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Ergodic transitions in continuous-time random walks

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We consider continuous-time random walk models described by arbitrary sojourn time probability density functions. We find a general expression for the distribution of time-averaged observables for such systems, generalizing some recent results presented in the literature. For the case where sojourn times are identically distributed independent random variables, our results shed some light on the recently proposed transitions between ergodic and weakly nonergodic regimes. On the other hand, for the case of nonidentical trapping time densities over the lattice points, the distribution of time-averaged observables reveals that such systems are typically nonergodic, in agreement with some recent experimental evidences on the statistics of blinking quantum dots. Some explicit examples are considered in detail. Our results are independent of the lattice topology and dimensionality.

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I. INTRODUCTION

Boltzmann's hypothesis of ergodicity is a central concept in statistical mechanics. Roughly speaking, in an ergodic system, for a long time observation, the residence time of a trajectory in a given region of the phase space is proportional to the volume measure of the region. Despite the high success of Boltzmann's description of large systems, the ergodic hypothesis cannot be used, for instance, for systems whose phase space can be subdivided in mutually inaccessible regions. A subtler physical scenario of nonergodicity was introduced by Bouchaud [1] in the context of glass dynamics: the so-called *weakly nonergodic* systems are also nonergodic, but their phase spaces are not subdivided in mutually inaccessible regions. Recently, there has been great interest [2–4], in particular, in the weak ergodicity breaking phenomenon, where a transition to an ergodic phase may occur. Notice that Bouchaud's ideas about weak ergodicity breaking are, in turn, closely related to concepts that have been previously considered in the mathematical literature about ergodic theory and stochastic processes, see, for instance [5]. In particular, a weakly nonergodic regime corresponds to a situation in which the state space of a (semi)-Markov process is connected, i.e., any state can be reached from any other state with finite probability in a finite number of steps, but the fraction of occupation time for a given state is not equal to its invariant spatial measure.

From the physical point of view, weakly nonergodic systems have proved to be relevant in many applications as, for instance, complex networks [6], weak turbulence [7], and, in particular, they are at the basis for the statistical modeling of atoms trapping by laser cooling devices [8,9]. In these models, the atom quantum dynamics are equivalent to a classical random walk in the momentum space, where the standard deviation of the jump lengths Δp is of the same order of the incident photon momentum. The trapping process consists effectively in successive frontal collisions, for some given time interval, between the atom and the resonant laser photons, in a process called subrecoil cooling. Statistically, it is equivalent to the application of a controllable potential jump rate $R(p)$, responsible for the trapping near $p=0$. The resulting scenario is a kind of continuous-time random walk where rare events play a dominant role, with Lévy type non-Gaussian probability density functions (PDFs) $\psi(\tau) \sim A_\alpha \tau^{-(1+\alpha)}$ governing the trapping times. The divergence of the average trapping time

$$\bar{\tau} = E(\tau) = \int_0^\infty \tau\psi(\tau)d\tau \tag{1}$$

for $0<\alpha<1$, with $E(\,)$ denoting the expectation value with respect to the PDF $\psi(\tau)$, is precisely at the root of the subrecoil cooling mechanism effectiveness. Such mechanisms are typically nonergodic, and the origin of the nonergodicity is usually attributed to the divergence of the average trapping time [8].

We investigate here the PDF of time-averaged observables for a general class of continuous-time random walk models (CTRW), extending considerably the classes already considered in [2]. The nonergodic properties of CTRW models are studied here by comparing PDF of ensemble averages with fluctuations of time averages. Initially, we consider CTRW models described by an arbitrary trapping time PDF $\psi(\tau)$, in which all the sojourn times τ of the L lattice points $1 \le x \le L$ are identically distributed independent random variables. We show that models with finite $\bar{\tau}$ are always ergodic. In the cases where $\bar{\tau}$ diverges mildly, one still has ergodicity, but for stronger divergences the dynamical regime is weakly nonergodic and characterized by a Lamperti type PDF of time-averaged observables, in agreement with the recent results announced in [2]. We also extend our approach to include CTRW models with non identical trapping time PDF over the lattice points. For such cases, we show that even models with $\bar{\tau}$ finite for all lattice points are typically nonergodic. The obtained PDF of time-averaged ob-

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servables clarify some nonergodic features reported recently for blinking quantum dots systems [10,11].

The paper is organized as follows. In Sec. II, we provide the basic concepts and definitions required to describe ergodic and nonergodic properties of CTRW models. In Sec. III, the distribution density of time-averaged observables for the CTRW model with identical trapping time PDF is obtained. Section IV is devoted to discuss the weak ergodicity breaking phenomenon, extending some of the results obtained in [2]. Some explicit examples of PDF are considered in Sec. V. In Sec. VI, we investigate the CTRW with non identical trapping time PDF. We conclude in Sec. VII with some remarks about possible universal aspect of nonergodic fluctuations close to the weak ergodicity breaking.

II. STATISTICAL APPROACH FOR CTRW MODELS

Here, we consider a general CTRW model on a lattice with points $x=1,\ldots,L$, subject to certain trapping statistics. The topology and the dimensionality of the lattice are irrelevant for our purposes. The notion of random waiting time between successive steps was originally introduced by Montroll and Weiss [12]. In the model discussed here, the particle can jump to one of its nearest neighbors after waiting for a random time τ. The trapping statistics is given by the waiting time $t_x=\sum_{i=1}^{n_x}\tau_i$, where τ_i is the ith sojourn time of a given lattice site x. The set $\{\tau_1,\ldots,\tau_{n_x}\}$ for $1\le x\le L$ is composed of non-negative and identically distributed independent random variables with a common and arbitrary PDF $\psi(\tau)$. As in Ref. [2], we also consider here $n_x=nP_x^{eq}$ for the equilibrium regime of the random walk, which is reached after a sufficiently large number of jumps n. The Laplace transform for the random variable t_x is given by

$$E[\exp(-ut_x)]=\prod_{i=1}^{n_x}\int_0^\infty d\tau_i\psi(\tau_i)e^{-u\tau_i}=\exp[-nP_x^{eq}h(u)]. \tag{2}$$

Evidently, $h(u)$ is a monotonically increasing function for $u>0$, with $h(0)=0$.

In order to investigate the ergodic phase transitions for the trapping process, it is interesting to introduce the generating function $\hat{\rho}(\xi)$ developed in Ref. [13] and successfully used in Ref. [2] for describing time-average observables. We will assume here, however, that the generating function depends on an additional parameter $\beta>0$ as follows:

$$\hat{\rho}(\beta,\xi)=E[(\beta+\xi[\mathcal{O}])^{-1}], \tag{3}$$

where

$$[\mathcal{O}]=\frac{\sum_{j=1}^{L}\mathcal{O}_j t_j}{\sum_{j=1}^{L}t_j}, \tag{4}$$

with the square bracket denoting, hereafter, the average of a given operator with respect to the set $\{t_1,\ldots,t_L\}$. The generating function (3) is related to the corresponding density function of time-average observable $\bar{\mathcal{O}}$ as

$$\rho(\bar{\mathcal{O}})=E[\delta(\bar{\mathcal{O}}-[\mathcal{O}])]$$
$$=-\frac{1}{\pi}\lim_{\epsilon\to 0}\mathrm{Im}\frac{1}{\bar{\mathcal{O}}+i\epsilon}\lim_{\beta\to 1}\hat{\rho}\left(\beta,\frac{-1}{\bar{\mathcal{O}}+i\epsilon}\right). \tag{5}$$

Notice that ξ must always be chosen in order to assure that $\beta+\xi[\mathcal{O}]>0$ for any set $\{t_1,\ldots,t_L\}$.

Ergodicity and weak nonergodicity

Physically, we introduce the idea of a weakly nonergodic behavior by demanding that the lattice be not subdivided, from the dynamical point of view, in mutually inaccessible regions. This is equivalent to impose that [2]

$$0<P_x^{eq}<1 \quad \text{for all} \quad 1\le x\le L. \tag{6}$$

This condition is common to ergodic and weakly nonergodic systems. In fact, we say that a CTRW is ergodic if the condition (6) holds and, besides, the PDF [Eq. (5)] could be written as $\rho(\bar{\mathcal{O}})=\rho_E(\bar{\mathcal{O}})\equiv\delta(\bar{\mathcal{O}}-\langle\mathcal{O}\rangle)$, where

$$\langle\mathcal{O}\rangle=\sum_{j=1}^{L}P_j^{eq}\mathcal{O}_j \tag{7}$$

will denote, hereafter, the equilibrium ensemble average, being $\mathcal{O}_j$ the value of the observable $\mathcal{O}$ when the particle is at the lattice point j. The PDF for the ergodic regime can be rewritten as

$$\rho_E(\bar{\mathcal{O}})=-\frac{1}{\pi}\lim_{\epsilon\to 0}\mathrm{Im}\frac{1}{\bar{\mathcal{O}}+i\epsilon}\lim_{\beta\to 1}\left(\beta-\frac{\langle\mathcal{O}\rangle}{\bar{\mathcal{O}}+i\epsilon}\right)^{-1}. \tag{8}$$

Combining Eqs. (5) and (8) we obtain the corresponding ergodic generating function,

$$\hat{\rho}_E(\beta,\xi)=\frac{1}{\beta+\xi\langle\mathcal{O}\rangle}. \tag{9}$$

The generating function (9) will be used hereafter as an unequivocal mark of ergodicity. In fact, we call weakly nonergodic a system which obeys [Eq. (6)] but for which the generating function (5) is not equivalent to Eq. (9). In other words, a weakly nonergodic system is a system where it is possible to get to any state from any state, but still time and ensemble averages do not coincide.

III. DISTRIBUTION DENSITIES OF TIME AVERAGED OBSERVABLES

The generating function defined by Eq. (3) can be rewritten in the following form:

$$\hat{\rho}(\beta,\xi) = \int_0^\infty ds \int_0^\infty dt \int_0^\infty dt_1 \Psi(t_1) \ldots \int_0^\infty dt_L \Psi(t_L) \times \delta\left(t - \sum_{j=1}^{L} t_j\right) e^{-(\beta+\xi[\mathcal{O}])s}, \quad (10)$$

where $\Psi(t_x)$ is the corresponding PDF for t_x. We also demand

$$\mathcal{U}_j = \beta + \xi \mathcal{O}_j > 0, \quad 1 \le j \le L, \quad (11)$$

in order to guarantee that $\beta+\xi[\mathcal{O}]>0$. Now, by using the Laplace transformation given by Eq. (2) and taking into account that

$$\delta\left(t - \sum_{j=1}^{L} t_j\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ik(t-\Sigma_{j=1}^{L} t_j)}. \quad (12)$$

Equation (10) can be cast, after successive changes of variables similar to those one performed in Ref. [2], in the form

$$\hat{\rho}(\beta,\xi) = -\int_0^\infty \int_{-\infty}^\infty ds dk \frac{e^{-n\Sigma_{j=1}^{L} P_j^{eq} h(-ik+\xi\mathcal{O}_j s)}}{2\pi(k-i\beta s)^2}. \quad (13)$$

The k-integration can be performed by using the Cauchy formula, leading finally to

$$\hat{\rho}(\mathcal{U}) = -\int_0^\infty ds \frac{1}{s}\frac{d}{d\beta} e^{-n\langle h(s\mathcal{U})\rangle}. \quad (14)$$

Equation (14) is similar to that one found in Ref. [2] for the particular, but very important, case of one-sided Lévy PDF, for which $h(u)=Cu^{\alpha}$, with $0<\alpha<1$. Here, however, we did not use either the generalized central limit theorem or scaling hypothesis.

In order to solve Eq. (14) for arbitrary functions $h(u)$, we introduce a new parametrization for the integral

$$\mu(s) = \langle h(s\mathcal{U})\rangle, \quad (15)$$

from which it is always possible to obtain $s=s(\mu)$ since

$$\frac{d\mu}{ds} = \langle \mathcal{U} h'(s\mathcal{U}\rangle \geqslant 0. \quad (16)$$

Notice that $\mu(0)=0$ and $\mu(\infty)=\mu_{\max}$. With the new parametrization, Eq. (14) reads

$$\hat{\rho}(\mathcal{U}) = n\int_0^{\mu_{\max}} f(\mu) e^{-n\mu} d\mu, \quad (17)$$

where

$$f(\mu) = \frac{\langle h'(s(\mu)\mathcal{U})\rangle}{\langle \mathcal{U} h'(s(\mu)\mathcal{U})\rangle}. \quad (18)$$

The integral [Eq. (17)] can be evaluated in the limit $n\to\infty$, leading to (see Appendix for details)

$$\hat{\rho}(\mathcal{U}) = \lim_{\mu\to 0^+} f(\mu) = \lim_{s\to 0^+} \frac{\langle h'(s\mathcal{U})\rangle}{\langle \mathcal{U} h'(s\mathcal{U})\rangle}, \quad (19)$$

from which the PDF of time-averaged observables $\rho(\bar{\mathcal{O}})$ can be obtained by using Eq. (5), generalizing the previous results obtained in [2]. Notice that Eq. (19) depends only on the behavior of $h(u)$ near $u=0^+$, confirming that only the asymptotic tail behavior of $\psi(\tau)$ is relevant in the limit $n\to\infty$. It should also be emphasized that the sojourn time τ is a non-negative unbounded random variable. Hence, only functions $\exp[-h(u)]$ that are Laplace transforms of PDF $\psi(\tau)$ with support on $[0,\infty)$ are relevant here. According to Bernstein's theorem [14], the functions $\exp(-h(u))$ must be completely monotonic, i.e., they should obey

$$(-1)^k \frac{d^k}{du^k}\exp[-h(u)] \ge 0, \quad u > 0. \quad (20)$$

IV. ERGODIC AND WEAKLY NONERGODIC REGIMES

The first conclusion that one can draw from Eq. (19) is that systems with finite average trapping time $\bar{\tau}$ are ergodic since for such cases $\lim_{u\to 0^+} h'(u)=\bar{\tau}$, implying that Eq. (19) reduces to Eq. (9). On the other hand, the relevant weakly nonergodic models with diverging average trapping time are those ones considered in [2], for which $h(u)=Cu^{\alpha}$, for small non negative u, with $0<\alpha<1$. The associate PDF in this case are the well known stable Lévy densities, for which the asymptotic behavior for large τ is given by

$$\psi(\tau) \sim A_\alpha \tau^{-(1+\alpha)}. \quad (21)$$

For such models, Eq. (19) reads simply

$$\hat{\rho}(\mathcal{U}) = \frac{\langle \mathcal{U}^{\alpha-1}\rangle}{\langle \mathcal{U}^{\alpha}\rangle}, \quad (22)$$

leading to a density function of Lamperti type [2], where the ergodic regime can be recovered in the limit $\alpha\to 1^-$. Interestingly enough, many other subtler ergodic solutions do also exist. This is the case, for instance, of the function $h(u)=-Cu\log u$ for small u, for which Eq. (19) also reduces to Eq. (9). This case corresponds namely to a PDF that asymptotically tends to the Lévy density [Eq. (21)] with $\alpha=1$.

Since PDF with finite average sojourn time $\bar{\tau}$ give rise to ergodic behavior, it would be worthy to classify the possible PDF $\psi(\tau)$ with diverging $\bar{\tau}$ in order to identify possible nonergodic regimes. Since one must demand $E(1)=1$, all PDF $\psi(\tau)$ shall asymptotically decrease faster than τ^{-1}. On the other hand, in order to have $\bar{\tau}=E(\tau)$ diverging, $\psi(\tau)$ cannot decrease faster than τ^{-2}. Hence, a PDF $\psi(\tau)$ with diverging average sojourn time $\bar{\tau}$ must obey

$$\frac{A_1}{\tau^2} \le \psi(\tau) < \frac{A_0}{\tau}, \quad (23)$$

for large τ, with A_0 and A_1 arbitrary positive constants. The central point here is that the class of powerlike functions like $h(u)=Cu^{\alpha}$, with $0<\alpha<1$, which correspond to the Levy distributions [Eq. (21)], does not exhaust all the interval [Eq. (23)]. In particular, the lower bound of the interval does not correspond to any of these functions. As we have already shown, such case (ergodic and with $\alpha=1$) corresponds indeed to the function $h(u)=-Cu\log u$ for small u. PDF with asymptotic behavior of the type

$$\psi(\tau) \sim \frac{B_\gamma}{\tau \log^\gamma \tau}, \tag{24}$$

where B_γ and $\gamma>1$ are constants, obey Eq. (23). They, in fact, accumulate in the upper bound. On the other hand, PDF of the type

$$\psi(\tau) \sim C_\nu \frac{\log^\nu \tau}{\tau^2}, \tag{25}$$

where C_ν and $\nu>0$ are constants, also belong to the interval and accumulate in the lower bound.

In order to extend our analysis to consider the PDF with logarithmic terms are those ones of Eqs. (24) and (25), one can make use of Karamata's Abelian and Tauberian theorems [14] for the Laplace-Stieltjes transform

$$e^{-h(u)} = \int_0^\infty e^{-u\tau} d\Upsilon(\tau), \tag{26}$$

for $u>0$, which states that

$$e^{-h(u)} \sim \Gamma(\rho+1)\Upsilon\left(\frac{1}{u}\right) \tag{27}$$

for $u\to 0+$, if

$$\lim_{\tau\to\infty} \frac{\Upsilon(\tau x)}{\Upsilon(\tau)} \to x^\rho. \tag{28}$$

In the present case, since $\Upsilon(\tau)$ is the cumulative distribution function associated to $\psi(\tau)$, the condition (28) is automatically fulfilled with $\rho=0$. Hence, from Eq. (27), we have

$$h(u) \sim 1 - \Upsilon\left(\frac{1}{u}\right) \tag{29}$$

for $u\to 0+$.

For the PDF [Eq. (24)], we have, by using Eq. (29)

$$h(u) \sim \frac{B_\gamma}{\gamma-1} \frac{1}{|\log u|^{\gamma-1}}, \tag{30}$$

for small u and $\gamma>1$. For this case, we have $h'(u) = B_\gamma/(u|\log u|^\gamma)$ for small u, implying, from Eq. (19), that

$$\hat{\rho}(\mathcal{U}) = \langle \mathcal{U}^{-1} \rangle, \tag{31}$$

for any value of $\gamma>1$, which, interestingly, coincides with the limit $\alpha\to 0$ of the Lampertian case given by Eq. (22). For the PDF [Eq. (25)], on the other hand, we have

$$h(u) \sim C_\nu u |\log u|^\nu, \tag{32}$$

for small u and $\nu>0$, which give rises to a ergodic generating function

$$\hat{\rho}(\mathcal{U}) = \langle \mathcal{U} \rangle^{-1}, \tag{33}$$

for any $\nu>0$, which, of course, also coincides with the limit $\alpha\to 1$ of the Lampertian case [Eq. (22)]. The Lampertian generating function (22), with $0\le\alpha\le 1$, seems to be enough to describe any CTRW of the type considered up to here.

V. STABLE PDF

Any physical application of the preceding sections results would require, of course, stable PDF $\psi(\tau)$. We recall that a stable PDF $S_\alpha(\tau;\sigma,\beta,\mu)$ is characterized by four parameters: $0<\alpha\le 2$, which determines the asymptotic falling tails; $\sigma\ge 0$ is the corresponding scale; $-1\le\beta\le 1$ being the skewness; and μ the shift parameter. The support of a generic stable density is given by [15]

$$\begin{cases} [\mu,\infty), & 0<\alpha<1, \quad \beta=1, \\ (-\infty,\mu], & 0<\alpha<1, \quad \beta=-1, \\ (-\infty,+\infty), & \text{otherwise}. \end{cases} \tag{34}$$

Since τ is an unbounded non-negative random variable we disregard the case $\beta=-1$. The natural stable PDF in this case are

$$\psi(\tau) = S_\alpha(\tau;\sigma,1,0), \tag{35}$$

with $0<\alpha<1$, which have the asymptotic behavior given by Eq. (21). There are, however, several other useful possibilities.

Let us redefine the support of a stable PDF by taking, first, $\mu=C>0$ and then $\tau\to|\tau-C|$ so that, for $\beta\ne -1$, one has

$$\psi(\tau) = \begin{cases} N S_\alpha(|\tau-C|;\sigma,\beta,0), & \text{for} \quad \tau\ge 0, \\ 0, & \text{for} \quad \tau<0, \end{cases} \tag{36}$$

where N is the pertinent normalization constant and $0<\alpha<2$. The general asymptotic behavior of the PDF [Eq. (36)] is given by Eq. (21) [15]. The average sojourn time $\bar{\tau}=E(\tau)$ diverges as $\tau^{1-\alpha}$ for $0<\alpha<1$, and logarithmically for $\alpha=1$. According to the results of the previous section, the cases for which $\bar{\tau}$ is finite are ergodic. Some explicit examples will help to illustrate our main results.

A. Non-negative delta sequence PDF

Consider the non-negative delta sequence PDF on $[0,\infty)$ given by

$$\psi(\tau) = N \frac{\sin^2 \gamma(\tau-C)}{\gamma(\tau-C)^2}, \tag{37}$$

where N is the appropriate normalization constant and γ and C are positive parameters. PDF [Eq. (37)] is the one-sided version of the density defined in Ref. [16]. In the limit $\gamma\to\infty$, it corresponds to a delta function centered in $\tau=C$. The function $h(u)$ associated to Eq. (37) is given, for small non-negative u, by

$$h(u) = \left(D - N\log\left(\frac{u}{2\gamma}\right)\right)\frac{u}{2\gamma} + O\left(\left(\frac{u}{2\gamma}\right)^2\right), \qquad (38)$$

where D is a constant depending on γ and C. Despite of having a diverging first moment $\bar{\tau}$, the PDF [Eq. (37)] gives rise to an ergodic regime since the dominant term in the function $h(u)$ for small u is precisely $-u\log u$, that it is known from the results of the last section to reduce Eq. (19) to the ergodic generating function (9).

B. Cauchy stable PDF

The Cauchy stable PDF, defined for $\tau \in [0,\infty)$ as

$$\psi(\tau) = N\frac{\sigma}{(\tau - C)^2 + \sigma^2}, \qquad (39)$$

where N is the appropriate normalization constant and σ and C are positive parameters, is another example of a PDF with diverging $\bar{\tau}$ but that, nevertheless, gives rise to an ergodic regime. The function $h(u)$ in this case is given by

$$h(u) = [D - N\log(\sigma u)]\sigma u + O[(\sigma u)^2], \qquad (40)$$

for small and non-negative u, where D is a constant depending on C and σ. Again, we have a situation where the logarithmic term dominates and renders Eq. (19) to the ergodic expression (9).

C. Lévy stable PDF

Based on the one-sided PDF [Eq. (36)], let us consider the (α=1/2) Lévy density with support on $[0,\infty)$ given by

$$\psi(\tau) = N\sqrt{\frac{\sigma}{2\pi}}\frac{\exp\left(\frac{-\sigma}{2|\tau - C|}\right)}{|\tau - C|^{3/2}}, \qquad (41)$$

where N is the appropriate normalization constant and σ and C are positive parameters. The corresponding $h(u)$ function can be calculated for $u \to 0^+$ as

$$h(u) = N\sqrt{2\sigma u} + (C - D)u + O[(\sigma u)^{3/2}], \qquad (42)$$

where

$$D = 2N\left(\sqrt{\frac{\sigma C}{2\pi}}e^{-\sigma/2C} + (1 - N)\sigma\right). \qquad (43)$$

For large times, the term $(\sigma u)^{1/2}$ prevails, leading to the (α=1/2) Lamperti's statistics [Eq. (22)]. This result coincides with that one obtained in Ref. [2] for the PDF [Eq. (35)]. Note, however, that $\sigma \to 0$ leads to an ergodic regime, regardless of having α=1/2! This is a somewhat surprising result.

VI. NONIDENTICAL TRAPPING TIME PDF

So far we have considered only situations where a single trapping time PDF $\psi(\tau)$ is sufficient to describe the random dynamics. However, there are situations where the trapping times are not identically distributed over the lattice points. This is the case, for instance, of the fluorescence blinking observed in some colloidal nanocrystals, namely, the case of certain quantum dots systems [10,11]. When a laser pulse reaches these systems, their dots fluorescence intensity randomly switch between bright (on) and dull (off) states. The blinking quantum dots are typically characterized by means of the statistics of on/off times, whose distributions exhibit power law decay $\psi_k(\tau) \sim A_k\tau^{-(1+\alpha_k)}$, where k corresponds to the on/off states and $0 < \alpha_k < 1$. Nonergodicity for such systems has been reported from experimental observations [10].

TABLE I. Some possible non-homogeneous PDF over the lattice. The first column depicts the behavior of $h_j(u)$ for small non-negative u, corresponding to the lattice point j. The second column has the associate PDF asymptotic behavior for large τ. The last column shows how the average sojourn time diverges. We assume that C_j are positive constants and $0 < \alpha < 1$.

$h_j(u)$	$\psi_j(\tau)$	$\bar{\tau}_j$
$C_j u$	Faster than τ^{-2}	Finite
$-C_j u \log u$	$A_j\tau^{-2}$	$\log\tau$
$C_j u^\alpha$	$A_j\tau^{-(1+\alpha)}$	$\tau^{1-\alpha}$

Our approach can be generalized to include such kind of situation. To this end, let us consider now a CTRW model described by a set of arbitrary trapping time PDF $\psi_j(\tau)$ governing the sojourn times τ_j of the L lattice points $1 \le j \le L$. In such non-homogeneous lattice, the sojourn times are not identically distributed anymore, but they are still independent random variables. We can extend all the results Secs. II and III essentially by replacing $h(u)$ by $h_j(u)$, resulting finally in

$$\hat{\rho}(\mathcal{U}) = \lim_{s \to 0^+}\frac{\sum_j^L P_j^{eq}h_j'(s\mathcal{U}_j)}{\sum_j^L P_j^{eq}\mathcal{U}_j h_j'(s\mathcal{U}_j)}, \qquad (44)$$

instead of Eq. (19). The first conclusion we get from Eq. (44) is that weak nonergodicity is also present in this case, since we might have nonergodic regimes for which $0 < P_j^{eq} < 1$ for all $1 \le j \le L$, assuring that the lattice is not dynamically subdivided in mutually inaccessible regions. Furthermore, we have also that the weak ergodicity breaking is not a structurally stable phenomenon since ergodic transitions are impossible for systems where the sojourn times τ are not identically distributed. Even for systems where the average values $\bar{\tau}_j$ are finite for all lattice points, but different, one has the predominance of nonergodicity. In order to illustrate this scenario, some explicit examples are useful. For any of the three cases depicted in Table I, we have that the expression for $\rho(\bar{\mathcal{O}})$ as defined by Eq. (5) are obtained from the corresponding single trapping time PDF cases by replacing $P_j^{eq} \to P_j^{eq}C_j$, i.e., by replacing

$$\langle\mathcal{O}\rangle \to \frac{\langle C\mathcal{O}\rangle}{\langle C\rangle}, \qquad (45)$$

what prevents any transition between weakly nonergodic and ergodic regimes. We can also conclude that the arising of narrow PDF does not necessarily mean that a system has reached the ergodic regime.

Another interesting dynamical aspect of the large time evolution of systems with nonhomogeneous PDF is the competition between the lattice points. Let us introduce the survival rate η_{kl} between two lattice points k and l, defined as

$$\eta_{kl} = \lim_{u\to 0^+} \frac{h_k'(u)}{h_l'(u)}. \quad (46)$$

The survival rate is, of course, related to the ratio $\bar{\tau}_k/\bar{\tau}_l$ for the lattices points k and l. Such relation, however, is not obvious for general PDF. Notice that

$$e^{-h_i(u)}h_i'(u) = \int_0^\infty e^{-u\tau} d\Lambda_i(\tau), \quad (47)$$

where

$$\Lambda_i(\tau) = \int_0^\tau s\psi_i(s)ds. \quad (48)$$

Provided that $\Lambda_i(\tau)$ obey a condition like Eq. (28), we have by using Karamata's theorem for Eq. (47)

$$h_i'(u) \sim \Gamma(\rho_i+1)\Lambda_i\left(\frac{1}{u}\right), \quad (49)$$

for $u\to 0^+$. Since $\Lambda_i(\infty)=\bar{\tau}_i$, we conclude that the survival rate [Eq. (46)] does not coincide, in general, with the ratio $\bar{\tau}_k/\bar{\tau}_l$. Nevertheless, the survival rate is indeed an appropriate quantity to compare the residence time of the CRWT dynamics in different points of the nonhomogeneous lattice.

According to Eq. (46), two types of behavior can occur. Both states coexist if η_{kl} is finite and non-negligible. If η_{kl} is infinite or vanishes, then, respectively, state k prevails over l, or vice versa. In the case where prevailing states exist, only them are relevant for the calculation of the density of time averages. Notice that even if a state k has a visitation fraction negligible when compared to another state l, i.e., $P_k^{eq} \ll P_l^{eq}$, state k can still prevail over state l if $\eta_{kl}=\infty$. Irrelevant states can be visited many times, but the time spent among them by the CTRW dynamics is negligible and, hence, they should not contribute to the time average. This is the mechanism behind the nonergodicity of nonhomogeneous lattices.

VII. CONCLUDING REMARKS

Weak ergodicity breaking has been investigated [2–4] by considering Levy PDF [Eq. (21)], for which the time-average densities are given by a Lamperti distribution by means of Eq. (22). For this class of PDF, the weakly nonergodic regime breaks up, giving origin to an ergodic one, in the limit $\alpha\to 1$. Despite that our results strongly indicate that Lamperti distributions seems to be enough to describe any time-average density for CTRW with PDF obeying Eq. (23), our results also show that the limit $\alpha\to 1$ is not the only way to obtain ergodic regimes. This fact is explicitly illustrated by the PDF given by Eq. (41), for which an ergodic regime arises for $\sigma\to 0$, irrespective of having $\alpha=1/2$ for such PDF. This results challenges the naive association between ergodicity/weak nonergodicity and diffusion/subdiffusion as suggested in [4], which indeed appears reasonable at first sight since models with PDF [Eq. (21)] typically exhibit anomalous diffusion characterized by $\langle x^2\rangle \sim t^\alpha$. The characterization of any possible universal behavior close to the weak ergodic breaking cannot by achieved by analyzing only the breaking associated with $\alpha\to 1$.

As to the case of nonidentical sojourn times over the lattice, the issue of weak ergodicity breaking is still more involved. Even for PDF with finite average sojourn times $\bar{\tau}_j$, we do not have an ergodic regime, in contrast with the case of homogeneous lattices, for which a finite $\bar{\tau}$ will necessarily imply in an ergodic regime. This nonergodic regime can be considered also as weakly nonergodic since, from Eq. (45), we have that the phase space is similar to the homogeneous case being, in particular, not subdivided in mutually inaccessible regions. Ergodicity, in this case, is recovered in the limit of equal PDF. This can be illustrated by a simple explicit example. Let us consider a Markov chain with only two states A and B and with the transition matrix

$$W = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \quad (50)$$

The corresponding invariant weights are $P_A^{eq}=P_B^{eq}=1/2$. Assume now that the sojourn times τ_A and τ_B are governed by the PDFs $\psi_A(\tau)=\lambda_A e^{-\lambda_A\tau}$ and $\psi_B(\tau)=\lambda_B e^{-\lambda_B\tau}$. After $n\gg 1$ steps, we will have $n_A\approx n_B\approx n/2$, but the visitation fraction of the dynamics in the two states will be in general different since

$$\frac{n_A E(\tau_A)}{n_B E(\tau_B)} \approx \frac{\lambda_B}{\lambda_A}. \quad (51)$$

For the case of diverging $\bar{\tau}_j$, besides of the consequences associated with Eq. (45), we have also new features associated with the possible divergence or vanishing of the survival rate [Eq. (46)]. This leads to the possibility of a new weakly nonergodic regime, for which, despite of the CTRW spreading over all lattice points, the time-averaged distributions will depend only on some points, namely the prevailing ones according to the survival rate [Eq. (46)]. Despite these points are not surprising from the probabilistic point of view, they certainly deserve a deeper physical investigation.

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APPENDIX: DERIVATION OF THE GENERATING FUNCTION

The first observation about the limit of large n of Eq. (17) is that one cannot apply Watson's lemma [17] directly since we cannot assure, *a priori*, if the derivatives of $f(\mu)$ are finite or not for $\mu\to 0^+$. Let us take, for instance, the first derivative

$$\lim_{\mu\to 0^+} f'(\mu) = \lim_{s\to 0^+}\left(\frac{\langle \mathcal{U}h''(s\mathcal{U})\rangle}{\langle \mathcal{U}h'(s\mathcal{U})\rangle^2} - \langle h'(s\mathcal{U})\rangle\frac{\langle \mathcal{U}^2h''(s\mathcal{U})\rangle}{\langle \mathcal{U}h'(s\mathcal{U})\rangle^3}\right) \tag{A1}$$

where Eq. (16) was used. Although, apparently, for all physically relevant functions $h(u)$ the limit [Eq. (A1)] is indeed finite, one cannot rule out, in principle, possible situations where it might diverge. Fortunately, thanks to the boundedness of $f(\mu)$, one can evaluate Eq. (17) without using Watson's lemma.

By introducing the new variable $v=n\mu$, Eq. (17) can be cast in the limit $n\to\infty$ as

$$\hat{\rho} = \lim_{n\to\infty}\int_0^{n\mu_{\max}} f\left(\frac{v}{n}\right)e^{-v}dv, \tag{A2}$$

with $f(\mu)\geq 0$ given by Eq. (18). First of all, let us suppose that $\mu_{\max}<\infty$. Equation (A2) can be decomposed in this case as

$$\hat{\rho} = \lim_{n\to\infty} I_0(n) + \lim_{n\to\infty} I_1(n), \tag{A3}$$

where

$$I_0(n) = \int_0^{\sqrt{n}\mu_{\max}} f\left(\frac{v}{n}\right)e^{-v}dv \tag{A4}$$

and

$$I_1(n) = \int_{\sqrt{n}\mu_{\max}}^{n\mu_{\max}} f\left(\frac{v}{n}\right)e^{-v}dv. \tag{A5}$$

For both integrals, we have

$$f_i^- g_i(n) \leq I_i(n) \leq f_i^+ g_i(n), \tag{A6}$$

with $i=0,1$, where

$$g_0(n) = 1 - e^{-\sqrt{n}\mu_{\max}}, \tag{A7}$$

$$g_1(n) = e^{-\sqrt{n}\mu_{\max}} - e^{-n\mu_{\max}}, \tag{A8}$$

$$f_i^- = \inf_{\mu\in\mathcal{I}_i} f(\mu), \tag{A9}$$

$$f_i^+ = \sup_{\mu\in\mathcal{I}_i} f(\mu), \tag{A10}$$

where $\mathcal{I}_0=[0,\frac{\mu_{\max}}{\sqrt{n}}]$ and $\mathcal{I}_1=[\frac{\mu_{\max}}{\sqrt{n}},\mu_{\max}]$. Since $f(\mu)$ is bounded, we have $f_1^-\leq f_1^+<\infty$. Taking into account that $\lim_{n\to\infty} g_1(n)=0$, one has from Eq. (A6) that $\lim_{n\to\infty} I_1(n)=0$. As to the integral $I_0(n)$, notice that $\lim_{n\to\infty} f_0^- =\lim_{n\to\infty} f_0^+=\lim_{\mu\to 0^+} f(\mu)$ and $\lim_{n\to\infty} g_0(n)=1$, implying finally from Eq. (A6) that

$$\hat{\rho} = \lim_{n\to\infty} n\int_0^{\mu_{\max}} f(\mu)e^{-n\mu}d\mu = \lim_{\mu\to 0^+} f(\mu). \tag{A11}$$

The cases for which $\mu_{\max}$ diverges can be treated in an analogous way, by choosing $\sqrt{n}\mu_*$ with finite μ_*, instead of $\sqrt{n}\mu_{\max}$, for the decomposition of Eq. (A2) into $I_0(n)$ and $I_1(n)$ integrals, leading to the same final result [Eq. (A11)].

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