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JAQUES DUBUCS AND MICHEL BOURDEAU (eds.) *Constructivity and Computability in Historical and Philosophical Perspective* (*Logic, Epistemology and the Unity of Science*, vol. 34) Various authors. Springer, 2014. xi + 214 pp. €83. ISBN 978-94-017-9216-5

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reviewed by
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THIS IS A LOOSE ASSORTMENT of historical, philosophical and/or expository papers dealing with themes of computability, computational complexity and constructiveness in mathematics.

The first part of Göran Sundholm's wide-ranging essay concerns itself with the question: can the notion of a *function*, as employed in constructive mathematics, be fruitfully identified with that of a *recursive function*, as understood in computability theory?

Roughly stated, a function is recursive if and only if instructions for its calculation can be issued in the form: *given the argument(s), proceed thus-and-so until such-and-such a termination condition obtains; then read off the value from the end-state of your computation.* The sorts of instruction allowed in spelling out the *thus-and-so* can be circumscribed in various ways – in terms of Turing machines, Kleene's primitive recursive T predicate, etc. – but on any standard account the resulting class of functions remains the same.

Now although specifying the rules of a process of computation in an unequivocal way presents no special difficulty (at any rate, none that needs to be addressed here), such a specification will only define a total function if, for every possible argument, the resulting computation will indeed lead eventually to a state where the designated termination condition obtains; and the specification will only be *known* to define a total function if this unfailing terminability can be *proved*. This, Sundholm argues, echoing Heyting and others, spells trouble for any attempt at basing constructive reasoning on the notion of a recursive function by way of the BHK interpretation of the logical operators. For according to that interpretation, a statement of the form $\forall x \exists y Rxy$ – such as the claim that, for any input, a certain process of computation will eventually reach a terminating state – will only qualify as assertable if a constructive function can be produced of which it can be recognized that for any x it will deliver a y such that Rxy. Thus, establishing an $\forall\exists$ claim requires producing a recognizably total *constructive* function, while recognizing a *recursive* function as total requires establishing an $\forall\exists$ claim, and so – so the argument goes if I have understood it correctly – an identification of constructivity with recursiveness will land us in an infinite regress.

It is not obvious to me, though, that declining to make this identification will be sufficient to deliver the constructivist from problems of circularity. On the BHK interpretation, a universally quantified claim $\forall x Px$ is assertable only on the basis of a function of which it can be recognized that, applied to any x, it will deliver a (canonical) proof that Px. Thus, establishing a universally quantified claim requires recognizing that a certain universally quantified claim holds good. Here again (the point will be familiar) one senses the rumblings of an infinite regress, and one that arises regardless of how the notion of a function is spelled out. If Sundholm has a reason for considering the regress described earlier more vicious than this latter one, I cannot say what it is; it would seem that in both cases there will be cause for concern if the BHK interpretation is expected to furnish a reductive *justification* of modes of inference that would otherwise stand unwarranted, but less so if it is seen as a purely descriptive account of the meanings of mathematical assertions.

Indeed, there seem to be at least three different issues at play here: (1) Are all functions used in constructive mathematics recursive? (2) Is a grasp of the technical notion of a recursive function helpful in, maybe even a prerequisite for, grasping the intuitionistic meanings of the logical connectives? (3) In order to justify deductions in constructive mathematics, is it necessary first to establish that certain computational procedures are guaranteed to terminate? It is not always clear to me which of these questions Sundholm is addressing.

In the second part of his paper, Sundholm considers the concepts of recursiveness and constructiveness in the light of Brouwer's theory of the Creative Subject (an idealized mathematical thinker), as axiomatized and investigated by Kreisel, Myhill and others; the axiomatic theory is labelled CS. He demonstrates how the axioms of CS can be derived from a variant of the principle known as Kripke's Schema, and recounts an argument of Kripke's to the effect that CS in turn implies the existence of a function amenable to effective evaluation by the Creative Subject (and hence constructive) despite not being recursive. On the other hand, Sundholm also raises serious *a priori* objections to one of the axioms of CS. I am unable to make out what moral he would have us draw from the fact that the identification of constructiveness with recursiveness is contradicted by a theory that he judges inherently implausible anyway.

Finally, Sundholm claims to demonstrate that CS is "classically valid", a result he describes as "somewhat surprising". "Classical validity" in this context turns out to mean, roughly, derivability from the hypothetical existence of a function guaranteed, for any proposition A, to deliver a proof either of A or of its negation. This strikes me as an abuse of the term "classical"; what moves the classical mathematician to adopt the law of Excluded Middle is not a belief that every proposition can be either proved or refuted, but a willingness to dissociate the notion of truth from that of provability. (Granted, a sufficiently epistemically diluted understanding of *function* and *proof* will make the premiss of Sundholm's proof acceptable to a classicist – but on such an understanding, given the way Sundholm defines the creative-subject notation in terms of the behaviour of certain functions, the argument's conclusion becomes classically trivial.)

Thierry Coquand's contribution to the volume also treats – albeit from a more expressly historical perspective than Sundholm's – of the question of constructiveness versus recursiveness in functions and the problem whether a given process of computation will ever reach completion and yield a value. In the course of this discussion, Coquand also brings up (as does Sundholm) the issue of functions that are defined in such a way as to guarantee computability in the eyes of a classical mathematician despite providing no guidance for actual computation. An example of such a definition might be

$$f(x) = \begin{cases} 1 & \text{if Goldbach's conjecture is true,} \\ 0 & \text{if Goldbach's conjecture is false.} \end{cases}$$

An extensionally inclined classicist will maintain that this f is either constant 1 or constant 0, and hence certainly a computable function – we just do not know which one. A constructivist, by contrast, will dispute that a function has been defined at all. Well, the difference in attitude is clear enough; but how exactly is this meant to bear on the termination issue? Not having been given proper instructions for computation is one thing; not having received assurance that following given instructions will lead to a desired result is another. Sundholm and Coquand both run these questions together in ways that leave this reader perplexed.

The issue of constructive versus recursive functions is encountered yet again in Marc van Atten's meticulously researched and philosophically perspicacious chronicle of Gödel's shifting positions vis-à-vis intuitionism. With special emphasis on the *Dialectica* interpretation, van Atten traces in Gödel a movement towards an acceptance of the concept of constructive function(al) as a primitive notion sufficiently intelligible for foundational work independently of any technical explication in terms of recursiveness – a historical development similar to that observed by Coquand in constructivist mathematicians.

The remaining four papers can all be described as concerning themselves with aspects of real-world computing, as opposed to computability in principle.

Jean Fichot puts an interesting, "feasibilist" twist on proof-theoretical semantics by imposing the requirement that deductive reasoning be underpinned by the existence, not merely of canonical proofs, but of canonical proofs of manageable size. Two systems of arithmetical deduction are analysed in this regard, the upshot being that “the main restriction that must be made to constructive arithmetic in order to obtain a feasible arithmetic is the adoption of a weak induction principle” (p. 155). One’s curiosity is piqued as to the implications of such a restriction: will standard results from elementary number theory still be attainable?

Jean Mosconi provides a historical survey, rich in enjoyable detail, of formal models of computation from Turing’s seminal work through the 1960’s; Marie Ferbus-Zanda and Serge Grigorieff give a technical introduction to Kolmogorov complexity theory; and Ferbus-Zanda applies the concepts of that theory to issues of information processing in the Internet age.

Ensuring proper English grammar throughout the volume does not appear to have been a primary editorial concern; in places, deficiencies in this regard are serious enough to impair readability.