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Characterization of multiscroll attractors using Lyapunov exponents and Lagrangian coherent structures

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The present work aims to apply a recently proposed method for estimating Lyapunov exponents to characterize—with the aid of the metric entropy and the fractal dimension—the degree of information and the topological structure associated with multiscroll attractors. In particular, the employed methodology offers the possibility of obtaining the whole Lyapunov spectrum directly from the state equations without employing any linearization procedure or time series-based analysis. As a main result, the predictability and the complexity associated with the phase trajectory were quantified as the number of scrolls are progressively increased for a particular piecewise linear model. In general, it is shown here that the trajectory tends to increase its complexity and unpredictability following an exponential behaviour with the addition of scrolls towards to an upper bound limit, except for some degenerated situations where a non-uniform grid of scrolls is attained. Moreover, the approach employed here also provides an easy way for estimating the finite time Lyapunov exponents of the dynamics and, consequently, the Lagrangian coherent structures for the vector field. These structures are particularly important to understand the stretching/folding behaviour underlying the chaotic multiscroll structure and can provide a better insight of phase space partition and exploration as new scrolls are progressively added to the attractor. [http://dx.doi.org/10.1063/1.4802428]

Multiscroll attractors for low-dimensional dynamical systems exhibit an interesting trade-off between regularity and randomness in phase space exploration. This particular topological solution follows from state equations that leads to a challenging scenario from the standpoint of estimation of invariant measures, which, up to now, has been solved in the context of a laborious piecewise analysis of phase space or in terms of the time series analysis of the obtained solutions. In the present work, a practical method for informational and topological characterization of such attractors based on the estimation of the Lyapunov spectrum is extensively employed, providing a better insight of the phase space exploration. In addition to that, the employed methodology also offers the possibility of computing the Lagrangian Coherent Structures (LCS) for the vector field and, consequently, to reveal how the mixing process underlying the chaotic multiscroll behaviour is organized around crucial geometrical objects (represented by fixed points of index-1).

I. INTRODUCTION

Classically, dynamical systems with chaotic behaviour can be described in terms of a deterministic state mapping that originates aperiodic solutions with strong dependence on the initial conditions.[36,38] Solutions of this kind establish an interesting trade-off between apparent randomness and regularity in the exploration of their possible states, giving rise to potentially useful information-processing features.[35,40,46]

This oscillatory versatility has been systematically investigated in the context of complex systems, i.e., in terms of mappings composed of several elementary dynamics connected by means of some neighbourhood relationship.[36] In fact, such complex dynamical systems exhibit several degrees of freedom, which implies an interesting oscillatory flexibility at the expense of the need for employing a mapping difficult to handle in practical terms (e.g., for control or even for general engineering purposes[2,26,27,52]).

In contrast with this paradigm, dynamical systems capable of generating multiscroll attractors can offer the advantage of requiring a relatively small number of state variables to exhibit an oscillatory versatility related to a chaotic attractor that visits the phase space with a specific "organized" scroll structure. This particular propriety of multiscroll attractors has already been explored within the engineering framework as, for instance, for mobile robot navigation purposes.[7]

In general, multiscroll attractors can be obtained from non-smooth piecewise linear dynamics (e.g., variants of the classical Chua's system[11,49]), where the number of scrolls is associated with the number of different linear domains

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exhibited by the dynamics. Furthermore, the insertion of new scrolls is given by the addition of new state equation solutions, which implies in new linear intervals and, consequently, in the division of the phase space into new partitions that are visited through the scrolls.[6,31,32,51]

Although such systems are composed of a low number of state variables, they still define a challenging scenario insofar as the estimation of the invariant measures is concerned. This is an issue of major relevance for characterizing their solutions and, consequently, the way that the phase space is "explored" by the various trajectories. To accomplish this task, the Lyapunov spectrum—the set of all Lyapunov exponents classically defined as the mean rate of divergence (or convergence) of initially close trajectories—is commonly used to infer the amount of information generated by the dynamics (or lost by an external observer) and the topological structure of the attractor, which is done by means of the fractal dimension obtained via the Kaplan-Yorke conjecture.[1,36,50]

Within this context, the classical calculation of the Lyapunov spectrum in non-smooth piecewise linear models would require a laborious analysis in order to obtain the suitable set of variational equations[38,50] for each linear interval defined by the dynamical system.[37] In fact, this could be virtually impossible in practical terms when the number of intervals increases too much. Another possible approach would be to rely on estimating the Lyapunov spectrum based on time series analysis, which would also introduce some unnecessary uncertainties (as that related to the estimation of embedding dimensions or the lag between the time series samples in order to reconstruct the attractor[1,3,4,42,50]) since the state equations are fully known.

To deal with problems of this nature, it was previously developed a method called Cloned Dynamics (ClDyn) approach[48] for estimating the Lyapunov spectrum based on the divergence (or the convergence) of small disturbed copies of the original trajectory. This approach does not require the construction of the tangent space associated with the variational equations and, as a consequence, it is attractive for being employed in non-smooth dynamical systems or even for models with a hard mathematical description.

Therefore, the present work aims to exploit the ClDyn features for characterizing the topological and the informational aspects for models capable of engendering multiscroll attractors. It is shown here, by means of the estimation of quantitative measures (such as the metric entropy and the fractal dimension) that, in general, the insertion of scrolls tends to lead to more "complex" attractors (i.e., attractors that are more unpredictable and with a higher fractal dimension), except for some degenerated situations, where the phase space is not filled with scrolls and exhibits a non-uniform visiting frequency. The obtained results also show that, for a grid of scrolls, the amount of generated information and the fractal dimension do not linearly increase with the addition of scrolls. This increasing behaviour actually seems to obey an exponential behaviour and points towards an upper bound for the trajectory predictability, something that is explained by a simple fit proposed here.

In addition to the topological and the informational characterization of multiscroll attractors for particular initial conditions in the phase space, this work also aims to characterize the mixing process underlying state equations capable of engendering such attractors. This is also done here with the aid of the ClDyn approach for evaluating the Finite-Time Lyapunov Exponents (FTLE) and the LCS, which are related to geometric objects in the phase space that maximize a measure of hyperbolicity within a finite time interval, acting as crucial "organizers" of the flux.[18–24,29,43] In the multiscroll context, it was shown that the insertion of fixed points of index-1 leads to separatrices (clearly located with the aid of the LCS), which is illustrated for the first time for this class of attractors in the present work. In fact, the LCS provide a better insight of the mechanism of phase space exploration for multiscroll attractors as time evolves, and it seems valid to expect that this information be relevant for engineering purposes.

This work is structured such as follows: Sec. II presents a brief review of the Cloned Dynamics approach for Lyapunov spectrum calculation. Section III brings a presentation of the multiscroll attractor model studied here, which is exposed in detail by Lü *et al.*[32] Section IV contains the results concerning Lyapunov spectrum, fractal dimension, and Kolmogorov-Sinai (KS) estimation for multiscroll attractors. Section V brings the estimation of Lagrangian Coherent Structures for one- and two-dimensional arrays of such scroll attractors. The article is closed by Sec. VI, which provides discussions and conclusions about the work.

II. THE CLONED DYNAMICS APPROACH: A BRIEF OVERVIEW

Lyapunov spectrum estimation has classically been performed in the context of the tangent map approach (i.e., by evaluating the divergence of close states in terms of the underlying linear system subjacent to the dynamics for each time step), which is best represented by the methodologies developed by Benettin *et al.*,[8] Eckmann and Ruelle,[15] Shimada and Nagashima,[44] and Wolf *et al.*[50] On the other hand, perturbation theory—that also has been used for estimating the largest Lyapunov exponent in different scenarios—has not been suitably employed to calculate the whole Lyapunov spectrum, specially in the case of non-smooth and complex dynamical systems.

To fulfil this gap, a specific methodology, called Cloned Dynamics (ClDyn) approach, was proposed by Soriano *et al.*[47] This strategy is based on the idea of analysing the evolution of the difference state vectors defined as the distance between the fiducial trajectory, and the copies (or "clones") of these motion equations initially disturbed by small values in the orthogonal directions. This methodology was developed in order to characterize nonlinear dynamical systems with a very general description (including, for instance, non-smoothness, as mentioned above), avoiding the construction of the tangent map or the use of procedures based on time series analysis. This methodology was successfully tested for a number of classical nonlinear dynamical systems and also for discontinuously excited neuronal models for stability analysis purposes.[47,48]

Formally, given an N-dimensional dynamical system described by $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t)$, with $\mathbf{x}_0$ being the initial condition, the first step to compute the Lyapunov spectrum is to define N "clones" of the dynamics in the form

$$\dot{\mathbf{x}}_{ci} = \mathbf{F}(\mathbf{x}_{ci}, t), \tag{1}$$

being $\mathbf{x}_{ci}$ the ith clone of the state vector, with $i = 1, 2,\ldots, N$. In this case, each clone is used to estimate the mean divergence (or convergence) rate of a small disturbance applied in a specific direction of the vector field, which is completely spanned by an orthogonal basis when all the clones are considered. To accomplish this purpose, each copy receives the original initial condition of the fiducial trajectory with a small perturbation such as described by,

$$\mathbf{x}_{0ci}^{(0)} = \mathbf{x}_0 + \boldsymbol{\delta}_{ix}^{(0)}. \tag{2}$$

Here, the superscript index denotes the current iteration of the algorithm, and $\{\boldsymbol{\delta}_{1x}^{(0)}, \boldsymbol{\delta}_{2x}^{(0)}, \ldots, \boldsymbol{\delta}_{Nx}^{(0)}\}$ represents an orthonormal basis initially defined by $\delta_{x0}\mathbf{I}_N$, being $\mathbf{I}_N$ the $N \times N$ identity matrix and δ_{x0} a constant that establishes the magnitude of the perturbation.

In the sequence, the whole system (the original motion equations and all clones) is allowed to evolve for T time units, and the difference state vectors can be evaluated using the following relation:

$$\boldsymbol{\delta}_{ix}^{(1)} = \mathbf{x}(T) - \mathbf{x}_{ci}(T). \tag{3}$$

After the set of difference state vectors has been determined, the Gram-Schmidt Reorthonormalization (GSR) procedure[38,48,50] can be applied in order to separate the contribution of the most expansive direction of the dynamics from the remaining ones, which also avoids the collapse of the clones into a single (i.e., the most expansive) direction. This procedure leads to a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N$ that correspond to corrected versions of the difference state vectors. Then, the GSR procedure establishes the corrected vectors for all directions as shown in the following equation:

$$\begin{aligned}
\mathbf{v}_1^{(1)} &= \boldsymbol{\delta}_{1x}^{(1)},\\
\mathbf{u}_1^{(1)} &= \frac{\mathbf{v}_1^{(1)}}{\|\mathbf{v}_1^{(1)}\|},\\
\mathbf{v}_2^{(1)} &= \boldsymbol{\delta}_{2x}^{(1)} - \langle \boldsymbol{\delta}_{2x}^{(1)}, \mathbf{u}_1^{(1)} \rangle \mathbf{u}_1^{(1)},\\
\mathbf{u}_2^{(1)} &= \frac{\mathbf{v}_2^{(1)}}{\|\mathbf{v}_2^{(1)}\|},\\
&\vdots\\
\mathbf{v}_N^{(1)} &= \boldsymbol{\delta}_{Nx}^{(1)} - \langle \boldsymbol{\delta}_{Nx}^{(1)}, \mathbf{u}_1^{(1)} \rangle \mathbf{u}_1^{(1)} - \ldots\\
&\quad \ldots - \langle \boldsymbol{\delta}_{Nx}^{(1)}, \mathbf{u}_{N-1}^{(1)} \rangle \mathbf{u}_{N-1}^{(1)},\\
\mathbf{u}_N^{(1)} &= \frac{\mathbf{v}_N^{(1)}}{\|\mathbf{v}_N^{(1)}\|},
\end{aligned} \tag{4}$$

where $\langle \mathbf{a}, \mathbf{b} \rangle$ denotes the inner product between vectors $\mathbf{a}$ and $\mathbf{b}$. Before starting another algorithm iteration, the clones receive an infinitesimal disturbed initial condition related to the fiducial system along the orthogonal directions, i.e.,

$$\mathbf{x}_{0ci}^{(1)} = \mathbf{x}(T) + \delta_{x0}\mathbf{u}_i^{(1)}. \tag{5}$$

After iterating the algorithm K times, the ith finite time Lyapunov exponent can be computed using the following equation:

$$\lambda_i = \frac{1}{KT} \sum_{k=1}^{K} \ln \left\| \frac{\mathbf{v}_i^{(k)}}{\delta_{x0}} \right\|, \tag{6}$$

and, for K large enough to capture the average behaviour of the whole attractor, the classical definition of the global Lyapunov exponent is obtained.

It is important to remark that the ClDyn approach does not require the construction of the tangent map, being an interesting procedure, for instance, with respect to the characterization of non-smooth models such as exposed in Sec. III. More details about the method can be found in Soriano *et al.*[48]

III. THE MULTISCROLL MODEL

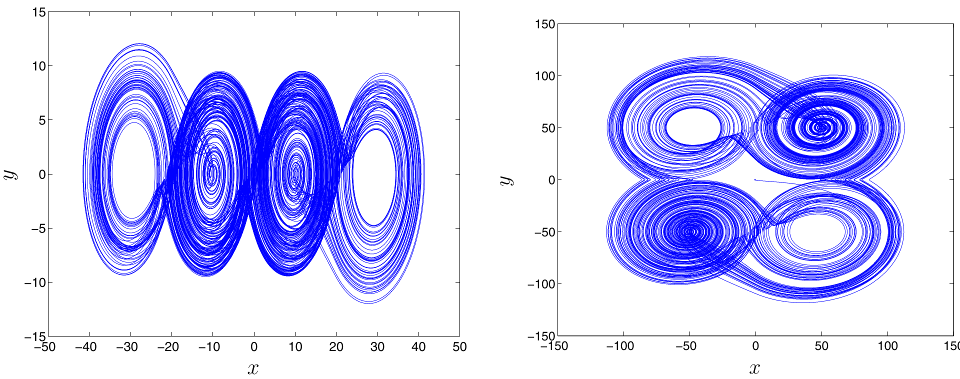
In general, multiscroll attractors can be obtained from dynamical systems based on piecewise linear functions, such as the step[51] and saturated[32] functions, and also can be related to the analysis of switched systems,[31] for instance, in accordance with the one related to the Unstable Dissipative Systems (UDS).[12] Particularly, the dynamical system analysed by Lü *et al.*[32] can exhibit chaotic behaviour with a number of scrolls varying according to the number of piecewise linear intervals and, furthermore, is capable to generate one-, two-, and three-dimensional multiscroll attractors (see Fig. 1 for a clearly picture of one- and two-dimensional arrays of such attractors). In this case, the dynamical system can be described by the following state equations:

$$\begin{aligned}
\dot{x} &= y - \left(\frac{d_2}{b}\right) f(y; k_2; h_2; p_2; q_2)\\
\dot{y} &= z\\
\dot{z} &= -ax - by - cz + d_1 f(x; k_1; h_1; p_1; q_1)\\
&\quad \ldots + d_2 f(y; k_2; h_2; p_2; q_2),
\end{aligned} \tag{7}$$

being the function $f(x; k; h; p; q)$ responsible for controlling the number of piecewise intervals and, consequently, the number of scrolls. This function assumes the form described in the following equation:

$$\begin{aligned}
&f(x; k; h; p; q)\\
&= \begin{cases}
(2q+1)k, & \text{if} \quad x > qh + 1\\
k(x - ih) + 2ik, & \text{if} \quad |x - ih| \le 1\\
& \qquad -p \le i \le q\\
(2i+1)k, & \text{if} \quad ih + 1 < x < (i+1)h - 1\\
& \qquad -p \le i \le q - 1\\
-(2p+1)k, & \text{if} \quad x < -ph - 1.
\end{cases}
\end{aligned} \tag{8}$$

Note that an one-dimensional multiscroll attractor can be understood as a particular case of a two-dimensional grid by setting $d_2 = 0$. Just as a matter of illustration, Fig. 1



(a) (b)

FIG. 1. Phase portrait for the state Eqs. (7). (a) One-dimensional and (b) Two-dimensional multiscroll attractors.

shows the phase portrait obtained when the parameter vector $\mathbf{p} = [p_1\ q_1\ p_2\ q_2]$ is changed in order to increase the number of scrolls. In this case, parameters p_1 and q_1 are related to the number of scrolls that are added, respectively, to the left and to the right sides of the fundamental double scroll attractor (obtained when $\mathbf{p} = \mathbf{0}$ and $d_2 = 0$), producing an one-dimensional array of scrolls (Fig. 1(a), with $a = b = c = d_1 = 0.7$, $d_2 = 0$, $k_1 = 10$, $h_1 = 20$, and $p_1 = q_1 = 1$). Similarly, p_2 and q_2 add pairs of scrolls in the negative and in the positive directions of the y-axis, producing a two-dimensional grid (Fig. 1(b), with $a = b = c = d_1 = d_2 = 0.7$, $k_1 = k_2 = 50$, $h_1 = h_2 = 100$, and $p_1 = p_2 = q_1 = q_2 = 0$), which exhibits a kind of organized partition of the phase space that is explored by a chaotic motion.

From the state Eqs. (7), it can be observed that the addition of new scrolls is accompanied by an increase in number of linear intervals, which can render the task of obtaining of the variational equations for each interval quite laborious for the classical Lyapunov spectrum estimation methodologies. This situation indicates a typical instance where the ClDyn approach is extremely convenient, as it is shown for the extensive characterization performed in Sec. IV.

IV. MULTISCROLL ATTRACTOR ANALYSIS EMPLOYING LYAPUNOV EXPONENTS

A. About the employed invariant measures

Phase space exploration is a major information processing task performed by a chaotic solution. In the multiscroll context, this process of phase exploration can be controlled by the introduction of saddle points of indexes-1 and -2 (see Refs. 13, 34, and 45) in the vector field, which is associated with the addition of plateaus and slopes in the saturated functions series described by (8).

The introduction of index-1 saddle points gives rise to separatrices in the vector field which organize the motion over different regions,[5] which may cause the trajectory more "complex" as the number of scrolls rises. Here, to become "complex" means that the predictability of the dynamical system[9] tends to decrease, since the introduction of separatrices implies in the insertion of "decision regions" in the phase space, i.e., regions of critical divergence rate between disturbed trajectories. In order to study this phenomenon in a quantitative way for one- and two-dimensional multiscroll attractors, the Kolmogorov-Sinai entropy and the fractal dimension of the system (7) were calculated by means of their relationship with the Lyapunov spectrum as estimated using the ClDyn approach.

The KS entropy (or metric entropy) quantifies the average information rate produced by the dynamical system (or lost by an observer).[17,36] The KS entropy is related to the Lyapunov spectrum by the Pesin formula[25,36] which, in this case, leads to

$$\mathrm{KS} = \sum_{i}^{N_p} \lambda_i, \tag{9}$$

where N_p is the number of positive Lyapunov exponents of the spectrum.

Additionally, the relation between the Lyapunov spectrum and the fractal dimension of the underlying attractor is given by the Kaplan-Yorke conjecture, which can be expressed in terms of the Kaplan-Yorke dimension ($\mathrm{D_{KY}}$), defined as[1,36,50]

$$\mathrm{D_{KY}} = j + \frac{\sum_{i=1}^{j} \lambda_i}{|\lambda_{j+1}|}. \tag{10}$$

In this case, j is considered as the highest integer such that

$$\sum_{i=1}^{j} \lambda_i > 0. \tag{11}$$

Thus, both the metric entropy and the fractal dimension can be estimated by means of the Lyapunov spectrum, which can be directly obtained for multiscroll attractors when the ClDyn approach is employed, as done in the following.

B. Characterization of an one-dimensional array of scrolls

As mentioned earlier, an one-dimensional array of scrolls can be achieved by taking $d_2 = 0$ in (7) and changing parameters p_1 and q_1 for adding scrolls, respectively, in the positive and in the negative x-direction of the phase space. For this sort of multiscroll attractor, Table I describes the values of the parameter vector $\mathbf{p}$ used in the saturated

TABLE I. For system (7), the KS entropy is equal to λ_1, since it is the sole positive exponent. The number of scrolls can be computed using the relation $(p_1 + q_1 + 2)$.[32]

$[p_1\ q_1\ p_2\ q_2]$	Scrolls	λ_1	λ_2	λ_3	D_{KY}
[0 0 0 0]	2	0.1012	0.0000	−0.8013	2.1264
[1 1 0 0]	4	0.1386	0.0001	−0.8387	2.1654
[1 2 0 0]	5	0.1485	0.0001	−0.8486	2.1751
[2 2 0 0]	6	0.1513	0.0001	−0.8514	2.1778
[3 3 0 0]	8	0.1577	0.0000	−0.8577	2.1838
[4 4 0 0]	10	0.1572	0.0001	−0.8573	2.1835
[5 5 0 0]	12	0.1622	0.0002	−0.8625	2.1884
[6 6 0 0]	14	0.1651	0.0000	−0.8651	2.1908
[7 7 0 0]	16	0.1598	0.0001	−0.8600	2.1860
[8 8 0 0]	18	0.1686	0.0000	−0.8686	2.1941
[9 9 0 0]	20	0.1640	0.0001	−0.8642	2.1900
[10 10 0 0]	22	0.1627	−0.0001	−0.8626	2.1885
[11 11 0 0]	24	0.1627	0.0000	−0.8627	2.1886
[12 12 0 0]	26	0.1634	0.0005	−0.8639	2.1897

function series (8) to obtain the one-dimensional array of scrolls, the respective number of obtained scrolls, the estimated Lyapunov spectrum, and the Kaplan-Yorke dimension. In all simulations, the same conditions applied to obtain the phase portrait presented in Fig. 1(a) were adopted.

From Table I, it can be noted that the introduction of separatrices and, consequently, of new scrolls increases the largest exponent according to a nonlinear rule. For instance, the addition of 2 new scrolls (e.g., $\mathbf{p} = [1\ 1\ 0\ 0]$) to the fundamental double scroll ($\mathbf{p} = \mathbf{0}$) increases the largest Lyapunov exponent and the fractal dimension by, approximately 37% and 1.8%, respectively, while the addition of 2 new scrolls (i.e., when the 4-scroll attractor becomes a 6-scroll attractor) increases $\lambda_1\,(D_{KY})$ by almost 9.17% (0.57%). This kind of saturation in the value of the largest exponent and in the fractal dimension is confirmed by a progressive increase of the number of scrolls along the x-axis positive and negative directions individually and jointly, as illustrated in Fig. 2.

In fact, it is clear from Fig. 2(a) that the number of scrolls defines the informational and the topological characteristics of the attractor and not its position in the phase space. Thus, it does not matter if scrolls are added in the positive or in the negative (or in both) direction of x-axis: it is the number of scrolls that defines the invariant measures of the attractor. This fact suggests an interesting isomorphism property,[9] where scrolls located at different places in the phase space leads to approximately the same way of "trajectory navigation," which can be of major relevance for information processing purposes.[7]

In addition to that Fig. 2(a) also reveals a kind of exponential increase for λ_1 (and, consequently, the KS entropy) as the number of scrolls increases, which clearly indicates an informational asymptotic limit. For instance, this exponential behaviour for the KS entropy (or for λ_1) can be well described by the following fit:

$$\kappa(n) = \kappa_{\lim} + (\kappa(2) - \kappa_{\lim})\exp[-\tau(n-2)], \qquad (12)$$

being $\kappa_{\lim}$ the informational asymptotic limit for the KS entropy, n is the number of scrolls (with $n \geq 2$), $\kappa(2)$ the Kolmogorov-Sinai entropy for the double scroll attractor, and τ the informational-scroll constant. It can be noted by (12) that, for $n = 2$, the expression provides approximately the value for the KS entropy for the double scroll attractor and, for $n \to \infty$, provides the informational asymptotic limit. Both $\kappa_{\lim}$, $\kappa(2)$ and τ can be estimated by (12), which is shown in Fig. 2(b) for the data given in Table I. In such a case, the values $\kappa_{\lim} = 0.1632$, $\kappa(2) = 0.1017$, and $\tau = 0.4385$ are obtained, providing a Root Mean Squared Error (RMSE) of 0.0025, which is indicative of the validity of the model. Table II also contains the coefficients for (12) when scrolls are added along the x-axis positive (increasing parameter p_1) and negative (increasing parameter q_1) directions.

C. Characterizing a two-dimensional multiscroll attractor

For a two-dimensional array of scrolls, the same exponential behaviour for the invariant measures is obtained under some considerations. In such a model, it can be observed that a pair of scrolls are added to the y-direction of the fundamental double scroll structure by means of the parameters p_2 and q_2 increase.[32] This fact can produce degenerated multiscroll attractors, i.e., solutions that are not covered by scrolls and exhibit a high non-uniform visiting frequency of the phase space, but still remain chaotic. These degenerated cases usually happen when pairs of scrolls are added only to the y-axis, without adding scrolls to the x-axis.

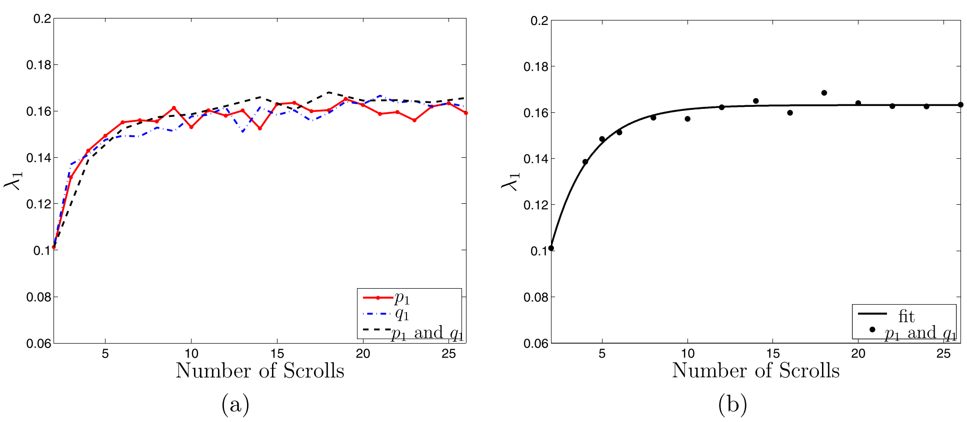


FIG. 2. (a) The positive Lyapunov exponent (λ_1) values for a progressive increase of scrolls. (b) Exponential curve fit for the relation between the largest Lyapunov exponent and the number of scrolls in both direction in x-axis.

TABLE II. Coefficients $\kappa_{\rm lim}$, $\kappa(2)$ and τ described in (12) for the one-dimensional multiscroll attractor case.

	$\kappa_{\rm lim}$	$\kappa(2)$	τ	RMSE
p_1	0.1601	0.1032	0.5632	0.0026
q_1	0.1606	0.1059	0.4529	0.0039
p_1 and q_1	0.1632	0.1017	0.4385	0.0025

On the other hand, when the number of scrolls rises in both axes or even just in x-axis, a well-behaved grid fulfilled by scrolls is generally obtained, and the proposed exponential fit in (12) is valid. In fact, this is a peculiar feature of the model that has to be carefully examined before using two-dimensional grids of scrolls for exploring the phase space. This feature could only be brought to light in the context of the extensive numerical analysis performed here with the aid of the ClDyn approach.

In order to better clarify this point, Fig. 3 shows the exponential behaviour of λ_1 when scrolls are added to the x-axis (by increasing p_1 and q_1) and the typical phase portrait of the obtained grid of scrolls (panel (b)). This situation illustrates an instance of a well-behaved grid of scrolls, for which the proposed fit (12) is valid. On the other hand, the same figure shows that this exponential behaviour of λ_1 does not hold when scrolls are added just in the y-direction (by means of increasing p_2 and q_2), and a classical example of a degenerated two-dimensional multiscroll attractor is obtained (panel (c)).

In addition to that, Table III shows that the same fit proposed for the one-dimensional case is valid for the well-behaved two-dimensional grid. Curiously, the obtained upper bound κ_{lim} for the two-dimensional array of scrolls is smaller than that obtained for the one-dimensional case, which suggests that the intuitive assumption that the complexity increases according to the number of scrolls has also to take in account the grid structure.

After characterizing multiscroll attractors, Sec. V shows the application of the ClDyn approach in the context of a study on the local properties of the vector field in the form of Lagrangian Coherent Structures.

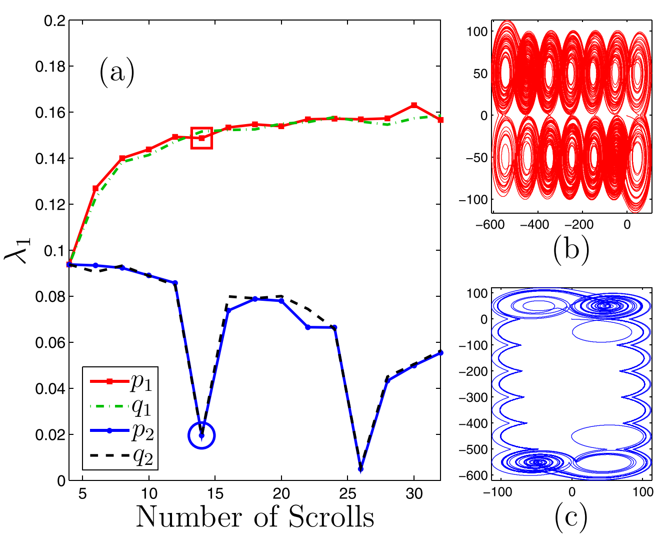


FIG. 3. (a) The positive Lyapunov exponent (λ_1) variation for a progressive increase of scrolls when the two-dimensional multiscroll attractor is considered. (b) and (c) contain the phase portrait projected on the plane (x, y) for system (7).

TABLE III. Coefficients $\kappa_{\rm lim}$, $\kappa(2)$ and τ described in (12) for the two-dimensional multiscroll attractor case.

	$\kappa_{\rm lim}$	$\kappa(2)$	τ	RMSE
p_1	0.1570	0.0484	0.2884	0.0030
q_1	0.1565	0.0532	0.2623	0.0023
p_2	0.0315	0.1068	0.0548	0.0217
q_2	0.0196	0.1029	0.0381	0.0223

V. FINITE-TIME LYAPUNOV EXPONENTS AND LAGRANGIAN COHERENT STRUCTURES IN MULTISCROLL ATTRACTORS

During the last decade, research about the structures that govern fluid flow has evolved and become increasingly important, aiming to formalize the concepts related to the fluid transportation. The development of these concepts led to the proposal of LCS, which have been successfully applied to the study of many natural and engineering systems.[10,30,39]

Pioneering works[18–24,29,43] have defined Lagrangian Coherent Structures over a finite-time interval ζ as the locally strongest repelling or attracting material surface in ζ.[22] These surfaces act as organizing trajectory patterns, since they define barriers or separatrices between different oscillatory behaviours. In their work, Shadden, Lekien, and Marsden[29,43] show that LCS can be identified by means of the maximization of some hyperbolicity measure, like that obtained from the FTLE field. In other words, LCS may be explicitly extracted or visualized by the ridges of the FTLE field.[30,41] By "FTLE field," we specifically mean the Lyapunov exponent calculated over a finite time ($t_{\rm LCS}$) for a grid of initial conditions that tends to cover a continuous part of the phase space. In this context, the possibility of calculating the FTLE applying the ClDyn approach for such a model that presents non-smoothness and several different linear parts opens an interesting perspective of forming a clearer view on the mixing process underlying the chaotic motion that rules the state equations (7). Moreover, this possibility makes feasible the analysing of how the introduction of new equilibrium points allows the addition of scrolls by a fragmentation of the phase space.

When the multiscroll attractor model defined by the system (7) is considered, the LCS are intrinsically associated with the separatrices introduced by index-1 equilibrium points, as shown by Lü *et al.*[32] These points will act as a major local divergence source and will define the global behaviour of multiscroll attractors. This fact can be clearly observed in Fig. 4(a), which shows the FTLE field related to the one-dimensional 4-scroll attractor (exactly as presented in Fig. 1(a)) for a finite-time interval $t_{\rm LCS} = 30$ s (from now on, consider $T = 0.5$ s in Eq. (6)) and initial conditions chosen in the domain $[-60, 60]$ for x_0 and $[-20, 20]$ for y_0 (with step size of 0.05 in both cases), and fixing $z_0 = 0.2$. In this case, the index-1 equilibrium points are located at $(\pm 20, 0, 0)$ and (0, 0, 0), i.e., define exactly the manifolds

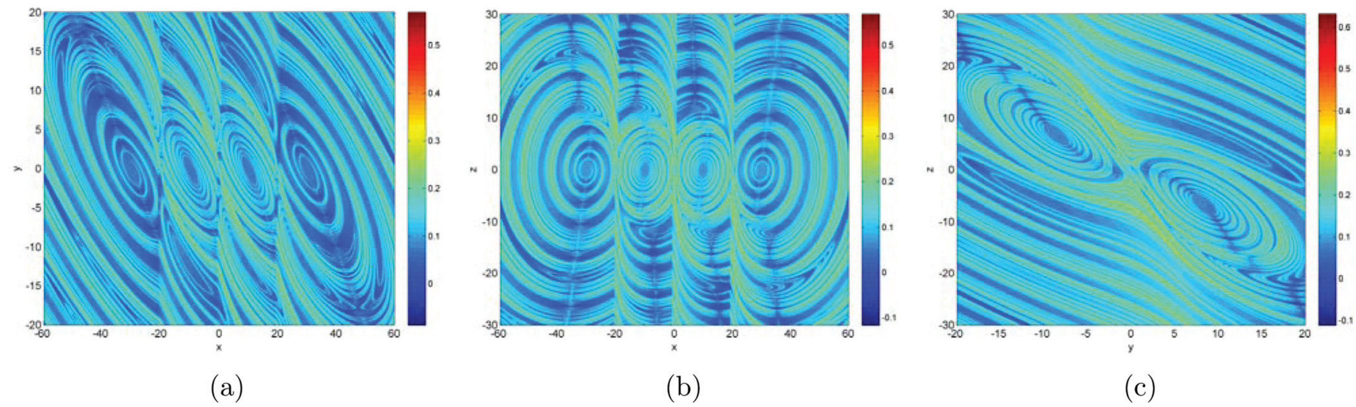


FIG. 4. Projection of the LCS on the planes (a) (x, y), (b) (x, z), and (c) (y, z). From now on, consider that the color bar represents the range of values for the maximum local Lyapunov exponent.

that give rise to the separatrices captured by the LCS. It can be clearly observed that these equilibrium points act as main "organizers" of the vector field,[5] introducing critical divergence structures that under a finite-time condition lead to the separatrices in the form of the LCS, which, in fact, establishes the fragmentation of the phase space. Applying the same procedure described to obtain Fig. 4(a), in Fig. 4(b), the FTLE field was calculated using initial conditions in the domain $[-60, 60]$ for x_0 and $[-30, 30]$ for z_0 (defining a grid with 2400×1200 points) and $y_0 = 0.1$. In Fig. 4(c), fixing $x_0 = 0.1$, it was adopted $y_0 \in [-20, 20]$ and $z_0 \in [-30, 30]$ (composing a grid with 800×1200 points). It is also quite important to remark that the orbits do not intersect the LCS, which could be erroneously concluded from two-dimensional projections shown, for instance, in Figs. 1(a) and 4(a), being the results in perfectly agreement with Ref. 43 that states that the flux across LCS is (approximately) null.

For a two-dimensional array of scrolls (as shown in Fig. 5(a), with $t_{LCS} = 30\,\mathrm{s}$), when the projection of the LCS in the (x, y) plane is considered, it can be noticed that the introduction of index-1 equilibria in both y-axis directions of the phase space also gives rise to separatrices in these directions. In this case, the separatrices are defined by a thin line that passes through the equilibrium points located at $(\pm 50, 0, 0)$ and (0, 0, 0) and over $(0, \pm 50, 0)$. Furthermore, applying the same procedure to obtain Fig. 4, the FTLE field shown in Fig. 5(a) was calculated using the initial conditions defined by $x_0 \in [-150, 150]$, $y_0 \in [-150, 150]$ (with a step size of 0.5 in both cases, defining a grid with 600×600 points) and $z_0 = 0.2$. Fig. 5(b) was obtained with $x_0 \in [-150, 150]$, $z_0 \in [-100, 100]$ (defining a grid with 600×400 points) and fixing $y_0 = 0.1$. In Fig. 5(c), it was adopted $y_0 \in [-150, 150]$, $z_0 \in [-100, 100]$ (composing a grid with 600×400 points) and $x_0 = 0.1$.

It is also important to remark that the LCS depend on the time evolution used to calculate the local divergence. For instance, consider the one-dimensional multiscroll attractor and the respective projection of the LCS in the (x, y) plane, such as illustrated in Fig. 4(a). At first, for small values of t_{LCS}, the LCS are defined by strong divergence regions in the phase space related to the separatrices (Figs. 6(a) and 6(b)). For higher values of t_{LCS}, it becomes possible to discriminate the structures that are related to the general mixing behaviour promoted by the application of the motion equations (Figs. 6(c) and 6(d)). As the value of t_{LCS} increases, the LCS become better outlined (Figs. 6(e) and 6(f)) and, if t_{LCS} is sufficiently large, the FTLE tends to the global Lyapunov exponents since the mean behaviour of the attractor is captured, i.e., the global average divergence behaviour is achieved.

The same analysis can be extended for the two-dimensional multiscroll attractor and the respective projection of the LCS onto the (x, y) plane (illustrated in Fig. 5(a)), as shown in Fig. 7. It can be noticed that the introduction of

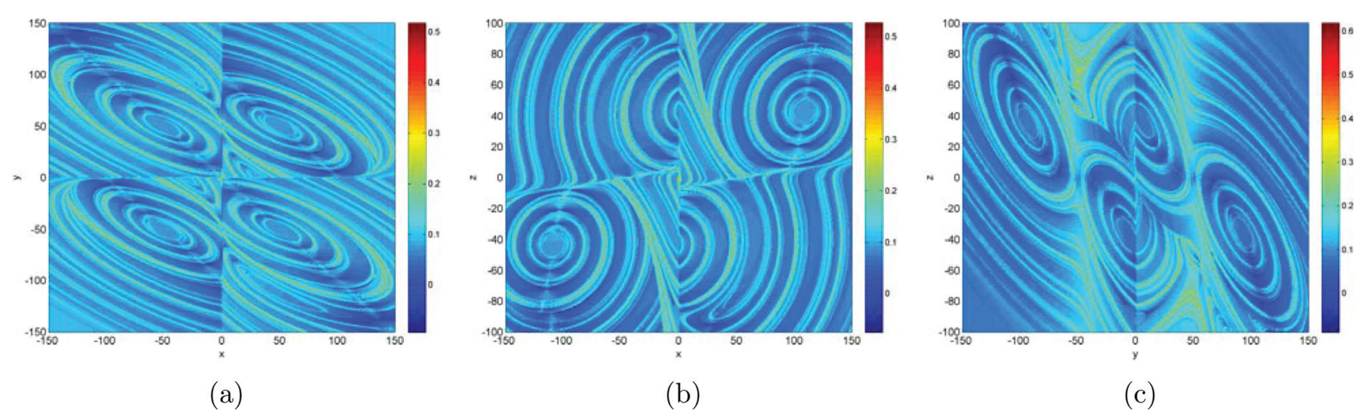


FIG. 5. Projection of the LCS on the planes (a) (x, y), (b) (x, z), and (c) (y, z). It was applied the same procedure to obtain Fig. 4.

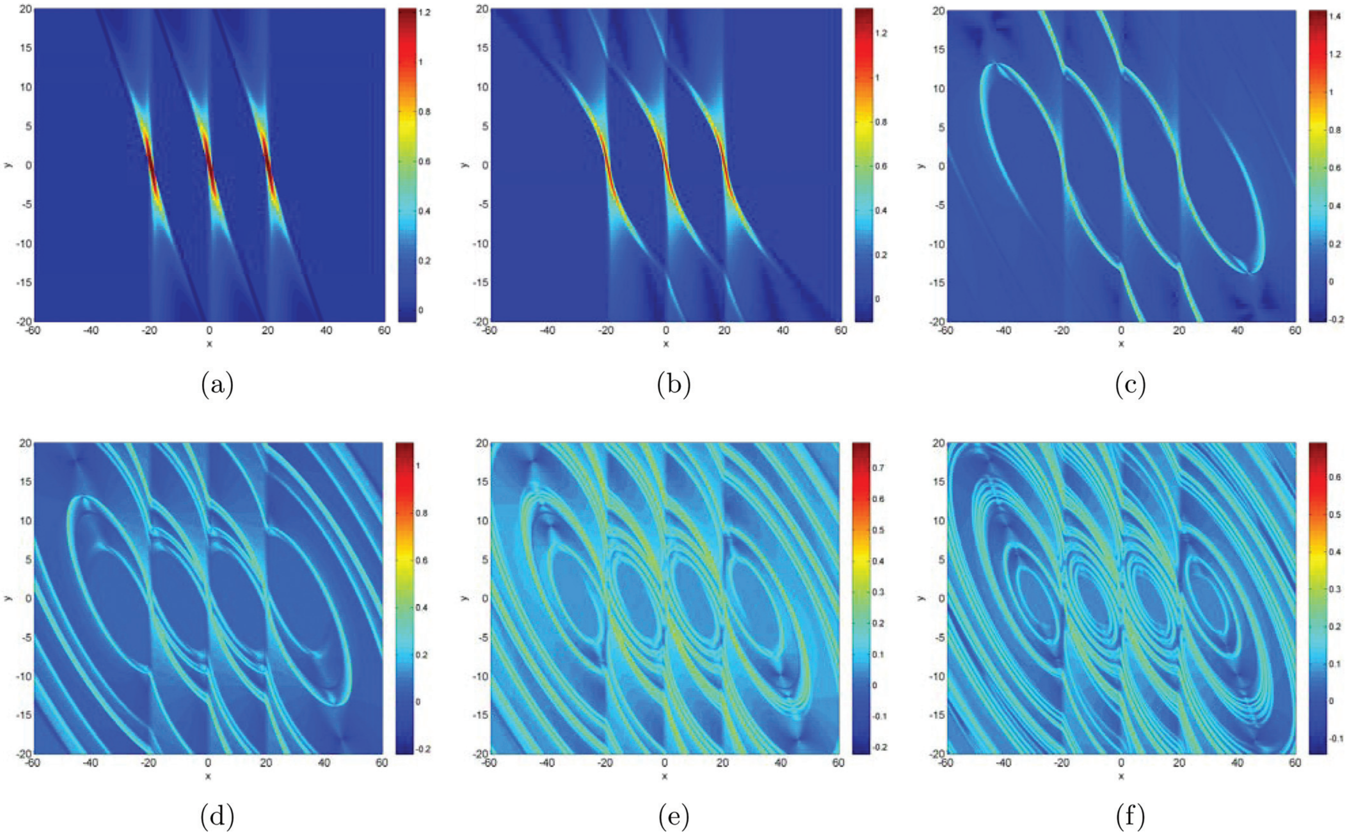


FIG. 6. LCS time evolution projected on (x, y) plane considering the one-dimensional multiscroll attractor illustrated in Fig. 1(a). (a) $t_{\mathrm{LCS}} = 1$ s. (b) $t_{\mathrm{LCS}} = 2$ s. (c) $t_{\mathrm{LCS}} = 5$ s. (d) $t_{\mathrm{LCS}} = 10$ s. (e) $t_{\mathrm{LCS}} = 15$ s. (f) $t_{\mathrm{LCS}} = 20$ s (enhanced online) [URL: http://dx.doi.org/10.1063/1.4802428.1].

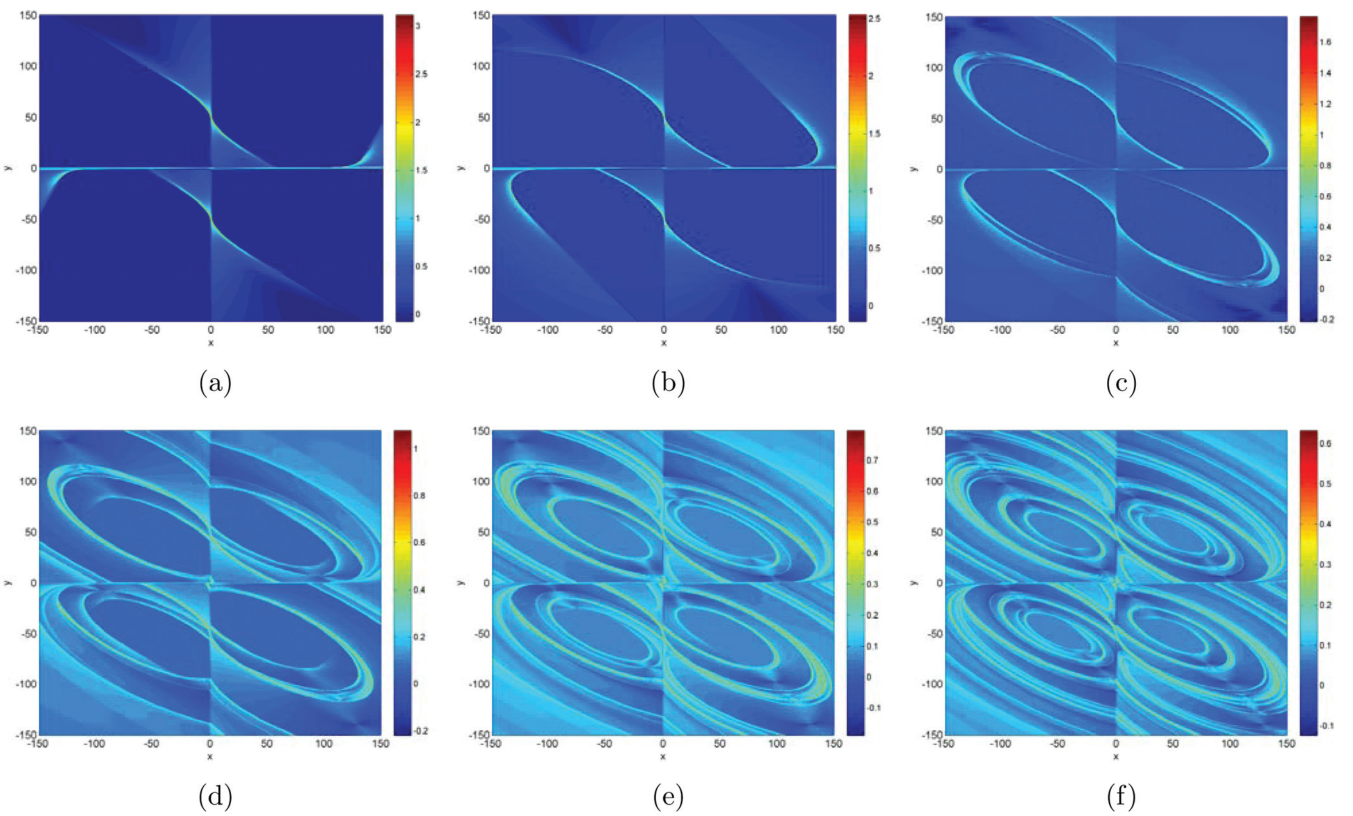


FIG. 7. LCS time evolution projected on (x, y) plane considering the two-dimensional multiscroll attractor illustrated in Fig. 1(b). (a) $t_{\mathrm{LCS}} = 1$ s. (b) $t_{\mathrm{LCS}} = 2$ s. (c) $t_{\mathrm{LCS}} = 5$ s. (d) $t_{\mathrm{LCS}} = 10$ s. (e) $t_{\mathrm{LCS}} = 15$ s. (f) $t_{\mathrm{LCS}} = 20$ s (enhanced online) [URL: http://dx.doi.org/10.1063/1.4802428.2].

index-1 equilibria in both *y*-axis directions of the phase space also gives rise to separatrices in these directions. In this case, the separatrices are defined by a thin line that passes through the equilibrium points located at $(\pm 50, 0, 0)$ and (0, 0, 0) and another line that passes through by $(0, \pm 50, 0)$ and (0, 0, 0).

VI. DISCUSSIONS AND CONCLUSIONS

In this work, we have extended the analysis previously discussed in Ref. 16 aiming to address the information generation and the complexity increase process in a multiscroll attractor model when the number of fundamental scrolls units rises. This task was accomplished by means of the evaluation of the Lyapunov spectrum using a method developed to overcome the drawbacks underlying this model[48] without the need to apply methodologies based on time series analysis or laborious partition of the phase space for obtaining the variational equation in the tangent map approach.[37,38,42,50]

It is also important to emphasize that there are different ways of generating multiscroll attractors,[12,14,28,33] being a general approach to analyse all of them virtually impossible. For instance, in Dana *et al.*,[14] the multiscroll structure is generated by specific couplings of dynamical systems originally operating in different steady states. This outlines a scenario not suitable for the performed analysis, as shown, for instance, in Fig. 2, since it would compare scrolls generated in quite different situations. Moreover, the multiscroll attractors are also obtained by increasing the number of state variables (coupling different dynamical systems), which rapidly increases the number of differential equations to be solved and imposes a computational cost constraint for any method of Lyapunov spectrum estimation. The ClDyn approach is not an exception.

As main results obtained here, it was shown that the addition of scrolls causes an exponential increase of the largest Lyapunov exponent (and, consequently, the Kolmogorov-Sinai entropy) and the fractal dimension towards to a upper bound for a well-behaved grid of scrolls. This suggests that the introduction of new separatrices does not necessarily implies that they will be frequently visited for the trajectory. Nevertheless, in this situation, the trajectory may be restricted mainly to more predictable regions of the phase space, which would justify the observed upper bound for the Kolmogorov-Sinai entropy and the fractal dimension.

Furthermore, following the modus operandi presented by Haller,[19,21] which was extensively developed by Shadden, Lekien and Marsden,[29,43] the ClDyn approach was applied to construct the FTLE field for a whole representative set of initial conditions. This strategy allowed the identification of the LCS and its correspondence to the introduction of fixed points of index-1, providing a better geometrical picture of how the phase space is divided and "explored."

However, the aforementioned strategy is valid for models capable of engendering multiscroll attractors with a relatively low number of state variables, since LCS are quite difficult to interpret for dynamical systems with dimensions higher than 3, such as discussed by Lekien, Shadden, and Marsden.[29] Notwithstanding, a quantitative characterization of multiscroll structure generated by coupling dynamical systems is an interesting perspective for future work, possibly requiring a specific approach.

Finally, the characterization provided here by means of invariant measures and LCS offers a clearer view on the dynamical system for application in different contexts. For instance, the local predictability of phase space given by the LCS can be useful to define critical regions to be controlled in order to stabilize unstable orbits, as proposed by Arena *et al.*[7] for robot navigation purposes. This could also be helpful in the context of designing scroll-based associative memories, specially in the definition of their basins of attraction, which defines a natural future work. Concerning the results related to the estimation of global invariant measures, the authors claim that the proposed fit for information generation in multiscroll attractors can reveal a desirable minimal number of scrolls to attain a certain trajectory "navigation behaviour" that is close to the informational upper bound, or complexity limit. Overall, the reported study can be understood as an investigation, from distinct theoretical standpoints, of key aspects regarding the dynamical behaviour of systems capable of engendering multiscroll structures.

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