ON THE SINGULAR SCHEME OF SPLIT FOLIATIONS

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Abstract. We prove that the tangent sheaf of a codimension one locally free distribution splits as a sum of line bundles if and only if its singular scheme is arithmetically Cohen-Macaulay. In addition, we show that a foliation by curves is given by an intersection of generically transversal holomorphic distributions of codimension one if and only if its singular scheme is arithmetically Buchsbaum. Finally, we establish that these foliations are determined by their singular schemes, and deduce that the Hilbert scheme of certain arithmetically Buchsbaum schemes of codimension 2 is birational to a Grassmannian.

1. Introduction

The goal of this paper is to characterize holomorphic distributions on projective spaces either whose tangent sheaf splits as a sum of line bundles, or which are globally defined as an intersection of generically transversal distributions of codimension one. The last case is equivalent to the splitting as a sum of line bundles of a certain reflexive sheaf canonically defined from the foliation, which we call *Pfaff sheaf*.

Concerning split tangent sheaves, we start by recalling J. P. Jouanolou's celebrated 1979 paper [21]. It is doubtless a landmark in the study of holomorphic foliations in projective spaces. By means of algebro-geometric tools, he described, for instance, the irreducible components of the space of one-codimensional foliations of degree 0 and 1. New components were found later on by Omegar [3] and D. Cerveau and A. Lins Neto [7].

In such a study, it turned out to be helpful deciding when the tangent sheaf of the foliation splits as a sum of line bundles. In fact, F. Cukierman and J. V. Pereira proved in [12] that the splitting conditon, along with some properties for the singular locus, provide stability under deformations and make it possible to characterize certain components of the space of foliations.

More recently, L. Giraldo and A. J. Pan-Collantes showed in [19] that the tangent sheaf of a foliation of dimension 2 on $\mathbb{P}^3$ splits if and only if its singular scheme is arithmetically Cohen Macaulay (ACM). The first part of the present article is devoted precisely to extend this result. More precisely, we prove the following result.

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Theorem 1. *Let $\mathscr{F}$ be a distribution on $\mathbb{P}^n$ of codimension k, such that the tangent sheaf $\mathcal{T}_{\mathscr{F}}$ is locally free and whose singular locus has the expected dimension $n-k-1$. If $\mathcal{T}_{\mathscr{F}}$ splits as a sum of line bundles, then $\operatorname{Sing}(\mathscr{F})$ is arithmetically Cohen–Macaulay. Conversely, if $k=1$ and $\operatorname{Sing}(\mathscr{F})$ is arithmetically Cohen–Macaulay, then $\mathcal{T}_{\mathscr{F}}$ splits as a sum of line bundles.*

Note that [19, Thm 3.3] corresponds to the case $n=3$ and $k=1$. We emphasize that when n is odd and greater than 3, a technical difficulty arises, which we resolve via a spectral sequence.

The *Pffaf sheaf* of a distribution $\mathscr{F}$ is defined to be the dual of its normal sheaf $\mathcal{N}_{\mathscr{F}}$. The second part of this paper concerns distributions whose Pfaff sheaves split as a sum of line bundles. For distributions with pure dimensional singular locus, this property is equivalent to saying that the distribution is an intersection of generically transversal holomorphic distributions of codimension one.

We say that $\mathscr{F}$ is a *complete intersection distribution* if there exist generically transversal holomorphic distributions $\mathscr{F}_1,\dots,\mathscr{F}_k$ of codimension one on $\mathbb{P}^n$ such that $\mathscr{F}=\mathscr{F}_1\cap\cdots\cap\mathscr{F}_k$ is a distribution of codimension k on $\mathbb{P}^n$. In this case, the Pfaff sheaf of $\mathscr{F}$ is a locally free sheaf of the form

$$\mathcal{P}_{\mathscr{F}}=\bigoplus_{i=1}^{k}\mathcal{O}_{\mathbb{P}^n}(-2-d_i)$$

where $d_i=\deg(\mathscr{F}_i)\geq 0$. Then $\mathscr{F}$ is induced by a k-form

$$\omega\in H^0(\mathbb{P}^n,\Omega^k_{\mathbb{P}^n}\otimes\det(\mathcal{P}^*_{\mathscr{F}}))$$

globally decomposable and $\deg(\mathscr{F})=d_1+\ldots+d_k+k-1$. In particular, $\deg(\mathscr{F})\geq k-1$.

Our strategy is to link this subject with arithmetically Buchsbaum schemes, using a characterization of codimension two arithmetically Buchsbaum schemes due to M. C. Chang [11]. The results we obtained are stated below.

Theorem 2. *Let $\mathscr{F}$ be a holomorphic distribution of dimension r on $\mathbb{P}^n$. Suppose that $\operatorname{cod}(\operatorname{Sing}(\mathscr{F}))=r+1$ and that the induced Pfaff system $\mathcal{P}_{\mathscr{F}}\to\Omega^1_{\mathbb{P}^n}$ is locally free. Then the following hold:*

(i) *if $r=1$ and $\operatorname{Sing}(\mathscr{F})$ is of pure dimension, then the following are equivalent:*
- *$\mathcal{P}_{\mathscr{F}}=\bigoplus_{i=1}^{n-1}\mathcal{O}_{\mathbb{P}^n}(-2-d_i)$, with $d_i\geq 0$ for all $i=1,\dots,n-1$.*
- *$\operatorname{Sing}(\mathscr{F})$ is arithmetically Buchsbaum; $h^1(\mathcal{I}_Z(d-1))=1$; and this is the only nonzero intermediate cohomology for $H^i_*(\mathcal{I}_Z)$ in the range $1\leq i\leq n-2$.*
- *$\mathscr{F}$ is a complete intersection foliation.*

Moreover, if $\operatorname{Sing}(\mathscr{F})$ is smooth and $d_i\geq n$ for some i, then $\operatorname{Sing}(\mathscr{F})$ is of general type.

(ii) *if* $r = 1$, *and* $\mathscr{F}'$ *is a foliation by curves on* $\mathbb{P}^n$, *with the same degree of* $\mathscr{F}$ *such that* $\mathrm{Sing}(\mathscr{F}) \subset \mathrm{Sing}(\mathscr{F}')$, *then* $\mathscr{F}' = \mathscr{F}$.

(iii) *if* $r = 2$ *and* $\mathcal{P}_{\mathscr{F}}$ *splits as a sum of line bundles, then* $\mathrm{Sing}(\mathscr{F})$ *is arithmetically Buchsbaum, but not arithmetically Cohen Macaulay;*

(iv) *if* $r = 3$, $\mathcal{P}_{\mathscr{F}}$ *splits as a sum of line bundles, the* $|d_i - d_j| \neq 1$, *and, for* $n \geq 7$, *the* $d_i \neq 1$ *as well, then* $\mathrm{Sing}(\mathscr{F})$ *is arithmetically Buchsbaum, but not arithmetically Cohen Macaulay.*

An interesting consequence is the construction of holomorphic foliation with prescribed singular scheme being arithmetically Buchsbaum.

Corollary 3. *Let* Z *be an arithmetically Buchsbaum scheme of codimension* 2 *such that* $h^1(\mathcal{I}_Z(d-1)) = 1$ *is the only nonzero intermediate cohomology for* $H^i_*(\mathcal{I}_Z)$ *in the range* $1 \leq i \leq n-2$. *If* $d \geq n-2$, *then there exists an unique one-dimensional foliation* $\mathscr{F}$, *of degree* d, *such that* $\mathrm{Sing}(\mathscr{F}) = Z$.

It is therefore worth pointing out that the latter topic appears implicitly in works of classical algebro-geometers such as Fano, Castelnuovo and Palatini, in their study of varieties given by the set of centers of complexes belonging to the linear systems of linear complexes (see [16] and references therein).

We believe that an approach via distributions may clarify the study of Hilbert schemes of such varieties. In fact, the Corollary 3 shows that there exists a one-to-one correspondence between a class of these varieties and distribution given by intersections of generically transversal holomorphic distributions of codimension one.

For instance, let $\mathcal{H}_\ell$ be the union of components of the Hilbert scheme of $\mathbb{P}^n$ containing the degeneracy locus of a general map of the form

$$\zeta : \mathcal{O}_{\mathbb{P}^n}(-\ell)^{\oplus(n-1)} \longrightarrow \Omega^1_{\mathbb{P}^n},$$

We have associated to ζ a globally decomposable $(n-1)$-form $\omega_\zeta = \omega_1 \wedge \cdots \wedge \omega_{n-1}$, where $\omega_i \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(\ell))$ for all $i = 1, \ldots, n-1$. Thus, we get a natural rational map

$$\begin{array}{rcl} \rho_\ell : \mathbf{G}(n-1, H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(\ell))) & \dashrightarrow & \mathcal{H}_\ell \\ \\ (\omega_\zeta = \omega_1 \wedge \cdots \wedge \omega_{n-1}) & \mapsto & \mathrm{Sing}(\omega_\zeta) \end{array}$$

where $\mathbf{G}(n-1, H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(\ell)))$ is the Grassmann variety parametrizing $(n-1)$-dimensional vector subspaces of $H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(\ell))$. It follows form Corollary 3 that ρ_ℓ is generically injective. In [16, Theorem, pg. 2] is showed the birationality of ρ in the case $\ell = 2$.

Corollary 4. *The map* ρ_ℓ *is generically injective, for all* $\ell \geq 2$.

In the section 5, we use the part (ii) of Theorem 2 and M. C. Chang's characterization of codimension two arithmetically Buchsbaum to classify complete intersection foliations by curves whose singular locus is smooth and non-general type.

Finally, we give some characterizations for the splitting type for distributions with (co)tangent sheaf locally free and globally generated.

Theorem 5. *Let $\mathscr{F}$ be a distribution on $\mathbb{P}^n$ of dimension r, degree d, and such that the tangent sheaf $\mathcal{T}_{\mathscr{F}}$ is locally free. Then the following hold:*

(i) *if $\mathcal{T}_{\mathscr{F}}^*$ is globally generated and ample, then $\mathrm{Sing}(\mathscr{F})$ is nonempty with pure dimension $r-1$. This holds in particular if $\mathcal{T}_{\mathscr{F}} = \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(-d_i)$ with $d_i > 0$ for all i. Moreover*

$$\deg(\mathrm{Sing}(\mathscr{F})) = \deg(c_{n-r+1}(\mathcal{T}_{\mathbb{P}^n} - \mathcal{T}_{\mathscr{F}})) \leq d^{n-r+1} + d^{n-r} + \cdots + d + 1.$$

(ii) *if $\mathscr{F}$ is a foliation, then $\mathcal{T}_{\mathscr{F}} = \mathcal{O}_{\mathbb{P}^n}(1-d) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r-1}$ if and only if there exists a generic linear projection $\pi : \mathbb{P}^n \to \mathbb{P}^{n-r+1}$ and a one-dimensional foliation $\mathscr{G}$ on $\mathbb{P}^{n-r+1}$, of degree d, with isolated singularities, such that $\mathscr{F} = \pi^*\mathscr{G}$.*

(iii) *$\mathcal{T}_{\mathscr{F}} = \mathcal{O}_{\mathbb{P}^n}(r-d) \oplus \mathcal{O}_{\mathbb{P}^n}^{\oplus r-1}$ if and only if $\mathcal{T}_{\mathscr{F}}^*$ is globally generated and $\mathscr{F}$ admits a locally free subdistribution of rank and degree $r-1$. The last condition can be dropped if $d = r+1$, and replaced by $c_r(\mathcal{T}_{\mathscr{F}}) = 0$ if $d = r$;*

(iv) *if $d = r-2$ and $\mathcal{T}_{\mathscr{F}}$ is globally generated and does not split, then $\mathcal{T}_{\mathscr{F}}$ is the tangent sheaf of the contact distribution on $\mathbb{P}^3$.*

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2. Preliminaries

We start by collecting relevant definitions and results from the literature.

2.1. **The Eagon-Northcott resolution.** Let $\mathcal{E}$ and $\mathcal{G}$ be locally free sheaves on X of rank e and g, respectively, and $\varphi : \mathcal{E} \to \mathcal{G}$ a generically surjective morphism. The induced map $\wedge^g\varphi : \bigwedge^g \mathcal{E} \to \det(\mathcal{G})$ corresponds to a global section $\omega_\varphi \in H^0\big(X, \bigwedge^g(\mathcal{E}^*) \otimes \det(\mathcal{G})\big)$.

Definition 2.1. The *degeneracy scheme* $\mathrm{Sing}(\varphi)$ of the map $\varphi : \mathcal{E} \to \mathcal{G}$ is the zero scheme of the associated global section $\omega_\varphi \in H^0\big(X, \bigwedge^g(\mathcal{E}^*) \otimes \det(\mathcal{G})\big)$.

Alternatively, ω_φ may also be regarded as a map $\bigwedge^g \mathcal{E} \otimes \det(\mathcal{G})^* \to \emptyset_X$; its image is the ideal sheaf of $\mathrm{Sing}(\varphi)$.

Suppose that $Z = \mathrm{Sing}(\varphi)$ has pure expected dimension, i.e., Z has pure codimension equal to $e-g+1$. Then the structure sheaf of Z admits a special resolution, called the *Eagon-Northcott resolution* (see for instance [13, A2.6]):

$$0 \to \bigwedge^{e} \mathcal{E} \otimes S_{e-g}(\mathcal{G}^*) \otimes \det(\mathcal{G}^*) \to \bigwedge^{e-1} \mathcal{E} \otimes S_{e-g-1}(\mathcal{G}^*) \otimes \det(\mathcal{G}^*) \to \dots$$
$$\to \bigwedge^{g+1} \mathcal{E} \otimes \mathcal{G}^* \otimes \det(\mathcal{G}^*) \to \bigwedge^{g} \mathcal{E} \otimes \det(\mathcal{G}^*) \to \mathcal{I}_Z \to 0.$$

2.2. **Holomorphic Distributions and Pfaff systems.** For the remainder, X is an irreducible, non-singular complex scheme of dimension n. Its tangent sheaf is denoted by $\mathcal{T}_X$, Ω^1_X denotes its sheaf of differentials, and ω_X is its canonical bundle.

A *(saturated) distribution $\mathscr{F}$ of codimension k* on X is a short exact sequence

$$0 \longrightarrow \mathcal{T}_{\mathscr{F}} \xrightarrow{\varphi} \mathcal{T}_X \xrightarrow{\varpi} \mathcal{N}_{\mathscr{F}} \longrightarrow 0 \tag{1}$$

such that $\mathcal{N}_{\mathscr{F}}$, called the *normal sheaf of $\mathscr{F}$*, is a nontrivial torsion free sheaf of rank k on X. It follows that the sheaf $\mathcal{T}_{\mathscr{F}}$, called the *tangent sheaf of $\mathscr{F}$*, is a reflexive sheaf. We may also refer to the *dimension* of $\mathscr{F}$, which is defined as $\mathrm{rank}(\mathcal{T}_{\mathscr{F}}) = n-k$.

A distribution $\mathscr{F}$ is said to be *locally free* if $\mathcal{T}_{\mathscr{F}}$ is a locally free sheaf, and one says that $\mathscr{F}$ is a *foliation* if it is integrable, that is, the stalks of $\mathcal{T}_{\mathscr{F}}$ are invariant under Lie Bracket.

A dual perspective can have been considered. Indeed, a *Pfaff system $\mathcal{S}$ of codimension k* on X is a short exact sequence

$$0 \longrightarrow P_{\mathcal{S}} \xrightarrow{\phi} \Omega^1_X \xrightarrow{\pi} \Omega_{\mathcal{S}} \longrightarrow 0 \tag{2}$$

such that $\Omega_{\mathcal{S}}$ is a nontrivial torsion free sheaf of rank $n-k$ on X. It follows that $P_{\mathcal{S}}$ is a reflexive sheaf of rank k.

Indeed, every distribution $\mathscr{F}$ induces a Pfaff system $\mathcal{S}(\mathscr{F})$ by duality as follows. Dualizing the sequence (1) one obtains the exact sequence

$$0 \to \mathcal{N}^*_{\mathscr{F}} \to \Omega^1_X \to \mathcal{T}^*_{\mathscr{F}} \to \mathscr{E}xt^1(\mathcal{N}_{\mathscr{F}}, \mathcal{O}_X) \to 0.$$

Breaking it into two short exact sequences we obtain

$$0 \longrightarrow \mathcal{N}^*_{\mathscr{F}} \longrightarrow \Omega^1_X \longrightarrow \Omega \longrightarrow 0 \tag{3}$$

and

$$0 \longrightarrow \Omega \longrightarrow \mathcal{T}^*_{\mathscr{F}} \to \mathscr{E}xt^1(\mathcal{N}_{\mathscr{F}}, \mathcal{O}_X) \to 0. \tag{4}$$

It is easy to see from sequence (4) that Ω is a torsion free sheaf, thus sequence (3) is a Pfaff system. We say that $\mathcal{P}_{\mathscr{F}} := \mathcal{N}^*_{\mathscr{F}}$ is the *Pffaf sheaf* of the distribution $\mathscr{F}$. We say that the Pfaff system is *locally free* if $\mathcal{P}_{\mathscr{F}}$ is locally free.

A third, closely related concept is that of a Pfaff field, cf. [14]. A *Pfaff field η of rank r* is a nonzero map $\eta : \Omega^r_X \to \mathcal{L}$; alternatively, since $\mathrm{Hom}(\Omega^r_X, \mathcal{L}) \simeq$

$H^0(\Omega_X^{n-r} \otimes \mathcal{L} \otimes \omega_X)$, η can also be regarded as a $(n-r)$-form ω_η with values on the line bundle $\mathcal{L} \otimes \omega_X$.

The *singular scheme* $\mathrm{Sing}(\eta)$ *of a Pfaff field* η is the zero locus of ω_η. Alternatively, ω_η may also be regarded as a map $\psi_\eta : \Omega_X^{n-r} \otimes \mathcal{L}^* \otimes \omega_X^* \to \mathcal{O}_X$, and $\mathrm{Sing}(\eta)$ is the subscheme of X whose ideal is precisely the sheaf $\mathrm{im}(\psi_\eta)$.

Every distribution $\mathscr{F}$ of dimension r induces a Pfaff field $\eta_{\mathscr{F}}$ of rank r in the following manner. Dualizing and taking determinants of the map $\varphi : \mathcal{T}_{\mathscr{F}} \to \mathcal{T}_X$, we obtain a map:

$$\eta_{\mathscr{F}} : \Lambda^r(\mathcal{T}_X^*) \simeq \Omega_X^r \to \det(\mathcal{T}_{\mathscr{F}}^*). \tag{5}$$

The *singular scheme* $\mathrm{Sing}(\mathscr{F})$ *of a foliation* $\mathscr{F}$ is defined to be the singular scheme of the corresponding Pfaff field $\eta_{\mathscr{F}}$.

Note that if $\mathcal{T}_{\mathscr{F}}$ is locally free, $\mathrm{Sing}(\mathscr{F})$ coincides with the degeneracy locus

$$\Delta_\varphi := \{x \in X \mid \varphi(x) \text{ not injective}\}$$

of the map $\varphi : \mathcal{T}_{\mathscr{F}} \to \mathcal{T}_X$, thus $\mathrm{Sing}(\mathscr{F})$ is a determinantal scheme. In this case, one also sees that $\mathrm{Sing}(\mathscr{F})$ coincides with the singular locus of its normal sheaf, i.e.

$$\mathrm{Sing}(\mathscr{F}) = \mathrm{Sing}(\mathcal{N}_{\mathscr{F}}) := \{x \in X \mid (\mathcal{N}_{\mathscr{F}})_x \text{is not free } \mathcal{O}_x\text{−module}\}.$$

Every distribution $\mathscr{F}$ of codimension k can also be associated to a k-form $\omega_{\mathscr{F}} \in H^0(\Omega_X^k \otimes \omega_X \otimes \det \mathcal{T}_{\mathscr{F}}^*)$, which is the form associated with the Pfaff field $\eta_{\mathscr{F}}$. Note that $\omega_{\mathscr{F}}$ is a *locally decomposable k-form* with coefficients in $\omega_X \otimes \det \mathcal{T}_{\mathscr{F}}^*$, i.e. for every point $x \in X$, there are 1-forms $\omega_1, \dots, \omega_k$ defined on an open neighbourhood U of x such that $\omega_{\mathscr{F}}|_U = \omega_1 \wedge \cdots \wedge \omega_k$.

The sheaf $\mathcal{T}_{\mathscr{F}}$ can be recovered as the kernel of the morphism

$$\mathcal{T}_X \longrightarrow \Omega_X^{k-1} \otimes \det \mathcal{N}_{\mathscr{F}}$$

and we have the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{T}_{\mathscr{F}} \arrow[r] & \mathcal{T}_X \arrow[r] \arrow[d] & \Omega_X^{k-1} \otimes \det \mathcal{N}_{\mathscr{F}} \\
 & 0 \arrow[r] & \mathcal{N}_{\mathscr{F}} \arrow[ur] \arrow[d] & \\
 & & 0 &
\end{tikzcd}$$

Now, let $\mathscr{F}$ be a distribution. If the Pfaff System $\phi : \mathcal{P}_{\mathscr{F}} \to \Omega_X^1$ induced by $\mathscr{F}$ is locally free, then $\mathrm{Sing}(\mathscr{F})$ coincides with the degeneracy locus

$$\Delta_\phi := \{x \in X \mid \phi(x) \text{ not injective}\}$$

of the map $\phi : \mathcal{P}_{\mathscr{F}} \to \Omega_X^1$.

Although different from the way we opted to set up concepts here, the reader can check for instance [14, Sec. 3], a very helpful source of how capturing the essence of this subject and put it within a general framework.

Proposition 2.2. *Let $\mathcal{L}$ be a line bundle and $\omega \in H^0(\Omega^k_X \otimes \mathcal{L})$ be a locally decomposable section. Then $\operatorname{cod}(\operatorname{Sing}(\omega)) \leq k+1$. In particular, if $\mathscr{F}$ is a distribution on X of codimension k, then $\operatorname{cod}(\operatorname{Sing}(\mathscr{F})) \leq k+1$. Moreover, if $\mathcal{N}_{\mathscr{F}}$ is a k-th syzygy sheaf, then $\operatorname{cod}(\operatorname{Sing}(\mathscr{F})) = k+1$.*

Proof. Since ω is locally decomposable, for each $p \in X$ there exist germs of polynomial 1-forms $\omega_1, \ldots, \omega_k$ on a neighborhood U of p, such that $\omega|_U = \omega_1 \wedge \ldots \wedge \omega_k$ and $\operatorname{Sing}(\omega) \cap U = \{\omega_1 \wedge \ldots \wedge \omega_k = 0\}$. Therefore $\operatorname{Sing}(\omega) \cap U$ is determinantal and has codimension at most $k+1$.

Since a distribution $\mathscr{F}$ of codimension k on X induces a locally decomposable section $\omega \in H^0(\Omega^k_X \otimes \det \mathcal{N}_{\mathscr{F}})$ with $\operatorname{Sing}(\mathscr{F}) = \operatorname{Sing}(\omega)$, it immediately follows that $\operatorname{cod}(\operatorname{Sing}(\mathscr{F})) \leq k+1$. On the other hand, if $\mathcal{N}_{\mathscr{F}}$ is a k-th syzygy sheaf then $\operatorname{cod}(\operatorname{Sing}(\mathscr{F})) \geq k+1$ owing to [24, Thm. 1.1.6, p. 145]. □

2.3. **ACM and Arithmetically Buchsbaum schemes.** A closed subscheme $Y \subset \mathbb{P}^n$ is *arithmetically Cohen–Macaulay (ACM)* if its homogeneous coordinate ring $S(Y) = k[x_0, \ldots, x_n]/I(Y)$ is a Cohen–Macaulay ring.

Equivalently, Y is ACM if $H^p_*(\mathcal{O}_Y) = 0$ for $1 \leq p \leq \dim Y - 1$ and $H^1_*(\mathcal{I}_Y) = 0$ (cf. [5]). From the long exact sequence of cohomology associated to the short exact sequence

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

one also deduces that Y is ACM if and only if $H^p_*(\mathcal{I}_Y) = 0$ for $1 \leq p \leq \dim Y$.

Note that every ACM closed subscheme in $\mathbb{P}^n$ is Cohen–Macaulay, but the converse is not true.

Similarly, a closed subscheme in $\mathbb{P}^n$ is *arithmetically Buchsbaum* if its homogeneous coordinate ring is a Buchsbaum ring. Clearly, every ACM scheme is arithmetically Buchsbaum, but the converse is not true: the union of two disjoint lines is arithmetically Buchsbaum, but not ACM.

We will use the following cohomological characterization of arithmetically Buchsbaum schemes [30], see also [9].

Proposition 2.3 (Stückrad, Vogel)**.** *If $Y \subset \mathbb{P}^n$ is closed subscheme such that*

(i) *the multiplication map $H^p(\mathcal{I}_Y(i)) \xrightarrow{x} H^p(\mathcal{I}_Y(i+1))$ is zero for every section $x \in H^0(\mathcal{O}_{\mathbb{P}^n}(1))$, $i \in \mathbb{Z}$ and $1 \leq p \leq \dim Y$;*

(ii) *$h^p(\mathcal{I}_Y(i)), h^q(\mathcal{I}_Y(j)) \neq 0$ for $1 \leq p < q \leq \dim Y$, implies $(p+i)-(q+j) \neq 1$;*

then Y is arithmetically Buchsbaum.

For two-codimensional subschemes, a more precise result is found in [11, p. 324].

Theorem 2.4 (Chang). *If $Y \subset \mathbb{P}^n$ $(n \geq 3)$ is a closed subscheme of codimension 2, then Y is arithmetically Buchsbaum if and only if the ideal sheaf $\mathcal{I}_Y$ admits a resolution of the form*

$$0 \to \oplus_i \mathcal{O}_{\mathbb{P}^n}(-a_i) \to (\oplus_j \Omega^{p_j}(-k_j)^{\oplus l_j}) \oplus (\oplus_s \mathcal{O}_{\mathbb{P}^n}(-c_s)) \to \mathcal{I}_Y \to 0, \tag{6}$$

where $h^{p_j}(\mathcal{I}_Z(k_j)) = l_j$ are the only nonzero intermediate cohomologies for $1 \leq p_j \leq n-2$.

2.4. **Splitting criteria for locally free sheaves on $\mathbb{P}^n$.** The well-known Horrocks splitting criterion says that a locally free sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is a sum of line bundles if and only if it has no intermediate cohomology, i.e., $H^p(\mathcal{F}(i)) = 0$ for every $1 \leq p \leq n-1$ and every $i \in \mathbb{Z}$.

However, there are stronger splitting criteria, due to G. Evans and P. Griffith [15] and N. M. Kumar, C. Peterson and A. P. Rao [22, Thm 1] which will be relevant here. For the convenience of the reader, let us briefly revise them.

We say that a sheaf $\mathcal{F}$ on X *splits* if it is a sum of line bundles.

Theorem 2.5 (Evans, Griffith). *Let $\mathcal{F}$ be a locally free sheaf on $\mathbb{P}^n$ of rank $r \leq n$. Then $\mathcal{F}$ splits if and only if $H^p_*(\mathcal{F}) = 0$ for $1 \leq p \leq r-1$.*

Theorem 2.6 (Kumar, Peterson, Rao). *Let $\mathcal{F}$ be a locally free sheaf on $\mathbb{P}^n$ of rank r. Then*

(i) *if n is even and $r \leq n-1$, then $\mathcal{F}$ splits iff $H^p_*(\mathcal{F}) = 0$ for $2 \leq p \leq n-2$;*
(ii) *if n is odd and $r \leq n-2$, then $\mathcal{F}$ splits iff $H^p_*(\mathcal{F}) = 0$ for $2 \leq p \leq n-2$.*

2.5. **Holomorphic distributions on projective spaces.** Our main concern in this work is about the holomorphic distributions on the projective space $\mathbb{P}^n$. In order to proceed, we need a better descripton of them. So let $\mathscr{F}$ be a codimension k distribution on $\mathbb{P}^n$ given by $\omega \in H^0(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n} \otimes \mathcal{L})$.

If $i : \mathbb{P}^k \to \mathbb{P}^n$ is a general linear immersion then $i^*\omega \in H^0(\mathbb{P}^k, \Omega^k_{\mathbb{P}^k} \otimes \mathcal{L})$ is a section of a line bundle, and its zero divisor reflects the tangencies between $\mathscr{F}$ and $i(\mathbb{P}^k)$. The *degree* of $\mathscr{F}$ is, by definition, the degree of such tangency divisor. Set $d := \deg(\mathscr{F})$.

Since $\Omega^k_{\mathbb{P}^k} \otimes \mathcal{L} = \mathcal{O}_{\mathbb{P}^k}(\deg(\mathcal{L}) - k - 1)$, one concludes that $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d+k+1)$. Observe also that $\bigwedge^k \mathcal{N}_{\mathscr{F}} = \mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}^n}(d+k+1)$ where $Z := \mathrm{Sing}(\mathscr{F})$. Besides, the Euler sequence implies that a section ω of $\Omega^k_{\mathbb{P}^n}(d+k+1)$ can be thought as a polynomial k-form on $\mathbb{C}^{n+1}$ with homogeneous coefficients of degree $d+1$, which we will still denote by ω, satisfying

$$i_R\omega = 0 \tag{7}$$

where

$$R = x_0 \frac{\partial}{\partial x_0} + \cdots + x_n \frac{\partial}{\partial x_n}$$

is the radial vector field. Thus the study of distributions of degree d on $\mathbb{P}^n$ reduces to the study of locally decomposable homogeneous k-forms of degree $d+1$ on $\mathbb{C}^{n+1}$ satisfying the relation (7).

3. SPLIT DISTRIBUTIONS ON PROJETIVE SPACES

In this Section we prove Theorem 1, which generalizes a result due to Giraldo and Pan-Collantes [19] in the case of 3-dimensional projective space.

Before moving to the main result of this section, we apply the splitting criteria of Section 2.4, we obtain the following characterization of split distributions in terms of its normal sheaf.

Proposition 3.1. *Let $\mathscr{F}$ be a locally free distribution of codimension k on $\mathbb{P}^n$.*

(i) *If $\mathcal{T}_{\mathscr{F}}$ splits, then $H^p_*(\mathcal{N}_{\mathscr{F}}) = 0$ for $1 \leq p \leq n-2$.*
(ii) *If n is even and $H^p_*(\mathcal{N}_{\mathscr{F}}) = 0$ for $1 \leq p \leq n-2$, then $\mathcal{T}_{\mathscr{F}}$ splits.*
(iii) *If n is odd, $k \geq 2$ and $H^p_*(\mathcal{N}_{\mathscr{F}}) = 0$ for $1 \leq p \leq n-2$, then $\mathcal{T}_{\mathscr{F}}$ splits.*
(iv) *If n is odd, $k = 1$ and $H^1_*(\mathcal{T}_{\mathscr{F}}) = H^p_*(\mathcal{N}_{\mathscr{F}}) = 0$ for $1 \leq p \leq n-2$, then $\mathcal{T}_{\mathscr{F}}$ splits.*

Proof. Suppose that $\mathcal{T}_{\mathscr{F}}$ splits, then $H^p(\mathcal{T}_{\mathscr{F}}(q)) = 0$ for $1 \leq p \leq n-1$ and all q. Consider, for each $q \in \mathbb{Z}$, the exact sequence

$$0 \longrightarrow \mathcal{T}_{\mathscr{F}}(q) \longrightarrow \mathcal{T}_{\mathbb{P}^n}(q) \longrightarrow \mathcal{N}_{\mathscr{F}}(q) \longrightarrow 0$$

from which we get

$$\ldots \longrightarrow H^p(\mathcal{T}_{\mathbb{P}^n}(q)) \longrightarrow H^p(\mathcal{N}_{\mathscr{F}}(q)) \longrightarrow H^{p+1}(\mathcal{T}_{\mathscr{F}}(q)) \longrightarrow \ldots$$

From Bott's formula, $H^p_*(\mathcal{T}_{\mathbb{P}^n}) = 0$ for $1 \leq p \leq n-2$. Hence $H^p_*(\mathcal{N}_{\mathscr{F}}) = 0$ for $1 \leq p \leq n-2$.

Now, suppose that $H^p_*(\mathcal{N}_{\mathscr{F}}) = 0$ for $1 \leq p \leq n-2$. Taking the long exact sequence of cohomology we have

$$\ldots \longrightarrow H^{p-1}(\mathcal{N}(q)) \longrightarrow H^p(\mathcal{T}_{\mathscr{F}}(q)) \longrightarrow H^p(\mathcal{T}_{\mathbb{P}^n}(q)) \longrightarrow \ldots$$

we conclude that $H^p_*(\mathcal{T}_{\mathscr{F}}) = 0$ for $2 \leq p \leq n-2$. Since $\operatorname{rank}(\mathcal{T}_{\mathscr{F}}) \leq n-1$, item *(ii)* follows from the first part of Theorem 2.6 above.

If $k \geq 2$, then $\operatorname{rank}(\mathcal{T}_{\mathscr{F}}) \leq n-2$, thus item *(iii)* follows from the second part of Theorem 2.6.

Finally, *(iv)* follows from Theorem 2.5. □

Next, we establish the first part of Theorem 1.

Theorem 3.2. *Let $\mathscr{F}$ be a locally free distribution on $\mathbb{P}^n$ and whose singular locus has the expected codimension $k+1$. If $\mathcal{T}_{\mathscr{F}}$ splits then $\operatorname{Sing}(\mathscr{F})$ is ACM.*

Proof. Let r, k, d and Z be, respectively, the rank, codimension, degree and singular set of $\mathscr{F}$. Consider the Eagon–Northcott complex associated to the morphism $\varphi^* : \Omega^1_{\mathbb{P}^n} \to \mathcal{T}^*_{\mathscr{F}}$:

$$0 \longrightarrow \Omega^n_{\mathbb{P}^n} \otimes S_k(\mathcal{T}_{\mathscr{F}})(r-d) \xrightarrow{\alpha_k} \Omega^{n-1}_{\mathbb{P}^n} \otimes S_{k-1}(\mathcal{T}_{\mathscr{F}})(r-d) \xrightarrow{\alpha_{k-1}} \dots \tag{8}$$
$$\dots \longrightarrow \Omega^{r+1}_{\mathbb{P}^n} \otimes \mathcal{T}_{\mathscr{F}}(r-d) \xrightarrow{\alpha_1} \Omega^r_{\mathbb{P}^n}(r-d) \xrightarrow{\alpha_0} \mathcal{I}_Z \longrightarrow 0.$$

Twist by $\mathcal{O}_{\mathbb{P}^n}(q)$ and break it down into the short exact sequences

$$0 \longrightarrow S_k(\mathcal{T}_{\mathscr{F}})(q-d-k-1) \longrightarrow \Omega^{n-1}_{\mathbb{P}^n} \otimes S_{k-1}(\mathcal{T}_{\mathscr{F}})(r-d+q) \longrightarrow \ker \alpha_{k-2}(q) \longrightarrow 0$$
$$\vdots$$
$$0 \longrightarrow \ker \alpha_i(q) \longrightarrow \Omega^{n-i}_{\mathbb{P}^n} \otimes S_{k-i}(\mathcal{T}_{\mathscr{F}})(r-d+q) \longrightarrow \ker \alpha_{k-i-1}(q) \longrightarrow 0$$
$$\vdots$$
$$0 \longrightarrow \ker \alpha_0(q) \longrightarrow \Omega^r_{\mathbb{P}^n}(r-d+q) \longrightarrow \mathcal{I}_Z(q) \longrightarrow 0.$$

If $\mathcal{T}_{\mathscr{F}}$ splits so does $S_k(\mathcal{T}_{\mathscr{F}})$ and hence $H^p_*(S_k(\mathcal{T}_{\mathscr{F}})) = 0$ for $1 \le p \le n-1$ by Horrocks splitting criterion. Therefore

$$H^p(\ker \alpha_{k-2}(q)) \simeq H^p(\Omega^{n-1}_{\mathbb{P}^n} \otimes S_{k-1}(\mathcal{T}_{\mathscr{F}})(r-d+q))$$

for $1 \le p \le n-2$ and all $q \in \mathbb{Z}$. But $H^p_*(\Omega^{n-1}_{\mathbb{P}^n} \otimes S_{k-1}(\mathcal{T}_{\mathscr{F}})) = 0$ for $1 \le p \le n-2$ by splitting $S_{k-1}(\mathcal{T}_{\mathscr{F}})$ and applying Bott's Formula term by term. It follows that $H^p_*(\ker \alpha_{k-2}) = 0$ for $1 \le p \le n-2$. Using this in the next sequence and proceeding with the argument we see that $H^p_*(\ker \alpha_{k-i}) = 0$ for $1 \le p \le n-i$. In particular, $H^p_*(\ker \alpha_0) = 0$ for $1 \le p \le n-k = r$. Thus, from the last sequence we get $H^p_*(\mathcal{I}_Z) = 0$ for $1 \le p \le r-1 = \dim Z$, that is, Z is ACM, as desired. □

4. Proof of Theorem 1

In order to establish the second part of Theorem 1, we now focus on one-codimensional distributions. We start with the following result.

Lemma 4.1. *If $\mathscr{F}$ is a distribution on $\mathbb{P}^n$ with $\operatorname{cod}(\mathscr{F}) = 1$, then*

(i) $H^0(\mathcal{T}_{\mathscr{F}}(p)) = 0$ *for* $p \le -2$*;*
(ii) $H^1(\mathcal{T}_{\mathscr{F}}(p)) = 0$ *for* $p \le -\deg(\mathscr{F}) - 3$.

If, in addition, $\operatorname{Sing}(\mathscr{F})$ *is ACM of codimension 2 then*

(iii) $H^q(\mathcal{T}_{\mathscr{F}}(p)) = 0$ *for* $2 \le q \le n-2$ *and all* p *when* $n \ge 4$*;*
(iv) $H^{n-1}(\mathcal{T}_{\mathscr{F}}(p)) = 0$ *for all* $p \neq -n-1$*, and* $h^{n-1}(\mathcal{T}_{\mathscr{F}}(-n-1)) \le 1$.

Proof. Set $d := \deg(\mathscr{F})$ and $Z := \operatorname{Sing}(\mathscr{F})$. Consider the sequence

$$0 \longrightarrow \mathcal{T}_{\mathscr{F}} \longrightarrow \mathcal{T}_{\mathbb{P}^n} \longrightarrow \mathcal{I}_Z(d+2) \longrightarrow 0. \tag{9}$$

Twisting it by $\mathcal{O}_{\mathbb{P}^n}(p)$ and passing to cohomology, we first have that

$$0 \longrightarrow H^0(\mathcal{T}_{\mathscr{F}}(p)) \longrightarrow H^0(\mathcal{T}_{\mathbb{P}^n}(p)) \longrightarrow \cdots$$

Since $H^0(\mathcal{T}_{\mathbb{P}^n}(p)) = 0$ for $p \le -2$, we also conclude that $H^0(\mathcal{T}_{\mathscr{F}}(p)) = 0$ for $p \le -2$.

We also have the sequence

$$H^0(\mathcal{I}_Z(d+2+p)) \longrightarrow H^1(\mathcal{T}_{\mathscr{F}}(p)) \longrightarrow H^1(\mathcal{T}_{\mathbb{P}^n}(p)).$$

Since $H^0(\mathcal{I}_Z(l)) = 0$ for $l \leq -1$, we conclude that $H^1(\mathcal{T}_{\mathscr{F}}(p)) = 0$ for $p \leq -d-3$.

For the second part of the Lemma, if Z is ACM then $H^q(\mathcal{I}_Z(d+2+p) = 0$ for $1 \leq q \leq n-2 = \dim Z$ and all p. Since $H^q(\mathcal{T}_{\mathbb{P}^n}(p)) = 0$ for $1 \leq q \leq n-2$, one concludes from the sequence in cohomology

$$H^{q-1}(\mathcal{I}_Z(d+2+p)) \longrightarrow H^q(\mathcal{T}_{\mathscr{F}}(p)) \longrightarrow H^q(\mathcal{T}_{\mathbb{P}^n}(p))$$

that $H^q(F(p)) = 0$ for $2 \leq q \leq n-2$ and all p.

Moreover, one also has the sequence

$$0 \to H^{n-1}(\mathcal{T}_{\mathscr{F}}(p)) \to H^{n-1}(\mathcal{T}_{\mathbb{P}^n}(p)) \to \cdots$$

from which one obtain item (iv), since $H^{n-1}(\mathcal{T}_{\mathbb{P}^n}(p)) = 0$ for all $p \neq -n-1$, and $h^{n-1}(\mathcal{T}_{\mathbb{P}^n}(-n-1)) = 1$. □

4.1. **Proof of even dimensional case.** Now let $\mathscr{F}$ be a locally free distribution of codimension one on an even dimensional projective space $\mathbb{P}^{2m}$. If $\mathrm{Sing}(\mathscr{F})$ is ACM, then $\mathcal{T}_{\mathscr{F}}$ is a locally free sheaf of rank $2m-1$ satisfying $H^q(\mathcal{T}_{\mathscr{F}}(p)) = 0$ for $2 \leq q \leq 2m-2$ and all p, by item (iii) of Lemma 4.1 above. By the Kumar–Peterson–Rao splitting criterion for even dimensional projective spaces, Theorem 2.6, it follows that $\mathcal{T}_{\mathscr{F}}$ must split.

4.2. **Proof of odd dimensional case.** For odd dimensional projective spaces the situation is more delicate, and we will need the following technical result.

Lemma 4.2. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$ ($n \geq 4$) such that*

(i) $H^0(F(s)) = 0$ *for* $s \leq -2$;
(ii) $H^q(F(s)) = 0$ *for* $-n-2 \leq s \leq -2$ *and* $2 \leq q \leq n-2$;
(iii) $H^{n-1}(F(s)) = 0$ *for* $s = -n-2$ *and* $-n \leq s \leq -2$.

If $h^{n-1}(\mathcal{F}(-n-1)) = a$, *then* $\mathrm{rank}(\mathcal{F}) \geq a \cdot n$.

Proof. By a theorem of Beilinson [24, Thm 3.1.3, p. 240], for any coherent sheaf E on $\mathbb{P}^n$ there is a spectral sequence with E_1-term

$$E_1^{p,q} = H^q(E(p)) \otimes \Omega^{-p}(-p) \text{ where } 0 \leq q \leq n \text{ and } -n \leq p \leq 0$$

which converges to the graded sheaf associated to a filtration of E.

Applying this to $E := \mathcal{F}(-2)$, we get that

(i) $E_1^{p,0} = 0$ for $-n \leq p \leq 0$;
(ii) $E_1^{p,q} = 0$ for $-n \leq p \leq 0$ and $0 \leq q \leq n-2$;
(iii) $E_1^{p,n-1} = 0$ for $p \neq 1-n$.

The only nontrivial E_1-terms are:

(i) $E_1^{p,1} = H^1(\mathcal{F}(p-2)) \otimes \Omega^{-p}(-p)$;

(ii) $E_1^{1-n,n-1} := H^{n-1}(\mathcal{F}(-n-1)) \otimes \Omega^{n-1}(n-1) \simeq \Omega^{n-1}(n-1)^{\oplus a}$;
(iii) $E^{p,n} = H^n(\mathcal{F}(p-2)) \otimes \Omega^{-p}(-p)$.

The E_1-terms $E_1^{p,1}$ and $E_1^{p,n}$, $-n \leq p \leq 0$, together with the d_1-differentials $d_1^{p,1} : E_1^{p,1} \to E_1^{p+1,1}$ and $d_1^{p,n} : E_1^{p,n} \to E_1^{p+1,n}$, respectively, form complexes, which we denote by $\mathcal{C}^\bullet$ and $\mathcal{D}^\bullet$, respectively; notice that all other d_1-differentials must vanish.

The E_2-term of this spectral sequence will have the same shape:

(i) $E_2^{p,0} = 0$ for $-n \leq p \leq 0$;
(ii) $E_2^{p,q} = 0$ for $-n \leq p \leq 0$ and $0 \leq q \leq n-2$;
(iii) $E_2^{p,n-1} = 0$ for $p \neq 1-n$,

while the only nontrivial E_2-terms are:

(i) $E_1^{p,1} = \mathcal{H}^p(\mathcal{C}^\bullet)$;
(ii) $E_1^{1-n,n-1} := H^{n-1}(\mathcal{F}(-n-1)) \otimes \Omega^{n-1}(n-1) \simeq \Omega^{n-1}(n-1)^{\oplus a}$;
(iii) $E^{p,n} = \mathcal{H}^p(\mathcal{D}^\bullet)$.

It is then easy to see that all d_2-differentials $d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$ must be zero, so the spectral sequence converges already at the E_2-term, i.e. $E_\infty^{p,q} = E_2^{p,q}$.

It follows from Beilinson's theorem that $\mathcal{H}^p(\mathcal{C}^\bullet) = 0$ for $p \neq -1$, $\mathcal{H}^p(\mathcal{D}^\bullet) = 0$ for $p > -n$ and

$$\oplus_p E_\infty^{p,-p} = \Omega^{n-1}(n-1)^{\oplus a} \oplus \mathcal{H}^{-1}(\mathcal{C}^\bullet) \oplus \mathcal{H}^{-n}(\mathcal{D}^\bullet)$$

is the graded sheaf associated to a filtration of $\mathcal{F}(-2)$. In particular, it follows that the rank of $\mathcal{F}$ must be at least equal to the rank of $\Omega^{n-1}(n-1)^{\oplus a}$, as desired. □

Now let $\mathcal{T}_{\mathscr{F}}$ be a locally free distribution of codimension one on an odd dimensional projective space $\mathbb{P}^{2m+1}$; the case $m = 1$ of locally free distribution of codimension one on $\mathbb{P}^3$ is proved by Giraldo and Pan-Collantes in [19].

Thus we set $n = 2m+1 \geq 5$. If $\mathrm{Sing}(\mathscr{F})$ is ACM, we get from the second part of Lemma 4.1, that $H^q(\mathcal{T}_{\mathscr{F}}(p)) = 0$ for all $2 \leq q \leq n-1$ and all p, except for $q = n-1$ and $p = -n-1$, in which case $h^{n-1}(\mathcal{T}_{\mathscr{F}}(-n-1)) \leq 1$.

We are then left with two possibilities. If $h^{n-1}(\mathcal{T}_{\mathscr{F}}(-n-1)) = 0$, it follows that $H^q(\mathcal{T}_{\mathscr{F}}(p)) = 0$ for all $2 \leq q \leq n-1$ and all p; applying Serre duality, we conclude that $H^q(\mathcal{T}_{\mathscr{F}}^*(p)) = 0$ for $1 \leq p \leq n-2$ and all p, thus $\mathcal{T}_{\mathscr{F}}^*$ splits by the Evans–Griffith spliting criterion, Theorem 2.5, and so does $\mathcal{T}_{\mathscr{F}}$.

The second possibility, $h^{n-1}(\mathcal{T}_{\mathscr{F}}(-n-1)) = 1$, leads to a contradiction: by Lemma 4.1, we get that $\mathcal{T}_{\mathscr{F}}$ satisfies all the hypotheses of Lemma 4.2 with $a = 1$; it then follows from that $\mathcal{T}_{\mathscr{F}}$ must have rank at least n, which contradicts the hypothesis that $\mathcal{T}_{\mathscr{F}}$ has rank $n-1$.

This completes the proof of Theorem 1 in the odd dimensional case.

5. Complete intersections holomorphic foliations

Let $\mathscr{F}_1,\dots,\mathscr{F}_k$ be generically transversal holomorphic distributions of codimension one on $\mathbb{P}^n$. Then $\mathscr{F}=\mathscr{F}_1\cap\cdots\cap\mathscr{F}_k$ is a foliation of codimension k on $\mathbb{P}^n$. Therefore, the Pfaff bundle $\mathcal{P}_{\mathscr{F}}$ of $\mathscr{F}$ splits, see [21, Cor. 4.2.7 pg. 133]. In this case we say that $\mathscr{F}$ is a *complete intersection* foliation.

Assuming $\dim(\mathscr{F})=1$, we have the sequence

$$0\longrightarrow \mathcal{P}_{\mathscr{F}}\longrightarrow \Omega^1_{\mathbb{P}^n}\longrightarrow \mathcal{I}_Z(c_1(\Omega^1_{\mathbb{P}^n})-c_1(\mathcal{P}_{\mathscr{F}}))\longrightarrow 0. \tag{10}$$

Set $s:=c_1(\mathcal{P}_{\mathscr{F}})$. Taking determinant we get a global section of

$$\Omega^{n-1}_{\mathbb{P}^n}\otimes\det\mathcal{P}^*_{\mathscr{F}}\simeq \mathcal{T}_{\mathbb{P}^n}\otimes\mathcal{O}_{\mathbb{P}^n}(-n-1-s)).$$

Then $s=-n-d$ and $c_1(\Omega^1_{\mathbb{P}^n})-c_1(\mathcal{P}_{\mathscr{F}})=-n-1+d+n=d-1$. Therefore we obtain

$$0\longrightarrow \mathcal{P}_{\mathscr{F}}\longrightarrow \Omega^1_{\mathbb{P}^n}\longrightarrow \mathcal{I}_Z(d-1)\longrightarrow 0. \tag{11}$$

We can use the result of [1] to show that complete intersection foliations of dimension one is determined by their singular schemes.

Theorem 5.1. *Let $\mathscr{F}$ be an one-dimensional complete intersection foliation on $\mathbb{P}^n$, of degree d, such that $\operatorname{cod}(\operatorname{Sing}(\mathscr{F}))=2$. If $\mathscr{F}'$ is a one-dimensional foliation on $\mathbb{P}^n$, of degree d, with $\operatorname{Sing}(\mathscr{F})\subset\operatorname{Sing}(\mathscr{F}')$, then $\mathscr{F}'=\mathscr{F}$.*

Proof. Since the Pfaff bundle splits, write $\mathcal{P}_{\mathscr{F}}=\bigoplus_{i=1}^{n-1}\mathcal{O}_{\mathbb{P}^n}(-d_i-2)$ and consider

$$0\longrightarrow\bigoplus_{i=1}^{n-1}\mathcal{O}_{\mathbb{P}^n}(-d_i-2)\longrightarrow\Omega^1_{\mathbb{P}^n}\longrightarrow\mathcal{I}_Z(d-1)\longrightarrow 0.$$

We have that $\operatorname{Sing}(\mathscr{F})$ is the degeneracy scheme of the induced dual map

$$\mathcal{T}_{\mathbb{P}^n}\to\bigoplus_{i=1}^{n-1}\mathcal{O}_{\mathbb{P}^n}(d_i+2).$$

Taking determinant we obtain a global section $\zeta_{\mathscr{F}}\in H^0(\mathcal{T}_{\mathbb{P}^n}(d-1))$ inducing $\mathscr{F}$, where $d-1=\sum_{i=1}^{n-1}(d_i+2)-n-1$. By Bott's formulae,

$$H^1\Big(\Omega^{n-1}_{\mathbb{P}^n}\otimes\bigwedge^n\mathcal{T}_{\mathbb{P}^n}\otimes\bigoplus_{i=1}^{n-1}\mathcal{O}_{\mathbb{P}^n}(-d_i-2)\Big)=\bigoplus_{i=1}^{n-1}H^1\Big(\Omega^{n-1}_{\mathbb{P}^n}\otimes\bigwedge^n T_{\mathbb{P}^n}(-d_i-2)\Big)=0.$$

Thus, from [1, Thm 1.1], if $\zeta_{\mathscr{F}'}\in H^0(\mathcal{T}_{\mathbb{P}^n}(d-1))$ induces a one-dimensional foliation $\mathscr{F}'$ of degree d with $\operatorname{Sing}(\mathscr{F})=\operatorname{Sing}(\zeta_{\mathscr{F}})\subset\operatorname{Sing}(\zeta_{\mathscr{F}'})=\operatorname{Sing}(\mathscr{F}')$, then we must have $\zeta_{\mathscr{F}'}=\lambda\zeta_{\mathscr{F}}$ for some $\lambda\in\mathbb{C}^*$, since $\operatorname{End}(\Omega^{n-1}_{\mathbb{P}^n})\cong\mathbb{C}$ (cf. [1, Lem 4.8]). Hence $\mathscr{F}=\mathscr{F}'$ as we wish. □

Theorem 5.2. *Let $\mathscr{F}$ be an one-dimensional complete intersection foliation on $\mathbb{P}^n$, of degree d, such that $\operatorname{cod}(\operatorname{Sing}(\mathscr{F}))=2$. Suppose that the induced Pfaff system*

$\mathcal{P}_{\mathscr{F}} \to \Omega^1_{\mathbb{P}^n}$ *is locally free. Then* $\mathcal{P}_{\mathscr{F}}$ *splits if and only if* $\mathrm{Sing}(\mathscr{F}) = Z$ *is arithmetically Buchsbaum with* $h^1(\mathcal{I}_Z(d-1)) = 1$ *being the only nonzero intermediate cohomology for* $H^i_*(\mathcal{I}_Z)$ *in the range* $1 \leq i \leq n-2$.

Proof. It follows from Theorem 2.4 that $\mathrm{Sing}(\mathscr{F})$ is arithmetically Buchsbaum with $h^1(\mathcal{I}_Z(d-1)) = 1$ being the only nonzero intermediate cohomology for $\mathcal{I}_Z$ if and only if the ideal sheaf $\mathcal{I}_Z$ has a resolution of the form

$$0 \longrightarrow \bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^n}(-b_i) \longrightarrow \Omega^1_{\mathbb{P}^n}(1-d) \longrightarrow \mathcal{I}_Z \longrightarrow 0.$$

By (11) we have that $\mathcal{P}_{\mathscr{F}} = \bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^n}(-b_i)$. □

We will use the characterization of Theorem 5.2 and the Theorem 5.1 to classify complete intersection foliations by curves whose singular locus is smooth and non-general type.

Proposition 5.3. *Let* $\mathscr{F}$ *be an one-dimensional complete intersection foliation on* $\mathbb{P}^n (n \geq 4)$*, of degree* d*, such that* $\mathrm{cod}(\mathrm{Sing}(\mathscr{F})) = 2$*. If* $\mathrm{Sing}(\mathscr{F})$ *is smooth and of nongeneral type, then the classification of* $\mathscr{F}$ *can be stated as:*

n	$\deg(\mathscr{F})$	$\mathcal{P}_{\mathscr{F}}$	$\mathrm{Sing}(\mathscr{F})$
4	2	$\mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 3}$	smooth projected Veronese surface
4	3	$\mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4}(-3)$	$K3$ surface of genus 7
5	3	$\mathcal{O}_{\mathbb{P}^5}(-2)^{\oplus 4}$	a scroll over a plane cubic surface
5	4	$\mathcal{O}_{\mathbb{P}^5}(-2)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^5}(-3)$	$\mathbb{P}(R_2) \cap Bl_{\mathbb{P}^2}\mathbb{P}^8$

where

$$R_2 := \left\{ v_i \wedge v_i \wedge \tau \;\middle|\; v_i \in \mathbb{C}^6,\ \tau \in \bigwedge^2 \mathbb{C}^6 \right\}$$

$$\mathbb{P}(R_2) \subset \mathbb{P}\left(\bigwedge^2 \mathbb{C}^6 \otimes \mathcal{O}_{\mathbb{P}^5} \right) = \mathbb{P}^{14} \times \mathbb{P}^5$$

and

$$Bl_{\mathbb{P}^2}\mathbb{P}^8 \subset \mathbb{P}^8 \times \mathbb{P}^5 \subset \mathbb{P}^{14} \times \mathbb{P}^5.$$

Proof. Since $\mathscr{F}$ is a complete intersection foliation, then $\mathrm{Sing}(\mathscr{F}) = Z$ has a resolution

$$0 \longrightarrow \bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^n}(-1-d-a_i) \longrightarrow \Omega^1_{\mathbb{P}^n}(1-d) \longrightarrow \mathcal{I}_Z \longrightarrow 0.$$

Now, the result follows from Chang's classification of smooth arithmetically Buchsbaum schemes [10, Prp 1.4, Tables I and IV] along with the fact just stated that complete intersection foliations by curves are determined by their singular loci(Theorem 5.1). □

The above proposition shows that if $n \geq 4$ and $\mathrm{Sing}(\mathscr{F})$ is smooth and of nongeneral type, then $2 \leq \deg(\mathscr{F}) \leq 4$ and $n = 4, 5$. In particular, if $\mathscr{F}$ is a complete intersection foliation on $\mathbb{P}^3$ such that $\mathrm{Sing}(\mathscr{F})$ is nongeneral type, then $\mathrm{Sing}(\mathscr{F})$ is not smooth.

We give the following example of a complete intersection foliation on $\mathbb{P}^3$ whose the singular set is not smooth and nongeneral type arithmetically Buchsbaum curve.

Example 5.4. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be one-codimensional foliations on $\mathbb{P}^3$ given, respectively, by the pencils $\{\alpha z_0 + \beta z_1 = 0\}$ and $\{\lambda z_3 + \mu z_4 = 0\}$. Then $\mathscr{F}_1$ and $\mathscr{F}_2$ are induced, respectively, by the 1-forms $\omega_1 = z_0 dz_1 - z_1 dz_0$ and $\omega_2 = z_3 dz_4 - z_4 dz_3$. We have that the complete intersection $\mathscr{F} = \mathscr{F}_1 \cap \mathscr{F}_2$, of degree one, is given by

$$\omega = \omega_1 \wedge \omega_2 = z_0 z_2 dz_1 \wedge dz_3 - z_0 z_3 dz_1 \wedge dz_2 - z_1 z_2 dz_0 \wedge dz_3 + z_1 z_3 dz_0 \wedge dz_2$$

with $\mathrm{Sing}(\mathscr{F}) = \{z_0 = z_1 = 0\} \cup \{z_2 = z_3 = 0\}$. Then $\mathrm{Sing}(\mathscr{F})$ is an arithmetically Buchsbaum curve which is not smooth and nongeneral type with $\mathcal{P}_{\mathscr{F}} = \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 2}$.

Theorem 5.5. *Let $\mathscr{F}$ be a holomorphic distribution of dimension r on $\mathbb{P}^n$. Suppose that $\mathrm{cod}(\mathrm{Sing}(\mathscr{F})) = r + 1$ and that the induced Pfaff system $\mathcal{P}_{\mathscr{F}} \to \Omega^1_{\mathbb{P}^n}$ is locally free. Then the following hold:*

(i) *if $r = 2$ and $\mathcal{P}_{\mathscr{F}}$ splits, then $\mathrm{Sing}(\mathscr{F})$ is arithmetically Buchsbaum, but not arithmetically Cohen Macaulay;*
(ii) *if $r = 3$, $\mathcal{P}_{\mathscr{F}}$ splits, the $|d_i - d_j| \neq 1$, and, for $n \geq 7$, the $d_i \neq 1$ as well, then $\mathrm{Sing}(\mathscr{F})$ is arithmetically Buchsbaum, but not arithmetically Cohen Macaulay.*

Proof. Assuming the Pfaff bundle splits, write $\mathcal{P}_{\mathscr{F}} = \oplus_i \mathcal{O}_{\mathbb{P}^n}(a_i)$, $c := \sum_i a_i$ and set $Z := \mathrm{Sing}(\mathscr{F})$.

To prove (ii), the Eagon-Northcott complex obtained from $\mathcal{T}_{\mathbb{P}^n} \to \mathcal{P}^*_{\mathscr{F}}$ is

$$0 \longrightarrow \bigwedge^{n} \mathcal{T}_{\mathbb{P}^n} \otimes S_2(\mathcal{P}_{\mathscr{F}})(c) \longrightarrow \bigwedge^{n-1} \mathcal{T}_{\mathbb{P}^n} \otimes \mathcal{P}_{\mathscr{F}}(c) \longrightarrow \bigwedge^{n-2} \mathcal{T}_{\mathbb{P}^n}(c) \xrightarrow{\alpha} \mathcal{I}_Z \longrightarrow 0. \tag{12}$$

Setting $\mathcal{S} := S_2(\mathcal{P}_{\mathscr{F}}) = \sum_{i,j} \mathcal{O}_{\mathbb{P}^n}(a_i a_j)$ and twisting (12) by $\mathcal{O}_{\mathbb{P}^n}(q)$ we get

$$0 \longrightarrow \mathcal{S}(n{+}1{+}c{+}q) \longrightarrow \bigoplus_{i=1}^{n-2} \left(\bigwedge^{n-1} \mathcal{T}_{\mathbb{P}^n}(a_i{+}c{+}q) \right) \longrightarrow \bigwedge^{n-2} \mathcal{T}_{\mathbb{P}^n}(c{+}q) \longrightarrow \mathcal{I}_Z(q) \longrightarrow 0.$$

which breaks down into the short exact sequences

$$0 \longrightarrow \mathcal{S}(n + 1 + c + q) \longrightarrow \bigoplus_{i=1}^{n-2} \left(\bigwedge^{n-1} \mathcal{T}_{\mathbb{P}^n}(a_i + c + q) \right) \longrightarrow \ker \alpha(q) \longrightarrow 0 \tag{13}$$

$$0 \longrightarrow \ker \alpha(q) \longrightarrow \bigwedge^{n-2} \mathcal{T}_{\mathbb{P}^n}(c + q) \longrightarrow \mathcal{I}_Z(q) \longrightarrow 0. \tag{14}$$

Now (refequse1) yields

$$H^1(\ker\alpha(q)) \cong \bigoplus_{i=1}^{n-2} H^1\left(\bigwedge^{n-1}\mathcal{T}_{\mathbb{P}^n}(a_i+c+q)\right) \ \text{ for } q\in\mathbb{Z}$$

$$H^p(\ker\alpha(q)) \cong \bigoplus_{i=1}^{n-2} H^p\left(\bigwedge^{n-1}\mathcal{T}_{\mathbb{P}^n}(a_i+c+q)\right) = 0 \ \text{ for } q\in\mathbb{Z},\ 2\le p\le n-2 \tag{15}$$

while from (14) we get

$$H^p(\ker\alpha(q)) \longrightarrow H^p\left(\bigwedge^{n-2}\mathcal{T}_{\mathbb{P}^n}(c+q)\right) \longrightarrow H^p(\mathcal{I}_Z(q)) \longrightarrow$$
$$\longrightarrow H^{p+1}(\ker\alpha(q)) \longrightarrow H^{p+1}\left(\bigwedge^{n-2}\mathcal{T}_{\mathbb{P}^n}(c+q)\right).$$

Now, for $p=1$ or $3\le p\le n-3=\dim Z$, we have that $H^p(\wedge^{n-2}\mathcal{T}_{\mathbb{P}^n}(c+q))$ vanishes for every $q\in\mathbb{Z}$ and so does $H^{p+1}(\ker\alpha(p))$ from (15). It follows that

$$H^p(\mathcal{I}_Z(q)) = 0 \ \text{ for } q\in\mathbb{Z} \text{ and } p=1 \text{ or } 3\le p\le n-3 \tag{16}$$

From (15) again, $H^3(\ker\alpha(q))=0$ because $n\ge 5$. Therefore

$$H^2\left(\bigwedge^{n-2}\mathcal{T}_{\mathbb{P}^n}(c+q)\right) \cong H^2(\mathcal{I}_Z(q))$$

and hence

$$H^2(\mathcal{I}_Z(q)) = 0 \text{ for } q\ne -c-n-1. \tag{17}$$

From (16) and (17) one rapidly sees that Z is arithmetically Buchsbaum though not ACM.

To prove (iii), set $\mathcal{S} := S_2(\mathcal{P}_{\mathscr{F}})$ and $\mathcal{S}' := S_3(\mathcal{P}_{\mathscr{F}})$. Twisting the Eagon-Northcott complex obtained from $\mathcal{T}_{\mathbb{P}^n}\to\mathcal{P}_{\mathscr{F}}^*$ by $\mathcal{O}_{\mathbb{P}^n}(q)$ we get

$$0\longrightarrow \mathcal{S}'(n+1+c+q) \longrightarrow \bigwedge^{n-1}\mathcal{T}_{\mathbb{P}^n}\otimes\mathcal{S}(c+q) \longrightarrow \bigwedge^{n-2}\mathcal{T}_{\mathbb{P}^n}\otimes\mathcal{P}_{\mathscr{F}}(c+q) \longrightarrow$$
$$\xrightarrow{\alpha_q'} \bigwedge^{n-3}\mathcal{T}_{\mathbb{P}^n}(c+q) \xrightarrow{\alpha_q} \mathcal{I}_Z(q) \longrightarrow 0$$

which breaks down into the short exact sequences

$$0\longrightarrow \mathcal{S}'(n+1+c+q) \longrightarrow \bigwedge^{n-1}\mathcal{T}_{\mathbb{P}^n}\otimes\mathcal{S}(c+q) \longrightarrow \ker\alpha_q' \longrightarrow 0 \tag{18}$$

$$0\longrightarrow \ker\alpha_q' \longrightarrow \bigwedge^{n-2}\mathcal{T}_{\mathbb{P}^n}\otimes\mathcal{P}_{\mathscr{F}}(c+q) \longrightarrow \ker\alpha_q \longrightarrow 0 \tag{19}$$

$$0\longrightarrow \ker\alpha_q \longrightarrow \bigwedge^{n-3}\mathcal{T}_{\mathbb{P}^n}(c+q) \longrightarrow \mathcal{I}_Z(q) \longrightarrow 0 \tag{20}$$

Now (20) yields

$$H^p(\mathcal{I}_Z(q)) \cong H^{p+1}(\ker\alpha_q) \ \text{ for } q\in\mathbb{Z},\ 1\le p\le n-2 \text{ but } p=2,3 \tag{21}$$

and also the exact sequence

$$0 \longrightarrow H^2(\mathcal{I}_Z(q)) \longrightarrow H^3(\ker \alpha_q) \longrightarrow H^3\Big(\bigwedge^{n-3} \mathcal{T}_{\mathbb{P}^n}(c+q)\Big) \longrightarrow \longrightarrow H^3(\mathcal{I}_Z(q)) \longrightarrow H^4(\ker \alpha_q) \tag{22}$$

while from (19) we get

$$H^{p+1}(\ker \alpha_q) \cong H^{p+2}(\ker \alpha'_q) \ \text{ for } q \in \mathbb{Z},\ 0 \le p \le n-3 \text{ but } p = 0,1 \tag{23}$$

and (18) leads to

$$H^p(\ker \alpha'_q) = 0 \ \text{ for } q \in \mathbb{Z},\ 3 \le p+1 \le n-1 \tag{24}$$

thus

$$H^p(\ker \alpha_q) = 0 \ \text{ for } q \in \mathbb{Z},\ 3 \le p \le n-1 \tag{25}$$

hence from (21) we get

$$H^p(\mathcal{I}_Z(q)) = 0 \ \text{ for } q \in \mathbb{Z},\ 1 \le p \le n-4 \text{ but } p = 1,3. \tag{26}$$

Besides, (19) and (21) yields

$$H^1(\mathcal{I}_Z(q)) \cong H^2(\ker \alpha_q) \cong H^2\Big(\bigwedge^{n-2} \mathcal{T}_{\mathbb{P}^n} \otimes \mathcal{P}_{\mathscr{F}}(c+q)\Big)$$

so

$$H^1(\mathcal{I}_Z(q)) \cong \bigoplus_{i=1}^{n-3} H^2\Big(\bigwedge^{n-2} \mathcal{T}_{\mathbb{P}^n}(a_i+c+q)\Big)$$

which implies

$$H^1(\mathcal{I}_Z(q)) = 0 \text{ for } q \in \mathbb{Z} \text{ but } q = -n-1-c-a_i \tag{27}$$

On the other hand, from (22) and (24) we have

$$H^3(\mathcal{I}_Z(q)) \cong H^3\Big(\bigwedge^{n-3} \mathcal{T}_{\mathbb{P}^n}(c+q)\Big)$$

so

$$H^3(\mathcal{I}_Z(q)) = 0 \text{ for } q \in \mathbb{Z} \text{ but } q = -n-1-c. \tag{28}$$

Now gather (26), (27), (28) and recall Proposition 2.3. If $|a_i - a_j| \neq 1$ its first condition is trivially satisfied. For the second, if $\dim Z \geq 3$, i.e., $n \geq 7$, we just need

$$(1-n-1-c-a_i) - (3-n-1-c) = -2-a_i \neq 1,$$

that is, $a_i \neq -3$, i.e., $d_i \neq 1$. □

5.1. **Proof of Theorem 2.** Now we are able to prove Theorem 2, in the Introduction. The first part of item (i) correspond to the Theorem 5.2 and the second part of (i) can be derived from [10, Lem 1.1]. Part (ii) is the Theorem 5.1. Parts (ii) and (iii) follows from Theorem 5.5.

6. Distribution with (co)tangent sheaf globally generated

In this section we prove the Theorem 4.

Proposition 6.1. *If $\mathscr{F}$ is a locally free distribution on $\mathbb{P}^n$, of dimension r, such that the cotangent sheaf $\mathcal{T}^*_{\mathscr{F}}$ is globally generated and ample, then* $\mathrm{Sing}(\mathscr{F})$ *is nonempty with pure dimension $r-1$. This holds in particular if $\mathcal{T}_{\mathscr{F}} = \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(-d_i)$ with $d_i > 0$ for all i. Moreover*

$$\deg(\mathrm{Sing}(\mathscr{F})) = \deg(c_{n-r+1}(\mathcal{T}_{\mathbb{P}^n} - \mathcal{T}_{\mathscr{F}})) \leq d^{n-r+1} + d^{n-r} + \cdots + d + 1..$$

Proof. Since $\mathcal{T}_{\mathbb{P}^n}$ and $\mathcal{T}^*_{\mathscr{F}}$ are globally generated and ample then $\mathcal{T}_{\mathbb{P}^n} \otimes \mathcal{T}^*_{\mathscr{F}}$ is so. The nonemptiness of $\mathrm{Sing}(\mathscr{F})$ follows from [18] and the expected dimension follows from Bertini type Theorem, since $\mathcal{T}_{\mathbb{P}^n} \otimes \mathcal{T}^*_{\mathscr{F}}$ is globally generated.

On the other hand, it follows from Kempf- Laksov theorem [23] that

$$c_{n-r+1}(\mathcal{T}_{\mathbb{P}^n} - \mathcal{T}_{\mathscr{F}}) = [\mathrm{Sing}(\mathscr{F})] = \sum_{j=1}^m [V_j],$$

where $V_1, \ldots, V_m$ are the irreducible components of $\mathrm{Sing}(\mathscr{F})$. The bound

$$\deg\left(\sum_{j=1}^m [V_j]\right) \leq d^{n-r+1} + d^{n-r} + \cdots + d + 1.$$

follows from [29, Corolarry 4.8]. $\square$

Proposition 6.2. *Let $\mathscr{F}$ be a foliation on $\mathbb{P}^n$ of dimension r and degree d. Then the tangent sheaf $\mathcal{T}_{\mathscr{F}} = \mathcal{O}_{\mathbb{P}^n}(1-d) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r-1}$ if and only if there exists a generic linear projection $\pi : \mathbb{P}^n \to \mathbb{P}^{n-r+1}$ and a one-dimensional foliation $\mathscr{G}$ on $\mathbb{P}^{n-r+1}$, of degree d, with isolated singularities, such that $\mathscr{F} = \pi^*\mathscr{G}$.*

Proof. We will use the idea of proof of [12, Cor. 5.1] and [8, Lem. 2.2]. In fact, suppose that $\mathcal{T}_{\mathscr{F}} = \mathcal{O}_{\mathbb{P}^n}(1-d) \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus r-1}$, then $\mathcal{T}_{\mathscr{F}}$ is induced by an r-form ω that may be written as

$$\omega = i_X i_{Z_1} \cdots i_{Z_{r-1}} i_R \Omega\,,$$

where the X is a homogeneous vector field of degree d, the Z_j are constant vector fields, R is the radial vector field and Ω is the canonical volume form of $\mathbb{C}^{n+1}$.

In suitable coordinate system $(z_0, \ldots, z_n)$ of $\mathbb{C}^{n+1}$ we can write

$$Z_j = \frac{\partial}{\partial z_{n-r+1+j}}$$

for $j = 1, \ldots, r-1$. It follows that the fibers of the linear projection

$$\pi(z_0, \ldots, z_n) = (z_0, \ldots, z_{n-r+1})$$

are everywhere tangent to the leaves of $\mathscr{F}$. It follows from [8, Lem. 2.2] that there exists a one-dimensional foliation $\mathscr{G}$ such that $\mathcal{T}_{\mathscr{F}} = \pi^*\mathscr{G}$. $\square$

Example 6.3. Let $\mathscr{F}$ be a foliation on $\mathbb{P}^n$ of dimension r and degree d. If tangent sheaf $\mathcal{T}_{\mathscr{F}} = \mathcal{O}_{\mathbb{P}^n}(1-d) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r-1}$. Then

$$\deg(\mathrm{Sing}(\mathscr{F})) = d^{n-r+1} + \cdots + d + 1.$$

This follows from proposition 6.2.

A *subdistribution* $\mathscr{G}$ of $\mathscr{F}$ is a distribution whose tangent sheaf $\mathscr{G} \subset \mathcal{T}_{\mathscr{F}}$.

Proposition 6.4. *Let $\mathscr{F}$ be a locally free distribution on $\mathbb{P}^n$ of rank $r > 2$, degree d, that admits a locally free subdistribution of rank and degree $r-1$. If $\mathcal{T}^*_{\mathscr{F}}$ is globally generated, then $\mathcal{T}_{\mathscr{F}} = \mathcal{O}_{\mathbb{P}^n}(r-d) \oplus \mathcal{O}_{\mathbb{P}^n}^{\oplus r-1}$.*

Proof. Let $\mathscr{G}$ be the locally free subsheaf of $\mathcal{T}_{\mathscr{F}}$ of rank $r-1$. We have an exact sequence

$$0 \to \mathscr{G} \to \mathcal{T}_{\mathscr{F}} \to \mathcal{O}_{\mathbb{P}^n}(c_1(\mathcal{T}_{\mathscr{F}}) - c_1(\mathscr{G})) \to 0.$$

The distribution determined by $\mathscr{G}$ has degree $r-1$, then $c_1(\mathscr{G}) = 0$. Therefore there is a non trivial morphism $\mathcal{O}_{\mathbb{P}^n}(d-r) \hookrightarrow \mathcal{T}^*_{\mathscr{F}}$. This implies that

$$H^0(\mathcal{T}^*_{\mathscr{F}} \otimes \mathcal{O}_{\mathbb{P}^n}(r-d)) \neq 0.$$

Since $\det(\mathcal{T}^*_{\mathscr{F}}) = \mathcal{O}_{\mathbb{P}^n}(d-r)$, the result follows from [27, Prp. 1] $\square$

Remark 6.5. Consider a distribution on $\mathbb{P}^n$ of rank $r > 1$, degree $r+1$, and such that the cotangent sheaf $\mathcal{T}^*_{\mathscr{F}}$ is globally generated. Then $\mathcal{T}_{\mathscr{F}} = \mathcal{O}_{\mathbb{P}^n}(-1) \oplus \mathcal{O}_{\mathbb{P}^n}^{\oplus r-1}$. In fact, it follows from [24, p. 53] that $c_1(\mathcal{T}^*_{\mathscr{F}}) = 1$ iff $\mathcal{T}^*_{\mathscr{F}} \simeq \mathcal{O}_{\mathbb{P}^n}(1) \oplus \mathcal{O}_{\mathbb{P}^n}^{\oplus r-1}$.

Proposition 6.6. *Let $\mathscr{F}$ be a distribution on $\mathbb{P}^n$ of rank r, degree d, such that $\mathcal{T}^*_{\mathscr{F}}$ is globally generated. If $c_r(\mathcal{T}_{\mathscr{F}}) = 0$ then $\mathscr{F}$ admits a locally free subdistribution of rank $r-1$ and degree $d-1$.*

Proof. It follows from [24, Lem. 4.3.2] that there exists a bundle $\mathscr{G}^*$ of rank $r-1$ and an exact sequence $0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{T}^*_{\mathscr{F}} \to \mathscr{G}^* \to 0$. In particular, $\mathscr{G}$ is a subbundle of $\mathcal{T}_{\mathscr{F}}$ and yields a locally free subdistribution $\mathscr{G}$. Besides,

$$r - \deg(\mathscr{F}) = c_1(\mathcal{T}_{\mathscr{F}}) = c_1(\mathscr{G}) = r - 1 - \deg(\mathscr{G})$$

and we are done. $\square$

Corollary 6.7. *Let $\mathscr{F}$ be a distribution on $\mathbb{P}^n$ of rank and degree r, such that $\mathcal{T}^*_{\mathscr{F}}$ is globally generated. If $c_r(\mathcal{T}_{\mathscr{F}}) = 0$ then $\mathcal{T}_{\mathscr{F}} = \mathcal{O}_{\mathbb{P}^n}^{\oplus r}$.*

Proof. Combine Propositions 6.4 and 6.6. $\square$

Recall that the twisted null-correlation bundle $\mathscr{N}(1)$ on $\mathbb{P}^n$, with n odd, is defined by the short exact sequence (cf. [24, p. 77])

$$0 \to \mathscr{N}(1) \to \mathcal{T}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(2) \to 0$$

and it is the tangent sheaf of the contact distribution on $\mathbb{P}^n$.

In [28] J. C. Sierra and L. Ugaglia proved that a globally generated vector bundle $\mathcal{F}$ on $\mathbb{P}^n$ such that $\mathrm{rank}(\mathcal{F}) < n$ and $c_1(\mathcal{F}) = 2$, always splits unless $\mathcal{F}$ is a twisted null-correlation bundle on $\mathbb{P}^3$. Using this we can prove the following.

Proposition 6.8. *Let $\mathscr{F}$ be a distribution on $\mathbb{P}^n$ of rank $r > 1$, degree $r-2$, and such that $T_{\mathscr{F}}$ is globally generated. Then*

(i) $T_{\mathscr{F}} = \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^n}^{\oplus r-2}$;
(ii) *$T_{\mathscr{F}}$ is the twisted null-correlation bundle on $\mathbb{P}^3$.*

In particular, if $T_{\mathscr{F}}$ is not split, then $n = 3$, $\mathscr{F}$ is not a foliation and $\mathrm{Sing}(\mathscr{F}) = \emptyset$.

Proof. It follows from the J. Sierra and L. Ugaglia classification [28] and the fact that $c_1(T_{\mathscr{F}}) = 2$. □

6.1. **Proof of Theorem 5.** Item (i) is part of Proposition 6.1 while (ii) corresponds to Proposition 6.2. Besides, Proposition 6.4, Remark 6.5 and Corollary 6.7 prove (iii), and (iv) is the conclusion of Proposition 6.8.

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