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Classification and Structure of certain Representations of Quantum Affine Algebras

Classificação e Estrutura de certas Representações de Álgebras Afim Quantizadas

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CLASSIFICATION AND STRUCTURE OF CERTAIN REPRESENTATIONS OF QUANTUM AFFINE ALGEBRAS

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Orientador: Adriano Adrega de Moura

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Abstract

We study finite–dimensional representations for a quantum affine algebra from two different points of view. In the first part of this work we study the graded limit of a certain subclass of irreducible representations. Let V be a finite–dimensional representation for a quantum affine algebra of type A and assume that V is isomorphic to the tensor product of a minimal affinization by parts whose highest weight is a sum of distinct fundamental weights by Kirillov-Reshetkhin modules whose highest weights are twice a fundamental weight. We prove that V admits a graded limit L and that L is isomorphic to a level-two Demazure module as well as to the fusion product of the graded limits of each of the aforementioned tensor factors of V. We also prove that if the quantum affine algebra is of classical type (resp. type G_2), the graded limit of (regular) minimal affinizations (resp. Kirillov–Reshetkin modules) are isomorphic to CV-modules for some R^+–partition explicitly described.

In the second part we show that a module for the quantum affine algebra of type B_n is tame if and only if it is thin. In other words, the Cartan currents are diagonalizable if and only if all joint generalized eigenspaces have dimension one. We classify all such modules and describe their q-characters. In some cases, the q-characters are described by super standard Young tableaux of type $(2n|1)$.

Resumo

Estudamos representações de dimensão finita para uma álgebra afim quantizada a partir de dois pontos de vista distintos. Na primeira parte deste trabalho estudamos o limite graduado de uma certa subclasse de representações irredutíveis. Seja V uma representação de dimensão finita para uma álgebra do tipo A e suponha que V é isomorfa ao produto tensorial de uma afinização minimal por partes cujo peso máximo é a soma de distintos pesos fundamentais por módulos de Kirillov–Reshetikhin cujos pesos máximos são o dobro de um peso fundamental. Provamos que V admite limite graduado L e que L é isomorfo a um módulo de Demazure de nível dois bem como ao produto de fusão dos limites graduados de cada um dos supramencionados fatores tensoriais de V. Provamos ainda que, se a álgebra for do tipo clássica (resp. G_2), o limite graduado das afinizações minimais (regulares) (resp. módulos de Kirillov–Reshetikhin) são isomorfos ao módulos CV para alguma R^+–partição descrita explicitamente.

Na segunda parte provamos que um módulo para a álgebra afim quantizada do tipo B_n é manso se, e somente se, ele é fino. Em outras palavras, os geradores da subálgebra de Cartan afim são diagonalizáveis se, e somente se, os autoespaços generalizados associados têm dimensão um. Classificamos tais módulos e descrevemos seus respectivos q-caracteres. Em alguns casos, o q-caracter é descrito por super standard Young tableaux do tipo $(2n|1)$.

Contents

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"E aqueles que foram vistos dançando foram julgados insanos por aqueles que não podiam escutar a música."

Friedrich Nietzsche

Introduction

The finite-dimensional representation theory of Kac-Moody algebras and its quantizations has been the subject of intense research for almost three decades partially motivated by its applications to Mathematical-Physics but, by now, the rich and intricate structure of the underlying category of modules and its interconnections with other areas such as Combinatorics draws attention to it as a worthy object of study by itself. One of the standard goals when studying such representations is to understand the simple objects in that category. In the case of affine Lie algebras, it was relatively easy to describe the simple objects [CP86].

In the quantum situation, it was shown in [CP91, CP94b] that the irreducible objects are parametrized by the multiplicative monoid $\mathcal{P}^+$ of n–tuples of polynomials with constant term one (also referred to as Drinfeld polynomials), where n denotes the rank of the associated simple finite-dimensional complex Lie algebra $\mathfrak{g}$. Although the classification in terms of Drinfeld polynomials is analogous to the classical context, understanding the structure of the irreducible module has proved to be a very difficult task and an extensive list of references dedicated to studying them can be found in [CH10]. However, there are families of irreducible representations which are better understood. For example, in type A, one has evaluation representations. Their analogs in other types, called minimal affinizations, received a lot of attention and, for $\mathfrak{g}$ of types B and C, the structure of all members of the family is understood, see [Cha95, CP95, CP96a, CP96b, Her07, LM13, MTZ04, Mou10, Nao13, Nao14].

An important tool in the study of finite-dimensional representations is the q-character map. The quantum affine algebra contains an infinite-dimensional commutative subalgebra which plays a role similar to that of the Cartan subalgebra of $\mathfrak{g}$ in the study of finite-dimensional representations. Hence, when we restrict the action, on a representation V, to this subalgebra we have a decomposition of V in generalized eigenspaces for the joint action of this subalgebra. The generalized eigenvalues (linear functionals on this subalgebra) associated to this decomposition are called ℓ-weights and the q-character of V is then the collection of the dimension of the generalized eigenspace associated to the ℓ-weights of V. In particular, in many cases the q-character of a simple module V can be computed recursively by an algorithm [FM01] starting from its highest ℓ-weight, i.e., the ℓ-weight associated to the Drinfeld polynomial of V. The algorithm works for the class of Kirillov–Reshetikhin modules (minimal affinizations whose highest weight is a positive integer of a fundamental weight) see [Nak11] for ADE types, [Her06] for other types. It also works for some minimal affinizations [Her07]. However, the straightforward application of the algorithm is known to fail in some cases – see for instance Example 5.6 of [HL10].

Another method used to study the simple objects in the quantum case is to understand the classical limit of these representations, i.e., their specialization at 1 of the quantum parameter q associated to the quantum algebra. The more general philosophy behind this approach is that many interesting families of irreducible representations of the quantum affine algebra (associated to $\mathfrak{g}$), when we consider their classical limit, give rise to indecomposable representations of a certain maximal parabolic subalgebra of the affine Lie algebra $\widehat{\mathfrak{g}}$, namely, $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbf{C}[t]$, the current algebra associated to $\mathfrak{g}$.

The families of representations satisfying the aforementioned property include the Kirillov-Reshetikhin modules, the minimal affinizations and the more general modules studied in [MY12]. The defining relations of the classical limit of Kirillov-Reshetikhin modules were given in [CM06] and it is clear from the definition that they are graded by the non-negative integers. The graded $\mathfrak{g}$-module decomposition was also given in that paper, and it is remarkable that the Hilbert series of this decomposition coincides with the deformed character formulae for the Kirillov-Reshetikhin modules given in [HKOTY99] where the powers of q appear for entirely different reasons from the study of solvable lattice models. As a consequence of this approach, it was proved in [CM06] and also in [FL07], that these graded limits of the Kirillov-Reshetikhin modules are isomorphic to certain Demazure modules in positive level representations of the affine Lie algebra.

Towards the study of the general minimal affinizations there still is no such results in general type. In [Mou10], the author presented several conjectures for the graded limits of minimal affinizations and partially proved them. Years later, graded limits of minimal affinizations in types ABC and D (regular case) were further studied in [Nao13, Nao14], where it was proved that they are isomorphic to some generalization of Demazure modules. We take another step in this direction and prove that a certain subclass of simple finite-dimensional modules for the quantum affine algebras $U_q(\widetilde{\mathfrak{g}})$ with $\mathfrak{g}$ of type A can be studied by considering such limit construction and that the resulting modules are isomorphic to level-two $\mathfrak{g}$-stable Demazure modules (Theorem 4.1.1). It is interesting to note that every $\mathfrak{g}$-stable Demazure module appears in this way.

Assume that $\mathfrak{g}$ is of type A_n. We now describe the subclass of simple $U_q(\widetilde{\mathfrak{g}})$-modules that we consider in the first part of this work. To each quiver whose underlying graph is the Dynkin diagram of $\mathfrak{g}$, Hernandez and Leclerc [HL10, HL13] associated a subcategory of that of finite-dimensional representations of $U_q(\widetilde{\mathfrak{g}})$ which gives rise to a monoidal categorification of certain cluster algebras (see also [Nak11]). We consider quivers such that the orientation of the arrows change exactly on the nodes of J, for a fixed subset J of the set $\{1, \ldots, n\}$. For each J there are only two such quivers, determined by the orientation of the first arrow. The prime objects of the Hernandez-Leclerc subcategories are either Kirillov-Reshetikhin modules whose highest weight is twice a fundamental weight or its highest weight is the sum of fundamental weights associated to the set J and its Drinfeld polynomial satisfies condition (ii) of [CMY13, Theorem 2]. The prime objects of this latter type can be naturally regarded as minimal affinizations by parts. Our subclass of simple modules consists exactly of those which are tensor products of the prime objects of some Hernandez-Leclerc subcategory with at most one tensor factor being a minimal affinizations by parts.

Let D be a $\mathfrak{g}$-stable level-two Demazure module and V be a simple $U_q(\tilde{\mathfrak{g}})$-module in some Hernandez-Leclerc subcategory giving rise to D in the sense described in the last paragraph (Theorem 4.1.1). It follows from [CL06, CSVW14] that D is isomorphic to the fusion product of the graded limits of the prime tensor factors of V (Theorem 4.2.8). We also show (Theorem 4.2.1) that there exists an injective map of D in a tensor product of appropriate level-one Demazure modules. More precisely, the map is determined by sending the highest weight vector of D in the tensor product of the highest-weight vectors of the two level-one Demazure modules. The submodule generated by the top weight space of a tensor product of $\mathfrak{g}$-stable Demazure modules was referred to as a generalized Demazure module in [Nao13]. Thus, Theorem 4.2.1 says that all level-two $\mathfrak{g}$-stable Demazure modules can be constructed as a generalized Demazure module from level-one Demazure modules. The results of this first part of the Thesis are part of the upcoming paper [BCM].

Before moving to the second part of the Thesis, we address two other questions related to the above. First, we establish a connection between graded limits of minimal affinizations and a new class of modules defined in [CV14], which we shall refer to as CV-modules. The CV-modules are defined by generators and relations encoded by R^+-tuples of partitions attached to a dominant weight. Since it was shown in [CV14] that all $\mathfrak{g}$-stable affine Demazure modules are CV-modules, the CV-modules can also be regarded as a generalization of Demazure modules. Based on the aforementioned results of [Nao13, Nao14], we prove that, for $\mathfrak{g}$ of classical type, the graded limits of (regular) minimal affinizations are isomorphic to CV-modules for an explicitly described partition. The second question we address was actually the original goal of this project. Namely, to describe the structure (graded character) of the graded limits of the minimal affinizations in the case that $\mathfrak{g}$ is of type G_2. We first approached the problem by using the techniques of [Mou10], at first, and [Nao13, Nao14] later. However, these techniques were not sufficient to reach our goal so far. As other papers with new techniques for related problems were surfacing ([CV14, LM13, LQ14]), we started studying them and, eventually, we were lead to the questions that became the core of the present work. The original goal of describing the structure of minimal affinizations for type G_2 remains incomplete, but we present here all the partial results we have obtained and state a conjecture relating them to CV-modules.

The motivation for our second object of study comes from the works [NT98], [MY12] and [KOS95]. It is shown in [NT98] in type A, that if the Cartan generators are diagonalizable on an irreducible module (we call this property "tame"), then their joint spectrum is necessarily simple (that is "thin"). Moreover, all such modules are pull-backs with respect to the evaluation homomorphism from a natural class of $U_q(\widetilde{\mathfrak{sl}}_{n+1})$-modules and their q-characters are described by the semistandard Young tableaux corresponding to fixed skew Young diagram. We extend these results to algebras of type B_n. We are assisted by [MY12], where the q-characters of a large family of thin B_n modules are described combinatorially in terms of certain paths and by [KOS95], where some of the q-characters are given in terms of certain Young tableaux.

We define explicitly a family of sets of Drinfeld polynomials which we call "extended snakes", and consider the corresponding irreducible finite-dimensional modules of quantum affine algebra of type B_n. This family contains all snake modules of [MY12], in particular, it contains all minimal affinizations. We extend the methods of [MY12] and describe the q-characters of the extended snake modules via explicit combinatorics of paths, see Theorem 5.1.2. This is done by using the recursive algorithm of [FM01], since the extended snake modules are thin and special (meaning that there is only one dominant ℓ-weight).

From this we show that a simple tame module of B_n type has to be an extended snake module (more precisely, a tensor product of extended snake modules), see Theorem 5.1.3. This is done by the reduction to the results of [NT98] and by induction on n. It turns out that it is sufficient to control only a small part of the q-character near the highest ℓ-weight associated to its Drinfeld polynomial. Therefore, we obtain the main result of the second part of this Thesis: an irreducible module in type B is tame if and only if it is thin. All such modules are special and antispecial (meaning that there is only one anti-dominant ℓ-weight). Moreover, thin modules are (tensor products of) extended snake modules and their q-characters are described explicitly.

Finally, we study the combinatorics of the q-character of tame B_n-modules in terms of Young tableaux. We observe a curious coincidence with the representation theory of the superalgebra $\mathfrak{gl}(2n|1)$. The irreducible representations of the latter algebra are parametrized by Young diagrams which do not contain the box with coordinates $(2n+1, 2)$. More generically, one can construct representations of $\mathfrak{gl}(2n|1)$ corresponding to skew Young diagrams which do not contain a rectangle with vertical side of length $2n+1$ and horizontal side of length 2. The character of such representations is given by super semistandard Young tableaux, see [BR83]. We find that each such skew Young diagram also corresponds to an irreducible snake module of the affine quantum algebra of type B_n. Moreover, the q-character of this module is described by the same super standard Young tableaux, see Theorem 7.3.2. We note that not all snake modules appear that way and there are cases when two different skew Young diagrams correspond to the same snake module, see Section 7.4. We have no conceptual explanation for this coincidence.

We expect that a similar analysis by the same methods can be done in other types and that the properties of being thin and tame are equivalent in general. In particular, one has Young tableaux description of certain modules in types C and D, see [NN06, NN07a, NN07b]. However, in other types, minimal affinizations are neither thin nor special in general, see [Her07, LM13]. This part of the Thesis was submitted to publication [BM14].

This work is divided in seven chapters. In the first chapter we briefly review the classical and quantum algebras as well as its affine versions. In the second chapter we review the basic concepts of Lie algebra representation theory and recall the definition of Demazure and CV modules which will play an important role along this text. In the third chapter we turn to the representation theory of the quantum affine algebra. We review the main results such as classification of finite-dimensional simple objects, main properties and tools such as the q-character and the classical limit ($q = 1$ limit). We also recall the definition of minimal affinizations by parts. In the fourth chapter we state and prove our main results which involve the graded limit of modules. In chapter 5 we define the extended snake modules and prove the classification theorem of tame modules for type B. In the sixth chapter we follow the techniques of [MY12] to compute their q-characters in terms of non-overlapping paths. Finally, in Chapter 7 we study the bijection between non-overlapping paths associated to snakes and super standard skew Young tableaux.

Index of notation

We provide for the reader's convenience a brief index of the notation which is used repeatedly in this thesis:

1.1 $\mathfrak{g}, \mathfrak{h}, \mathfrak{n}^{\pm}, I, R^+, \alpha_i, \omega_i, Q, Q^+, P, P^+, x_\alpha^{\pm}, h_\alpha, (\cdot,\cdot), \mathrm{supp}(\alpha), \theta, r^\vee, d_i, d_\alpha, \check{d}_\alpha, C = (c_{ij})_{i,j\in I}$, $\widetilde{\mathfrak{g}}, \mathfrak{g}[t], \widehat{\mathfrak{g}}, \widehat{\mathfrak{h}}, \widehat{\mathfrak{n}}^{\pm}, \widehat{\mathfrak{b}}, \widehat{R}, \widehat{R}^+, \widehat{\Delta}, \widehat{I}, \delta, h_0, \alpha_0, x_0^{\pm}, \widehat{Q}, \widehat{Q}^+, W, \widehat{W}, \widetilde{W}, t_\mu\ (\mu \in P), w_0$, $\ell(w)\ (w \in \widehat{W}), \Lambda_i\ (i \in \hat{I}), \widehat{P}, \widehat{P}^+$;

1.2 $\mathbb{F}, q_i, [m]_p, \left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]_p, U_q(\mathfrak{g}), U_q(\widetilde{\mathfrak{g}}), U_q(\widetilde{\mathfrak{h}}), U_q(\widetilde{\mathfrak{n}}^{\pm}), x_{i,r}^{\pm}, h_{i,s}, \phi_i^{\pm}(u)$;

1.3 $\mathbb{A}, (x_{i,r}^{\pm})^{(k)}, U_{\mathbb{A}}(\widetilde{\mathfrak{g}}), U_{\mathbb{A}}(\mathfrak{g})$;

2.1 $V(\lambda), V_\mu, \mathrm{ch}(V), \widehat{V}(\Lambda), V[r]\ (r \in \mathbb{Z}), V(\lambda, r)$;

2.2 $D(\Psi), D(\ell, \lambda)$;

2.3 $D(\Psi_1, \ldots, \Psi_m), \mathrm{ch}_{\widehat{\mathfrak{h}}}, \mathcal{D}_i\ (i \in \widehat{I})$;

2.4 $W(\lambda), \boldsymbol{\xi} = (\xi^\alpha)_{\alpha\in R^+}, V(\boldsymbol{\xi}), {}_k\mathbf{S}(r,s), \mathbf{S}(r,s)_k, {}_k\mathbf{x}_\alpha{}^{\pm}(r,s), \mathbf{x}_\alpha^{\pm}(r,s)_k$;

2.5 $\boldsymbol{\xi}(\ell, \lambda)$;

2.6 $V * W$;

3.1 $V_q(\lambda)$;

3.2 $V_{\boldsymbol{\gamma}}, Y_{i,a}, \mathcal{P}, \mathcal{P}^+, \mathcal{M}(V), L_q(\boldsymbol{\pi})$;

3.3 $\chi_q, \mathrm{wt}, A_{i,a}, \mathcal{Q}, \mathcal{Q}^+, \boldsymbol{\varpi} \le \boldsymbol{\pi}, u_{i,a}, \beta_J, U_q(\widetilde{\mathfrak{g}}_J), \mathcal{P}_J, \mathrm{res}_J$;

3.4 $\mathcal{P}^+_{\mathbb{A}}, \mathcal{P}^+_{\mathbb{Z}}, L(\boldsymbol{\pi})$;

3.5 $\boldsymbol{\omega}_{i,a,m}$,

4.1 $P(1)^+$;

5.1 $\mathcal{X}, \mathcal{W}, \mathcal{P}^+_{\mathcal{X}}, \mathcal{X}(\boldsymbol{\pi})$;

5.3 $S_k(a), \mathbf{S}_q(\boldsymbol{\pi})$;

6.1 $\mathscr{P}_{i,k}, C_{p,\pm}, p^{\pm}_{i,k}, \mathsf{m}$;

6.2 $p\mathscr{A}^{\pm 1}_{j,l}$;

6.3 $\overline{\mathscr{P}}_{(i_t,k_t)_{1\leq t\leq T}}$;

6.4 $\mathrm{top}(p,p'), \mathscr{R}(p,p')$;

7.1 $\mathsf{A}, B^{[k]}, {}^{[k]}B, R_p, \overline{R}_p, S_p, \overline{S}_p$ $(p \in \mathscr{P}_{i,k})$;

7.2 $(\lambda/\mu), \mathcal{T}, M(\mathcal{T}), \mathcal{T}_+$,

Chapter 1

Definitions and Notations

Throughout the thesis, let $\mathbb{C}, \mathbb{R}, \mathbb{Z}, \mathbb{Z}_{\geq m}$ denote the sets of complex numbers, reals, integers, and integers bigger or equal to m, respectively. Given any complex Lie algebra $\mathfrak{a}$ we let $U(\mathfrak{a})$ be the universal enveloping algebra of $\mathfrak{a}$.

1.1 Classical Lie algebras

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over $\mathbb{C}$ with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. Let I be an indexing set of the vertices of the Dynkin diagram of $\mathfrak{g}$ and R^+ the set of positive roots. The simple roots and fundamental weights are denoted by α_i and $\omega_i, i \in I$, respectively, while Q, P, Q^+, P^+ denote the root and weight lattices with corresponding positive cones.

We identify $\mathfrak{h}$ and $\mathfrak{h}^*$ by means of the invariant inner product $(\cdot, \cdot)$ on $\mathfrak{g}$ normalized such that the square length of the maximal root equals 2. Notice that

$$(\alpha, \alpha) = \begin{cases} 2, & \text{if } \alpha \text{ is long,} \\ 2/r^\vee, & \text{if } \alpha \text{ is short,} \end{cases}$$

where $r^\vee \in \{1, 2, 3\}$ is the lacing number of $\mathfrak{g}$. Let also

$$\check{d}_\alpha = \frac{2}{(\alpha, \alpha)}, \quad d_\alpha = \frac{1}{2} r^\vee (\alpha, \alpha), \quad \text{and} \quad d_i = d_{\alpha_i}, \ i \in I.$$

We fix a Chevalley basis of the Lie algebra $\mathfrak{g}$ consisting of $x_\alpha^\pm \in \mathfrak{g}_{\pm\alpha}$, for each $\alpha \in R^+$, and $h_i \in \mathfrak{h}$, for each $i \in I$. We also define $h_\alpha \in \mathfrak{h}, \alpha \in R^+$, by $h_\alpha = [x_\alpha^+, x_\alpha^-]$ (in particular, $h_i = h_{\alpha_i}, i \in I$). We often simplify notation and write $x_i^\pm$ in place of $x_{\alpha_i}^\pm, i \in I$.

If $\alpha = \sum_{i \in I} n_i \alpha_i \in R^+$, the *height* of α is defined by $\operatorname{ht} \alpha = \sum n_i$ and the *support* of α by $\operatorname{supp}(\alpha) = \{i \in I | n_i > 0\}$. Note that

$$h_\alpha = \sum_{i \in I} \frac{(\alpha_i, \alpha_i)}{(\alpha, \alpha)} n_i h_i,$$

Recall that, if $C = (c_{ij})$ is the Cartan matrix of $\mathfrak{g}$, i.e., $c_{ij} = \alpha_j(h_i)$, then $d_i c_{ij} = d_j c_{ji}$.

Consider the *loop algebra* $\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, with Lie bracket given by

$$[x \otimes f(t), y \otimes g(t)] = [x, y] \otimes f(t)g(t), \quad x, y \in \mathfrak{g},\ f, g \in \mathbb{C}[t, t^{-1}].$$

We identify $\mathfrak{g}$ with the subalgebra $\mathfrak{g} \otimes 1$ of $\widetilde{\mathfrak{g}}$. The subalgebra $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ is the *current algebra* of $\mathfrak{g}$. If $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$, let $\mathfrak{a}[t] = \mathfrak{a} \otimes \mathbb{C}[t]$ and $\mathfrak{a}[t]_\pm := \mathfrak{a} \otimes (t^{\pm 1}\mathbb{C}[t^{\pm 1}])$. In particular, as vector spaces,

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{n}}^- \oplus \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{n}}^+ \qquad \text{and} \qquad \mathfrak{g}[t] = \mathfrak{n}^-[t] \oplus \mathfrak{h}[t] \oplus \mathfrak{n}^+[t].$$

The *affine Kac-Moody algebra* $\widehat{\mathfrak{g}}$ is the Lie algebra with underlying vector space $\widetilde{\mathfrak{g}} \oplus \mathbb{C}c \oplus \mathbb{C}d$ equipped with the Lie bracket given by

$$[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s} + r\ \delta_{r,-s}\ (x, y)\ c, \quad [c, \widehat{\mathfrak{g}}] = \{0\}, \quad \text{and} \quad [d, x \otimes t^r] = r\ x \otimes t^r$$

for any $x, y \in \mathfrak{g}, r, s \in \mathbb{Z}$. In particular, $\widehat{\mathfrak{g}}$ is naturally a $\mathbb{Z}$-graded Lie algebra. Notice that $\mathfrak{g}$, $\mathfrak{g}[t]$, and $\mathfrak{g}[t]_\pm$ remain subalgebras of $\widehat{\mathfrak{g}}$. Set

$$\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d, \quad \widehat{\mathfrak{n}}^\pm = \mathfrak{n}^\pm \oplus \mathfrak{g}[t]_\pm \quad \text{and} \quad \widehat{\mathfrak{b}} = \mathfrak{h} \oplus \mathfrak{n}^+.$$

The root system, positive root system, and set of simple roots associated to the triangular decomposition $\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}^- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}^+$ will be denoted by $\widehat{R}$, $\widehat{R}^+$ and $\widehat{\Delta}$, respectively. Let $\widehat{I} = I \sqcup \{0\}$ and $h_0 = c - h_\theta$, so that $\{h_i | i \in \widehat{I}\} \cup \{d\}$ is a basis of $\widehat{\mathfrak{h}}$. Identify $\mathfrak{h}^*$ with the subspace $\{\lambda \in \widehat{\mathfrak{h}}^* : \lambda(c) = \lambda(d) = 0\}$. Let also $\delta \in \widehat{\mathfrak{h}}^*$ be such that $\delta(d) = 1$ and $\delta(h_i) = 0$, for all $i \in \widehat{I}$, and define $\alpha_0 = \delta - \theta$. Then $\widehat{\Delta} = \{\alpha_i | i \in \widehat{I}\}$ and $\widehat{R}^+ = R^+ \cup \{\alpha + r\delta | \alpha \in R \cup \{0\}, r \in \mathbb{Z}_{>0}\}$.

Set $x^\pm_{\alpha,r} = x^\pm_\alpha \otimes t^r, h_{\alpha,r} = h_\alpha \otimes t^r, \alpha \in R^+, r \in \mathbb{Z}$. We often simplify notation and write $x^\pm_{i,r}$ and $h_{i,r}$ in place of $x^\pm_{\alpha_i,r}$ and $h_{\alpha_i,r}, i \in I, r \in \mathbb{Z}$. Set also

$$x^\pm_0 = x^\mp_{\theta,\pm 1}.$$

Then, $[x^+_0, x^-_0] = h_0$ and $\widehat{\mathfrak{n}}^\pm$ is generated by $x^\pm_i, i \in \widehat{I}$, respectively.

Let $\widehat{Q} = \oplus_{i \in \widehat{I}} \mathbb{Z}\alpha_i$ and $\widehat{Q}^+ = \oplus_{i \in \widehat{I}} \mathbb{Z}_{\geq 0}\alpha_i$. Equip $\widehat{\mathfrak{h}}^*$ with the partial order $\lambda \leq \mu$ iff $\mu - \lambda \in \widehat{Q}^+$.

Define also $\Lambda_i \in \widehat{\mathfrak{h}}^*, i \in \widehat{I}$, by the requirement $\Lambda_i(d) = 0, \Lambda_i(h_j) = \delta_{ij}$, for all $i, j \in \widehat{I}$. Notice that

$$\Lambda_i - \omega_i = \omega_i(h_\theta)\Lambda_0, \quad \text{for all } i \in I. \tag{1.1.1}$$

Let $\widehat{P} = \mathbb{Z}\Lambda_0 \oplus P \oplus \mathbb{Z}\delta$ and $\widehat{P}^+ = \oplus_{i \in \widehat{I}} \mathbb{Z}_{\geq 0}\Lambda_i \oplus \mathbb{Z}\delta$. Given $\Lambda \in \widehat{P}$, the number $\Lambda(c)$ is called the *level* of Λ.

Let $\widehat{W}$ denote the *affine Weyl group*, which is generated by the simple reflections $\{s_i, i \in \widehat{I}\}$, which are defined by $s_i\mu = \mu - \mu(h_i)\alpha_i$ for all $\mu \in \widehat{\mathfrak{h}}^*$. The subgroup generated by $s_i, i \in I$, the *Weyl group* of $\mathfrak{g}$, will be denoted by W and its longest element by w_0. In general, the *length* of $w \in \widehat{W}$ will be denoted by $\ell(w)$.

Recall that the coroot lattice M (resp. coweight lattice L) is the sublattice of Q (resp. P) spanned by the elements $\check{d}_i\alpha_i$ (resp. $\check{d}_i\omega_i$), $1 \leq i \leq n$. The group W preserves M and L and we have an isomorphism of groups

$$\widehat{W} \cong W \ltimes M.$$

The *extended affine Weyl group* $\widetilde{W}$ is the semi–direct product $W \ltimes L$. The affine Weyl group is a normal subgroup of $\widetilde{W}$ and if $\mathcal{T}$ is the group of diagram automorphisms of $\widehat{\mathfrak{g}}$, we have

$$\widetilde{W} \cong \mathcal{T} \ltimes \widehat{W}.$$

Since $\mathcal{T}$ preserves $\widehat{P}$ and $\widehat{P}^+$, we see that $\widetilde{W}$ preserves $\widehat{P}$. The following formulae make explicit the action of $\mu \in L$ on $\widehat{\mathfrak{h}}^*$:

$$t_\mu(\lambda) = \lambda - (\lambda,\mu)\delta, \quad \lambda \in \mathfrak{h}^* \oplus \mathbb{C}\delta, \quad t_\mu(\Lambda_0) = \Lambda_0 + \mu - \frac{1}{2}(\mu,\mu)\delta. \tag{1.1.2}$$

For any $\Psi \in \widehat{P}$ and $\mu \in P$ we have

$$\#(\widehat{W}\Psi \cap \widehat{P}^+) \leq 1 \quad \text{and} \quad \#(W\mu \cap P^+) = 1. \tag{1.1.3}$$

1.2 Quantum affine algebra

Let q be a formal parameter and let $\mathbb{F}$ be the algebraic closure of $\mathbb{C}(q)$, field of rational functions on q. For $p = q^k$, $k \in \mathbb{Z}_{\geq 1}$, define

$$[m]_p = \frac{p^m - p^{-m}}{p - p^{-1}}, \qquad [m]_p! = [m]_p[m-1]_p \ldots [2]_p[1]_p, \qquad \left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]_p = \frac{[m]_p!}{[r]_p![m-r]_p!},$$

for $r, m \in \mathbb{Z}_{\geq 0}$, $m \geq r$. Let $q_i := q^{d_i}$ for $i \in I$.

The *quantum loop algebra* $U_q(\widetilde{\mathfrak{g}})$ in Drinfeld's new realization, [Dri88], is the $\mathbb{F}$-associative algebra with unit given by generators $x_{i,r}^{\pm}$, $k_i^{\pm}$, $h_{i,s}$, for $i \in I$, $r \in \mathbb{Z}$ and $s \in \mathbb{Z}\setminus\{0\}$, subject to the following relations:

$$\begin{gathered}
k_i k_i^{-1} = k_i^{-1}k_i = 1, \quad k_i k_j = k_j k_i, \\
k_i h_{j,r} = h_{j,r} k_i, \\
k_i x_{j,r}^{\pm} k_i^{-1} = q_i^{\pm c_{ij}} x_{j,r}^{\pm}, \\
[h_{i,n}, h_{j,m}] = 0 \quad [h_{i,r}\, x_{j,s}^{\pm}] = \pm\frac{1}{r}[rc_{ij}]_{q_i} x_{j,r+s}^{\pm}, \\
x_{i,r+1}^{\pm}x_{j,s}^{\pm} - q_i^{\pm c_{ij}} x_{j,s}^{\pm}x_{i,r+1}^{\pm} = q_i^{\pm c_{ij}} x_{i,r}^{\pm}x_{j,s+1}^{\pm} - x_{j,s+1}^{\pm}x_{i,r}^{\pm}, \\
[x_{i,r}^{+}, x_{j,s}^{-}] = \delta_{i,j}\, \frac{\phi_{i,r+s}^{+} - \phi_{i,r+s}^{-}}{q_i - q_i^{-1}}, \\
\sum_{\sigma \in S_m}\sum_{k=0}^{m}(-1)^k \left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]_{q_i} x_{i,n_{\sigma(1)}}^{\pm} \ldots x_{i,n_{\sigma(k)}}^{\pm} x_{j,s}^{\pm} x_{i,n_{\sigma(k+1)}}^{\pm} \ldots x_{i,n_{\sigma(m)}}^{\pm} = 0, \quad \text{if } i \neq j,
\end{gathered}$$

for all sequences of integers $n_1, \ldots, n_m$, where $m = 1 - c_{ij}$, $i, j \in I$, S_m is the symmetric group on m letters, and the $\phi^{\pm}_{i,r}$ are determined by equating powers of u in the formal power series

$$\phi^{\pm}_i(u) = \sum_{r=0}^{\infty} \phi^{\pm}_{i,\pm r} u^r = k_i^{\pm 1} \exp\left(\pm (q - q^{-1}) \sum_{s=1}^{\infty} h_{i,\pm s} u^s \right). \tag{1.2.1}$$

Remark 1.2.1. The algebra $U_q(\widetilde{\mathfrak{g}})$ is isomorphic to the quotient of the quantum affine algebra $U_q(\widehat{\mathfrak{g}}')$ by the ideal generated by some central element [Dri88, Bec94], where $\widehat{\mathfrak{g}}' = [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]$. Therefore the study of finite-dimensional $U_q(\widehat{\mathfrak{g}}')$ reduces to that of $U_q(\widetilde{\mathfrak{g}})$-modules.

Consider the subalgebras $U_q(\widetilde{\mathfrak{n}}^{\pm})$ and $U_q(\widetilde{\mathfrak{h}})$ of $U_q(\widetilde{\mathfrak{g}})$ generated, respectively, by $(x^{\pm}_{i,r})_{i \in I, r \in \mathbb{Z}}$ and $(h_{i,s})_{i \in I, s \in \mathbb{Z} \setminus \{0\}}, (k_i)_{i \in I}$ and $c^{\pm 1/2}$. As vector space we have the following isomorphism

$$U_q(\widetilde{\mathfrak{g}}) \cong U_q(\widetilde{\mathfrak{n}}^-) \otimes U_q(\widetilde{\mathfrak{h}}) \otimes U_q(\widetilde{\mathfrak{n}}^+).$$

There exist coproduct, counit, and antipode making $U_q(\widetilde{\mathfrak{g}})$ a Hopf algebra. Moreover, the subalgebra of $U_q(\widetilde{\mathfrak{g}})$ generated by $(k_i)_{i \in I}$, $(x^{\pm}_{i,0})_{i \in I}$ is a Hopf sublagebra of $U_q(\widetilde{\mathfrak{g}})$ and is isomorphic as a Hopf algebra to $U_q(\mathfrak{g})$, the quantized enveloping algebra of $\mathfrak{g}$. On $U_q(\mathfrak{g})$ we have, for all $i \in I$,

$$\Delta(x^+_{i,0}) = x^+_{i,0} \otimes 1 + k_i \otimes x^+_{i,0}, \quad \Delta(x^-_{i,0}) = x^-_{i,0} \otimes k_i^{-1} + 1 \otimes x^-_{i,0}, \quad \Delta(k_i) = k_i \otimes k_i.$$

An explicit formula for the comultiplication of the current generators of $U_q(\widetilde{\mathfrak{g}})$ is not known. However, we have the following useful lemma.

Lemma 1.2.2 ([Dam98])**.** *Modulo $U_q(\widetilde{\mathfrak{g}})X^- \otimes U_q(\widetilde{\mathfrak{g}})X^+$, we have*

$$\Delta(\phi^{\pm}_i(u)) = \phi^{\pm}_i(u) \otimes \phi^{\pm}_i(u),$$

where $X^{\pm}$ is the $\mathbb{F}$-span of the elements $x^{\pm}_{j,r}$, $j \in I, r \in \mathbb{Z}$. □

1.3 Restricted Specialization

In this section we recall the definition of restricted specialization of $U_q(\mathfrak{g})$ and $U_q(\widetilde{\mathfrak{g}})$. Let $\mathbb{A} = \mathbb{C}[q, q^{-1}]$, and for $i \in I, r \in \mathbb{Z}, k \in \mathbb{Z}_{\geq 0}$, define $(x^{\pm}_{i,r})^{(k)} = \frac{(x^{\pm}_{i,r})^k}{[k]_{q_i}!}$.

Let $U_{\mathbb{A}}(\widetilde{\mathfrak{g}})$ be the $\mathbb{A}$-subalgebra of $U_q(\widetilde{\mathfrak{g}})$ generated by elements $(x^{\pm}_{i,r})^{(k)}$, $k_i^{\pm}$ for $i \in I, r \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$. Define $U_{\mathbb{A}}(\mathfrak{g})$ similarly and notice that $U_{\mathbb{A}}(\mathfrak{g}) = U_{\mathbb{A}}(\widetilde{\mathfrak{g}}) \cap U_q(\mathfrak{g})$. Set $U_{\mathbb{A}}(\widetilde{\mathfrak{n}}^{\pm}) = U_{\mathbb{A}}(\widetilde{\mathfrak{g}}) \cap U_q(\widetilde{\mathfrak{n}}^{\pm})$ and $U_{\mathbb{A}}(\widetilde{\mathfrak{h}}) = U_{\mathbb{A}}(\widetilde{\mathfrak{g}}) \cap U_q(\widetilde{\mathfrak{h}})$. The multiplication establishes an isomorphism of $\mathbb{A}$-modules

Proposition 1.3.1 ([Cha01], Lemma 2.1)**.** *We have $U_q(\widetilde{\mathfrak{g}}) = \mathbb{F} \otimes_{\mathbb{A}} U_{\mathbb{A}}(\widetilde{\mathfrak{g}})$ and $U_q(\mathfrak{g}) = \mathbb{F} \otimes_{\mathbb{A}} U_{\mathbb{A}}(\mathfrak{g})$. Moreover, the multiplication establishes an isomorphism of $\mathbb{A}$-modules*

$$U_{\mathbb{A}}(\widetilde{\mathfrak{g}}) \cong U_{\mathbb{A}}(\widetilde{\mathfrak{n}}^-) \otimes U_{\mathbb{A}}(\widetilde{\mathfrak{h}}) \otimes U_{\mathbb{A}}(\widetilde{\mathfrak{n}}^+).$$

□

Given $\xi \in \mathbb{C}^\times$, denote by ϵ_ξ the evaluation map $\mathbb{A} \to \mathbb{C}$ sending q to ξ and by $\mathbb{C}_\xi$ the $\mathbb{A}$-module obtained by pulling-back ϵ_ξ. Set

$$U_\xi(\mathfrak{a}) = \mathbb{C}_\xi \otimes U_\mathbb{A}(\mathfrak{a}),$$

for $\mathfrak{a} = \mathfrak{g}, \mathfrak{n}^\pm, \mathfrak{h}, \widetilde{\mathfrak{g}}, \widetilde{\mathfrak{n}}^\pm, \widetilde{\mathfrak{h}}$.

The algebra $U_\xi(\mathfrak{a})$ is called the *restricted specialization* of $U_q(\mathfrak{a})$ at $q = \xi$. We shall denote an element of the form $1 \otimes x \in U_\xi(\mathfrak{a})$ with $x \in U_\mathbb{A}(\mathfrak{a})$ simply by x. If ξ is not a root of unity, the algebra $U_\xi(\widetilde{\mathfrak{g}})$ is isomorphic to the algebra given by generators and relations analogous to those of $U_q(\widetilde{\mathfrak{g}})$ with ξ in place of q and its representation theory is parallel to that of $U_q(\widetilde{\mathfrak{g}})$. In this work we shall be particularly interested in the case that $\xi = 1$, for which we have the following proposition.

Proposition 1.3.2 ([CP94a], Proposition 9.2.3)**.** *The algebra $U(\widetilde{\mathfrak{g}})$ is isomorphic to the quotient of $U_1(\widetilde{\mathfrak{g}})$ by the ideal generated by $k_i - 1$, $i \in I$. In particular, the category of $U_1(\widetilde{\mathfrak{g}})$-modules on which k_i acts as the identity operator, for all $i \in I$, is equivalent to the category of all $\widetilde{\mathfrak{g}}$-modules.*

The following partial information about the comultiplication will suffice for us.

Lemma 1.3.3 ([Mou10], Lemma 1.6)**.** *Let $i \in I$. Then*

$$\Delta(x_{i,1}^-) = x_{i,1}^- \otimes k_i + 1 \otimes x_{i,1}^- + x,$$

for some $x \in U_\mathbb{A}(\widetilde{\mathfrak{g}}) \otimes U_\mathbb{A}(\widetilde{\mathfrak{g}})$ such that $\bar{x} = 0$.

Chapter 2

Representations of Lie algebras

In this chapter we first state the basic properties of the representation theory of $\mathfrak{g}$ and $\widehat{\mathfrak{g}}$. In a second moment, we recall the definition of Demazure modules and its generalizations along with the recent class of modules for the current algebra introduced in [CV14]. We also collect several known results about both class of representations which will be useful for our main theorems.

2.1 Basic definitions

Given $\lambda \in P^+$, we denote by $V(\lambda)$ the associated finite-dimensional simple highest weight $\mathfrak{g}$-module with highest weight λ. If V is a $\mathfrak{g}$-module and $\mu \in P$, V_μ will denote the associated weight space of V. If V is a finite-dimensional $\mathfrak{g}$-module, then

$$V = \oplus_{\mu \in P} V_\mu,$$

and the *character* of V is given by $\mathrm{ch}(V) = \sum_{\mu \in P} \dim V_\mu \cdot e^\mu \in \mathbb{Z}[P]$.

Similarly, for $\Lambda \in \widehat{P}^+$, we denote by $\widehat{V}(\Lambda)$ the simple integrable $\widehat{\mathfrak{g}}$-module with highest weight Λ and V_Ψ, $\Psi \in \widehat{P}$, will denote the weight space of the $\widehat{\mathfrak{g}}$-module V associated to Ψ. Then, if V is integrable, we have $V_\Psi \neq 0$ only if $\Psi \in \widehat{P} + \mathbb{C}\delta$. Henceforth, when we say integrable $\widehat{\mathfrak{g}}$-module, we will assume that the weights are in $\widehat{P}$.

The following proposition is well–known (see [Kac83, Chapters 10,11] for instance).

Proposition 2.1.1. *(i) Let $\Lambda \in \widehat{P}^+$. We have*

$$\dim \widehat{V}(\Lambda)_{w\Lambda} = 1, \quad \text{for all} \quad w \in \widehat{W}.$$

(ii) Given $\Lambda', \Lambda'' \in \widehat{P}^+$, we have

$$\widehat{V}(\Lambda') \otimes \widehat{V}(\Lambda'') \cong \bigoplus_{\Lambda \in \widehat{P}^+} \dim \mathrm{Hom}_{\widehat{\mathfrak{g}}}\left(\widehat{V}(\Lambda), \widehat{V}(\Lambda') \otimes \widehat{V}(\Lambda'')\right) \widehat{V}(\Lambda).$$

Moreover,

$$\dim \operatorname{Hom}_{\widehat{\mathfrak{g}}}\Big(\widehat{V}(\Lambda), \widehat{V}(\Lambda') \otimes \widehat{V}(\Lambda'')\Big) = \begin{cases} 1, & \Lambda = \Lambda' + \Lambda'', \\ 0, & \Lambda \notin \Lambda' + \Lambda'' - \widehat{Q}^+. \end{cases} \tag{2.1.1}$$

In particular, if $\Lambda = \Lambda' + \Lambda''$*, and* $w \in \widehat{W}$*, we have*

$$\Big(\widehat{V}(\Lambda') \otimes \widehat{V}(\Lambda'')\Big)_{w\Lambda} = \widehat{V}(\Lambda)_{w\Lambda}, \tag{2.1.2}$$

where we have identified $\widehat{V}(\Lambda)$ *with its image in* $\widehat{V}(\Lambda') \otimes \widehat{V}(\Lambda'')$*.*

□

Recall that the action of $d \in \widehat{\mathfrak{g}}$ on an integrable $\widehat{\mathfrak{g}}$-module V induces a $\mathbb{Z}$-gradation on V. Then, if V is an integrable $\widehat{\mathfrak{g}}$-module, we have

$$V[r] = \{v \in V | dv = rv\}.$$

In particular,

$$U(\mathfrak{g})\widehat{V}(\Lambda)_\Lambda = \widehat{V}(\Lambda)[\Lambda(d)] \cong_{\mathfrak{g}} V(\Lambda|_{\mathfrak{h}}).$$

Let also τ_s be the grading shift functor by s. More precisely, $\tau_s V$ has the same underlying vector space V with graded shift by

$$(\tau_s V)[r] = V[r+s].$$

Any $\mathfrak{g}$-module V can be turned into a $\mathfrak{g}[t]$-module by letting $\mathfrak{g}[t]_+$ act trivially. Then, given $\lambda \in P^+$ and $r \in \mathbb{Z}$, regard $V(\lambda)$ as a $\mathfrak{g}[t]$-module in this manner and set

$$V(\lambda, r) = \tau_r V(\lambda).$$

Every simple $\mathbb{Z}$-graded finite-dimensional $\mathfrak{g}[t]$-module is isomorphic to a unique module of the form $V(\lambda, r)$ [CKR12, Proposition 2.3].

2.2 Demazure modules

Given $\Lambda \in \widehat{P}^+$ and $w\tau \in \widetilde{W}$, where $w \in \widehat{W}$ and $\tau \in \mathcal{T}$, the Demazure module $D(w\tau\Lambda)$ is the $\widehat{\mathfrak{b}}$–submodule of $\widehat{V}(\tau\Lambda)$ generated by a non–zero element $v_{w\tau\Lambda}$ of $\widehat{V}(\tau\Lambda)_{w\tau\Lambda}$. We say that $D(w\tau\Lambda)$ is a level ℓ–Demazure module if $\Lambda(c) = \ell$.

The following is immediate from Proposition 2.1.1(ii).

Lemma 2.2.1. *Let* $w\tau \in \widetilde{W}$ *and* $\Lambda', \Lambda'' \in \widehat{P}^+$*. We have an isomorphism of* $\widehat{\mathfrak{b}}$*–modules,*

$$D(w\tau(\Lambda' + \Lambda'')) \cong U(\widehat{\mathfrak{b}})(v_{w\tau\Lambda'} \otimes v_{w\tau\Lambda''}) \subset \widehat{V}(\tau\Lambda') \otimes \widehat{V}(\tau\Lambda'').$$

□

Let $\Lambda \in \widehat{P}^+$, $\tau \in \mathcal{T}$ and $w \in \widehat{W}$ such that $w\tau\Lambda(h_i) \leq 0$, for all $i \in I$. In this case, we have $\mathfrak{n}^- v_{w\tau\Lambda} = 0$ and $D(w\tau\Lambda)$ is a module for the parabolic subalgebra $\widehat{\mathfrak{b}} \oplus \mathfrak{n}^-$, i.e.,

$$D(w\tau\Lambda) = U(\widehat{\mathfrak{b}} \oplus \mathfrak{n}^-)v_{w\tau\Lambda} = U(\mathfrak{g}[t])v_{w\tau\Lambda}.$$

We shall also refer to such modules as the $\mathfrak{g}$–stable Demazure modules. Notice that we can write $w\tau\Lambda = w_0\lambda + \Lambda(c)\Lambda_0 + r\delta$ for a unique $\lambda \in P^+$ and $r \in \mathbb{Z}$. The action of d defines a $\mathbb{Z}$–grading on these modules which is compatible with the $\mathfrak{g}[t]$–action. Moreover, for fixed λ and $\Lambda(c)$ the modules for varying $r \in \mathbb{Z}$ are just graded shifts. Hence we shall denote the module corresponding to $w_0\lambda + \ell\Lambda_0$ by $D(\ell, \lambda)$; the module for arbitrary r is then $\tau_r^* D(\ell, \lambda)$. Moreover,

$$D(\ell, \lambda)[r] = D(\ell, \lambda) \cap \widehat{V}(\Lambda)[r], \quad \text{for all} \quad r \in \mathbb{Z}.$$

In particular,

$$D(\ell, \lambda)[0] = U(\mathfrak{g})D(\ell, \lambda)_\lambda \cong_{\mathfrak{g}} V(\lambda),$$

and

$$\operatorname{soc}(D(\ell, \lambda)) = D(\ell, \lambda)[r] = \widehat{V}(\Lambda)[r] = U(\mathfrak{g})\widehat{V}(\Lambda)_\Lambda \cong V(\mu, r), \tag{2.2.1}$$

where $r = \Lambda(d)$ and $\mu = \Lambda|_{\mathfrak{h}}$.

2.3 Generalized Demazure modules

A natural generalization of the Demazure modules is as follows. Let $m \in \mathbb{Z}_{>0}$. Given $(w_r, \Lambda^r) \in \widehat{W} \times \widehat{P}^+$, $1 \leq r \leq m$, let

$$D(w_1\Lambda^1, \ldots, w_m\Lambda^m) = U(\widehat{\mathfrak{b}})(v_{w_1\Lambda^1} \otimes \cdots \otimes v_{w_m\Lambda^m}) \subseteq \widehat{V}(\Lambda^1) \otimes \cdots \otimes \widehat{V}(\Lambda^m).$$

These modules are referred to as *generalized Demazure modules* in [Nao13]. Evidently, $D(w_1\Lambda^1, \ldots, w_m\Lambda^m)$ is $\mathfrak{g}$–stable if each $D(w_r\Lambda^r)$ is so.

Let $\mathbb{Z}[\widehat{P}]$ be the group ring of $\widehat{P}$ with basis $\{e^\Lambda, \Lambda \in \widehat{P}\}$. The action of $\widehat{W}$ on $\widehat{P}$ defines an action of $\widehat{W}$ on $\mathbb{Z}[\widehat{P}]$. Given any finite–dimensional weight module V for $\widehat{\mathfrak{h}}$ such that $V = \oplus_{\Lambda \in P} V_\Lambda$, set

$$\operatorname{ch}_{\widehat{\mathfrak{h}}} V = \sum_{\Lambda \in P} \dim V_\Lambda \; e^\Lambda \in \mathbb{Z}[\widehat{P}].$$

For $i \in \widehat{I}$, the *Demazure operator* $\mathcal{D}_i : \mathbb{Z}[\widehat{P}] \to \mathbb{Z}[\widehat{P}]$ is defined by

$$\mathcal{D}_i(f) = \frac{f - e^{-\alpha_i} s_i(f)}{1 - e^{-\alpha_i}}. \tag{2.3.1}$$

If $w = s_{j_1} \cdots s_{j_\ell}$ is a reduced expression for $w \in \widehat{\mathcal{W}}$, it is well-known that the operator $\mathcal{D}_w = \mathcal{D}_{j_1} \ldots \mathcal{D}_{j_\ell}$ is independent of the choice of the reduced expression and

$$\ell(w_1w_2) = \ell(w_1) + \ell(w_2) \quad \Rightarrow \quad \mathcal{D}_{w_1w_2} = \mathcal{D}_{w_1}\mathcal{D}_{w_2}. \tag{2.3.2}$$

The following is [Nao13, Theorem 2.2] and gives the character of a generalized Demazure modules under certain conditions on $(w_r, \Lambda^r),\ 1 \leq r \leq m$.

Theorem 2.3.1. *Let $m \in \mathbb{Z}_{\geq 1}$ and $(w_r, \Lambda^r) \in \widehat{W} \times \widehat{P}^+$, for $1 \leq r \leq m$. Assume moreover that $\ell(w_r) = \ell(w_{r-1}) + \ell(w_{r-1}^{-1} w_r)$, for all $1 < r \leq m$. Then*

$$\mathrm{ch}_{\widehat{\mathfrak{h}}} D(w_1\Lambda^1, \cdots, w_m\Lambda^m) = \mathcal{D}_{\tilde{w}_1}\left(e^{\Lambda^1}\left(\mathcal{D}_{\tilde{w}_2} e^{\Lambda^2}\left(\cdots \mathcal{D}_{\tilde{w}_{m-1}}\left(e^{\Lambda^{m-1}}\mathcal{D}_{\tilde{w}_m}(e^{\Lambda^m})\right)\cdots\right)\right)\right),$$

where $\tilde{w}_1 = w_1$ and $\tilde{w}_r = w_{r-1}^{-1} w_r$ for $1 < r \leq m$.

2.4 CV-modules

To present the definition of CV-modules, introduced in [CV14], we first recall the definition of graded local Weyl modules.

Given $\lambda \in P^+$, the *graded local Weyl module* $W(\lambda)$ is the $\mathfrak{g}[t]$-module generated by a vector w_λ satisfying the following defining relations:

$$\mathfrak{n}^+[t] w_\lambda = \mathfrak{h}[t]_+ w_\lambda = 0, \quad h w_\lambda = \lambda(h) w_\lambda, \quad (x_i^-)^{\lambda(h_i)+1} w_\lambda = 0,$$

for all $h \in \mathfrak{h}, i \in I$. It is clear from the definition that $W(\lambda)$ can be equipped with a $\mathbb{Z}$-grading by setting the degree of w_λ to be zero. Moreover, it follows from [CP01] that $W(\lambda)$ is finite-dimensional and that every $\mathbb{Z}$-graded finite-dimensional $\mathfrak{g}[t]$-module generated by a vector v of weight λ satisfying $\mathfrak{n}^+[t]v = \mathfrak{h}[t]_+ v = 0$ is a quotient of $W(\lambda)$. We also recall that the following relations holds in $W(\lambda)$:

$$x^-_{i,\lambda(h_i)} w_\lambda = 0, \quad \text{for all} \quad i \in I. \tag{2.4.1}$$

Let $\boldsymbol{\xi}$ be an R^+-tuple of partitions $\boldsymbol{\xi} = (\xi^\alpha)_{\alpha \in R^+}$. We use the notation $\xi^\alpha = (\xi^\alpha_1, \xi^\alpha_2, \dots)$ with $\xi^\alpha_1 \geq \xi^\alpha_2 \geq \cdots \geq 0$. Given $\lambda \in P^+$, say that $\boldsymbol{\xi}$ is λ-*compatible* if ξ^α is a partition of $\lambda(h_\alpha)$ for every $\alpha \in R^+$. For each such R^+-tuple of partitions $\boldsymbol{\xi}$, let $V(\boldsymbol{\xi})$ be the quotient of $W(\lambda)$ by the submodule generated by

$$(x^+_{\alpha,1})^s (x^-_{\alpha,0})^{s+r} w_\lambda,\ \alpha \in R^+,\ s, r \in \mathbb{Z}_{\geq 1},\ s + r \geq 1 + rk + \sum_{j \geq k+1} \xi^\alpha_j \text{ for some } k \in \mathbb{Z}_{\geq 1}. \tag{2.4.2}$$

We denote by $v_{\boldsymbol{\xi}}$ the image of w_λ in $V(\boldsymbol{\xi})$.

Other two presentations of the modules $V(\boldsymbol{\xi})$ in terms of generators and relations were obtained in [CV14]. These other presentations played an important role in the proof of Theorem 2.5.4 below – which establishes the relation of these modules with Demazure modules – and will also be used in the proof of Theorems 4.1.2 and 4.5.1. Thus, we now recall them.

Let $\mathbf{S}$ be the set of sequences of nonnegative integers. Given $\mathbf{b} \in \mathbf{S}$ we will use the notation $\mathbf{b} = (b_p)_{p \geq 0}$, where $b_p \in \mathbb{Z}_{\geq 0}$, for all $p \geq 0$. For $k \in \mathbb{Z}_{\geq 0}$, let

$$\mathbf{S}_k = \{\mathbf{b} \in \mathbf{S} | b_p = 0 \text{ for } p \geq k\} \quad \text{and} \quad {}_k\mathbf{S} = \{\mathbf{b} \in \mathbf{S} | b_p = 0 \text{ for } p < k\}.$$

For $s, r \in \mathbb{Z}_{\geq 0}$, let

$$\mathbf{S}(r,s) = \left\{ \mathbf{b} \in \mathbf{S} | b_p \in \mathbb{Z}_{\geq 0},\ \sum_{p \geq 0} b_p = r,\ \sum_{p \geq 0} p\ b_p = s \right\}.$$

Set also

$$\mathbf{S}(r,s)_k = \mathbf{S}_k \cap \mathbf{S}(r,s) \quad \text{and} \quad {}_k\mathbf{S}(r,s) = {}_k\mathbf{S} \cap \mathbf{S}(r,s).$$

Then, for each $\alpha \in R^+$, define the following elements of $U(\mathfrak{g}[t])$:

$$\mathbf{x}^{\pm}_{\alpha}(r,s)_k = \sum_{\mathbf{b} \in \mathbf{S}(r,s)_k} (x^{\pm}_{\alpha,0})^{(b_0)} \dots (x^{\pm}_{\alpha,k-1})^{(b_{k-1})},$$

$${}_k\mathbf{x}^{\pm}_{\alpha}(r,s) = \sum_{\mathbf{b} \in {}_k\mathbf{S}(r,s)} (x^{\pm}_{\alpha,k})^{(b_k)} \dots (x^{\pm}_{\alpha,s})^{(b_s)},$$

where $x^{(b)} = x^b/b!$.

Given $\lambda \in P^+$ and a λ-compatible partition $\boldsymbol{\xi}$, let $M(\boldsymbol{\xi})$ be the submodule of $W(\lambda)$ defined by (2.4.2). Let also $M'(\boldsymbol{\xi})$ be the submodule generated by

$$\mathbf{x}^{-}_{\alpha}(r,s)w_{\lambda},\ \alpha \in R^+,\ r,s \in \mathbb{Z}_{\geq 1},\ s+r \geq 1+kr+\sum_{j \geq k+1} \xi^{\alpha}_j,\ \text{for some } k \in \mathbb{Z}_{\geq 1} \tag{2.4.3}$$

and $M''(\boldsymbol{\xi})$ be the submodule generated by

$${}_k\mathbf{x}^{-}_{\alpha}(r,s)w_{\lambda},\ \alpha \in R^+,\ r,s,k \in \mathbb{Z}_{\geq 1},\ s+r \geq 1+kr+\sum_{j \geq k+1} \xi^{\alpha}_j. \tag{2.4.4}$$

It was proved in [CV14] (see Equation (2.10) and Proposition 2.7) that

$$M(\boldsymbol{\xi}) = M'(\boldsymbol{\xi}) = M''(\boldsymbol{\xi}).$$

In fact, the proof shows the following stronger statement. For each fixed $\alpha \in R^+$, let $\mathfrak{sl}_\alpha[t]$ be the subalgebra of $\mathfrak{g}[t]$ generated by $x^{\pm}_{\alpha,r}, r \in \mathbb{Z}_{\geq 0}$. Let also $M_\alpha(\boldsymbol{\xi})$ be the $\mathfrak{sl}_\alpha[t]$-submodule of $W(\lambda)$ generated by the corresponding elements in (2.4.2) and similarly define $M'_\alpha(\boldsymbol{\xi}) = M''_\alpha(\boldsymbol{\xi})$. Then

$$M_\alpha(\boldsymbol{\xi}) = M'_\alpha(\boldsymbol{\xi}) = M''_\alpha(\boldsymbol{\xi}). \tag{2.4.5}$$

2.5 Connection between CV and Demazure modules

In the remainder of this section, we review several results on the modules $V(\boldsymbol{\xi})$ and how such modules are connected with $\mathfrak{g}$–stable Demazure modules.

For a partition $\xi = (\xi_1, \xi_2, \dots)$, let $\#\xi$ be the number of nonzero parts of ξ. Given $k_j, s_j \in \mathbb{Z}_{\geq 0}, 1 \leq j \leq m$, such that $k_{j+1} \leq k_j$, for all j, the notation $(k_1^{s_1}, k_2^{s_2}, \dots, k_m^{s_m})$ will stand for the partition whose first s_1 parts are equal to k_1 and so on. Recall that a partition of the form (k^s) is said to be *rectangular*. Following [CV14], we will refer to a partition of the form (k^s, k') with $0 \neq k' \neq k$ and $s \neq 0$ as a *special fat hook*.

Theorem 2.5.1 ([CV14, Theorem 1]). *Let $\lambda \in P^+$ and suppose $\boldsymbol{\xi}$ is λ-compatible such that, for each $\alpha \in R^+$, ξ^α is either rectangular or a special fat hook. Set $s_\alpha = \#\xi^\alpha$. Then $V(\boldsymbol{\xi})$ is the quotient of $W(\lambda)$ by the submodule generated by*

$$\{x^-_{\alpha,s_\alpha} w_\lambda | \alpha \in R^+\} \cup \{(x^-_{\alpha,s_\alpha-1})^{\xi^\alpha_{s_\alpha}+1} w_\lambda | \alpha \in R^+ \text{ with } \xi^\alpha \text{ a special fat hook}\}. \tag{2.5.1}$$

Remark 2.5.2. Let $\boldsymbol{\xi}$ be λ-compatible. In the spirit of (2.4.5), for each $\alpha \in R^+$ such that ξ^α is either rectangular or a special fat hook, define $M'''_\alpha(\boldsymbol{\xi})$ as the $\mathfrak{sl}_2^\alpha[t]$-submodule of $W(\lambda)$ generated by the elements in (2.5.1). Then the proof of the above theorem actually shows that $M_\alpha(\boldsymbol{\xi}) = M'''_\alpha(\boldsymbol{\xi})$, for every such α.

The case of special fat hooks of the form $\xi = (k^s, 1)$ will be of relevance for us. We will say that a partition is *essentially rectangular* if it is either rectangular or a special fat hook of this form. The following proposition is a corollary of [CV14, Theorem 1].

Proposition 2.5.3. *Let $\lambda \in P^+$ and suppose $\boldsymbol{\xi}$ is λ-compatible such that, for each $\alpha \in R^+$, ξ^α is essentially rectangular. Then $V(\boldsymbol{\xi})$ is the quotient of $W(\lambda)$ by the submodule generated by $\{x^-_{\alpha,s_\alpha} w_\lambda | \alpha \in R^+\}$ where $s_\alpha = \#\xi^\alpha$.* □

Given $\ell \in \mathbb{Z}_>0$ and $\lambda \in P^+$, consider the element $\boldsymbol{\xi}(\ell, \lambda)$ defined in the following way [CV14, Section 3.2]. For each $\alpha \in R^+$, let $s_\alpha, m_\alpha \in \mathbb{Z}_{>0}$ be determined by

$$\lambda(h_\alpha) = (s_\alpha - 1)\check{d}_\alpha \ell + m_\alpha \quad \text{and} \quad 1 \le m_\alpha \le \check{d}_\alpha \ell.$$

Now, let $\boldsymbol{\xi}(\ell, \lambda) = (\xi^\alpha)_{\alpha \in R^+}$ with $\xi^\alpha = ((\check{d}_\alpha \ell)^{s_\alpha - 1}, m_\alpha)$, if $s_\alpha > 1$, and $\xi^\alpha = (m_\alpha)$, otherwise. In particular, Theorem 2.5.1 applies to $\boldsymbol{\xi}(\ell, \lambda)$. Notice that

$$\check{d}_\alpha \ell = 1 \quad \Rightarrow \quad \xi^\alpha = (1^{\lambda(h_\alpha)}), \tag{2.5.2}$$

and

$$\check{d}_\alpha \ell = 2 \quad \Rightarrow \quad \xi^\alpha \text{ is essentially rectangular.} \tag{2.5.3}$$

The relation between Demazure modules and the modules $V(\boldsymbol{\xi})$ is established by:

Theorem 2.5.4 ([CV14, Theorem 2]). *For every $\lambda \in P^+$ and $\ell \in \mathbb{Z}_{\ge 0}$, we have $V(\boldsymbol{\xi}(\ell, \lambda)) \cong D(\ell, \lambda)$.* □

In particular, Theorem 2.5.1 gives a simple presentation of $D(\ell, \lambda)$ by generator and relations and, if $\check{d}_\alpha \ell \le 2$ for all $\alpha \in R^+$, Proposition 2.5.3 gives an even simpler one. We also recall the following special case.

Proposition 2.5.5 ([CV14, Lemma 3.3]). *Suppose $\check{d}_i \ell$ divides $\lambda(h_i)$ for all $i \in I$, say $\lambda(h_i) = s_i \check{d}_i \ell$. Then, $V(\boldsymbol{\xi}(\ell, \lambda))$ is the quotient of $W(\lambda)$ by the submodule generated by $x^-_{i,s_i} w_\lambda, i \in I$.* □

In particular, together with (2.4.1), Proposition 2.5.5 recovers the following result proved originally in [FL07]:

$$\lambda(h_i) = 0 \text{ for all } i \text{ such that } \check{d}_i > 1 \quad \Rightarrow \quad D(1, \lambda) \cong W(\lambda).$$

2.6 Fusion Products

We recall the notion of the fusion product of $\mathfrak{g}[t]$–modules introduced in [FL99]. Let V be a finite–dimensional cyclic $\mathfrak{g}[t]$–module generated by an element v and, for $r \in \mathbb{Z}_{\geq 0}$, set

$$F^r V = \left(\bigoplus_{0 \leq s \leq r} U(\mathfrak{g}[t])[s] \right) \cdot v.$$

Clearly $F^r V$ is a $\mathfrak{g}$–submodule of V and we have a finite $\mathfrak{g}$–module filtration

$$0 \subset F^0 V \subset F^1 V \subset \cdots \subset F^p V = V,$$

for some $p \in \mathbb{Z}_{\geq 0}$. The associated graded vector space $\mathrm{gr} V$ acquires a graded $\mathfrak{g}[t]$–module structure in a natural way and is generated by the image of v in $\mathrm{gr} V$.

Given a $\mathfrak{g}[t]$–module V and $z \in \mathbb{C}$, let V^z be the $\mathfrak{g}[t]$–module with action

$$(x \otimes t^r)w = (x \otimes (t+z)^r)w, \quad x \in \mathfrak{g}, \quad r \in \mathbb{Z}_{\geq 0}, \quad w \in V.$$

If V_s, $1 \leq s \leq k$, are cyclic finite–dimensional $\mathfrak{g}[t]$–modules with cyclic vectors v_s, $1 \leq s \leq k$ and $z_1, \cdots, z_k$ are distinct complex numbers, then the fusion product $V_1^{z_1} * \cdots * V_k^{z_k}$ is defined to be $\mathrm{gr}\boldsymbol{V}(\boldsymbol{z})$, where $\boldsymbol{V}(\boldsymbol{z})$ is the tensor product

$$\boldsymbol{V}(\boldsymbol{z}) = V_1^{z_1} \otimes \cdots \otimes V_k^{z_k}.$$

It was proved in [FL99] that in fact $\boldsymbol{V}(\boldsymbol{z})$ is cyclic and generated by $v_1 \otimes \cdots \otimes v_m$ and hence the fusion product is cyclic on the image $v_1 * \cdots * v_m$ of this element. Clearly the definition of the fusion product depends on the parameters z_s, $1 \leq s \leq k$. However it is conjectured in [FL99] and proved in certain cases (see [CL06], [FF02], [FL99] [FL07], [Ked11], for instance) that under suitable conditions on V_s and v_s, the fusion product is independent of the choice of the complex numbers. For ease of notation we shall often suppress the dependence on the complex numbers and write $V_1 * \cdots * V_k$ for $V_1^{z_1} * \cdots * V_k^{z_k}$.

Chapter 3

Representations of quantum algebras

This chapter is dedicated to recall the basic of the finite-dimensional representation theory of the quantum algebra along with the description of some tools used in their study, such as q–character and classical limit of representations.

3.1 Representations of $U_q(\mathfrak{g})$

To fix notation, we review some basic facts about the representation theory of $U_q(\mathfrak{g})$. For details see [CP94a] for instance.

Given a $U_q(\mathfrak{g})$-module V and $\mu \in P$, the *weight space* of V of weight μ is the subspace

$$V_\mu := \{v \in V | textrm for all\ i \in I,\ k_i v = q_i^{\mu(h_i)} v\}. \tag{3.1.1}$$

A nonzero vector $v \in V_\mu$ is called a *weight vector* of weight μ. If v is a weight vector such that $x_i^+ v = 0$, for all $i \in I$, then v is called a *highest-weight vector*. If V is generated by a highest weight vector of weight λ, then V is said to be a *highest-weight module* of highest-weight λ. An $U_q(\mathfrak{g})$-module V is called a *weight module* if $V = \bigoplus_{\mu \in P} V_\mu$. The following theorem summarizes the basic facts about finite-dimensional $U_q(\mathfrak{g})$-modules.

Theorem 3.1.1. *Let V be a finite-dimensional $U_q(\mathfrak{g})$-module. Then:*

(i) $\dim V_\mu = \dim V_{w\mu}$, *for all* $w \in \mathcal{W}$,

(ii) V is completely reducible,

(iii) for each $\lambda \in P^+$, the $U_q(\mathfrak{g})$-module $V_q(\lambda)$ generated by a vector v satisfying

$$x_i^+ v = 0, \quad k_i v = q^{\lambda(h_i)}, \quad (x_i^-)^{\lambda(h_i)+1} v = 0, \quad \textit{for all } i \in I,$$

is irreducible and finite-dimensional. Moreover, if V is irreducible then $V \cong V_q(\lambda)$, for some $\lambda \in P^+$.

3.2 Representations of $U_q(\widetilde{\mathfrak{g}})$

Let V be a finite-dimensional representation of $U_q(\widetilde{\mathfrak{g}})$. Regarding V as a representation of $U_q(\mathfrak{g})$, it has a decomposition in weight spaces. Such decomposition can be refined by decomposing it into Jordan subspaces of mutually commuting $\phi^{\pm}_{i,\pm r}$ defined in (1.2.1), [FR98]:

$$V = \bigoplus_{\boldsymbol{\gamma}} V_{\boldsymbol{\gamma}}, \qquad \boldsymbol{\gamma} = (\gamma^{\pm}_{i,\pm r})_{i\in I,\ r\in\mathbb{Z}_{\geq 0}}, \quad \gamma^{\pm}_{i,\pm r} \in \mathbb{C}, \tag{3.2.1}$$

where

$$V_{\boldsymbol{\gamma}} = \left\{ v \in V \,|\, \exists k \in \mathbb{N}, \forall i \in I,\ m \geq 0,\ \left(\phi^{\pm}_{i,\pm m} - \gamma^{\pm}_{i,\pm m}\right)^k v = 0 \right\}.$$

If $\dim(V_{\boldsymbol{\gamma}}) > 0$, $\boldsymbol{\gamma}$ is called an *ℓ-weight* of V.

For every finite-dimensional representation of $U_q(\widetilde{\mathfrak{g}})$, according to [FR98], the ℓ-weights are known to be of the form

$$\gamma_i^{\pm} := \sum_{r=0}^{\infty} \gamma^{\pm}_{i,\pm r} u^{\pm r} = q_i^{\deg Q_i - \deg R_i} \frac{Q_i(uq_i^{-1}) R_i(uq_i)}{Q_i(uq_i) R_i(uq_i^{-1})},$$

where the right hand side is to be treated as a formal series in positive (resp. negative) integer powers of u, and Q_i and R_i are polynomials of the form

$$Q_i(u) = \prod_{a\in\mathbb{F}^{\times}} (1-ua)^{w_{i,a}}, \qquad R_i(u) = \prod_{a\in\mathbb{F}^{\times}} (1-ua)^{x_{i,a}},$$

for some $w_{i,a}, x_{i,a} \geq 0$, $i \in I$. For all $i \in I, a \in \mathbb{F}^{\times}$, define the *fundamental ℓ-weights* $Y_{i,a}$ to be the n-tuple of polynomials

$$Y_{i,a} := (1 - \delta_{ij} a u)_{j\in I}.$$

Let $\mathcal{P}$ denote the free abelian multiplicative group of monomials in the variables $(Y_{i,a})_{i\in I,\ a\in\mathbb{F}^{\times}}$. The group $\mathcal{P}$ is in bijection with the set of ℓ-weights $\boldsymbol{\gamma}$ of the form above according to

$$\boldsymbol{\gamma} \leftrightarrow \prod_{i\in I, a\in\mathbb{F}^{\times}} Y_{i,a}^{w_{i,a} - x_{i,a}}. \tag{3.2.2}$$

We identify elements of $\mathcal{P}$ with ℓ-weights of finite-dimensional representations in this way, and henceforth write an ℓ-weight $\boldsymbol{\gamma}$ as an element of $\mathcal{P}$.

For each $j \in I$, an ℓ-weight $\boldsymbol{\varpi} = \prod_{i\in I, a\in\mathbb{F}^{\times}} Y_{i,a}^{u_{i,a}}$ is said to be *j-dominant* (resp. *j-anti-dominant*) if and only if $u_{j,a} \geq 0$ (resp. $u_{j,a} \leq 0$), for all $a \in \mathbb{F}^{\times}$. An ℓ-weight is *(anti-)dominant* if and only if it is i-(anti-)dominant, for all $i \in I$. Let $\mathcal{P}^+ \subseteq \mathcal{P}$ denote the set of all dominant ℓ-weights. The elements of $\mathcal{P}^+$ are also called *Drinfeld polynomials*.

For any finite-dimensional representation V of $U_q(\widetilde{\mathfrak{g}})$, define

$$\mathcal{M}(V) := \{\boldsymbol{\varpi} \in \mathcal{P} \,|\, \dim(V_{\boldsymbol{\varpi}}) > 0\}.$$

If $\boldsymbol{\pi} \in \mathcal{M}(V)$ is dominant, then a non-zero vector $v \in V_{\boldsymbol{\pi}}$ is called a *highest ℓ-weight vector*, with *highest ℓ-weight* $\boldsymbol{\pi} = (\pi^{\pm}_{i,\pm s})_{i\in I, k\in\mathbb{Z}_{\geq 0}}$, if

$$\phi^{\pm}_{i,\pm s} v = \pi^{\pm}_{i,\pm s} v \quad \text{and} \quad x^{+}_{i,r} v = 0, \quad \text{for all } i \in I, r \in \mathbb{Z}, s \in \mathbb{Z}_{\geq 0}.$$

A finite-dimensional representation V of $U_q(\widetilde{\mathfrak{g}})$ is said to be a *highest ℓ-weight representation* if $V = U_q(\widetilde{\mathfrak{g}})v$, for some highest ℓ-weight vector $v \in V$.

For every $\boldsymbol{\pi} \in \mathcal{P}^+$ there is a unique finite-dimensional irreducible higest ℓ-weight representation of $U_q(\widetilde{\mathfrak{g}})$ with highest ℓ-weight $\boldsymbol{\pi}$, and moreover every finite-dimensional irreducible $U_q(\widetilde{\mathfrak{g}})$-module is of this form, for some $\boldsymbol{\pi} \in \mathcal{P}^+$ [CP94b]. We denote the irreducible module corresponding to $\boldsymbol{\pi} \in \mathcal{P}^+$ by $L_q(\boldsymbol{\pi})$. Moreover in the case when $\mathfrak{g}$ is of type $\mathfrak{sl}_{n+1}$ (see [CP95] and also [FM01] for the general statement) we have

$$L_q(\boldsymbol{\pi})^* \cong L_q(\boldsymbol{\pi}^*), \quad \boldsymbol{\pi}^* = (\pi_1^*, \cdots, \pi_n^*), \quad \pi_i^*(u) = \pi_{n+1-i}(q^{-n-1}u), \tag{3.2.3}$$

3.3 q-characters

Let $\operatorname{Rep} U_q(\widetilde{\mathfrak{g}})$ be the Grothendieck ring of finite-dimensional representations of $U_q(\widetilde{\mathfrak{g}})$, and $\mathbb{Z}\left[Y_{i,a}^{\pm 1}\right]_{i\in I, a\in\mathbb{F}^\times}$ be the ring of Laurent polynomials in the variables $Y_{i,a}$ with integer coefficients.

The *q-character* map χ_q, defined in [FR98], is the injective homomorphism of rings

$$\chi_q : \operatorname{Rep} U_q(\widetilde{\mathfrak{g}}) \to \mathbb{Z}\left[Y_{i,a}^{\pm 1}\right]_{i\in I, a\in\mathbb{F}^\times},$$

defined by

$$\chi_q(V) = \sum_{\boldsymbol{\varpi}\in\mathcal{P}} \dim(V_{\boldsymbol{\varpi}})\boldsymbol{\varpi}.$$

Definition 3.3.1. An $U_q(\widetilde{\mathfrak{g}})$-module V is said to be *special* if $\chi_q(V)$ has exactly one dominant ℓ-weight. It is *anti-special* if $\chi_q(V)$ has exactly one anti-dominant ℓ-weight. It is *tame* if the action of $U_q(\widetilde{\mathfrak{h}})$ on V is semisimple. It is *thin* if $\dim(V_{\boldsymbol{\varpi}}) \leq 1$, for all ℓ-weights $\boldsymbol{\varpi}$. We observe that if V is thin, then it is also tame.

We let $\mathrm{wt} : \mathcal{P} \to P$ be the group homomorphism defined by $\mathrm{wt}(Y_{i,a}) = \omega_i$, $i \in I, a \in \mathbb{F}^\times$. Define $A_{i,a} \in \mathcal{P}$, $i \in I, a \in \mathbb{F}^\times$, by

$$A_{i,a} = Y_{i,aq_i} Y_{i,aq_i^{-1}} \prod_{c_{ji}=-1} Y_{j,a}^{-1} \prod_{c_{ji}=-2} Y_{j,aq}^{-1} Y_{j,aq^{-1}}^{-1} \prod_{c_{ji}=-3} Y_{j,aq^2}^{-1} Y_{j,a}^{-1} Y_{j,aq^{-2}}^{-1}. \tag{3.3.1}$$

Let $\mathcal{Q}$ be the subgroup of $\mathcal{P}$ generated by $A_{i,a}$, $i \in I, a \in \mathbb{F}^\times$. Let $\mathcal{Q}^{\pm}$ be the monoid generated by $A_{i,a}^{\pm 1}$, $i \in I, a \in \mathbb{F}^\times$. Note that $\mathrm{wt}(A_{i,a}) = \alpha_i$, for all $i \in I, a \in \mathbb{F}^\times$. There is a partial order $\leq$ on $\mathcal{P}$ in which $\boldsymbol{\varpi} \leq \boldsymbol{\pi}$ implies $\boldsymbol{\pi}\boldsymbol{\varpi}^{-1} \in \mathcal{Q}^+$. Moreover, this partial order is compatible with the partial order on P in the sense that $\boldsymbol{\varpi} \leq \boldsymbol{\pi}$ implies $\mathrm{wt}(\boldsymbol{\varpi}) \leq \mathrm{wt}(\boldsymbol{\pi})$.

According to [FM01], we have, for all $\boldsymbol{\lambda} \in \mathcal{P}^+$,

$$\mathcal{M}(L_q(\boldsymbol{\lambda})) \subseteq \boldsymbol{\lambda}\mathcal{Q}^-. \tag{3.3.2}$$

For all $i \in I, a \in \mathbb{F}^\times$, let $u_{i,a}$ be the homomorphism of abelian groups $\mathcal{P} \to \mathbb{Z}$ such that

$$u_{i,a}(Y_{j,b}) = \begin{cases} 1, & i = j \text{ and } a = b, \\ 0, & \text{otherwise.} \end{cases} \tag{3.3.3}$$

For each $J \subseteq I$, we denote by $U_q(\widehat{\mathfrak{g}}_J)$ the subalgebra of $U_q(\widetilde{\mathfrak{g}})$ generated by $(x_{j,r}^\pm)_{j\in J, r\in\mathbb{Z}}$, $(\phi_{j,\pm r}^\pm)_{j\in J, r\in\mathbb{Z}_{\geq 0}}$ and let $\mathcal{P}_J$ be the subgroup of $\mathcal{P}$ generated by $(Y_{j,a}^{\pm 1})_{j\in J, a\in\mathbb{F}^\times}$ and $\mathcal{P}_J^+ \subset \mathcal{P}_J$, the set of J-dominant ℓ-weights. Let

$$\mathrm{res_J} : \operatorname{Rep} U_q(\widetilde{\mathfrak{g}}) \to \operatorname{Rep} U_q(\widetilde{\mathfrak{g}}_J)$$

be the restriction map and β_J be the homomorphism $\mathcal{P} \to \mathcal{P}_J$ sending $Y_{i,a}^\pm$ to itself, for $i \in J$, and to 1, for $i \notin J$. It is well known [FR98] that the following diagram is commutative

$$\begin{tikzcd} \operatorname{Rep} U_q(\widetilde{\mathfrak{g}}) \arrow[r, "\chi_q"] \arrow[d, "\mathrm{res}_J"] & \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{F}^\times} \arrow[d, "\beta_J"] \\ \operatorname{Rep} U_q(\widetilde{\mathfrak{g}}_J) \arrow[r, "\chi_q"] & \mathbb{Z}[Y_{j,b}^{\pm 1}]_{j\in J, b\in\mathbb{F}^\times}. \end{tikzcd}$$

The following result will be useful for us and can be found in [CP96a].

Lemma 3.3.2. *Let $\emptyset \neq J \subseteq I$ be a connected subdiagram of the Dynkin diagram of $\mathfrak{g}$. Let $\boldsymbol{\pi} \in \mathcal{P}^+$ and v be the highest weight of $L_q(\boldsymbol{\pi})$. Then $U_q(\widetilde{\mathfrak{g}}_J)v \cong_{U_q(\widetilde{\mathfrak{g}}_J)} L_q(\beta_J(\boldsymbol{\pi}))$.* □

For each $J \subseteq I$, by [FM01], there exists a ring homomorphism

$$\tau_J : \mathcal{P} \to \mathcal{P}_J \otimes \mathbb{Z}[Z_{k,c}^\pm]_{k\in I\setminus J, c\in\mathbb{F}^\times},$$

where $(Z_{k,c}^\pm)_{k\in I\setminus J, c\in\mathbb{F}^\times}$ are certain new formal variables, with the following properties:

(i) τ_J is injective,

(ii) τ_J refines β_J in the sense that β_J is the composition of τ_J with the homomorphism

$$\mathcal{P}_J \otimes \mathbb{Z}[Z_{k,c}^{\pm 1}]_{k\in I\setminus J, c\in\mathbb{F}^\times} \to \mathcal{P}_J,$$

which sends $Z_{k,c} \mapsto 1$, for all $k \notin J$, $c \in \mathbb{F}^\times$. Moreover, the restriction of τ_J to the image of $\operatorname{Rep} U_q(\widetilde{\mathfrak{g}})$ in $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{F}^\times}$ is a refinement of the restriction homomorphism res_J,

(iii) in the diagram

$$\begin{tikzcd}
\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I,a\in\mathbb{F}^\times} \arrow[r, "\tau_j"] \arrow[d] & \mathbb{Z}[Y_{j,b}^{\pm}]_{j\in J,b\in\mathbb{F}^\times} \otimes \mathbb{Z}[Z_{k,c}^{\pm}]_{k\in I\setminus J,c\in\mathbb{F}^\times} \arrow[d] \\
\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I,a\in\mathbb{F}^\times} \arrow[r, "\tau_j"] & \mathbb{Z}[Y_{j,b}^{\pm}]_{j\in J,b\in\mathbb{F}^\times} \otimes \mathbb{Z}[Z_{k,c}^{\pm}]_{k\in I\setminus J,c\in\mathbb{F}^\times}
\end{tikzcd} \tag{3.3.4}$$

if the right vertical arrow is multiplication by $\beta_j(A_{j,c}^{-1}) \otimes 1$, then the diagram commutes if and only if the left vertical arrow is multiplication by $A_{j,c}^{-1}$, where $\tau_j = \tau_{\{j\}}$, for some $j \in I$.

The following lemma is a straightforward consequence of the above properties of τ_J, when $J = \{j\}$.

Lemma 3.3.3. *Let V be a $U_q(\widetilde{\mathfrak{g}})$-module, $\boldsymbol{\varpi} \in \chi_q(V)$ such that $\beta_j(\boldsymbol{\varpi}) \in \mathcal{P}_j^+$, and $v \in V_{\boldsymbol{\gamma}} \setminus \{0\}$ such that v is a highest ℓ-weight vector for the action of $U_q(\widehat{\mathfrak{g}}_j)$. Then $\boldsymbol{\gamma\alpha} \in \chi_q(V)$, for all $\boldsymbol{\alpha} \in \mathcal{Q}^-$ such that $\beta_j(\boldsymbol{\gamma\alpha}) \in \chi_q(L_q(\beta_j(\boldsymbol{\gamma})))$.*

In what follows we often use the following lemma which follows from properties of β_J described above together with (3.3.2) and the algebraic independence of $A_{j,a}$.

Lemma 3.3.4. *Let $\boldsymbol{\pi} \in \mathcal{P}^+$ and $\{i_1, i_2, \ldots, i_N\} = I$. Let $k \in \{1, \ldots, n-1\}$ and $\boldsymbol{\alpha}_j \in \mathcal{Q}_{i_j}^-$, $j = 1, \ldots, k$. Let $\boldsymbol{\varpi} = \boldsymbol{\pi} \prod_{j=1}^k \boldsymbol{\alpha}_j$. Then $x_{i_j,r}^+ \cdot v = 0$, for all $v \in L_q(\boldsymbol{\pi})_{\boldsymbol{\varpi}}$, $j = k+1, \ldots, n$, and $r \in \mathbb{Z}$.* $\square$

3.4 Classical limits

An $\mathbb{A}$-*lattice* of an $\mathbb{F}$-vector space V is a free $\mathbb{A}$-submodule L of V such that $\mathbb{F} \otimes_{\mathbb{A}} L = V$. If V is a $U_q(\widetilde{\mathfrak{g}})$-module, a $U_{\mathbb{A}}(\widetilde{\mathfrak{g}})$-*admissible lattice* of V is an $\mathbb{A}$-lattice of V which is also an $U_{\mathbb{A}}(\widetilde{\mathfrak{g}})$-submodule of V. Given an $U_{\mathbb{A}}(\widetilde{\mathfrak{g}})$-admissible lattice of $U_q(\widetilde{\mathfrak{g}})$-module V, define

$$\overline{L} = \mathbb{C} \otimes_{\mathbb{A}} L, \tag{3.4.1}$$

where $\mathbb{C}$ is regarded as an $\mathbb{A}$-module by letting q act as 1, which is called the *classical limit* of V.

Let $\mathcal{P}_{\mathbb{A}}^+$ (resp. $\mathcal{P}_{\mathbb{Z}}^+$) be the subset of $\mathcal{P}^+$ consisting of n-tuples of polynomials $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$, where $\pi_i(u)$ has its roots in $\mathbb{A}$ (resp. $q^{\mathbb{Z}}$) for all $i \in I$, i.e., $\mathcal{P}_{\mathbb{A}}^+$ (resp. $\mathcal{P}_{\mathbb{Z}}^+$) is the monoid generated by the fundamental ℓ-weights $Y_{i,a}$, $i \in I$, $a \in \mathbb{A}$ (resp. $a \in q^{\mathbb{Z}}$).

The following was proved in [CP01, Section 4], [Cha01, Section 2] if $\mathfrak{g}$ is not of type G_2 and [JM14, Section 2.4] for $\mathfrak{g}$ of any type.

Proposition 3.4.1. *Let $\boldsymbol{\pi} \in \mathcal{P}_{\mathbb{A}}^+$.*

(i) The $U_q(\widetilde{\mathfrak{g}})$–module $L_q(\boldsymbol{\pi})$ admits a $\mathbb{A}$–lattice and hence $\overline{L_q(\boldsymbol{\pi})}$ is a module for $U(\widetilde{\mathfrak{g}})$ on which the center acts trivially.

(ii) If $\boldsymbol{\pi} = (\pi_1(u), \cdots, \pi_n(u)) \in \mathcal{P}_{\mathbb{Z}}^+$ then there exists $N \in \mathbb{N}$ such that $\overline{L_q(\boldsymbol{\pi})}$ is a module for the finite–dimensional Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]/(t-1)^N$. Moreover $\overline{L_q(\boldsymbol{\pi})}$ is generated as an $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$–module by an element $v_{\boldsymbol{\pi}}$ which satisfies the (not necessarily defining) relations:

$$(\mathfrak{n}^+ \otimes \mathbb{C}[t, t^{-1}])v_{\boldsymbol{\pi}} = 0, \qquad (h_i \otimes (t-1)^r)v_{\boldsymbol{\pi}} = \delta_{r,0} (\deg \pi_i)\, v_{\boldsymbol{\pi}}, \quad 1 \le i \le n,$$
$$(x_{\alpha_i}^- \otimes 1)^{\deg \pi_i + 1} v_{\boldsymbol{\pi}} = 0, \quad 1 \le i \le n.$$

□

The Lie algebras $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]/(t-1)^N$ and $\mathfrak{g} \otimes \mathbb{C}[t]/(t-1)^N$ are isomorphic. Hence if $\boldsymbol{\pi} \in \mathcal{P}_{\mathbb{Z}}^+$ we can regard $\overline{L_q(\boldsymbol{\pi})}$ as a $\mathfrak{g}[t]$–module and we let $L(\boldsymbol{\pi})$ be the $\mathfrak{g}[t]$–module obtained by pulling back $\overline{L_q(\boldsymbol{\pi})}$ by the Lie algebra automorphism $\mathfrak{g}[t] \to \mathfrak{g}[t]$ given by $x \otimes f \to x \otimes f(t-1)$. We continue denoting by $v_{\boldsymbol{\pi}}$ the generator of $L(\boldsymbol{\pi})$ as $\mathfrak{g}[t]$–module. The following is immediate.

Corollary 3.4.2. *For $\boldsymbol{\pi} \in \mathcal{P}_{\mathbb{Z}}^+$, we have a surjective map of $\mathfrak{g}[t]$–modules $W(\mathrm{wt}(\boldsymbol{\pi})) \to L(\boldsymbol{\pi}) \to 0$ mapping $w_{\mathrm{wt}(\boldsymbol{\pi})} \to v_{\boldsymbol{\pi}}$.* □

Remark 3.4.3. The defining relations of $L(\boldsymbol{\pi})$ are not known in general. However, in the cases treated in this work (see Chapter 4) we will obtain them and, moreover, $L(\boldsymbol{\pi})$ admits a $\mathbb{Z}$-grading compatible with the grading on $\mathfrak{g}[t]$ and, hence, we refer to the module $L(\boldsymbol{\pi})$ as the *graded limit* of $L_q(\boldsymbol{\pi})$.

3.5 Affinizations of $U_q(\mathfrak{g})$–modules

Given $\lambda \in P^+$, a finite dimensional $U_q(\widetilde{\mathfrak{g}})$-module V is said to be an affinization of $V_q(\lambda)$ if

$$[V : V_q(\lambda)] = 1 \quad \text{and} \quad [V : V_q(\mu)] \neq 0 \ \Rightarrow\ \mu \le \lambda, \tag{3.5.1}$$

where $[V : V_q(\mu)]$ denotes the multiplicity of $V_q(\mu)$ as a simple factor of V.

Two affinizations V and W are said to be equivalent if they are isomorphic as $U_q(\mathfrak{g})$-modules. The partial order on P^+ induces a natural partial order on the set of (equivalence classes of) affinizations of $V_q(\lambda)$ as follows. Say $V \le W$ if one of the following conditions hold:

(i) $[V : V_q(\mu)] \le [W : V_q(\mu)]$, for all $\mu \in P^+$,

(ii) for all $\mu \in P^+$ such that $[V : V_q(\mu)] > [W : V_q(\mu)]$, there exists $\nu > \mu$ such that $[V : V_q(\nu)] < [W : V_q(\nu)]$.

A minimal element with respect to this partial order is said to be a *minimal affinization*. Clearly, a minimal affinization of $V_q(\lambda)$ must be irreducible as an $U_q(\widetilde{\mathfrak{g}})$-module and, hence, is of the form $L_q(\boldsymbol{\pi})$, for some $\boldsymbol{\pi} \in \mathcal{P}^+$ such that $\mathrm{wt}(\boldsymbol{\pi}) = \lambda$. The notion of minimal affinizations was originally introduced in [Cha95]. The classification of the Drinfeld polynomials of the minimal affinizations is still not complete when $\mathfrak{g}$ is of type E. For the other types the classification was obtained in [CP95, CP96a, CP96b, Per14].

Given $i \in I, a \in \mathbb{F}^\times, m \in \mathbb{Z}_{\geq 0}$, define $\boldsymbol{\omega}_{i,a,m} \in \mathcal{P}^+$ by

$$\boldsymbol{\omega}_{i,a,m} = \prod_{r=0}^{m-1} Y_{i,aq_i^{m-1-2r}}. \tag{3.5.2}$$

Notice that $\mathrm{wt}(\boldsymbol{\omega}_{i,a,m}) = m\omega_i$. The modules $L_q(\boldsymbol{\omega}_{i,a,m})$ are called *Kirillov-Reshetikhin modules*.

We now review the classification of minimal affinizations for $\mathfrak{g}$ not of types D or E.

Theorem 3.5.1. *[] Let $\boldsymbol{\pi} \in \mathcal{P}^+, \lambda = \mathrm{wt}(\boldsymbol{\pi})$, and $V = L_q(\boldsymbol{\pi})$. Then, V is a minimal affinization if and only if there exist $a \in \mathbb{F}^\times$ and $\epsilon \in \{\pm 1\}$ such that*

$$\boldsymbol{\pi} = \prod_{i=1}^{n} \boldsymbol{\omega}_{i,a_i,\lambda(h_i)}, \quad \textit{with} \quad a_1 = a \quad \textit{and} \quad a_{i+1} = a_i q^{\epsilon(d_i\lambda(h_i)+d_{i+1}\lambda(h_{i+1})+d_i-1-c_{i,i+1})},$$

for all $i \in I, i < n$. Moreover, all minimal affinizations of $V_q(\lambda)$ are equivalent. $\square$

In particular, Kirillov-Reshetikhin modules are minimal affinizations of simple highest weight $U_q(\mathfrak{g})$-modules whose highest weight is an integer multiple of a fundamental weight.

Notice that $d_i\lambda(h_i) + d_{i+1}\lambda(h_{i+1}) + d_i - 1 - c_{i,i+1} \in \mathbb{Z}_{>0}$, for all $1 \leq i < n$. Because of this, we shall say that $L_q(\boldsymbol{\pi})$ is an *increasing* minimal affinization if $\boldsymbol{\pi}$ is as in Theorem 3.5.1, with $\epsilon = 1$. Otherwise, if $\epsilon = -1$, we say that $V_q(\boldsymbol{\pi})$ is a *decreasing* minimal affinization .

Lemma 3.5.2. *[] Let $\mathfrak{g}$ be of type A_n. Let $\lambda \in P^+$ and $\boldsymbol{\pi} \in \mathcal{P}_{\mathbb{Z}}^+$ be such that $L_q(\boldsymbol{\pi})$ is a minimal affinization of $V_q(\lambda)$. Then*

$$L(\boldsymbol{\pi}) \cong_{\mathfrak{g}[t]} V(\lambda, 0).$$

In particular, $x^-_{\alpha,r}v_{\boldsymbol{\pi}} = 0$, for all $r \in \mathbb{Z}_{\geq 1},\ \alpha \in R^+$. $\square$

We shall say that $L_q(\boldsymbol{\pi})$ is a *minimal affinization by parts* if, for every connected subdiagram $J \subseteq I$ such that $\#\ J \cap \mathrm{supp}(\lambda) \leq 2$, either $L_q(\beta_J(\boldsymbol{\pi}))$ is a minimal affinization for $U_q(\widetilde{\mathfrak{g}}_J)$ or there exist connected subdiagrams J_1, J_2 satisfying:

(i) $J = (J_1 \cap J) \cup (J_2 \cap J)$,

(ii) $\#\ J_1 \cap J_2 \cap \mathrm{supp}_P(\lambda) = 1$,

(iii) $\#\ J_i \cap \mathrm{supp}_P(\lambda) = 2$ for $i = 1, 2$,

(iv) $\#\ J \cap J_i \cap \mathrm{supp}_P(\lambda) = 1$ for $i = 1, 2$,

(v) $L_q(\beta_{J_i}(\boldsymbol{\pi}))$ is a minimal affinization for $U_q(\widetilde{\mathfrak{g}}_{J_i})$ for $i = 1, 2$,

where $\mathrm{supp}_P(\lambda) = \{i \in I | \lambda(h_i) \neq 0\}$.

Remark 3.5.3. If either $\mathrm{supp}_P(\lambda)$ does not bound a subdiagram of type D (which is the case if $\mathfrak{g}$ is not of types D or E) or $\mathrm{supp}(\lambda)$ contains the trivalent node, then subdiagrams J_1 and J_2 satisfying conditions (i)–(v) do not exist. In that case, every minimal affinization of $L_q(\lambda)$ is a minimal affinization by parts. Otherwise, there exist minimal affinizations which are not minimal affinizations by parts (cf. [CP96b, Per14]).

An $U_q(\widetilde{\mathfrak{g}})$-module V is said to be *prime* if V is not the trivial representation and

$$V \cong V_1 \otimes V_2 \;\Rightarrow\; V_1 \cong V \quad \text{or} \quad V_2 \cong V.$$

The following is a consequence of the main results of [CMY13].

Theorem 3.5.4. *[] Every minimal affinization by parts is prime.* □

For $i, j \in I$, let $[i, j]$ be the minimal connected subdiagram containing i, j. Suppose $\boldsymbol{\pi} \in \mathcal{P}^+$ is such that $L_q(\boldsymbol{\pi})$ is a minimal affinization by parts and let $\lambda = \mathrm{wt}(\boldsymbol{\pi})$. We say that $L_q(\boldsymbol{\pi})$ is *alternating* if, for every $i, j \in \mathrm{supp}_P(\lambda)$ such that $\#\mathrm{supp}_P(\lambda) \cap [i, j] = 3$, say $\mathrm{supp}_P(\lambda) \cap [i, j] = J_1 \cup J_2$ with $J_1 = [i, k], J_2 = [k, j], k \in \mathrm{supp}_P(\lambda)$, we have

$$L_q(\beta_{J_1}(\boldsymbol{\pi})) \quad \text{increasing} \quad \Rightarrow \quad L_q(\beta_{j_2}(\boldsymbol{\pi})) \quad \text{decreasing, or vice-versa.}$$

Chapter 4

Graded limit of affinizations

In this chapter we prove the main results of this work related to the study of modules for the quantum affine algebra by means of their classical limit.

4.1 Statement of results and motivations

Set

$$P^+(1) = \{\lambda \in P^+ | \lambda(h_i) \leq 1, \text{ for all } 1 \leq i \leq n\}.$$

Theorem 4.1.1. *Let $\mathfrak{g}$ be of type A_n. Let $\mu = 2\nu + \lambda \in P^+$, with $\nu \in P^+$ and $\lambda \in P(1)^+$. Let also $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2 \in \mathcal{P}_{\mathbb{Z}}^+$ be such that:*

(i) $\mathrm{wt}(\boldsymbol{\pi}_1) = 2\nu$ *and* $\mathrm{wt}(\boldsymbol{\pi}_2) = \lambda$,

(ii) there exists $j_s \in I$ and $z_s \in q^{\mathbb{Z}}$, for all $1 \leq s \leq r = \nu(h_\theta)$, such that

$$L_q(\boldsymbol{\pi}_1) \cong_{U_q(\tilde{\mathfrak{g}})} L_q(\boldsymbol{\omega}_{j_1,z_1,2}) \otimes \cdots \otimes L_q(\boldsymbol{\omega}_{j_r,z_r,2}),$$

(iii) $L_q(\boldsymbol{\pi}_2)$ is an alternating minimal affinization by parts of $V_q(\lambda)$,

(iv) $L_q(\boldsymbol{\pi}_1\boldsymbol{\pi}_2) \cong_{U_q(\tilde{\mathfrak{g}})} L_q(\boldsymbol{\pi}_1) \otimes L_q(\boldsymbol{\pi}_2)$.

Then

$$L(\boldsymbol{\pi}_1\boldsymbol{\pi}_2) \cong_{\mathfrak{g}[t]} D(2, \mu).$$

Theorem 4.1.2. *Let $\mathfrak{g}$ be of classical type, $\boldsymbol{\pi} \in \mathcal{P}_{\mathbb{Z}}^+$, and $\lambda = \mathrm{wt}(\boldsymbol{\pi})$. Suppose that $L_q(\boldsymbol{\pi})$ is a minimal affinization and λ is regular. Then there exists a λ-compatible $\boldsymbol{\xi}$ such that $L(\boldsymbol{\pi}) \cong V(\boldsymbol{\xi})$.*

We now explain our motivation for restricting our attention to the modules considered in Theorem 4.1.1.

In [HL10] and [HL13] the authors identified certain interesting subcategories with motivations coming from cluster algebras and monoidal categorification. For this, they consider the notion of a height function, $\kappa : I \to \mathbb{Z}$, i.e., a function satisfying $|\kappa(i+1) - \kappa(i)| \leq 1$. (Here we continue to assume that $\mathfrak{g}$ is of type A_n and that $I = \{1, \cdots, n\}$ is the standard enumeration of the vertices of the Dynkin diagram). The objects of the category $\mathcal{C}_\kappa$ are finite–dimensional representations of $U_q(\widetilde{\mathfrak{g}})$, whose Jordan–Holder series have composition factors of the form $L_q(\boldsymbol{\pi})$, where $\boldsymbol{\pi} \in \mathcal{P}^+$ is a product of elements of the form $Y_{i,q^{\kappa(i)}}$ and $Y_{i,q^{\kappa(i)+2}}$, for $1 \leq i \leq n$.

Let $\mathbf{Q}_\kappa$ be the quiver where the vertices are the elements of I with edge $i \to i+1$, if $\kappa(i+1) < \kappa(i)$, and $i+1 \to i$, otherwise. It is shown in [HL10, HL13, HL14] that the prime representations in $\mathcal{C}_\kappa$ are of the following form:

$$L_q(Y_{i,q^{\kappa(i)}}), \quad L_q(Y_{i,q^{\kappa(i)+2}}), \quad L_q(Y_{i,q^{\kappa(i)}}Y_{i,q^{\kappa(i)+2}}), \tag{4.1.1}$$

$$L_q\left(\prod_{j\in I_{1,<}} Y_{j,q^{\kappa(j)}} \prod_{j\in I_{1,>}} Y_{j,q^{\kappa(j)+2}}\right), \tag{4.1.2}$$

where $1 \leq i \leq n$, I_1 is a connected subset of I and $I_{1,<}$ (resp. $I_{1,>}$) is the set of sinks (resp. sources) of the subquiver of $\mathbf{Q}_\kappa$ associated to I_1. It is known that the objects in (4.1.1) are irreducible as $U_q(\mathfrak{g})$–modules and more generally, the q–characters of these representations are also known. On the other hand very little is known about the objects in (4.1.2). Note that these modules are easily identified with alternating minimal affinizations by parts of the module $V_q(\lambda)$, for $\lambda = \sum_{i\in I_1} \omega_i$.

It is not difficult to write down all possible height functions and we describe this here. Given an arbitrary subset $J = \{i_1, \cdots, i_k\}$ of I with $1 \leq i_1 < \cdots < i_k \leq n$ consider the quiver with edges

$$1 \to 2 \cdots \to i_1 \leftarrow i_1 + 1 \leftarrow \cdots \leftarrow i_2 \to i_2 + 1 \cdots \to i_3 \leftarrow \cdots .$$

For $r \in \mathbb{Z}$ and setting $\kappa_{J,r}(1) = 0$, one can define the corresponding height function $\kappa_{J,r}$ and it is trivially seen that any height function is of this form. For a fixed J the categories $\mathcal{C}_{\kappa_{J,r}}$ are all essentially the same: they are obtained from each other by pulling back by a suitable automorphism of $U_q(\widetilde{\mathfrak{g}})$. Hence it suffices to understand the category $\mathcal{C}_J = \mathcal{C}_{\kappa_{J,0}}$, where $\kappa_J = \kappa_{J,0}$.

The main object of this chapter is understanding the $q \to 1$ specialization of the objects in (4.1.1) an (4.1.2).

Remark 4.1.3. As noted in [HL13, Remark 1(c)], the sink-source and monotonic quivers give rise to nonequivalent subcategories (with the same number of prime objects).

Regarding the last two theorems, it is well–known that, in general, even graded limits of Kirilov–Reshetkhin modules are not isomorphic to Demazure modules. However, recent results (see [Nao13, Nao14]) identify the graded limit of (regular) minimal affinizations with generalized Demazure modules, when $\mathfrak{g}$ is of classical type. Since CV-modules can also be regarded as a generalization of ($\mathfrak{g}$-stable) affine Demazure modules, it is natural to raise the question whether such modules are also isomorphic to CV-modules.

4.2 Level two $\mathfrak{g}$-stable Demazure modules

Assume that $\mathfrak{g}$ is of type A_n. Our primary focus in this chapter is the study of level two $\mathfrak{g}$-stable Demazure modules. In this section we prove several properties of these modules which will be fundamental in the proof of Theorem 4.1.1.

4.2.1 A realization of level two $\mathfrak{g}$–stable Demazure modules

The goal of this section is to prove the following:

Theorem 4.2.1. *Let $\mathfrak{g}$ be of type A_n. Given $\mu \in P^+$ there exists $\mu^o, \mu^e \in P^+$ with $\mu = \mu^o + \mu^e$ such that we have an injective map of $\widehat{\mathfrak{b}} \oplus \mathfrak{n}^-$–modules*

$$D(2, \mu) \hookrightarrow D(1, \mu^o) \otimes D(1, \mu^e), \quad v_\mu \mapsto v_{\mu^o} \otimes v_{\mu^e}.$$

Given $\lambda = \sum_{j=1}^k \omega_{i_j} \in P^+(1)$, with $1 \le i_1 < i_2 < \cdots < i_k \le n$, define $\lambda^o, \lambda^e \in P^+(1)$ by:

$$\lambda^o = \begin{cases} \omega_{i_1} + \omega_{i_3} + \cdots + \omega_{i_k}, & k \text{ odd}, \\ \omega_{i_1} + \omega_{i_3} + \cdots + \omega_{i_{k-1}}, & k \text{ even}, \end{cases} \qquad \lambda^e = \lambda - \lambda^o.$$

We shall prove

Proposition 4.2.2. *Given $\lambda \in P^+(1)$ and $\nu \in P^+$ there exists $w \in \widetilde{W}$ such that*

$$\Lambda^o = w(\nu + \lambda^o + \Lambda_0) \in \widehat{P}^+, \quad \Lambda^e = w(\nu + \lambda^e + \Lambda_0) \in \widehat{P}^+.$$

Assuming Proposition 4.2.2 the proof of Theorem 4.2.1 is completed as follows. Write $\mu = 2\nu + \lambda$, where $\nu \in P^+$ and $\lambda \in P^+(1)$, and set $\mu^o = \nu + \lambda^o$ and $\mu^e = \nu + \lambda^e$. Choose Λ^o and Λ^e to be as in Proposition 4.2.2 and take $\Lambda = w(\lambda + 2\Lambda_0) = \Lambda^o + \Lambda^e \in \widehat{P}^+$. Then

$$D(2, \lambda) = D(w_0 w^{-1} \Lambda), \quad D(1, \lambda^o) = D(w_0 w^{-1} \Lambda^o), \quad D(1, \lambda^e) = D(w_0 w^{-1} \Lambda^e).$$

Theorem 4.2.1 is now immediate from Lemma 2.2.1.

Proof of Proposition 4.2.2. We claim that for a fixed λ, it suffices to prove the result when $\nu = 0$. Thus suppose that we have chosen $w \in \widetilde{W}$ such that $w(\lambda^o + \Lambda_0)$ and $w(\lambda^e + \Lambda_0)$ are in $\widehat{P}^+$. Since $(\lambda^o + \Lambda_0)(c) = 1 = (\lambda^e + \Lambda_0)(c)$, we may write

$$w(\lambda^o + \Lambda_0) = \omega_i + \Lambda_0 + p^o \delta, \quad w(\lambda^e + \Lambda_0) = \omega_j + \Lambda_0 + p^e \delta,$$

for some $p^o, p^e \in \mathbb{Z}$ and $0 \le i, j \le n$, where we have the convention that $\omega_0 = 0 = \omega_{n+1}$. Using the formulae in (1.1.2), we get

$$t_{-w\nu} w(\lambda^o + \Lambda_0 + \nu) = \omega_i + \Lambda_0 + (p^o + \frac{1}{2}(\nu, \nu) + (\omega_i, w\nu))\delta,$$

$$t_{-w\nu}w(\lambda^e + \Lambda_0 + \nu) = \omega_j + \Lambda_0 + (p^e + \frac{1}{2}(\nu,\nu) + (\omega_j, w\nu))\delta,$$

and the claim is established.

Consider the partial order on $P^+(1)$ given by $\mu \le \nu$ if and only if $\nu - \mu \in Q^+$. The minimal elements of this order are ω_i, $1 \le i \le n$, and $\omega_i + \Lambda_0 \in \widehat{P}^+$, i.e., the proposition is trivially true for these elements. Let $\lambda \in P^+(1)$ and suppose that we have proved the result for all elements $\mu \in P^+(1)$ with $\mu < \lambda$. To prove the result for λ it suffices to show that there exists $w \in \widetilde{W}$ and $\mu \in P^+$ with $\mu < \lambda$ and $p^o, p^e \in \mathbb{Z}$ such that

$$w(\lambda^o + \Lambda_0) = \mu^o + \Lambda_0 + p^o\delta, \quad w(\lambda^e + \Lambda_0) = \mu^e + \Lambda_0 + p^e\delta. \tag{4.2.1}$$

This is done as follows: writing $\lambda = \sum_{j=1}^k \omega_{i_j}$, $i_1 < \cdots < i_k$, we take

$$w = \begin{cases} s_{i_3}s_{i_3+1}\cdots s_n s_{i_{k-2}-1}s_{i_{k-2}-2}\cdots s_1 s_0, & k > 3, \text{ or } k = 3, \quad i_1 > 1, \\ s_{i_3}s_{i_3+1}\cdots s_n s_0, & k = 3, \quad i_1 = 1, \end{cases}$$

and

$$\mu = \begin{cases} \omega_{i_1-1} + \omega_{i_2} + \omega_{i_3+1}, & k = 3, \\ \omega_{i_1-1} + \omega_{i_2-1} + \omega_{i_3} + \cdots + \omega_{i_{k-2}} + \omega_{i_{k-1}+1} + \omega_{i_k+1}, & k > 3. \end{cases}$$

Note that $\mu < \lambda$, since

$$k = 3 \implies \lambda - \mu = \alpha_{i_1} + \cdots + \alpha_{i_3}, \quad k > 3 \implies \lambda - \mu = (\alpha_{i_1} + \cdots + \alpha_{i_k}) + (\alpha_{i_2} + \cdots + \alpha_{i_{k-1}}).$$

It remains to establish that equation (4.2.1) is satisfied. For this, it is more convenient to deal with the cases $k = 2, 3, 4$ separately. If $k = 2$ then taking $w = \mathrm{Id}$ gives the result.

If $k = 4$ or if $k = 3$ and $i_1 > 1$, a simple calculation gives

$$w(\lambda^o + \Lambda_0) = w(\Lambda_{i_1} + \Lambda_{i_3} - \Lambda_0) = \Lambda_{i_1} + \Lambda_{i_3} - \Lambda_0 + \sum_{j=0}^{i_1-1}\alpha_j + \sum_{j=i_3}^{n}\alpha_j = \mu^o + \Lambda_0 + \delta.$$

Moreovoer, if $k = 3$, we have

$$w(\lambda^e + \Lambda_0) = w(\Lambda_{i_2}) = \Lambda_{i_2} = \mu^e + \Lambda_0,$$

while if $k = 4$, we have

$$w(\lambda^e + \Lambda_0) = w(\Lambda_{i_2} + \Lambda_{i_4} - \Lambda_0) = \Lambda_{i_2} + \Lambda_{i_4} - \Lambda_0 + \sum_{j=0}^{i_2-1}\alpha_j + \sum_{j=i_4+1}^{n}\alpha_j = \mu^e + \Lambda_0 + \delta.$$

The case $k = 3$ and $i_1 = 1$ is identical and we omit the details. For $k \ge 5$. we write

$$w(\lambda^o + \Lambda_0) = w(\lambda^o + \Lambda_0)(d)\delta + \sum_{j=0}^{n}(w(\lambda^o + \Lambda_0), \alpha_j)\Lambda_j$$

and similarly for $\lambda^e+\Lambda_0$. We have

$$w(\lambda^o+\Lambda_0)(d)=(\lambda^o+\Lambda_0)(w^{-1}d)=(\lambda^o+\Lambda_0)(d-h_0)=\lambda^o(h_\theta)-1.$$

To prove that $(w(\lambda^o+\Lambda_0),\alpha_j)=(\mu^o+\Lambda_0,\alpha_j)$ it is enough to prove that

$$(\lambda^o+\Lambda_0)(w^{-1}h_j)=(\mu^o+\Lambda_0)(h_j)$$

and this is done by using the following easily established formulae:

$$\begin{gathered}
w^{-1}(\alpha_0)=\alpha_0+\alpha_1+\alpha_n, \qquad w^{-1}(\alpha_j)=\alpha_j,\quad i_3<j<i_{k-2},\\
w^{-1}(\alpha_j)=\alpha_{j+1},\quad 0<j<i_3-1, \qquad w^{-1}(\alpha_j)=\alpha_{j-1},\quad j>i_{k-2}+1,\\
w^{-1}(\alpha_{i_3-1})=\alpha_{i_3}+\cdots+\alpha_n+\alpha_0, \qquad w^{-1}(\alpha_{i_{k-2}+1})=\alpha_{i_{k-2}}+\cdots+\alpha_1+\alpha_0,\\
w^{-1}(\alpha_{i_{k-2}})=-(\alpha_0+\alpha_1+\cdots+\alpha_{i_{k-2}-1}),
\end{gathered}$$

and

$$w^{-1}(\alpha_{i_3})=\begin{cases}-(\alpha_{i_3+1}+\cdots+\alpha_n+\alpha_0), & k>5,\\ -(\alpha_1+\cdots+\alpha_n+2\alpha_0), & k=5.\end{cases}$$

The case of $w(\lambda^e+\Lambda_0)$ is identical and the proof of Proposition 4.2.2 is complete. $\square$

4.2.2 A presentation of level two $\mathfrak{g}$–stable Demazure modules.

In this section, we show that in the case of $\ell=2$ one can further whittle down the set of defining relations given in Theorem 2.5.4. This will be crucial for our study of graded limits of representations of quantum affine $\mathfrak{sl}_{n+1}$.

Recall our assumption that $\mathfrak{g}$ is of type A_n. Set

$$\alpha_{i,j}=\sum_{p=i}^{j}\alpha_p,\quad 1\le i\le j\le n,$$

and note that $R^+=\{\alpha_{i,j}|1\le i\le j\le n\}$. This section is devoted to prove

Proposition 4.2.3. *For $\mu=2\nu+\lambda\in P^+$, with $\lambda\in P^+(1)$ and $\nu\in P^+$, we have that $D(2,\mu)$ is the $U(\mathfrak{g}[t])$–module generated by an element v_μ with defining relations:*

$$x_{\alpha_i}^+v_\mu=0,\qquad (h_i\otimes t^s)v_\mu=\delta_{s,0}\mu(h_i)v_\mu,\qquad (x_{\alpha_i}^-)^{\mu(h_i)+1}v_\mu=0, \tag{4.2.2}$$

where $1\le i\le n$ and $0\le s\le\mu(h_i)$, and if $\lambda=\omega_{i_1}+\cdots+\omega_{i_k}$, $1\le i_1<\cdots<i_k\le n$, then

$$(x_{\alpha_i}^-\otimes t^{(\nu+\lambda)(h_i)})v_\mu=0,\quad 1\le i\le n,\qquad (x_{\alpha_{i_p,i_{p+1}}}^-\otimes t^{\nu(h_{\alpha_{i_p,i_{p+1}}})+1})v_\mu=0,\quad 1\le p<k. \tag{4.2.3}$$

The following is straightforward.

Lemma 4.2.4. *For all $\ell \in \mathbb{Z}_{\geq 0}$ and $\mu \in P^+$we have* $\mathrm{wt}D(\ell,\mu)_\nu \neq \{0\}$ *only if* $\nu \in \mu - Q^+$. *Moreover,*

$$\dim \mathrm{Hom}_{\mathfrak{g}}(V(\mu), D(\ell,\mu)) = \dim \mathrm{Hom}_{\mathfrak{g}[t]}(D(\ell,\mu), V(\mu,0)) = 1.$$

Finally, if $\lambda(h_\theta) \leq \ell$ *then*

$$D(\ell,\lambda) \cong_{\mathfrak{g}[t]} V(\lambda,0).$$

□

In the case when $\ell = 2$, a further simplification of Theorem 2.5.4 is possible and this uses the following consequence of [CV14, Theorem 5].

Proposition 4.2.5. *Assume that $\mathfrak{g}$ is of type $\mathfrak{sl}_2$, let $\mu \in P^+$ and consider the local Weyl module $W(\mu)$. For all $1 \leq s \leq \mu(h_\alpha)$, we have $(x^- \otimes t^{s-1})^2 w_\mu \in U(\mathfrak{g}[t])(x^- \otimes t^s)w_\mu$.*

Proof. We indicate briefly how to obtain this proposition from [CV14, Theorem 5]. In the case when $s = \mu(h_\alpha)$, the result is a consequence of the identification of $W(\mu)$ with $D(1,\mu)$. The quotient of $W(\mu)$ by the submodule generated by the element $(x^- \otimes t^{\mu(h_\alpha)-1})w_\mu$ is denoted as $V(2, 1^{\mu(h_\alpha)-2})$ in [CV14] and it is shown that $(x^- \otimes t^{\mu(h_\alpha)-2})^2 w_\mu = 0$ in $V(2,1^{\mu(h_\alpha)-2})$. Using [CV14, Theorem 5], we have an inclusion $V(2,1^{\mu(h_\alpha)-4}) \hookrightarrow V(2,1^{\mu(h_\alpha)-2})$ which maps $w_{\mu-2} \to (x^- \otimes t^{\mu(h_\alpha)-2})w_\mu$, whose quotient is denoted as $V(2^2 1^{\mu(h_\alpha)-4})$ and $(x^- \otimes t^{\mu(h_\alpha)-3})^2 w_\mu = 0$ in $V(2^2, 1^{\mu(h_\alpha)-4})$. Repeating this, we get the proposition. □

Given any proper connected subset J of I, we let $\mathfrak{g}_J$ be the simple subalgebra of $\mathfrak{g}$ generated by the elements $x_i^\pm$, $i \in J$, and let $\mathfrak{h}_J$ and P_J^+ be defined in the obvious way. Given $\mu \in P^+$, let $\mu_J \in P_J^+$ be the restriction to $\mathfrak{h}_J$ and denote by $D_J(\ell,\mu_J)$ the level ℓ–Demazure module for $\mathfrak{g}_J[t]$ associated to μ_J. It follows from Theorem 2.5.4 that there exists a non–zero map of $\mathfrak{g}_J[t]$–modules from $D_J(\ell,\mu_J) \to D(\ell,\mu)$ which maps $v_{\mu_J} \to v_\mu$. Taking $J = \{\alpha\}$, we see that the following Lemma is immediate from Proposition 4.2.5 and Theorem 2.5.4.

Lemma 4.2.6. *For $\mu \in P^+$, the module $D(2,\mu)$ is the quotient of $W(\mu)$ by the additional relations $(x_\alpha^- \otimes t^{\lceil \mu(h_\alpha)/2 \rceil})w_\mu = 0$, for all $\alpha \in R^+$.* □

Proof of Proposition 4.2.3. Let $\bar{D}(2,\mu)$ be the $U(\mathfrak{g}[t])$–module generated by an element v_μ with defining relations given by (4.2.2) and (4.2.3). By Lemma 4.2.6, the proposition follows if we prove that

$$(x_\alpha^- \otimes t^{\lceil \mu(h_{\alpha_{i,j}})/2 \rceil})v_\mu = 0, \quad 1 \leq i \leq j \leq n. \tag{4.2.4}$$

We proceed by induction on $i - j$, with induction beginning at $i = j$ by equation (4.2.3). Assume that we have proved the result for all $\alpha_{i,j}$ with $i - j < s$ and consider the case when $j = i + s$.

Suppose first that there does not exist $1 \le p \le k$ such that $i \le i_p \le i+s$. In this case, we have $\mu(h_{\alpha_{i,j}}) = 2\nu(h_{\alpha_{i,j}}) = 2\nu(h_i) + 2\nu(h_{\alpha_{i+1,i_s}})$. The induction hypothesis implies

$$(x^-_{\alpha_i} \otimes t^{\nu(h_i)})w_\mu = 0 = (x^-_{\alpha_{i+1,i+s}} \otimes t^{\nu(h_{\alpha_{i+1,i+s}})})w_\mu.$$

Since $[x^-_{\alpha_i}, x^-_{\alpha_{i+1,i+s}}] = x^-_{\alpha_{i,i+s}}$, we get (4.2.4) in this case. We consider the other case, when we can choose $1 \le p \le k$ be minimal and $1 \le r \le k$ maximal so that $i \le i_p \le i_r \le i+s$. If $i < i_p$ we have again by the inductive hypothesis that

$$(x^-_{\alpha_i} \otimes t^{\nu(h_i)})w_\mu = 0 = (x^-_{\alpha_{i+1,i+s}} \otimes t^{(\nu+\lambda)(h_{\alpha_{i+1,i+s}})})w_\mu,$$

and the inductive step is completed as before. If $j > i_r$ then the proof is similar where we write $\alpha_{i,i+s} = \alpha_{i,i+s-1} + \alpha_{i,i+s}$. Finally, suppose $i = i_p$ and $i+s = i_r$. If $r = p+1$ then the inductive step is the hypothesis in equation (4.2.3). If $r \ge p+2$, then we write $\alpha_{i,i+s} = \alpha_{i_p,i_{p+1}} + \alpha_{i_{p+1}+1,i_r}$. This time the induction hypothesis gives,

$$(x^-_{\alpha_{i_p,i_{p+1}}} \otimes t^{\nu(h_{\alpha_{i_p,i_{p+1}}})+1})w_\mu \quad = \quad 0 \quad = \quad (x^-_{\alpha_{i_{p+1}+1,i_r}} \otimes t^{(\nu+\lambda)(h_{\alpha_{i_{p+1}+1,i_r}})})w_\mu,$$

and the inductive step is completed as before. □

4.2.3 Fusion products and a characterization of $\mathfrak{g}$–stable level two Demazure modules

Our goal in this section is to prove a result which will help us to study that the graded limit of a tensor product of certain prime representations of quantum affine $\mathfrak{sl}_{n+1}$ is a level two Demazure module. As we have remarked before, it is very difficult to check relations in the classical limit of representations of quantum affine algebras and the result in this section circumvents this difficulty. Another interesting feature of this result is that it provides a new family of examples with the following property. Suppose that $V_1, \cdots, V_r$ are modules for quantum affine $\mathfrak{sl}_{n+1}$ each admitting a graded limit. Then the tensor product has a graded limit which is the fusion product in the sense of [FL99] of the graded limits. This section is devoted to prove

Proposition 4.2.7. *Let $\mu \in P^+$ and let V be a (not necessarily graded) $\mathfrak{g}[t]$–module quotient of $W(\mu)$ such that V is isomorphic to $D(2,\mu)$ as $\mathfrak{g}$–modules. Then V is isomorphic to $D(2,\mu)$ as $\mathfrak{g}[t]$–modules and hence $\mathbb{Z}$–graded.*

This result will be used to prove that a level two Demazure module is isomorphic to the graded limit of an irreducible module for quantum affine $\mathfrak{sl}_{n+1}$ and also that the graded limit of certain tensor products of quantum affine algebras give rise to fusion products of representations of $\mathfrak{g}[t]$.

The first isomorphism in the next result was proved in [CL06] and the second is a special case of [CSVW14, Theorem 1]. These isomorphisms give two families of examples where the fusion product is independent of the parameters which will be needed in this chapter.

Theorem 4.2.8. *Let $\mu = 2\nu + \lambda \in P^+$ where $\nu = \sum_{i=1}^n r_i\omega_i \in P^+$ and $\lambda = \sum_{j=1}^k \omega_{i_j} \in P^+(1)$, for some $1 \le i_1 < \cdots < i_k \le n$. We have isomorphisms of $\mathfrak{g}[t]$–modules, as follows:*

$$D(1,\mu) \cong_{\mathfrak{g}[t]} V(\omega_1)^{*2r_1} * \cdots * V(\omega_n)^{*2r_n} * V(\omega_{i_1}) * \cdots * V(\omega_{i_k}) \tag{4.2.5}$$

$$D(2,\mu) \cong_{\mathfrak{g}[t]} V(2\omega_1)^{*r_1} * \cdots * V(2\omega_n)^{*r_n} * D(2,\lambda). \tag{4.2.6}$$

□

We shall need the following corollary.

Corollary 4.2.9. *For $\mu = 2\nu + \lambda \in P^+$ with $\nu \in P^+$ and $\lambda = \sum_{j=1}^k \omega_{i_j}$ with $1 \le j \le k$, we have,*

$$\dim D(2,\mu)_{\mu-\alpha_i} = (\nu+\lambda)(h_i), \quad 1 \le i \le n, \tag{4.2.7}$$

$$\dim D(1,\lambda)_{\lambda-\alpha_{i_p,i_{p+1}}} = i_{p+1} - i_p + 2, \quad \dim D(2,\lambda)_{\lambda-\alpha_{i_p,i_{p+1}}} = i_{p+1} - i_p + 1, \quad 1 \le p < k. \tag{4.2.8}$$

Proof. Since $\dim V(2\omega_j)_{2\omega_j-\alpha_i} = \delta_{i,j}$, by using the isomorphism in (4.2.6), we have

$$\dim D(2,\mu)_{\mu-\alpha_i} = r_i + \dim D(2,\lambda)_{\lambda-\alpha_i}.$$

Moreover, by Lemma 4.2.4, we have $\dim \mathrm{Hom}_{\mathfrak{g}}(V(\lambda), D(2,\lambda)) = 1$, hence

$$\dim D(2,\lambda)_{\lambda-\alpha_i} \ge \dim V(\lambda)_{\lambda-\alpha_i} = \sum_{p=1}^k \delta_{i,i_p}.$$

On the other hand,

$$\dim D(2,\lambda)_{\lambda-\alpha_i} \le \dim D(1,\lambda)_{\lambda-\alpha_i} \le \sum_{p=1}^k \delta_{i,i_p},$$

where the first inequality is by Theorem 2.5.4 and the second inequality follows from (4.2.5). The first equality in (4.2.8) is also a simple consequence of (4.2.5); note moreover that it also proves that $(x^-_{\alpha_{i_p,i_{p+1}}} \otimes t)w_\lambda$ along with elements from the $\mathfrak{g}$–submodule $V(\lambda)$ of $D(1,\lambda)$ forms a basis for $D(1,\lambda)_{\lambda-\alpha_{i_p,i_{p+1}}}$. On the other hand since $\lambda(h_{\alpha_{i_p,i_{p+1}}}) = 2$, it follows in the language of Theorem 2.5.4 that $s_\alpha = 1$ and hence $(x^-_{\alpha_{i_p,i_{p+1}}} \otimes t)w_\lambda = 0$ in $D(2,\lambda)$. The second equality in (4.2.8) is established. □

Let V be as in Proposition 4.2.7 and $\varphi : W(\mu) \to V \to 0$ be a map of $\mathfrak{g}[t]$–modules. We shall prove that $\bar{v}_\mu = \varphi(w_\mu)$ satisfies (4.2.3). Proposition 4.2.3 thus implies that φ factors through $D(2,\mu)$ to give a surjective map $D(2,\mu) \to V$ of $\mathfrak{g}[t]$–modules. Since $V \cong D(2,\mu)$ as $\mathfrak{g}$–modules it follows that φ induces an isomorphism of $\mathfrak{g}[t]$–modules as required.

Write $\mu = 2\nu + \lambda$ with $\nu = \sum_{i=1}^n r_i\omega_i \in P^+$ and $\lambda = \sum_{j=1}^k \omega_{i_j} \in P^+(1)$. To prove that $\bar{v}_\mu$ satisfies the first equality in (4.2.3), observe that

$$(h_i \otimes t)(x^-_{\alpha_i} \otimes t^s)\bar{v}_\mu = 2(x^-_{\alpha_i} \otimes t^{s+1})\bar{v}_\mu.$$

Hence if $(x^-_{\alpha_i} \otimes t^s)\bar{v}_\mu = 0$, then $(x^-_{\alpha_i} \otimes t^{s+1})\bar{v}_\mu = 0$. In particular, if $(x^-_{\alpha_i} \otimes t^{(\nu+\lambda)(h_i)})\bar{v}_\mu \neq 0$ then Corollary 4.2.9 gives that the non–zero elements $\{(x^-_{\alpha_i} \otimes t^s)\bar{v}_\mu | 0 \leq s \leq (\nu+\lambda)(h_i)\}$ must be linearly dependent. Choose $m < (\nu + \lambda)(h_i)$ so that

$$\sum_{s=m}^{(\nu+\lambda)(h_i)} z_s(x^-_{\alpha_i} \otimes t^s)\bar{v}_\mu = 0, \quad z_s \in \mathbb{C},\ z_m \neq 0.$$

Since $(x^-_{\alpha_i} \otimes t^{\mu(h_i)})w_\mu = 0$ we have $(x^-_{\alpha_i} \otimes t^{\mu(h_i)})\bar{v}_\mu = 0$. Applying $(h_i \otimes t^{\mu(h_i)-m-1})$ to the preceding equation we get $(x^-_{\alpha_i} \otimes t^{\mu(h_i)-1})\bar{v}_\mu = 0$. Repeating this argument with $(h_i \otimes t^{\mu(h_i)-m-2})$ then gives $(x^-_{\alpha_i} \otimes t^{\mu(h_i)-2})\bar{v}_\mu = 0$ and further iterations eventually gives

$$(x^-_{\alpha_i} \otimes t^m)\bar{v}_\mu = 0$$

which contradicts our assumptions. Hence we have $(x^-_{\alpha_i} \otimes t^{(\nu+\lambda)(h_i)})\bar{v}_\mu = 0$, as required.

Denote by $\tilde{V}$ the graded quotient of $W(\mu)$ by the $\mathfrak{g}[t]$–submodule generated by the set

$$\{(x^-_{\alpha_i} \otimes t^{(\nu+\lambda)(h_i)})w_\mu | 1 \leq i \leq n\}.$$

We denote the image of w_μ in $\tilde{V}$ by $\tilde{v}_\mu$, and

$$(x^-_{\alpha_i} \otimes t^{(\nu+\lambda)(h_i)})\tilde{v}_\mu = 0.$$

The previous discussion shows that φ factors through to a surjective map (which we continue to denote by φ) from $\tilde{V}$ to V. The standard argument of writing $x^-_{\alpha_{i_p,i_{p+1}}}$ as a commutator of simple root vectors gives

$$(x^-_{\alpha_{i_p,i_{p+1}}} \otimes t^{\nu(h_{\alpha_{i_p,i_{p+1}}})+2})\tilde{v}_\mu = 0, \tag{4.2.9}$$

for all $1 \leq p < k$. More generally, it also proves that if $\alpha = \alpha_{i_p} + \alpha_{i_p+1} + \cdots + \alpha_s$ with $s < i_{p+1}$ or if $\alpha = \alpha_r + \alpha_{r+1} + \cdots \alpha_{i_{p+1}}$ with $r > i_p$ and $\beta = \alpha_r + \cdots + \alpha_s$ with $i_p < r < s < i_{p+1}$ for some $1 \leq p < k$, then

$$(x^-_\alpha \otimes t^{\nu(h_\alpha)+1})\tilde{v}_\mu = 0 = (x^-_\beta \otimes t^{\nu(h_\alpha)})\tilde{v}_\mu = 0. \tag{4.2.10}$$

We shall prove

Proposition 4.2.10. *(i) We have*

$$\dim \tilde{V}_{\mu-\alpha} > \dim V_{\mu-\alpha},$$

for all $\alpha = \alpha_{i_p} + \cdots + \alpha_{i_{p+1}}$ with $1 \leq p < k$.

(ii) Let U be any submodule of $\tilde{V}$ with $U_{\mu-\alpha} \neq 0$, for some $\alpha = \alpha_{i_p} + \cdots + \alpha_{i_{p+1}}$ with $1 \leq p < k$. Then

$$(x^-_\alpha \otimes t^{\nu(h_\alpha)+1})\tilde{v}_\mu \in U.$$

Assuming this proposition, the proof of Proposition 4.2.7 is completed as follows. Consider the submodule $\ker\varphi$ of $\tilde{V}$. The first assertion of Proposition 4.2.10 shows that $(\ker\varphi)_{\mu-\alpha} \neq 0$, for all $\alpha = \alpha_{i_p} + \cdots + \alpha_{i_{p+1}}$ with $1 \leq p < k$, and the second assertion proves that for such α,we have

$$(x_\alpha^- \otimes t^{\nu(h_\alpha)+1})\tilde{v}_\mu \in \ker\varphi, \quad i.e., \ (x_\alpha^- \otimes t^{\nu(h_\alpha)+1})\bar{v}_\mu = 0,$$

as needed.

Proof of Proposition 4.2.10. We first prove item (i).

Let $\{z_s | \ 1 \leq s \leq r_1 + \cdots + r_n + k = (\nu+\lambda)(h_\theta)\}$ be a set of distinct complex numbers and consider the fusion product

$$\tilde{V}_1 = V(2\omega_1)^{*r_1} * \cdots * V(2\omega_n)^{*r_n} * V(\omega_{i_1}) * \cdots * V(\omega_{i_k}),$$

defined by these parameters. In other words, $\tilde{V}_1$ is the graded module associated as in Section 2.6 to the tensor product

$$\left(\bigotimes_{j=1}^{n} \bigotimes_{r=s_j+1}^{s_{j+1}} V(2\omega_j)^{z_r}\right) \bigotimes \left(\bigotimes_{s=\nu(\theta^\vee)+1}^{(\nu+\lambda)(h_\theta)} V(\omega_{i_{s-\nu(h_\theta)}})^{z_s}\right),$$

where $s_1 = 0$ and $s_j = r_1 + \cdots + r_{j-1}$ for $2 \leq j \leq n$. For $1 \leq i \leq n$, define $f_i \in \mathbb{C}[t]$ by,

$$f_i = \begin{cases} (t - z_{s_i+1}) \cdots (t - z_{s_{i+1}}) & i \neq i_j, \quad 1 \leq j \leq k, \\ (t - z_{s_i+1}) \cdots (t - z_{s_{i+1}})(t - z_{\nu(h_\theta)+j}), & i = i_j, \quad 1 \leq j \leq k. \end{cases}$$

It is trivial to check that

$$(x_{\alpha_i}^- \otimes f_i)\left(v_{2\omega_1}^{\otimes r_1} \otimes \cdots \otimes v_{2\omega_n}^{\otimes r_n} \otimes v_{\omega_{i_1}} \otimes \cdots \otimes v_{\omega_{i_k}}\right) = 0,$$

which implies that

$$(x_{\alpha_i}^- \otimes t^{(\nu+\lambda)(h_i)})\left(v_{2\omega_1}^{*r_1} * \cdots * v_{2\omega_n}^{*r_n} * v_{\omega_{i_1}} * \cdots * v_{\omega_{i_k}}\right) = 0,$$

as needed. Summarizing, we have proved: there exists a non–zero surjective map of $\mathfrak{g}[t]$–modules $\tilde{\varphi} : \tilde{V} \to \tilde{V}_1$.

By Propositon 4.2.3 we have that $D(2,\lambda)$ is a $\mathfrak{g}[t]$–quotient of $D(1,\lambda)$ (or equivalently by Theorem 4.2.8 of $V(\omega_{i_1}) \cdots * V(\omega_{i_k})$) by the submodule U generated by the elements of the subset $\{(x_{\alpha_{i_p,i_{p+1}}}^- \otimes t)w_\lambda\}$ of $D(1,\lambda)$. In particular we have a splitting of $\mathfrak{g}$–modules

$$D(1,\lambda) \cong_{\mathfrak{g}} D(2,\lambda) \oplus U.$$

Using Theorem 4.2.8, it follows that the assumption $V \cong D(2,\mu)$ as $\mathfrak{g}$–modules implies that V is a $\mathfrak{g}$–module quotient of $\tilde{V}_1$, hence we have a splitting of $\mathfrak{g}$–modules

$$\tilde{V}_1 \cong_{\mathfrak{g}} V \bigoplus (V(2\omega_1)^{*r_1} * \cdots * V(2\omega_n)^{*r_n} * U).$$

Since equation (4.2.8) implies that $U_{\lambda-\alpha_{i_p,i_{p+1}}} \neq 0$ for all $1 \leq p < k$, we get

$$\dim \tilde{V}_{\mu-\alpha} \geq \dim(\tilde{V}_1)_{\mu-\alpha} > \dim V_{\mu-\alpha},$$

for all $\alpha = \alpha_{i_p} + \cdots + \alpha_{i_{p+1}}$, with $1 \leq p < k$ as required.

For Proposition 4.2.10(ii) we fix $1 \leq p < k$ and set $i_p = i$, $i_{p+1} = j$ and $\alpha = \alpha_{i_p,i_{p+1}}$. The result follows if we prove the implication

$$v \in \tilde{V}_{\mu-\alpha}, v \neq 0 \implies (x_\alpha^- \otimes t^{\nu(h_\alpha)+1})\tilde{v}_\mu \in U(\mathfrak{h}[t])v.$$

We shall need a spanning set for $\tilde{V}_{\mu-\alpha}$. For this, we take S to be the subset of $U(\mathfrak{n}^-[t])$ consisting of linearly independent monomials of the form

$$\{(x_\alpha^- \otimes t^r) |\ 0 \leq s \leq \nu(h_\alpha) + 1\},$$

(we call these monomials of length one) and the following monomials of length $s > 1$,

$$(x_{\beta_s}^- \otimes t^{p_s}) \cdots (x_{\beta_1}^- \otimes t^{p_1}),$$

where $\beta_j \in R^+$, $p_j \in \mathbb{Z}_{\geq 0}$ satisfy:

(i) $\beta_1 = \alpha_i + \alpha_{i+1} + \cdots + \alpha_p$ for some $p \geq i$, $\beta_1 + \cdots + \beta_r \in R^+$ for all $1 \leq r \leq s$ and $\beta_1 + \cdots + \beta_s = \alpha$,

(ii) $0 \leq p_r \leq \nu(h_{\beta_r})$ for all $1 \leq r \leq s$ with both inequalities being strict if $1 < r < s$.

Lemma 4.2.11. *The elements $\{\mathbf{x}\tilde{v}_\mu | \mathbf{x} \in S\}$ span $\tilde{V}_{\mu-\alpha}$.*

Proof. A straightforward application of the Poincare–Birkhoff–Witt theorem shows that the monomials of the form given above with no restriction on the powers of t span $\tilde{V}_{\mu-\alpha}$. The restriction on r in the case of monomials of length one and on p_1 for monomials of length $s > 1$ comes from equations (4.2.9) and (4.2.10). Suppose that $\beta_1 + \beta_2 = \alpha$. Then, by (4.2.10), we have $(x_{\beta_2}^- \otimes t^{\nu(h_{\beta_2})+1})\tilde{v}_\mu = 0$ and

$$(x_{\beta_2}^- \otimes t^{\nu(h_{\beta_2})+1})(x_{\beta_1}^- \otimes t^{\nu(h_{\beta_1})+1})\tilde{v}_\mu = (x_\alpha^- \otimes t^{\nu(h_\alpha)+2})\tilde{v}_\mu = 0,$$

and the bound on p_2 is established in this case. On the other hand, if $\beta_1 + \beta_2 \neq \alpha$ and $p_2 \geq \nu(h_{\beta_2})$ then (4.2.10) gives $(x_{\beta_2}^- \otimes t^{\nu(h_{\beta_2})})\tilde{v}_\mu = 0$ and we get

$$\begin{aligned}&(x_{\beta_s}^- \otimes t^{p_s}) \cdots (x_{\beta_3}^- \otimes t^{p_3})(x_{\beta_2}^- \otimes t^{p_2})(x_{\beta_1}^- \otimes t^{\nu(h_{\beta_1})+1})\tilde{v}_\mu \\ &= (x_{\beta_s}^- \otimes t^{p_s}) \cdots (x_{\beta_3}^- \otimes t^{p_3})(x_{\beta_1+\beta_2}^- \otimes t^{\nu(h_{\beta_1})+p_2+1})\tilde{v}_\mu.\end{aligned}$$

Therefore, by induction on s, it follows $(x_{\beta_s}^- \otimes t^{p_s}) \cdots (x_{\beta_3}^- \otimes t^{p_3})(x_{\beta_1+\beta_2}^- \otimes t^{\nu(h_{\beta_1})+p_2+1})\bar{v}_\mu = 0$, and thus the bound on p_2 is established. An iteration of the argument establishes the bounds in general. ∎

Let $\mathbf{m} = \sum_{\mathbf{x}\in S} z_{\mathbf{x}}\mathbf{x}$ be a non–trivial linear combination of elements of S and let

$$S(\mathbf{m}) = \{\mathbf{x} \in S | z_{\mathbf{x}} \neq 0\}.$$

Assume that the maximal length of a monomial in $S(\mathbf{m})$ is s. We proceed by induction on s to prove that

$$(x_\alpha^- \otimes t^{\nu(h_\alpha)+1})\tilde{v}_\mu \in U(\mathfrak{h}[t])\mathbf{m}\tilde{v}_\mu. \tag{4.2.11}$$

If $s = 1$, then

$$\mathbf{m} = \sum_{p=0}^{\nu(h_\alpha)+1} z_p(x_\alpha^- \otimes t^p), \quad z_p \in \mathbb{C}.$$

Let r be minimal such that $z_r \neq 0$. If $r = \nu(h_\alpha)+1$ there is nothing to prove. Otherwise, by using (4.2.9), we get

$$(h_\alpha \otimes t^{\nu(h_\alpha)+1-r})\mathbf{m}\tilde{v}_\mu = A(x_\alpha^- \otimes t^{\nu(h_\alpha)+1})\tilde{v}_\mu,$$

for some non–zero $A \in \mathbb{C}$ which shows that induction begins. For the inductive step, let S_1 be the subset of S consisting of monomials

$$\{x_\alpha^- \otimes t^{\nu(h_\alpha)+1}\} \ \cup \ \{(x_{\beta_s}^- \otimes t^{p_s}) \cdots (x_{\beta_1}^- \otimes t^{\nu(h_{\beta_1})}) |\ s \geq 2\}.$$

If $S(\mathbf{m})$ is not a subset of S_1, we claim that there exists $r \in \mathbb{Z}_{\geq 1}$ such that

$$[(\omega_i^\vee \otimes t^r), \mathbf{m}]\tilde{v}_\mu = \sum_{\mathbf{x}\in S_1} c_x\mathbf{x}\tilde{v}_\mu,$$

for some $c_{\mathbf{x}} \in \mathbb{C}$ not all zero. For the claim, note that there must exist

$$(x_{\beta_s}^- \otimes t^{p_s}) \cdots (x_{\beta_2}^- \otimes t^{p_2})(x_{\beta_1}^- \otimes t^{p_1}) \in S(\mathbf{m}),$$

with $\nu(h_{\beta_1}) - p_1 > 0$ if $s > 1$ or $\nu(h_{\beta_1}) + 1 - p_1 > 0$ if $s = 1$ and take r to be the maximum of these numbers. Then we have

$$[\omega_i^\vee \otimes t^r, (x_{\beta_s}^- \otimes t^{p_s}) \cdots (x_{\beta_2}^- \otimes t^{p_2})(x_{\beta_1}^- \otimes t^{p_1})]\tilde{v}_\mu = (x_{\beta_s}^- \otimes t^{p_s}) \cdots (x_{\beta_2}^- \otimes t^{p_2})(x_{\beta_1}^- \otimes t^{p_1+r})\tilde{v}_\mu,$$

where $\omega_i^\vee$ is such that $\alpha_j(\omega_i^\vee) = \delta_{i,j}$ for all $1 \leq i, j \leq n$. The right hand side is zero unless $(x_{\beta_s}^- \otimes t^{p_s}) \cdots (x_{\beta_2}^- \otimes t^{p_2})(x_{\beta_1}^- \otimes t^{p_1+r}) \in S_1$. This proves that the element $[\omega_1^\vee \otimes t^r, \mathbf{m}]$ acts on $\tilde{v}_\mu$ by a non–trivial linear combination of elements from S_1; i.e. there exists $\mathbf{m}_1 = \sum_{\mathbf{x}\in S_1} c_{\mathbf{x}}\mathbf{x}$ with not all $c_{\mathbf{x}} = 0$ so that $\mathbf{m}_1\tilde{v}_\mu \in U(\mathfrak{h}[t])\mathbf{m}\tilde{v}_\mu$. Define $S(\mathbf{m}_1)$ in the obvious way. If the maximal length of a monomial in $S(\mathbf{m}_1)$ is less than the maximal length in $S(\mathbf{m})$ the inductive hypothesis applies to $\mathbf{m}_1$ and hence we get the inductive step for $\mathbf{m}$.

Otherwise, choose $j_1 > i$ minimal so that there exists an element in $S(\mathbf{m}_1)$ with $\beta_1 = \alpha_i + \cdots + \alpha_{j_1-1}$. Since the maximal length of a monomial in $S(\mathbf{m}_1)$ is $s > 1$, we also have $j_1 - 1 < j$.

If $(x_\alpha^- \otimes t^{\nu(h_\alpha)+1}) \in S(\mathbf{m}_1)$, or if $\mathbf{x} \in S(\mathbf{m}_1)$ with $\beta_1 = \alpha_i + \cdots + \alpha_{j_2}$ with $j_1 - 1 < j_2 < j$, then for all $r > 0$ we have,

$$(\omega_{j_1}^\vee \otimes t^r)(x_\alpha^- \otimes t^{\nu(h_\alpha)+1})\tilde{v}_\mu = (x_\alpha^- \otimes t^{\nu(h_\alpha)+1+r})\tilde{v}_\mu = 0,$$
$$(\omega_{j_1}^\vee \otimes t^r)(x_{\beta_s}^- \otimes t^{p_s}) \cdots (x_{\beta_1}^- \otimes t^{\nu(h_{\beta_1})})\tilde{v}_\mu = (x_{\beta_s}^- \otimes t^{p_s}) \cdots (x_{\beta_1}^- \otimes t^{\nu(h_{\beta_1})+r})\tilde{v}_\mu = 0.$$

This means that we can write

$$\begin{aligned}(\omega_{j_1}^\vee \otimes t^r)\mathbf{m}_1\tilde{v}_\mu &= (\omega_{j_1}^\vee \otimes t^r)\mathbf{m}_2(x_{\alpha_{i,j_1-1}}^- \otimes t^{\nu(h_{\alpha_{i,j_1-1}})})\tilde{v}_\mu \\ &= [\omega_{j_1}^\vee \otimes t^r, \mathbf{m}_2](x_{\alpha_{i,j_1-1}}^- \otimes t^{\nu(\alpha_{i,j_1-1}^\vee)})\tilde{v}_\mu.\end{aligned}$$

Here $\mathbf{m}_2$ is a non–trivial linear combination of terms of the form $(x_{\beta_s}^- \otimes t^{p_s}) \cdots (x_{\beta_2}^- \otimes t^{p_2})$, where the p_m are such that $p_m \le \nu(h_{\beta_m})$ with equality only if m is equal to the length of the monomial. Choose r_2 so that $r_2 + p_2 \ge \nu(\beta_2)$ for all the terms occurring in $\mathbf{m}_2$, if $s \ge 3$, and $r_2 + p_2 \ge \nu(\beta_2) + 1$, if $s = 2$, with equality holding for at least one term, and note that $r_2 > 0$. We have

$$\begin{aligned}&[(\omega_{j_1}^\vee \otimes t^{r_2}), (x_{\beta_s}^- \otimes t^{p_s}) \cdots (x_{\beta_2}^- \otimes t^{p_2})](x_{\alpha_{i,j_1-1}}^- \otimes t^{\nu(h_{\alpha_{i,j_1-1}})})\tilde{v}_\mu \\ &= (x_{\beta_s}^- \otimes t^{p_s}) \cdots (x_{\beta_2}^- \otimes t^{p_2+r_2})(x_{\alpha_{i,j_1-1}}^- \otimes t^{\nu(h_{\alpha_{i,j_1-1}})})\tilde{v}_\mu \\ &= (x_{\beta_s}^- \otimes t^{p_s}) \cdots (x_{\beta_2+\alpha_{i,j_1-1}}^- \otimes t^{p_2+r_2+\nu(h_{\alpha_{i,j_1-1}})})\tilde{v}_\mu,\end{aligned}$$

where the last equality is because the choice of r_2 gives $(x_{\beta_2}^- \otimes t^{p_2+r_2})\tilde{v}_\mu = 0$. Moreover if $p_2 + r_2 > \nu(h_{\beta_2})$ the right hand side is zero and when equality holds the right hand side is a monomial in S acting on $\tilde{v}_\mu$. To summarize, this argument proves that $(\omega_{j_1}^\vee \otimes t^{r_2})\mathbf{m}_1\tilde{v}_\mu$ can be written as a non–trivial linear combination of terms of the form $\mathbf{x}\tilde{v}_\mu$ with $\mathbf{x} \in S$, where the maximal length of the monomials is one less than that of $\mathbf{m}$ which completes the inductive step. $\square$

4.3 Proof of Theorem 4.1.1

We now turn to the proof of Theorem 4.1.1. The next result is simple but very useful and is a consequence of [Mou10, Lemma 2.20].

Lemma 4.3.1. *Let $\boldsymbol{\pi}, \boldsymbol{\varpi} \in \mathcal{P}_{\mathbb{Z}}^+$. Assume that there exists a map of $U_q(\widetilde{\mathfrak{g}})$–modules*

$$L_q(\boldsymbol{\pi}\boldsymbol{\varpi}) \to L_q(\boldsymbol{\pi}) \otimes L_q(\boldsymbol{\varpi}).$$

Then there exists a map of $\mathfrak{g}[t]$–modules $L(\boldsymbol{\pi}\boldsymbol{\varpi}) \to L(\boldsymbol{\pi}) \otimes L(\boldsymbol{\varpi})$ mapping $v_{\boldsymbol{\pi}\boldsymbol{\varpi}} \to v_{\boldsymbol{\pi}} \otimes v_{\boldsymbol{\varpi}}$. $\square$

We first prove that it suffices to prove the theorem in the case when $\boldsymbol{\pi}_1 = 1$. To see this, assume that $\boldsymbol{\pi}_1$ is as in Theorem 4.1.1, in which case, since $L(\boldsymbol{\omega}_{j,a,2}) \cong_{\mathfrak{g}} V_q(2\omega_j)$, $j \in I$, $a \in \mathbb{C}^\times$, by Lemma 3.5.2, we have

$$L(\boldsymbol{\pi}_1\boldsymbol{\pi}_2) \cong_{\mathfrak{g}} V(2\omega_{j_1}) \otimes \cdots \otimes V(2\omega_{j_r}) \otimes D(2, \lambda).$$

Since $L(\boldsymbol{\pi}_1\boldsymbol{\pi}_2)$ is a quotient of $W(\mu)$ by Proposition 3.4.1, Theorem 4.1.1 follows from Proposition 4.2.7 and Theorem 4.2.8.

Proposition 4.3.2. *Let $\mathfrak{g}$ be of type A_n. Then if $\boldsymbol{\pi} \in \mathcal{P}_{\mathbb{Z}}^+$ is such that*

$$L_q(\boldsymbol{\pi}) \cong_{U_q(\widehat{\mathfrak{g}})} L_q(Y_{j_1,a_1}) \otimes \cdots \otimes L_q(Y_{j_r,a_r}),$$

for some $r \in \mathbb{Z}_{\geq 0}$ and $1 \leq j_1, \cdots, j_r \leq n$, then

$$L(\boldsymbol{\pi}) \cong_{\mathfrak{g}[t]} W(\mathrm{wt}(\boldsymbol{\pi})) \cong_{\mathfrak{g}[t]} D(1, \mathrm{wt}(\boldsymbol{\pi})).$$

Proof. It was shown in [CL06] that $\dim W(\mathrm{wt}(\boldsymbol{\pi})) = \dim \prod_{s=1}^r \dim V(\omega_{j_s})$. Since $\dim_{\mathbb{C}} L(\boldsymbol{\pi}) = \dim_{\mathbb{F}} L_q(\boldsymbol{\pi})$ the proposition is immediate from Corollary 3.4.1. $\square$

From now on, let $\boldsymbol{\pi} = \boldsymbol{\pi}_2$ as in Theorem 4.1.1. Let $\boldsymbol{\pi}^o, \boldsymbol{\pi}^e \in \mathcal{P}_{\mathbb{Z}}^+$ be such that $\mathrm{wt}(\boldsymbol{\pi}^o) = \lambda^o$ and $\boldsymbol{\pi}^e = \lambda^e$, in the notations of Section 4.2.1.

Proposition 4.3.3. *Let $\boldsymbol{\pi}$, $\boldsymbol{\pi}^o$ and $\boldsymbol{\pi}^e$ be as in the preceding discussion.*

(i) The module $L_q(\boldsymbol{\pi})$ is isomorphic to a submodule of either $L_q(\boldsymbol{\pi}^o) \otimes L_q(\boldsymbol{\pi}^e)$ or $L_q(\boldsymbol{\pi}^e) \otimes L_q(\boldsymbol{\pi}^o)$.

(ii) We have

$$L(\boldsymbol{\pi}^o) \cong_{\mathfrak{g}[t]} W(\lambda^o), \quad L(\boldsymbol{\pi}^e) \cong_{\mathfrak{g}[t]} W(\lambda^e).$$

We prove Proposition 4.3.3 in the next section. Assuming the result, the proof of Theorem 4.1.1 is completed as follows. By Lemma 4.3.1 there exists a map of $\mathfrak{g}[t]$–modules

$$L(\boldsymbol{\pi}) \to W(\lambda^o) \otimes W(\lambda^e), \quad v_{\boldsymbol{\pi}} \to v_{\lambda^o} \otimes v_{\lambda^e}.$$

We claim that there exists a surjective map of $\mathfrak{g}[t]$–modules $D(2, \lambda) \to L(\boldsymbol{\pi})$ mapping $v_\lambda \to v_{\boldsymbol{\pi}}$ and hence we have a non–zero map $D(2, \lambda) \to W(\lambda^o) \otimes W(\lambda^e)$, mapping $v_\lambda \to v_{\lambda^o} \otimes v_{\lambda^e}$. This map is injective by Theorem 4.2.1 and hence so is the surjective map $D(2, \lambda) \to L(\boldsymbol{\pi})$ which proves the theorem.

To prove the claim it suffices to show that $v_{\boldsymbol{\pi}}$ satisfies the relations given in Proposition 4.2.3. Let $1 \leq p < k$ and let $J_p = \{i_p, i_p + 1, \ldots i_{p+1}\} \subseteq I$. Then Lemma 3.3.2 implies that $L_q(\beta_{J_p}(\boldsymbol{\pi}))$ is a minimal affinization of $V_q(\omega_{i_p} + \omega_{i_p+1})$ (as $U_q(\mathfrak{g}_J)$–module), and, hence, the aforementioned relations follows by Lemma 3.5.2.

4.3.1 Cyclicity criteria of the tensor product

Let $\lambda = \sum_{j=1}^k \omega_{i_j} \in P(1)^+$, with $i_1 < i_2 < \cdots < i_k$. Define integers r_j, $1 \leq j \leq k$ by

$$r_1 = 0, \quad r_2 = i_2 - i_1 + 2,$$
$$r_{2s+1} = -i_1 + 2(i_2 - i_3 + \cdots - i_{2s-1} + i_{2s}) - i_{2s+1}, \quad s \geq 1,$$
$$r_{2s+2} = -i_1 + 2(i_2 - i_3 + \cdots + i_{2s} - i_{2s+1}) + i_{2s+2} + 2, \quad s \geq 1.$$

For $\epsilon \in \{\pm 1\}$, let $\boldsymbol{\lambda}_\epsilon = \prod_{j=1}^m Y_{i_j,q^{\epsilon r_j}} \in \mathcal{P}_\mathbb{Z}^+$. Observe that, if $\boldsymbol{\pi} \in \mathcal{P}_\mathbb{Z}^+$ is such that $L_q(\boldsymbol{\pi})$ is an alternating minimal affinization by parts of $V_q(\lambda)$, then

$$\boldsymbol{\pi} = \tau_b \boldsymbol{\lambda}_\epsilon, \quad \text{for some } \epsilon \in \{\pm 1\},\ b \in q^\mathbb{Z},$$

where τ_b is the *shift of spectral parameters*, defined by the homomorphism of rings $\tau_b : \mathbb{Z}\mathcal{P} \to \mathbb{Z}\mathcal{P}$ sending $Y_{i,a}$ to $Y_{i,ab}$, $i \in I$, $a \in \mathbb{F}^\times$. In particular, we have

$$L(\boldsymbol{\pi}) \cong_{\mathfrak{g}[t]} L(\boldsymbol{\lambda}_\epsilon),$$

and, hence, there is no loss of generality in assuming that $\boldsymbol{\pi} = \boldsymbol{\lambda}_\epsilon$ for some $\epsilon \in \{\pm 1\}$. For the remainder of the section we suppose $\epsilon = 1$. Similar arguments prove the case $\epsilon = -1$.

Observe that the dual $\boldsymbol{\pi}^*$ of $\boldsymbol{\pi}$ is also of the same kind, i.e., $L(\boldsymbol{\pi}^*)$ is a minimal affinization by parts. In fact, by (3.2.3) it follows that

$$\boldsymbol{\pi}^* = Y_{j_1,b_1} Y_{j_2,b_2} \dots Y_{j_k,b_k},$$

where $j_p = n+1-i_{k-p+1}$ and $b_{j_p} = q^{r_{i_{k+1-p}}} q^{-n-1}$, for all $1 \le p \le k$. It is straightforward to check that

$$b_{j_{2p}}/b_{j_{2p-1}} = \begin{cases} q^{j_{2p}-j_{2p-1}+2} & k \text{ odd}, \\ q^{j_{2p-1}-j_{2p}-2} & k \text{ even}, \end{cases} \quad b_{j_{2p+1}}/b_{j_{2p}} = \begin{cases} q^{j_{2p}-j_{2p+1}-2} & k \text{ odd}, \\ q^{j_{2p+1}-j_{2p}+2} & k \text{ even}. \end{cases}$$

In particular, we have $(\boldsymbol{\pi}^*)^o = (\boldsymbol{\pi}^o)^*$ if k is odd, and $(\boldsymbol{\pi}^*)^o = (\boldsymbol{\pi}^e)^*$ if k is even. Similar statements hold for $(\boldsymbol{\pi}^*)^e$.

Hence, it suffices to prove the dual statement in Proposition 4.3.3; namely to show that $L_q(\boldsymbol{\pi})$ is a quotient of $L_q(\boldsymbol{\pi}^e) \otimes L(\boldsymbol{\pi}^o)$ (see the proof of [Mou10, Corollary 4.4]).

Theorem 4.3.4. *Assume that k is even. We have isomorphisms of $U_q(\widetilde{\mathfrak{sl}}_{n+1})$–modules*

$$L_q(\boldsymbol{\pi}^o) \cong L_q(Y_{i_1,q^{r_1}}) \otimes \cdots \otimes L_q(Y_{i_{k-1},q^{r_{k-1}}}), \qquad \text{and} \quad L_q(\boldsymbol{\pi}^e) \cong L_q(Y_{i_2,q^{r_2}}) \otimes \cdots \otimes L_q(Y_{i_k,q^{r_k}}).$$

Moreover $L_q(\boldsymbol{\pi})$ is a quotient of $L_q(\boldsymbol{\pi}^e) \otimes L_q(\boldsymbol{\pi}^o)$. Analogous statements hold if k is odd.

Remark 4.3.5. For $\epsilon = -1$ one proves that $L_q(\boldsymbol{\pi})$ is a quotient of $L_q(\boldsymbol{\pi}^o) \otimes L_q(\boldsymbol{\pi}^e)$.

Assuming Theorem 4.3.4, Proposition 4.3.3 is immediate from Proposition 4.3.2 and Lemma 4.3.1.

The proof of the theorem depends on Theorem 3 and Corollary 5.1 of [Cha02] and we recall the result in the case of interest to us.

Proposition 4.3.6. *Let $m \ge 1$ and suppose that $1 \le j_1, \cdots, j_m \le n$ and $b_1, \cdots, b_m \in \mathbb{C}^\times$ are such*

$$s < r \implies b_r/b_s \notin \{q^{2p+2-j_s-j_r} \,|\, \max\{j_r, j_s\} < p+1 \le \min\{j_r+j_s, n+1\}\}. \tag{4.3.1}$$

Then $L_q(Y_{j_1,b_1})\otimes\cdots\otimes L_q(Y_{j_m,b_m})$ has a unique irreducible quotient $L_1(\boldsymbol{\pi})$ where $\boldsymbol{\pi} = Y_{j_1,b_1}\cdots Y_{j_m,b_m}$. Moreover, if (4.3.1) holds for all $1 \le r, s \le m$, then we have an isomorphism

$$L_q(Y_{j_1,b_1}) \otimes \cdots \otimes L_q(Y_{j_m,b_m}) \cong L_q(\boldsymbol{\pi}).$$

□

The proof of Theorem 4.3.4 is now a straightforward checking to see that the conditions of Proposition 4.3.6 are satisfied. We first prove that $L_q(\boldsymbol{\pi}^o)$ is irreducible. For this, assume that $s > j$ and note that

$$r_{2s+1} - r_{2j+1} = -i_{2j+1} + 2(i_{2j+2} - i_{2j+3} + \cdots - i_{2s-1} + i_{2s}) - i_{2s+1}.$$

If in addition we have

$$r_{2s+1} - r_{2j+1} = 2p + 2 - i_{2s+1} - i_{2j+1}, \qquad p + 1 > i_{2s+1},$$

then we get

$$p + 1 = i_{2j+2} - i_{2j+3} + \cdots - i_{2s-1} + i_{2s} = i_{2s} - (i_{2s-1} - i_{2s-2}) - \cdots - (i_{2j+3} - i_{2j+2}) < i_{2s},$$

which is an absurd since $i_{2s} < i_{2s+1}$. On the other hand, if

$$r_{2s+1} - r_{2j+1} = -2p - 2 + i_{2s+1} + i_{2j+1}, \qquad p + 1 > i_{2s+1},$$

then we get

$$i_{2s+1} + i_{2j+1} = p + 1 + (i_{2j+2} - i_{2j+3} + \cdots - i_{2s-1} + i_{2s}).$$

Since $p + 1 > i_{2s+1}$, it follows that

$$i_{2s} - i_{2s-1} + \cdots + i_{2j+2} - i_{2j+1} < 0,$$

which is an absurd. Hence $L_q(\boldsymbol{\pi}^o)$ is irreducible, as required. A similar argument proves the result for $L_q(\boldsymbol{\pi}^e)$.

To prove that $L_q(\boldsymbol{\pi}^e) \otimes L_q(\boldsymbol{\pi}^o)$ has $L_q(\boldsymbol{\pi})$ as its quotient it is enough now to check that (4.3.1) is satisfied for pairs of the form $q^{r_{2s}}/q^{r_{2j-1}}$ for all s and j. This amounts to proving

$$r_{2j-1} - r_{2s} \notin \{2 + 2p - i_{2s} - i_{2j-1} | \max\{i_{2j-1}, i_{2s}\} < p + 1 \le \min\{i_{2j-1} + i_{2s}, n + 1\}\}. \quad (4.3.2)$$

For clarity, we prove this by breaking up the checking into several cases. If $s \ge j \ge 1$ and $i_{2s} + i_{2j-1} \le n + 1$, we have

$$r_{2s} - r_{2j-1} = i_{2s} + i_{2j-1} + 2 - 2(i_{2j-1} - i_{2j} + \cdots - i_{2s-2} + i_{2s-1}),$$

i.e.,

$$r_{2j-1} - r_{2s} = -i_{2s} - i_{2j-1} + 2(-1 + (i_{2j-1} - i_{2j}) + \cdots + (i_{2s-1} - i_{2s}) + i_{2s}).$$

Since $(-1 + (i_{2j-1} - i_{2j}) + \cdots + (i_{2s-1} - i_{2s}) + i_{2s}) < i_{2s}$, we see that (4.3.2) is satisfied. On the other hand, if $j \geq s \geq 1$ and $i_{2s} + i_{2j-1} \leq n+1$, we have

$$r_{2s} - r_{2j-1} = i_{2s} + i_{2j-1} - 2(-1 + (i_{2s} - i_{2s+1}) + \cdots + (i_{2j-2} - i_{2j-1}) + i_{2j-1}),$$

and this time we need the expression in parentheses to be bigger than i_{2j-1} and this is clearly not the case. The other two cases are similar and we omit the details.

4.4 Proof of Theorem 4.1.2

In this section we prove Theorem 4.1.2. For convenience we split the proof in cases, depending on the type of $\mathfrak{g}$. We first fix some notations and recall the main results of [Nao13] and [Nao14].

For $\eta = \sum_i m_i \alpha_i \in Q$, let $\epsilon_i(\eta) = m_i$. Recall that if $\mathfrak{g}$ is of classical type, then $\epsilon_i(\alpha) \leq 2$ for every $\alpha \in R^+$ and $i \in I$. Set

$$R_1^+ = \{\alpha \in R^+ | \epsilon_i(\alpha) \leq 1 \text{ for all } i \in i\} \quad \text{and} \quad R_2^+ = R^+ \setminus R_1^+.$$

4.4.1 Types B and C

Assume that $\mathfrak{g}$ is either of type B_n or C_n. Then $R_1^+ = \{\alpha_{i,j} : 1 \leq i \leq j \leq n\}$ where $\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$. For $\mathfrak{g}$ of type B_n, we have

$$R_2^+ = \{\beta_{i,j} | 1 \leq i < j \leq n\}, \quad \text{where} \quad \beta_{i,j} = \alpha_{i,n} + \alpha_{j,n}.$$

For $\mathfrak{g}$ of type C_n, we have

$$R_2^+ = \{\beta_{i,j} | 1 \leq i \leq j < n\}, \quad \text{where} \quad \beta_{i,j} = \alpha_{i,n} + \alpha_{j,n-1}.$$

Let $\boldsymbol{\pi}$ and λ be as in Theorem 4.1.2. The following is

Theorem 4.4.1 ([Nao13, Theorem 4.6]). *The module $L(\boldsymbol{\pi})$ is isomorphic to the quotient of $W(\lambda)$ by the submodule generated by $x^-_{\alpha,1} w_\lambda, \alpha \in R_1^+$.* □

In order to explicitly describe the partition $\boldsymbol{\xi}$ mentioned in the statement of Theorem 4.1.2, we will need the function $\rho : R^+ \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ defined as follows (cf. [Nao13, Section 5.1]). Set

$$\rho(\alpha, 0) = \lambda(h_\alpha), \qquad \text{for all } \alpha \in R^+.$$

We shall use the following notation in the remaining definitions. Given a connected subdiagram $J \subseteq I$, set

$$|\lambda|_J = \sum_{j \in J} \lambda(h_j).$$

For $i, j \in I$, let $[i, j]$ be the minimal connected subdiagram containing i, j. Then, for $\mathfrak{g}$ of type B_n and $r > 0$, set

$$\rho(\alpha, r) = \begin{cases} |\lambda|_{[j,n-1]} + \lfloor \lambda(h_n)/2 \rfloor, & \text{if } r = 1 \text{ and } \alpha = \beta_{i,j} \text{ for some } i < j < n, \\ \lfloor \lambda(h_n)/2 \rfloor, & \text{if } r = 1 \text{ and } \alpha = \beta_{i,n} \text{ for some } i, \\ 0, & \text{otherwise.} \end{cases}$$

If $\mathfrak{g}$ is of type C_n, set

$$\rho(\beta_{i,j}, 1) = \begin{cases} \left\lfloor \frac{|\lambda|_{[j,n-1]}}{2} \right\rfloor, & \text{if } i = j, \\ |\lambda|_{[j,n-1]} - 1, & \text{if } i < j,\ |\lambda|_{[1,n-1]} \text{ is odd and } |\lambda|_{[i,j-1]} = 0, \\ |\lambda|_{[j,n-1]}, & \text{otherwise,} \end{cases}$$

and $\rho(\alpha, r) = 0$, if (α, r) does not satisfy any of the above listed conditions. Finally, let $\boldsymbol{\xi}$ be the R^+-tuple of partitions defined by

$$\xi_j^\alpha = \rho(\alpha, j-1) - \rho(\alpha, j) \quad \text{for all} \quad j \geq 1. \tag{4.4.1}$$

Equivalently,

$$\xi_1^\alpha = \lambda(h_\alpha) - \rho(\alpha, 1), \quad \xi_2^\alpha = \rho(\alpha, 1), \quad \xi_j^\alpha = 0 \quad \text{for all} \quad j > 2.$$

One easily checks that $\xi_1^\alpha \geq \xi_2^\alpha$, for all $\alpha \in R^+$ and that

$$\alpha \in R_1^+ \ \Rightarrow\ \xi_2^\alpha = 0.$$

In particular, by Theorems 2.5.1 and 4.4.1, we have an epimorphism of $\mathfrak{g}[t]$-module $L(\boldsymbol{\pi}) \to V(\boldsymbol{\xi})$. Furthermore, these theorems also imply that Theorem 4.1.2 follows if one shows that

$$(x_{\alpha,1}^-)^{\rho(\alpha,1)+1} v_{\boldsymbol{\pi}} = 0, \quad \text{for all} \quad \alpha \in R_2^+.$$

But this was proved in [Nao13, Proposition 5.1] which is part of the proof of [Nao13, Theorem 4.6] mentioned in Section 4.1.2.

Before moving to type D_n, we examine some basic examples for type B_n. A straightforward but tedious computation using the previous subsection shows that, if $\mathfrak{g}$ is of type B_2, then

$$L(\boldsymbol{\pi}) \cong V(\boldsymbol{\xi}(\ell, \lambda)), \quad \text{where} \quad \ell = \lambda(h_1) + \lfloor \lambda(h_2)/2 \rfloor.$$

Therefore, by Theorem 2.5.4, $L(\boldsymbol{\pi})$ is a Demazure module. Let us now show that this property disappears in higher rank. Let $\mathfrak{g}$ be of type B_3, $\lambda = \omega_1 + 2\omega_3$, and $\boldsymbol{\xi}$ as in (4.4.1). One then checks that we have the following table:

Root	$\boldsymbol{\xi}$	$\boldsymbol{\xi}(1,\lambda)$	$\boldsymbol{\xi}(2,\lambda)$
α_1	(1)	(1)	(1)
α_2	$\emptyset$	$\emptyset$	$\emptyset$
α_3	(2)	(2)	(2)
$\alpha_{[1,2]}$	(1)	(1)	(1)
$\alpha_{[2,3]}$	(2)	(2)	(2)
$\alpha_{[1,3]}$	(4)	(2, 2)	(4)
$\beta_{2,3}$	(1, 1)	(1, 1)	(2)
$\beta_{1,3}$	(2, 1)	(1, 1, 1)	(2, 1)
$\beta_{1,2}$	(2, 1)	(1, 1, 1)	(2, 1)

Using Proposition 2.5.3, we see from the line corresponding to $\alpha_{[1,3]}$ in the above table that $L(\boldsymbol{\pi})$ is a quotient of $D(1,\lambda)$. On the other hand, the line corresponding to $\beta_{2,3}$ implies that $D(2,\lambda)$ is a quotient of $L(\boldsymbol{\pi})$. However, $L(\boldsymbol{\pi})$ is not isomorphic to either of these modules. Indeed, it follows from [Mou10, Proposition 5.7] (see also [Nao13, Example 4.4]) that $L(\boldsymbol{\pi})[r]=0$ for $r>1$ and

$$L(\boldsymbol{\pi})[1] \cong_{\mathfrak{g}[t]} V(\omega_2,1)\oplus V(2\omega_1,1).$$

Hence, the socle of $L(\boldsymbol{\pi})$ is not simple showing it cannot be a Demazure module (see (2.2.1)). It may also be interesting to recall that, by [Nao13, Theorem 4.5], we have

$$L(\boldsymbol{\pi}) \cong D(w_0\Phi_1, w_0\Phi_2), \quad \text{where} \quad \Phi_1=\Lambda_0+\omega_1 \ \text{ and } \ \Phi_2=\Lambda_0+2\omega_3.$$

4.4.2 Type D

Henceforth, assume that $\mathfrak{g}$ is of type D_n and let $E=\{1,n-1,n\}\subseteq I$. Given a connected subdiagram J of I, let Q_J be the subgroup of Q generated by $\alpha_j, j\in J$, and set $R_J^+=R^+\cap Q_J$ and

$$\alpha_J=\sum_{j\in J}\alpha_j \ \in R_J^+.$$

$$R_1^+=\{\alpha_{[i,j]}|i,j\in I\}\sqcup\{\beta_i|1\le i<n-2\}, \quad \text{where} \quad \beta_i=\alpha_{[i,n]}+\alpha_{n-1},$$

Observe that and

$$R_2^+=\{\beta_{i,j}|1\le i<j\le n-2\}, \quad \text{where} \quad \beta_{i,j}=\alpha_{[i,n-1]}+\alpha_{[j,n]}.$$

The hypothesis of regularity of λ means that one of the following holds:

A $\mathrm{supp}_P(\lambda)\subseteq[i,j]$, for some $i,j\in E$,

B $\mathrm{supp}_P(\lambda)$ intersects all three connected components of $I\setminus\{n-2\}$ and $n-2\in\mathrm{supp}(\lambda)$, where $\operatorname{supp}_P(\lambda) := \{i \in I | \lambda(h_i) > 0\}$. We will say that λ is of type A or D according to whether (**A**) or (**B**) holds .

It was proved in [CP96a] that there is one equivalence class of minimal affinizations if λ is of type A while there are three equivalence classes of minimal affinizations if λ is of type D. For our purposes, it will suffice to recall the following characterization of the Drinfeld polynomials $\boldsymbol{\pi}$ such that $L_q(\boldsymbol{\pi})$ is a minimal affinization of $L_q(\lambda)$.

Theorem 4.4.2. *Let $\boldsymbol{\pi} \in \mathcal{P}^+$ be such that $\mathrm{wt}(\boldsymbol{\pi}) = \lambda$.*

a *If λ is of type A, then $L_q(\boldsymbol{\pi})$ is a minimal affinization of $V_q(\lambda)$ if and only if $L_q(\beta_{[i,j]}(\boldsymbol{\pi}))$ is a minimal affinization for $U_q(\widetilde{\mathfrak{g}}_{[i,j]})$, for all $i, j \in E, i \neq j$.*

b *If λ is of type D, then $L_q(\boldsymbol{\pi})$ is a minimal affinization of $V_q(\lambda)$ if and only if there exists $i \in E$ such that $L_q(\beta_{[i,j]}(\boldsymbol{\pi}))$ is a minimal affinization for $U_q(\widetilde{\mathfrak{g}}_{[i,j]})$, for both $j \in E \setminus \{i\}$. Moreover, two minimal affinizations $L_q(\boldsymbol{\pi})$ and $L_q(\boldsymbol{\varpi})$ of $V_q(\lambda)$ are equivalent if and only if $\boldsymbol{\pi}$ and $\boldsymbol{\varpi}$ satisfy this property for the same $i \in E$.* □

Remark 4.4.3. This theorem was proved in [CP96a] and it holds also for algebras of type E_n with the obvious modifications: $n-2$ is replaced by the trivalent node, say i_0, and the set E is replaced by the set of extreme nodes of the diagram (or any set containing one node for each connected component of $I \setminus \{i_0\}$). The classification of minimal affinizations for irregular λ is not complete. It was mostly obtained for $\mathfrak{g}$ of type D_4 in [CP96b]. In the upcoming paper [HMP], the authors completed the classification of minimal affinizations for algebras of type $D_n, n \geq 4$. We shall study the classical limits of the minimal affinizations for irregular λ from the perspective of generalized Demazure modules elsewhere.

Assume λ is of type D and, given $i \in E$, fix a polynomial $\boldsymbol{\pi}^{(i)}$ satisfying the property described in part (b) of Theorem 4.4.2. If λ is of type A, we fix $\boldsymbol{\pi}$ such that $L_q(\boldsymbol{\pi})$ is a minimal affinization of $V_q(\lambda)$ and, for convenience, we set $\boldsymbol{\pi}^{(i)} = \boldsymbol{\pi}$, for all $i \in E$.

Define

$$R^{(i)} = \bigcup_{j \in E \setminus \{i\}} R^+_{[i,j]} \subseteq R^+_1.$$

The following is proved in [Nao14].

Theorem 4.4.4 ([Nao14, Theorem 3.2])**.** *The module $L(\boldsymbol{\pi}^{(i)})$ is isomorphic to the quotient of $W(\lambda)$ by the submodule $N^{(i)}(\lambda)$ generated by $x^-_{\alpha,1} w_\lambda, \alpha \in R^{(i)}$.* □

As in the previous cases, we define a map $\rho^{(i)} : R^+ \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ as follows. For $r = 0$ we define

$$\rho^{(i)}(\alpha, 0) = \lambda(h_\alpha), \quad \text{for all} \quad \alpha \in R^+.$$

Otherwise, if $i=1$, define

$$\rho^{(1)}(\alpha,r)=\begin{cases}|\lambda|_{[k,n-2]}\;+\min\{\lambda(h_{n-1}),\lambda(h_n)\}, & \text{if } r=1 \text{ and } \beta_{j,k} \text{ for some } j,k,\\ \min\{\lambda(h_{n-1}),\lambda(h_n)\}, & \text{if } r=1 \text{ and } \alpha=\beta_j \text{ for some } j \text{ or } \alpha=\alpha_{[n-1,n]},\\ 0, & \text{otherwise.}\end{cases}$$

If $i\in E\setminus\{1\}$, let $i'\in E\setminus\{1,i\}$ and define

$$\rho^{(i)}(\alpha,1)=\begin{cases}|\lambda|_{[k,n-2]}+\min\{|\lambda|_{[j,n-3]},\lambda(h_{i'})\}, & \text{if } \ \alpha=\beta_{j,k} \text{ for some } j<k\le n-2,\\ \min\{|\lambda|_{[k,n-3]},\lambda(h_{i'})\}, & \text{if } \ \alpha=\alpha_{[k,i']} \text{ for some } k<n-2,\\ 0, & \text{otherwise,}\end{cases} \tag{4.4.2}$$

$$\rho^{(i)}(\alpha,2)=\begin{cases}\min\{|\lambda|_{[k,n-3]},\lambda(h_{i'})\}, & \text{if } \ \alpha=\beta_{j,k} \text{ for some } j<k<n-2,\\ 0, & \text{otherwise,}\end{cases} \tag{4.4.3}$$

$$\rho^{(i)}(\alpha,r)=0 \ \text{ if } \ r>2.$$

We now fix $i\in E$ and simplify notation writing $\rho,\boldsymbol{\pi},N(\lambda)$ in place of $\rho^{(i)},\boldsymbol{\pi}^{(i)},N^{(i)}(\lambda)$. We then define $\boldsymbol{\xi}$ by (4.4.1) as before. This time, this is equivalent to

$$\xi_1^\alpha=\lambda(h_\alpha)-\rho(\alpha,1),\quad \xi_2^\alpha=\rho(\alpha,1)-\rho(\alpha,2),\quad \xi_3^\alpha=\rho(\alpha,2),\quad \xi_j^\alpha=0\quad \text{for all}\quad j>3.$$

One easily checks that $\xi_1^\alpha\ge\xi_2^\alpha\ge\xi_3^\alpha$, for all $\alpha\in R^+$, and that

$$\alpha\in R^{(i)}\ \Rightarrow\ \xi_2^\alpha=0.$$

In particular, Remark 2.5.1 and Theorem 4.4.4 imply that we have an epimorphism of $\mathfrak{g}[t]$-modules

$$L(\boldsymbol{\pi})\to V(\boldsymbol{\xi}). \tag{4.4.4}$$

On the other hand, it follows from [Nao14, Proposition 4.3 and Theorem 3.1] that

$$(x_{\alpha,r}^-)^{\rho(\alpha,r)+1}w_\lambda\in N(\lambda),\quad \alpha\in R^+,\ r\ge 0. \tag{4.4.5}$$

If λ is of type A, one easily checks using (4.4.2) and (4.4.3) that

$$\rho(\alpha,1)=0,\quad \text{for all } \alpha\in R_1^+,\quad \text{and}\quad \#\xi^\alpha\le 2,\quad \text{for all}\quad \alpha\in R^+.$$

In particular, together with (4.4.5), it follows that $N(\lambda)$ is generated by $x_{\alpha,1}^-w_\lambda$, $\alpha\in R_1^+$. The proof of Theorem 4.1.2 is completed as in the previous cases by another application of Remark 2.5.1 and Theorem 4.4.4.

For the remainder of this section we assume that λ is of type D. We prove Theorem 4.1.2 for each choice of $i\in E$. Assume first that $i=1$, in which case $\#\xi^\alpha\le 2$, for all $\alpha\in R^+$. In light of (4.4.5), the proof of Theorem 4.1.2 is completed by the same arguments used before.

In what follows we will prove Theorem 4.1.2 in the case that i is a spin node. In that case, one can easily check that, if $n > 4$, there always exists $\alpha \in R^+$ such that $\#\xi^\alpha = 3$. In particular, ξ^α may not be rectangular nor a special fat hook and the completion of the proof of Theorem 4.1.2 cannot be performed using Theorem 2.5.1 (or its remark) as in the previous cases. For $n = 4$, it is easy to see that the definition of $\rho^{(i)}$ is obtained from that of $\rho^{(1)}$ by rotational symmetry of the diagram and, hence, we can consider that Theorem 4.1.2 is proved for $n = 4$ as well. Thus, we assume $n > 4$ from now on.

Remark 4.4.5. The fact that $\#\xi^\alpha \leq 2$, for any $\alpha \in R^+$, and any λ when $i = 1$ while there always exists $\alpha \in R^+$ with $\#\xi^\alpha = 3$ when i is a spin node suggests that the minimal affinization corresponding to $\boldsymbol{\pi}^{(1)}$ is, from a certain point of view, more minimal than those corresponding to $\boldsymbol{\pi}^{(n)}$ and $\boldsymbol{\pi}^{(n-1)}$.

For simplicity, we assume $i = n$ in (4.4.2) and (4.4.3). Although, as mentioned at the of the previous subsection, the proof of Theorem 4.1.2 requires extra steps in this case, the spirit of the proof will be the same. Namely, by Theorem 4.4.4 and (4.4.4), Theorem 4.1.2 follows if we prove the inclusion

$$M(\boldsymbol{\xi}) \subseteq N(\lambda).$$

Observe the implication

$$\alpha \notin R_2^+ \quad \Rightarrow \quad \#\xi^\alpha \leq 2.$$

Therefore, by an application of Remark 2.5.2 as in the previous cases, it follows

$$M_\alpha(\boldsymbol{\xi}) \subseteq N(\lambda), \quad \text{for all} \quad \alpha \notin R_2^+.$$

Since $M(\boldsymbol{\xi})$ is generated by $\cup_{\alpha \in R^+} M_\alpha(\boldsymbol{\xi})$, it remains to show

$$M_\alpha(\boldsymbol{\xi}) \subseteq N(\lambda), \quad \text{for all} \quad \alpha \in R_2^+. \tag{4.4.6}$$

Evidently, if $\#\xi^\alpha \leq 2$, the proof can be completed as in the previous cases. Thus, henceforth we fix $\alpha \in R_2^+$ such that $\#\xi^\alpha = 3$.

To prove (4.4.6), in light of (2.4.5), it suffices to show that, for all $k, s, r \in \mathbb{Z}_{\geq 1}$ such that

$$s + r \geq 1 + rk + \sum_{j \geq k+1} \xi_j^\alpha, \tag{4.4.7}$$

one of the following holds:

$$(x_{\alpha,1}^+)^s (x_{\alpha,0}^-)^{s+r} w_\lambda \in N(\lambda), \tag{4.4.8}$$

$$\mathbf{x}_\alpha^-(r, s) w_\lambda \in N(\lambda), \tag{4.4.9}$$

$$_k\mathbf{x}_\alpha^-(r, s) w_\lambda \in N(\lambda). \tag{4.4.10}$$

We split the argument in the following (intersecting) list of subcases:

(i) $r \geq \xi_1^\alpha$,

(ii) $r \leq \xi_3^\alpha$,

(iii) $\xi_3^\alpha < r < \xi_1^\alpha, k \geq 2$,

(iv) $k = 1$.

We prove that (4.4.8) is satisfied in case (i), (4.4.9) is satisfied in cases (ii) and (iii), and (4.4.10) is satisfied in case (iv).

Suppose we are in case (i). Then

$$s + r \geq 1 + rk + \sum_{j \geq k+1} \xi_j^\alpha \geq 1 + \xi_1^\alpha k + \sum_{j \geq k+1} \xi_j^\alpha \geq 1 + |\xi^\alpha| = 1 + \lambda(h_\alpha).$$

Hence, $(x_{\alpha,0}^-)^{s+r} w_\lambda = 0$ by definition of $W(\lambda)$ and (4.4.8) holds.

For cases (ii)-(iv) we will make use of the following argument: (4.4.5) implies $x_{\alpha,r}^- w_\lambda \in N(\lambda)$ for all $\alpha \in R^+$ and $r \geq 3$. Therefore, if $\mathbf{b} = (b_p)_{p \geq 0} \in \mathbf{S}(r,s)$ is such that

$$b_m > 0, \quad \text{for some} \quad m \geq 3, \quad \text{then} \quad \left((x_\alpha^-)^{(b_0)} \dots (x_{\alpha,m}^-)^{(b_m)} \dots\right) w_\lambda \in N(\lambda). \tag{4.4.11}$$

From this, one easily deduces that

$$\left(\mathbf{x}_\alpha^-(r,s) - \mathbf{x}_\alpha^-(r,s)_3\right) w_\lambda \in N(\lambda). \tag{4.4.12}$$

Suppose we are in case (ii). Then,

$$s + r \geq 1 + kr + \sum_{j \geq k+1} \xi_j^\alpha \geq 1 + kr + (3-k)r = 1 + 3r.$$

This means that, if $\mathbf{b} \in \mathbf{S}(r,s)$, we must have $b_m > 0$ for some $m \geq 3$ since, otherwise, we would have $s = \sum_{p<3} p b_p \leq 2r$. In particular, $\mathbf{x}_\alpha^-(r,s)_3 = 0$ and (4.4.12) simplifies to (4.4.9).

For case (iii), we claim that it suffices to prove that

$$b_2 \geq \xi_3^\alpha + 1 = \rho(\alpha, 2) + 1, \quad \text{for all} \quad \mathbf{b} \in \mathbf{S}(r,s)_3. \tag{4.4.13}$$

Indeed, together with (4.4.5), this implies $\mathbf{x}_\alpha^-(r,s)_3 w_\lambda \in N(\lambda)$ which, together with (4.4.12), implies (4.4.9). Thus, let $\mathbf{b} \in \mathbf{S}(r,s)_3$ and note that

$$2b_2 + b_1 = s \geq 1 + r(k-1) + \sum_{j \geq k+1} \xi_j^\alpha,$$

where the equality follows from the definition of $\mathbf{S}(r,s)_3$ and the inequality follows from (4.4.7). Since $b_1 + b_2 \leq r$, by definition of $\mathbf{S}(r,s)_3$, we get

$$b_2 \geq 1 + r(k-2) + \sum_{j \geq k+1} \xi_j^\alpha. \tag{4.4.14}$$

Under the assumptions of (iii), it is clear that (4.4.14) implies (4.4.13).

Finally, we consider case (iv). Notice that, in this case, (4.4.10) simplifies to

$$ {}_1\mathbf{x}_\alpha^-(r,s)w_\lambda \in N(\lambda), \quad \text{for all} \quad s \geq 1 + \xi_2^\alpha + \xi_3^\alpha = 1 + \rho(\alpha,1). $$

Recall that

$$ {}_1\mathbf{x}_\alpha^-(r,s)w_\lambda = \sum_{\mathbf{b}\in {}_1\mathbf{S}(r,s)} (x_{\alpha,1}^-)^{(b_1)} \dots (x_{\alpha,s}^-)^{(b_s)} w_\lambda $$

and that (4.4.11) implies that the summands corresponding to $\mathbf{b} \in {}_1\mathbf{S}(r,s) \setminus \mathbf{S}(r,s)_2$ are in $N(\lambda)$. It remains to consider the summands corresponding to $\mathbf{S}(r,s) \cap \mathbf{S}(r,s)_2$, i.e., to consider

$$ \mathbf{b} = (0, b_1, b_2, 0, \dots), \quad \text{with} \qquad b_1 + b_2 = r, \quad b_1 + 2b_2 = s, $$

or, equivalently, with

$$ b_1 = 2r - s \quad \text{and} \quad b_2 = s - r. $$

In particular, if either $2r < s$ or $s < r$, then $\mathbf{S}(r,s) \cap \mathbf{S}(r,s)_2 = \emptyset$ and we are done. Otherwise, if $r \leq s \leq 2r$, we are left to show that

$$ (x_{\alpha,1}^-)^{2r-s}(x_{\alpha,2}^-)^{s-r} w_\lambda \in N(\lambda). $$

We will prove this by induction on $t = s - r$. For convenience, we rewrite the above as

$$ (x_{\alpha,1}^-)^{s-2t}(x_{\alpha,2}^-)^{t} w_\lambda \in N(\lambda), \quad \text{for all} \quad s \geq 1 + \rho(\alpha,1), \ 0 \leq t \leq s/2. \tag{4.4.15} $$

Note that (4.4.5) implies that induction starts when $t = 0$.

Recall that, since $\alpha \in R_2^+$, we must have $\alpha = \alpha_{i,n-1} + \alpha_{j,n}$, for some $1 \leq i < j \leq n-2$. To simplify notation, we set $\alpha = \alpha_{i,n-1}$ and $\beta = \alpha_{j,n}$. We claim that, for $m \in \mathbb{Z}_{\geq 2}$ and $\ell \in \mathbb{Z}_{\geq 0}$,

$$ (x_{\alpha,1}^-)^{m-2}(x_{\alpha,2}^-)^{\ell+1} w_\lambda \ \in \ \mathbb{C}\ x_\beta^+ x_\alpha^+ (x_{\alpha,1}^-)^{m}(x_{\alpha,2}^-)^{\ell} w_\lambda \ + \ N(\lambda). \tag{4.4.16} $$

Assuming (4.4.16), we finish the proof of (4.4.15) as follows. Let $t > 0$ and suppose that (4.4.15) holds for $t-1$. Setting $m = s - 2(t-1)$ and $\ell = t-1$ in (4.4.16), the right-hand-side is in $N(\lambda)$ by the induction hypothesis while the left-hand-side is the element in (4.4.15).

It remains to prove (4.4.16). Since $[x_{\alpha,p}^-, x_{\alpha,q}^-] = 0$, for all $p, q \in \mathbb{Z}_{\geq 0}$, there exists $c_p \in \mathbb{C}$ such that

$$ [x_\beta^+, (x_{\alpha,p}^-)^m] = c_p (x_{\alpha,p}^-)^{m-1} x_{\alpha,p}^-, \quad p \in \mathbb{Z}_{\geq 0}. $$

Therefore,

$$ x_\beta^+ (x_{\alpha,1}^-)^m (x_{\alpha,2}^-)^\ell w_\lambda = c_1 (x_{\alpha,1}^-)^{m-1}(x_{\alpha,2}^-)^\ell x_{\alpha,1}^- w_\lambda + c_2 (x_{\alpha,1}^-)^m (x_{\alpha,2}^-)^{\ell-1} x_{\alpha,2}^- w_\lambda. $$

Since $x_{\alpha,2}^- w_\lambda \in N(\lambda)$ by (4.4.5), it follows that

$$ x_\beta^+ (x_{\alpha,1}^-)^m (x_{\alpha,2}^-)^\ell w_\lambda \ \in \ \mathbb{C}(x_{\alpha,1}^-)^{m-1}(x_{\alpha,2}^-)^\ell x_{\alpha,1}^- w_\lambda \ + \ N(\lambda). \tag{4.4.17} $$

Similar arguments show that

$$\begin{aligned}x_\alpha^+(x_{\alpha,1}^-)^{m-1}(x_{\alpha,2}^-)^\ell x_{\alpha,1}^- w_\lambda &= c_1(x_{\alpha,1}^-)^{m-2}(x_{\alpha,2}^-)^\ell x_{\beta,1}^- x_{\alpha,1}^- w_\lambda \\ &\quad + c_2(x_{\alpha,1})^{m-1}(x_{\alpha,2}^-)^{\ell-1}x_{\beta,2}^- x_{\alpha,1}^- w_\lambda + (x_{\alpha,1}^-)^{m-1}(x_{\alpha,2}^-)^\ell x_\alpha^+ x_{\alpha,1}^- w_\lambda,\end{aligned}$$

for some $c_1, c_2 \in \mathbb{C}$. Using the commutators $[x_{\beta,1}^-, x_{\alpha,1}^-] \in \mathbb{C}x_{\alpha,2}^-$, $[x_{\beta,1}^-, x_{\alpha,1}^-] \in \mathbb{C}x_{\alpha,3}^-$, and $[x_\alpha^+, x_{\alpha,1}^-] = h_{\alpha,1}$, we get

$$\begin{aligned}x_\alpha^+(x_{\alpha,1}^-)^{m-1}(x_{\alpha,2}^-)^\ell x_{\alpha,1}^- w_\lambda &= c_1(x_{\alpha,1}^-)^{m-2}(x_{\alpha,2}^-)^{\ell+1} w_\lambda + c_1'(x_{\alpha,1}^-)^{m-2}(x_{\alpha,2}^-)^\ell x_{\alpha,1}^- x_{\beta,1}^- w_\lambda \\ &\quad + c_2(x_{\alpha,1})^{m-1}(x_{\alpha,2}^-)^{\ell-1}x_{\alpha,1}^- x_{\beta,2}^- w_\lambda + c_2'(x_{\alpha,1})^{m-1}(x_{\alpha,2}^-)^{\ell-1}x_{\alpha,3}^- w_\lambda \\ &\quad + (x_{\alpha,1}^-)^{m-1}(x_{\alpha,2}^-)^\ell h_{\alpha,1} w_\lambda.\end{aligned}$$

for some $c_1', c_2' \in \mathbb{C}$. The definition of $W(\lambda)$ implies that $h_{\alpha,1}w_\lambda = 0$ while (4.4.5) implies that

$$x_{\beta,1}^- w_\lambda,\ x_{\beta,2}^- w_\lambda,\ x_{\alpha,3}^- w_\lambda \ \in\ N(\lambda).$$

Therefore,

$$x_\alpha^+(x_{\alpha,1}^-)^{m-1}(x_{\alpha,2}^-)^\ell x_{\alpha,1}^- w_\lambda \ \in\ \mathbb{C}(x_{\alpha,1}^-)^{m-2}(x_{\alpha,2}^-)^{\ell+1} w_\lambda \ +\ N(\lambda).$$

Combining this with (4.4.17) we get (4.4.16). Theorem 4.1.2 is proved.

4.5 Appendix: Minimal affinizations of G_2

As mentioned in Introduction, the original goal of this project was the study of the structure of minimal affinizations when the underlying Lie algebra is of type G. In [CM07], Chari and Moura gave an explicit graded decomposition of the graded limit of Kirillov–Reshetkhin modules. Moreover, they found a presentation by generator and relations for these objects. Motivated by this work, our project aimed to extended their methods to general minimal affinizations in the spirit of [Mou10]. However, these techniques were not sufficient to reach our goal so far, even with the advances in this theory ([CV14, LM13, LQ14, Nao13, Nao14]) and, the original goal remains incomplete. Nevertheless, we present in this section the partial results we have obtained towards the study of these modules and state a conjecture relating them with CV-modules (see Conjecture 4.5.10).

Assume that $\mathfrak{g}$ is of type G_2. Let $\alpha_1 \in R^+$ be the simple short root. Then $\check{d}_1 = 3$ and $\check{d}_2 = 1$. Recall that

$$R^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2\}.$$

4.5.1 Graded limit of KR-modules as CV-modules

In this section we connect the well–known graded limits of Kirillov-Reshetkhin modules, or KR-modules for short, with CV-modules, by the following theorem.

Theorem 4.5.1. *Let $\mathfrak{g}$ be of type G_2. Let $m \in \mathbb{Z}_{\geq 1}$, $i \in I$ and set $\lambda = m\omega_i$. Assume that $\boldsymbol{\pi} \in \mathcal{P}_{\mathbb{Z}}^+$ is such that $L_q(\boldsymbol{\pi})$ is a minimal affinization of $V_q(m\omega_i)$. Then there exists a λ-compatible $\boldsymbol{\xi}$ such that $L(\boldsymbol{\pi}) \cong V(\boldsymbol{\xi})$.*

Let $\boldsymbol{\pi}$ be as in Theorem 4.5.1. Write $m = \check{d}_i m_0 + m_1$, $0 \leq m_1 < \check{d}_i$. The following is well known (see [FL07], for instance).

Proposition 4.5.2. *If $m_1 = 0$, then*

$$L(\boldsymbol{\pi}) \cong_{\mathfrak{g}[t]} D(m_0, m\omega_i).$$

□

In particular, if $m_1 = 0$ Theorem 4.5.1 follows from Proposition 4.5.2 and Theorem 2.5.4 by setting $\boldsymbol{\xi} = \boldsymbol{\xi}(m_0, m\omega_i)$. Therefore it remains to study the cases when $i = 1$ and $m = 3m_0 + m_1$, $m_1 = 1, 2$.

Define the $\mathfrak{g}[t]$-module $M(m\omega_i)$ to be the quotient of $W(m\omega_i)$ by the submodule generated by $x^-_{\alpha_i,1} w_\lambda$, and denote by $v_{i,m}$ the image of $w_{m\omega_i}$ on $M(m\omega_i)$. Let also $D(m\omega_i)$ be the $\mathfrak{g}$-stable generalized Demazure module defined by

$$D(m\omega_i) = \begin{cases} D(-m_1\omega_i + \Lambda_0, -\check{d}_i m_0\omega_i + m_0\Lambda_0), & \text{if } m_1 > 0, \\ D(-\check{d}_i m_0 \omega_i + m_0\Lambda_0), & \text{otherwise.} \end{cases}$$

It follows from Proposition 4.5.2 and Lemma 2.2.1 that,

$$D(m\omega_i) \cong_{\mathfrak{g}[t]} U(\mathfrak{g}[t])(v_{i,\check{d}_i}^{\otimes m_0} \otimes v_{i,m_1}) \subseteq M(\check{d}_i\omega_i)^{\otimes m_0} \otimes M(m_1\omega_i), \tag{4.5.1}$$

and, moreover

Theorem 4.5.3 ([CM07, Corollary 2.3])**.** *The $\mathfrak{g}[t]$-modules $L(\boldsymbol{\pi})$, $M(m\omega_i)$ and $D(m\omega_i)$ are all isomorphic.* □

The following lemma will be very helpful in the proof of Theorem 4.5.1.

Lemma 4.5.4 ([CM07, Lemma 4.4])**.** *Let $\alpha \in R^+$ and $r, s \in \mathbb{Z}_{\geq 1}$. Then*

(i) $(x^-_{\alpha,r})^s v_{2,1} = 0$, *if and only if* $(\alpha, r, s) \neq (\theta, 1, 1)$,

(ii) $(x^-_{\alpha,r})^s v_{1,1} = 0$, *for all* $\alpha \in R^+,\ r, s \in \mathbb{Z}_{\geq 1}$,

(iii) $(x^-_{\alpha,r})^s v_{1,2} = 0$, *if and only if,*

$$(\alpha, r, s) \notin \{(2\alpha_1 + \alpha_2, 1, 1), (3\alpha_1 + \alpha_2, 1, 1), (3\alpha_1 + 2\alpha_2, 1, 1)\},$$

(iv) $(x^-_{\alpha,r})^s v_{1,3} = 0$, *if and only if,*

$$(\alpha, r, s) \notin \{(2\alpha_1 + \alpha_2, 1, k) | 0 \leq k \leq 3\} \cup \{(\beta, 1, 2), (\beta, 2, 1) | \beta = 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}.$$

Define the following λ-compatible partitions $\boldsymbol{\xi}$, for $m_1 = 1$ and $m_1 = 2$, respectively

Root	$m_1 = 1$	$m_1 = 2$
α_1	(m)	(m)
α_2	$\emptyset$	$\emptyset$
$\alpha_1 + \alpha_2$	(m)	(m)
$2\alpha_1 + \alpha_2$	$(3m_0 + 2, 3m_0)$	$(3m_0 + 3, 3m_0 + 1)$
$3\alpha_1 + \alpha_2$	$(m_0 + 1, m_0, m_0)$	$(m_0 + 1, m_0 + 1, m_0)$
$3\alpha_1 + 2\alpha_2$	$(m_0 + 1, m_0, m_0)$	$(m_0 + 1, m_0 + 1, m_0)$

It is straightforward from the definitions that $V(\boldsymbol{\xi})$ is a quotient of $M(\lambda)$. Using the identification (4.5.1)We prove that exists a homomorphism of $\mathfrak{g}[t]$-modules

$$V(\boldsymbol{\xi}) \to D(m\omega_i), \quad v_{\boldsymbol{\xi}} \to v_D := \otimes_{j=1}^{m_0} v_{1,3} \otimes v_{1,m_1}, \tag{4.5.2}$$

and, hence, by Theorem 4.5.3, the proof of Theorem 4.5.1 is complete. We show that v_D satisfies the defining relations of $V(\boldsymbol{\xi})$.

Recall from Proposition 2.5.3 that, if ξ^α is essentially rectangular, it suffices to show that

$$x^-_{\alpha,s_\alpha} v_D = (x^-_{\alpha,s_\alpha - 1})^{\xi^\alpha_{s_\alpha}+1} v_D = 0,$$

where $s_\alpha = \#\xi^\alpha$. This is proved by inspection, using comultiplication rules and Lemma 4.5.4, for each $\alpha \in R^+$, such that ξ^α is essentially rectangular. In particular this argument proves (4.5.2) when $m_1 = 2$.

Remark 4.5.5. The above discussion together with Theorem 2.5.4 proves that $L(\boldsymbol{\pi})$ is isomorphic to the Demazure module $D(m_0 + 1, m\omega_1)$ if $m_1 = 2$.

The only case not covered by the previous argument is when $m_1 = 1$ and $\alpha \in \{3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$. For the remainder of this section let α be one of those roots. If we prove that, for each $k \in \mathbb{Z}_{\geq 1}$,

$$_k\mathbf{x}^-_\alpha(r, s) v_D = 0, \quad s, r \in \mathbb{Z}_{\geq 1}, \ s + r \geq 1 + kr + \sum_{j \geq k+1} \xi^\alpha_j, \tag{4.5.3}$$

by Remark 2.5.2, Proposition 2.4.2 and (2.4.5), we prove (4.5.2).

Let $k \in \mathbb{Z}_{\geq 1}$ and $(b_p)_{p \geq 0} \in_k \mathbf{S}(r, s)$. Then $(b_p) = (0, \ldots, 0, b_k, b_{k+1}, \ldots)$. In particular, since $x^-_{\alpha,3} v_T = 0$, by Lemma 4.5.4, it implies that $_k\mathbf{x}^-_\alpha(r, s) v_D = 0$, $k \geq 3$, $r, s \in \mathbb{Z}_{\geq 1}$. Suppose $k \leq 2$. Using the previous argument, it suffices to consider the sum over the sequences $(b_p)_{p \geq 0} \in_k \mathbf{S}(r, s)$ such that $b_p \neq 0$ only if $k \leq p \leq 2$. We split the argument in cases, depending on k.

Case $k = 2$**.**

Let $(0, 0, b_2, 0, \ldots) \in {}_2\mathbf{S}(r, s)$. Then $r = b_2$ and $s = 2b_2$. Therefore, we are left to prove that

$$ {}_2\mathbf{x}_\alpha^-(r, 2r)v_D = 0, \quad r = s - r \geq 1 + m_0, $$

or equivalently,

$$ (x_{\alpha,2}^-)^r v_D = 0, \quad r \geq 1 + m_0, $$

which follows by Lemma 4.5.4.

Case $k = 1$**.**

Let $(0, b_1, b_2, 0, \ldots) \in {}_1\mathbf{S}(r, s)$. Then $r = b_1 + b_2$ and $s = b_1 + 2b_2$. Equivalently, $b_1 = 2r - s$ and $b_2 = s - r$. Therefore, we have to prove that

$$ (x_{\alpha,1}^-)^{2r-s}(x_{\alpha,2}^-)^{s-r}v_D = 0, \quad \text{for } s \geq 1 + 2m_0,\ r \in \mathbb{Z}_{\geq 1}. $$

This is proved by induction on $t = s - r$. For convenience, we rewrite the above as follows

$$ (x_{\alpha,1}^-)^{s-2t}(x_{\alpha,2}^-)^t v_D = 0. \tag{4.5.4} $$

Note that the induction starts for $t = 0$, by Lemma 4.5.4, since $s \geq 1 + 2m_0$.

We claim that

$$ (x_{\alpha,1}^-)^{p-2}(x_{\alpha,2}^-)^{\ell+1}v_D \in \mathbb{C}x_\alpha^+(x_{\alpha,1}^-)^p(x_{\alpha,2}^-)^\ell v_D, \quad p, \ell \in \mathbb{Z}_{\geq 0},\ p \geq 2. \tag{4.5.5} $$

Assuming the claim we finish the proof as follows. Let $t > 0$ and suppose that (4.5.4) holds for $t - 1$. Setting $p = s - 2(t - 1)$ and $\ell = t - 1$ in (4.5.5) we have that (4.5.4) follows for t.

To prove the claim we make use of the following Lemma, which is straightforward from the relations of $\mathfrak{g}[t]$.

Lemma 4.5.6. *Let $\alpha \in R^+$, $k, r \in \mathbb{Z}_{\geq 0}$, $k \geq 2$. There exists $c_1, c_2 \in \mathbb{C}$ such that*

$$ [x_\alpha^+, (x_{\alpha,r}^-)^k] = c_1 x_{\alpha,2r}^-(x_{\alpha,r}^-)^{k-2} + c_2(x_{\alpha,r}^-)^{k-1}h_{\alpha,r}. \quad \square $$

By the above lemma we have that

$$ x_\alpha^+(x_{\alpha,1}^-)^p(x_{\alpha,2}^-)^\ell v_D = \left(c_1 x_{\alpha,2}^-(x_{\alpha,1}^-)^{p-2} + c_2(x_{\alpha,1}^-)^{p-1}h_{\alpha,1} - (x_{\alpha,1}^-)^p x_\alpha^+\right)(x_{\alpha,2}^-)^\ell v_D. \tag{4.5.6} $$

Taking proper commutators it follows that $h_{\alpha,1}(x_{\alpha,2}^-)^\ell v_D = 0$, and then, (4.5.6) is equivalent to

$$ x_\alpha^+(x_{\alpha,1}^-)^p(x_{\alpha,2}^-)^\ell v_D = c_1(x_{\alpha,1}^-)^{p-2}(x_{\alpha,2}^-)^{\ell+1} - (x_{\alpha,1}^-)^p\left(c_3(x_{\alpha,2}^-)^{\ell-2}x_{\alpha,4}^- + c_4(x_{\alpha,2}^-)^{\ell-1}h_{\alpha,2}\right)v_D $$

for some $c_3, c_4 \in \mathbb{C}^\times$. Since $\left(c_3(x_{\alpha,2}^-)^{\ell-2}x_{\alpha,4}^- + c_4(x_{\alpha,2}^-)^{\ell-1}h_{\alpha,2}\right)v_D = 0$, the claim follows.

Remark 4.5.7. By Theorem 2.5.4, Lemma 4.5.4 and Theorem 4.5.1 it is quite clear to prove that, if $m_1 = 1$, $L(\boldsymbol{\pi})$ cannot be isomorphic $\mathfrak{g}$-stable Demazure module of any level (unless $m_0 = 0$, in which case it is isomorphic (as $\mathfrak{g}$-module) to a simple $\mathfrak{g}$-module).

4.5.2 General minimal affinations

In this section we present the advances obtained towards the study of the structure of the graded limit of general minimal affinizations. For all $i \in I, \ell \in \mathbb{Z}_{\geq 0}$ and $r \in \mathbb{Z}_{\geq 1}$ let

$$R(i,m,r) := \{\alpha \in R^+ | x^-_{\alpha,r} v_{i,m} = 0\},$$

where we recall that $v_{i,m}$ is the image of $w_{m\omega_i}$ in $M(m\omega_i)$. Let $\lambda \in P^+$ and define

$$R(\lambda, r) = \bigcap_{i \in I} R(i, \lambda(h_i), r).$$

Since $t\mathfrak{h}[t]w_\lambda = 0$, we have $R(\lambda,r) \subseteq R(\lambda,s)$ for all $r \geq s$. Let $M(\lambda)$ be the quotient of $W(\lambda)$ by the submodule generated by

$$x_{\alpha,r} w_\lambda, \quad \text{for } \alpha \in R(\lambda,r),\ r \in \mathbb{Z}_{\geq 1},$$

and denote by $\bar{w}_\lambda$ the image of w_λ in $M(\lambda)$. Also, define

$$D(\lambda) := U(\mathfrak{g}[t])(v_{1,\lambda(h_1)} \otimes v_{2,\lambda(h_2)}) \subseteq M(\lambda(h_1)\omega_1) \otimes M(\lambda(h_2)\omega_2).$$

Note that $D(\lambda)$ is isomorphic to a $\mathfrak{g}$-stable generalized Demazure module. In fact, write $\lambda(h_1) = 3m_0 + m_1$, $0 \leq m_1 < 3$ and, then Theorem 4.5.1 implies

$$M(\lambda(h_1)\omega_1) \cong_{\mathfrak{g}[t]} D(-m_1\omega_1 + \Lambda_0, -3m_0\omega_1 + m_0\Lambda_0) \text{ and } M(\lambda(h_2)\omega_2) \cong_{\mathfrak{g}[t]} D(-\lambda(h_2)\omega_2 + \lambda(h_2)\Lambda_0).$$

Therefore, it is clear that

$$D(\lambda) \cong_{\mathfrak{g}[t]} D(-m_1\omega_1 + \Lambda_0, -3m_0\omega_1 + m_0\Lambda_0, -\lambda(h_2)\omega_2 + \lambda(h_2)\Lambda_0).$$

Quite easily we check that there exists an epimorphism of $\mathfrak{g}[t]$-modules $M(\lambda) \to D(\lambda)$ which maps $\bar{w}_\lambda$ to $v_{1,\lambda(h_1)} \otimes v_{2,\lambda(h_2)}$.

Our goal was to prove [Mou10, Conjecture 3.20], which we recall now for our case of interest.

Conjecture 4.5.8. *Let $\lambda \in P^+$ and $\boldsymbol{\pi} \in \mathcal{P}^+_{\mathbb{Z}}$ such that $L_q(\boldsymbol{\pi})$ is a minimal affinization of $V_q(\lambda)$. Then*

$$M(\lambda) \cong_{\mathfrak{g}[t]} L(\boldsymbol{\pi}) \cong_{\mathfrak{g}[t]} D(\lambda). \tag{4.5.7}$$

Let $\boldsymbol{\pi}$ as in Conjecture 4.5.8. Since $L_q(\boldsymbol{\pi})$ is a minimal affinization of $V_q(\lambda)$ it follows that exists $a_1, a_2 \in q^{\mathbb{Z}}$ such that $\boldsymbol{\pi} = \boldsymbol{\omega}_{1,a_1,\lambda(h_1)}\boldsymbol{\omega}_{2,a_2,\lambda(h_2)}$. Moreover, as a consequence of [Cha02, Theorem 3], we have that exists a homomorphism of $U_q(\widetilde{\mathfrak{g}})$-modules

$$L_q(\boldsymbol{\pi}) \to L_q(\boldsymbol{\omega}_{\sigma(1),a_{\sigma(1)},\lambda(h_{\sigma(1)})}) \otimes L_q(\boldsymbol{\omega}_{\sigma(2),a_{\sigma(2)},\lambda(h_{\sigma(2)})}),$$

for some permutation $\sigma \in S_2$. Since, $L_q(\boldsymbol{\omega}_{i,a_i,\lambda(h_i)})$ is a minimal affinization of $V_q(\lambda(h_i))$, Proposition 4.3.1 and Theorem 4.5.3 implies that there exists an epimorphism of $\mathfrak{g}[t]$-modules

$$L(\boldsymbol{\pi}) \to D(\lambda), \tag{4.5.8}$$

mapping $v_{\boldsymbol{\pi}}$ to $v_{1,\lambda(h_1)} \otimes v_{2,\lambda(h_2)}$.

Lemma 4.5.9 ([Mou10], Lemma 4.1.8). *Let $\mathfrak{g}$ be any simple Lie algebra. Let $i \in I, a \in \mathbb{F}^\times$ and $m \in \mathbb{Z}_{\geq 0}$. Suppose that v is a highest ℓ-weight vector of $L_q(\boldsymbol{\omega}_{i,a,m})$. Then*

$$x_{i,1}^- v = aq_i^m x_i^- v.$$

For each $\lambda \in P^+$ write $\lambda = (3\ell + k)\omega_1 + m\omega_2$, for some $\ell, k, m \in \mathbb{Z}_{\geq 0}$, with $0 \leq k < 3$, and define a λ-compatible partition $\boldsymbol{\xi}$ according the following table.

Root	$k=0$	$k=1$	$k=2$
α_1	(3ℓ)	$(3\ell+1)$	$(3\ell+2)$
α_2	(m)	(m)	(m)
$\alpha_1+\alpha_2$	$(3\ell+3m)$	$(3\ell+3m+1)$	$(3\ell+3m+2)$
$2\alpha_1+\alpha_2$	$(3\ell+3m, 3\ell)$	$(3\ell+3m+2, 3\ell)$	$(3\ell+3m+3, 3\ell+1)$
$3\alpha_1+\alpha_2$	$(\ell+m, \ell, \ell)$	$(\ell+m+1, \ell, \ell)$	$(\ell+m+1, \ell+1, \ell)$
$3\alpha_1+2\alpha_2$	$(\ell+m, \ell+m, \ell)$	$(\ell+m+1, \ell+m, \ell)$	$(\ell+m+1, \ell+m+1, \ell)$

Conjecture 4.5.10. *Let $\lambda \in P^+$ and $\boldsymbol{\pi} \in \mathcal{P}_{\mathbb{Z}}^+$ be such that $L_q(\boldsymbol{\pi})$ is a minimal affinizations of $V_q(\lambda)$. Then*

$$L(\boldsymbol{\pi}) \cong_{\mathfrak{g}[t]} V(\boldsymbol{\xi}).$$

Observe that Conjecture 4.5.10 holds for Kirrilov-Reshetkhin modules by Theorem 4.5.1.

In what follows we show the advances towards the proof of Conjectures 4.5.8 and 4.5.10 for general minimal affinizations. It will be convenient to break the argument in cases $\lambda(h_1) = 1, 2$ and $\lambda(h_1) \geq 3$.

Case $\lambda(h_1) = 1$**.** We prove Conjectures 4.5.8 and 4.5.10 in this case.

We show that there exists an epimomorphism of $\mathfrak{g}[t]$-modules

$$M(\lambda) \to L(\boldsymbol{\pi}), \tag{4.5.9}$$

and that $D(\lambda) \cong_{\mathfrak{g}[t]} M(\lambda)$. Those together with (4.5.8) prove the Conjecture 4.5.8 in this case.

We start proving (4.5.9). By Lemma 4.5.4, it follows

$$R(\lambda, 1) = R^+ \setminus \{\theta\}, \quad R(\lambda, s) = R^+,\ s \geq 2.$$

Since $x_{\theta,2}^- = [x_{2\alpha_1+\alpha_2}^-, x_{\alpha_1+\alpha_2}^-]$, it suffices to show that

$$x_{\alpha,1}^- v_{\boldsymbol{\pi}} = 0, \quad \text{for all } \alpha \in R^+ \setminus \{\theta\}. \tag{4.5.10}$$

This is clear for the simple roots, by Lemma 3.5.2. For non simple roots α, $\alpha \in R^+ \setminus \{\theta\}$, it suffices find elements $X_{\alpha,1}, X_\alpha, Y \in U_{\mathbb{A}}(\widetilde{\mathfrak{g}})$ such that

$$X_{\alpha,1} v \in q^{\mathbb{Z}} X_\alpha v + Yv, \quad \overline{X_{\alpha,1}} = x_{\alpha,1}^-, \quad \overline{X_\alpha} = x_\alpha^-, \ \text{ and } \ \overline{Y} = 0, \tag{4.5.11}$$

where v is the highest ℓ-weight of $L_q(\boldsymbol{\pi})$

Let $\boldsymbol{\pi} = Y_{1,1}\boldsymbol{\omega}_{2,q^{3m+4},n} \in \mathcal{P}_{\mathbb{Z}}^{+}$. It follows by Theorem 3.5.1 that $L_q(\boldsymbol{\pi})$ is a (increasing) minimal affinization of $V_q(\lambda)$. Let v_1 and v_2 be the highest ℓ-weight vectors of $L_q(Y_{1,1})$ and $L_q(\boldsymbol{\omega}_{2,q^{3m+4},n})$, respectively. By [Cha02, Theorem 3], we have

$$L_q(\boldsymbol{\pi}) \cong U_q(\tilde{\mathfrak{g}})(v_1 \otimes v_2) \subseteq L_q(Y_{1,1}) \otimes L_q(\boldsymbol{\omega}_{2,q^{3m+4},n}).$$

Let $\alpha = \alpha_1 + \alpha_2$, and set $X_\alpha = [x_1^-, x_2^-]$ and $X_{\alpha,1} = [x_1^-, x_{2,1}^-]$ in $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$. Then

$$\begin{aligned}
X_\alpha(v_1 \otimes v_2) &= x_1^- x_2^-(v_1 \otimes v_2) - x_2^- x_1^-(v_1 \otimes v_2)\\
&= x_1^-(v_1 \otimes x_2^- v_2) - x_2^-(x_1^- v_1 \otimes v_2)\\
&= x_1^- v_1 \otimes k_1^{-1} x_2^- v_2 + v_1 \otimes x_1^- x_2^- v_2 - x_2^- x_1^- v_1 \otimes k^2 v_2 - x_1^- v_1 \otimes x_2^- v_2\\
&= (q^{-3} - 1) x_1^- v_1 \otimes x_2^- v_2 + v_1 \otimes x_1^- x_2^- v_2 - q^{-3m} x_2^- x_1^- v_1 \otimes v_2.
\end{aligned}$$

On the other hand, by Lemma 1.3.3 modulo elements of the form $x(v_1 \otimes v_2)$ with $Y \in U_{\mathbb{A}}(\tilde{\mathfrak{g}}) \otimes U_{\mathbb{A}}(\tilde{\mathfrak{g}})$ such that $\bar{Y} = 0$, we have

$$\begin{aligned}
X_{\alpha,1}(v_1 \otimes v_2) &= x_1^- x_{2,1}^-(v_1 \otimes v_2) - x_{2,1}^- x_1^-(v_1 \otimes v_2)\\
&= x_1^-(v_1 \otimes x_{2,1}^- v_2) - x_{2,1}^-(x_1^- v_1 \otimes v_2)\\
&= x_1^- v_1 \otimes k_1^{-1} x_{2,1}^- v_2 + v_1 \otimes x_1^- x_{2,1}^- v_2 - x_{2,1}^- x_1^- v_1 \otimes k_2 v_2 - x_1^- v_1 \otimes x_{2,1}^- v_2\\
&= (q^{-3} - 1) x_1^- v_1 \otimes x_{2,1}^- v_2 + v_1 \otimes x_1 x_{2,1}^- v_2 - q^{3m} x_{2,1}^- x_1^- v_1 \otimes v_2.
\end{aligned}$$

Since $L_q(\beta_2(\boldsymbol{\pi}))$ is a minimal affinization of $V_q(n\omega_2)$ with respect the subalgebra $U_q(\tilde{\mathfrak{g}}_2)$ (which is isomorphic to $U_{q^3}(\widetilde{\mathfrak{sl}}_2)$), Lemma 4.5.9 implies

$$x_{2,1}^- v_2 = q^{3m+4} q_2^m x_2^- v_2 = q^{6m+4} x_2^- v_2.$$

Therefore,

$$\begin{aligned}
X_{\alpha,1}(v_1 \otimes v_2) &= (q^{-3} - 1) q^{6m+4} x_1^- v_1 \otimes x_2^- v_2 + q^{6m+4} v_1 \otimes x_1 x_2^- v_2 - q^{3m} x_{2,1}^- x_1^- v_1 \otimes v_2\\
&= q^{6m+4} X_\alpha(v_1 \otimes v_2) - q^{3m} x_{2,1}^- x_1^- v_1 \otimes v_2 + q^{3m+4} x_2^- x_1^- v_1 \otimes v_2. \qquad (4.5.12)
\end{aligned}$$

We claim that $q^{3m} x_{2,1}^- x_1^- v_1 \otimes v_2 = q^{3m+4} x_2^- x_1^- v_1 \otimes v_2$, and then (4.5.11) follows from (4.5.12) for the root α. For the claim, note that $x_{2,r}^+ x_1^- v_1 = 0$, for all $r \geq 0$. In particular, Lemma 3.3.3 implies

$$U_q(\tilde{\mathfrak{g}}_2) x_1^- v_1 \cong_{U_q(\tilde{\mathfrak{g}}_2)} L_q(Y_{2,q}),$$

and, hence, $x_{2,1}^- x_1^- v_1 = q^4 x_2^- x_1^- v_1$, by Lemma 4.5.9. The claim is immediate now.

Similar arguments work for $\vartheta = 2\alpha_1 + \alpha_2$, by setting $X_\vartheta = [x_1^-, X_\alpha]$ and $X_{\vartheta,1} = [x_1^-, X_{\alpha,1}]$, and for $\beta = 3\alpha_1 + \alpha_2$, by setting $X_\beta = [x_1^-, X_\vartheta]$ and $X_{\beta,1} = [x_1^-, X_{\vartheta,1}]$, which proves (4.5.11) for all elements of $R^+ \setminus \{\theta\}$. We omit the details.

We now prove that $D(\lambda) \cong_{\mathfrak{g}[t]} M(\lambda)$. By the defining relations of $M(\lambda)$ it follows

$$M(\lambda) = \sum_{r \in \mathbb{Z}_{\geq 0}} U(\mathfrak{g})(x^-_{\theta,1})^r \bar{w}_\lambda.$$

In particular, $M(\lambda)$ is a graded quotient of $W(\lambda)$, thus finite-dimensional. Since $\mathfrak{n}^+(x^-_{\theta,1})^r \bar{w}_\lambda = 0$, for each $r \in \mathbb{Z}_{\geq 0}$, it follows that $M(\lambda)[r] \cong_{\mathfrak{g}} V(\lambda - r\theta)^{\oplus m_r}$, for $m_r \leq 1$. Therefore,

$$\lambda - r\theta \notin P^+ \Rightarrow (x^-_{\theta,1})^r \bar{w}_\mu = 0.$$

Since $\theta = \omega_2$, it follows $(x^-_{\theta,1})^r \bar{w}_\mu \neq 0$ only if $0 \leq r \leq m$.

On the other hand, by Lemma 4.5.4, we have

$$x^-_{\theta,s}(v_{1,1} \otimes v_{2,m}) = 0, \quad x^-_{\alpha,r}(v_{1,1} \otimes v_{2,m}) = 0, \quad \alpha \in R^+ \setminus \{\theta\},\ r \geq 1,\ s \geq 2,$$

and then

$$D(\lambda) = \sum_{r \geq 0} U(\mathfrak{g})(x^-_{\theta,1})^r(v_{1,1} \otimes v_{2,m}) = 0.$$

Also by Lemma 4.5.4, we have

$$(x^-_{\theta,1})^r(v_{1,1} \otimes v_{2,m}) = v_{1,1} \otimes (x^-_{\theta,1})^r v_{2,m} \neq 0 \Leftrightarrow 0 \leq r \leq m,$$

and, hence, together with (4.5.7), $M(\lambda) \cong_{\mathfrak{g}[t]} D(\lambda)$, as desired, which proves Conjecture 4.5.8 in this case. Then, the proof of Conjecture 4.5.10 in this case follows by observing that we have a chain of epimorphisms of $\mathfrak{g}[t]$-modules

$$M(\lambda) \to V(\boldsymbol{\xi}) \to D(\lambda),$$

since $v_{\boldsymbol{\xi}}$ and v_D satisfy the defining relations of $M(\lambda)$ and $V(\boldsymbol{\xi})$, respectively.

Remark 4.5.11. It follows, by Theorem 2.5.4, that

$$L(\boldsymbol{\pi}) \cong_{\mathfrak{g}[t]} D(m+1, \lambda).$$

Case $\lambda(h_1) = 2$. In this case we have

$$R(\lambda, 1) = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \}, \quad \text{and} \quad R(\lambda, 2) = R^+.$$

Therefore,

$$M(\lambda) = \sum_{(r,s,t) \in \mathbb{Z}^3_{\geq 0}} U(\mathfrak{g})(x^-_{2\alpha_1+\alpha_2,1})^r (x^-_{3\alpha_1+\alpha_2,1})^s (x^-_{3\alpha_1+2\alpha_2,1})^t \bar{w}_\lambda. \tag{4.5.13}$$

By similar arguments used in [CM07] based on the study of Heisenberg subalgebras of $\mathfrak{g}$ and the fact that $M(\lambda)$ is finite-dimensional we can restrict the sum of (4.5.13) to the set

$$\{(r, 0, t) \in \mathbb{Z}^3_{\geq 0} | t \leq m,\ r \leq 1\}.$$

Similarly to the previous case, we use these restrictions to prove that $D(\lambda) \cong_{\mathfrak{g}[t]} M(\lambda)$.

To prove Conjecture 4.5.8 in this case it remains to show that $v_{\boldsymbol{\pi}}$ satisfies the defining relations of $M(\lambda)$. As usual, $x_{i,1}v_{\boldsymbol{\pi}} = 0$, $i \in I$, by Lemma 4.5.9. Similarly to the previous case we show that $x^-_{\alpha_1+\alpha_2,1}v_{\boldsymbol{\pi}} = 0$. Then we have $x^-_{2\alpha_1+\alpha_2,2}v_{\boldsymbol{\pi}}$, since $x^-_{2\alpha_1+\alpha_2,2} \in \mathbb{C}^\times[x^-_{\alpha_1+\alpha_2,1}, x^-_{1,1}]$. Therefore, it remains to show that

$$x^-_{3\alpha_1+\alpha_2,2}v_{\boldsymbol{\pi}} \;=\; 0 \;=\; x^-_{3\alpha_1+2\alpha_2,2}v_{\boldsymbol{\pi}}. \tag{4.5.14}$$

Moreover, we also have that the proof of Conjecture 4.5.8 implies the proof of Conjecture 4.5.10, by similar arguments as in the previous case.

Differently form the case $\lambda(h_1) = 1$, in this case the relations involving elements of $\mathfrak{g}[t]$ with degree 2 do not follow immediate from the degree 1 ones. Therefore, the challenge imposed to prove (4.5.14) demands a deeper study of the module $L_{\mathbb{A}}(\boldsymbol{\pi})$, which we do not have enough tools to deal with.

Case $\lambda(h_1) \geq 3$. Note that in this case we have

$$R(\lambda, 1) = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \}, \quad R(\lambda, 2) = \{2\alpha_1 + \alpha_2\}, \;\text{ and }\; R(\lambda, 3) = R^+.$$

Lemma 3.5.2 implies

$$x^-_{1,1}v_{\boldsymbol{\pi}} = 0 = x^-_{2,1}v_{\boldsymbol{\pi}}.$$

Since

$$x^-_{2\alpha_1+\alpha_2,2} \in \mathbb{C}^\times[x^-_{\alpha_1+\alpha_2,1}, x^-_{\alpha_1,1}], \quad x^-_{3\alpha_1+\alpha_2,3} \in \mathbb{C}^\times[x^-_{2\alpha_1+\alpha_2,2}, x^-_{\alpha_1,1}]$$
$$\text{and } \; x^-_{3\alpha_1+2\alpha_2,3} \in \mathbb{C}^\times[x^-_{2\alpha_1+\alpha_2,2}, x^-_{\alpha_1+\alpha_2,1}],$$

we prove that exists an epimorphism of $\mathfrak{g}[t]$-modules

$$M(\lambda) \to L(\boldsymbol{\pi}),$$

if we show that $x^-_{\alpha_1+\alpha_2,1}v_{\boldsymbol{\pi}} = 0$. The proof the latter is analogous of the case $m = 1$.

Therefore, together with (4.5.8), we have the following chain of epimorphisms of $\mathfrak{g}[t]$-modules

$$M(\lambda) \to L(\boldsymbol{\pi}) \to D(\lambda).$$

To prove Conjecture 4.5.8 in this case it remains to show that $M(\lambda) \cong_{\mathfrak{g}[t]} D(\lambda)$. Differently from the previous case, the challenge here is to find a restriction of $M(\lambda)$, similar to (4.5.13) as in the case $\lambda(h_2) = 2$, so we can identify each simple $\mathfrak{g}$-module in the decomposition of $M(\lambda)$ to one of $D(\lambda)$.

Chapter 5

Tame modules

Recall the definitions of tame, thin, special and anti-special modules given in Section 3.3. In this chapter we define explicitly a family of sets of Drinfeld polynomials which we call "extended snakes", and consider the corresponding irreducible finite-dimensional modules of a quantum affine algebra of type B_n. We show that a simple tame module of B_n type has to be an extended snake module (more precisely, a tensor product of snake modules, see Remark 5.2.5), see Theorem 5.1.3.

Thus, we obtain the main result of the chapter: an irreducible module in type B is tame if and only if it is thin. All such modules are special and antispecial.

Throughout this chapter we work only with simple finite-dimensional representations of $U_q(\widetilde{\mathfrak{g}})$, for $\mathfrak{g}$ of types A_n and B_n.

5.1 Statement of results

Define the subset $\mathcal{X} \subset I \times \mathbb{Z}$ by

Type A $\mathcal{X} := \{(i,k) \in I \times \mathbb{Z} | \ i - k \equiv 1 \mod 2\}$.

Type B $\mathcal{X} := \{(n, 2k+1) | \ k \in Z\} \bigsqcup \{(i,k) \in I \times \mathbb{Z} | \ i < n \text{ and } k \equiv 0 \mod 2\}$.

Let $c \in \mathbb{C}^\times$ and consider only the representations whose q-characters lie in the subring $\mathbb{Z}[Y_{i,cq^k}^{\pm 1}]_{(i,k)\in\mathcal{X}}$.

Recall that we have $d_i = 1$, for type A_n, and, for type B_n, we have $d_i = 2$, $i < n$, and $d_n = 1$. We also define

$$\mathcal{W} := \{(i,k) | (i, k - d_i) \in \mathcal{X}\}$$

in order to have a refinement of (3.3.2), such that $\mathcal{M}(L_q(\boldsymbol{\pi})) \subseteq \boldsymbol{\pi}\mathbb{Z}[A_{i,cq^k}^{-1}]_{(i,k)\in\mathcal{W}}$, for all $\boldsymbol{\pi} \in \mathbb{Z}[Y_{i,cq^k}]_{(i,k)\in\mathcal{X}}$.

Henceforth, by an abuse of notation, we write

$$Y_{i,k} := Y_{i,cq^k}, \quad A_{i,k} := A_{i,cq^k}, \quad u_{i,k} := u_{i,cq^k},$$

for all $(i,k) \in \mathcal{X}$.

We denote by $\mathcal{P}_{\mathcal{X}}$ (resp. $\mathcal{P}_{\mathcal{X}}^+$) the subgroup (resp. submonoid) of $\mathcal{P}$ generated by $Y_{i,k}$, $(i,k) \in \mathcal{X}$.

Given $\boldsymbol{\pi} \in \mathcal{P}_{\mathcal{X}}^+$, we always write $\boldsymbol{\pi} = \prod_{t=1}^{T} Y_{i_t,k_t}$ in such a way that $k_{t+1} \geq k_t$ and $i_{t+1} \geq i_t$ whenever $k_{t+1} = k_t$. We denote the ordered sequence $(i_t, k_t)_{1 \leq t \leq T}$ by $\mathcal{X}(\boldsymbol{\pi})$.

Definition 5.1.1. Let $\mathfrak{g}$ be of type B_n and let $(i,k) \in \mathcal{X}$. A point $(i',k') \in \mathcal{X}$ is in *extended snake position* with respect to (i,k) if one of the following holds.

(i) $k' - k \geq 4 + 2|i'-i| - \delta_{ni} - \delta_{ni'}$ and $k' - k \equiv 2|i'-i| - \delta_{ni} - \delta_{ni'} \mod 4$,

(ii) $k' - k \geq 2n + 2 + 2|n - i - i'| - \delta_{ni} - \delta_{ni'}$.

Pictorially, the extended snake position is shown in Figure 6.1 under the image of ι defined in (6.0.1).

Let (i_t, k_t), $1 \leq t \leq T$, $T \in \mathbb{Z}_{\geq 1}$, be a sequence of points in $\mathcal{X}$. The sequence $(i_t,k_t)_{1 \leq t \leq T}$ is said to be an *extended snake* if (i_t, k_t) is in extended snake position to (i_{t-1}, k_{t-1}) for all $2 \leq t \leq T$. We call the simple module $L_q(\boldsymbol{\pi})$ an *extended snake module* if $\mathcal{X}(\boldsymbol{\pi})$ is an extended snake.

We now state the main theorems of the second part of this Thesis.

Theorem 5.1.2. *Let $\mathfrak{g}$ be of type B_n. Let $(i_t, k_t) \in \mathcal{X}$, $1 \leq t \leq T$ be an extended snake of length $T \in \mathbb{Z}_{\geq 1}$ and $\boldsymbol{\pi} := \prod_{t=1}^{T} Y_{i_t,k_t}$. Then $L_q(\boldsymbol{\pi})$ is thin, tame, special and anti-special.*

Theorem 5.1.3. *Let $\mathfrak{g}$ be of type B_n and let $\boldsymbol{\pi} \in \mathcal{P}_{\mathcal{X}}^+$. The module $L_q(\boldsymbol{\pi})$ is tame if and only if $\mathcal{X}(\boldsymbol{\pi})$ is an extended snake. In particular, all irreducible tame modules are thin, special and anti-special.*

The following is immediate from the specialty property.

Corollary 5.1.4. *Let V be a tame representation. For all $\boldsymbol{\pi} \in \mathcal{M}(V) \cap \mathcal{P}^+$, $L_q(\boldsymbol{\pi})$ is a tame subfactor of V.* □

We prove Theorem 5.1.3 in this chapter. The proof of Theorem 5.1.2 is essentially combinatorial and, therefore, it will be treated in the following chapter.

5.2 First properties

The main objects of this chapter are the irreducible tame representations. In all cases they turn out to be also thin. We start with simple remarks about tame and thin modules.

Lemma 5.2.1. *The restriction of a tame module to any diagram subalgebra is tame. A subfactor of a tame module is tame.* □

Lemma 5.2.2. *A subfactor of a thin module is thin.* □

Next we consider tensor products.

Lemma 5.2.3. *Let $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2 \in \mathcal{P}^+$. If the module $L_q(\boldsymbol{\pi}_1) \otimes L_q(\boldsymbol{\pi}_2)$ is thin, then both $L_q(\boldsymbol{\pi}_1)$ and $L_q(\boldsymbol{\pi}_2)$ are thin.* $\square$

We also have the tame analogue.

Lemma 5.2.4. *Let $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2 \in \mathcal{P}^+$. If the module $L_q(\boldsymbol{\pi}_1) \otimes L_q(\boldsymbol{\pi}_2)$ is tame, then both $L_q(\boldsymbol{\pi}_1)$ and $L_q(\boldsymbol{\pi}_2)$ are tame.*

Proof. Let $\{v_j\}_{j=1}^{n_1}$ be a basis of $L_q(\boldsymbol{\pi}_1)$ and $\{w_j\}_{j=1}^{n_2}$ be a basis of $L_q(\boldsymbol{\pi}_2)$ such that $\phi_i^\pm(u)$, $i \in I$, act diagonally in the basis $v_j \otimes w_k$. We also assume that w_1 is the highest weight vector and v_{n_1} is the lowest weight vector.

Let $f_i^\pm(u)$ and $g_i^\pm(u)$ be series in $u^{\pm 1}$ defined by $\phi_i^\pm(u) w_1 = f_i^\pm(u) w_1$ and $\phi_i^\pm(u) v_{n_1} = g_i^\pm(u) v_{n_1}$. By Lemma (1.2.2), we have

$$f_i^\pm(u)\ (\phi_i^\pm(u) v_j) \otimes w_1 = \phi_i^\pm(u)(v_j \otimes w_1) \in \mathbb{C}[[u^{\pm 1}]]\ v_j \otimes w_1,$$

$i \in I$, $j = 1, \ldots, n_1$, and

$$g_i^\pm(u)\ v_{n_1} \otimes (\phi_i^\pm(u) w_j) = \phi_i^\pm(u)(v_{n_1} \otimes w_j) \in \mathbb{C}[[u^{\pm 1}]]\ v_{n_1} \otimes w_j,$$

$i \in I$, $j = 1, \ldots, n_2$.

The lemma follows. $\square$

The converse statements of Lemma 5.2.3 and Lemma 5.2.4 are false. For example, a two dimensional irreducible evaluation module of $U_q(\widetilde{\mathfrak{sl}}_2)$ is tame and thin, but its tensor square is not tame nor thin.

The statements similar to Lemma 5.2.3 and Lemma 5.2.4, for tensor products with more than two factors easily follow by induction.

It is known that for all $\boldsymbol{\pi} \in \mathcal{P}^+$, the module $L_q(\boldsymbol{\pi})$ can be written as a tensor product $L_q(\boldsymbol{\pi}) = \otimes_{a \in \mathbb{C}^*} L_q(\boldsymbol{\pi}_a)$ where each $L_q(\boldsymbol{\pi}_a)$ is a module such that

$$\chi_q(L_q(\boldsymbol{\pi}_a)) \in \mathbb{Z}[Y_{i,aq^k}^{\pm 1}]_{(i,k) \in \mathcal{X}}. \tag{5.2.1}$$

Remark 5.2.5. By Lemma 5.2.3 and (5.2.1), $L_q(\boldsymbol{\pi})$ is thin if and only if each $L_q(\boldsymbol{\pi}_a)$ is thin. Therefore, to classify all tame modules, it is sufficient to classify all tame modules for modules $L_q(\boldsymbol{\pi}_a)$ satisfying (5.2.1) and to show that these modules are thin. Then it follows that $L_q(\boldsymbol{\pi})$ is tame if and only if all $L_q(\boldsymbol{\pi}_a)$ are tame.

5.3 Thin special q-characters and tame modules

5.3.1 Thin $U_q(\widetilde{\mathfrak{sl}}_2)$-modules

In this subsection we assume $\mathfrak{g} = \mathfrak{sl}_2$ and let $I = \{1\}$.

Recall the definition of $\boldsymbol{\omega}_{i,a,k}$, $i \in I, a \in \mathbb{F}^\times, k \in \mathbb{Z}_{\geq 1}$, (see (3.5.2)). The set $S_k(a) := \{aq^{-k+1}, aq^{-k+3}, \ldots, aq^{k-1}\}$ is called the *q-string* of length $k \in \mathbb{Z}_{\geq 0}$ centered on $a \in \mathbb{F}^\times$. Two q-strings are said to be in *general position* if one contains the other or their union is not a q-string. For each $\boldsymbol{\pi} \in \mathcal{P}^+$ there is a unique multiset of q-strings in pairwise general position, denoted by $\mathbf{S}_q(\boldsymbol{\pi})$, such that

$$\mathbf{S}_q(\boldsymbol{\pi}) = \{S_{k_t}(b_t) | 1 \leq t \leq m\} \quad \text{and} \quad \boldsymbol{\pi} = \prod_{t=1}^{m} \boldsymbol{\omega}_{1,b_t,k_t}, \tag{5.3.1}$$

for some $m \in \mathbb{Z}_{\geq 1}$. Moreover,

$$L_q(\boldsymbol{\pi}) \cong \bigotimes_{t=1}^{m} L_q(\boldsymbol{\omega}_{1,b_t,k_t}),$$

where $L_q(\boldsymbol{\omega}_{1,a,k})$ is an evaluation module, and

$$\chi_q(L_q(\boldsymbol{\omega}_{1,a,k})) = \boldsymbol{\omega}_{1,a,k} \left(1 + \sum_{t=0}^{k-1} A^{-1}_{1,aq^k} A^{-1}_{1,aq^{k-2}} \ldots A^{-1}_{1,aq^{k-2t}} \right). \tag{5.3.2}$$

The module $L_q(\boldsymbol{\pi})$ is a thin simple finite-dimensional representation if and only if each two q-strings in $\mathbf{S}_q(\boldsymbol{\pi})$ are in pairwise position and pairwise disjoint, see [NT98]. Moreover, we have the following.

Lemma 5.3.1 ([NT98, Theorem 4.1])**.** *Let V be a finite-dimensional simple $U_q(\widehat{\mathfrak{sl}}_2)$-module. Then the V is tame if and only if V is thin. All thin modules are special.* ☐

The following useful lemma describes the ℓ-weights which can be found in a simple thin module.

Lemma 5.3.2 ([MY12, Lemma 3.1])**.** *Let $\boldsymbol{\gamma} \in \mathcal{P}$. There exists $\boldsymbol{\pi} \in \mathcal{P}^+$ such that $L_q(\boldsymbol{\pi})$ is thin and $\boldsymbol{\gamma} \in \mathcal{M}(L_q(\boldsymbol{\pi}))$ if and only if, for all $a \in \mathbb{C}^*$, $|u_{1,a}(\boldsymbol{\gamma})| \leq 1$ and $u_{1,a}(\boldsymbol{\gamma}) - u_{1,aq^2}(\boldsymbol{\gamma}) \neq 2$. Moreover, $\boldsymbol{\gamma} A^{-1}_{1,aq}$ is a ℓ-weight of $L_q(\boldsymbol{\pi})$ if and only if $u_{1,a}(\boldsymbol{\gamma}) = 1$ and $u_{1,aq^2}(\boldsymbol{\gamma}) = 0$.* ☐

Remark 5.3.3. It follows from the previous discussions that if V is a thin $U_q(\widetilde{\mathfrak{g}})$-module such that $\mathcal{M}(V) \subseteq \mathbb{Z}[Y^{\pm 1}_{1,aq^{2k}}]_{k \in Z}$, for some $a \in \mathbb{C}^\times$, and $\boldsymbol{\varpi} \in \mathcal{M}(V)$, then all ℓ-weight of V can be obtained from $\boldsymbol{\varpi}$ by several applications of Lemma 5.3.2.

5.3.2 Sufficient criteria for a correct thin special q-character.

Given a set of ℓ-weights $\mathcal{M}$, the followig result gives sufficient conditions to guarantee it corresponds to the q-character of a thin special finite-dimensional $U_q(\widetilde{\mathfrak{g}})$-module.

Theorem 5.3.4 ([MY12, Theorem 3.4]). *Let $\boldsymbol{\pi} \in \mathcal{P}^+$. Suppose that $\mathcal{M} \subseteq \mathcal{P}$ is a finite set of distinct ℓ-weights such that*

(i) $\{\boldsymbol{\pi}\} = \mathcal{P}^+ \cap \mathcal{M}$,

(ii) for all $\boldsymbol{\omega} \in \mathcal{M}$ and all $(i,a) \in I \times \mathbb{C}^*$, if $\boldsymbol{\omega} A_{i,a}^{-1} \notin \mathcal{M}$ then $\boldsymbol{\omega} A_{i,a}^{-1} A_{j,b} \notin \mathcal{M}$ unless $(j,b) = (i,a)$,

(iii) for all $\boldsymbol{\omega} \in \mathcal{M}$ and all $i \in I$ there exists $\boldsymbol{\varpi} \in \mathcal{M}$, i-dominant, such that

$$\chi_q(L_q(\beta_i(\boldsymbol{\varpi}))) = \sum_{\boldsymbol{\gamma} \in \boldsymbol{\omega}\mathbb{Z}[A_{i,a}^{\pm 1}]_{a \in \mathbb{C}^*} \cap \mathcal{M}} \beta_i(\boldsymbol{\gamma}).$$

Then

$$\chi_q(L_q(\boldsymbol{\pi})) = \sum_{\boldsymbol{\omega} \in \mathcal{M}} \boldsymbol{\omega},$$

and in particular $L_q(\boldsymbol{\pi})$ is thin and special.

We use this theorem below to compute q-characters of all thin irreducible modules for types A and B.

5.3.3 The $\mathfrak{sl}_{n+1}$ case.

In this subsection we assume $\mathfrak{g} = \mathfrak{sl}_{n+1}$. We recall well–known results about tame simple representations of $U_q(\widetilde{\mathfrak{g}})$.

A point $(i', k') \in \mathcal{X}$ is said to be in *snake position* to $(i,k) \in \mathcal{X}$ if

$$k' - k \geq 2 + |i' - i|.$$

A sequence of points $(i_t, k_t) \in \mathcal{X}$, $1 \leq t \leq T$, $T \in \mathbb{Z}_{\geq 1}$ is called a *snake* if (i_t, k_t) is in snake position to (i_{t-1}, k_{t-1}), for all $2 \leq t \leq T$. The following is essentially a result of [NT98], see also [MY12, Theorem 6.1].

Theorem 5.3.5 ([NT98, Theorem 4.1]). *Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$. Let $\boldsymbol{\pi} \in \mathcal{P}^+_{\mathcal{X}}$ and let $\mathcal{X}(\boldsymbol{\pi})$ be a snake. Then $L_q(\boldsymbol{\pi})$ is special, thin and therefore tame.* □

This theorem can be proved by explicit computation of the q-character. The converse statement is also essentially [NT98, Theorem 4.1]. We give a proof to illustrate methods we use in the proof of Theorem 5.1.3.

Theorem 5.3.6 ([NT98, Theorem 4.1]). *Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$. Let $\boldsymbol{\pi} \in \mathcal{P}^+_{\mathcal{X}}$. Then $L_q(\boldsymbol{\pi})$ is a tame representation if and only if $\mathcal{X}(\boldsymbol{\pi})$ is a snake.*

Proof. The *if* part follows from Theorem 5.3.5. For the *only if* part we proceed by induction on n which begins in $n = 2$, by Lemma 5.3.1.

Write $\boldsymbol{\pi} = \prod_{t=1}^{T} Y_{i_t,k_t}$, where $T \in \mathbb{Z}_{\geq 0}$, $(i_t, k_t) \in \mathcal{X}$, $t = 1, \ldots, T$. For simplicity write $V = L_q(\boldsymbol{\pi})$. Let $J = \{1, 2, \ldots, n-1\}$ and $K = \{2, 3, \ldots, n\}$ be subsets of I. By Lemma 5.2.1, the $U_q(\widetilde{\mathfrak{g}}_J)$-module $L_q(\beta_J(\boldsymbol{\pi}))$ is tame. Note that $U_q(\widetilde{\mathfrak{g}}_J)$ is isomorphic to $U_q(\tilde{\mathfrak{sl}}_{n-1})$. Therefore by the induction hypothesis,

$$k_{t+1} - k_t \geq 2 + |i_{t+1} - i_t|,$$

whenever $N \notin \{i_t, i_{t+1}\}$.

Similar arguments applied to $\mathrm{res}_K V$ show that

$$k_{t+1} - k_t \geq 2 + |i_{t+1} - i_t|,$$

whenever $1 \notin \{i_t, i_{t+1}\}$.

Therefore it remains to consider the case when $\{i_t, i_{t+1}\} = \{1, n\}$. Suppose, by contradiction, $0 \leq k_{t+1} - k_t < 2 + |n-1|$. By the definition of the set $\mathcal{X}$, this is equivalent to $k_{t+1} - k_t \leq n - 1$. By Lemma 3.3.3, we have

$$m = m_+ A^{-1}_{i_t, k_t+1} \in \chi_q(V).$$

Explicitly,

$$\boldsymbol{\omega} = \begin{cases} \boldsymbol{\pi} Y^{-1}_{1,k_t} Y^{-1}_{1,k_t+2} Y_{2,k_t+1} & \text{if } i_t = 1 \text{ and } i_{t+1} = n \\ \boldsymbol{\pi} Y^{-1}_{n,k_t} Y^{-1}_{n,k_t+2} Y_{n-1,k_t+1} & \text{if } i_t = n \text{ and } i_{t+1} = 1. \end{cases}$$

In the first case, by Lemma 3.3.4, $L_q(\beta_J(\boldsymbol{\omega}))$ is a subfactor of $\mathrm{res}_J V$. However,

$$k_{t+1} - (k_t + 1) \leq |n-1| - 1 = |n-2|,$$

yields a contradiction with the inductive hypothesis. A similar argument proves that the second case cannot hold either. □

Since all simple tame $U_q(\widetilde{\mathfrak{sl}}_n)$-modules are special, the following is immediate from the previous theorem.

Corollary 5.3.7. *Let V be a tame $U_q(\widetilde{\mathfrak{sl}}_{n+1})$-module. Then, for each dominant ℓ-weight $\boldsymbol{\pi} \in \mathcal{M}(V)$, $L_q(\boldsymbol{\pi})$ is a subfactor of V.* □

5.4 Proof of Theorem 5.1.3

Assume that $\mathfrak{g}$ is of type B_n and recall the definition of extended snake (Definition 5.1.1). We observe that if we drop the condition (ii), we have the definition of *snake position* given in [MY12]. A point $(i, k) \in \mathcal{X}$ is said to be in *minimal snake position* to (i', k') if it is in snake position and $k - k'$ is equal to the given lower bound in (i).

We extend the definition of snakes to include all cases where the q-character formula of Theorem 6.1 in [MY12] pertains, see Theorem 6.4.8.

Let (i_t, k_t), $1 \leq t \leq T$, $T \in \mathbb{Z}_{\geq 1}$, be a sequence of points in $\mathcal{X}$. The sequence $(i_t, k_t)_{1\leq t\leq T}$ is said to be a *snake* (resp. *minimal snake*) if (i_t, k_t) is in snake (resp. *minimal snake*) position to (i_{t-1}, k_{t-1}) for all $2 \leq t \leq T$. We call the simple module $L_q(\boldsymbol{\pi})$ an *snake module* (resp. minimal snake module) if $\mathcal{X}(\boldsymbol{\pi})$ is a snake (resp. minimal snake).

Observe that the minimal affinizations (see Section 3.5) are minimal snake modules where the sequence of i_t, $t = 1, \ldots, T$, is monotone. Finally recall that the *Kirillov-Reshetikhin modules* are minimal affinizations where the sequence i_t, $t = 1, \ldots, T$, is constant. Therefore we have the following families of representations, in order of increasing generality:

$$\text{KR modules} \subset \text{minimal affinizations} \subset \text{minimal snakes} \subset \text{snakes} \subset \text{extended snakes.}$$

Note also that, in the definition of extended snake position, there is no upper bound on the gap $k' - k$. Suppose $(i_t, k_t)_{1\leq t\leq T} \subseteq \mathcal{X}$ is an extended snake and $k_{s+1} - k_s$ is sufficiently large for some s. Then we have a tensor product decomposition: $L_q(\prod_{t=1}^{T} Y_{i_t,k_t}) \cong L_q(\prod_{t=1}^{s} Y_{i_t,k_t}) \otimes L_q(\prod_{t=s+1}^{T} Y_{i_t,k_t})$, see Corollary 6.4.9 bellow.

Remark 5.4.1. The extended snake position is not a transitive concept. Namely, if (i', k') is in extended snake position to (i, k) and (i'', k'') is in extended snake position to (i', k'), then it does not follow in general that (i'', k'') is in snake position to (i, k). However, in the cases when not, we necessarily have $i' = n$, $i + i'' > n - 2$, (i', k') is in snake position to (i, k) and (i'', k'') is in snake position to (i', k').

Assume Theorem 5.1.2 for a moment and let us prove Theorem 5.1.3. We first prove the case $n = 2$.

Proposition 5.4.2. *Let $\mathfrak{g}$ be of type B_2 and let $\boldsymbol{\pi} \in \mathcal{P}^+_{\mathcal{X}}$. The module $L_q(\boldsymbol{\pi})$ is tame if and only if $\mathcal{X}(\boldsymbol{\pi})$ is an extended snake.*

Proof. The *if* part follows by Theorem 5.1.2. We prove the *only if* part. Let $V = L_q(\boldsymbol{\pi})$ be tame. Let $\mathcal{X}(\boldsymbol{\pi}) = (i_t, k_t)_{1\leq t\leq T}$, $T \in \mathbb{Z}_{\geq 1}$, $(i_t, k_t) \in \mathcal{X}$, $1 \leq t \leq T$.

By Lemma 5.2.1, the $U_q(\widetilde{\mathfrak{g}}_j)$-module $L_q(\beta_j(\boldsymbol{\pi}))$ is tame, $j = 1, 2$. Note that $U_q(\widetilde{\mathfrak{g}}_j)$ is isomorphic to $U_{q_j}(\widetilde{\mathfrak{sl}}_2^{(j)})$. Therefore, by Theorem 5.3.6, $k_{t+1} \neq k_t$, whenever $i_{t+1} = i_t$, $1 \leq t \leq T - 1$. Hence, it remains to discard the cases

(i) $i_t = i_{t+1} = 1$ and $k_{t+1} - k_t = 2$,

(ii) $i_t = i_{t+1} = 2$ and $k_{t+1} - k_t = 4$,

(iii) $i_t \neq i_{t+1}$ and $k_{t+1} - k_t \in \{1, 3\}$.

where $1 \leq t \leq T-1$.

Suppose, by contradiction, that t is maximal such that one of the above conditions holds. In each case, using Lemma 3.3.3 and (5.3.2), we find an ℓ-weight $\boldsymbol{\varpi} \in \chi_q(V)$ such that $\boldsymbol{\varpi}$ is j-dominant and $u_{j,l}(\boldsymbol{\varpi}) \geq 2$ for some $(j,l) \in \mathcal{X}$. By Corollary 5.3.7, $L_q(\beta_j(\boldsymbol{\omega}))$ is a subfactor of $\mathrm{res}_j V$ and not tame, by Theorem 5.3.6, which yields a contradiction with Lemma 5.2.1.

Suppose that (i) holds. Let $r \in \mathbb{Z}_{\geq 0}$ be maximal such that $u_{1,k_{t+1}+4j}(\boldsymbol{\pi}) > 0$, for all $0 \leq j \leq r$. Then

$$\boldsymbol{\varpi} = \boldsymbol{\pi} A^{-1}_{1,k_t+2} \left(\prod_{j=0}^{r} A^{-1}_{1,k_{t+1}+4(r-j)+2} \right) \in \chi_q(V).$$

One easily checks using (3.3.1) that $\boldsymbol{\varpi}$ is 2-dominant and $u_{2,k_t+3}(\boldsymbol{\varpi}) \geq 2$.

Suppose that (ii) holds. Then

$$\boldsymbol{\varpi} = \boldsymbol{\pi} A^{-1}_{2,k_t+1} A^{-1}_{1,k_t+3} \in \chi_q(V),$$

where $\boldsymbol{\varpi}$ is 2-dominant and $u_{2,k_t+4}(\boldsymbol{\varpi}) \geq 2$.

Suppose that (iii) holds. Set

$$\boldsymbol{\varpi} = \boldsymbol{\pi} A^{-1}_{i_t,k_t+d_{i_t}} \in \chi_q(V).$$

Explicitly,

$$\boldsymbol{\varpi} = \begin{cases} \boldsymbol{\pi} Y^{-1}_{1,k_t} Y^{-1}_{1,k_t+4} Y_{2,k_t+1} Y_{2,k_t+3} & \text{if} \quad i_t = 1, \\ \boldsymbol{\pi} Y^{-1}_{2,k_t} Y^{-1}_{2,k_t+2} Y_{1,k_t+1} & \text{if} \quad i_t = 2. \end{cases}$$

If $i_t = 1$, then $\boldsymbol{\varpi}$ is 2-dominant and either $u_{2,k_t+1}(\boldsymbol{\varpi}) \geq 2$ or $u_{2,k_t+3}(\boldsymbol{\varpi}) \geq 2$. If $i_t = 2$ and $k_{t+1} = k_t + 1$ then $\boldsymbol{\varpi}$ is 1-dominant and $u_{1,k_t+1}(\boldsymbol{\varpi}) \geq 2$. Therefore, it remains to consider the case when $i_t = 2$ and $k_{t+1} = k_t + 3$. Let $r \in \mathbb{Z}_{\geq 0}$ be maximal such that $u_{1,k_{t+1}+4j}(\boldsymbol{\varpi}) > 0$, for all $0 \leq j \leq r$. Then

$$\boldsymbol{\omega} = \boldsymbol{\varpi} A^{-1}_{1,k+3} \left(\prod_{j=0}^{r} A^{-1}_{1,k_{t+1}+4(r-j)+2} \right) \in \chi_q(V),$$

$\boldsymbol{\omega}$ is 2-dominant and $u_{2,k_{t+1}+1}(\boldsymbol{\omega}) \geq 2$. □

We now prove Theorem 5.1.3 by induction on n, beginning on $n = 2$, which is Proposition 5.4.2.

Assume $n \geq 3$. Let $V = L_q(\boldsymbol{\pi})$ and $\mathcal{X}(\boldsymbol{\pi}) = (i_t, k_t)_{1 \leq t \leq T}$, $T \in \mathbb{Z}_{\geq 1}$.

Set $J_1 = \{2, 3, \ldots, n\}$ and $J_2 = \{1, 2, \ldots, n-1\}$, subsets of I, and consider the subalgebras $U_q(\widetilde{\mathfrak{g}}_{J_1})$ and $U_q(\widetilde{\mathfrak{g}}_{J_2})$ of $U_q(\widetilde{\mathfrak{g}})$. These subalgebras are isomorphic to quantum affine algebras of type B_{n-1} and A_{n-1}, respectively.

By Lemma 5.2.1, $L_q(\beta_{J_1}(\boldsymbol{\pi}))$ is a subfactor of $\mathrm{res}_{J_1} V$, and by induction hypothesis,

$$k_{t+1} - k_t \geq 4 + 2|i_{t+1} - i_t| - \delta_{n,i_{t+1}} - \delta_{n,i_t} \quad \text{if} \quad k_{t+1} - k_t \equiv 2|i_{t+1} - i_t| - \delta_{n,i_{t+1}} - \delta_{n,i_t} \mod 4, \quad (5.4.1)$$

or

$$k_{t+1} - k_t \geq 2n + 2|n - i_{t+1} - i_t + 1| - \delta_{n,i_{t+1}} - \delta_{n,i_t} \tag{5.4.2}$$

for all t, $1 \leq t \leq T-1$, such that $1 \notin \{i_t, i_{t+1}\}$.

Similarly, $L_q(\beta_{J_2}(\boldsymbol{\pi}))$ is a subfactor of $\mathrm{res}_{J_2} V$, hence, by Theorem 5.3.6,

$$k_{t+1} - k_t \geq 4 + 2|i_{t+1} - i_t| - \delta_{n,i_{t+1}} - \delta_{n,i_t} \quad \text{if} \quad k_{t+1} - k_t \equiv 2|i_{t+1} - i_t| - \delta_{n,i_{t+1}} - \delta_{n,i_t} \mod 4, \tag{5.4.3}$$

for all t, $1 \leq t \leq T-1$, such that $n \notin \{i_t, i_{t+1}\}$.

Suppose, by contradiction, that (i_{t+1}, k_{t+1}) is not in extended snake position to (i_t, k_t), for some t, $1 \leq t \leq T-1$. Without loss of generality, we can assume that (i_{s+1}, k_{s+1}) is in extended snake position to (i_s, k_s), for $t < s \leq T-1$. For convenience, denote

$$(i, k) = (i_t, k_t) \quad \text{and} \quad (i', k') = (i_{t+1}, k_{t+1}).$$

We divide the argument in the following two cases.

(i) $k' - k \equiv 2|i' - i| - \delta_{n,i'} - \delta_{n,i} \mod 4$,

(ii) $k' - k \equiv 2 + 2|i' - i| - \delta_{n,i'} - \delta_{n,i} \mod 4$.

In each case, using Lemma 3.3.3 and (5.3.2), we find a J_ℓ-dominant ℓ-weight $\boldsymbol{\varpi} \in \chi_q(V)$, for some $\ell = 1, 2$, and consecutive points $(j, l), (j', l') \in \mathcal{X}(\beta_{J_\ell}(\boldsymbol{\varpi}))$, $l' \geq l$, such that one of the following holds:

- $\ell = 1$ and (j', l') is not in extended snake position to (j, l) with respect to the algebra $U_q(\widetilde{\mathfrak{g}}_{J_1})$,
- $\ell = 2$ and (j', l') is not in snake position to (j, l) with respect to the algebra $U_q(\widetilde{\mathfrak{g}}_{J_2})$.

By induction hypothesis and Corollary 5.1.4, if $\ell = 1$, and by Theorem 5.3.6 and Corollary 5.3.7, if $\ell = 2$, $L_q(\beta_{J_\ell})$ is a subfactor of $\mathrm{res}_{J_\ell} V$ which is not tame, thus a contradiction with Lemma 5.2.1.

Let (i) hold. By (5.4.1) and (5.4.3), it remains to consider the case $\{i, i'\} = \{1, n\}$. Our assumption implies $0 \leq k' - k < 2 + 2|n-1|$. By the definition of $\mathcal{X}$, this is equivalent to

$$1 \leq k' - k \leq 2n - 3.$$

Therefore,

$$\boldsymbol{\varpi} = \boldsymbol{\pi} A^{-1}_{i,k+d_i} \in \chi_q(V).$$

If $i = 1$, then $\boldsymbol{\varpi}$ is J_1-dominant and let $(j, l) = (2, k+2)$ and $(j', l') = (n, k')$. If $i = n$, then $\boldsymbol{\varpi}$ is J_2-dominant and let $(j, l) = (n-1, k+1)$ and $(j', k') = (1, k')$. In each case we have $u_{j,l}(\boldsymbol{\varpi}) = 1$, $u_{j',l'}(\boldsymbol{\varpi}) = 1$ and

$$-1 \leq l' - l \leq 2|j' - j|.$$

This finishes the case (i).

Let (ii) hold. Our assumption implies

$$0 \le k' - k < 2n + 2 + 2|n - i - i'| - \delta_{n,i} - \delta_{n,i'}.$$

By the definition of $\mathcal{X}$ this is equivalent to

$$0 \le k' - k \le 2n - 2 + 2|n - i' - i| - \delta_{n,i'} - \delta_{n,i}. \tag{5.4.4}$$

We split the argument further in the following subcases:

a $i' \ne 1$ and $i = 1$,

b $i' \ne 1$ and $i \ne 1$,

c $i' = 1$.

Consider the case **a**. Set

$$\boldsymbol{\varpi} = \boldsymbol{\pi} A_{1,k+2}^{-1} \in \chi_q(V).$$

Let $(j,l) = (2, k+2)$ and $(j', l') = (i', k')$. Then $\boldsymbol{\varpi}$ is J_1-dominant, $u_{j,l}(\boldsymbol{\varpi}) = 1$, $u_{j',l'}(\boldsymbol{\varpi}) = 1$. Moreover, (5.4.4) implies

$$-2 \le l' - l \le 2n - 4 + 2|n - j' - j + 1| - \delta_{n,j'},$$

completing the proof in this subcase.

Consider the case **b**. By (5.4.2) and (5.4.4) it follows that

$$2n + 2|n - i' - i + 1| - \delta_{n,i'} - \delta_{n,i} \le k' - k \le 2n - 2 + 2|n - i' - i| - \delta_{n,i'} - \delta_{n,i}. \tag{5.4.5}$$

If $n - i' - i \ge 0$, there is no k and k' satisfying (5.4.5). On the other hand, if $n - i' - i < 0$, then (5.4.5) is equivalent to

$$k' - k = 2i' + 2i - 2 - \delta_{n,i'} - \delta_{n,i}. \tag{5.4.6}$$

In particular, (5.4.6) implies

$$\boldsymbol{\varpi} = \boldsymbol{\pi} \prod_{j=0}^{i-1} A_{i-j,k+d_i+2j}^{-1} \in \chi_q(V),$$

Explicitly, by (3.3.1),

$$\boldsymbol{\varpi} = \begin{cases} \boldsymbol{\pi} Y_{i,k}^{-1} Y_{i+1,k+2} Y_{1,k+2+2i}^{-1} & \text{if} \quad i \le n-2, \\ \boldsymbol{\pi} Y_{n-1,k}^{-1} Y_{n,k+1} Y_{n,k+3} Y_{1,k+2+2i}^{-1} & \text{if} \quad i = n-1, \\ \boldsymbol{\pi} Y_{n,k}^{-1} Y_{n,k+4} Y_{1,k+1+2i}^{-1} & \text{if} \quad i = n. \end{cases} \tag{5.4.7}$$

Let

$$(j,l) = \begin{cases} (i+1, k+2+\delta_{n,i+1}) & \text{if } i < n, \\ (n, k+4) & \text{if } i = n, \end{cases} \quad \text{and} \quad (j', l') = (i', k').$$

One readily checks that $\boldsymbol{\varpi}$ is J_1-dominant, $u_{j,l}(\boldsymbol{\varpi}) = 1$, $u_{j',l'}(\boldsymbol{\varpi}) = 1$ and, by (5.4.6), it follows that

$$l' - l = 2n - 4 + 2|n - j' - j + 1| - \delta_{n,j'} - \delta_{n,j},$$

finishing the proof in this subcase.

Consider the case **c**. Let $r \in \mathbb{Z}_{\geq 0}$ be maximal such that $(1, k' + 4j) \in \mathcal{X}(\boldsymbol{\pi})$, for all $0 \leq j \leq r$. Then

$$\boldsymbol{\omega} = \begin{cases} \boldsymbol{\pi} \prod_{j=0}^{r} A^{-1}_{1,k'+4(r-j)+2} \in \chi_q(V) & \text{if } i > 1, \\ \boldsymbol{\pi} A^{-1}_{1,k+2} \prod_{j=0}^{r} A^{-1}_{1,k'+4(r-j)+2} \in \chi_q(V) & \text{if } i = 1. \end{cases} \tag{5.4.8}$$

Let

$$(j, l) = \begin{cases} (i, k) & \text{if } i > 1, \\ (2, k+2) & \text{if } i = 1, \end{cases} \quad \text{and} \quad (j', l') = (2, k' + 2).$$

One easily checks that $\boldsymbol{\omega}$ is J_1-dominant, $u_{j,l}(\boldsymbol{\omega}) = 1$ and $u_{j',l'}(\boldsymbol{\omega}) = 1$. By induction hypothesis, it follows that

$$l' - l \geq 2n + 2|n - j' - j + 1| - \delta_{n,j}.$$

Equivalently,

$$\begin{cases} k' - k \geq 2n - 2 + 2|n - i - 2 + 1| - \delta_{n,i} & \text{if } i > 1, \\ k' - k \geq 4n - 6 & \text{if } i = 1. \end{cases} \tag{5.4.9}$$

Relations (5.4.4) and (5.4.9) together are equivalent to

$$k' - k = 2n - 2 + 2|n - i - 1| - \delta_{n,i}. \tag{5.4.10}$$

If $i < n$, (5.4.10) implies

$$\boldsymbol{\varpi} = \boldsymbol{\pi} \left(\prod_{j=0}^{n-i} A^{-1}_{i+j,k+2+2j} \right) A^{-1}_{n,k+2(n-1-i)} \in \chi_q(V).$$

Explicitly,

$$\boldsymbol{\varpi} = \boldsymbol{\pi} Y^{-1}_{i,k} Y_{i-1,k+2} Y_{n-1,k+2(n-1-i)+2} Y^{-1}_{n,k+2(n-1-i)+3} Y^{-1}_{n,k+2(n-1-i)+1}.$$

Let $(j, l) = (n - 1, k + 2(n - i))$ and $(j', l') = (1, k')$. Then $\boldsymbol{\varpi}$ is J_2-dominant, $u_{j,l}(\boldsymbol{\varpi}) = 1$, $u_{j',l'}(\boldsymbol{\varpi}) = 1$ and

$$l' - l = 2|j' - j|.$$

If $i = n$, (5.4.10) implies

$$\boldsymbol{\varpi} = \boldsymbol{\pi} \prod_{j=0}^{n-1} A^{-1}_{n-j,k+1+2j} \prod_{j=0}^{r} A^{-1}_{1,k'+4(r-j)+2} \in \chi_q(V).$$

Let $(j,l)=(n,k{+}4)$ and $(j',l')=(2,k'{+}2)$. One easily checks using (3.3.1) that $\boldsymbol{\varpi}$ is J_1-dominant, $u_{j,l}(\boldsymbol{\varpi})=1$, $u_{j',l'}(\boldsymbol{\varpi})=1$. Moreover, (5.4.10) implies

$$l'-l=2n-4+2|n-j'-j+1|-1.$$

This completes the proof in this subcase. The proof of Theorem 5.1.3 is finished. $\square$

Chapter 6

Combinatorics of paths and moves

In this chapter we recall the combinatorial system of *non-overlapping paths*, introduced in [MY12, Section 5] and extend their results to one which our extended snakes will pertain. Then, we use such results to prove Theorem 6.4.8 bellow, which proves, in particular, Theorem 5.1.2 and gives a closed formulae for the q-character of extended snake modules in terms of non-overlapping paths.

Assume the notations and definitions given in the previous chapters. We draw the images of points in $\mathcal{X}$ under the injective map $\iota : \mathcal{X} \to \mathbb{Z} \times \mathbb{Z}$ defined as follows

$$\iota : (i,k) \mapsto \begin{cases} (2i,k), & \text{if } i < n \text{ and } 2n+k-2i \equiv 2 \mod 4 \\ (4n-2-2i,k), & \text{if } i < n \text{ and } 2n+k-2i \equiv 0 \mod 4 \\ (2n-1,k), & \text{if } i = n. \end{cases} \tag{6.0.1}$$

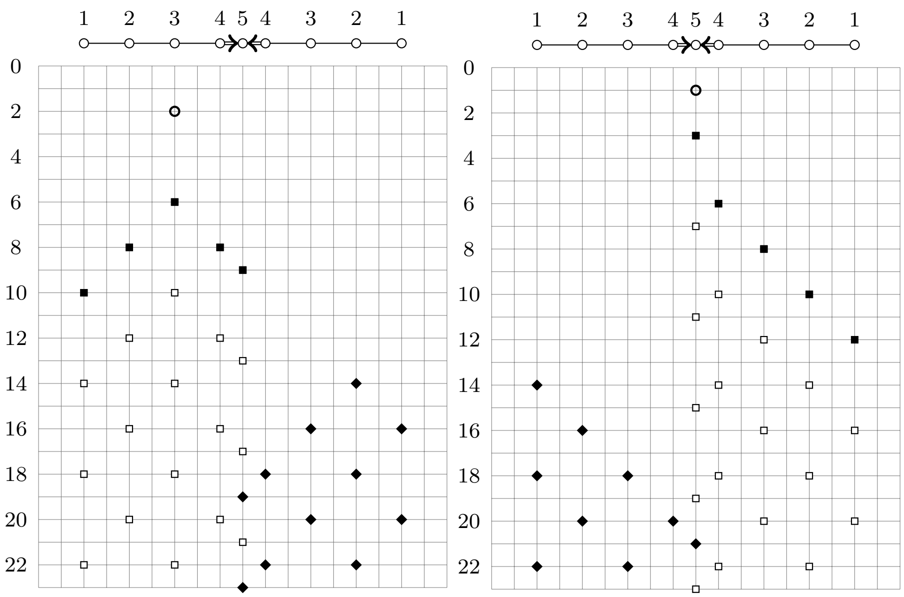
6.1 Paths and corners

A *path* p is a finite sequence of points in the plane $\mathbb{R}^2$. We write $(i,k) \in p$, if (i,k) is a point of the path p. In our diagrams, we connect consecutive points of the path by line segments, for illustrative purposes only. For each $(i,k) \in \mathcal{X}$, we define a set $\mathscr{P}_{i,k}$ of paths. Pick and fix an ϵ, $0 < \epsilon < 1/2$. Define $\mathscr{P}_{n,k}$ for all $k \in 2\mathbb{Z}+1$ in the following way

- For all $k \equiv 3 \mod 4$,

$$\begin{aligned} \mathscr{P}_{n,k} := \{ & ((0,y_0),(2,y_1),\ldots,(2n-4,y_{n-2}),(2n-2,y_{n-1}),(2n-1,y_n)) \mid \\ & y_0 = k+2n-1, y_{j+1}-y_j \in \{2,-2\} \text{ for all } 0 \le j \le n-2 \\ & \text{and } y_n - y_{n-1} \in \{1+\epsilon, -1-\epsilon\}\}. \end{aligned}$$

Figure 6.1: Each of the points $\blacklozenge$, $\square$ and $\blacksquare$ are, respectively, in extended snake, snake and minimal snake position to the point marked $\circ$.



- For all $k \equiv 1 \mod 4$,

$$\begin{aligned}\mathscr{P}_{n,k} := \{&((4n-2, y_0), (4n-4, y_1), \ldots, (2n+2, y_{n-2}), (2n, y_{n-1}), (2n-1, y_n)) \mid \\ &y_0 = k + 2n - 1, y_{j+1} - y_j \in \{2, -2\} \text{ for all } 0 \leq j \leq n-2 \\ &\text{and } y_n - y_{n-1} \in \{1+\epsilon, -1-\epsilon\}\}.\end{aligned}$$

Next, $\mathscr{P}_{i,k}$ is defined, for all $(i,k) \in \mathcal{X}$, as follows.

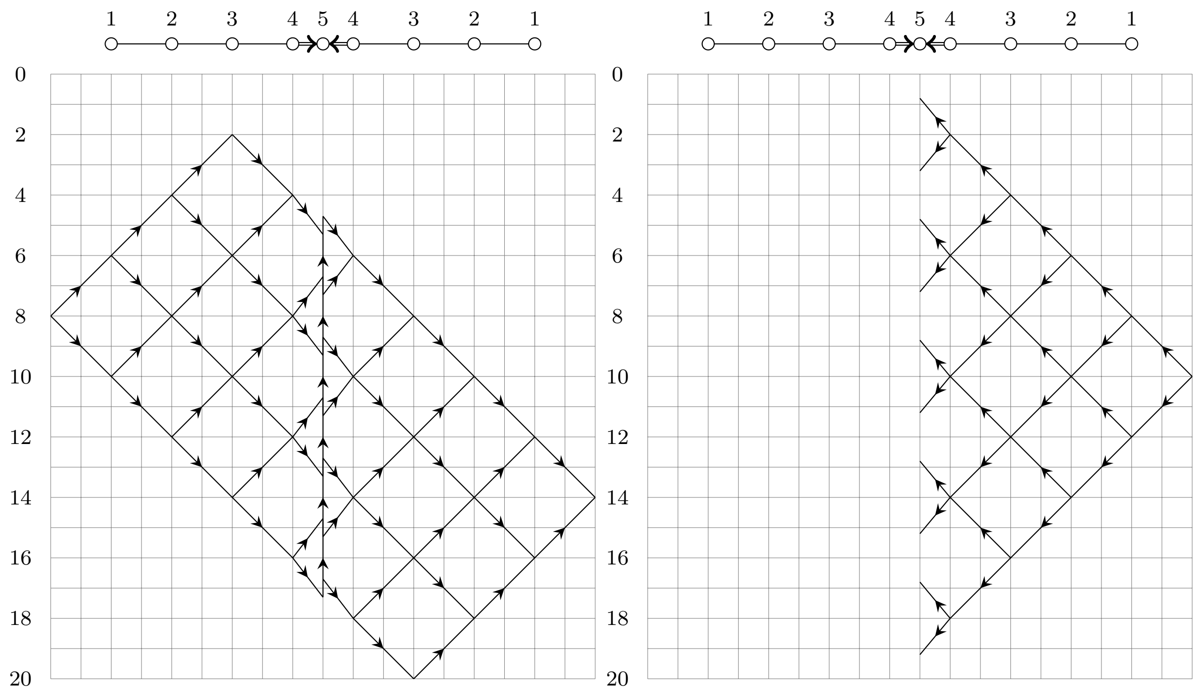
$$\begin{aligned}\mathscr{P}_{i,k} := \{&(a_0, a_1, \ldots, a_n, \overline{a_n}, \ldots, \overline{a_1}, \overline{a_0}) \mid \\ &(a_0, a_1 \ldots, a_n) \in \mathscr{P}_{n,k-(2n-2i-1)}, (\overline{a_0}, \overline{a_1}, \ldots, \overline{a_n}) \in \mathscr{P}_{n,k+(2n-2i-1)}, \\ &\text{and } a_n - \overline{a_n} = (0, y) \text{ where } y > 0\}.\end{aligned}$$

For all $(i,k) \in \mathcal{X}$, we define the sets of *upper* and *lower corners* $C_{p,\pm}$ of a path $p = ((j_r, l_r))_{0 \leq r \leq |p|-1} \in \mathscr{P}_{i,k}$ as follows:

$$\begin{aligned}C_{p,+} := \iota^{-1}&\{(j_r, l_r) \in p | j_r \notin \{0, 2n-1, 4n-2\},\ l_{r-1} > l_r, l_{r+1} > l_r\} \\ &\sqcup \{(n, l) \in \mathcal{X} | (2n-1, l-\epsilon) \in p \text{ and } (2n-1, l+\epsilon) \notin p\},\end{aligned}$$

$$\begin{aligned}C_{p,-} := \iota^{-1}&\{(j_r, l_r) \in p | j_r \notin \{0, 2n-1, 4n-2\},\ l_{r-1} < l_r, l_{r+1} < l_r\} \\ &\sqcup \{(n, l) \in \mathcal{X} | (2n-1, l+\epsilon) \in p \text{ and } (2n-1, l-\epsilon) \notin p\},\end{aligned}$$

Figure 6.2: In type B_5 illustration of the paths in $\mathscr{P}_{3,2}$ (left) and $\mathscr{P}_{5,1}$ (right).



We define a map m sending paths to ℓ-weights, as follows:

$$\begin{aligned}
\mathsf{m}: \bigsqcup_{(i,k)\in\mathcal{X}} \mathscr{P}_{i,k} &\to \mathbb{Z}\left[Y_{j,l}^{\pm 1}\right]_{(j,l)\in\mathcal{X}} \\
p &\mapsto \mathsf{m}(p) := \prod_{(j,l)\in C_{p,+}} Y_{j,l} \prod_{(j,l)\in C_{p,-}} Y_{j,l}^{-1}.
\end{aligned} \tag{6.1.1}$$

For all $(i,k)\in\mathcal{X}$ we define $p_{i,k}^{+}$, the *highest path* to be the unique element of $\mathscr{P}_{i,k}$ with no lower corners. Equivalently, $p_{i,k}^{+}$ is the unique path such that $\iota(i,k)-\delta_{n,i}(0,\epsilon)\in p_{i,k}^{+}$.

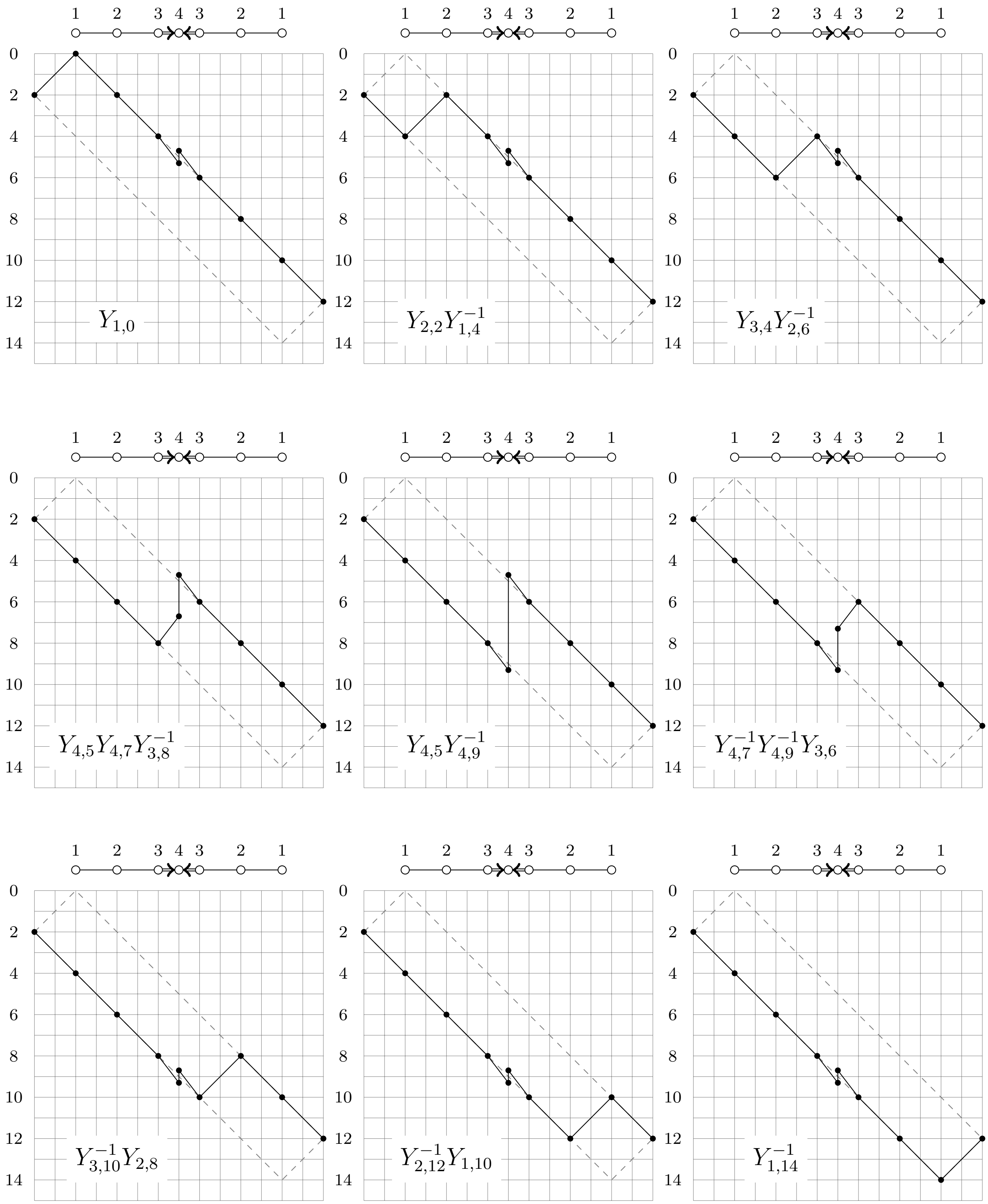
Analogously, we define $p_{i,k}^{-}$, the *lowest path*, to be the unique element of $\mathscr{P}_{i,k}$ with no upper corners. Equivalently, $p_{i,k}^{-}$ is the unique path such that $\iota(i,k+4n-2)+\delta_{n,i}(0,\epsilon)\in p_{i,k}^{-}$.

6.2 Lowering and raising moves

Let $(i,k)\in\mathcal{X}$ and $(j,l)\in\mathcal{W}$. We say a path $p\in\mathscr{P}_{i,k}$ can be *lowered* at (j,l) if $(j,l-d_j)\in C_{p,+}$ and $(j,l+d_j)\notin C_{p,+}$. If so, a new path is defined, called the *lowering move* on p at (j,l) and denoted by $p\mathscr{A}_{j,l}^{-1}$, as follows.

We first define the lowering moves on paths in $\mathscr{P}_{n,k}$, $k\equiv 3 \mod 4$. For any path $p = ((0,y_0),(2,y_1),\dots,(2n-1,y_n))$ in $\mathscr{P}_{k,l}$ we define the lowering moves case-by-case.

Figure 6.3: In type B_4, the paths corresponding to the ℓ-weights of $\chi_q(L_q(Y_{1,0}))$, by Theorem 6.4.8.

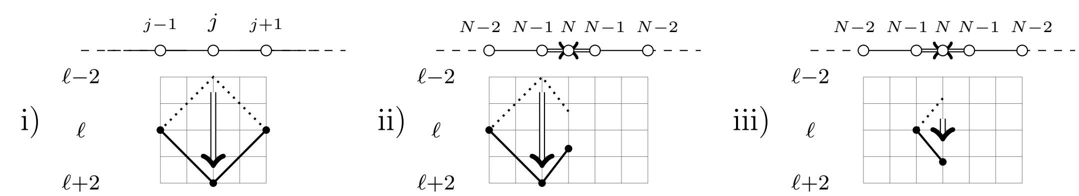


(i) If $(j, l-2) \in C_{p,+}$ for some $j < n-1$, we have $l = y_{j-1} = y_j + 2 = y_{j+1}$ and we define $p\mathscr{A}_{j,l}^{-1} := ((0, y_0), (2, y_1), \ldots, (2j-2, y_{j-1}), (2j, y_j + 4), (2j+2, y_{j+1}), \ldots (2n-2, y_{n-1}), (2n-1, y_n))$.

(ii) If $(n-1, l-2) \in C_{p,+}$, we have $l = y_{n-2} = y_{n-1} + 2 = y_n + 1 - \epsilon$ and we define $p\mathscr{A}_{j,l}^{-1} := ((0, y_0), (2, y_1), \ldots (2n-4, y_{n-2}), (2n-2, y_{n-1} + 4), (2n-1, y_n + 2 - 2\epsilon))$.

(iii) If $(n, l-1) \in C_{p,+}$, we have $l = y_{n-1} = y_n + 1 + \epsilon$ and we define $p\mathscr{A}_{j,l}^{-1} := ((0, y_0), (2, y_1), \ldots, (2n-2, y_{n-1}), (2n-1, y_n + 2 + 2\epsilon))$.

In each case, $p\mathscr{A}_{j,l}^{-1} \in \mathscr{P}_{n,k}$. Pictorially, these moves are



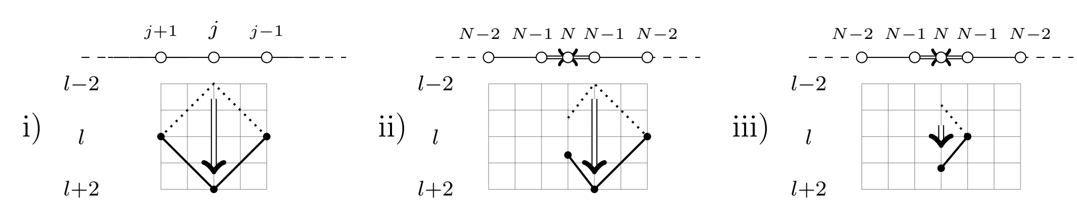
When $k \equiv 1 \mod 4$ we define the lowering moves on $\mathscr{P}_{n,k}$ simply as mirror images of the moves above. Formally, for any path $p = ((4n-2, y_0), (4n-4, y_1), \ldots, (2n, y_{n-1}), (2n-1, y_n))$ in $\mathscr{P}_{n,k}$ we define the lowering move case-by-case

(i) If $(j, l-2) \in C_{p,+}$ for some $j < n-1$, we have $l = y_{j+1} = y_j + 2 = y_{j-1}$ and we define $p\mathscr{A}_{j,l}^{-1} := ((4n-2, y_0), \ldots, (2j+2, y_{j-1}), (2j, y_j + 4), (2j-2, y_{j+1}), \ldots, (2n, y_{n-1}), (2n-1, y_n))$.

(ii) If $(n-1, l-2) \in C_{p,+}$, we have $l = y_n + 1 - \epsilon = y_{n-1} + 2 = y_{n-2}$ and we define $p\mathscr{A}_{j,l}^{-1} := ((4n-2, y_0), (4n-4, y_1), \ldots, (2n+2, y_{n-2}), (2n, y_{n-1} + 4), (2n-1, y_n + 2 - 2\epsilon))$.

(iii) If $(n, l) \in C_{p,+}$, we have $l = y_n + 1 + \epsilon = y_{n-1}$ and we define $p\mathscr{A}_{j,l}^{-1} := ((4n-2, y_0), (4n-4, y_1), \ldots, (2n, y_{n-1}), (2n-1, y_n + 2 + 2\epsilon))$.

In each case $p\mathscr{A}_{j,l}^{-1} \in \mathscr{P}_{i,k}$. Pictorially, these moves are



Finally, for $(i, k) \in \mathcal{X}$ such that $i < n$, we have that every path $p \in \mathscr{P}_{i,k}$ is equivalent to a pair $(a, \overline{a}) \in \mathscr{P}_{n,k-(2n-2i-1)} \times \mathscr{P}_{n,k+(2n-2i-1)}$. Since a and $\overline{a}$ can share no upper corners the set of upper corners of p is a subset of disjoint union of the sets of upper corners of a and $\overline{a}$. Therefore, if p can be lowered at (j, l), either a or $\overline{a}$ can be lowered at (j, l), and we define $p\mathscr{A}_{j,l}^{-1}$ to be the given by whichever of the pairs $(a\mathscr{A}_{j,l}^{-1}, \overline{a})$ and $(a, \overline{a}\mathscr{A}_{j,l}^{-1})$ is defined.

Analogously, given $(i, k) \in \mathcal{X}$ and $(j, l) \in \mathcal{W}$ we say that a path $p \in \mathscr{P}_{i,k}$ can be raised at (j, l) if $p = p'\mathscr{A}_{j,l}^{-1}$ for some $p' \in \mathscr{P}_{i,k}$. If such p' exists it is unique, and we define $p\mathscr{A}_{j,l} := p'$.

6.3 Non-overlapping paths

A path p is said to be *strictly above* a path p', and p' *strictly bellow* p, if

$$(x, y) \in p \quad \text{and} \quad (x, z) \in p' \Rightarrow y < z.$$

If a path p is strictly above a path p', then we also say p' is *strictly bellow* p. A T-tuple of paths $(p_1, \dots, p_T)$ is said *non-overlapping* if p_s is strictly above p_t for all $s < t$. Otherwise, for some $s < t$ there exist $(x, y) \in p_s$ and $(x, z) \in p_t$ such that $y \geq z$, and we say p_s *overlaps* p_t in column x.

Let $(i_t, k_t) \in \mathcal{X}, 1 \leq t \leq T$, be an extended snake. Define

$$\overline{\mathscr{P}}_{(i_t,k_t)_{1\leq t\leq T}} := \{(p_1, \dots, p_T) | p_t \in \mathscr{P}_{i_t,k_t}, 1 \leq t \leq T, (p_1, \dots, p_T) \text{ is non-overlapping}\}.$$

Lemma 6.3.1. *If $(i_t, k_t) \in \mathcal{X}, 1 \leq t \leq T$, $T \in \mathbb{Z}_{\geq 1}$, is an extended snake, then $(p^+_{i_1,k_1}, \dots, p^+_{i_T,k_T}) \in \overline{\mathscr{P}}_{(i_t,k_t)_{1\leq t\leq T}}$ and $(p^-_{i_1,k_1}, \dots, p^-_{i_T,k_T}) \in \overline{\mathscr{P}}_{(i_t,k_t)_{1\leq t\leq T}}$.*

Proof. We argue about highest weight paths. Lowest weight paths are treated similarly.

If

$$k' - k \equiv 2 + 2(i' - i) - \delta_{n,i'} - \delta_{n,i} \mod 4, \tag{6.3.1}$$

then $p^+_{i,k}$ is strictly above $p^+_{i',k'}$ because of the inequality

$$k' - k \geq 4n + 2 - 2i - 2i' - \delta_{n,i} - \delta_{n,i'}. \tag{6.3.2}$$

And if

$$k' - k \equiv 2(i' - i) - \delta_{n,i'} - \delta_{n,i} \mod 4, \tag{6.3.3}$$

then $p^+_{i,k}$ is strictly above $p^+_{i',k'}$, because of the inequality

$$k' - k \geq 4 + 2|i' - i| - \delta_{n,i} - \delta_{n,i'}. \tag{6.3.4}$$

Therefore $p^+_{i_s,k_s}$ and $p^+_{i_{s+1},k_{s+1}}$ are non-overlapping, for $s = 1, \dots, T - 1$. The proof that non-adjacent paths are non-overlapping, see Remark 5.4.1, is also straightforward. ☐

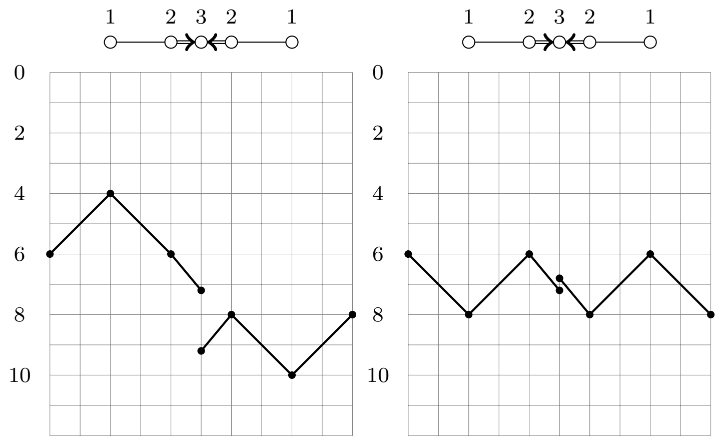
Lemma 6.3.2. *Let $(i_t, k_t) \in \mathcal{X}, 1 \leq t \leq T$, be an extended snake of length $T \in \mathbb{Z}_{\geq 1}$ and $(p_1, \dots, p_T) \in \overline{\mathscr{P}}_{(i_t,k_t)_{1\leq t\leq T}}$. Suppose $(i, k) \in C_{p_t,\pm}$, for some $1 \leq t \leq T$. Then*

(i) $(i, k) \notin C_{p_s,\pm}$, for any $s \neq t$, $1 \leq s \leq T$, and

(ii) if $(i, k) \in C_{p_s,\mp}$, for some s, $1 \leq s \leq T$, then $s = t \pm 1$ and $i = n$.

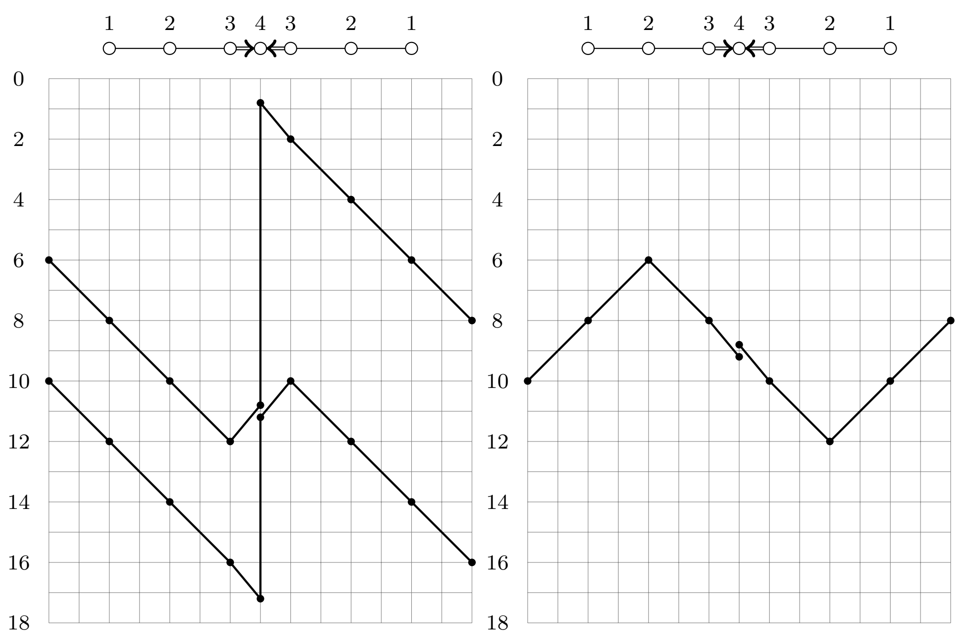
Proof. This follows from the definitions of non-overlapping paths, cf. Figure 6.4. Examples of (ii) can be shown in Figure 6.5 ☐

Figure 6.4: Illustration of the definition of overlapping paths in type B_3. By Theorem 5.1.2, $L_q(Y_{3,1}Y_{3,3})$ contains the ℓ-weight $(Y_{1,4}Y_{3,7}^{-1})(Y_{3,9}^{-1}Y_{2,8}Y_{1,10}^{-1})$ (left) but *n*ot the ℓ-weight $Y_{1,8}^{-1}Y_{2,6}Y_{2,8}^{-1}Y_{1,6} = (Y_{1,8}^{-1}Y_{2,6}Y_{3,7}^{-1})(Y_{3,7}Y_{2,8}^{-1}Y_{1,6})$(right).



(6.3.5)

Figure 6.5: Illustration of Lemma 6.3.2. By Theorem 5.1.2, $L_q(Y_{3,0}Y_{2,6})$ contains the ℓ-weight $Y_{3,12}^{-1}Y_{4,17}^{-1}Y_{3,10} = (Y_{3,12}^{-1}Y_{4,11})(Y_{4,17}^{-1}Y_{4,11}Y_{3,10})$ (left) and $L_q(Y_{4,1}Y_{4,5})$ contains the ℓ-weight $Y_{2,6}Y_{2,12}^{-1} = (Y_{2,6}Y_{4,9}^{-1})(Y_{4,9}Y_{2,12}^{-1})$(right).



(6.3.6)

It follows that for any $(p_1,\ldots,p_T) \in \overline{\mathscr{P}}_{(i_t,k_t)_{1\le t\le T}}$ and $(j,l) \in \mathcal{W}$, at most one of the paths can be lowered at (j,l) and at most one path can be raised at (j,l). Therefore, there is no ambiguity in performing a raising or a lowering move at (j,l) on a non-overlapping tuple of paths $(p_1,\ldots,p_T) \in \overline{\mathscr{P}}_{(i_t,k_t)_{1\le t\le T}}$, to yield a new tuple $(p_1,\ldots,p_T)\mathscr{A}_{j,l}^{\pm 1}$.

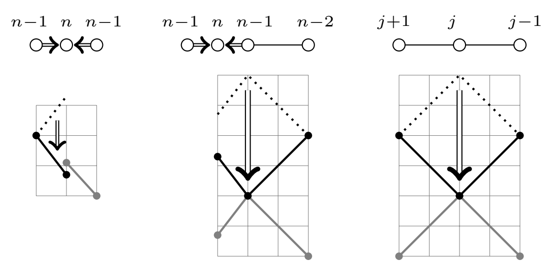
The following lemma easily follows from the definitions.

Lemma 6.3.3. *Let $(i_t,k_t) \in \mathcal{X}$, $1 \le t \le T$, be an extended snake of length $T \in \mathbb{Z}_{\ge 1}$ and $(p_1,\ldots,p_T) \in \overline{\mathscr{P}}_{(i_t,k_t)_{1\le t\le T}}$. Then $\prod_{t=1}^{T} \mathsf{m}(p_t)$ is dominant if and only if p_t is the highest path of $\mathscr{P}_{i_t,k_t}$ for all $1 \le t \le T$. Analogously, $\prod_{t=1}^{T} \mathsf{m}(p_t)$ is anti-dominant if and only if p_t is the lowest path of $\mathscr{P}_{i_t,k_t}$ for all $1 \le t \le T$.* □

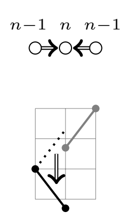
The following lemma gives information about how overlaps arise when performing a lowering move on a tuple of non-overlapping paths.

Lemma 6.3.4. *Let $(i_t,k_t)_{1\le t\le T} \subseteq \mathcal{X}$ an extended snake of length $T \in \mathbb{Z}_{\ge 1}$ and $(p_1,\ldots,p_T) \in \overline{\mathscr{P}}_{(i_t,k_t)_{1\le t\le T}}$. Let $1 \le t \le T$ and $(j,l) \in \mathcal{X}$ be such that the path p_t can be lowered at (j,l). This move introduces an overlap if and only if there is an s, $t < s \le T$, such that p_s has an upper corner at $(j,l+d_j)$ or a lower corner at $(j,l-d_j)$.*

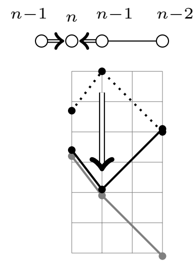
Proof. This is seen by inspection of the definitions above of paths and moves. We illustrate the distinct cases, up to symmetry. In the most cases the overlap occurs at an upper corner $(j,l+d_j)$ of p_s:



The exception is when the upper corner $(j,l-d_j)$ of p_t is also a lower corner of p_s, which can happen only when $j=n$, cf. Lemma 6.3.2(ii):



The only other possible overlapping scenario is as shown.



This situation occurs if and only if $(n-1,k)$ is an upper corner of some path p, $(n-1,k+4) \in p'$ for some p' and $(n-1,k+4)$ is not an upper corner of p'. However, we claim this does not happen for extended snakes.

Indeed, let $(i,k) \in \mathcal{X}$ and $(i',k') \in \mathcal{X}$ such that $k' \geq k$. Let $p \in \mathscr{P}_{i,k}, p' \in \mathscr{P}_{i'k'}$.

If the extended snake also has a point (i'',k'') with $k' > k'' > k$ then the overlap above is impossible. Otherwise, we use the definition of the extended snake modules. In the case of (6.3.1), we have:

$$k' - k \geq 2 + 2i' + 2i - \delta_{n,i} - \delta_{n,i'}. \tag{6.3.7}$$

It follows that if, for some $\ell \in \mathbb{Z}$, $(n-1,\ell) \in C_{p,+}$, for some $p \in \mathscr{P}_{i,k}$, then $(n, \ell+3) \notin p'$, for all $p' \in \mathscr{P}_{i',k'}$.

The case of (6.3.3), is similar with the use of the equation

$$k' - k \geq 4 + 2i' - 2i - \delta_{n,i} - \delta_{n,i'}. \tag{6.3.8}$$

□

The property of the tuples of paths described by this lemma is used for the proof of Theorem 5.1.2. Informally, it means that the first overlap between paths always corresponds, in the q-character, to an illegal lowering step in some $U_q(\widehat{\mathfrak{sl}}_2)$ evaluation module.

Remark 6.3.5. The definition of the extended snakes is exactly the combination of (6.3.2), (6.3.7) in the case of (6.3.1) and of (6.3.4) in the case of (6.3.3). Observe that (6.3.4) implies (6.3.8) in the latter case.

Our definitions of paths and moves are so constructed that we have the following.

Lemma 6.3.6. *Let $(i_t,k_t) \in \mathcal{X}$, $1 \leq t \leq T$ be an extended snake and $(p_1,\ldots,p_T) \in \overline{\mathscr{P}}_{(i_t,k_t)_{1\leq t\leq T}}$. If $(p'_1,\ldots,p'_t) = (p_1,\ldots,p_T)\mathscr{A}^{\pm1}_{j,l}$, where $(j,l) \in \mathcal{W}$ is any point at $(p_1,\ldots,p_T)$ that can be raised/lowered, then $\prod_{t=1}^{T} \mathsf{m}(p'_t) = A^{\pm1}_{j,l} \prod_{t=1}^{T} \mathsf{m}(p_t)$.*

6.4 Properties of paths and moves

Given two paths $p = (x_r, y_r)_{1 \le r \le m}$ and $p' = (x_r, y'_r)_{1 \le r \le m}$, $m \in \mathbb{Z}_{\ge 1}$, in $\mathscr{P}_{i,k}$ we say p is *weakly above* (resp. *weakly below*) p', if $y_r \le y'_r$ (resp. $y_r \ge y'_r$), for all $1 \le r \le m$. Let

$$\mathrm{top}(p, p') := (x_r, \min(y_r, y'_r))_{1 \le r \le m}.$$

The following lemma gives properties of paths and moves that can be checked by inspection.

Lemma 6.4.1. *Let p and p' be paths in $\mathscr{P}_{i,k}$. Then*

(i) p is uniquely defined by its set of lower corners,

(ii) suppose p is weakly below p'. If $p \ne p'$, then there is a $(j, l) \in \mathcal{W}$ such that p can be raised at (j, l) and $p\mathscr{A}_{j,l}$ is weakly below p',

(iii) $\mathrm{top}(p, p') \in \mathscr{P}_{i,k}$ and $\mathrm{top}(p, p')$ is weakly above both p and p'.

Lemma 6.4.2. *Let p and p' be paths in $\mathscr{P}_{i,k}$. Then the path p can be obtained from p' by a sequence of moves containing no inverse pair of raising/lowering moves.*

Proof. By applying Lemma 6.4.1(ii) a finite number of times we construct a sequence $\mathscr{R}(p, p')$ of distinct points in $\mathcal{W}$ such that, starting with p, performing raising moves at these points, in order, yields $\mathrm{top}(p, p')$. Similarly, we construct a sequence $\mathscr{R}(p', p)$ of raising moves taking p' to $\mathrm{top}(p, p')$. By reversing the sequence $\mathscr{R}(p', p)$ we have a sequence of lowering moves taking $\mathrm{top}(p, p')$ to p'. It suffices to show that $\mathscr{R}(p, p')$ and $\mathscr{R}(p', p)$ have no element in common.

Suppose for a contradiction that $(j, l) \in \mathcal{W}$ occurs in both sequences $\mathscr{R}(p, p')$ and $\mathscr{R}(p', p)$. Let $\tilde{p}$ be the path obtained from p, by performing raising moves at the points in $\mathscr{R}(p, p')$ preceding (j, l), in order. Similarly, let $\tilde{p}'$ be the path obtained from p' by performing raising moves at the points in $\mathscr{R}(p', p)$ preceding (j, l), in order. Observe that $\mathrm{top}(p, p') = \mathrm{top}(\tilde{p}, \tilde{p}')$. But both paths $\tilde{p}$ and $\tilde{p}'$ have a lower corner at $(j, l + d_j)$. Therefore, $\mathrm{top}(\tilde{p}, \tilde{p}')$ has a lower corner at $(j, l + d_j)$, while $\mathrm{top}(p, p')$ does not, hence a contradiction. □

Lemma 6.4.3. *Let $(i_t, k_t) \in \mathcal{X}$, $1 \le t \le T$, be an extended snake and let $(p_1, \dots, p_T)$ and $(p_1, \dots, p'_T)$ paths in $\overline{\mathscr{P}}_{(i_t,k_t)_{1 \le t \le T}}$. Then $(p_1, \dots, p_T)$ can be obtained from $(p'_1, \dots, p'_T)$ by a sequence of moves containing no inverse pair of raising/lowering moves, such that no move introduces any overlaps.*

Proof. For each $1 \le t \le T$, consider the sequences $\mathscr{R}(p_t, p'_t)$ and $\mathscr{R}(p'_t, p_t)$ of points of $\mathcal{W}$ as in the previous lemma. We claim that the following is a sequence of moves obeying the requirements of the lemma. We first perform raising moves on p_1 at the points in $\mathscr{R}(p_1, p'_1)$, in order to reach $\mathrm{top}(p_1, p'_1)$, then on p_2 we perform raising moves at the points in $\mathscr{R}(p_2, p'_2)$, in order to reach $\mathrm{top}(p_2, p'_2)$ and so on until we raise p_T to $\mathrm{top}(p_T, p'_T)$. We now perform lowering moves on $\mathrm{top}(p_T, p'_T)$ at the points in $\mathscr{R}(p'_T, p_T)$, in reverse order to reach p'_T, then on $\text{top}(p_{T-1}, p'_{T-1})$ at the points in $\mathscr{R}(p'_{T-1}, p_{T-1})$, in reverse order to reach p'_{T-1} and so on until we lower $\text{top}(p_1, p'_1)$ to p'_1. In fact, by Lemma 6.4.2, $\mathscr{R}(p_t, p'_t) \cap \mathscr{R}(p'_t, p_t) = \emptyset$ for all t, $1 \leq t \leq T$. It remains to check that $\mathscr{R}(p_t, p'_t) \cap \mathscr{R}(p'_u, p_u) = \emptyset$ for all $1 \leq t \neq u \leq T$. Suppose, by contradiction, $(j, l) \in \mathscr{R}(p_t, p'_t) \cap \mathscr{R}(p'_u, p_u)$, for some $1 \leq t \neq u \leq T$ and, without loss of generality, suppose $t < u$. Let $(x, y) := \iota(j, l + d_j)$. Then p'_u has a lower corner at $(j, l + d_j)$ and $\text{top}(p_u, p'_u)$ does not. So p_u contains a point (x, y'), $y' < y$. But p_t also has a lower corner at $(j, l + d_j)$ and then, p_u and p_t overlap in a column x which is a contradiction since p_t and p_u are non-overlapping. □

Lemma 6.4.4. *Let $(i_t, k_t) \in \mathcal{X}$, $1 \leq t \leq T$ be an extended snake and (j_r, l_r), $1 \leq r \leq R$ be a sequence of points in $\mathcal{W}$. For all $(p_1, \ldots, p_T) \in \overline{\mathscr{P}}_{(i_t,k_t)_{1\leq t\leq T}}$ and $(p'_1, \ldots, p'_T) \in \overline{\mathscr{P}}_{(i_t,k_t)_{1\leq t\leq T}}$, the following are equivalent:*

(i) $\prod_{t=1}^T \mathsf{m}(p'_t) = \prod_{t=1}^T \mathsf{m}(p_t) \prod_{r=1}^R A^{-1}_{j_r,l_r}$,

(ii) there is a permutation $\sigma \in \Sigma_R$ such that $\big((j_{\sigma(1)}, l_{\sigma(1)}), \ldots, (j_{\sigma(R)}, l_{\sigma(R)})\big)$ is a sequence of lowering moves that can be performed on $(p_1, \ldots, p_T)$, without ever introducing overlaps, to yield $(p'_1, \ldots, p'_T)$.

Proof. It is clear that (ii) implies (i), by Lemma 6.3.6. To see that (i) implies (ii), note that since $(p_1, \ldots, p_T)$ and $(p'_1, \ldots, p'_T)$ are elements of $\overline{\mathscr{P}}_{(i_t,k_t)_{1\leq t\leq T}}$, by Lemma 6.4.3, there is a sequence of moves that takes $(p_1, \ldots, p_T)$ to $(p'_1, \ldots, p'_T)$ without introducing overlaps and without ever performing a move and its inverse. By Lemma 6.3.6 and the fact that the $(A_{j,l})_{(j,l)\in\mathcal{W}}$ are algebraically independent, these moves must indeed be lowering moves at the points $(j_r, l_r)_{1\leq r\leq R}$ arranged in some order. □

Corollary 6.4.5. *Let $(i_t, k_t) \in \mathcal{X}$, $1 \leq t \leq T$ be an extended snake. The map*

$$\begin{array}{rcl} \overline{\mathscr{P}}_{(i_t,k_t)_{1\leq t\leq T}} & \to & \mathbb{Z}[Y^{\pm 1}_{i,k}]_{(i,k)\in\mathcal{X}}, \\ (p_1, \ldots, p_T) & \mapsto & \prod_{t=1}^T \mathsf{m}(p_t), \end{array}$$

is injective.

Lemma 6.4.6. *Let $(i_t, k_t) \in \mathcal{X}$, $1 \leq t \leq T$ be an extended snake and $(p_1, \ldots, p_T) \in \overline{\mathscr{P}}_{(i_t,k_t)_{1\leq t\leq T}}$. Let $\varpi = \prod_{t=1}^T \mathsf{m}(p_t)$. If $\varpi A^{-1}_{i,k}$ is not of the form $\prod_{t=1}^T \mathsf{m}(p'_t)$, for any $(p'_1, \ldots, p'_T) \in \overline{\mathscr{P}}_{(i_t,k_t)_{1\leq t\leq T}}$, then neither $\varpi A^{-1}_{i,k} A_{j,l}$ is so, unless $(j, l) = (i, k)$.*

Proof. If $\varpi A^{-1}_{i,k} A_{j,l} = \prod_{t=1}^T \mathsf{m}(p'_t)$ for some $(p'_1, \ldots, p'_T) \in \overline{\mathscr{P}}_{(i_t,k_t)_{1\leq t\leq T}}$ then, by Lemma 6.4.4, $(p'_1, \ldots, p'_T)$ can be obtained from $(p_1, \ldots, p_T)$ without ever introducing overlaps. Since $\varpi A^{-1}_{i,k}$ does not correspond to a non-overlapping path, either it is not possible to lower $(p_1, \ldots, p_T)$ at (i, k) or this is a valid lowering move but one which introduces an overlap. Therefore, $(p'_1, \ldots, p'_T)$ is obtained by raising $(p_1, \ldots, p_T)$ at (j, l) and then lowering at (i, k). Suppose the raising move is on p_s and the lowering move is on p_t, $1 \leq s, t \leq T$. If $s \neq t$, then this requires that p_t can be lowered at (i,k), so we must have r such that when we lower p_t at (i,k) it overlaps with p_r. If $s \notin \{t, r\}$ then, after the rising move at (j,l) on p_s, it is still true that when p_t is lowered at (i,k) it overlaps with p_r. If $s = r$, then note that p_t lowered at (i,k) overlaps with p_r raised at (j,l). Therefore, in fact, $s = t$, i.e. both moves must be on the same path. By inspection we see that it is necessary that $(j,l) = (i,k)$. □

Lemma 6.4.7. *Let $(i_t, k_t) \in \mathcal{X}$, $1 \leq t \leq T$ be an extended snake and $(p_1, \ldots, p_T) \in \overline{\mathscr{P}}_{(i_t,k_t)_{1\leq t\leq T}}$. Pick and fix an $i \in I$. Let $\overline{\mathscr{P}} \subseteq \overline{\mathscr{P}}_{(i_t,k_t)_{1\leq t\leq T}}$ be the set of those non-overlapping tuples of paths that can be obtained from $(p_1, \ldots, p_T)$ by performing a sequence of raising or lowering moves at points of the form $(i,l) \in \mathcal{W}$. Then $\sum_{(p_1,\ldots,p_T)\in\overline{\mathscr{P}}} \beta_i(\prod_{t=1}^T \mathsf{m}(p_t))$ is the q-character of a simple finite-dimensional $U_q(\widehat{\mathfrak{sl}}_2)$-module.*

Proof. Let $\boldsymbol{\varpi} := \prod_{t=1}^T \mathsf{m}(p_t)$. It follows that $u_{i,l}(\boldsymbol{\varpi}) \leq 1$, by Lemma 6.3.2; since there cannot be an upper corner at $(i, l - d_i)$ and a lower corner at $(i, l + d_i)$ without paths overlapping, we have $u_{i,l-d_i}(\boldsymbol{\varpi}) - u_{i,l+d_i}(\boldsymbol{\varpi}) \neq 2$. By Lemma 5.3.2, these two conditions imply that $\beta_i(\boldsymbol{\varpi})$ is part of a thin simple finite-dimensional $U_{q_i}(\widehat{\mathfrak{sl}}_2^{(i)})$-module. Let us call it V.

We now prove that the elements of $\overline{\mathscr{P}}$ are in bijection with the set $\mathcal{M}(V)$. By Lemma 6.3.4, it is possible to lower $(p_1, \ldots, p_T)$ at (i,l) without introducing an overlap, if and only if

(i) $(i, l - d_i)$ is an upper corner of some path in $(p_1, \ldots, p_T)$, and

(ii) $(i, l - d_i)$ is not a lower corner of any path in $(p_1, \ldots, p_T)$, and

(iii) $(i, l + d_i)$ is not an upper corner of any path in $(p_1, \ldots, p_T)$.

By Lemma 6.4.4,

$$\boldsymbol{\varpi} A_{i,l}^{-1} \in \sum_{(p'_1,\ldots,p'_T)\in\overline{\mathscr{P}}} \prod_{t=1}^{T} \mathsf{m}(p'_t)$$

if and only if (i)-(iii) hold; that is, cf (6.1.1), if and only if $u_{i,l-d_i}(\boldsymbol{\varpi}) = 1$ and $u_{i,l+d_i}(\boldsymbol{\varpi}) = 0$. These are precisely the conditions of Lemma 5.3.2 under which

$$\beta_i(\boldsymbol{\varpi} A_{i,l}^{-1}) \in \mathcal{M}(V).$$

Similar statements hold for raising moves. Moreover, by a finite sequence of moves of this type we obtain every element of $\mathcal{M}(V)$ and no others, by Remark 5.3.3. We also generate all elements of $\overline{\mathscr{P}}$ and no other tuple of paths. □

We are now prepared to state the main result of this chapter, which proves in particular Theorem 5.1.2.

Theorem 6.4.8. *Let $(i_t, k_t) \in \mathcal{X}$, $1 \le t \le T$ be an extended snake of length $T \in \mathbb{Z}_{\ge 1}$ and set $\boldsymbol{\pi} = \prod_{t=1}^{T} Y_{i_t,k_t}$. Then*

$$\chi_q(L_q(\boldsymbol{\pi})) = \sum_{(p_1,\ldots,p_T)\in\overline{\mathscr{P}}_{(i_t,k_t)_{1\le t\le T}}} \prod_{t=1}^{T} \mathsf{m}(p_t).$$

In particular $L_q(\boldsymbol{\pi})$ is thin, special and anti-special.

Proof. By Lemma 6.3.3 and the definition of highest path, we have $\boldsymbol{\pi} = \prod_{t=1}^{T} \mathsf{m}(p^{+}_{i_t,k_t})$. Define

$$\mathcal{M} := \left\{ \prod_{t=1}^{T} \mathsf{m}(p_t) | (p_1, \ldots, p_T) \in \overline{\mathscr{P}}_{(i_t,k_t)_{1\le t\le T}} \right\}$$

and observe that $\mathcal{M}$ has exactly one anti-dominant ℓ-weight, by Lemma 6.3.3. We shall show that the conditions of Theorem 5.3.4 apply to the pair $(\boldsymbol{\pi}, \mathcal{M})$. In fact, property (i) of Theorem 5.3.4 follows from Lemma 6.3.3. Property (ii) is Lemma 6.4.6 and property (iii) is Lemma 6.4.7. Since, by Corollary 6.4.5,

$$\sum_{\boldsymbol{\varpi}\in\mathcal{M}} \boldsymbol{\varpi} = \sum_{(p_1,\ldots,p_T)\in\overline{\mathscr{P}}_{(i_t,k_t)_{1\le t\le T}}} \prod_{t=1}^{T} \mathsf{m}(p_t),$$

the theorem follows from Theorem 5.3.4. □

Corollary 6.4.9. *Let $(i_t, k_t) \in \mathcal{X}$, $1 \le t \le T$ be an extended snake of length $T \in \mathbb{Z}_{\ge 1}$ and $\boldsymbol{\pi} := \prod_{t=1}^{T} Y_{i_t,k_t}$. Let $1 \le s < T$ and let $\boldsymbol{\pi}_1 := \prod_{t=1}^{s} Y_{i_t,k_t}$, $\boldsymbol{\pi}_2 := \prod_{t=s+1}^{T} Y_{i_t,k_t}$. Then $L_q(\boldsymbol{\pi}_+) = L_q(\boldsymbol{\pi}_1) \otimes L_q(\boldsymbol{\pi}_2)$ if and only if*

$$k_{s+1} - k_s \ge 4 + 2i_s + 2i_{s+1} - \delta_{n,i_s} - \delta_{n,i_{s+1}}, \tag{6.4.1}$$

when $k_{s+1} - k_s \equiv 2(i_s - i_{s+1}) - \delta_{n,i_s} - \delta_{n,i_{s+1}} \mod 4$, or

$$k_{s+1} - k_s \ge 4n + 2 - 2|i_s - i_{s+1}| - \delta_{n,i_s} - \delta_{n,i_{s+1}}, \tag{6.4.2}$$

when $k_{s+1} - k_s \equiv 2 + 2(i_s - i_{s+1}) - \delta_{n,i_s} - \delta_{n,i_{s+1}} \mod 4$.

In particular, the extended snake module is prime if and only if (6.4.1) and (6.4.2) fails for all $s = 1, \ldots, T-1$.

Proof. The inequalities (6.4.1) and (6.4.2) are equivalent to the requirement that any path $p \in \mathscr{P}_{i_s,k_s}$ is strictly above any path $p' \in \mathscr{P}_{i_{s+1},k_{s+1}}$, in their respective parity cases. The corollary follows from Theorem 6.4.8. □

Chapter 7

Tableaux description of snake modules

In this section we define a bijection between super standard skew Young tableaux and paths of some associated snake. We freely use the notation of the previous chapters.

7.1 Combinatorial properties of non-overlapping paths

Define a set A (the *alphabet*) equipped with a total ordering $<$ (*alphabetical ordering*) as follows:

$$\mathsf{A} := \{1, 2, \ldots, n, 0, \overline{n}, \ldots, \overline{2}, \overline{1}\}, \quad 1 < 2 < \cdots < n < 0 < \overline{n} < \cdots < \overline{2} < \overline{1}.$$

Given a subset $B \subset \mathsf{A}$, assume $B = \{a_1 \leq a_2 \leq \ldots \leq a_m\}$ for some $m \in \mathbb{Z}_{\geq 0}$. For $k \in \mathbb{Z}_{\geq 0}$ define the subsets of B

$$^{[k]}B := \{a_{k+1}, a_{k+2}, \ldots, a_m\} \quad \text{and} \quad B^{[k]} := \{a_1, a_2, \ldots, a_{m-k}\}, \tag{7.1.1}$$

with the convention $^{[k]}B = B^{[k]} = \emptyset$, if $k \geq m$.

Let $(i, k) \in \mathcal{X}$ and $p \in \mathscr{P}_{i,k}$. If $i = n$ and $p =: ((x_r, y_r))_{0 \leq r \leq n}$, define

$$R_p := \{r | 1 \leq r \leq n,\ y_r - y_{r-1} < 0\} \subseteq \mathsf{A}, \quad \text{and} \quad \overline{R}_p := \{\overline{r} | 1 \leq r \leq n,\ y_r - y_{r-1} > 0\} \subseteq \mathsf{A}.$$

If $i < n$, recall that p is given by a pair $(a, \overline{a})$, where

$$a \in \mathscr{P}_{n,k-(2n-2i-1)} \quad \text{and} \quad \overline{a} \in \mathscr{P}_{n,k+(2n-2i-1)}.$$

Let

$$a =: ((x_r, y_r))_{0 \leq r \leq n} \quad \text{and} \quad \overline{a} =: ((\overline{x}_r, \overline{y}_r))_{0 \leq r \leq n},$$

and define

$$R_p := \{r | 1 \leq r \leq n,\ \overline{y}_r - \overline{y}_{r-1} < 0\} \subseteq \mathsf{A}, \quad \text{and} \quad \overline{R_p} := \{\overline{r} | 1 \leq r \leq n,\ y_r - y_{r-1} > 0\} \subseteq \mathsf{A}.$$

If $i < n$, we also define

$$S_p := \{r | 1 \leq r \leq n,\ y_r - y_{r-1} < 0\} \subseteq \mathsf{A}, \quad \text{and} \quad \overline{S_p} := \{\overline{r} | 1 \leq r \leq n,\ \overline{y}_r - \overline{y}_{r-1} > 0\} \subseteq \mathsf{A}.$$

Note that

$$S_p = \{r | 1 \leq r \leq n,\ \overline{r} \notin \overline{R_p}\} \quad \text{and} \quad \overline{S_p} = \{\overline{r} | 1 \leq r \leq n,\ r \notin R_p\}.$$

Clearly p is completely described by the pair of sets $R_p, \overline{R}_p$, and equally so by $S_p, \overline{S}_p$, when $i < n$.

Example. If $p = p^+_{i,k}$, for some $(i,k) \in \mathcal{X}$, $i < n$, then $S_p = \{1, 2, \ldots, i\}$, $\overline{S_p} = \emptyset$, $R_p = \{1, 2, \ldots, n\}$ and $\overline{R_p} = \{\overline{i+1}, \ldots, \overline{n}\}$. ◇

We denote cardinality of a finite set A by $\#A$. The next lemma follows from the definition of paths.

Lemma 7.1.1. *Let $(i,k) \in \mathcal{X}$ and let $p \in \mathscr{P}_{i,k}$. Then*

$$\#R_p + \#\overline{R}_p \geq 2n - i \quad \text{and} \quad \#S_p + \#\overline{S}_p \leq i. \tag{7.1.2}$$

□

Let $(i', k') \in \mathcal{X}$ be in snake position to $(i,k) \in \mathcal{X}$. We say that (i', k') and (i,k) are *shifted by* $\sigma \in \mathbb{Z}_{\geq 0}$ if

$$k' - k = 4 + 2|i' - i| + 4\sigma - \delta_{n,i} - \delta_{n,i'}.$$

Observe that $\sigma = 0$ corresponds to the minimal snake position. If $(i_t, k_t)_{1 \leq t \leq T}$, $T \in \mathbb{Z}_{\geq 1}$ is a snake, we denote by σ_t the shift between (i_t, k_t) and (i_{t+1}, k_{t+1}).

Lemma 7.1.2. *Let $(i', k') \in \mathcal{X}$ be in snake position to $(i,k) \in \mathcal{X}$ shifted by $\sigma \in \mathbb{Z}_{\geq 0}$. Let $p \in \mathscr{P}_{i,k}, p' \in \mathscr{P}_{i',k'}$. If p is strictly above p' then*

$$\#\overline{R}_p + \#R_{p'} \leq 2n - i + \max\{i - i', 0\} + \sigma, \tag{7.1.3}$$

$$\#\overline{R}_p^{[\sigma + \max\{i' - i, 0\}]} \cap \{\overline{1}, \ldots, \overline{r}\} \leq \#\overline{R}_{p'} \cap \{\overline{1}, \ldots, \overline{r}\}, \quad r = 1, \ldots, n-1, \tag{7.1.4}$$

$$\#R_p \cap \{1, \ldots, r\} \geq \#^{[\sigma + \max\{i - i', 0\}]} R_{p'} \cap \{1, \ldots, r\}, \quad r = 1, \ldots, n-1. \tag{7.1.5}$$

Proof. Let $y \in \mathbb{Z} \pm \epsilon$ be such that $(2n-1, y) \in p$ and $(2n-1, z) \notin p$ for $z > y$. Let $y' \in \mathbb{Z} \pm \epsilon$ be such that $(2n-1, y') \in p'$ and $(2n-1, z) \notin p'$ for $z < y$.

Let s, s' such that $0 \leq s \leq i$, $0 \leq s' \leq i'$,

$$4(s-1) \leq y - k - 2(n - i + 1) + \delta_{n,i} \leq 4s \quad \text{and} \quad 4(s'-1) \leq y' - k' - 2(n - i' + 1) + \delta_{n,i'} \leq 4s'.$$

Since p and p' do not overlap, we have $y' > y$. It follows that $s - s' \leq \sigma + \max\{i - i', 0\}$. We also have $\#R'_p = n - s'$ and $\overline{R}_p = n - i + s$. Therefore, (7.1.3) follows.

Suppose that r is the minimal integer such that (7.1.4) does not hold. Let $s = \#\overline{R}_{p'} \cap \{\overline{1}, \dots, \overline{r}\}$. Then

$$\#\overline{R}_p \cap \{\overline{1}, \dots, \overline{r}\} = s + \sigma + \max\{i' - i, 0\} + 1.$$

In particular, it follows that

$$\iota(r, k + 2i - \delta_{n,i} + 4(s + \sigma + \max\{i - i', 0\} + 1) - 2r) \in p \quad \text{and} \quad \iota(r, k' + 2i' - \delta_{n,i'} + 4s - 2r) \in p'.$$

Then paths p and p' overlap at r, hence, a contradiction.

Equation (7.1.5) is proved similarly. □

Corollary 7.1.3. *Let $(i', k') \in \mathcal{X}$ be in snake position to $(i, k) \in \mathcal{X}$ shifted by $\sigma \in \mathbb{Z}_{\geq 0}$. Let $p \in \mathscr{P}_{i,k}$, $p' \in \mathscr{P}_{i',k'}$ and assume $i, i' < n$. If p and p' do not overlap, then*

$$\#S_p + \#\overline{S}_{p'} \geq i - \max\{i - i', 0\} - \sigma, \tag{7.1.6}$$

$$\#S_p \cap \{1, \dots, r\} \geq \#^{[\sigma + \max\{i' - i, 0\}]} S_{p'} \cap \{1, \dots, r\}, \quad r = 1, \dots, n-1, \text{ and} \tag{7.1.7}$$

$$\#\overline{S}_p^{[\sigma + \max\{i - i', 0\}]} \cap \{\overline{1}, \dots, \overline{r}\} \leq \#\overline{S}_{p'} \cap \{\overline{1}, \dots, \overline{r}\}, \quad r = 1, \dots, n-1. \tag{7.1.8}$$

□

7.2 Tableaux

In this chapter a *skew diagram* (λ/μ) is a finite subset $(\lambda/\mu) \subset \mathbb{Z} \times \mathbb{Z}_{>0}$ such that

(i) if $(\lambda/\mu) \neq \emptyset$, then there is a $j \in \mathbb{Z}$ such that $(j, 1) \in (\lambda/\mu)$, and,

(ii) if $(i, j) \notin (\lambda/\mu)$, then either $\forall i' \geq i, \forall j' \geq j$, $(i', j') \notin (\lambda/\mu)$ or $\forall i' \leq i, \forall j' \leq j$, $(i', j') \notin (\lambda/\mu)$.

If $(i, j) \in (\lambda/\mu)$ we say (λ/μ) has a *box* in *row* i, *column* j. For each $j \in \mathbb{Z}_{>0}$, let $b_j = \max\{i \in \mathbb{Z} | (i, j) \in (\lambda/\mu)\}$, be the *bottom box* in the column j and $t_j = \min\{i \in \mathbb{Z} | (i, j) \in (\lambda/\mu)\}$ be the *top box* in the column j. Define also the *length* of the column j by $l_j = \#\{i \in \mathbb{Z} | (i, j) \in (\lambda/\mu)\}$, and observe that $l_j = b_j - t_j + 1$.

A *skew tableau* $\mathcal{T}$ with shape (λ/μ) is then any function $\mathcal{T} : (\lambda/\mu) \to \mathsf{A}$ that obeys the following horizontal rule (H) and vertical rule (V):

(H) $\mathcal{T}(i, j) \leq \mathcal{T}(i, j+1)$ and $(\mathcal{T}(i, j), \mathcal{T}(i, j+1)) \neq (0, 0)$,

(V) $\mathcal{T}(i, j) < \mathcal{T}(i+1, j)$ or $(\mathcal{T}(i, j), \mathcal{T}(i+1, j)) = (0, 0)$.

Let $\mathrm{Tab}(\lambda/\mu)$ denote the set of tableaux of shape (λ/μ). If a skew diagram contains a rectangle of size $(2N+1) \times 2$ then horizontal rule implies that there exists no skew tableau of shape (λ/μ).

We call a skew diagram *super skew diagram* if $\#\{i \in \mathbb{Z} | (i, j) \in (\lambda/\mu),\ (i, j+1) \in (\lambda/\mu)\} \leq 2n$, for all $j \in \mathbb{Z}_{>0}$.

From now on we consider only super skew diagrams (λ/μ).

We call a skew diagram *generic super skew diagram* if $\#\{i \in \mathbb{Z} | (i,j) \in (\lambda/\mu),\ (i,j+1) \in (\lambda/\mu)\} < 2n$, for all $j \in \mathbb{Z}_{>0}$. In other words, a generic skew diagram contains no rectangles of size $2n \times 2$.

Let (λ/μ) be a super skew diagram. For each $\mathcal{T} \in \mathrm{Tab}(\lambda/\mu)$, we associate an ℓ-weight in $\mathcal{P}_q$ as follows:

$$M(\mathcal{T}) := \prod_{(i,j)\in\lambda/\mu} \mathsf{m}(\mathcal{T}(i,j), 4(j-i)), \tag{7.2.1}$$

where the contribution of each box is given by

$$\begin{aligned}
\mathsf{m} : \mathsf{A} \times \mathbb{Z} &\to \mathbb{Z}[Y_{i,k}^{\pm 1}]_{(i,k)\in I\times\mathbb{Z}},\\
(i,k) &\mapsto Y_{i-1,2i+k}^{-1} Y_{i,2i-2+k}, 1 \le i \le n-1,\\
(n,k) &\mapsto Y_{n-1,2n+k}^{-1} Y_{n,2n-3+k} Y_{n,2n-1+k},\\
(0,k) &\mapsto Y_{n,2n+1+k}^{-1} Y_{n,2n-3+k},\\
(\overline{n},k) &\mapsto Y_{n,2n-1+k}^{-1} Y_{n,2n+1+k}^{-1} Y_{n-1,2n-2+k},\\
(\bar{i},k) &\mapsto Y_{i,4n-2i+k}^{-1} Y_{i-1,4n-2-2i+k}, 1 \le i \le n-1,
\end{aligned}$$

with convention, $Y_{0,k} := 1$ and $Y_{n+1,k} := 1$ for all $k \in \mathbb{Z}$. Note that

$$\chi_q(L_q(Y_{1,0})) = \sum_{i\in\mathsf{A}} \mathsf{m}(i,0).$$

Given a super skew diagram (λ/μ), define $\mathcal{T}_+ : (\lambda/\mu) \to \mathsf{A}$, by filling up the boxes of (λ/μ) with letters in A according to the following rule. Starting from column 1 and going from the column j to the column $j+1$ in (λ/μ), always from the most top empty box and working downwards in the alphabetical order, fill up the column j as follows:

(i) enter letters $1, \ldots, r \le n$, for the maximum r possible,

(ii) enter as many successive 0's as possible, respecting the horizontal rule for the column $j-1$,

(iii) enter letters $\overline{n}, \ldots, \overline{r}$, for the maximum $\overline{r}$ possible.

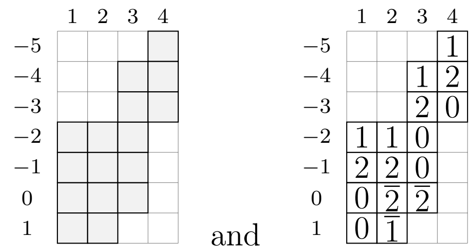
By construction, one easily checks that

Lemma 7.2.1. *The map $\mathcal{T}_+$ is a skew tableau.* □

The tableau $\mathcal{T}_+$ is called the *dominant tableau* of shape (λ/μ). Let $\boldsymbol{\pi} = \boldsymbol{\pi}(\lambda/\mu) := M(\mathcal{T}_+)$ obtained by (7.2.1). We give an alternative way of computing $\boldsymbol{\pi}$. For each column j of (λ/μ) such that $l_j \ge n$, let $s_j = t_j + n - 1$, and observe that $\mathcal{T}_+(s_j, j) = n$. A column j of (λ/μ) is said to be *special* if $l_j \ge N$ and $(s_j+1, j+1) \notin (\lambda/\mu)$. Define

$$\mathscr{S} = \mathscr{S}(\lambda/\mu) := \{j \mid \text{the column } j \text{ of } (\lambda/\mu) \text{ is special}\}.$$

Figure 7.1: A non-generic super skew diagram and its dominant tableau in type B_2.



Lemma 7.2.2. *Let (λ/μ) be a skew diagram. Then*

$$\boldsymbol{\pi}(\lambda/\mu) = \prod_{j\in\mathbb{Z}_{>0}} \mathsf{b}(\mathcal{T}_+(b_j, j), 4(j-b_j)) \prod_{j\in\mathscr{S}} \mathsf{b}(n, 4(j-s_j)+2), \tag{7.2.2}$$

where $\mathsf{b} : \mathsf{A}\times\mathbb{Z} \to \mathbb{Z}[Y_{i,k}]_{(i,k)\in I\times\mathbb{Z}}$ *maps*

$$\begin{array}{rcl} (i,k) & \mapsto & Y_{i,2i-2+k},\ 1\le i\le n-1, \\ (n,k) & \mapsto & Y_{n,2n-3+k}, \\ (0,k) & \mapsto & Y_{n,2n-3+k}, \\ (\bar{i},k) & \mapsto & Y_{i-1,4n-2-2i+k},\ 1\le i\le n. \end{array}$$

In particular, $\boldsymbol{\pi}(\lambda/\mu)\in\mathcal{P}_{\mathcal{X}}$. □

Example. In type B_2, the non-generic super skew diagram shown in Figure 7.1 has $\mathscr{S}=\{3,4\}$, the column 2 is non-generic (cf. Section 7.4), and the dominant ℓ-weight associated to $\mathcal{T}_+$ is

$$Y_{2,1}Y_{1,14}Y_{2,27}Y_{2,29}Y_{2,35}.$$

Later we will show that, for each $\mathcal{T}\in\mathrm{Tab}(\lambda/\mu)$, $M(\mathcal{T})\in\mathcal{P}^+$ if and only if $\mathcal{T}=\mathcal{T}_+$, see Theorem 7.3.2.

Note that a super skew diagram is non-generic if and only if there exist j such that $\mathcal{T}_+(b_j,j)=\bar{1}$. We now focus on generic skew diagrams. The non-generic ones are treated in Section 7.4.

Let (λ/μ) be a generic super skew diagram and let column j be non-empty. Define

$$\varsigma_j = j + \#\{k\in\mathscr{S}(\lambda/\mu)\mid k<j\}.$$

Then the column j contributes to $\boldsymbol{\pi}(\lambda/\mu)$ the fundamental ℓ-weight $Y_{i_{\varsigma_j},k_{\varsigma_j}}$, and if column j is special then it also contributes $Y_{n,k_{\varsigma_j+1}}$.

The following lemma is a consequence of Lemma 7.2.2.

Lemma 7.2.3. *Let (λ/μ) be a generic super skew diagram and j, j' columns of (λ/μ). If $j' > j$ then*

$$k_{\varsigma_{j'}} \geq k_{\varsigma_j+1} > k_{\varsigma_j}.$$

Moreover, the equality holds only if $j' = j+1$ and j is not special. □

Lemma 7.2.4. *Let (λ/μ) be a generic super skew diagram. Then the sequence $\mathcal{X}(\boldsymbol{\pi}(\lambda/\mu))$ is a snake.*

Proof. By Lemmas 7.2.2 and 7.2.3, it suffices to prove that the ℓ-weights corresponding to two consecutive columns are in snake position. Let j and $j+1$ two columns of (λ/μ). Using the definition of $\mathcal{T}_+$, by inspection we prove that:

If j is not special, then $(i_{\varsigma_{j+1}}, k_{\varsigma_{j+1}})$ is in snake position to $(i_{\varsigma_{j+1}}, k_{\varsigma_{j+1}})$, and

$$\sigma_{\varsigma_j} = \begin{cases} b_j - b_{j+1} - \max\{i_{\varsigma_j} - i_{\varsigma_{j+1}}, 0\} & \text{if } l_j < n, \\ b_j - b_{j+1} - \max\{i_{\varsigma_{j+1}} - i_{\varsigma_j}, 0\} & \text{if } l_j > n. \end{cases}$$

If j is special, then $(i_{\varsigma_{j+1}}, k_{\varsigma_{j+1}})$ is in snake position to (n, k_{ς_j+1}), and the latter is in snake position to $(i_{\varsigma_j}, k_{\varsigma_j})$. Moreover,

$$\sigma_{\varsigma_j} = l_j - 2n + i_{\varsigma_j} \quad \text{and} \quad \sigma_{\varsigma_j+1} = s_j - b_{j+1} - n + i_{\varsigma_{j+1}}.$$

□

7.3 Bijection between paths and tableaux

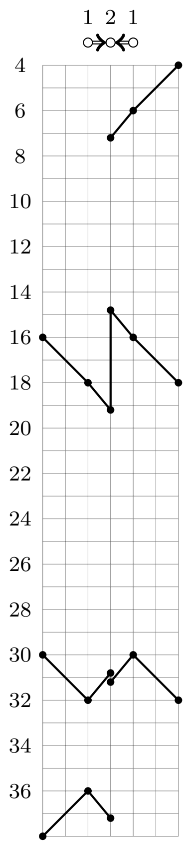
Let (λ/μ) be a generic super skew diagram. Let $(i_t, k_t)_{1\leq t\leq T} := \mathcal{X}(\boldsymbol{\pi}(\lambda/\mu))$. By Lemma 7.2.2, T is the number of non-empty columns plus the number of special columns. Given $(p_1, \dots, p_T) \in \overline{\mathscr{P}}_{(i_t,k_t)_{1\leq t\leq T}}$, we write simply R_t instead of R_{p_t} and similarly so for $\overline{R}_t$, S_t and $\overline{S}_t$.

For $(p_1, \dots, p_T) \in \overline{\mathscr{P}}_{(i_t,k_t)_{1\leq t\leq T}}$, we define the map $\mathcal{T}_{(p_1,\dots,p_T)} : (\lambda/\mu) \to \mathsf{A}$ by filling up the boxes of (λ/μ). Each column j is filled up by the following way:

(i) if $l_j < n$, starting from the box (t_j, j) and working downwards, enter the letters of S_{ς_j} in alphabetical order. Then starting from the box (b_j, j) and working upwards enter the letters of $\overline{S}_{\varsigma_j}$ in reverse alphabetical order. Enter the letter 0 into all the boxes in the j^{th} column that remain unfilled,

(ii) if $l_j \geq n$, start from the box (t_j, j) and working downwards, enter the letters of R_{ς_j+1} in alphabetical order. Then starting from the box (b_j, j) and working upwards enter the letters of $\overline{R}_{\varsigma_j}$ in reverse alphabetical order. Enter the letter 0 into all the boxes in column j that remain unfilled.

Figure 7.2: Non-overlapping paths and its associated non-generic skew tableau in type B_2.



	1	2	3	4
-5				1
-4			2	0
-3			0	$\overline{2}$
-2	1	1	0	
-1	2	2	$\overline{2}$	
0	$\overline{2}$	$\overline{2}$	$\overline{1}$	
1	$\overline{1}$	$\overline{1}$		

Example Figure 7.3 corresponds to the non-generic skew tableau obtained from the non-overlapping paths in the same figure by using Proposition 7.4.3. In particular, the ℓ-weight $Y_{2,7}^{-1}Y_{2,15}Y_{2,19}^{-1}Y_{1,30}Y_{1,32}^{-1}Y_{1,36}Y_{2,37}^{-1}$ belongs to the q-character of $L_q(Y_{2,1}Y_{1,14}Y_{2,27}Y_{2,29}Y_{2,35})$, for $\mathfrak{g}$ is of type B_2, by Theorem 7.3.2.

Proposition 7.3.1. *The map $\overline{\mathscr{P}}_{(i_t,k_t)_{1\le t\le T}} \to \mathrm{Tab}(\lambda/\mu)$ sending $(p_1,\dots,p_T)\mapsto \mathcal{T}_{(p_1,\dots,p_T)}$ is a bijection and*

$$\mathsf{m}((p_1,\dots,p_T)) = M(\mathcal{T}_{(p_1,\dots,p_T)}). \tag{7.3.1}$$

Proof. Let for brevity $\mathcal{T}=\mathcal{T}_{(p_1,\dots,p_T)}$. We show that $\mathcal{T}$ is, in fact, a skew tableau. First, observe that no box of the column j is filled up twice. If $l_j<n$, it follows from (7.1.2). If $l_j\ge n$, it follows from (7.1.3) and from relation

$$l_j = 2n - i_{\varsigma_j} + \max\{i_{\varsigma_j} - i_{\varsigma_j+1}, 0\} + \sigma_{\varsigma_j}.$$

Moreover, $\mathcal{T}$ respects the vertical rule (V), by construction. To prove that $\mathcal{T}$ respects the horizontal rules (H), it suffices to study $\mathcal{T}(i,j)$ and $\mathcal{T}(i,j+1)$ for each $(i,j)\in(\lambda/\mu)$. We split the analysis in the following cases:

(a) $l_j<n$ and $l_{j+1}<n$,

(b) $l_j>n$ and $j\notin\mathscr{S}$,

(c) $j\in\mathscr{S}$ and $l_j\ge n$,

(d) $j\in\mathscr{S}$ and $l_{j+1}<n$,

(e) $l_j<n$ and $l_{j+1}\ge n$.

Let $B_j\subseteq\{1,\dots,n\}$ and $\overline{B}_j\subset\{\overline{1},\dots,\overline{n}\}$ (resp. B_{j+1} and $\overline{B}_{j+1}$) be the sets which fill up the column j (resp. $j+1$) by the above procedure. In each case, we prove that

(i) $\#B_j\cap\{1,\dots,r\}\ge \#^{[t_j-t_{j+1}]}B_{j+1}\cap\{1,\dots,r\}$, $r=1,\dots,n-1$,

(ii) $\#\overline{B}_j^{[b_j-b_{j+1}]}\cap\{\overline{1},\dots,\overline{r}\}\le \#\overline{B}_{j+1}\cap\{\overline{1},\dots,\overline{r}\}$, $r=1,\dots,n-1$,

(iii) $\#B_j+\#\overline{B}_{j+1}\ge b_{j+1}-t_j+1$.

In particular, (i)-(iii) implies that j and $j+1$ respect the horizontal rule.

Consider the case (a). By Lemma 7.2.4, we have $b_j-b_{j+1}=\sigma_{\varsigma_j}+\max\{i_{\varsigma_j}-i_{\varsigma_j+1},0\}$. Then $t_j-t_{j+1}=\sigma_{i_{\varsigma_j}}+\max\{i_{\varsigma_j+1}-i_{\varsigma_j},0\}$, and $b_{j+1}-t_j=i_{\varsigma_j}-\max\{i_{\varsigma_j}-i_{\varsigma_j+1},0\}-\sigma_{\varsigma_j}-1$. Therefore (i), (ii) and (iii) follow, respectively, by (7.1.7), (7.1.8) and (7.1.6).

Consider the case (b). By Lemma 7.2.4, we see that $b_j - b_{j+1} = \sigma_{\varsigma_j} + \max\{i_{\varsigma_{j+1}} - i_{\varsigma_j}, 0\}$, $t_j - t_{j+1} = \sigma_{\varsigma_j+1} + \max\{i_{\varsigma_{j+1}} - i_{\varsigma_{j+2}}, 0\}$, and $b_{j+1} - t_j = 2n - i_{\varsigma_{j+1}} - 1$. Therefore, (i), (ii) and (iii) follow, respectively, by (7.1.5), (7.1.4) and (7.1.2).

Consider the case (c). By Lemma 7.1.2, we have

$$\#\overline{R}_{\varsigma_j}^{[\sigma_{\varsigma_j}]} \cap \{\overline{1}, \dots, \overline{r}\} \leq \#\overline{R}_{\varsigma_j+1} \cap \{\overline{1}, \dots, \overline{r}\}, \tag{7.3.2}$$

and

$$\#\overline{R}_{\varsigma_j+1}^{[\sigma_{\varsigma_j+1}]} \cap \{\overline{1}, \dots, \overline{r}\} \leq \#\overline{R}_{\varsigma_{j+1}} \cap \{\overline{1}, \dots, \overline{r}\}, \tag{7.3.3}$$

for $r = 1, \dots, n-1$. Therefore, inequalities (7.3.2) and (7.3.3) combined imply

$$\#\overline{R}_{\varsigma_j}^{[\sigma_{\varsigma_j}+\sigma_{\varsigma_j+1}]} \cap \{\overline{1}, \dots, \overline{r}\} \leq \#\overline{R}_{\varsigma_{j+1}} \cap \{\overline{1}, \dots, \overline{r}\}, \quad r = 1, \dots, n-1. \tag{7.3.4}$$

Moreover, since $l_j = b_j - t_j + 1$ and $s_j - t_j = n - 1$, Lemma 7.2.4 implies

$$\sigma_{\varsigma_j} + \sigma_{\varsigma_j+1} + n - i_{\varsigma_j} = b_j - b_{j+1},$$

hence, (ii) is proved in this case. We prove (i) similarly, using

$$\#R_{\varsigma_j+1} \cap \{1, \dots, r\} \geq \#^{[\sigma_{\varsigma_j+1}+\sigma_{\varsigma_{j+1}}+n-i_{\varsigma_{j+1}+1}]} R_{\varsigma_{j+1}+1} \cap \{1, \dots, r\}, \quad r = 1, \dots, n-1, \tag{7.3.5}$$

and $\sigma_{\varsigma_j+1} + \sigma_{\varsigma_{j+1}} + n - i_{\varsigma_{j+1}+1} = t_j - t_{j+1}$.

Since $i_{\varsigma_j+1} = i_{\varsigma_{j+1}} = n$, by (7.1.3) it follows that $\#\overline{R}_{\varsigma_j+1} + \#R_{\varsigma_{j+1}} \geq n + \sigma_{\varsigma_j+1}$. Therefore (iii) follows by observing that $b_{j+1} - t_j \leq n - 1$.

Consider the case (d). By Lemma 7.1.2, we have

$$\#\overline{R}_{\varsigma_j+1}^{[\sigma_{\varsigma_j+1}]} \cap \{\overline{1}, \dots, \overline{r}\} \leq \#\overline{R}_{\varsigma_{j+1}} \cap \{\overline{1}, \dots, \overline{r}\}, \quad r = 1, \dots, n-1.$$

However, since $i_{\varsigma_j+1} = n$, the above inequality is equivalent to

$$\#R_{\varsigma_j+1} \cap \{1, \dots, r\} \geq \#^{[\sigma_{\varsigma_j+1}]} S_{\varsigma_{j+1}} \cap \{1, \dots, r\}, \quad r = 1, \dots, n-1. \tag{7.3.6}$$

Moreover, by Lemma 7.2.4, we prove that $t_j - t_{j+1} = \sigma_{\varsigma_j+1}$, using $s_j = t_j + n - 1$, $b_{j+1} - i_{\varsigma_{j+1}} = t_{j+1} - 1$. Item (i) follows in this case.

By Lemma 7.1.2, we also have

$$\#\overline{R}_{\varsigma_j}^{[\sigma_{\varsigma_j}+n-i_{\varsigma_j}]} \cap \{\overline{1}, \dots, \overline{r}\} \leq \#\overline{R}_{\varsigma_j+1} \cap \{\overline{1}, \dots, \overline{r}\}, \tag{7.3.7}$$

and

$$\#R_{\varsigma_j+1} \cap \{1, \dots, r\} \geq \#^{[\sigma_{\varsigma_j+1}+n-i_{\varsigma_{j+1}}]} R_{\varsigma_{j+1}} \cap \{1, \dots, r\}, \tag{7.3.8}$$

for $r = 1, \ldots, n-1$. Since $i_{\varsigma_j+1} = n$, (7.3.8) is equivalent to

$$\#\overline{R}_{\varsigma_j+1}^{[\sigma_{\varsigma_j+1}+n-i_{\varsigma_{j+1}}]} \cap \{\overline{1}, \ldots, \overline{r}\} \leq \#\overline{S}_{\varsigma_{j+1}} \cap \{\overline{1}, \ldots, \overline{r}\}, \quad r = 1, \ldots, n-1. \tag{7.3.9}$$

Inequalities (7.3.7) and (7.3.9) combined imply

$$\#\overline{R}_{\varsigma_j}^{[\sigma_{\varsigma_j}+n-i_{\varsigma_j}+\sigma_{\varsigma_j+1}+n-i_{\varsigma_{j+1}}]} \cap \{\overline{1}, \ldots, \overline{r}\} \leq \#\overline{S}_{\varsigma_{j+1}} \cap \{\overline{1}, \ldots, \overline{r}\}, \quad r = 1, \ldots, n-1.$$

Moreover, by Lemma 7.2.4, we have

$$b_j - b_{j+1} = \sigma_{\varsigma_j} + n - i_{\varsigma_j} + \sigma_{\varsigma_j+1} + n - i_{\varsigma_{j+1}},$$

using $l_j = b_j - t_j + 1$ and $s_j = t_j + n - 1$, and then, (ii) holds in this case.

By (7.1.3), we have

$$\#\overline{R}_{\varsigma_j+1} + \#R_{\varsigma_{j+1}} \leq 2n - i_{\varsigma_j+1} + (n - i_{\varsigma_{j+1}}) + \sigma_{\varsigma_j+1},$$

which is equivalent to

$$n - \#R_{\varsigma_j+1} + n - \#\overline{S}_{\varsigma_{j+1}} \leq 2n - i_{\varsigma_j+1} + (n - i_{\varsigma_{j+1}}) + \sigma_{\varsigma_j+1}.$$

Since $i_{\varsigma_j+1} = n$, we have

$$\#R_{\varsigma_j+1} + \#\overline{S}_{\varsigma_{j+1}} \geq i_{\varsigma_{j+1}} - \sigma_{\varsigma_j+1}.$$

Since $b_{j+1} - t_j + 1 = i_{\varsigma_{j+1}} - \sigma_{\varsigma_j+1}$, (iii) holds in this case

Case (e) is proved similarly to case (d).

Thus, for $(p_1, \ldots, p_T) \in \overline{\mathscr{P}}_{(i_t,k_t)_{1\leq t\leq T}}$, the map $\mathcal{T} = \mathcal{T}_{(p_1,\ldots,p_T)}$, as defined above, is a skew tableau. Since each path p_t is completely described by the pair of sets R_t and $\overline{R}_t$, and the above description can be made backwards in order to obtain the sequence $(p_1, \ldots, p_T)_{\mathcal{T}}$ from a tableaux $\mathcal{T}$, we have a bijection between $\overline{\mathscr{P}}_{(i_t,k_t)_{1\leq t\leq T}}$ and $\mathrm{Tab}(\lambda/\mu)$.

It remains to show that (7.3.1) holds. By Lemma 6.3.3 and the definition of highest path we have $\boldsymbol{\pi}(\lambda/\mu) = \prod_{t=1}^{T} \mathsf{m}(p^+_{i_t,k_t})$. Since each path $(p_1, \ldots, p_T)$ is obtained from $(p^+_{i_1,k_1}, \ldots, p^+_{i_T,k_T})$ by applying a sequence of lowering moves $\mathscr{A}^{-1}_{j_s,l_s}, 0 \leq s \leq S$, for some $S \in \mathbb{Z}_{\geq 0}$, it suffices to check that if $(i,k) \in \mathcal{W}$ is such that $(p_1, \ldots, p_T)\mathscr{A}^{\pm 1}_{i,k} \in \overline{\mathscr{P}}_{(i_t,k_t)_{1\leq t\leq T}}$, then

$$M(\mathcal{T}_{(p_1,\ldots,p_T)\mathscr{A}^{\pm 1}_{i,k}}) = M(\mathcal{T}_{(p_1,\ldots,p_T)})A^{\pm 1}_{i,k}.$$

It is easily done by inspection.

The proof of the proposition is finished.

□

Theorem 7.3.2. *Let (λ/μ) be a generic super skew diagram and $\boldsymbol{\pi} = \boldsymbol{\pi}(\lambda/\mu)$. Then*

$$\chi_q(L_q(\boldsymbol{\pi})) = \sum_{\mathcal{T}\in \mathrm{Tab}((\lambda/\mu))} M(\mathcal{T}).$$

Proof. Given Proposition 7.3.1, this is immediate from Theorem 5.1.2. □

7.4 Non-generic super skew diagrams

In this section we discuss non-generic skew diagrams.

We call a column j of diagram (λ/μ) *non-generic* if

$$\#\{i \in \mathbb{Z} | (i,j) \in (\lambda/\mu), (i, j+1) \in (\lambda/\mu)\} = 2n. \tag{7.4.1}$$

Let (λ/μ) be a non-generic super skew diagram. Let column j' be such that it is non-generic and all columns j with $j > j'$ are generic. Define

$$\begin{aligned}(\lambda'/\mu') = \{&(i,j) \in (\lambda/\mu) | j \leq j'\} \cup \\ &\{(t_{j'} - r, j') | 1 \leq r \leq l_{j'+1} - 2n + 1\} \cup \{(i-1, j-1) | (i,j) \in (\lambda/\mu), j > j'+1\}.\end{aligned}$$

The following lemma is straightforward.

Lemma 7.4.1. *The shape (λ'/μ') is a super skew diagram. The number of non-empty columns of (λ'/μ') is one less than that of (λ/μ). The number of boxes of (λ'/μ') is $2n - 1$ less than that of (λ/μ). The number of non-generic columns of (λ'/μ') is one less than that of (λ/μ).* □

We call the super skew diagram (λ'/μ') *closely related to* (λ/μ). We call a generic super skew diagram (λ'/μ') *related* to a non-generic super skew diagram (λ/μ) if there exists a sequence of super skew diagrams (λ_i/μ_i), $1 \leq i \leq S$, $S \in \mathbb{Z}_{\geq 1}$, such that $(\lambda_1/\mu_1) = (\lambda/\mu)$, $(\lambda_S/\mu_S) = (\lambda'/\mu')$ and (λ_i/μ_i) is closely related to $(\lambda_{i-1}/\mu_{i-1})$ for $2 \leq i \leq S$.

Corollary 7.4.2. *Let (λ/μ) be a non-generic super skew diagram. Then there exist unique generic skew diagram related to (λ/μ).* □

Finally, we observe that the related super skew diagrams correspond to the same module over the affine quantum algebra.

Proposition 7.4.3. *Let super skew diagram (λ'/μ') be related to super skew diagram (λ/μ). Then there exist a bijection $\tau : \mathrm{Tab}(\lambda/\mu) \to \mathrm{Tab}(\lambda'/\mu')$ such that $M(\mathcal{T}) = M(\tau\mathcal{T})$ for all $\mathcal{T} \in \mathrm{Tab}(\lambda/\mu)$.*

Proof. It is sufficient to give such a bijection for closely related super skew diagrams. Let (λ'/μ') be related to a non-generic super skew diagram (λ/μ) and let $\mathcal{T} \in \mathrm{Tab}(\lambda/\mu)$. We define $\tau\mathcal{T}$ as follows:

$$(\tau\mathcal{T})(i,j) = \begin{cases} \mathcal{T}(i,j), & j < j' \text{ or } j = j',\ i > b'_j - n, \\ \mathcal{T}(i+1, j+1), & j > j' \text{ or } j = j',\ i < t_{j'} + n, \\ 0, & j = j' \text{ and } t_{j'} + N \leq i \leq b_{j'} - n, \end{cases}$$

where $t_{j'}$ and $b_{j'}$ are, respectively, the top and the bottom box of the column j' in (λ'/μ').

All the checks are straightforward. □

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