Energy dissipation via coupling with a finite chaotic environment

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We study the flow of energy between a harmonic oscillator (HO) and an external environment consisting of N two-degrees-of-freedom nonlinear oscillators, ranging from integrable to chaotic according to a control parameter. The coupling between the HO and the environment is bilinear in the coordinates and scales with system size as $1/\sqrt{N}$. We study the conditions for energy dissipation and thermalization as a function of N and of the dynamical regime of the nonlinear oscillators. The study is classical and based on a single realization of the dynamics, as opposed to ensemble averages over many realizations. We find that dissipation occurs in the chaotic regime for fairly small values of N, leading to the thermalization of the HO and the environment in a Boltzmann distribution of energies for a well-defined temperature. We develop a simple analytical treatment, based on the linear response theory, that justifies the coupling scaling and reproduces the numerical simulations when the environment is in the chaotic regime.

PACS number(s): 05.20.Gg, 05.45.Ac, 05.40.Jc

I. INTRODUCTION

Dissipation is usually modeled by coupling a *central system* to an external *environment*. The environment can be represented by an infinite collection of modes (either harmonic or spinlike) at a given temperature or by a few chaotic degrees of freedom. Models in the first category include the Caldeira-Leggett (CL) model [1], where the environment is represented by N harmonic oscillators with a linear distribution of frequencies. In the thermodynamic limit $N \to \infty$, the central system obeys the classical Langevin equation, exhibiting exponential dissipation subjected to fluctuating forces.

In the case of small environments, the exchange of energy with the central system exhibits large fluctuations, and equilibration is achieved only after averaging over an ensemble of realizations, usually of the microcanonical type. Wilkinson [2], and later Berry and Robbins [3], studied the coupling of a fast chaotic environment to a slow central system using the properties of adiabatic invariants. They showed that the adiabatic invariance of the chaotic energy shell [4–6] leads, in first order, to a Born-Oppenheimer conservative type of reaction on the central system. Second order corrections lead to a force proportional to the velocity whose symmetric part corresponds to friction and whose antisymmetric part acts as a geometric magnetism [2,3]. Since the publication of these pioneering works, a large number of papers have explored the possibility of dissipation caused by small environments, both classically [7–11] and quantum mechanically [12–28], where decoherence is the main focus.

Although the approach to dissipation via ensemble average over small environments has lead to several interesting results, it cannot account for situations where a single experiment is performed and no averages take place. This is, for example, the case of the C_{60} molecule interacting with its own internal degrees of freedom. As pointed out in [29], couplings of these molecules to the environment or to their own internal degrees of freedom play an essential role in the appearance of decoherence, destroying self-interferences in a double slit experiment, and deserve a careful analysis. When the environment has relatively few degrees of freedom, it is not

clear whether they lead to standard dissipation effects over the collective degrees of freedom.

The main goal of this paper is to study dissipation from the classical point of view as induced by small environments without resorting to ensemble averages. To distinguish this process from those obtained after averaging, we term it *effective dissipation*. We follow the general approach of Feynman and Vernon [30], where the system of interest, or *central system*, has many fewer degrees of freedom than the environment, which, however, is also small. In our simulations the central system is a harmonic oscillator with a single degree of freedom, whereas the environment has $2N$ degrees of freedom. We study the system as a function of N and of its dynamical character, regular or chaotic. We wish to determine the minimum value of N for which the environment behaves as a thermal bath, inducing irreversible dissipation on the harmonic oscillator.

II. MODEL

We consider a globally conservative system governed by the Hamiltonian

$$H = H_{HO} + H_E + \lambda_N H_I, \tag{1}$$

where

$$H_{HO} = \frac{p^2}{2m} + \frac{m\omega_o^2 q^2}{2}, \tag{2}$$

$$H_E = \sum_{n=1}^{N} \left[\frac{p_{x_n}^2 + p_{y_n}^2}{2} + \frac{a}{4}\left(x_n^4 + y_n^4\right) + \frac{x_n^2 y_n^2}{2} \right] \equiv \sum_{n=1}^{N} H_{QS}^{(n)}, \tag{3}$$

$$H_I = \sum_{n=1}^{N} q x_n. \tag{4}$$

The environment H_E is modeled by the finite sum of identical quartic systems (QS's) [31]. Each QS has two degrees of freedom and is invariant under a scaling transformation in such a way that the dynamics in all energy shells are the same. This property allows us to adjust the QS's energy and time scales without changing its dynamical regime, which is

solely controlled by the parameter a. The QS is integrable for $a = 1.0$, strongly chaotic for $a \lesssim 0.1$, and mixed for intermediate values.

In addition to the dynamical regime, the number of QS's coupled to the harmonic oscillator (HO) plays a crucial role in the dynamics of dissipation. In order to compare results for different values of N, the coupling parameter λ_N, as seen in Eq. (1), has to be properly normalized with the size of the environment, so that the effective coupling becomes independent of N for large N. We use $\lambda_N = \lambda/\sqrt{N}$ and derive this scaling later using linear response theory.

We want to study the behavior of the system for a single global initial condition, and we consider two types of initialization. In both cases the HO starts from $q = 0$ and $p = \sqrt{2mE_o}$, with $m = 1$, $\omega_o = 0.3$, and $E_o = 10.0$. In the first type, the initial condition for each QS is chosen randomly at a fixed energy shell E_{QS}. This choice is related to the microcanonical distribution

$$\rho = \frac{1}{Z} \prod_{n=1}^{N} \delta\big(H_{QS}^{(n)} - E_{QS}^{(n)}\big), \qquad (5)$$

where $E_{QS}^{(n)} = E_{QS}$, for $n = 1, \ldots, N$. We refer to it as *pseudomicrocanonical* to distinguish it from the real micro-canonical distribution, which is uniform over the entire energy shell of the N QS's.

The second type of initialization corresponds to pick a different energy shell $E_{QS}^{(n)}$ for each QS from the Boltzmann distribution $p(E) = \exp(-E/\overline{E}_{QS})/\overline{E}_{QS}$, and we call it *pseudocanonical*. The initial condition for each QS is again chosen randomly at the corresponding energy $E_{QS}^{(n)}$.

III. NUMERICAL RESULTS

In all simulations the coupling parameter will be fixed at $\lambda = 0.01$. We consider initially the pseudomicrocanonical distribution with $E_{QS} = 0.01$.

We first study the role of chaos in promoting dissipation. Fixing $N = 100$, we integrate Hamilton's equations for different values of the parameter a and compute the energy of the harmonic oscillator H_O as a function of the time. Time is measured in units of the HO's period, $\tau = 2\pi/\omega_o \approx 20.94$. Figure 1 shows the results of simulations for a single numerical realization for three values of a.

In the integrable regime, $a = 1.0$, the effect of the environment is minimal, causing a small decrease in the oscillator's energy and inducing small amplitude, noisy, oscillations. In the mixed case, $a = 0.5$, the energy loss is slightly larger, and the oscillations are more clearly visible, particularly at shorter times. At long times the energy fluctuates around $0.8E_o$. We refer to Refs. [10,11] for numerical results on analogous systems, computed via ensemble averages, in the regular and in mixed regimes. The authors describe in great detail the dependence of the dissipation rate on the number of degrees of freedom of the environment in these regimes.

In the chaotic regime the energy of the HO shows a clear exponential decay for $N = 100$ QS's. Similar results were obtained in [9] with $N = 1$ but performing ensemble averages over several thousands of trajectories. The results displayed in Fig. 1 are for a single trajectory. Changing the initial conditions
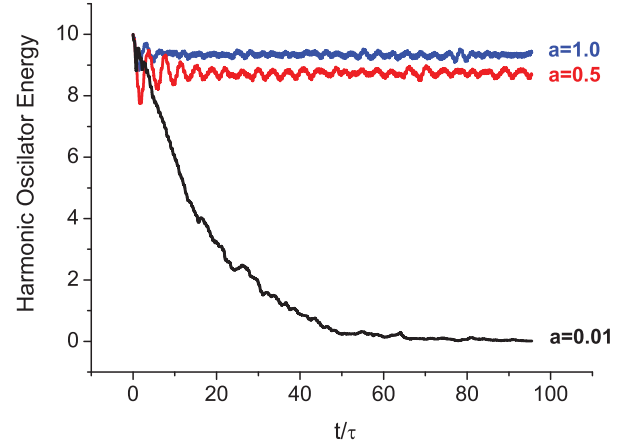

FIG. 1. (Color online) Energy of the HO when coupled to $N = 100$ QS's in distinct dynamical regimes: integrable, $a = 1.0$ (top blue line); mixed, $a = 0.5$ (middle red line); chaotic, $a = 0.01$ (bottom black line).

in the environment but keeping the same microcanonical constraint produces energy curves that differ only in the details of the fluctuations but not in their main qualitative features.

The number of degrees of freedom of the environment is of fundamental importance to the effective dissipative behavior. The exponential decay observed in Fig. 1 occurs only if N is sufficiently large. Figure 2 shows the typical behavior of $H_O(t)$ for different values of N.

For small environments ($1 \leqslant N \lesssim 6$), the fluctuations are so large that dissipation cannot be characterized for each numerical realization. In these cases, and in particular for $N = 1$, exponential dissipation can be observed only in an ensemble averaged treatment of the energy [9].

Increasing the number of QS's to $7 \lesssim N \lesssim 20$ makes the decreasing behavior of the HO energy more evident, although the fluctuations are still large. Finally, for the parameters of this simulation, the curves converge to an N-independent exponential decay $H_O(t) \simeq e^{-\gamma t}$ for $N \gtrsim 100$. This can be
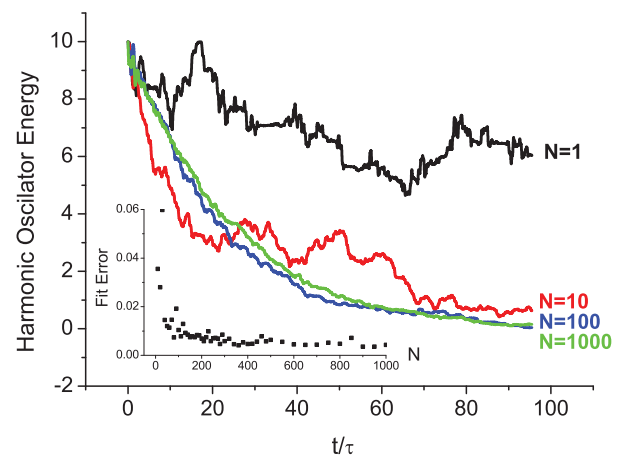

FIG. 2. (Color online) Harmonic oscillator energy for different numbers of QS's in the environment. For large N the harmonic oscillator energy saturates at $2E_{QS}/3 \approx 0.0067$. The inset shows standard error in the parameter t_d in a fit to $\exp(-t/t_d)$ as a function of N.

seen in the inset in Fig. 2, which shows the standard error in the exponential fit as a function of N. This behavior is observed only if the coupling constant scales properly with N. Clearly, λ_N has to decrease with the number of QS's in the environment; otherwise, the interaction Hamiltonian H_I would diverge as $N \to \infty$. Only if $\lambda_N = \lambda/\sqrt{N}$, the effect of the environment over the HO converges as N becomes large, as shown in Fig. 2. This scaling of the coupling constant will be demonstrated using linear response theory.

Environments with sufficiently large N behave as infinite reservoirs. However, how large N needs to be depends on the model parameters and, in particular, on a, which controls the degree of chaos in the QS. Indeed, dissipation is intimately associated with correlation functions that contain information about the memory of the environment. Increasing the phase space region covered by chaos (by decreasing a) makes these correlation functions decay faster, leading to a faster transfer of energy from the oscillator to the environment, even for small values of N. This makes the number of chaotic systems needed to mimic a reservoir smaller for smaller values of a.

In the regular regime, for $a \approx 1$, the behavior of the system is very different, and we have not observed dissipation for N up to 10 000. This is in part because an environment composed by integrable QS's is quite different from the usual Caldeira-Leggett environment, composed by harmonic oscillators [1]. In that case, the oscillators in the bath have a distribution of frequencies, whereas here the QS's are all identical. An estimate of the number of harmonic oscillators needed to mimic a thermodynamic bath was performed in [11] and turns out to be of the order of 1000.

The flow of energy between the HO and environment also depends on the relative time scales of the HO and QS. Increasing the HO frequency ω_o decreases the relative velocity of the two subsystems, leading to a less efficient energy dissipation. However, since the QS is scalable, we can always change its energy in such a way to readjust its relative velocity.

The decay rate γ, or the characteristic time $t_d = 1/\gamma$, also changes with λ, as shown in Fig. 3 for $N = 100$ and $a = 0.01$. The dependence is well fitted by a quadratic law, a result that will also be demonstrated using linear response theory.
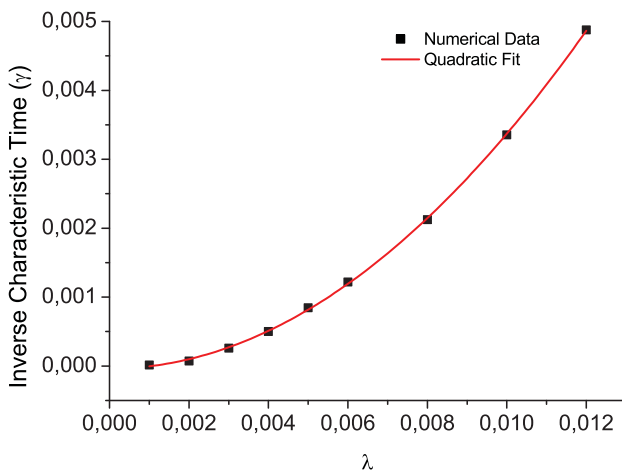
The numerical results in Fig. 2 show that small chaotic environments may behave as infinite reservoirs, absorbing energy from the HO. This behavior can be further characterized by associating a temperature to the full system after its equilibration. In order to do so, we look at the energy distribution of the HO and the environment at equilibrium. This can be done in two different ways: the first is to count, at a fixed time, the number of QS's with energy in the interval $(E, E + dE)$ for a large number of initial conditions (ensemble average). The second possibility is to count, for a single realization, the number os QS's with energy in the interval $(E, E + dE)$, starting at some time after equilibrium and repeating the procedure several times at a prespecified time step (time average). The same procedures can be applied to the energy of the HO. The statistical distribution of these results will be the same if the system is ergodic. We performed numerical calculations using both the time and ensemble averages and found very similar results, indicating that both the environment and the HO are indeed ergodic in equilibrium.

In order to define a temperature for the environment we use the general equipartition theorem [32], which states that every quadratic degree of freedom contributes $K_B T/2$ to the total energy E. The contribution of the quartic potential $a(x^4 + y^4)/4 + x^2 y^2/2$ can also be calculated and also results $K_B T/2$. Therefore

$$E = \overline{E}_{HO} + N\overline{E}_{QS} = K_B T + N\frac{3K_B T}{2}. \quad (6)$$

This shows that $\overline{E}_{QS} = \frac{3}{2}\overline{E}_{HO}$. The energy distribution of the environment for $N = 100$ is well fitted by $\exp(-E/\overline{E}_{QS})$ with $\overline{E}_{QS} \approx 0.1094$. The value of $\overline{E}_{QS}$ is a measure of the environment's temperature. Figure 4 shows that the energy distribution of the HO also follows a Boltzmann-like distribution,

$$p_{HO}(E) = \frac{1}{\overline{E}_{HO}} e^{-E/\overline{E}_{HO}}, \quad (7)$$

where $\overline{E}_{HO} = 0.0723 \approx 2/3 E_{QS}$, showing that the systems are indeed in thermal equilibrium.

FIG. 3. (Color online) Inverse characteristic time (γ) as a function of the coupling constant (λ) for $N = 100$ and $a = 0.01$. The curve is well fitted by a quadratic function.
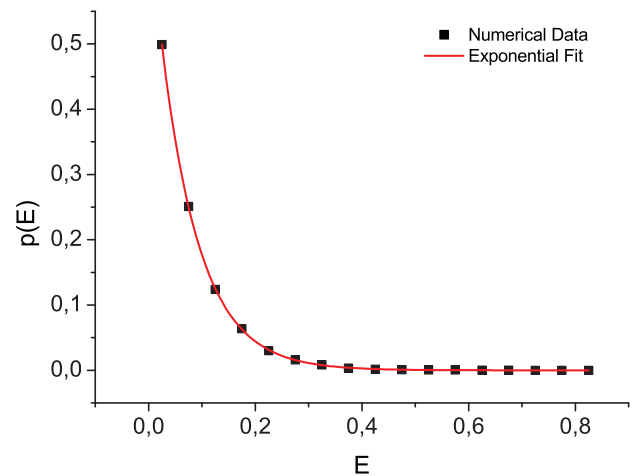

FIG. 4. (Color online) Energy distribution for HO with $N = 100$ QS's. The temperature is $\overline{E}_{HO} = 0.0723 \pm 0.0002$. Energy was sampled 100 000 times at equal intervals.

Using $E = E_{HO} + NE_{QS}$ (its value at time zero) in Eq. (6), we also obtain that, for large N, the equilibrium value of $\overline{E}_{HO}$ converges to $\overline{E}_{HO} = K_B T = 2E_{QS}/3 + \mathcal{O}(1/N)$, as illustrated in Fig. 2.

This is an important result of this paper, which distinguishes between the coupling of the HO with a single QS (where results are averaged over many realizations with different initial conditions) [9] and the coupling with many ($N \simeq 100$) simultaneous QS's (and no average over realizations). In the former case a temperature can be defined only by modifying the definition of entropy [33,34]. When several QS's are present, the indirect, and therefore weak, interaction between the members of the environment plays a key role in the dynamics of the system. It is this weak interaction that allows for their eventual equilibration.

These numerical results suggest that classically irreversible processes may indeed occur in nature if the system interacts with a small, but not too small, chaotic environment. The primary coupling between the system and the environment leads to dissipation, whereas the secondary interactions between the few, but chaotic, degrees of freedom of the environment lead to thermalization.

IV. LINEAR RESPONSE

We can understand the basic mechanisms of dissipation using linear response theory (LRT) [35]. We start by writing an approximation to the full Hamiltonian of the system as

$$H = H_E(Q,P) + H_I(Q,P,t),$$

where Q and P represent the full set of canonical variables of the environment, $\{x_n, y_n, p_{x_n}, p_{y_n}\}$, $n = 1, \ldots, N$. The second term $H_I(Q,P,t) \equiv A(Q,P)\chi(t)$ is a perturbation that includes the central system through the time dependent function $\chi(t)$. This approximation to (1) is valid in the limit of weak coupling since the central system is treated as an external source that does not respond to the environment. The feedback of the environment on the central system will be considered in a moment.

The dynamics can be described by the Liouville equation:

$$\frac{\partial \rho}{\partial t} = iL_E \rho(t) + iL_I(t)\rho(t), \tag{8}$$

where the Liouville operators L_E and L_I are given by

$$iL\rho = \sum_j \left[\frac{\partial H}{\partial Q_j} \frac{\partial \rho}{\partial P_j} - \frac{\partial H}{\partial P_j} \frac{\partial \rho}{\partial Q_j} \right] = i\{H,\rho\}, \tag{9}$$

with H replaced by H_E and H_I, respectively. Equation (8) can also be written in integral form as

$$\rho(t) = e^{i(t-t_0)L_0} \rho(t_0) + i \int_{t_0}^t e^{i(t-s)L_0} L_I(s)\rho(s)ds. \tag{10}$$

The initial distribution $\rho(t_0)$ is assumed to be invariant under H_E, like, for example, the pseudomicrocanonical or pseudocanonical distributions given by Eq. (5). The action of $H_I(t)$ removes the environment from this initial equilibrium.

If the perturbation is small, Eq. (10) can be expanded to first order in L_I as

$$\rho(t) = \rho(t_0) + \int_{t_0}^t e^{i(t-s)L_0} \{H_I(s), \rho(t_0)\} ds. \tag{11}$$

The ensemble average of a general function $B(Q,P)$ can then be calculated as

$$\langle B(Q,P)\rangle(t) = \langle B(Q(t),P(t))\rangle_0$$
$$+ \int_{t_0}^t \phi_{BA}(t-s)\chi(s)dQdPds, \tag{12}$$

where

$$\phi_{BA}(t) = \langle \{A(Q,P), B(Q(t),P(t))\}\rangle_0 \tag{13}$$

is the *response function* of $B(Q,P)$ when the environment is under the influence of the perturbation $A(Q,P)$. The subscript 0 indicates that the averages are computed with the initial invariant distribution $\rho(Q,P,t_0)$.

We can now compute the response of the central system to the environment. For the Hamiltonian (1) the HO satisfies the equation

$$\ddot{q} + \omega_0^2 q = -\frac{\lambda_N}{m} \sum_{n=1}^N x_n(t) \equiv -\frac{\lambda_N}{m} X(t) \tag{14}$$

and is perturbed by the "external force" $F(t) = -\frac{\lambda_N}{m} X(t)$. If the dynamics of the bath coordinates $x_n(t)$ are chaotic, we may replace $X(t)$ by its average $\langle X(t)\rangle$:

$$\ddot{q} + \omega_0^2 q \approx -\frac{\lambda_N}{m} \langle X(t)\rangle. \tag{15}$$

Using Eq. (12),

$$\langle X(t)\rangle = \left\langle \sum_n x_n(t) \right\rangle_0 - \lambda_N \int_0^t \Phi_{XX}(t-s)q(s)ds, \tag{16}$$

where $\Phi_{XX}(t-s) = \langle \{X(t),X(s)\}\rangle_0$. The first term on the right-hand side of Eq. (16) is zero due to the parity of H_E. Using the pseudomicrocanonical as the initial invariant distribution, it follows that

$$\Phi_{XX}(t-s) = \sum_{n=1}^N \Phi_{xx}^{(n)}(t-s) = N \Phi_{xx}, \tag{17}$$

where

$$\Phi_{xx}(t-s) = \frac{5}{4E_{QS}(0)} \langle x(t)P_x(s)\rangle_0$$
$$+ \frac{t-s}{4E_{QS}(0)} \langle P_x(t)P_x(s)\rangle_0 \tag{18}$$

is the response function obtained by Bonanca [9] for the case of a single QS coupled with the oscillator. Substituting (17) in (16) and (15), we obtain

$$\ddot{q} + \omega_0^2 q \approx \frac{\lambda_N^2 N}{m} \int_0^t \Phi_{xx}(t-s)q(s)ds, \tag{19}$$

which shows that it is necessary to rescale the coupling constant as $\lambda_N = \lambda/\sqrt{N}$ in order for the external driving $F(t)$ to be independent of the size of the environment in the limit of large N.
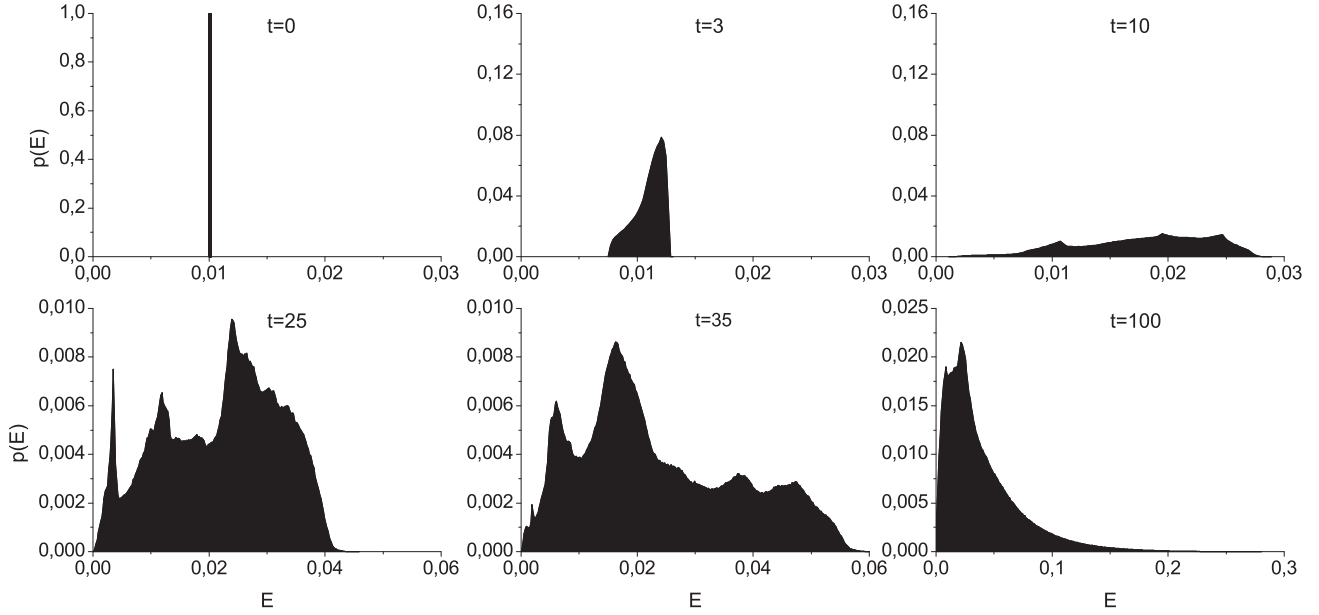
FIG. 5. Time evolution of the environment energy distribution. At $t = 0$, ρ_E is given by Eq. (5), with $E_{Cn} = 0.01$ for all QS's and $N = 100$. Note the different scales in the panels.

The approximation where the N chaotic systems forming the environment are treated independently correspond to ensemble averages of a single QS coupled to the central system. As we have seen, this approximation cannot describe the long time equilibration of the environment since this requires some interaction between the chaotic systems. Linear response theory cannot, therefore, describe the long-time behavior of the system. In particular, it cannot describe the change from an initial pseudomicrocanonical distribution to the final canonical distribution.

Figure 5 shows the breakdown of the pseudomicrocanonical distribution and its consecutive rearrangement to a Boltzmann-like distribution as a function of time. The transition between the distributions occurs at about 1/10 of the characteristic time for dissipation, as can be seen from Fig. 5 (see also Fig. 10). As pointed out in the last section, this redistribution of energy is promoted by the secondary interactions between the QS's.

This rapid equilibration on the environment hints at the possibility of using LRT to describe the long time behavior of the system by changing from the initial pseudomicrocanonical to the pseudocanonical distribution. If the initial temperature, as given by $\beta_{QS} = \overline{E}_{QS}$, does not change much during the time evolution, the environment will not be largely affected by the central system, and the interaction between the QS's can be ignored. Figure 6 shows an example with $N = 100$ and $\overline{E}_{QS}(0) = 0.01$ where the distribution of energy of the environment does not change significantly with time.

Therefore, for N sufficiently large, we can establish a regime where the central system dissipates energy exponentially but the environment's temperature remains nearly constant. This allows us to use the LRT to describe the system dynamics for much longer times. In order to do this we need to obtain explicit expressions for Eqs. (17) and (18)

for the pseudocanonical distribution. We write the correlation functions in (18) and the response function as

$$\sum_n x_n(t) P_{x_n}(s) = \frac{d}{ds} \sum_n x_n(t) x_n(s), \tag{20}$$

$$\sum_n P_{x_n}(t) P_{x_n}(s) = \frac{d^2}{dt\,ds} \sum_n x_n(t) x_n(s), \tag{21}$$

$$\Phi_{XX}(t-s) = \frac{5}{4} \frac{d}{ds} \left\langle \sum_n \frac{1}{E_{QS}^{(n)}(0)} x_n(t) x_n(s) \right\rangle_0$$

$$+ \frac{(t-s)}{4} \frac{d^2}{dt\,ds} \left\langle \sum_n \frac{1}{E_{QS}^{(n)}(0)} x_n(t) x_n(s) \right\rangle_0. \tag{22}$$
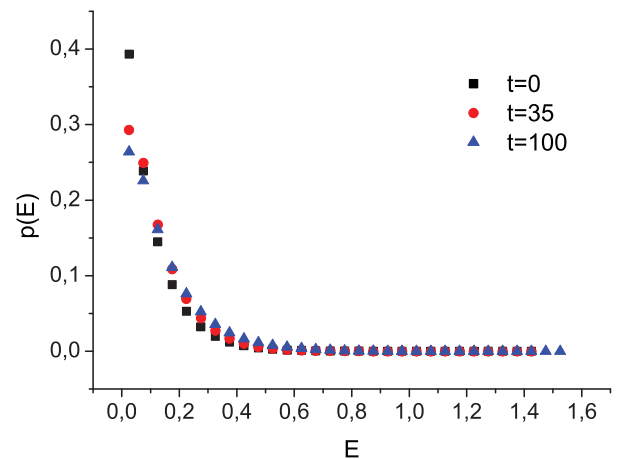

FIG. 6. (Color online) Energy distribution of the environment at different times for $N = 100$. At $t = 0$, $p(E) = \exp(-E/\overline{E}_{QS})/\overline{E}_{QS}$, with $\overline{E}_{QS} = 0.01$.
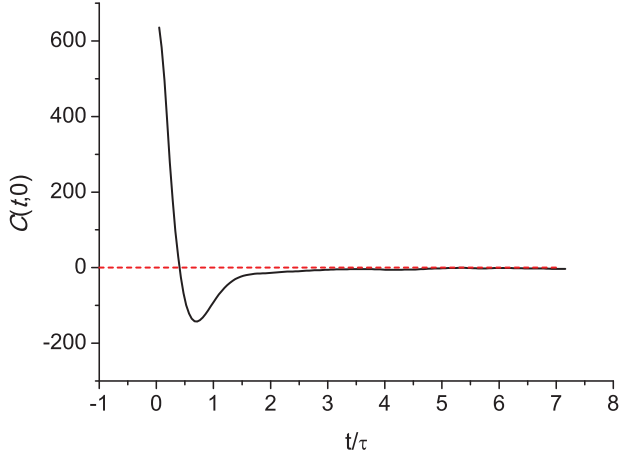
FIG. 7. (Color online) The total correlation function $C(t,0) = \sum_n c_n(t,0)$, with $n = 1, \ldots, N = 100$ QS's obeying a pseudocanonical distribution with $\overline{E}_{QS} = 0.01$.

The individual autocorrelation functions $c_n(t,s) = \frac{1}{E_{QS}^{(n)}} \langle x_n(t)x_n(s) \rangle_0$ were obtained numerically and follow the typical behavior expected for chaotic systems [36], which can be fitted by $c_n(t,s) = \sigma_n e^{-\xi_n(t-s)} \cos[\nu_n(t-s)]$. However, the total correlation function, $C(t,s) = \langle \sum_n \frac{1}{E_{QS}^{(n)}(0)} x_n(t)x_n(s) \rangle_0$, smooths out the oscillatory behavior exhibited by a single QS, as shown in Fig. 7. It is important to note that the time scale where $C(t,s)$ is significant is much smaller than the time scale of the dissipation, as can be seen by comparison with Fig. 2.

Therefore, in the time scale of dissipation, we can make the approximation

$$\left\langle \sum_{n=1}^{N} \frac{1}{E_{QS}^{(n)}(0)} x_n(t)x_n(s) \right\rangle_0 \approx \mu_{\overline{E}} N \delta(t-s), \qquad (23)$$

where $A_N = \mu_{\overline{E}} N$ is the maximal amplitude of $C_N(t,s)$. It turns out that A_N increases linearly with N, as shown in Fig. 8, and we call $\mu_{\overline{E}}$ the slope of such a linear function. The index $\overline{E}$ on $\mu_{\overline{E}}$ expresses the dependence with the mean energy of the reservoir, as we will see in a moment.

Equation (23) indicates that the presence of chaos introduces a fast memory loss in the microscopic dynamics of the reservoir, so that its dynamics can be described by a Markovian process, unlike the integrable and mixed regimes.

Using (23), we obtain the response function

$$\phi_{XX}(t-s) = \frac{5\mu_{\overline{E}} N}{4} \frac{d}{ds} \delta(t-s)$$

$$+ \frac{\mu_{\overline{E}} N (t-s)}{4 E_{QS}^{(n)}(0)} \frac{d^2}{dt\,ds} \delta(t-s). \qquad (24)$$

Substituting $\phi_{XX}(t-s)$ in Eqs. (15) and (16) and computing the integrals, we obtain

$$\ddot{q} + \omega_o^2 q + \gamma_T \dot{q} = 0, \qquad (25)$$

where

$$\gamma_T = \frac{7\lambda_N^2 \mu_{\overline{E}} N}{4m} = \frac{7\lambda^2 \mu_{\overline{E}}}{4m}. \qquad (26)$$
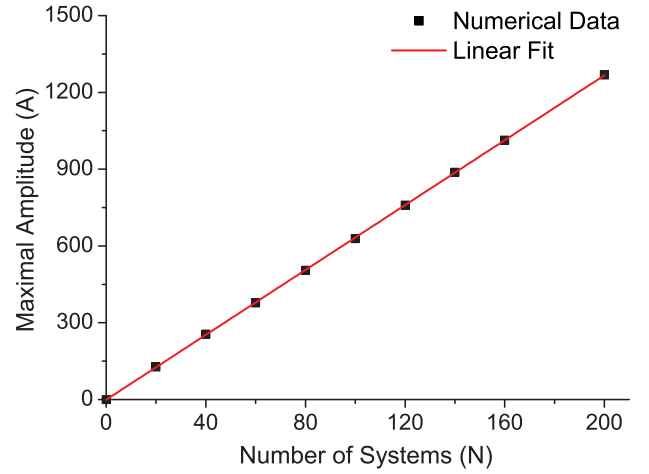

FIG. 8. (Color online) Linear dependence between the maximal amplitude of the total correlation function $C(t,0)$ and the number of QS's in the environment. The slope depends on the environment's mean energy.

The theoretical value γ_T of the inverse characteristic time decreases exponentially with $\overline{E}$, as shown in Fig. 9 (red circles). The numerically obtained values, on the other hand, display a more complex behavior, as shown by the black squares in Fig. 9. The numerical results can be divided into two regimes: in the first, $0 < \overline{E} < 0.01$ and $\gamma(\overline{E})$ increases with $\overline{E}$ and, in the second regime, $\overline{E} > 0.01$ and $\gamma(\overline{E})$ decreases with $\overline{E}$.

The discrepancies between the numerical and LRT results can be understood if we note that, in the first regime, where $\overline{E}$ is small, the QS's are very slow. Since the value of γ_T is related to the maximal amplitude of the correlation $C(t,s)$, this value increases considerably in these regimes because changes in the state of the quartic oscillators are slow. This, in turn, implies a superestimated value for γ_T. In this regime, as $\overline{E}$ increases, the influence of the environment on the HO increases, facilitating the flow of energy and increasing γ_T. This is a nontrivial effect since an environment with lower mean energy should correspond to a cooler thermal bath, with a greater dissipation
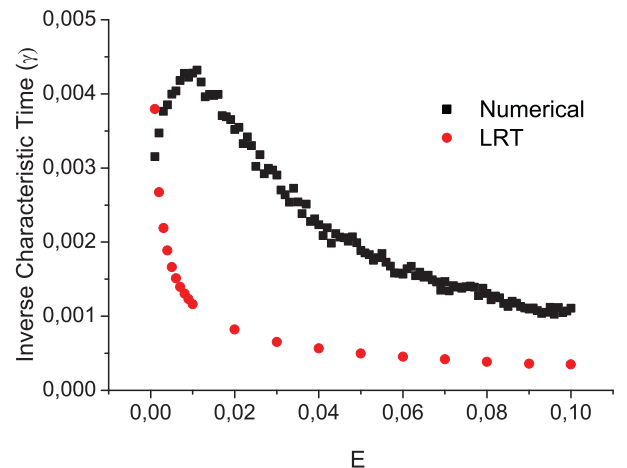

FIG. 9. (Color online) The inverse characteristic time as a function of the environment's mean energy for the theoretical model (γ_T, red circles) and numerical data (γ, black squares).
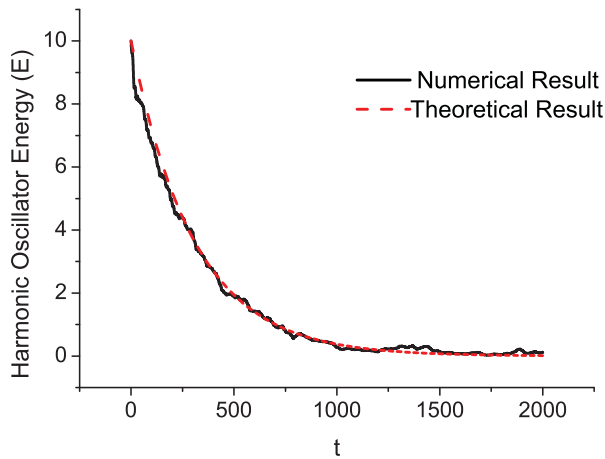
FIG. 10. (Color online) Comparison between the numerical results (solid black line) and the theoretical model (dashed red line) for $N = 100$ and $\kappa_{\overline{E}} = 3.68$.

rate. This shows that the finite set of QS's cannot be traded by a thermal bath at very low energies.

After a threshold is passed, γ recovers the expected decreasing behavior, and the association between the mean energy and temperature is restored. Even in this case, where the qualitative behavior of γ_T and γ is the same, the numerical value of the coefficients needs some correction. We define the parameter $\kappa_{\overline{E}} = \gamma/\gamma_T$ in order to adjust the theoretical model with numerical results. When this is done, the numerical and theoretical curves fall on top of each other, as shown in Fig. 10.

V. CONCLUSIONS

We constructed a simple model of classical dissipation where a harmonic oscillator interacts with an environment consisting of N identical quartic subsystems with two degrees of freedom. Interactions among the subsystems occur via the harmonic oscillator and are, therefore, of second order in the coupling constant. In order to compare results with different numbers of QS's we rescaled the coupling constant according to $\lambda_N = \lambda/\sqrt{N}$.

The main result of this paper is that a small chaotic environment can indeed behave as an effective infinite reservoir, promoting energy flow and equilibration with smaller systems in a single realization of the dynamics. This is to be contrasted with previous results where a single chaotic system plays the role of the environment and thermodynamical behavior is achieved via ensemble averaging over many realizations with random initial conditions [9]. For the same values of parameters the results with $N = 100$ are roughly equivalent to averaging over 30 000 initial conditions. Comparing our model environment with those composed of harmonic oscillators with linear distribution of frequencies [10,11], the number of oscillators needed to mimic an infinite reservoir is at least an order of magnitude larger than needed for chaotic QS's.

At long times the environment and the HO equilibrate, and their energy distributions converge to Boltzmann-like curves with the same temperature. The equilibration of the environment depends on the indirect interactions among its members that occur via the HO.

We have also developed a linear response theory to describe the system. We have shown, in particular, that the scaling of the coupling constant with N is indeed correct and that the dissipation constant γ depends quadratically on λ.

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