# Entanglement Irreversibility from Quantum Discord and Quantum Deficit

Marcio F. Cornelio,[1,*] Marcos C. de Oliveira,[1,†] and Felipe F. Fanchini[1,2]

[1]Instituto de Física Gleb Wataghin, Universidade Estadual de Campinas, CEP 13083-859, Campinas, São Paulo, Brazil

[2]Departamento de Física, Universidade Federal de Ouro Preto, CEP 35400-000, Ouro Preto, MG, Brazil

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We relate the problem of irreversibility of entanglement with the recently defined measures of quantum correlation—quantum discord and one-way quantum deficit. We show that the entanglement of formation is always strictly larger than the coherent information and the entanglement cost is also larger in most cases. We prove irreversibility of entanglement under local operations and classical communication for a family of entangled states. This family is a generalization of the maximally correlated states for which we also give an analytic expression for the distillable entanglement, the relative entropy of entanglement, the distillable secret key, and the quantum discord.

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Two complementary and among the most important tasks in quantum information theory (QIT) are entanglement dilution and entanglement distillation [1,2]. These tasks are performed in a scenario where two spatially separated observers, usually called Alice and Bob, share some quantum states and are able to manipulate their respective parties through local operations and classical communication (LOCC) [2]. In the first task, Alice and Bob share a large number of copies of a standard pure maximally entangled state,

$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \tag{1}$$

which is associated with a unit of entanglement called $e$-bit. Their task is to construct many copies of an arbitrary, generally mixed, state $\rho$ from many copies of $|\Phi\rangle$ using only LOCC (see Fig. 1). In the second task, Alice and Bob want to perform the reverse operation, i.e., to extract from many copies of an arbitrary state, generally mixed, the maximal possible amount of $e$-bits using only LOCC. Those tasks naturally raise the two most important measures of entanglement-entanglement cost ($E^{\mathcal{C}}$) and distillable entanglement ($E^{\mathcal{D}}$) [2]. For a given state $\rho_{ab}$, $E^{\mathcal{C}}(\rho_{ab})$ is the optimal rate for converting a large number of $e$-bits into a large number of copies of the mixed state $\rho_{ab}$ under LOCC by Alice and Bob. Similarly $E^{\mathcal{D}}(\rho_{ab})$ is the optimal rate for converting a large number of $\rho_{ab}$ into $e$-bits under LOCC [3].

When Alice and Bob can build a large number of copies of an arbitrary state $\rho_{ab}$ and can get the same amount of $e$-bits back through LOCC, it is said that there is entanglement reversibility. Conversely, the entanglement is said irreversible. To understand the aspects leading to entanglement irreversibility is one of the most important open problems in QIT [2] with practical implications. Particularly, entanglement dilution is connected to the problem of classical communication over a noise quantum channel [4] and entanglement distillation is connected to quantum communication and quantum key distribution [3,5–7] for secure cryptography. It is known that the task of building an entangled state and extracting back the $e$-bits is reversible if Alice and Bob are limited to build and to distill pure entangled states [1]. For a pure state $\varphi$, $E^{\mathcal{C}}$ and $E^{\mathcal{D}}$ are equal to the von Neumann entropy $S(\rho_r)$ of the reduced density matrix $\rho_r$ of one of the subsystems. Moreover, it is a long-standing conjecture that the only states with $E^{\mathcal{C}} = E^{\mathcal{D}}$ are pure states and the so-called pseudopure (PP) [3,8] states,

$$\rho_{\mathrm{PP}} = \sum p_i |\varphi^i_{ab}\rangle\langle\varphi^i_{ab}| \otimes |f_i\rangle\langle f_i|, \tag{2}$$

where $|f_i\rangle$ is an ancilla, locally accessible for Alice or Bob, working as a flag that indicates which pure entangled state $|\varphi^i_{ab}\rangle$ is in the mixture. Although widely believed, there are few concrete evidences for this conjecture. To understand irreversibility for mixed states has revealed itself to be a very difficult question and the first examples were given only some years later in Refs. [9–12]. Particularly, in Ref. [12] it is shown that one can find mixed states that consume entanglement to be created but no entanglement can be extracted from it, the so-called bound entanglement.

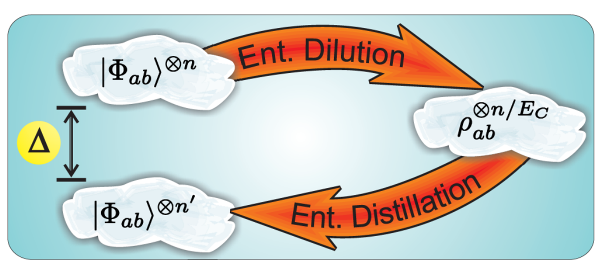


FIG. 1 (color online). Entanglement dilution-distillation cycle. The entanglement loss is given by Δ. In the case of reversible entanglement, Δ vanishes. In the irreversible case of Eqs. (8) and (9), Δ is the regularized quantum discord.

 

One of the main reasons why it is so difficult to understand irreversibility for mixed states is that $E^{\mathcal{C}}$ and $E^{\mathcal{D}}$ are given by formal limits that are very hard to evaluate in general. The first attempt to quantify the entanglement cost was given by Bennett *et al.* [2] introducing the entanglement of formation (EOF),

$$E^{\mathcal{F}}(\rho) = \min_{\mathcal{E}}\left\{\sum_i p_i E^{\mathcal{C}}(\varphi_i)\right\},$$

where the minimization is over the set $\mathcal{E}$ of all ensembles of pure states $\{p_i, \varphi_i\}$ such that $\rho = \sum_i p_i \varphi_i$. EOF is the cost of diluting the $e$-bits in the pure states of the ensemble of $\rho$ and mixing them. As there are many ensembles that realizes $\rho$, one can always choose the ensemble that gives the minimal cost, hence the minimization in the formula of $E^{\mathcal{F}}$. For a long time, it was generally believed that this method was the best dilution protocol and that $E^{\mathcal{C}} = E^{\mathcal{F}}$. Indeed, it was shown by Hayden *et al.* [13] that $E^{\mathcal{C}}$ is the regularization of the EOF:

$$E^{\mathcal{C}}(\rho) = \lim_{n\to\infty} \frac{1}{n} E^{\mathcal{F}}(\rho^{\otimes n}). \tag{3}$$

So the question was reduced to whether $E^{\mathcal{F}}(\rho^{\otimes n}) = nE^{\mathcal{F}}(\rho)$ or not, that is, whether EOF is an additive measure [4]. However, recently it was shown that EOF is not additive in general [14], implying that there are states for which better dilution protocols exist than the one given the EOF. For such states, $E^{\mathcal{F}}(\rho^{\otimes n}) < nE^{\mathcal{F}}(\rho)$ for some $n$ and $E^{\mathcal{C}}$ is strictly smaller than $E^{\mathcal{F}}$. Since $E^{\mathcal{F}}$ is known to be additive only for very particular states [7,10], it is not generally known when one can take $E^{\mathcal{F}}$ for $E^{\mathcal{C}}$.

The difficulty is similar for evaluating $E^{\mathcal{D}}$. In fact, $E^{\mathcal{D}}$ is only known in the particular case of maximally correlated (MC) states [15]. There is an important lower bound, however. When one of the conditional entropies $S_{a|b}$ or $S_{b|a}$ is negative ($S_{a|b} = S_{ab} - S_b$), there is a protocol called hashing which can distill $-S_{a|b}$ $e$-bits from $\rho$ [2,5]. Then the coherent information, $I_C = \max\{0, -S_{a|b}, -S_{b|a}\}$, captures this negative part and is a lower bound for $E^{\mathcal{D}}$. Indeed it is known that $I_C$ can be increased by LOCC, and notably an optimal distillation protocol can always be achieved performing the optimization of $I_C$ followed by hashing [5]. That is,

$$E^{\mathcal{D}}(\rho) = \lim_{n\to\infty} \sup_V \frac{1}{k} I_C(V\rho^{\otimes k}), \tag{4}$$

where $V$ is some LOCC operating on $k$ copies of $\rho$. There is no bound on the number of copies $V$ can act. So $E^{\mathcal{D}}$ might in fact exist only as the limit of $V$ acting on a very large number of copies of $\rho$. In the end, it is very difficult to know or to efficiently bound $E^{\mathcal{C}}$ and $E^{\mathcal{D}}$ simultaneously for answering the reversibility question. The difficulty in calculating these quantities is the main reason for this questioning to be open for 14 years [2]. Here we will be able to calculate $E^{\mathcal{D}}$ for a new family of states.

In this context, it is convenient to introduce our first formal results in the form of an important theorem and a lemma. In what follows, when we say a mixed state, we mean a not pure and not PP state.

*Lemma 1.*—For every mixed entangled state $\rho_{ab}$

$$E^{\mathcal{F}}(\rho_{ab}) > I_C(\rho_{ab});$$

i.e., the EOF is strictly larger than the coherent information for every mixed $\rho_{ab}$.

*Theorem 1.*—Let $\rho_{ab}$ be a mixed entangled state, if

$$E^{\mathcal{C}}(\rho_{ab}) = \frac{1}{n} E^{\mathcal{F}}(\rho_{ab}^{\otimes n}), \tag{5}$$

$$E^{\mathcal{D}}(\rho_{ab}) = \max_V \frac{1}{k} I_C(V\rho^{\otimes k}), \tag{6}$$

for some finite $n$ and $k$, then the entanglement is irreversible for $\rho_{ab}$, i.e., $E^{\mathcal{C}}(\rho_{ab}) > E^{\mathcal{D}}(\rho_{ab})$.

The technical details of the proofs of Lemma 1 and Theorem 1 are left to the supplementary material [16]. Here we limit ourselves to discuss their meaning in the context of entanglement irreversibility and the main concepts involved. First we notice that Eqs. (5) and (6) differ from Eqs. (3) and (4) only by the lack of the limits. So entangled states satisfying condition (5) will be called type $A$ and those satisfying condition (6) will be called type $B$. The states satisfying both conditions will be called type $AB$ and, to complete the analogy, states satisfying none will be called type $O$. In this way, Theorem 1 simply says that states of type $AB$ are irreversible. It is important to notice that for all states that $E^{\mathcal{C}}$ and/or $E^{\mathcal{D}}$ are known, the conditions (5) and/or (6) are satisfied.

The central concept behind Lemma 1 and Theorem 1 is the quantum discord [17]. It is defined as the difference between two ways of defining mutual information,

$$\delta_{a|c}(\rho_{ac}) = I(\rho_{ac}) - J_{a|c}(\rho_{ac}),$$

where $I(\rho_{ac}) = S(\rho_a) + S(\rho_c) - S(\rho_{ac})$ is the quantum mutual information and $J_{a|c}(\rho_{ac})$ is a measure of the amount of classical correlations present in quantum states,

$$J_{a|c}(\rho_{ac}) = \max_{\{\Pi_i\}}\left[S(\rho_a) - \sum_i p_i S(\rho_a^i|\Pi_i)\right],$$

where $\{\Pi_i\}$ is a complete positive operator valued measure (POVM) on subsystem $c$ and $p_i$ are the respective probabilities, so $S(\rho_a^i|\Pi_i)$ is the entropy of subsystem $a$ conditioned to the output $\Pi_i$ on $c$. So $I(\rho_{ac})$ measure the total amount of correlations in $\rho_{ac}$ while $J_{a|c}(\rho_{ac})$ measures the amount of classical correlations when the POVM $\{\Pi_i\}$ is performed on $c$. In this way, $\delta_{a|c}(\rho_{ac})$ gives a distinct notion of nonclassicality from entanglement.

It is easy to relate quantum discord with the EOF. For every pure tripartite state $|\psi_{abc}\rangle$ holds [18]

$$E^{\mathcal{F}}(\rho_{ab}) = \delta_{a|c}(\rho_{ac}) - S_{a|b}(\rho_{ab}), \tag{7}$$

where $\rho_{ab}$ and $\rho_{ac}$ are the reduced states of the respective subsystems. From Eq. (7) it is easy to see that $\delta_{a|c}$ is not additive only when $E^{\mathcal{F}}(\rho_{ab})$ is not additive as well. Then it is necessary to define the regularized quantum discord (RQD) in the same way as for $E^{\mathcal{F}}$,

$$\Delta_{a|c}(\rho_{ac}) = \lim_{n\to\infty}\frac{1}{n}\delta_{a|c}(\rho_{ac}).$$

Similarly to Eq. (7), we have for the regularized quantities

$$E^{\mathcal{C}}(\rho_{ab}) = \Delta_{a|c}(\rho_{ac}) - S_{a|b}(\rho_{ab}), \tag{8}$$

Equation (8) is relating three fundamental quantities in QIT with a clear operational meaning. It is known that when the conditional entropy is negative it is possible to distill $-S_{a|b}$ $e$-bits out of the state $\rho_{ab}$. Then Eq. (8) is telling us that the amount of entanglement lost in the process of creating a mixed state $\rho_{ab}$ and distilling it by hashing is equivalent to the RQD with a complementary system $c$. Thus Eq. (8) gives a new operational meaning to $\Delta_{a|c}$ as a measure of the amount of entanglement loss when Alice and Bob distill entanglement by hashing.

For states of type $B$ and $AB$, i.e., all those satisfying condition (6), the connection between the RQD with the purifying subsystem $c$ and entanglement loss in distillation will turn clear. For every $\rho_{ab}$ of type $B$ there is a finite $k$ and a LOCC $V'$ giving the maximum in Eq. (6) such that

$$\Delta_{a|c}(\sigma_{ac}) = E^{\mathcal{C}}(\sigma_{ab}) - E^{\mathcal{D}}(\sigma_{ab}), \tag{9}$$

where $\sigma_{ab} = V'\rho_{ab}^{\otimes k}$ and $E^{\mathcal{D}}(\sigma_{ab}) = kE^{\mathcal{D}}(\rho_{ab})$. In this way, we say that $\sigma_{ab}$ is the optimized distillable state (ODS) of $\rho_{ab}$. We notice that $\sigma_{ab}$ can be the ODS of many distinct states, being the result of also distinct $V'$'s. Therefore each $\sigma_{ab}$ satisfying Eq. (9) defines a class of states $\rho_{ab}$ for which it is the ODS. For each class, $\Delta_{a|c}(\sigma_{ac})$ is the minimal amount of entanglement lost in any distillation protocol for all states belonging to the class. In the case of $\rho_{ab}$ being bound entangled, we have for any $\sigma_{ab} = V\rho_{ab}$, with an arbitrary LOCC $V$, that

$$\Delta_{a|c}(\sigma_{ab}) \geq E^{\mathcal{C}}(\rho_{ab}).$$

We have stated our more general results. Now we apply these results for an important case—we consider the tripartite state

$$|\psi_{abc}\rangle = \sum_{i=1}^{N}\alpha_i|a_i, i_b, c_i\rangle,$$

where $N$ is the dimension of the subsystems, $\{|i_b\rangle\}$ is an orthonormal basis for $b$, $\{|a_i\rangle\}$ and $\{|c_i\rangle\}$ are arbitrary (usually nonorthogonal) states of $a$ and $c$. The subsystem $ab$ results in the density matrix

$$\rho_{ab} = \sum_{ij}\beta_{ij}|a_i i_b\rangle\langle a_j j_b|, \tag{10}$$

where $\beta_{ij} = \alpha_i\alpha_j^*\langle c_j|c_i\rangle$. We call these states one-way maximally correlated (1-MC) since, despite $\rho_{ab}$ being mixed, the result of a measurement in the basis $\{|i_b\rangle\}$ is perfectly correlated with a definite state $|a_i\rangle$.

*Theorem 2.*—For every mixed 1-MC $\rho_{ab}$ the entanglement is irreversible. Equation (9) holds and

$$E_{ab}^{\mathcal{C}} > E_{ab}^{\mathcal{D}} = \delta_{a|b} = \Delta_{a|b} = -S_{a|b}.$$

Further, $\rho_{ab}$ is separable if and only if $-S_{a|b} = 0$ and there is no bound entangled state in this family.

In fact, 1-MC states are examples of type $AB$ states. The essential elements of the proof are the fact that EOF is additive for them and the distillable entanglement turns out to be exactly $-S_{a|b}$. Furthermore, Theorem 2 gives us also the quantum discord in one direction, $\delta_{a|b}$ (as well as other measures, like the relative entropy of entanglement and the distillable secret key; see the supplementary material [16] for details) for these states. From the fact that $\delta_{a|b} \geq 0$, one can deduce that $-S_{a|b} \geq 0$. We know also that $\delta_{a|b} = 0$ implies that $\rho_{ab}$ is separable. So $-S_{a|b} = 0$ is a necessary and sufficient separability criteria for 1-MC states and there is no bound entangled state belonging to this family.

In addition, we should notice that the only examples of irreversibility previously known [11] with $E^{\mathcal{D}} > 0$ are very particular cases of 1-MC states. Furthermore, the examples for which we knew $E^{\mathcal{D}}$ [5,15] are also a subset of null measure of 1-MC states. Therefore, the only states proved irreversible are the bound entangled and 1-MC correlated states.

*Example.*—A tripartite pure state satisfying the condition of Theorem 2, i.e., such that the reduced state $\rho_{ac}$ is separable, can be written as

$$|\psi_{abc}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |\theta 1\varphi\rangle),$$

where $|\theta\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$ and $|\varphi\rangle = \cos\varphi|0\rangle + \sin\varphi|1\rangle$. The resulting 1-MC state is given by

$$\sigma_{ab} = \frac{1}{2}[|00\rangle\langle 00| + |\theta 1\rangle\langle\theta 1| + \cos\varphi(|00\rangle\langle\theta 1| + |\theta 1\rangle\langle 00|)]. \tag{11}$$

Notice that a measurement in the basis $\{|0\rangle, |1\rangle\}$ has a perfect correlation with the states $|0\rangle$ and $|\theta\rangle$. The angle $\theta$ gives how far $\sigma_{ab}$ is from a usual MC state [15] belonging to this class when $|\theta\rangle = |1\rangle$. The angle $\varphi$ gives the amount of mixedness of $\rho_{ab}$. For $\varphi = 0$ the state is pure and for $\varphi = \pi/2$ the state is separable. Figure 2 shows the behavior of $E_{ab}^{\mathcal{C}}$ and $E_{ab}^{\mathcal{D}}$ and how the loss of entanglement is equal to $\Delta_{a|c}$. It is remarkable that this class is now the only one for which we know both $E^{\mathcal{C}}$ and $E^{\mathcal{D}}$.

Combining Theorems 1 and 2 we easily get the following.

*Corollary 1.*—A type $B$ reversible mixed state $\rho_{ab}$ exists if and only if there exists a bound entangled state $\rho_{ac}$ such that $\delta_{a|c} > 0$ and $\Delta_{a|c} = 0$.

The question about existence of states with $\Delta_{a|c} = 0$, but $\delta_{a|c} > 0$, was raised in 2005 [19] and is directly related

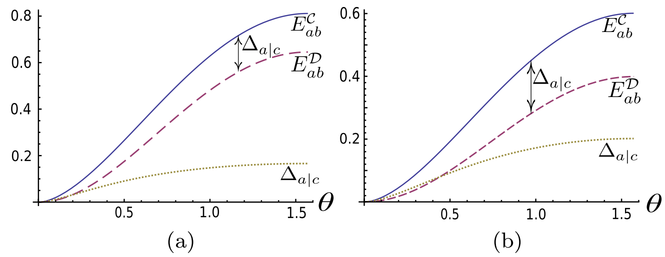


FIG. 2 (color online). $E^{\mathcal{C}}$, $E^{\mathcal{D}}$, and $\Delta_{a|c}$ for 1-MC states of 2-qubits $\sigma_{ab}$ with (a) $\varphi = \pi/6$ and (b) $\varphi = \pi/4$ in Eq. (11). The values satisfy Eq. (9), and $E^{\mathcal{D}}$ was previously known only for the case $\theta = \pi/2$.

to the question of additivity of EOF by Eq. (7). So our results tell us exactly in which situation the nonadditivity of EOF could be responsible for irreversibility providing a strong link between these two fundamentals questions.

*Thermodynamical analogy.*—Since the beginning of entanglement theory, it has been compared with thermodynamics. Dilution and distillation of pure into pure states are reversible operation under LOCC. This is analogous to a reversible process in classical thermodynamics where entropy remains constant and all the energy that is put in the system can be recovered without losses. Mixedness is caused by some noise and is associated with the increasing of entropy. Then our intuition tells us that noise will probably result in some irreversible loss of entanglement that cannot be recovered by LOCC only. However, this connection has never been done explicitly. Our work provides the desired connection directly between that noise and entanglement loss.

Zurek [20] has shown that QD can be interpreted as some amount of thermodynamical work that Alice and Bob must pay when they operate only locally on their respective subsystems. The same operational interpretation was developed independently [19,21], generating many kinds of a similar quantity called quantum deficit. In the asymptotic limit, the regularized expressions for QD and one-way quantum deficit are equivalent. The quantum deficit measures the following task: Suppose that Alice and Bob share many copies of $\rho_{ab}$. From that they can use the information they have about this state to produce work through a Szilard engine [19,21]. However, there is a difference between the amount of work Alice and Bob can perform whether they operate globally with the two subsystems or they can operate only locally on its respective subsystems. This difference in the amount of information they can use to perform work is the quantum deficit. We have seen that in the process of diluting $e$-bits inevitably some information corresponding to the entropy $S(\rho_{ab})$ is lost to the environment. In our approach, the environment is represented by $c$. Therefore the loss of entanglement is, *de facto*, associated with part of this information lost to the environment and is quantified by $\Delta_{a|c}$.

To summarize, our results provide strong evidences that irreversibility must happen for all mixed, not PP, entangled states. We have shown that such a counterintuitive possibility would necessarily imply other very counterintuitive properties. For instance, one possibility is having $\delta_{a|c} > 0$ and $\Delta_{a|c} = 0$. In this case the nonadditivity of EOF would be responsible for irreversibility. Another possibility is that, to obtain $E^{\mathcal{D}}$, it is necessary to optimize the coherent information over an arbitrary large number of copies of the entangled state. Moreover we have shown irreversibility for the important family of 1-MC states. In addition we calculate $E^{\mathcal{D}}$, quantum discord, and the relative entropy of entanglement for them, and, further, we have shown that there is no bound entangled and that $S_{a|b} = 0$ is a necessary and sufficient separability criteria for this family.

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*mfc@ifi.unicamp.br
†marcos@ifi.unicamp.br

[1] C. H. Bennett *et al.*, Phys. Rev. A **53**, 2046 (1996).
[2] C. H. Bennett *et al.*, Phys. Rev. A **54**, 3824 (1996).
[3] R. Horodecki *et al.*, Rev. Mod. Phys. **81**, 865 (2009).
[4] P. W. Shor, Commun. Math. Phys. **246**, 453 (2004); M. F. Cornelio and M. C. de Oliveira, Phys. Rev. A **81**, 032332 (2010).
[5] I. Devetak and A. Winter, Phys. Rev. Lett. **93**, 080501 (2004); Proc. R. Soc. A **461**, 207 (2005).
[6] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **85**, 433 (2000).
[7] I. Devetak and A. Winter, IEEE Trans. Inf. Theory **50**, 3183 (2004).
[8] P. Horodecki, R. Horodecki, and M. Horodecki, Acta Phys. Slovaca **48**, 141 (1998).
[9] G. Vidal and J. I. Cirac, Phys. Rev. Lett. **86**, 5803 (2001); Phys. Rev. A **65**, 012323 (2001).
[10] G. Vidal, W. Dür, and J. I. Cirac, Phys. Rev. Lett. **89**, 027901 (2002).
[11] K. G. H. Vollbrecht, R. F. Werner, and M. M. Wolf, Phys. Rev. A **69**, 062304 (2004).
[12] D. Yang *et al.*, Phys. Rev. Lett. **95**, 190501 (2005).
[13] P. M. Hayden, M. Horodecki, and B. M. Terhal, J. Phys. A **34**, 6891 (2001).
[14] M. B. Hastings, Nature Phys. **5**, 255 (2009).
[15] E. M. Rains, Phys. Rev. A **60**, 179 (1999).
[16] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.107.020502 for the proofs of the theorems and some particular extensions of our results.
[17] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. **88**, 017901 (2001); L. Henderson and V. Vedral, J. Phys. A **34**, 6899 (2001).
[18] M. Koashi and A. Winter, Phys. Rev. A **69**, 022309 (2004); F. F. Fanchini *et al.*, arXiv:1006.2460.
[19] M. Horodecki *et al.*, Phys. Rev. A **71**, 062307 (2005).
[20] W. H. Zurek, Phys. Rev. A **67**, 012320 (2003).
[21] J. Oppenheim *et al.*, Phys. Rev. Lett. **89**, 180402 (2002); I. Devetak, Phys. Rev. A **71**, 062303 (2005).