda}{2} f^+(x) + \frac{1-\lambda}{2} f^-(x) + \left(\Psi(\lambda) - \lambda\right) \frac{f^+(x) - f^-(x)}{2} \ .$$

If we define $G(x;\lambda) = \left(\Psi(\lambda) - \lambda\right)\left(f^+(x) - f^-(x)\right)/2$, we obtain the nonlinear combination (6.1.4), and taking $\lambda = \phi_\delta(h(x))$ we obtain its ϕ–regularization on $\lambda \in \mathcal{I}$. Since for $|s| \geq 1$ we have $\lambda = \phi(s) = \psi(s) = \text{sign}(s)$, this implies $G(x;\pm 1) = 0$ as required by (6.1.5). □

A simple consequence of this is that the family of ϕ–regularized nonlinear combinations (6.1.4) is larger than the family of ψ–regularized linear combinations (6.1.3), as shown by the following.

Corollary 6.3.1. If ϕ is a monotonic transition function, then there exists a non–monotonic transition function ψ such that the ϕ–regularization of (6.1.4) is a ψ–regularization of (6.1.3), if and only if $G(x;\lambda) = \gamma(\lambda)\left(f^+(x) - f^-(x)\right)/2$ such that $h(x)\gamma(\lambda) = 0$.

Proof. The proof follows directly by substituting G into (6.3.2) and applying Theorem P. □

Figure 6.2 provides the resulting schematic of how the discontinuous systems and their regularizations considered above fit together.

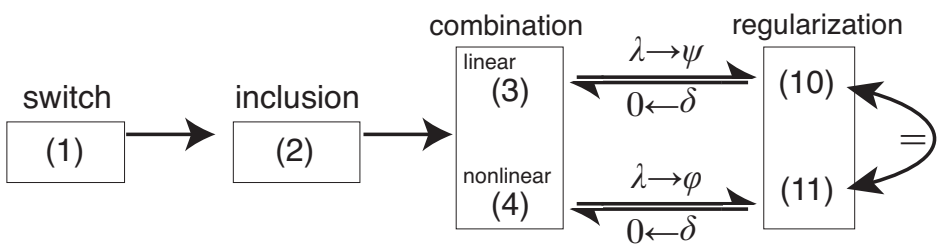

Figure 6.2: The discontinuous differential equation (6.1.1) is not defined on Σ, so is replaced by the inclusion (6.1.2), representing all possible systems at Σ. A solvable form for these is provided by the Filippov systems in the linear form (6.1.3) or more general nonlinear form (6.1.4). In the following sections we applying a regularization of nonlinear or linear kind, yielding the differentiable systems (6.3.1) and (6.3.2) respectively, which are equivalent for some choice of transition functions ϕ_δ and ψ_δ, and conversely whose singular limits as $\delta \to 0$ are (6.1.3) and (6.1.4).

In the next theorem we extend the results of [4, 23, 66, 67] showing that the nonlinear regularization (6.3.2) exhibits slow invariant dynamics that is conjugate to the sliding modes of the discontinuous system (6.1.3). The remainder of this section will consist of the proof of this theorem. First let us see how slow-fast dynamics arises in an example.

Example 6.3.1. Consider the system

$$(\dot{x}_1, \dot{x}_2) = \frac{1+\lambda}{2}(1,-2) + \frac{1-\lambda}{2}(1,1) + (\lambda^2 - 1)(2,0) \ ,$$

which is discontinuous if $\lambda = \text{sign}(x_1)$ for $x_1 \neq 0$. The regularization is obtained by replacing $\lambda \mapsto \phi_\delta(x_1)$ for small $\delta > 0$. Figure 6.3 shows the discontinuous system (left) with a nonlinear sliding region on which two sliding modes exist (one traveling upwards, the other downwards), and conjugate to each sliding mode. Compare this to the discontinuous linear and nonlinear systems in Example 6.1.1.

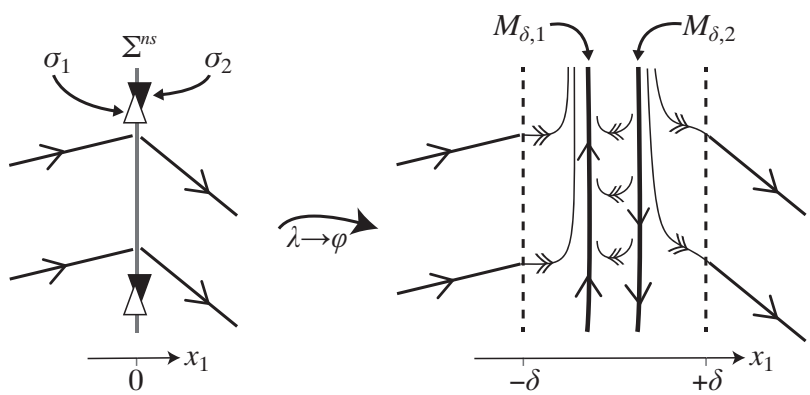

Figure 6.3: Left: a discontinuous system (6.1.4) with nonlinear sliding region with branches σ_r for $r = 1, 2$ (white and black filled arrows). Right: the regularization in which each sliding branch σ_k is conjugate to an invariant manifold $M_{\delta,k}$ of a slow-fast system (6.3.2).

Theorem Q. Let the region $\sigma \subset \Sigma^{ns}$ be expressible as a graph $x_1 = 0$ in coordinates $x = (x_1, x_2, .., x_2)$, on which there exists a C^r function $\lambda^*(p)$, $r \geq 0$, such that $K(p; \lambda^*(p)) = 0$ in (6.2.1) for every $p \in \sigma$. Then for any C^r (or continuous) function ϕ, the ϕ–regularization contains a slow manifold C^r–diffeomorphic (homeomorphic) to σ, on which the slow dynamics is C^r–conjugated (topologically conjugated) to the nonlinear sliding dynamics (6.2.3). Moreover, if $\partial K(p; \lambda^*(p))/\partial\lambda \neq 0$ then for $\delta > 0$ sufficiently small the nonlinear sliding dynamics defined on Σ^{ns} persists to order δ, on a manifold M_δ which is δ–close to Σ^{ns}.

Proof. In the coordinates given, $\sigma \subset \Sigma^{ns}$ is an open subset of the hyperplane $\{p = (0, x_2, x_3, \ldots, x_n) \in \mathcal{D}\}$. Writing vector components as $f = (f_1, f_2, ..., f_n)$ for any function f, the normal component (6.2.1) of the nonlinear combination (6.1.4) is

$$K(p; \lambda) = \frac{1+\lambda}{2} f_1^+(p) + \frac{1-\lambda}{2} f_1^-(p) + G_1(p; \lambda).$$

Sliding modes by (6.2.2)-(6.2.3) satisfy the differential-algebraic system

$$\begin{aligned} 0 &= f_1(p; \lambda^*(p)) \\ \dot{p}_i &= \frac{1+\lambda^*(p)}{2} f_i^+(p) + \frac{1-\lambda^*(p)}{2} f_i^-(p) + G_i(p; \lambda^*(p)) \end{aligned} \tag{6.3.3}$$

for $i = 2, 3, ..., n$.

Now consider the ϕ–regularization of (6.1.4), given by

$$\dot{x}_i = \frac{1+\phi_\delta(x_1)}{2} f_i^+(x) + \frac{1-\phi_\delta(x_1)}{2} f_i^+(x) + G_i(x;\phi_\delta(x_1))$$

for $i = 1, ..., n$. By a change of variables to $u = x_1/\delta$ and $v = (x_2, x_3, \ldots, x_n)$ for small $\delta > 0$, we obtain

$$\begin{aligned} \delta\dot{u} = & \frac{1+\phi(u)}{2} f_1^+(u\delta, v) + \frac{1-\phi(u)}{2} f_1^+(u\delta, v) + G_1(u\delta, v; \phi(u)), \\ \dot{x}_i = & \frac{1+\phi(u)}{2} f_i^+(u\delta, v) + \frac{1-\phi(u)}{2} f_i^+(u\delta, v) + G_i(u\delta, v; \phi(u)), \end{aligned} \tag{6.3.4}$$

where δ is a singular perturbation parameter. In the limit $\delta = 0$ we obtain the so-called reduced problem (using the notation $x = p$ on Σ)

$$\begin{aligned} 0 = & \frac{1+\phi(u)}{2} f_1^+(p) + \frac{1-\phi(u)}{2} f_1^+(p) + G_1(p;\phi(u)) = K(p;\phi(u)), \\ \dot{p}_i = & \frac{1+\phi(u)}{2} f_i^+(p) + \frac{1-\phi(u)}{2} f_i^+(p) + G_i(p;\phi(u)), \quad i = 2, ..., n, \end{aligned} \tag{6.3.5}$$

which describes dynamics on the 'slow' timescale t (for standard concepts of singularly perturbed or slow-fast systems see [33, 59]). This dynamics inhabits a hypersurface called the *slow critical manifold*, defined implicitly by $0 = K(p;\phi(u))$ in the first row of (6.3.5).

By hypothesis there exists at least one function $\lambda^*(p)$ satisfying (6.2.2), and therefore there exists at least one slow critical manifold M_0 given by the restriction $\phi(u) = \lambda^*(p)$. Since ϕ is invertible in $\mathcal{I}$ and $\lambda^*(p) \in \mathcal{I}$ for every $p \in \sigma$ we conclude that M_0 is the graph $u(p) = \phi^{-1} \circ \lambda^*(p)$. This is homeomorphic to σ as we can let $H : \sigma \to M_0$ be the bijective function $H(0, v) = (\phi^{-1} \circ \lambda^*(0, v), v)$, for which $H(\sigma) = M_0$. The function H is invertible and its order of differentiability is the same as that of ϕ and λ^*, that is r.

Substituting $\phi(u) = \lambda^*(p)$ into (6.3.5), the reduced problem on $x_1 = 0$ becomes

$$\dot{p}_i = \frac{1+\lambda^*(p)}{2} f_i^+(0) + \frac{1-\lambda^*(p)}{2} f_i^+(0) + G_i(0;\lambda^*(p)) = f_i(p;\lambda^*(p)),$$

for $i = 2, 3, ..., n$. Now let $\overline{p} = (0, \overline{v})$, so if $t \mapsto x_t(\overline{p}) = (0, v(t, \overline{p}))$ is the solution of the nonlinear sliding mode (6.3.3) such that $x_0(\overline{p}) = \overline{p} \in \sigma$, then the solution $t \mapsto X_t(H(\overline{p}))$ of the reduced problem (6.3.5) on the slow manifold such that $X_0(H(\overline{p})) = H(\overline{p})$ is given by

$$X_t(H(\overline{p})) = (\phi^{-1} \circ \lambda^*(v(t, \overline{v})), v(t, \overline{v})) = H(x_t(\overline{p})).$$

The flows of the regularized reduced (slow manifold) system and the discontinuous sliding system are therefore C^r(topologically)–conjugated.

It remains to show the persistence of the slow-fast dynamics for $\delta > 0$. By rescaling time in (6.3.4) by $t = \delta\tau$ and taking $\delta \to 0$, we obtain the so-called layer problem

$$\begin{aligned} u' = & \frac{1+\phi(u)}{2} f_1^+(p) + \frac{1-\phi(u)}{2} f_1^+(p) + G_1(p;\phi(u)) = K(p;\phi(u)), \\ p_i' = & \ 0, \qquad i = 2, 3, ..., n. \end{aligned}$$

which prescribes dynamics on the fast timescale τ external to the slow manifolds. The slow manifold M_0 is a manifold of critical points of the layer problem, which is normally hyperbolic if $(\partial K/\partial\lambda)(p;\lambda^*(p)) \neq 0$. The existence of slow manifolds δ–close to the slow critical manifold, with dynamics δ–close to the reduced problem (6.3.4), then follows by Fenichel's theorem [33]. $\square$

6.4 Pinching

Pinching, introduced in [14] and developed further in [28], can be thought of as an inverse to regularization, providing a method of deriving a discontinuous system as an approximation to a continuous system. A region of state space is chosen, say some $|h| \leq \varepsilon$ for $\varepsilon > 0$, to be collapsed down to a manifold Σ by means of a discontinuous transformation, resulting in a system of the form (6.1.1).

In considering nonlinear switching systems we are able to put the notion of pinching on a more rigorous footing. To do so we must distinguish between *intrinsic pinching*, where the pinching parameter ε is a small parameter of the original continuous system, and *extrinsic pinching* where the original problem is ε–independent. Before venturing into the technicalities, let us illustrate them with an example.

Example 6.4.1. Take a system

$$(\dot{x}_1, \dot{x}_2) = (-x_1, 2\mathcal{H}(x_1/\alpha; b) - 1) \ , \qquad \mathcal{H}(u; b) = \frac{u^b}{1+u^b} \ . \tag{6.4.1}$$

The Hill function $\mathcal{H}$ is a sigmoid graph with a switch about $h = x_1 = 0$, and is a function prevalent in biological applications (starting with [45]). There is an invariant manifold along $x_1 = 0$ with dynamics $(\dot{x}_1, \dot{x}_2) = (0, -1)$.

Let $b \gg 1$ be fixed. We shall take discontinuous approximations of this system. First, assuming α and b are constants, let us make an *extrinsic* pinching with respect to a small parameter ε by transforming to a coordinate $\tilde{x}_1 = h - \varepsilon \mathrm{sign}(h)$, creating a discontinuous system

$$(\dot{\tilde{x}}_1, \dot{x}_2) = \left(-\tilde{x}_1 \mp \varepsilon,\ 2\mathcal{H}\left(\frac{\tilde{x}_1 \pm \varepsilon}{\alpha}; b\right) - 1\right) = (-\tilde{x}_1 \mp \varepsilon,\ 2c_\pm - 1 + \mathsf{O}\left(\tilde{x}_1\right)) \tag{6.4.2}$$

where $c_\pm = \mathcal{H}\left(\pm\frac{\varepsilon}{\alpha}; b\right)$, with (6.4.2) taking the upper signs for $\tilde{x}_1 > 0$ and lower signs for $\tilde{x}_1 < 0$. If we fix α and pinch with respect to a small parameter ε that is extrinsic to the smooth system (6.4.1), then expanding for small ε/α gives $c_\pm = \mathsf{O}\left(\varepsilon/\alpha\right)$ and we can neglect it for small enough ε, giving the system in Figure 6.4. Solving (6.2.2) and (6.2.3) we obtain $\lambda^* = 0$ and a sliding vector field $\dot{p} = (0, -1) + G_\varepsilon$ on $\tilde{x}_1 = 0$, which is equivalent to the dynamics on the invariant manifold $x_1 = 0$ of (6.4.1) with $G_\varepsilon \equiv 0$.

Although the sliding mode captures the correction dynamics at $\tilde{x}_1 = 0$, the approximation outside is valid only for very small $\tilde{x}_1$ because is does not capture the turning around of the flow (the thin curves in the right of Figure 6.4). To capture these we must use the exact expression in (6.4.2), so this approximation is quite weak.

We can do something more powerful by pinching with respect to a parameter that is intrinsic to the system (6.4.1). If we set $\varepsilon = \alpha\sqrt{2}$ as an intrinsic pinching parameter, then expanding

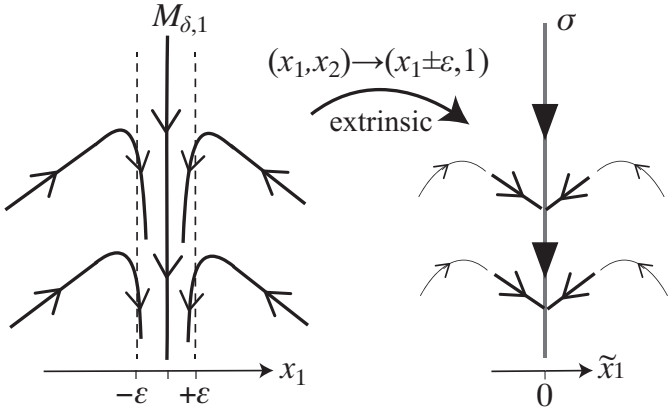

Figure 6.4: Differentiable systems with an invariant manifold $x_1 = 0$ (left), which we pinch by removing the region $|x_1| \leq \varepsilon$, with ε a small parameter extrinsic to (i.e. not appearing in) the smooth system.

$\mathcal{H}\left(\pm\frac{\varepsilon}{\alpha}; b\right)$ for small α/ε gives $c_\pm = 1 + \mathsf{O}\left((\alpha/\varepsilon)^b\right)$, and we have the simple piecewise linear approximation $(-\tilde{x}_1 \mp \varepsilon, 1)$ for the righthand side of (6.4.2), as shown in the bottom row of Figure 6.5. The arrangement of the vector fields in the bottom right figure would give a linear sliding mode $\dot{p} = (0, 1)$, which would be an incorrect representation of the dynamics of (6.4.1). Instead we need to find the nonlinear sliding mode, solving (6.2.2) and (6.2.3) we obtain $\lambda^* = 0$ and a sliding vector field $\dot{p} = (0, 1) + G_\varepsilon$ on $\tilde{x}_1 = 0$, which is equivalent to dynamics on the invariant manifold $x_1 = 0$ in (6.4.1) if we set $G_\varepsilon = (0, -2)$, correctly capturing the dynamics of the smooth system.

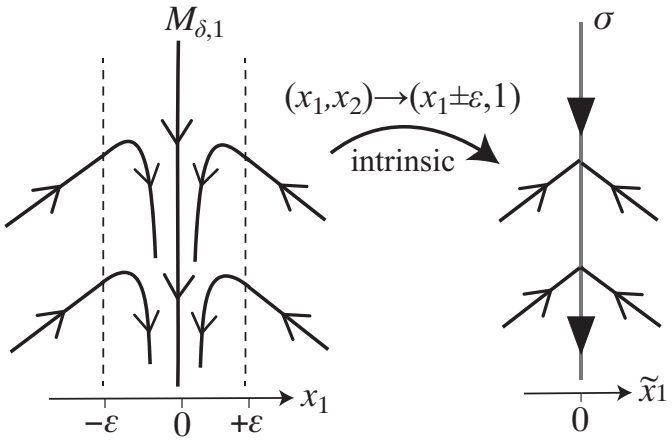

Figure 6.5: Starting from the same smooth system (left), we pinch by removing the region $|x_1| \leq \varepsilon$, with $\varepsilon = \alpha\sqrt{2}$ and hence intrinsic to the smooth system.

We say in these cases that $G_\varepsilon = (0, 0)$ and $G_\varepsilon = (0, -2)$ *complete* the extrinsic and intrinsic systems, respectively. Below we generalize these ideas.

6.4.1 Extrinsic pinching

Let U be a open bounded subset of $\mathbb{R}^n$ and consider the dynamical system

$$\dot{x} = F(x), \quad x \in U \tag{6.4.3}$$

where F is a C^1 function. Assume that the manifold $\Sigma = \{x \in \mathcal{D} : h(x) = 0\}$ is invariant under the flow, that is $F(p) \cdot \nabla h(p) = 0$ for every $p \in \Sigma$.

For small $\varepsilon > 0$ consider the discontinuous system

$$\dot{x} = \begin{cases} F(x + \varepsilon \nabla h(x)) & \text{if} \quad h(x) > 0, \\ F(x - \varepsilon \nabla h(x)) & \text{if} \quad h(x) < 0, \end{cases} \tag{6.4.4}$$

in which the manifold Σ becomes a switching manifold between some $F^+(x;\varepsilon) = F(x + \varepsilon \nabla h(x))$ and some $F^-(x;\varepsilon) = F(x - \varepsilon \nabla h(x))$. We call (6.4.4) the *incomplete* extrinsically pinched system, "incomplete" because like (6.1.1) it is not yet well defined on Σ.

We then ask whether it is possible to complete the pinched system (6.4.4) using a nonlinear combination (6.1.4), such that its nonlinear sliding modes (6.2.3) agree with the dynamics of (6.4.3) on the invariant manifold Σ. When this is possible for some family of functions G^ε (G^ε being the nonlinear part for (6.1.4) now dependent on ε) we say that G^ε *completes* the pinched system, and we call

$$\begin{aligned} &\dot{x} = f^\varepsilon(x;\lambda) = \frac{1+\lambda}{2} F(x + \varepsilon \nabla h(x)) + \frac{1-\lambda}{2} F(x - \varepsilon \nabla h(x)) + G^\varepsilon(x;\lambda), \\ &\lambda \in \mathcal{I}, \qquad h(x) G^\varepsilon(x;\lambda) = 0 \ , \end{aligned} \tag{6.4.5}$$

the *complete* extrinsically pinched system. In order to obtain $\lim_{\varepsilon \to 0} f^\varepsilon(x;\lambda) = F(x)$ we assume that the function $\varepsilon \mapsto G^\varepsilon(x;\lambda)$ is sufficiently differentiable and that $G^0(x;\lambda) = 0$.

Completing the pinched system in this way is possible provided that (6.4.3) restricted to the manifold Σ is structurally stable (see [94]). The function G that completes the pinched system is not unique.

Theorem R. For $\varepsilon > 0$ sufficiently small in (6.4.5), if there exists a continuous family $\lambda^*_\varepsilon(p) \in \mathcal{I}$ of C^1 functions such that $K(p;\lambda^*_\varepsilon(p)) = 0$ by (6.2.2) for every $p \in \Sigma$, then the nonlinear sliding mode by (6.2.3) satisfies

$$\dot{p} = f^{ns}(p) = F(p) + r(p;\varepsilon) \qquad \text{on} \quad \Sigma^{ns},$$

where $r(p;\varepsilon)$ is a continuous function that is C^1 in the first variable, and where $r(p;\varepsilon) \to 0$ as $\varepsilon \to 0$. Moreover if we assume that (6.4.3) restricted to the invariant manifold Σ is structurally stable, then it is topologically equivalent to the nonlinear sliding dynamics.

Proof. Direct application of (6.2.3) to (6.4.5) gives

$$\begin{aligned} f^{ns}(p) &= \frac{1+\lambda_\varepsilon^*(p)}{2}F(p+\varepsilon\nabla h(p)) + \frac{1-\lambda_\varepsilon^*(p)}{2}F(p-\varepsilon\nabla h(p)) + G^\varepsilon(p;\lambda_\varepsilon^*(p)) \\ &= F(p) + r(p;\varepsilon), \end{aligned}$$

the second line following because $\lambda_\varepsilon^*(p)$ is a continuous family of functions. Since the system $\dot{p} = F(p)$ is structurally stable it must therefore be topologically equivalent to $\dot{p} = f^{ns}(p)$. □

We shall assume now that the function F is of class C^{k+1}, and that

$$G^\varepsilon(p;\lambda) = \sum_{i=1}^{k} \varepsilon^i \gamma_i(p;\lambda) + \mathcal{O}(\varepsilon^{k+1})$$

for some functions γ_i. Similar to (6.2.1) we define the ε–family of functions $K^\varepsilon(p;\lambda) = f^\varepsilon(p;\lambda)\cdot\nabla h(p)$, and expand

$$K^\varepsilon(p;\lambda) = \sum_{i=r}^{k} \varepsilon^i \kappa_i(p;\lambda)/i! \tag{6.4.6}$$

in terms of functions κ_i given by

$$\begin{aligned} \kappa_i(p;\lambda) = &\quad i!\,\frac{\partial^i}{\partial\varepsilon^i}K^\varepsilon(p;\lambda)\Big|_{\varepsilon=0} \\ = &\quad \lambda^{\frac{1-(-1)^i}{2}}\left[(\nabla h(p)\cdot\nabla)^i F(p)\right]\cdot\nabla h(p) + \gamma_i(p;\lambda)\cdot\nabla h(p), \end{aligned}$$

for $i = 1, 2, \ldots, k$. Here $(\nabla h(p)\cdot\nabla)^i F(p) \in \mathcal{D}$ denotes the scalar derivative $\nabla h\cdot\nabla = \sum_{j=1}^{n}\frac{\partial h}{\partial x_j}\frac{\partial}{\partial x_j}$ applied i times to F and evaluated at p.

Theorem S. For $r \le k$ assume that $\kappa_i = 0$ for $i = 1, 2, \ldots, r-1$ and $\kappa_r \neq 0$. Suppose that there exists $\ell(p) \in (-1, 1)$ such that $\kappa_r(p;\ell(p)) = 0$ and $(\partial\kappa_r/\partial\lambda)(p;\ell(p)) \neq 0$ for every $p \in \Sigma$. Then for $\varepsilon > 0$ sufficiently small there exists a continuous family $\lambda_\varepsilon^*(p) \in \mathcal{I}$ of C^1 functions such that $K^\varepsilon(p;\lambda_\varepsilon^*(p)) = 0$ for every $p \in \Sigma$. Moreover if we assume that the system (6.4.3) restricted to the invariant manifold Σ is structurally stable, then on Σ it is topologically equivalent to the nonlinear sliding mode defined by (6.2.3).

Proof. Assuming that $\kappa_i = 0$ for $i = 1, 2, \ldots, r-1$ we write using (6.4.6)

$$K^\varepsilon(p;\lambda) = \varepsilon^r\frac{\kappa_r(p;\lambda)}{r!} + \mathcal{O}(\varepsilon^{r+1}).$$

Since $\kappa_r(p;\ell(p)) = 0$ and $(\partial\kappa_r/\partial\lambda)(p;\ell(p)) \neq 0$, applying the implicit function theorem for the function $K^\varepsilon(p;\lambda)/\varepsilon^r$ we obtain, for $\varepsilon > 0$ sufficiently small, the existence of a differentiable family $\lambda_\varepsilon^*(p) \in \mathcal{I}$ of C^1 functions such that $K^\varepsilon(p;\lambda_\varepsilon^*(p)) = 0$ for every $p \in \Sigma$. The result follows by applying Theorem R. □

In some cases it is sufficient to take $G^\varepsilon \equiv 0$ (i.e. a linear combination) to complete the pinched system (6.4.4). The following corollary concerns cases, as in Example 6.4.1, for which G^ε cannot be zero everywhere.

Corollary 6.4.1. Assume in (6.4.4) that F is a C^3 function. The following statements hold:

(a) If $[\nabla h(p) \cdot \nabla F(p)] \cdot \nabla h(p) \neq 0$ then $G^\varepsilon \equiv 0$ completes the pinched system (6.4.5).

(b) If $[(\nabla h(p) \cdot \nabla)F(p)] \cdot \nabla h(p) = 0$ and $[(\nabla h(p) \cdot \nabla)^2 F(p)] \cdot \nabla h(p) \neq 0$ then the function $G^\varepsilon \equiv 0$ does not complete the pinched system. In this case $G^\varepsilon = \varepsilon^2(\lambda^2 - 1)C(p)$ with $C(p) \neq [(\nabla h(p) \cdot \nabla)^2 F(p)] \cdot \nabla h(p)$ completes the system.

Proof. Taking $G^\varepsilon \equiv 0$ we have from above that

$$K^\varepsilon(p;\lambda) = \varepsilon\lambda[(\nabla h(p) \cdot \nabla)F(p)] \cdot \nabla h(p) + \varepsilon^2 \frac{1}{2}[(\nabla h(p) \cdot \nabla)^2 F(p)] \cdot \nabla h(p) + \mathcal{O}(\varepsilon^3).$$

If $[\nabla h(p) \cdot \nabla F(p)] \cdot \nabla h(p) \neq 0$ we can choose $\ell(p) = 0$, thus $\kappa_1(p, 0) = 0$ and $(\partial\kappa_1/\partial\lambda)(p; \ell(p)) = [\nabla h(p) \cdot \nabla F(p)] \cdot \nabla h(p) \neq 0$. Hence applying Theorem (S) we have statement (a).

If instead $[\nabla h(p) \cdot \nabla F(p)] \cdot \nabla h(p) = 0$ and $[(\nabla h(p) \cdot \nabla)^2 F(p)] \cdot \nabla h(p) \neq 0$, there is no bounded family of solutions $\lambda^*_\varepsilon(p)$ of the equation $K^\varepsilon(p; \lambda^*_\varepsilon(p)) = 0$ for $G^\varepsilon \equiv 0$. Taking instead $G^\varepsilon = \varepsilon^2(\lambda^2 - 1)C(p)$ such that $C(p) \neq [(\nabla h(p) \cdot \nabla)^2 F(p)] \cdot \nabla h(p)$ we have that

$$K^\varepsilon(p;\lambda) = \varepsilon^2 C(p).$$

So $\lambda^*_\varepsilon(p) = 0 \in \mathcal{I}$ is a family of solutions of $K^\varepsilon(p; \lambda^*_\varepsilon(p)) = 0$. Applying Theorem S we then have statement (b). □

A simple example is given by $\dot{x}_1 = -x_1$ with $(\dot{x}_2, .., \dot{x}_n) = q(x_2, ..., x_n)$ where q is any smooth function; this would give a complete pinched system with Filippov (i.e. $G^\varepsilon \equiv 0$) sliding dynamics equivalent to the smooth system's invariant dynamics on $x_1 = 0$. Instead consider the following more interesting system.

Example 6.4.2. For $x_1 \in \mathbb{R}$ and $\mathbf{y} = (x_2, x_3, ..., x_n) \in \mathbb{R}^{n-1}$ consider the system

$$\dot{x}_1 = x_1^2, \quad \dot{\mathbf{y}} = q(\mathbf{y}).$$

Taking $h(x_1, \mathbf{y}) = x_1$ the manifold $\Sigma = \{x \in \mathcal{D} : x_1 = 0\}$ is invariant under the flow. The dynamics defined on Σ is given by $\dot{\mathbf{y}} = q(\mathbf{y})$, and the incomplete pinched system is given by

$$\dot{x}_1 = \left\{ \begin{array}{ll} (x_1 + \varepsilon)^2 & \text{if} \quad x_1 > 0 \\ \\ (x_1 - \varepsilon)^2 & \text{if} \quad x_1 < 0 \end{array} \right\}, \quad \dot{\mathbf{y}} = q(\mathbf{y}). \tag{6.4.7}$$

Computing the function $K^\varepsilon(0, \mathbf{y}; \lambda)$ we obtain $K^\varepsilon(0, \mathbf{y}; \lambda) = G^\varepsilon(0, \mathbf{y}; \lambda) \cdot \nabla h + \varepsilon^2$. Clearly for $G^\varepsilon \equiv 0$ (the linear/Filippov case) with $\varepsilon > 0$ the equation $K^\varepsilon(0, \mathbf{y}; \lambda) = 0$ has no solutions, instead (6.4.4) has only crossing solutions, and this does not represent the dynamics of the smooth system (6.4.7). Taking instead $G^\varepsilon(x_1, \mathbf{y}; \lambda) = (\varepsilon^2(\lambda^2 - 1), 0, 0, ...)$ we find that, for $\varepsilon > 0$ sufficiently small, $\lambda^*_\varepsilon(x_1, \mathbf{y}) = 0 \in \mathcal{I}$ is a family of solutions of $K^\varepsilon(0, \mathbf{y}; \lambda) = 0$, and produces a nonlinear sliding mode given from (6.2.3) by $\dot{\mathbf{y}} = f^{ns}(0, \mathbf{y}) = (0, q(\mathbf{y}))$.

In this example, therefore, we can complete the pinched system, but we cannot use Theorem S to prove equivalence between the pinched sliding dynamics and the original invariant dynamics on Σ, because the original continuous system, in particular the term $\dot{x}_1 = x_1^2$, is structurally unstable. To handle such cases it is necessary to perturb the original system by a small quantity. It is then natural to pinch with respect to that small quantity, giving a pinching parameter that is intrinsic to the system.

6.4.2 Intrinsic pinching

Let I and U be open bounded subsets of $\mathbb{R}$ and $\mathbb{R}^{n-1}$, respectively. For $x_1 \in I$ and $\mathbf{y} = (x_2, x_3, ..., x_n) \in U$ consider the system

$$\dot{x}_1 = F_1(x_1, \mathbf{y}; \mu), \quad \dot{\mathbf{y}} = \mu\, F_{\mathbf{y}}(x_1, \mathbf{y}; \mu). \tag{6.4.8}$$

where $F = (F_1, F_{\mathbf{y}})$ is a C^1 function and μ is a small parameter. We assume that for $\mu = 0$ the graph $\Sigma = \{(0, \mathbf{y}) : \mathbf{y} \in U\}$ is a critical invariant manifold of (6.4.8), that is $F_1(0, \mathbf{y}; 0) = 0$ for every $\mathbf{y} \in U$.

We also assume that, for $\mu > 0$ sufficiently small, the graphs $\Sigma_\varepsilon^i = \{(m_\varepsilon^i(\mathbf{y}), \mathbf{y}) : \mathbf{y} \in U\}$ for $i = 1, 2, \ldots, k$, are invariant manifolds of (6.4.8), where $m_\varepsilon^i(\mathbf{y}) = \varepsilon\, m_i(\mathbf{y}) + \mathcal{O}(\varepsilon^2)$ for some differentiable functions $m_i : \overline{U} \to \mathbb{R}$, such that the Σ_ε^i are order ε-perturbations of Σ. We assume that $\mu = \mathcal{O}(\varepsilon^r)$ where $r \geq 1$, so that taking $\mu = \mu(\varepsilon)$ we have that $\mu(0) = 0$. System (6.4.8) induces dynamics on each Σ_ε^i, namely

$$\dot{\mathbf{y}} = \mu(\varepsilon) F_{\mathbf{y}}(m_\varepsilon^i(\mathbf{y}), \mathbf{y}; \mu(\varepsilon)) \qquad \text{on} \qquad x_1 = m_\varepsilon^i(\mathbf{y})\ . \tag{6.4.9}$$

Now let R be a positive real number such that $R > \max\{|m_i(\mathbf{y})|\colon \mathbf{y} \in \overline{U},\ i = 1, 2, \ldots, k\}$. For $\varepsilon > 0$ sufficiently small we consider the following discontinuous system,

$$\begin{aligned} \dot{x}_1 &= \begin{cases} F_1(x_1 + \varepsilon R, \mathbf{y}; \mu(\varepsilon)) & \text{if} \quad x_1 > 0, \\ F_1(x_1 - \varepsilon R, \mathbf{y}; \mu(\varepsilon)) & \text{if} \quad x_1 < 0, \end{cases} \\ \dot{\mathbf{y}} &= \begin{cases} \mu(\varepsilon) F_{\mathbf{y}}(x_1 + \varepsilon R, \mathbf{y}; \mu(\varepsilon)) & \text{if} \quad x_1 > 0, \\ \mu(\varepsilon) F_{\mathbf{y}}(x_1 - \varepsilon R, \mathbf{y}; \mu(\varepsilon)) & \text{if} \quad x_1 < 0. \end{cases} \end{aligned} \tag{6.4.10}$$

We call a *incomplete* intrinsically pinched system, where Σ is now the switching manifold where the dynamics is not well defined. The discontinuous vector field $(F_1, \mu F_{\mathbf{y}})$ on the righthand side of (6.4.10) will be denoted by $F(x_1, \mathbf{y}; \mu(\varepsilon))$.

As we did for extrinsic pinching, we must now attempt to complete the system. In this case we must ask whether the pinched system (6.4.10) can be completed in the form (6.1.4) such that there exist k nonlinear sliding modes, each of which agrees with the dynamics of (6.4.9) for $i = 1, 2, \ldots, k$.

When this is possible for some family of functions G^ε we say that G^ε *completes* the pinched system, and we call

$$\begin{aligned}
\dot{x}_1 &= f^\varepsilon(x_1, \mathbf{y}; \lambda) \\
&= \frac{1+\lambda}{2} F(x_1 + \varepsilon R, \mathbf{y}; \mu(\varepsilon)) + \frac{1-\lambda}{2} F(x_1 - \varepsilon R, \mathbf{y}; \mu(\varepsilon)) + G^\varepsilon(x_1, \mathbf{y}; \lambda), \\
\lambda \in \mathcal{I}, &\qquad G^\varepsilon(x_1, \mathbf{y}; \pm 1) = 0,
\end{aligned}$$

the *complete* intrinsically pinched system. As before we impose $G^0(x; \lambda) = 0$.

Theorem T. Suppose that the system (6.4.8) has an invariant manifold defined as the graph of the function $m_\varepsilon(\mathbf{y}) = \varepsilon m(\mathbf{y}) + \mathcal{O}(\varepsilon^2)$. If the system

$$\begin{aligned}
\dot{\mathbf{y}} = \varepsilon \mu'(0) F_{\mathbf{y}}(0, \mathbf{y}; 0) + \frac{\varepsilon^2}{2} \Big(&\mu''(0) F_{\mathbf{y}}(0, \mathbf{y}; 0) + 2\mu'(0)^2 \frac{\partial F_{\mathbf{y}}}{\partial \mu}(0, \mathbf{y}; 0) \\
&+ 2\mu'(0) m_i(\mathbf{y}) \frac{\partial F_{\mathbf{y}}}{\partial x_1}(0, \mathbf{y}; 0) \Big)
\end{aligned}$$

is structurally stable and

$$\mu'(0) \frac{\partial F_1}{\partial \mu}(0, \mathbf{y}; 0) \frac{\partial F_1}{\partial x_1} F_1(0, \mathbf{y}; 0) \neq 0,$$

then the function $G^\varepsilon(x_1, \mathbf{y}; \lambda) = (0, 0)$ completes the system.

Proof. The graph $\Sigma_\varepsilon = \{(m_\varepsilon(\mathbf{y}), \mathbf{y}) : \mathbf{y} \in U\}$ is an invariant manifold for system (6.4.8), so taking $h_\varepsilon(x_1, \mathbf{y}) = x_1 - m_\varepsilon(\mathbf{y})$ we have

$$\begin{aligned}
0 = &\ \nabla h_\varepsilon(m_\varepsilon(\mathbf{y}), \mathbf{y}) F(m_\varepsilon(\mathbf{y}), \mathbf{y}; \mu(\varepsilon)) \\
= &\ F_1(m_\varepsilon(\mathbf{y}), \mathbf{y}; \mu(\varepsilon)) - \mu(\varepsilon) m'_\varepsilon(\mathbf{y}) F_{\mathbf{y}}(\varepsilon, m(\mathbf{y}), \mathbf{y}; \mu(\varepsilon)),
\end{aligned}$$

for $\varepsilon > 0$ sufficiently small. Thus taking the derivative in $\varepsilon = 0$ we obtain

$$\mu'(0) \frac{\partial F_1}{\partial \mu}(0, \mathbf{y}; 0) + m(\mathbf{y}) \frac{\partial F_1}{\partial x_1}(0, \mathbf{y}; 0) = 0. \tag{6.4.11}$$

As previously we define

$$\begin{aligned}
K^\varepsilon(0, \mathbf{y}; \lambda) = &\ \nabla h(0, \mathbf{y}) f^\varepsilon(0, \mathbf{y}; \lambda) \\
= &\ \frac{1+\lambda}{2} F_1(\varepsilon R, \mathbf{y}; \mu(\varepsilon)) + \frac{1-\lambda}{2} F_1(-\varepsilon R, \mathbf{y}; \mu(\varepsilon)) \\
= &\ \varepsilon \left(\mu'(0) \frac{\partial F_1}{\partial \mu}(0, \mathbf{y}; 0) + R \lambda \frac{\partial F_1}{\partial x_1}(0, \mathbf{y}; 0) \right) + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

Now let $\mathcal{K}(\mathbf{y};\lambda,\varepsilon)=\dfrac{K^\varepsilon(0,\mathbf{y};\lambda)}{\varepsilon}$. From (6.4.11) we have that $\mathcal{K}\left(\mathbf{y};\dfrac{m(\mathbf{y})}{R},0\right)=0$, and by hypothesis

$$\frac{\partial\mathcal{K}}{\partial\lambda}(\mathbf{y};\lambda,0)\Big|_{(\lambda,\varepsilon)=(m_i(\mathbf{y})/R,0)}=R\frac{\partial F_1}{\partial x_1}(0,\mathbf{y};0)\neq 0.$$

Hence from the implicit function theorem we have that for $\varepsilon>0$ sufficiently small there exists $\lambda(0,\mathbf{y};\varepsilon)=\dfrac{m(\mathbf{y})}{R}+\varepsilon\overline{\lambda}+\mathcal{O}(\varepsilon^2)$ such that $\lambda(0,\mathbf{y};\varepsilon)\in\mathcal{I}$ and $K^\varepsilon(0,\mathbf{y};\lambda(0,\mathbf{y};\varepsilon))=0$ for every $\mathbf{y}\in U$ and for $\varepsilon>0$ sufficiently small. It is easy to obtain an expression for $\overline{\lambda}$, but we do not require it here.

Writing $f=(f_1,f_{\mathbf{y}})$, the nonlinear sliding mode $f(0,\mathbf{y};\lambda(0,\mathbf{y};\varepsilon))=(0,f_{\mathbf{y}}(0,\mathbf{y};\lambda(0,\mathbf{y};\varepsilon)))$ is given by

$$\begin{aligned}f_{\mathbf{y}}(0,\mathbf{y};\lambda_i(0,\mathbf{y};\varepsilon))=\;&\varepsilon\mu'(0)F_{\mathbf{y}}(0,\mathbf{y};0)+\varepsilon^2\Big(\frac{\mu''(0)}{2}F_{\mathbf{y}}(0,\mathbf{y};0)\\&+\mu'(0)^2\frac{\partial F_{\mathbf{y}}}{\partial\mu}(0,\mathbf{y};0)+\mu'(0)m(\mathbf{y})\frac{\partial F_{\mathbf{y}}}{\partial x_1}(0,\mathbf{y};0)\Big)+\mathcal{O}(\varepsilon^3).\end{aligned}\tag{6.4.12}$$

Hence, expanding system (6.4.9) about $\varepsilon=0$ in a Taylor series up to second order in ε, we conclude that the nonlinear sliding mode (6.4.12) is equivalent to the system (6.4.9). □

A prototype for systems satisfying the hypotheses of Theorem T is $\dot{x}_1=x_1-\mu$, $\dot{x}_2=\mu x_2$, with a slow invariant manifold $x_1=\mu m_\mu(x_2)$ that becomes the critical manifold $x_1=0$ when $\mu=0$.

It is clear that the function $G^\varepsilon\equiv 0$ does not complete the system if $k>1$. In particular we have the following.

Theorem U. Suppose that system (6.4.8) has two invariant manifolds defined as the graphs of the functions $m^i_\varepsilon(\mathbf{y})=\varepsilon m_i(\mathbf{y})+\mathcal{O}(\varepsilon^2)$ for $i=1,2$ where $\mu(\varepsilon)=\mathcal{O}(\varepsilon^2)$. We assume $m_1\neq m_2$ and that

$$\mu''(0)\frac{\partial F_1}{\partial\mu}(0,\mathbf{y};0)\frac{\partial^2F_1}{\partial x_1^2}F_1(0,\mathbf{y};0)\neq 0.$$

If for $\varepsilon>0$ sufficiently small the system

$$\dot{\mathbf{y}}=\frac{\mu''(0)}{2}F_{\mathbf{y}}(0,\mathbf{y};0)+\frac{\varepsilon}{6}\left(\mu'''(0)F_{\mathbf{y}}(0,\mathbf{y};0)+3\mu''(0)m_i(\mathbf{y})\frac{\partial F_{\mathbf{y}}}{\partial x_1}(0,\mathbf{y};0)\right)$$

is structurally stable for $i=1,2$, then the function

$$G^\varepsilon(x_1,\mathbf{y};\lambda)=\varepsilon^2(\lambda^2-1)\left(\frac{R^2}{2}\frac{\partial^2F_1}{\partial x_1^2}(0,\mathbf{y};0)\,,\,0\right)$$

completes the system.

Proof. The graph $\Sigma_\varepsilon^i = \{(m_\varepsilon^i(\mathbf{y}), \mathbf{y}) : \mathbf{y} \in U\}$ is an invariant manifold for system (6.4.8), so taking $h_\varepsilon^i(x_1, \mathbf{y}) = x_1 - m_\varepsilon^i(\mathbf{y})$ we have that

$$\begin{aligned} 0 = & \; F(m_i(\mathbf{y}), \mathbf{y}; \mu(\varepsilon)) \cdot \nabla h_i(m_i(\mathbf{y}), \mathbf{y}) \\ = & \; F_1(\varepsilon m_i(\mathbf{y}), \mathbf{y}; \mu(\varepsilon)) - \varepsilon m_i'(\mathbf{y}) F_{\mathbf{y}}(\varepsilon, m(\mathbf{y}), \mathbf{y}; \mu), \end{aligned}$$

for $\varepsilon > 0$ sufficiently small. Thus taking the second derivative at $\varepsilon = 0$ we obtain

$$\mu''(0) \frac{\partial F_1}{\partial \mu}(0, \mathbf{y}; 0) + m_i(\mathbf{y})^2 \frac{\partial^2 F_1}{\partial x_1^2}(0, \mathbf{y}; 0) = 0. \tag{6.4.13}$$

As previously we define

$$\begin{aligned} K^\varepsilon(0, \mathbf{y}; \lambda) = & \; f^\varepsilon(0, \mathbf{y}; \lambda) \cdot \nabla h(0, \mathbf{y}) \\ = & \; \frac{1+\lambda}{2} F_1(\varepsilon R, \mathbf{y}; \mu(\varepsilon)) + \frac{1-\lambda}{2} F_1(-\varepsilon R, \mathbf{y}; \mu(\varepsilon)) + G^\varepsilon(0, \mathbf{y}; \lambda) \\ = & \; \frac{\varepsilon^2}{2} \left(\mu''(0) \frac{\partial F_1}{\partial \mu}(0, \mathbf{y}; 0) + R^2 \lambda^2 \frac{\partial^2 F_1}{\partial x_1^2}(0, \mathbf{y}; 0) \right) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Now let $\mathcal{K}(\mathbf{y}; \lambda, \varepsilon) = \dfrac{K^\varepsilon(0, \mathbf{y}; \lambda)}{\varepsilon^2}$. From (6.4.13) we have $\mathcal{K}\left(\mathbf{y}; \dfrac{m_i(\mathbf{y})}{R}, 0\right) = 0$, and by hypothesis

$$\frac{\partial \mathcal{K}}{\partial \lambda}(\mathbf{y}; \lambda, \varepsilon)\Big|_{(\lambda,\varepsilon)=(m_i(\mathbf{y})/R, 0)} = R\, m_i(\mathbf{y}) \frac{\partial^2 F_1}{\partial x_1^2}(0, \mathbf{y}; 0) \neq 0.$$

Hence from the implicit function theorem, for $\varepsilon > 0$ sufficiently small there exists $\lambda_i(0, \mathbf{y}; \varepsilon) = \dfrac{m_i(\mathbf{y})}{R} + \mathcal{O}(\varepsilon)$ such that $\lambda_i(0, \mathbf{y}; \varepsilon) \in \mathcal{I}$ and $K^\varepsilon(0, \mathbf{y}; \lambda_i(0, \mathbf{y}; \varepsilon)) = 0$ for every $\mathbf{y} \in U$ and for $i = 1, 2$.

The nonlinear sliding mode $f(0, \mathbf{y}; \lambda_i(0, \mathbf{y}; \varepsilon)) = (0, f_{\mathbf{y}}(0, \mathbf{y}; \lambda_i(0, \mathbf{y}; \varepsilon)))$ is given by

$$\begin{aligned} f_{\mathbf{y}}(0, \mathbf{y}; \lambda_i(0, \mathbf{y}; \varepsilon)) = & \; \frac{\varepsilon^2 \mu''(0)}{2} F_{\mathbf{y}}(0, \mathbf{y}; 0) + \frac{\varepsilon^2}{6} \Big(\mu'''(0) F_{\mathbf{y}}(0, \mathbf{y}; 0) \\ & + 3\mu''(0) m_i(\mathbf{y}) \frac{\partial F_{\mathbf{y}}}{\partial x_1}(0, \mathbf{y}; 0) \Big) + \mathcal{O}(\varepsilon^4), \end{aligned} \tag{6.4.14}$$

for $i = 1, 2$. Hence, expanding system (6.4.9) around $\varepsilon = 0$ in Taylor series up to third order in ε, we conclude that the nonlinear sliding mode (6.4.14) is equivalent to the system (6.4.9) for each $i = 1, 2$. □

A prototype for systems satisfying the hypotheses of Theorem U is $\dot{x}_1 = x_1^2 - \mu$, $\dot{x}_2 = \mu x_2$, with slow manifolds $x_1 = \pm\sqrt{\mu} m(x_2)$ which are normally hyperbolic for $\mu > 0$, but which coalesce onto a non-hyperbolic critical manifold $x_1 = 0$ for $\mu = 0$.

Chapter 7

Further directions

Regarding Chapters 1 and 2, possible directions for further investigations are the extensions and generalizations, for non–smooth systems, of the averaging theory in multifrequency systems (see, for instance, [30]). As far as we know, up to now there are no results in this line of research.

Regarding Chapters 3 and 4, it remains an open problem to estimate the upper bound for the maximum number of limit cycles allowed in planar piecewise linear differential systems with two zones separated by a straight line. So it represents an obviously direction for further investigations.

Regarding Chapter 5, higher dimensional vector fields allows the existence of many other kinds of sliding homoclinic connections. So the study of typical sliding homoclinic connection in higher dimensions seems to be a very fertile theme of research. Another possible direction for further investigation is to apply the techniques from ergodic theory to provide deeper results on this kind of sliding connection. For instance, the existence of symbolic extensions, conjugation with Bernoulli shifts, and existence of Smale horseshoes.

Regarding Chapter 6, particular forms for the function G^ε that complete an intrinsically pinched system are given for slow-fast dynamics with one or two slow critical invariant manifolds, but the result can certainly be extended in such a way that a general theory may proceed along similar lines to normal forms of singularities.

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Appendix A

Basic results on the Brouwer degree

In this appendix we present some results of the degree theory in finite dimensional spaces. We follow the Browder's paper [15], where the properties of the classical Brouwer degree are formalized.

Proposition A.0.1. Let $X = \mathbb{R}^n = Y$ for a given positive integer n. For bounded open subsets V of X, consider the continuous map $f : \overline{V} \to Y$, and a point y_0 in Y such that y_0 does not lie in $f(\partial V)$ (as usual ∂V denotes the boundary of V). Then to each triple (f, V, y_0), there exists an integer $d(f, V, y_0)$ having the following properties.

(i) If $d(f, V, y_0) \neq 0$, then $y_0 \in f(V)$. If f_0 is the identity map of X onto Y, then for every bounded open set V and $y_0 \in V$, we have

$$d\left(f_0|_V, V, y_0\right) = \pm 1.$$

(ii) (Additivity) If $f : \overline{V} \to Y$ is a continuous map with V a bounded open set in X, and V_1 and V_2 are a pair of disjoint open subsets of V such that

$$y_0 \notin f(\overline{V} \backslash (V_1 \cup V_2)),$$

then,

$$d\left(f_0, V, y_0\right) = d\left(f_0, V_1, y_0\right) + d\left(f_0, V_1, y_0\right).$$

(iii) (Invariance under homotopy) Let V be a bounded open set in X, and consider a continuous homotopy $\{f_t : 0 \leq t \leq 1\}$ of maps of $\overline{V}$ in to Y. Let $\{y_t : 0 \leq t \leq 1\}$ be a continuous curve in Y such that $y_t \notin f_t(\partial V)$ for any $t \in [0, 1]$. Then $d(f_t, V, y_t)$ is constant in t on $[0, 1]$.

Proposition A.0.2. The degree function $d(f, V, y_0)$ is uniquely determined by the conditions of Theorem A.0.1.

For the proofs of Theorems A.0.1 and A.0.2 see [15].

Lemma A.0.1. We consider the continuous functions $f_i : \overline{V} \to \mathbb{R}^n$, for $i = 0, 1, \cdots, k$, and $f, g, r : \overline{V} \times [\varepsilon_0, \varepsilon_0] \to \mathbb{R}^n$, given by

$$g(\cdot, \varepsilon) = f_1(\cdot) + \varepsilon f_2(\cdot) + \varepsilon^2 f_3(\cdot) + \cdots + \varepsilon^{k-1} f_k(\cdot),$$

$$f(\cdot,\varepsilon) = g(\cdot,\varepsilon) + \varepsilon^k r(\cdot,\varepsilon).$$

Assume that $g(z,\varepsilon) \neq 0$ for all $z \in \partial V$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. If for $|\varepsilon| > 0$ sufficiently small $d_B\left(f(\cdot,\varepsilon), V, y_0\right)$ is well defined, then

$$d_B\left(f(\cdot,\varepsilon), V, y_0\right) = d_B\left(g(\cdot,\varepsilon), V, y_0\right).$$

For a proof of Proposition A.0.1 see Lemma 2.1 in [19].

Appendix B

Chebyshev systems

In this appendix we introduce important tools of the Chebyshev theory that we shall use to prove the main results of chapter 4 (Theorems J, K and L). For more details about Chebyshev systems, see for instance, the book of Karlin and Studden [61].

Consider an ordered set of smooth real functions $\mathcal{F} = (f_0, f_1, \ldots, f_n)$ defined on a interval I. The maximum number of zeros counting multiplicity admitted by any nontrivial linear combination of functions in $\mathcal{F}$ is denoted as $Z(\mathcal{F})$.

Definition B.0.1. We say that $\mathcal{F}$ is an *Extended Chebyshev* system or ET–system on I if and only if $Z(\mathcal{F}) \leq n$. We say that $\mathcal{F}$ is an *Extended Complete Chebyshev* system or an ECT–system on I if and only if for any k, $0 \leq k \leq n$, $(f_0, f_1, \ldots, f_k)$ is an ET–system.

The next proposition relates the property of an ordered set of functions $(f_0, f_1, \ldots, f_k)$ being an ECT–system with the nonvanishing property of their Wronskians

$$W(f_0, f_1, \ldots, f_k)(t) = \begin{vmatrix} f_0(t) & f_1(t) & \cdots & f_k(t) \\ f_0'(t) & f_1'(t) & \cdots & f_k'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(k)}(t) & f_1^{(k)}(t) & \cdots & f_k^{(k)}(t) \end{vmatrix}.$$

Proposition B.0.3 ([61])**.** A ordered set of functions $\mathcal{F} = (f_0, f_1, \ldots, f_k)$ is an ECT–system on I if and only if $W(f_0, f_1, \ldots, f_i)(t) \neq 0$ on I for $0 \leq i \leq k$.

The next result has been proved by Novaes and Torregrosa in [93].

Proposition B.0.4 ([93])**.** Let $\mathcal{F} = (u_0, u_1, \ldots, u_n)$ be an ordered set of smooth functions on $[a, b]$. Assume that all the Wronskians are nonvanishing except $W_n(x)$ which have $\ell \geq 0$ zeros on (a, b) and all these zeros are simple. Then $Z(\mathcal{F}) = n$ when $\ell = 0$, and $n + 1 \leq Z(\mathcal{F}) \leq n + \ell$ when $\ell \neq 0$.