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A faster algorithm for packing branchings in digraphs

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ABSTRACT

We consider the problem of finding an integral (and fractional) packing of branchings in a capacitated digraph with root-set demands. Schrijver described an algorithm that returns a packing with at most $m + n^3 + r$ branchings that makes at most $m(m + n^3 + r)$ calls to an oracle that basically computes a minimum cut, where n is the number of vertices, m is the number of arcs and r is the number of root-sets of the input digraph. Leston-Rey and Wakabayashi described an algorithm that returns a packing with at most $m + r - 1$ branchings but makes a large number of oracle calls. In this work we provide an algorithm, inspired on ideas of Schrijver and in a paper of Gabow and Manu, that returns a packing with at most $m+r-1$ branchings and makes at most $(m+r+2)n$ oracle calls. Moreover, for the arborescence packing problem our algorithm provides a packing with at most $m-n+2$ arborescences – thus improving the bound of m of Leston-Rey and Wakabayashi – and makes at most $(m - n + 5)n$ oracle calls.

1. Introduction

Integral packing problems form an important class of combinatorial optimization problems. A general algebraic version of a packing problem can be formalized as follows. Let $\mathcal{B}$ be a set of vectors in an m-dimensional vector space. Suppose we are given a non-negative integral vector c and we are asked to find (if possible) positive integers $y_1, \ldots, y_k$ and vectors $b_1, \ldots, b_k \in \mathcal{B}$ such that

$$y_1 b_1 + \cdots + y_k b_k \leqslant c.$$

When we require equality above, we are asking whether c belongs to the integer cone generated by the vectors in $\mathcal{B}$. In combinatorial applications, the set $\mathcal{B}$ is typically much larger than m. For example, $\mathcal{B}$ could be the set of (incidence vectors of) bases of a matroid whose ground set has m elements or could be the set of r-arborescences of a digraph with m arcs. From the perspective of an algorithmic designer, it is essential to find a packing with a polynomial number of members of $\mathcal{B}$. Moreover, from a theoretical point of view it would be very interesting to find the smallest upperbound k for which it is always possible to pack at most k elements of $\mathcal{B}$ independently of the choice of c. This latter problem is related to the study of Hilbert bases introduced by Cook, Fonlupt and Schrijver [2].

In this paper we are concerned with the case in which $\mathcal{B}$ corresponds to the set of branchings (with many root-sets) of digraphs. We note that the problem we consider here does not fit exactly in the general setting we have defined.

In a seminal paper Edmonds [4] characterized when a capacitated digraph with root-set demands has an integral packing of branchings. Lovász [12] established the fundamental ideas and proof techniques for problems involving packing

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of arborescences and branchings [1,6,7,10]. In this paper we investigate this problem from the point of view of finding efficiently a packing with few branchings.

Schrijver [15] described an algorithm that returns a packing with at most $m + n^3 + r$ distinct branchings that makes at most $m(m + n^3 + r)$ calls to an oracle that basically computes a minimum cut, where n is the number of vertices, m is the number of arcs, and r is the number of root-sets of the input digraph. Pevzner [14] considered the special case of finding a maximum integral packing of r-arborescences in a capacitated digraph and proved there exists such a packing with $O(nm)$ distinct arborescences. Gabow and Manu [8], also for this special case, provided an algorithm that returns a packing with at most $m + n - 2$ distinct arborescences whose time complexity is $O(\min\{n, \lg C\}n^2 m \lg(n^2/m))$, where C is the largest capacity of an arc. They also showed an upperbound of m for the number of distinct arborescences in a fractional packing. One of the key ideas in their algorithm is to keep a laminar family of cuts in order to bound the number of arborescences used in the final packing. Leston-Rey and Wakabayashi [11] described an algorithm for a general framework that implies an algorithm that returns an integral packing of branchings using at most $m+r-1$ distinct branchings, which in turn implies that there exists an integral packing of arborescences using at most m distinct arborescences. Though their algorithm is oracle-polynomial time and improves on the best known upper bounds for the packing size, it also requires a large number of oracle calls.

In this work we provide an algorithm for packing branchings for both the fractional and the integral version. The algorithm returns a packing with at most $m + r - 1$ branchings and makes at most $(m + r + 2)n$ oracle calls.

This paper is organized as follows. In the remaining of this section we introduce some basic notation, recall Edmonds' theorem and present our main result (Theorem 2) and its consequences. In Section 2, we discuss some concepts and auxiliary results which we use throughout the paper. In Section 3 we present our main contribution: a new algorithm for packing branchings in a network. Finally, in Section 4 we describe the pre-processing step of our algorithm — this is required to ensure that the packing produced by the algorithm is "small".

Preliminaries Before we begin, let us state a few preliminary definitions. For a function $f : X \to \mathbb{R}_+$, and $Y \subseteq X$, we write $f(Y)$ to denote the sum $\sum[f(y) \mid y \in Y]$. The *support* of f, denoted by f^+, is the set $\{x \in X \mid f(x) > 0\}$. Let B be a subset of some set E. The *characteristic function* of B is the function $\chi^B : E \to \{0, 1\}$ defined by

$$\chi^B(e) := \begin{cases} 1 & \text{if } e \in B, \\ 0 & \text{otherwise} \end{cases}$$

for each $e \in E$. We use characteristic functions without explicitly stating its domain, since the context will imply to which domain we refer. A function is *integral* if its image is a subset of the set $\mathbb{N}$ of nonnegative integers.

For a set V, a subset $\mathcal{P}$ of 2^V, we write $\bigcup \mathcal{P}$ to denote the set $\{u \mid u \in U \text{ for some } U \in \mathcal{P}\}$.

For a property P, we write

$$[P] := \begin{cases} 1, & \text{if } P \text{ is true} \\ 0, & \text{otherwise.} \end{cases}$$

A *digraph* is a pair $D = (V, A)$, where V is a finite set of *vertices*, and A is a finite set of *arcs*. Each arc is associated with two vertices, its *ends*, which are called its *tail* and its *head*. Let S be a subset of V. We write $\bar{S}$ to denote $V \setminus S$. We say that an arc $a \in A$ *enters* S if the tail of a is in $\bar{S}$ and the head of a is in S, otherwise we say that a *avoids* S. We write $\rho(S)$, or simply ρS, to denote the set of arcs of D that enter S. Finally, set

$$\mathcal{C}_S := \{\emptyset \neq U \subseteq V \mid U \cap S = \emptyset\}.$$

An S-*cover* is a subset B of arcs such that $B \cap \rho U \neq \emptyset$ for each $U \in \mathcal{C}_S$. An S-*branching* is a minimal S-cover. For most of the proofs in this paper we only require that B is an S-cover rather than an S-branching. We also say that $B \subseteq A$ *avoids* $\mathcal{P} \subseteq 2^V$ if $B \cap \rho U = \emptyset$ for each $U \in \mathcal{P}$. When $B = \{a\}$ we just say that a *avoids* $\mathcal{P}$.

Let $D = (V, A)$ be a digraph, $c : A \to \mathbb{R}_+$ an *arc capacity function* and $\mu : 2^V \setminus \{\emptyset\} \to \mathbb{R}_+$ a *demand function*. The triple $\mathbb{F} = (D, c, \mu)$ is called a *network*. Each set in μ^+ is called a *root-set*. It is convenient to set

$$c(U) := \sum[c(a) \mid a \in \rho U], \text{ and}$$
$$c(U, W) := \sum[c(a) \mid a \in A \text{ has one end in } U \text{ and the other in } W]$$

for each $U, W \subseteq V$. It is well-known that

$$c(U) + c(W) = c(U \cup W) + c(U \cap W) + c(U \setminus W, W \setminus U), \tag{1}$$

for each $U, W \subseteq V$. In other words, c is a *submodular* function.

For each nonempty $U \subseteq V$, we define the *demand induced* by μ by setting

$$p(U) := \sum[\mu(R) \mid R \in \mu^+, U \in \mathcal{C}_R].$$

It is straightforward to verify that p is a *supermodular* function, that is,

$$p(U) + p(W) \leqslant p(U \cup W) + p(U \cap W), \tag{2}$$

for each $U, W \subseteq V$ with $U \cap W \neq \emptyset$. Note that if $\emptyset \neq U \subseteq W$, then

$$p(U) \geqslant p(W), \tag{3}$$

or, in words, p is nonincreasing. For a vertex v, we write for brevity $c(v)$ and $p(v)$ instead of $c(\{v\})$ and $p(\{v\})$.

Let $S \in 2^V \setminus \{\emptyset\}$, and denote by $\mathcal{B}_S$ the set of all S-covers of D. We say that a function $y_S : \mathcal{B}_S \to \mathbb{R}_+$ is a *packing* of S if

$$\sum[y_S(B) \mid B \in \mathcal{B}_S] = \mu(S).$$

A function y that assigns a packing y_S to each $S \in 2^V \setminus \{\emptyset\}$ is a *packing* of $\mathbb{F}$ if

$$\sum[y_S(B)\chi^B \mid S \in 2^V \setminus \{\emptyset\}, B \in \mathcal{B}_S] \leqslant c.$$

Furthermore, y is *integral* if y_S is integral for each $S \in 2^V$. The *size* of y is the number

$$|y^+| := \sum[|y_S^+| \mid S \in \mu^+].$$

In this setting, the problem of finding a packing of branchings consists in given a network $\mathbb{F} = (D, c, \mu)$ finding a packing y of $\mathbb{F}$. Note that $|y^+|$ is an upper bound on the number of distinct covers used in y.

We say that $\mathbb{F}$ is *feasible* if $c(U) \geqslant p(U)$ for each nonempty $U \subseteq V$. The following theorem is a classical result of Edmonds [4].

Theorem 1 (*Edmonds [4]*). *A network has a packing if and only if it is feasible.*

In the proof of our main result, *tight* sets and laminar families play an important role. We say that a nonempty $U \subseteq V$ is *tight* in $\mathbb{F}$ if $c(U) = p(U)$. The set of all tight sets of $\mathbb{F}$ is denoted by $\Gamma\mathbb{F}$. We highlight a certain subset of $\Gamma\mathbb{F}$ and write

$$\Gamma_+\mathbb{F} := \{U \subseteq V \mid c(U) = p(U) > 0\},$$

that is, $\Gamma_+\mathbb{F}$ is just the set of tight sets with positive induced demand. We say that two subsets U, W of V *intersect* if

$$U \cap W \neq \emptyset, \qquad U \setminus W \neq \emptyset, \quad \text{and} \quad W \setminus U \neq \emptyset.$$

We say that a subset $\mathcal{L}$ of 2^V is *laminar* if no two of its members intersect, that is,

$$U \subseteq W, \quad \text{or} \quad W \subseteq U, \quad \text{or} \quad U \cap W = \emptyset$$

for each $U, W \in \mathcal{L}$. For $T \subseteq V$ let $\mathcal{L}_T$ be the set of all maximal elements L of $\mathcal{L}$ such that L and T intersect. For each $\mathcal{P} \subseteq 2^V$, we write $\gamma(\mathcal{P})$ to denote the size of a maximum laminar subset of $\mathcal{P}$ such that $\emptyset, V \notin \mathcal{P}$. It is easy to verify that $\gamma(\mathcal{P}) \leqslant 2|V| - 2$ if V is nonempty.

Oracle Henceforth, we assume that we have an oracle that given a feasible network $\mathbb{F} = (D, c, \mu)$, a root-set $S \in \mu^+$ and an arc $a \in \rho(\bar{S})$, it returns a set X in

$$\operatorname{argmin}\{c(W) - p(W) \mid W \subseteq V, W \cap S \neq \emptyset, a \in \rho W\}. \tag{4}$$

We also write

$$\alpha_S(a) \tag{5}$$

to denote the number $c(X) - p(X)$. Informally, this number is the maximum amount that we can subtract from the capacity of a while keeping the feasibility of the network (this is described more precisely in Section 2). Schrijver [15] showed that this quantity can be computed in polynomial time.

In order to describe our main result, we need one more definition. For a network $\mathbb{F} = (D, c, \mu)$ let $H\mathbb{F} := \{u \in V \mid p(u) > 0\}$.

Theorem 2. *There is an oracle-polynomial time algorithm that given a feasible network $\mathbb{F} = (D, c, \mu)$, returns a packing y such that $|y^+| \leqslant |A| - |H\mathbb{F}| + |\mu^+|$ and makes no more than $(|A| - |H\mathbb{F}| + |\mu^+| + 3)|V|$ oracle calls. Moreover, if c is integral, then y is integral.*

Some implications Theorem 2 has some interesting implications which we next describe.

The *integer cone* of a set $L := \{b_1, b_2, \ldots, b_k\}$ of vectors is defined as

$$\left\{\sum_{i=1}^{k} \lambda_i b_i \mid \lambda_i \in \mathbb{N}, i = 1, 2, \ldots, k\right\}.$$

The Carathéodory rank of L is the smallest integer t such that every element in the integer cone of L can be written as a nonnegative integral linear combination of at most t elements from L. Let $D = (V, A)$ be a digraph and consider a nonempty $S \subseteq V$. Let $\mathcal{B}_S$ denote the set of all S-branchings of D. Consider now a fixed nonempty $\mathcal{S} \subseteq 2^V \setminus \{\emptyset, V\}$. Suppose that $\mathcal{B}_S$ is nonempty for each $S \in \mathcal{S}$. Let Q denote the integer cone of incidence vectors of S-branchings for $S \in \mathcal{S}$. Let $c \in Q$ with $c \neq 0$, that is, $c = \sum[\lambda_B^S \chi^B \mid S \in \mathcal{S}, B \in \mathcal{B}_S]$ for nonnegative integers λ_B^S with $S \in \mathcal{S}, B \in \mathcal{B}_S$. Consider the network $\mathbb{F} = (D, c, \mu)$, where $\mu(S) := \sum[\lambda_B^S \mid B \in \mathcal{B}_S]$ if $S \in \mathcal{S}$, and $\mu(S) := 0$ otherwise. It is clear that $\mathbb{F}$ is feasible, $|\mu^+| = |\mathcal{S}|$ and $|H\mathbb{F}| \geqslant 1$. Moreover, since B is an S-branching (and thus minimal) for each $S \in \mathcal{S}$ and each $B \in \mathcal{B}_S$, we have that $c(u) = p(u)$ for each $u \in V$. Theorem 2 together with these last observations imply that c is a nonnegative integral linear combination of at most $|A| + |\mathcal{S}| - 1$ covers. In other words, the Carathéodory rank of the set of incidence vectors of minimal S-covers for $S \in \mathcal{S}$ is at most $|A| + |\mathcal{S}| - 1$.

The next consequence is the special case in which $\mu^+ = \{r\}$ for some $r \in V$, that is, we want to pack r-arborescences. An *r-arborescence* is an $\{r\}$-branching. It improves, with respect to the size of the packing, a previous result of Leston-Rey and Wakabayashi [11].

Corollary 1. *There exists an oracle-polynomial time algorithm that given a digraph D, a root $r \in V$ such that D has at least one r-arborescence, and an arc capacity function $c : A \to \mathbb{R}_+$, returns r-arborescences $B_1, \ldots, B_k$ and positive reals $y_1, \ldots, y_k$ such that*

$$k \leqslant |A| - |V| + 2, \qquad \sum_{i=1}^{k} y_i \chi^{B_i} \leqslant c, \quad \text{and} \quad \sum_{i=1}^{k} y_i = \min\{c(X) \mid X \in \mathcal{C}_{\{r\}}\}$$

by making no more that $(|A| - |V| + 5)|V|$ oracle calls. Moreover, if c is integral, then $y_1, \ldots, y_k$ are integers. □

So the Carathéodory rank of the set of incidence vectors of r-arborescences is at most $|A| - |V| + 2$.

Another byproduct of our main result concerns packing of spanning trees in undirected graphs. Nash-Williams [13] and Edmonds [3] characterized when an undirected graph contains k edge-disjoint spanning trees. Frank [6] provided an alternative proof of this result by finding an appropriate orientation of the graph and applying Theorem 1. Using the same approach we can show that Theorem 2 implies that given a graph $G = (V, E)$ with edge-capacities there is always a packing of spanning trees of size at most $2|E| - |V| + 2$. Bounds on the size of packing of spanning trees (in fact, of bases of a matroid) were previously known. de Pina and Soares [5] showed that the Carathéodory rank of the set of incidence vectors of bases of a matroid is at most $m + r - 1$ where m is size of the ground set and r is the rank of the matroid. Gijswijt and Regts [9] extended their result showing that the Carathéodory rank of the set of incidence vectors of bases of a polymatroid is at most m where m is the size of the ground set of the polymatroid.

2. Some auxiliary results

Let us begin with a simple lemma, which generalizes a lemma of Gabow and Manu [8] involving arborescences, and whose proof follows the idea of Lovász proof [12] of Edmonds theorem.

Lemma 1. *Let $\mathbb{F}$ be a feasible network, and $\mathcal{L}$ be a laminar subset of $\Gamma\mathbb{F}$. If there exists $S \in \mu^+$ with $S \neq V$, then there exists an arc $a \in c^+ \cap \rho\bar{S}$ which avoids $\mathcal{L} \setminus \mathcal{C}_S$.*

Proof. Let $S \in \mu^+$ with $S \neq V$. Suppose first that $L \cap \bar{S} = \emptyset$ for each $L \in \mathcal{L} \setminus \mathcal{C}_S$. Since $\mu(S) > 0$, it follows that $p(\bar{S}) > 0$. But $c(\bar{S}) \geqslant p(\bar{S})$, and thus there exists a in $c^+ \cap \rho\bar{S}$, and clearly a avoids $\mathcal{L} \setminus \mathcal{C}_S$.

Suppose now that $L \cap \bar{S} \neq \emptyset$ for some $L \in \mathcal{L} \setminus \mathcal{C}_S$. Choose such a minimal L. Observe that $L \cap S \neq \emptyset$ by the definition of $\mathcal{C}_S$. Now $c(L \cap \bar{S}) \geqslant p(L \cap \bar{S}) > p(L) = c(L)$, and so there exists an arc $a \in c^+$ whose tail is in $L \cap S$ and whose head is in $L \cap \bar{S}$. To finish we establish that a avoids $\mathcal{L} \setminus \mathcal{C}_S$. Indeed, let $T \in \mathcal{L} \setminus \mathcal{C}_S$ and suppose that $a \in \rho T$. In this case, $T \cap L \neq \emptyset$, since both sets contain the head of a, and $T \cap \bar{S} \neq \emptyset$. But the tail of a is in $L \setminus T$, and hence, the laminarity of $\mathcal{L}$ imply that $T \subset L$, which contradicts the minimality of L. □

Decreasing the capacity of a set of arcs Let $\mathbb{F}$ be a network, $S \in \mu^+$ a root-set, and $B \subseteq c^+$. Let H be the set of heads of the arcs in B and assume that $\bar{S} \cap H \neq \emptyset$. Moreover, let k be a number with $0 \leqslant k \leqslant \min\{\mu(S)\} \cup \{c(a) \mid a \in B\}$. The network obtained from $\mathbb{F}$ by decreasing k units the capacity of each arc in B, denoted by $\mathbb{F}^{S,kB}$, is the network $(D, c^{kB}, \mu^{S,kB})$, where

$$c^{kB}(a) := \begin{cases} c(a) - k, & \text{if a } \in \text{B}, \\ c(a), & \text{otherwise,} \end{cases} \quad \text{for each } a \in A.$$

$$\mu^{S,kB}(T) := \begin{cases} \mu(T) - k, & \text{if } T = S \\ \mu(T) + k, & \text{if } T = S \cup H, \\ \mu(T), & \text{otherwise,} \end{cases} \quad \text{for each } \emptyset \neq T \subseteq V.$$

Note that the demand induced by $\mu^{S,kB}$, denoted $p^{S,kB}$, satisfies

$$p^{S,kB}(X) = \begin{cases} p(X) - k, & \text{if } B \text{ enters } X \text{ and } X \in \mathcal{C}_S, \\ p(X), & \text{otherwise,} \end{cases} \quad \text{for each } \emptyset \neq X \subseteq V. \tag{6}$$

We write $\mathbb{F}^{kB}$ instead of $\mathbb{F}^{S,kB}$ when no confusion can arise. Moreover, if $B = \{a\}$ for some arc a, then we write $\mathbb{F}^{S,ka}$ instead of $\mathbb{F}^{S,k\{a\}}$. Finally, when $k = 1$, we write $\mathbb{F}^{S,B}$ instead of $\mathbb{F}^{S,1B}$.

Observe that if $\mathbb{F}$ is feasible and $0 \leqslant k \leqslant \alpha_S(a)$, then $\mathbb{F}^{ka}$ is feasible.

Let us roughly describe the main ideas of our algorithm—a precise formulation is given in the next section. The algorithm is recursive and has as input a feasible network $\mathbb{F}$ and a laminar family $\mathcal{L}$ of tight sets. It begins by choosing a root-set S with positive demand. Using Lemma 1 the algorithm finds an arc a that avoids $\mathcal{L} \setminus \mathcal{C}_S$. Then it decreases the capacity of a as much as possible keeping feasibility (using the oracle). Depending on the resulting capacity of a, the algorithm may increase $\mathcal{L}$, and then recursively finds a packing of the resulting network. The returned packing is then extended to a packing of the original network. As we will show, the laminar family $\mathcal{L}$ imposes an upperbound on the number of oracle calls.

The next lemma establishes in which conditions we can extend a laminar family of tight sets.

Lemma 2. *Let $\mathbb{F}$ be a feasible network, $S \in \mu^+$ with $S \neq V$, $\mathcal{L} \subseteq \Gamma\mathbb{F}$ laminar, and $a \in c^+ \cap \rho\bar{S}$. If a avoids $\mathcal{L} \setminus \mathcal{C}_S$ and $\alpha_S(a) = 0$, then there exists $L^* \in \Gamma_{\!+}\mathbb{F}$ such that $L^* \notin \mathcal{L}$ and $\mathcal{L} \cup \{L^*\} \subseteq \Gamma\mathbb{F}$ is laminar.*

Proof. Suppose that a avoids $\mathcal{L} \setminus \mathcal{C}_S$ and $\alpha_S(a) = 0$. Let k be such that $0 < k < \min\{c(a), \mu(S)\}$. Observe that $\mathbb{F}^{ka}$ is infeasible and

(1⋄) for each $\emptyset \neq U \subseteq V : c^{ka}(U) \geqslant p^{ka}(U) - k$, and
(2⋄) for each $L \in \mathcal{L} : c^{ka}(L) = p^{ka}(L)$,

since (1⋄) follows from the feasibility of $\mathbb{F}$, and (2⋄) holds because a avoids $\mathcal{L} \setminus \mathcal{C}_S$. We claim that there exists $T \subseteq V$ such that

(3⋄) $c^{ka}(T) = p^{ka}(T) - k$, and
(4⋄) for each $L \in \mathcal{L}$: if L intersects T, then $c^{ka}(L \cap T) \geqslant p^{ka}(L \cap T)$.

Since $a \in c^+$ and $\alpha_S(a) = 0$, then there exists $U \in \Gamma\mathbb{F} \setminus \mathcal{C}_S$ with $a \in \rho U$. Hence $c^{ka}(U) = p^{ka}(U) - k$. If (4⋄) holds with respect to U, then let $T := U$. Suppose that (4⋄) does not hold with respect to U. We show that there exists $L \in \mathcal{L}$ such that L intersects U and $c^{ka}(L \cap U) = p^{ka}(L \cap U) - k$. Indeed, since (4⋄) does not hold, then there exists $L \in \mathcal{L}$ such that L intersects U and $c^{ka}(L \cap U) < p^{ka}(L \cap U)$. Note that a enters $L \cap U$. In this case, $L \cap U \notin \mathcal{C}_S$, and so $L \notin \mathcal{C}_S$. But a avoids L, because a avoids $\mathcal{L} \setminus \mathcal{C}_S$ which, combined with $a \in \rho(L \cap U)$, implies that a avoids $L \cup U$, whence $c^{ka}(L \cup U) \geqslant p^{ka}(L \cup U)$. Now the submodularity of c^{ka}, and the supermodularity of p^{ka} imply that

$$\begin{aligned} p^{ka}(U) - k + p^{ka}(L) &= c^{ka}(U) + c^{ka}(L) \geqslant c^{ka}(L \cap U) + c^{ka}(L \cup U) \\ &\geqslant p^{ka}(L \cap U) - k + p^{ka}(L \cup U) \geqslant p^{ka}(U) - k + p^{ka}(L). \end{aligned}$$

Thus equality holds throughout, and $c^{ka}(L \cap U) = p^{ka}(L \cap U) - k$, as required. Select a minimal L such that L intersects U and $c^{ka}(L \cap U) = p^{ka}(L \cap U) - k$, and set $T := L \cap U$. Then the laminarity of $\mathcal{L}$ and the choice of L imply that T satisfies (3⋄) and (4⋄).

Let $L^* := T \cup \bigcup \mathcal{L}_T$. Observe that $\mathcal{L} \cup \{L^*\}$ is laminar. The following claim proves that $L^* \notin \mathcal{L}$.

Claim. $c^{ka}(T \cup \bigcup \mathcal{L}_T) = p^{ka}(T \cup \bigcup \mathcal{L}_T) - k$.

Proof. We prove that $c^{ka}(T \cup \bigcup \mathcal{P}) = p^{ka}(T \cup \bigcup \mathcal{P}) - k$ for each subset $\mathcal{P}$ of $\mathcal{L}_T$, by induction on $|\mathcal{P}|$. The result is obvious if $\mathcal{P} = \emptyset$. Suppose that $\mathcal{P} \neq \emptyset$. Let $L \in \mathcal{P}$ and set $\mathcal{P}' := \mathcal{P} \setminus \{L\}$. Note that L intersects $T \cup \bigcup \mathcal{P}'$. By induction hypothesis, $c^{ka}(T \cup \bigcup \mathcal{P}') = p^{ka}(T \cup \bigcup \mathcal{P}') - \varepsilon$. Hence

$$\begin{aligned} p^{ka}\left(T \cup \bigcup \mathcal{P}'\right) - k + p^{ka}(L) &= c^{ka}\left(T \cup \bigcup \mathcal{P}'\right) + c^{ka}(L) \\ &\geqslant c^{ka}\left(\left(T \cup \bigcup \mathcal{P}'\right) \cup L\right) + c^{ka}\left(\left(T \cup \bigcup \mathcal{P}'\right) \cap L\right) \\ &\geqslant p^{ka}\left(\left(T \cup \bigcup \mathcal{P}'\right) \cup L\right) - k + p^{ka}\left(\left(T \cup \bigcup \mathcal{P}'\right) \cap L\right) \\ &\geqslant p^{ka}\left(T \cup \bigcup \mathcal{P}'\right) - k + p^{ka}(L), \end{aligned}$$

where the first equality holds by induction hypothesis and by (2⋄); the first inequality, by submodularity of c^{ka}; the second inequality, by (1⋄) and by (4⋄); finally, the third inequality, by supermodularity of p^{ka}. Thus equality holds throughout and $c^{ka}(T \cup \bigcup \mathcal{P}) = p^{ka}(T \cup \bigcup \mathcal{P}) - \varepsilon$. This finishes the proof of the claim. •

Now, $c^{ka}(L^*) = p^{ka}(L^*) - k$ and $\mathbb{F}$ feasible imply that $L^* \in \Gamma_{\!+}\mathbb{F}$, and hence $\mathcal{L} \cup \{L^*\} \subseteq \Gamma\mathbb{F}$, which establishes the lemma. □

The following theorem is a reformulation of a classical theorem of Edmonds [4].

Theorem 3 (*Edmonds [4]*)**.** *Let $\mathbb{F}$ be a feasible network and $S \in \mu^+$ with $S \neq V$. Then there exists an arc $a \in c^+ \cap \rho\bar{S}$ which avoids $\Gamma\mathbb{F} \setminus \mathcal{C}_S$ and such that $\mathbb{F}^{ka}$ is a feasible network, where $k := \min\{c(a), \mu(S), \min\{c(W) - p(W) \mid W \subseteq V, W \cap S \neq \emptyset, a \in \rho W\}\}$.*

Proof. Choose a laminar subfamily $\mathcal{L}$ of $\Gamma\mathbb{F}$ of maximum cardinality. By Lemma 1, there is an arc $a \in c^+ \cap \rho\bar{S}$ that avoids $\mathcal{L} \setminus \mathcal{C}_S$. If a does not avoid $\Gamma\mathbb{F} \setminus \mathcal{C}_S$ then $\mathbb{F}^{k'a}$ is infeasible for each $0 < k' \leqslant \min\{c(a), \mu(S)\}$. Lemma 2 now implies the existence of $L^* \in \Gamma_{\!+}\mathbb{F}$ such that $\mathcal{L} \cup \{L^*\} \subseteq \Gamma\mathbb{F}$ is laminar and $L^* \notin \mathcal{L}$, which contradicts the maximality of $\mathcal{L}$. So a avoids $\Gamma\mathbb{F} \setminus \mathcal{C}_S$, and therefore $\mathbb{F}^{ka}$ is a feasible network. □

We note that Theorem 3 easily implies Theorem 1. This provides an alternative proof of Edmonds' result.

The following corollary will be useful.

Corollary 2. *Let $\mathbb{F}$ be a feasible network, and $S \in \mu^+$. Then there exists an S-cover $B \subseteq c^+$ and a real $\varepsilon > 0$ such that $\mathbb{F}^{S,\varepsilon B}$ is a feasible network. Moreover, if $\mathbb{F}^{S,ka}$ is a feasible network for some arc $a \in c^+$ and some real $k > 0$, then there exists an S-cover $B \subseteq c^+$ and a real $\varepsilon > 0$ such that $a \in B$ and $\mathbb{F}^{S,\varepsilon B}$ is a feasible network.* $\square$

3. The algorithm

In this section we present our algorithm for packing branchings in a network. Before that let us describe roughly Schrijver's algorithm for this problem. The algorithm finds a *safe* arc a, that is, for which $\alpha_S(a) > 0$—the existence of this arc is guaranteed by Theorem 3. Then it decreases the capacity of a by $\alpha_S(a)$ and recursively finds a packing in the resulting network. Finally, it extends this solution to a packing in the original network. Schrijver [15] showed that the algorithm returns a packing with at most $m + n^3 + r$ distinct branchings and makes at most $m(m + n^3 + r)$ calls to an oracle that basically computes a minimum cut, where n is the number of vertices, m is the number of arcs, and r is the number of root-sets of the input digraph.

Our algorithm uses essentially the same idea but in order to obtain better bounds we rely on a laminar family of tight cuts. The idea of using a laminar family was used by Gabow and Manu [8] in their algorithm for packing arborescences with a same root in a network. Roughly speaking at each iteration the algorithm either finds a *removable* arborescence or enlarges the laminar family. Since the size of a laminar family on V is bounded by $2n - 1$, this implies a bound on the number of arborescences used in the packing. In our algorithm we select a suitable arc a. If $\alpha_S(a) > 0$ then we decrease the capacity of a and recursively find a packing. On the other hand, if $\alpha_S(a) = 0$ then we enlarge the laminar family. We note however that in our algorithm, the bound on the number of branchings in the packing is related to the **rank** of a certain subset $Z\mathbb{F}$ of vectors. The laminar family plays a slightly different role when compared to Gabow and Manu's algorithm. This allows us to obtain better bounds. As far as we know, the idea of using the rank of $Z\mathbb{F}$ to bound the size of the packing and the number of oracle calls has not been used before.

For a network $\mathbb{F} = (D, c, \mu)$ set

$$Z\mathbb{F} := \{\chi^{\rho U} \mid U \in \Gamma\mathbb{F}\} \cup \{\chi^{\{e\}} \mid e \in A, c(e) = 0\}.$$

For a subset M of vectors, let $\mathrm{rank}\,(M)$ denote the size of a maximum linearly independent subset of M.

Our packing algorithm consists of two phases. In the first phase we construct a suitable laminar family of tight sets which is used in the second phase. We postpone the presentation of this phase for the last section.

Here we describe the algorithm RPACK $(\mathbb{F}, \mathcal{L})$ for the second phase. It receives a feasible network $\mathbb{F}$ and a laminar $\mathcal{L} \subseteq \Gamma\mathbb{F}$ and returns a packing y of $\mathbb{F}$ with $|y^+| \leqslant |A| - \mathrm{rank}\,(Z\mathbb{F}) + |\mu^+|$. Furthermore, the number of oracle calls is at most $(|A| - \mathrm{rank}\,(Z\mathbb{F}) + |\mu^+|)(|V| - 1) + \gamma(p^+) - |\mathcal{L}|$. See Theorem 2.

We remark that, in the description of the algorithm, we use Claim 1 (in the proof of Theorem 4) which is responsible for the construction of a packing of a feasible network $\mathbb{F}$ by using a recursively obtained packing of a feasible network $\mathbb{F}^{ka}$.

In the algorithm RPACK we use the following notation: for each $\emptyset \neq S \subseteq V$ and each integer k, let $\underline{k_S} : \mathcal{B}_S \to \mathbb{N}$ be the function defined by $\underline{k_S}(B) := k$ for each $B \in \mathcal{B}_S$.

Algorithm RPACK $(\mathbb{F}, \mathcal{L})$

if $\mu^+ = \emptyset$ **then**
  return the packing $\{(S, \underline{0_S}) \mid \emptyset \neq S \subseteq V\}$
if $\mu^+ = \{V\}$ **then**
  return the packing $\{(S, \underline{0_S}) \mid \emptyset \neq S \subset V\} \cup \{(V, \underline{\mu(V)_V})\}$
select $S \in \mu^+ \setminus \{V\}$ and $a \in c^+ \cap \rho\bar{S}$ such that a avoids $\mathcal{L} \setminus \mathcal{C}_S$
if $\alpha_S(a) = 0$ **then**
  select L^* as in Lemma 2
  return $y :=$ RPACK $(\mathbb{F}, \mathcal{L} \cup \{L^*\})$
$y' :=$ RPACK $(\mathbb{F}^{\alpha_S(a)a}, \mathcal{L})$
return the packing y of $\mathbb{F}$ obtained from y', using Claim 1

Analysis of the packing algorithm

The following result, which follows from elementary linear algebra, will be useful.

Lemma 3. *Let M be a finite set, and for each $i \in M$, let $a_i \in \mathbb{R}^n$. If there exist $x, z, d \in \mathbb{R}^n$, and $\eta \in \mathbb{R}$ such that $dx \leqslant \eta$, $dz > \eta$, and $a_i x = a_i z$ for each $i \in M$, then $d \notin \mathrm{span}\,(\{a_i \mid i \in M\})$.* $\square$

Recall that the size of a packing y of $\mathbb{F}$ is $|y^+| := \sum[|y_S^+| \mid S \in \mu]$. Our main result, Theorem 2, follows from the next result by picking $\mathcal{P} := p^+$.

Theorem 4. *Let $\mathbb{F}$ be a feasible network, and $\mathcal{P}, \mathcal{L} \subseteq 2^V$. If $\mathcal{P} \supseteq p^+$ and $\mathcal{L} \subseteq \{X \in \mathcal{P} \mid c(X) = p(X)\}$ is laminar, then* RPACK$(\mathbb{F}, \mathcal{L})$ *returns a packing y of $\mathbb{F}$ such that $|y^+| \leqslant \varphi(\mathbb{F})$, and makes no more than $q(\mathbb{F}, \mathcal{L})$ oracle calls, where*

$$\varphi(\mathbb{F}) := |A| - \text{rank}\,(Z\mathbb{F}) + |\mu^+|, \quad \textit{and}$$
$$q(\mathbb{F}, \mathcal{L}) := (|A| - \text{rank}\,(Z\mathbb{F}))(|V| - 1) + \gamma(\mathcal{P}) - |\mathcal{L}| + \sum_{S \in \mu^+} |\bar{S}|.$$

Moreover, if c is integral, then y is integral.

Proof. Suppose that $\mathcal{P} \supseteq p^+$ and $\mathcal{L} \subseteq \{X \in \mathcal{P} \mid c(X) = p(X)\}$ is laminar. The proof is by induction on $q(\mathbb{F}, \mathcal{L})$. If $\mu^+ = \emptyset$ or $\mu^+ = \{V\}$, then the result is immediate, since in this case the algorithm returns a packing y such that $|y^+| = |\mu^+| \leqslant \varphi(\mathbb{F})$, and makes no oracle calls.

Suppose now that there exists $S \in \mu^+$ with $S \neq V$ (so $|V| \geqslant 2$) and we are at line 5 of the algorithm. By Lemma 1, there exists an arc a as in line 5.

Assume first that $\alpha_S(a) = 0$. In this case, we are at line 7, and by Lemma 2, there exists $L^* \in \Gamma_{\!+}\mathbb{F}$ such that $\mathcal{L} \cup \{L^*\}$ is laminar. Note that $\mathcal{L} \cup \{L^*\} \subseteq \{X \in \mathcal{P} \mid c(X) = p(X)\}$. Furthermore, $q(\mathbb{F}, \mathcal{L} \cup \{L^*\}) + 1 \leqslant q(\mathbb{F}, \mathcal{L})$, and therefore, by induction hypothesis, the call in line 8 returns a packing y of $\mathbb{F}$ such that $|y^+| \leqslant \varphi(\mathbb{F})$, and makes no more than $q(\mathbb{F}, \mathcal{L} \cup \{L^*\}) + 1 \leqslant q(\mathbb{F}, \mathcal{L})$ oracle calls. This completes the proof in this case.

In what follows, we assume that $\alpha_S(a) > 0$. To simplify, we write $\mathbb{F}' := \mathbb{F}^{\alpha_S(a)a}$; similarly for the components of $\mathbb{F}'$, for instance, $\mu' := \mu^{\alpha_S(a)a}$. Let t be the head of a and $S' := S \cup \{t\}$. Observe that, by (6), $\mathcal{P} \supseteq (p')^+$, and furthermore $\mu'(S') \geqslant \alpha_S(a)$.

Before we proceed, we prove the following.

Claim 1. *If y' is a packing of $\mathbb{F}'$, then there exists a packing y of $\mathbb{F}$ with $|y^+| \leqslant |(y')^+| + [S' \in \mu^+]$.*

Proof. Let $k := \alpha_S(a)$. Let y' be a packing of $\mathbb{F}'$ and $B_1, \ldots, B_e$ be the covers in $(y'_{S'})^+$. Then $\sum_{i=1}^{e} y'_{S'}(B_i) = \mu'(S') \geqslant k$. Choose i in $\{1, \ldots, e\}$ such that

$$\sum_{j=1}^{i-1} y'_{S'}(B_j) < k \quad \text{and} \quad \sum_{j=1}^{i} y'_{S'}(B_j) \geqslant k.$$

Set $k' := k - \sum_{j=1}^{i-1} y'_{S'}(B_j)$. Define now $z : \mathcal{B}_S \to \mathbb{N}$, a packing of S, by setting

$$z(J) := \begin{cases} y'_S(J) + y'_{S'}(B_j), & \text{if } J = \{a\} \cup B_j \text{ and } j \in \{1, \ldots, i-1\}, \\ y'_S(J) + k', & \text{if } J = \{a\} \cup B_i, \\ y'_S(J), & \text{otherwise,} \end{cases}$$

for each $J \in \mathcal{B}_S$.

Moreover, we now define $z' : \mathcal{B}_{S'} \to \mathbb{N}$, a packing of S', by setting

$$z'(J) := \begin{cases} y'_{S'}(J) - k', & \text{if } J = B_i, \\ y'_{S'}(J), & \text{if } J = B_j \text{ and } j \in \{i+1, \ldots, e\}, \\ y'_{S'}(J), & \text{otherwise} \end{cases}$$

for each $J \in \mathcal{B}_{S'}$.

Finally, we define a packing y of $\mathbb{F}$ by setting

$$y_T := \begin{cases} z, & \text{if } T = S, \\ z', & \text{if } T = S', \\ y'_T, & \text{otherwise,} \end{cases} \quad \text{for each } \emptyset \neq T \subseteq V.$$

It is easy to see that y is a packing of $\mathbb{F}$ with $|y^+| \leqslant |(y')^+| + [S' \in \mu^+]$. This finishes the proof of the claim. •

Now we deal with lines 9 and 10 of the algorithm. We divide the rest of the proof in the following cases:

Case 1 $\alpha_S(a) = \mu(S)$.
Clearly, $S \notin (\mu')^+$. Then $\sum_{T \in (\mu')^+} |\bar{T}| \leqslant \sum_{T \in \mu^+} |\bar{T}| - 1$, and so $q(\mathbb{F}', \mathcal{L}) + 1 \leqslant q(\mathbb{F}, \mathcal{L})$. Thus, by induction hypothesis, the call in line 9 returns a packing y' of $\mathbb{F}'$ such that $|(y')^+| \leqslant \varphi(\mathbb{F}')$, and makes no more than $q(\mathbb{F}', \mathcal{L})$ oracle calls.
Now, it is clear that $|(\mu')^+| \leqslant |\mu^+| - [S' \in \mu^+]$, and so $\varphi(\mathbb{F}') \leqslant \varphi(\mathbb{F}) - [S' \in \mu^+]$. Hence, by Claim 1, $\mathbb{F}$ has a packing y such that $|y^+| \leqslant |(y')^+| + [S' \in \mu^+]$, and thus $|y^+| \leqslant \varphi(\mathbb{F})$.
Since only one oracle call is made in line 7, then the number of oracle calls is at most $q(\mathbb{F}', \mathcal{L}) + 1 \leqslant q(\mathbb{F}, \mathcal{L})$, as we wanted.

Case2 $\alpha_S(a) < \mu(S)$.
Recall that we are assuming $\alpha_S(a) > 0$. The following claim contains the main step of the proof for Case 2.

Claim 2. $\mathrm{rank}(Z\mathbb{F}') \geqslant \mathrm{rank}(Z\mathbb{F}) + 1$.

Proof. Suppose first that $\alpha_S(a) = c(a)$. Corollary 2 implies that there exists an S-cover $B \subseteq c^+$ in $\mathbb{F}$ and a real $\varepsilon > 0$ such that $a \in B$ and $\mathbb{F}^{\varepsilon B}$ is a feasible network. Moreover, since $\mu'(S) > 0$ and $\mathbb{F}'$ is feasible, it also implies that there exists an S-cover $J \subseteq (c')^+$ in $\mathbb{F}'$ and a real $\varepsilon > 0$ such that $(\mathbb{F}')^{\varepsilon J}$ is feasible, and thus $a \notin J$. Then

$$\text{for each } U \in \Gamma\mathbb{F} : \chi^B(\rho U) = \chi^{e_S}(U) = \chi^J(\rho U), \text{ and}$$
$$\text{for each } e \in E \text{ with } c(e) = 0 : \chi^B(e) = 0 = \chi^J(e).$$

But $\chi^B(a) = 1$ and $\chi^J(a) = 0$, whence, by Lemma 3, $\chi^{\{a\}} \notin \mathrm{span}\,(Z\mathbb{F})$. Since $\chi^{\{a\}} \in Z\mathbb{F}'$, we conclude that $\mathrm{rank}\,(Z\mathbb{F}') \geqslant \mathrm{rank}\,(Z\mathbb{F}) + 1$.
Now, assume that $\alpha_S(a) < c(a)$. In this case, $\alpha_S(a) = c(W) - p(W)$ for some $W \subseteq V$ such that $W \cap S \neq \emptyset$ and $a \in \rho W$. Since $\mathbb{F}$ is feasible, then Corollary 2 implies that there exists an S-cover $B \subseteq c^+$ in $\mathbb{F}$ and a real $\varepsilon > 0$ such that $a \in B$, and $\mathbb{F}^{\varepsilon B}$ is a feasible network. Moreover, since $\mathbb{F}'$ is feasible and $\mu'(S) > 0$, then Corollary 2 also implies that there exists an S-cover $J \subseteq (c')^+$ in $\mathbb{F}'$ and a real $\varepsilon > 0$ such that $(\mathbb{F}')^{\varepsilon J}$ is a feasible network. Note that no arc in J enters W, because $W \in \Gamma\mathbb{F}'$ and $W \cap S \neq \emptyset$. Then

$$\text{for each } U \in \Gamma\mathbb{F} : \chi^B(\rho U) = \chi^{e_S}(U) = \chi^J(\rho U), \text{ and}$$
$$\text{for each } e \in E \text{ with } c(e) = 0 : \chi^B(e) = 0 = \chi^J(e).$$

Furthermore, $\chi^B(\rho W) \geqslant 1$, since $a \in \rho W$, and $\chi^J(\rho W) = 0$ which, by Lemma 3, implies that $\chi^{\rho W} \notin \mathrm{span}\,(Z\mathbb{F})$. But $\chi^{\rho W} \in Z\mathbb{F}'$, and thus $\mathrm{rank}\,(Z\mathbb{F}') \geqslant \mathrm{rank}\,(Z\mathbb{F}) + 1$. This completes the proof of the claim. •

Now we establish that $q(\mathbb{F}', \mathcal{L}) + 1 \leqslant q(\mathbb{F}, \mathcal{L})$. By the claim, $\mathrm{rank}\,(Z\mathbb{F}') \geqslant \mathrm{rank}\,(Z\mathbb{F}) + 1$. However, $(\mu')^+ = \mu^+ \cup \{S'\}$, and $|V| - 1 \geqslant |\bar{S'}| + 1$. Thus

$$\begin{aligned} q(\mathbb{F}', \mathcal{L}) + 1 &= (|A| - \mathrm{rank}\,(Z\mathbb{F}'))(|V| - 1) + \sum_{T \in (\mu^{ka})^+} |\bar{T}| + \gamma(\mathcal{P}) - |\mathcal{L}'| + 1 \\ &\leqslant (|A| - \mathrm{rank}\,(Z\mathbb{F}))(|V| - 1) - (|V| - 1) + |\bar{S'}| + 1 + \sum_{T \in \mu^+} |\bar{T}| + \gamma(\mathcal{P}) - |\mathcal{L}| \\ &\leqslant (|A| - \mathrm{rank}\,(Z\mathbb{F}))(|V| - 1) + \sum_{T \in \mu^+} |\bar{T}| + \gamma(\mathcal{P}) - |\mathcal{L}| = q(\mathbb{F}, \mathcal{L}), \end{aligned}$$

as we wanted.
Since $\mathcal{L} \subseteq \{X \in \mathcal{P} \mid c'(X) = p'(X)\}$ and $q(\mathbb{F}', \mathcal{L}) + 1 \leqslant q(\mathbb{F}, \mathcal{L})$, then by induction hypothesis, the call in line 9 returns a packing y' of $\mathbb{F}'$ such that $|(y')^+| \leqslant \varphi(\mathbb{F}')$, and makes no more than $q(\mathbb{F}', \mathcal{L})$ oracle calls. But the number of oracle calls is at most $q(\mathbb{F}', \mathcal{L}) + 1 \leqslant q(\mathbb{F}, \mathcal{L})$, as required.
Clearly, $|(\mu')^+| = |\mu^+| + 1 - [S' \in \mu^+]$, which together with $\mathrm{rank}\,(Z\mathbb{F}') \geqslant \mathrm{rank}\,(Z\mathbb{F}) + 1$, implies that $\varphi(\mathbb{F}') \leqslant \varphi(\mathbb{F}) - [S' \in \mu^+]$. By Claim 1, $\mathbb{F}$ has a packing y such that $|y^+| \leqslant |(y')^+| + [S' \in \mu^+]$, and thus $|y^+| \leqslant \varphi(\mathbb{F})$.

This finishes the proof of the theorem. □

4. Pre-processing

If we recall the bounds on the size of the packing and on the number of oracle calls in the last section, we see that they are better when $\mathrm{rank}\,(Z\mathbb{F})$ is large. This motivates the following idea: given a feasible network $\mathbb{F}$, find a new feasible network $\mathbb{G}$ in such a way that a packing of $\mathbb{G}$ is also a packing of $\mathbb{F}$, and $\mathrm{rank}\,(Z\mathbb{G})$ is "large". This is the pre-processing step (first phase) of our algorithm.

It is possible to find such a network $\mathbb{G}$ with $\mathrm{rank}\,(Z\mathbb{G})$ "large", as in [11], by considering a critical feasible network. A feasible network $\mathbb{F}$ is *critical* if each arc enters a tight set. Recall that $H\mathbb{F} := \{u \in V \mid p(u) > 0\}$. It is not difficult to show (see [11]) that if $\mathbb{F}$ is critical, then $\mathrm{rank}\,(Z\mathbb{F}) \geqslant |H\mathbb{F}|$. Note, however, that such a straightforward approach would require $|A|$ oracle calls to turn $\mathbb{F}$ into a critical feasible network. In this section, by using the pre-processing algorithm, we show how to turn a feasible network $\mathbb{F}$ into a feasible network $\mathbb{G}$ for which $\mathrm{rank}\,(Z\mathbb{G}) \geq |H\mathbb{G}|$, by making no more than $|H\mathbb{F}|$ oracle calls. The pre-processing algorithm will not change the demand function, and so $H\mathbb{F} = H\mathbb{G}$. As a byproduct of this pre-processing, we will also obtain a laminar family of tight sets that can be used as an initial laminar family in our packing algorithm. Clearly, this is asymptotically better than reducing to a critical feasible network, since $|H\mathbb{F}| \leqslant |V|$. While, for the overall procedure of constructing a packing, this might not be asymptotically better than reducing to a critical feasible network – since number of oracle calls is dominated by $|V||A|$ – we believe that the procedure is interesting in its own.

The pre-processing algorithm uses the uncrossing procedure Uncross $(\mathbb{F}, \mathcal{L}, X, a)$ that receives a feasible network $\mathbb{F}$, a laminar subfamily $\mathcal{L}$ of $\Gamma_{\!+}\mathbb{F}$, a subset X of V and an arc $a \in \rho X$ and returns a set $T \in \Gamma_{\!+}\mathbb{F}$ such that $a \in \rho T$ and $\mathcal{L} \cup \{T\}$ is laminar. Recall that for $U \subseteq V$, $\mathcal{L}_U$ is the set of all maximal elements L of $\mathcal{L}$ such that L and U intersect.

Algorithm Uncross $(\mathbb{F}, \mathcal{L}, X, a)$

1 **if** a enters $X \cup \bigcup \mathcal{L}_X$ **then**
2   **return** $X \cup \bigcup \mathcal{L}_X$
3 choose $M \in \mathcal{L}$ minimal that contains both ends of a
4 $Y := X \cap M$
5 **return** $Y \cup \bigcup \mathcal{L}_Y$.

Lemma 5 describes more precisely what this routine does, but first we need the following simple lemma.

Lemma 4. *Let $\mathbb{F}$ be a feasible network. If $X, Y \in \Gamma\mathbb{F}$ and $X \cap Y \neq \emptyset$, then $X \cup Y, X \cap Y \in \Gamma\mathbb{F}$ and $c(X \setminus Y, Y \setminus X) = 0$. Moreover, if $X \in \Gamma\mathbb{F}$ and $\mathcal{P} \subseteq \Gamma\mathbb{F}$ is such that $Y \in \mathcal{P}$ implies $X \cap Y \neq \emptyset$, then $X \cup \bigcup \mathcal{P} \in \Gamma\mathbb{F}$.*

Proof. The proof is a simple application of submodularity of c and the supermodularity of p. Indeed, let $X, Y \in \Gamma\mathbb{F}$ be such that $X \cap Y \neq \emptyset$. Then

$$\begin{aligned} p(X) + p(Y) = c(X) + c(Y) &= c(X \cup Y) + c(X \cap Y) + c(X \setminus Y, Y \setminus X) \\ &\geqslant p(X \cup Y) + p(X \cap Y) \geqslant p(X) + p(Y), \end{aligned}$$

where the first equality holds because $X, Y \in \Gamma\mathbb{F}$; the first inequality, because of submodularity; the second, since $\mathbb{F}$ is feasible; and the third, since p is supermodular. Thus equality holds throughout, whence $X \cup Y, X \cap Y \in \Gamma\mathbb{F}$ and $c(X \setminus Y, Y \setminus X) = 0$. For the second part, just notice that it follows from the first by induction on $|\mathcal{P}|$. $\square$

Lemma 5. *Let $\mathbb{F} = (D, c, \mu)$ be a feasible network, $\mathcal{L}$ a laminar subfamily of $\Gamma_{+}\mathbb{F}$, $X \in \Gamma_{+}\mathbb{F}$, and $a \in c^{+} \cap \rho X$ be such that a avoids $\mathcal{L}$. Then* Uncross $(\mathbb{F}, \mathcal{L}, X, a)$ *returns a set $T \in \Gamma_{+}\mathbb{F}$ such that $a \in \rho T$, and $\mathcal{L} \cup \{T\}$ is laminar.*

Proof. By Lemma 4, $c(X \cup \bigcup \mathcal{L}_X) = p(X \cup \bigcup \mathcal{L}_X)$. First suppose that a enters $X \cup \bigcup \mathcal{L}_X$. Then $c(X \cup \bigcup \mathcal{L}_X) > 0$. Moreover, $\mathcal{L} \cup \{X \cup \bigcup \mathcal{L}_X\}$ is laminar, and we are done in this case. Suppose then that a avoids $X \cup \bigcup \mathcal{L}_X$. Since a enters X, then there exists $N \in \mathcal{L}_X$ such that the tail of a is in N. By Lemma 4, we have that $c(X \setminus N, N \setminus X) = 0$. Now $c(a) > 0$, and then the head of a is also in N. Therefore we can choose a set M as in line 3 of Uncross. Now $Y = X \cap M \in \Gamma_{+}\mathbb{F}$ and a enters Y. By the minimality of M and the laminarity of $\mathcal{L}$, we have that a enters $T = Y \cup \bigcup \mathcal{L}_Y$. By Lemma 4, $c(T) = p(T)$. But a enters T, and so $c(T) > 0$. Therefore $T \in \Gamma_{+}\mathbb{F}$. Furthermore $\mathcal{L} \cup \{T\}$ is laminar, and this completes the proof of the lemma. $\square$

We present now the algorithm Pre-Process $((D, c, \mu), P)$. The algorithm receives a feasible network $\mathbb{F} = (D, c, \mu)$ and a subset P of $H\mathbb{F}$, and returns a pair $(\mathcal{L}, g)$, where $\mathcal{L} \subseteq 2^V$ and $g : A \to \mathbb{R}_{+}$, such that

(i) $0 \leqslant g \leqslant c$, and $\mathbb{G} = (D, g, \mu)$ is a feasible network,
(ii) $\mathcal{L} \subseteq \Gamma_{+}\mathbb{G}$ is laminar and $\{\chi^{\rho L} \mid L \in \mathcal{L}\}$ is linearly independent, and
(iii) $\mathrm{rank}\,(Z\mathbb{G}) \geqslant |H\mathbb{G}|$.

and furthermore makes no more than $|H\mathbb{F}|$ oracle calls. Observe that during this procedure the demand function is not changed, and thus $H\mathbb{F} = H\mathbb{G}$. In the main algorithm we use Pre-Process with $P = H\mathbb{F}$, but for technical reasons, in the proof we assume that P is an arbitrary subset of $H\mathbb{F}$.

Algorithm Pre-Process $(\mathbb{F}, P)$

if $P = \emptyset$ **then**
    return $(\emptyset, c)$
choose $u \in P$
$(\mathcal{M}, g) :=$ Pre-Process $(\mathbb{F}, P \setminus \{u\})$
choose L minimal in $\mathcal{M} \cup \{V\}$ such that $u \in L$
if there exists $a \in g^{+} \cap \rho(u)$ with both ends in L **then**
    let $X \in \mathrm{argmin}\{g(W) - p(W) \mid W \subseteq V, a \in \rho W\}$.
    $k := g(X) - p(X)$
    if $g^{ka}(a) > 0$ **then**
        return $(\mathcal{M} \cup \{$Uncross $((D, g^{ka}, \mu), \mathcal{M}, X, a)\}, g^{ka})$
    else
        return $(\mathcal{M}, g^{ka})$
else
    return $(\mathcal{M} \cup \{\{u\}\}, g)$

Proposition 1. *If $\mathbb{F} = (D, c, \mu)$ is a feasible network and $P \subseteq H\mathbb{F}$, then* Pre-process $(\mathbb{F}, P)$ *returns a pair $(\mathcal{L}, g)$, where $\mathcal{L} \subseteq 2^V$ and $g : A \to \mathbb{R}_{+}$, such that*

(i) $0 \leqslant g \leqslant c$, and $\mathbb{G} = (D, g, \mu)$ is a feasible network,
(ii) $\mathcal{L} \subseteq \Gamma_{+}\mathbb{G}$ is laminar and $\{\chi^{\rho L} \mid L \in \mathcal{L}\}$ is linearly independent, and
(iii) $\mathrm{rank}\,(\{\chi^{\rho L} \mid L \in \mathcal{L}\} \cup \{\chi^{\{e\}} \mid e \in A; g(e) = 0\}) \geqslant |H\mathbb{F}|$.

by making no more than $|H\mathbb{F}|$ oracle calls. Moreover, if c is integral, then g is also integral.

Proof. We prove, by induction on $|P|$, that if $\mathbb{F}$ is a feasible network, and $P \subseteq H\mathbb{F}$, then PRE-PROCESS $(\mathbb{F}, P)$ returns a pair $(\mathcal{M}, g)$, where $\mathcal{M} \subseteq 2^V$ and $g : A \to \mathbb{R}_+$, such that

(i) $0 \leqslant g \leqslant c$, *and* $\mathbb{G} := (D, g, \mu)$ *is a feasible network,*

(ii) $\mathcal{M} \subseteq \Gamma_+\mathbb{G}$ *is laminar and* $\{v\} \in \mathcal{M} \Rightarrow v \in P$,

(iii) *for each nonempty* $\mathcal{N} \subseteq \mathcal{M}$ *there exist a* $\in g^+ \cap \bigcup\{\rho(v) \mid v \in P\}$ *and* $N \in \mathcal{N}$ *such that a enters N and avoids* $\mathcal{N} \setminus \{N\}$, *and* (7)

(iv) $\operatorname{rank}(\{\chi^{\rho L} \mid L \in \mathcal{M}\} \cup \{\chi^{\{e\}} \mid e \in A, g(e) = 0\}) \geqslant |P|$.

This, combined with the following claim which proves that truth of (7iii) implies that a certain set of vectors is linearly independent, implies the Proposition.

Claim 3. *If* $\mathcal{M} \subseteq 2^V, g : A \to \mathbb{R}_+$ *and* $P \subseteq V$ *satisfy condition (7iii), then the set*

$$\{\chi^{\rho L} \mid L \in \mathcal{M}\} \cup \{\chi^{\{e\}} \mid e \in A, g(e) = 0\}$$

is linearly independent.

Proof. Indeed, assume that $\mathcal{M} \subseteq 2^V, g : A \to \mathbb{R}_+$ and $P \subseteq V$ satisfy (7iii). Let $Z := \{e \in A \mid g(e) = 0\}$. Consider any nontrivial linear combination

$$\mathbf{v} := \sum[\lambda_L \chi^{\rho L} \mid L \in \mathcal{M}] + \sum[\lambda_e \chi^{\{e\}} \mid e \in Z].$$

Let $\mathcal{N} := \{L \in \mathcal{M} \mid \lambda_L \neq 0\}$. If $\mathcal{N} = \emptyset$, then $\lambda_e \neq 0$ for some $e \in Z$, and so $\mathbf{v} \neq 0$. Assume that $\mathcal{N} \neq \emptyset$. By (7iii), there exist $a \in g^+ \cap \bigcup\{\rho(v) \mid v \in P\}$ and $N \in \mathcal{N}$ such that a enters N and avoids $\mathcal{N} \setminus \{N\}$. This, is turn, combined with $a \notin Z$, implies that $\mathbf{v}(a) = \lambda_N \neq 0$, and therefore $\mathbf{v} \neq 0$. This establishes the linear independence of the set $\{\chi^{\rho L} \mid L \in \mathcal{M}\} \cup \{\chi^{\{e\}} \in A \mid g(e) = 0\}$. ∙

Now we return to the proof of the correctness of (7). Observe that (7i) is trivial, so we turn to the proof of (7ii) to (7iv). Let $\mathbb{F} = (D, c, \mu)$ be a feasible network, and $P \subseteq H\mathbb{F}$. If $P = \emptyset$, then it is clear that (7) holds for the pair $(\emptyset, c)$. Assume that $P \neq \emptyset$. By induction hypothesis, the call PRE-PROCESS $(\mathbb{F}, P \setminus \{u\})$ returns a pair $(\mathcal{M}, g)$ for which (7) holds. Let $\mathbb{G} := (D, g, \mu)$. We divide the proof in two cases:

Case1. There exists $a \in g^+ \cap \rho(u)$ with both ends in L (line 6).
In this case, by the choice of L, the arc a avoids $\mathcal{M}$.
Suppose first that $g^{ka}(a) > 0$ (line 9). Let $\mathbb{G}' := (D, g^{ka}, \mu)$ and notice that $\mathcal{M} \cup \{X\} \subseteq \Gamma_+\mathbb{G}'$. Thus, by Lemma 5, UNCROSS $((D, g^{ka}, \mu), \mathcal{M}, X, a)$ returns a set T such that $T \in \Gamma_+\mathbb{G}'$, a enters T and $\mathcal{L} := \mathcal{M} \cup \{T\}$ is laminar. Moreover, $T \notin \mathcal{M}$, because a enters T and avoids $\mathcal{M}$. So $\mathcal{L} \subseteq \Gamma_+\mathbb{G}'$, and the first part of (7ii) is valid. Since $u \in T$, then the second part of (7ii) also holds. Thus $(\mathcal{L}, g^{ka})$ satisfies (7ii).
We prove now that (7iii) holds. Let $\emptyset \neq \mathcal{N} \subseteq \mathcal{L}$. If $\mathcal{N} \subseteq \mathcal{M}$, then we are done, since $g^+ \cap \bigcup\{\rho(v) \mid v \in P \setminus \{u\}\} \subseteq (g^{ka})^+ \cap \bigcup\{\rho(v) \mid v \in P\}$. Assume that $T \in \mathcal{N}$. By the choice of a and T, we have that $a \in (g^{ka})^+ \cap \rho(T) \cap \rho(u)$, and since a avoids $\mathcal{M}$, then a avoids $\mathcal{N} \setminus \{T\}$. Therefore $(\mathcal{L}, g^{ka})$ satisfies (7iii).
For (7iv), observe that, by Claim 3, the sets $\{\chi^{\rho L} \mid L \in \mathcal{M}\} \cup \{\chi^{\{e\}} \mid e \in A, g(e) = 0\}$ and $\{\chi^{\rho L} \mid L \in \mathcal{L}\} \cup \{\chi^{\{e\}} \mid e \in A; g^{ka}(e) = 0\}$ are linearly independent. Now $|\mathcal{L}| = |\mathcal{M}| + 1, g^+ = (g^{ka})^+$, and the induction hypothesis imply that

$$\begin{aligned}\operatorname{rank}(\{\chi^{\rho L} \mid L \in \mathcal{L}\} \cup \{\chi^{\{e\}} \mid e \in A, g^{ka}(e) = 0\}) &= \operatorname{rank}(\{\chi^{\rho L} \mid L \in \mathcal{M}\} \cup \{\chi^{\{e\}} \mid e \in A, g(e) = 0\}) + 1 \\ &\geqslant |P \setminus \{u\}| + 1 = |P|.\end{aligned}$$

Thus $(\mathcal{L}, g^{ka})$ satisfies (7).
Suppose now that $g^{ka}(a) = 0$. We show that $(\mathcal{M}, g^{ka})$ satisfy (7). It is clear that (7ii) holds, and since a avoids $P \setminus \{u\}$, then (7iii) also holds. For the proof of (7iv), note that, by Claim 3, the sets $\{\chi^{\rho L} \mid L \in \mathcal{M}\} \cup \{\chi^{\{e\}} \mid e \in A, g(e) = 0\}$ and $\{\chi^{\rho L} \mid L \in \mathcal{M}\} \cup \{\chi^{\{e\}} \mid e \in A, g^{ka}(e) = 0\}$ are linearly independent. Now $(g^{ka})^+ = g^+ \setminus \{a\}$ and induction hypothesis imply that

$$\begin{aligned}\operatorname{rank}(\{\chi^{\rho L} \mid L \in \mathcal{L}\} \cup \{\chi^{\{e\}} \mid e \in A, g^{ka}(e) = 0\}) &= \operatorname{rank}(\{\chi^{\rho L} \mid L \in \mathcal{M}\} \cup \{\chi^{\{e\}} \mid e \in A, g(e) = 0\}) + 1 \\ &\geqslant |P \setminus \{u\}| + 1 = |P|.\end{aligned}$$

Thus $(\mathcal{M}, g^{ka})$ satisfies (7).

Case2. There does not exist $a \in g^+ \cap \rho(u)$ with both ends in L (line 13).
Observe that $\{u\}$ is tight in $\mathbb{G}$. Indeed,

$$p(u) \geqslant p(L) = g(L) \geqslant g(u) \geqslant p(u),$$

where the first inequality follows from (3); the first equality, because $L \in \Gamma\mathbb{G}$; the second inequality because $g^+ \cap \rho(u) \subseteq g^+ \cap \rho(L)$, and the third inequality, because $\mathbb{G}$ is a feasible network. Let $\mathcal{L} := \mathcal{M} \cup \{\{u\}\}$. We show that

$(\mathcal{L}, g)$ satisfy (7). Clearly, $\mathcal{L}$ is laminar and $\mathcal{L} \subseteq \Gamma_{+}\mathbb{G}$, and so $(\mathcal{L}, g)$ satisfy the first part of (7ii). Since $\{u\} \in \mathcal{L}$, then the second part of (7ii) is also satisfied.

Let us establish that (7iii) holds. Let $\emptyset \neq \mathcal{N} \subseteq \mathcal{L}$. If $\mathcal{N} \subseteq \mathcal{M}$, then we are done. Suppose that $\{u\} \in \mathcal{N}$. There is nothing to prove if $\mathcal{N} \setminus \{\{u\}\} = \emptyset$. Thus assume $\mathcal{N} \setminus \{\{u\}\} \neq \emptyset$. Then $\mathcal{N} \setminus \{\{u\}\} \subseteq \mathcal{M}$, and so there exists $b \in g^{+} \cap \bigcup\{\rho(v) \mid v \in P \setminus \{u\}\}$ and $N \in \mathcal{N} \setminus \{\{u\}\}$ such that b enters N and avoids $\mathcal{N} \setminus \{N, \{u\}\}$. But $b \notin \rho(u)$, and this proves (7iii).

For the proof of (7iv), first observe that since $(\mathcal{M}, g)$ satisfies (7ii), then $\{u\} \notin \mathcal{M}$, whence $|\mathcal{L}| = |\mathcal{M}| + 1$. We have, by Claim 3, that the sets $\{\chi^{\rho L} \mid L \in \mathcal{M}\} \cup \{\chi^{\{e\}} \mid e \in A, g(e) = 0\}$ and $\{\chi^{\rho L} \mid L \in \mathcal{L}\} \cup \{\chi^{\{e\}} \mid e \in A, g(e) = 0\}$ are linearly independent. Now $|\mathcal{L}| = |\mathcal{M}| + 1$ and induction hypothesis imply that

$$\begin{aligned}\operatorname{rank}(\{\chi^{\rho L} \mid L \in \mathcal{L}\} \cup \{\chi^{\{e\}} \mid e \in A, g(e) = 0\}) &= \operatorname{rank}(\{\chi^{\rho L} \mid L \in \mathcal{M}\} \cup \{\chi^{\{e\}} \mid e \in A, g(e) = 0\}) + 1 \\ &\geqslant |P \setminus \{u\}| + 1 = |P|.\end{aligned}$$

Thus $(\mathcal{L}, g)$ satisfies (7).

Finally, it is clear that if c is integral, then g is also integral, and this completes the proof of the Proposition. □

Finally, to obtain the algorithm with the desired bounds, we just compose the previous algorithms.

Algorithm Pack $(\mathbb{F} = (D, c, \mu))$

$(\mathcal{L}, g) :=$ Pre-Process $(\mathbb{F}, H\mathbb{F})$

return RPack$((D, g, \mu), \mathcal{L})$.

Theorem 5. *For each feasible network $\mathbb{F} = (D, c, \mu)$, algorithm* Pack (D, c, μ) *returns a packing y such that $|y^{+}| \leqslant |A| - |H\mathbb{F}| + |\mu^{+}|$ and makes no more than $(|A| - |H\mathbb{F}| + |\mu^{+}| + 3)|V|$ oracle calls. Moreover, if c is integral, then y is integral.*

Proof. Let $\mathbb{F} = (D, c, \mu)$ be a feasible network. The call to Pre-Process $(\mathbb{F}, H\mathbb{F})$, by Proposition 1, returns a pair $(\mathcal{L}, g)$ – by making no more than $|H\mathbb{F}|$ oracle calls – such that $\operatorname{rank}(\{\chi^{\rho L} \mid L \in \mathcal{L}\} \cup \{\chi^{\{e\}} \mid e \in A; g(e) = 0\}) \geqslant |H\mathbb{F}|$. Let $\mathbb{G} = (D, g, \mu)$. Then $\mathcal{L} \subseteq \Gamma_{+}\mathbb{G}$ and $\operatorname{rank}(Z\mathbb{G}) \geqslant |H\mathbb{F}| = |H\mathbb{G}|$. Moreover, $\gamma(p^{+}) \leqslant 2|H\mathbb{G}| - 2$. Hence, by Theorem 2, the call to RPack$(\mathbb{G}, \mathcal{L})$ returns a packing y of $\mathbb{G}$ such that $|y^{+}| \leqslant |A| - |H\mathbb{F}| + |\mu^{+}|$, by making no more than $(|A| - |H\mathbb{F}| + |\mu^{+}|)(|V| - 1) + 2|H\mathbb{F}| - 2$ oracle calls. Therefore the total number of oracle calls of the algorithm Pack is at most $(|A| - |H\mathbb{F}| + |\mu^{+}| + 3)|V|$. Since y is also a packing of $\mathbb{F}$, this implies the theorem. □

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