Effective Coupling between Two Brownian Particles

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(Received 30 August 2006; published 18 December 2006)

We use the system-plus-reservoir approach to study the dynamics of a system composed of two independent Brownian particles. We present an extension of the well-known model of a bath of oscillators which is capable of inducing an effective coupling between the two particles depending on the choice made for the spectral function of the bath oscillators. The coupling is nonlinear in the variables of interest, and an exponential dependence on these variables is imposed in order to guarantee the translational invariance of the model if the two particles are not subject to any external potential. The effective equations of motion for the particles are obtained by the Laplace transform method, and, besides recovering all the local dynamical properties for each particle, we end up with an effective interaction potential between them. We explicitly analyze one of its possible forms.

DOI: 10.1103/PhysRevLett.97.250601 PACS numbers: 05.40.Jc

Usually, the Brownian motion of a given dynamical variable is modeled by considering the system it describes—the system of interest—coupled to a thermal bath responsible for its energy loss. Assuming that any degree of freedom of this environment is only weakly perturbed by the system of interest, we can describe it as a set of independent harmonic oscillators with a coupling which is bilinear in the reservoir and system variables and endowed with a particular spectral function [1]. This model has been successfully used to describe general properties, classical or quantum mechanical, of dissipative systems with only 1 degree of freedom subject to arbitrary potentials [1–5]. Indeed, it has been extensively shown in the literature that, within the range of interest, other approaches to dealing with dissipative systems described by a single dynamical variable always furnish us with the same results as those obtained by the bath of oscillators with a properly chosen spectral function. It is the case, for example, of the application of the collective coordinate method to describing the damped motion of quantum solitons [6] or microscopic attempts to describe more realistic systems such as the electron gas of a metallic environment [7–9].

However, despite all its success, there are certain dissipative systems for which the usual model of the bath of oscillators can be shown to be inappropriate to account for the physics we expect from them. Here it should be stressed that by the *usual model* we mean independent oscillators coupled *bilinearly* in coordinates to the system of interest.

Suppose one immerses two independent particles in the same medium, where each of them would separately behave as a Brownian particle. Since for each individual particle we could mimic the effect of the medium by the bath of oscillators, it would be very natural to try to generalize the model to cope with the presence of those two particles. This generalization is quite straightforward, and the only point where one should be a bit cautious is when introducing the well-known counterterm [1,2] in the generalized model. In order to do that in an unambiguous way, all one has to do is employ the equivalent model for the system-bath Lagrangian where the interaction is described by a velocity-coordinate coupling and perform the usual canonical transformations to achieve the desired coordinate-coordinate coupling with the appropriate counterterm [1,2] (see below). It is then a simple matter to obtain the equations of motion for each particle under the influence of the environment using, for example, the Laplace transform method. These are coupled Langevin equations that, when written in terms of the center of mass and relative coordinates q and u of the two particles, present a somewhat bizarre result: namely, the latter obeys a free particle equation of motion. Therefore, if we give an initial velocity to one of those particles, the other must instantaneously reply in such a way that the relative coordinate evolves linearly in time ignoring completely the presence of the environment. This is a very unexpected result to say the least.

It is our intention in this Letter to propose an extension of the usual model of a bath of oscillators in order to fix this deficiency. As we will see, on top of succeeding in so doing, we will also be able to describe an effective coupling between the two particles mediated by the presence of the bath, and the resulting interaction potential depends on the specific form of the spectral function of the environment oscillators. This effect reminds us of the formation of Cooper pairs or bipolarons in material systems due to the electron-phonon interaction. Notice that we will not address effective interactions between localized objects like a spatially separated two-state system as in Ref. [10], although our model can also be applied to this kind of problem.

In our generalization of the usual model, the system of interest with a single degree of freedom will be represented

by the free particle Lagrangian

$$L_S = \tfrac{1}{2}M\dot{x}^2. \tag{1}$$

The heat bath will be described as a symmetrized collection of independent harmonic modes,

$$L_R = \frac{1}{2}\sum_k m_k(\dot{R}_k\dot{R}_{-k} - \omega_k^2 R_k R_{-k}), \tag{2}$$

and for the coupling term we initially assume the interaction Lagrangian

$$L_I = -\sum_k \tilde{C}_k(x)\dot{R}_k. \tag{3}$$

Switching to the Hamiltonian formulation and performing the canonical transformation $P_k \to m_k\omega_k R_k$ and $R_k \to P_k/m_k\omega_k$, we can show that the coupling term transforms into

$$L_I = -\frac{1}{2}\sum_k (C_{-k}(x)R_k + C_k(x)R_{-k}) - \sum_k \frac{C_k(x)C_{-k}(x)}{2m_k\omega_k^2}, \tag{4}$$

where $C_k(x) \equiv \tilde{C}_k(x)\omega_k$. In order to represent the effect of a local interaction of the particle with a spatially homogeneous environment, we choose

$$C_k(x) = \kappa_k e^{ikx}. \tag{5}$$

With this choice, it is easy to show that the entire system is translationally invariant and, moreover, the coupling to the environment is homogeneous. If the particle is displaced by a distance d, the coupling term transforms into $C_{-k}(x+d)R_k = C_{-k}(x)e^{-ikd}R_k$, which suggests the definition of a new set of canonical variables as $\bar{R}_k = e^{-ikd}R_k$ that renders the total Lagrangian invariant. It is important to notice that with a coupling like (5), the potential renormalization in (4) is a constant and, therefore, does not contribute to the particle dynamics. A coupling like (5) appears, for example, in the interaction of a particle with a fermionic bath [8].

Now we write the Euler-Lagrange equations for all of the variables involved in the problem, take their Laplace transforms, and eliminate the bath variables, in order to find the following equation of motion for the system of interest:

$$M\ddot{x} + \int_0^t K(x(t) - x(t'), t - t')\dot{x}(t')dt' = F(t). \tag{6}$$

Here the nonlinear dissipation kernel, given by

$$K(r,\tau) = \sum_k \frac{k^2\kappa_k\kappa_{-k}}{m_k\omega_k^2}\cos kr\cos\omega_k\tau, \tag{7}$$

shows that the interaction with the thermal bath induces a systematic influence on the system which is nonlocal and noninstantaneous. The function $F(t)$ can be interpreted as a fluctuating force

$$F(t) = -\frac{\partial}{\partial x}\sum_k\Big\{(C_{-k}(x)\tilde{R}_k(0) + C_k(x)\tilde{R}_{-k}(0))\frac{\cos\omega_k t}{2} + (C_{-k}(x)\dot{R}_k(0) + C_k(x)\dot{R}_{-k}(0))\frac{\sin\omega_k t}{2\omega_k}\Big\}, \tag{8}$$

where $\tilde{R}_k = R_k + [C_k(x_0)/m_k\omega_k^2]$. $F(t)$ depends explicitly on the initial conditions of the bath variables, and its statistical properties are obtained from the initial state of the total system. Now we need to choose a suitable distribution of oscillators which, in the continuum limit, leads us to the Brownian motion. The kernel in (6) can be written as

$$K = \sum_k \int_0^\infty d\omega 2k^2\kappa_k\kappa_{-k}\frac{\mathrm{Im}\chi_k^{(0)}(\omega)}{\pi\omega}\cos k(x(t) - x(t')) \times \cos\omega(t - t'), \tag{9}$$

where $\mathrm{Im}\chi_k^{(0)}(\omega) = (\pi/2m_k\omega_k)\delta(\omega - \omega_k)$ is the imaginary part of the dynamical response $\chi_k^{(0)}(\omega)$ of a harmonic oscillator with wave number k. If we now assume that the bath oscillators are, in fact, only approximately noninteracting and replace *only* their response functions by those of damped oscillators $\chi_k(\omega)$, one has

$$\mathrm{Im}\,\chi_k(\omega) = \frac{\gamma_k\omega}{m_k[(\omega^2 - \omega_k^2)^2 + \omega^2\gamma_k^2]}, \tag{10}$$

where γ_k is the relaxation frequency of that oscillator.

Our main interest is to study the system for times longer than the typical time scale of the reservoir. In other words, we wish to study the low-frequency limit of Eq. (10) in which $\mathrm{Im}\chi_k(\omega) \propto \omega$, and, therefore, we can assume

$$\mathrm{Im}\,\chi_k(\omega) \approx f(k)\omega\theta(\Omega - \omega). \tag{11}$$

Here we have, as usual, introduced a high frequency cutoff Ω as the characteristic frequency of the bath. A functional dependence like (11) for the dynamical response of the bath has been employed in Refs. [8,9] for fermionic environments. The particular choice of the dynamical susceptibility of the bath allows us to separate its time and length scales and to obtain a Markov dynamics when we replace (11) in (9) and integrate with respect to ω taking the limit $\Omega \to \infty$. With these considerations the equation of motion (6) reads

$$M\ddot{x}(t) + \eta\dot{x}(t) = F(t), \tag{12}$$

where we have identified $\eta = \sum_k k^2\kappa_k\kappa_{-k}f(k)$. Notice that, with this modification we obtain a relation between the damping constant and some microscopic parameters of the oscillator bath. With the current prescription and supposing that the bath is initially in thermal equilibrium, it is easy to show that, for *high temperatures*, the fluctuating force $F(t)$ satisfies the relations $\langle F(t)\rangle = 0$ and $\langle F(t)F(t')\rangle = 2\eta k_B T\delta(t - t')$, which are characteristics of white noise. Notice that this is valid only if we assume

a classical distribution of oscillators as the initial state of the bath.

In conclusion, the system-reservoir model with nonlinear coupling presented here allows us to reproduce the result one would have obtained by coupling the particle of interest bilinearly to a bath of noninteracting harmonic oscillators with the spectral function $J(\omega) = \eta\omega$ [1,2].

Now we are going to study the dynamics of a system with 2 degrees of freedom immersed in a dissipative environment. In this case, the Lagrangian of the system of interest is

$$L_S = \tfrac{1}{2}M\dot{x}_1^2 + \tfrac{1}{2}M\dot{x}_2^2, \tag{13}$$

and the coupling term

$$L_I = -\frac{1}{2}\sum_k [(C_{-k}(x_1) + C_{-k}(x_2))R_k + (C_k(x_1) + C_k(x_2))R_{-k}]. \tag{14}$$

Notice that we have not included any counterterm in (14) since our system is translationally invariant. The equations of motion for this Lagrangian are then

$$M\ddot{x}_i + \int_0^t K(x_i(t) - x_i(t'), t - t')\dot{x}_i(t')dt' + \int_0^t K(x_i(t) - x_j(t'), t - t')\dot{x}_j(t')dt' + \frac{\partial}{\partial x_i}V(x_i(t) - x_j(t)) = F_i(t), \tag{15}$$

where $i \neq j = 1, 2$, the fluctuating force $F_i(t)$ has the form given by (8) replacing $x(t)$ by $x_i(t)$, and

$$V(r(t)) = -\sum_k \int_0^\infty d\omega 2\kappa_k\kappa_{-k}\frac{\mathrm{Im}\chi_k^{(0)}(\omega)}{\pi\omega}\cos kr(t). \tag{16}$$

In the long-time limit, the second term in (15) can be written as $\eta\dot{x}_i$, providing us with the usual dissipative term. The third term represents a cross-dissipative term that depends on the velocity of the second particle and the relative distance between them. The last term is clearly an effective interaction induced by the coupling with the thermal reservoir. To see the explicit form of these terms, we need to evaluate expressions (9) and (16) with $\mathrm{Im}\chi_k^{(0)}(\omega)$ replaced by (11).

Performing the frequency integrals and assuming a one-dimensional environment of length L, we can transform the k summations into integrals with the usual replacement $\sum_k \to \frac{L}{2\pi}\int dk$. In order to proceed with the evaluation of those integrals, it is convenient to define a function $g(k)$,

$$\eta g(k) \equiv \frac{L}{2\pi}\kappa_k\kappa_{-k}f(k), \tag{17}$$

which satisfies the condition $\int_0^\infty g(k)k^2dk = 1$ derived from the definition of the damping constant η [see (12)].

A reasonable choice for this function is

$$g(k) = Ae^{-k/k_0}, \tag{18}$$

where $A = 1/(2k_0^3)$ is the normalization constant and k_0 determines the characteristic length scale of the environment. For example, in the case of fermionic environments, k_0 is of the order of k_F [8]. One should bear in mind that the choice of $g(k)$ is guided either by the knowledge of the microscopic details of the environment or by some phenomenological input about the effective interaction between the two particles.

With the form chosen for $g(k)$, the kernel of the third term in (15) is, again only for high temperatures,

$$K(x_1(t) - x_2(t'), t - t') = 2\delta(t - t')\eta(u(t)), \tag{19}$$

where $u(t) = x_1(t) - x_2(t)$,

$$\eta(u(t)) = \eta\left(\frac{1}{(k_0^2u^2 + 1)^2} - \frac{4u^2k_0^2}{(k_0^2u^2 + 1)^3}\right), \tag{20}$$

and the effective potential reads

$$V(u(t)) = -\frac{2\Omega\eta}{\pi k_0^2(k_0^2u^2(t) + 1)}. \tag{21}$$

The strength of the effective potential depends on the characteristic length and time scales. Therefore, the contribution of this coupling to the dynamics of the Brownian particles is important only for times longer than the typical time scale of the reservoir and distances of the order of (or shorter than) its characteristic length k_0^{-1}. The fluctuating forces still satisfy the white noise properties, but now they present an additional property associated with the distance between the particles. The forces $F_1(t)$ and $F_2(t)$ are also spatially correlated; that is,

$$\langle F_1(t)F_2(t')\rangle = 2\eta(u(t))k_BT\delta(t - t'). \tag{22}$$

In Fig. 1, we see the noise correlation strength as a function of the distance between the particles. For short distances, the noise correlation has the standard Brownian behavior. However, for longer distances, the correlation function becomes negative, and this *anticorrelation* indu-

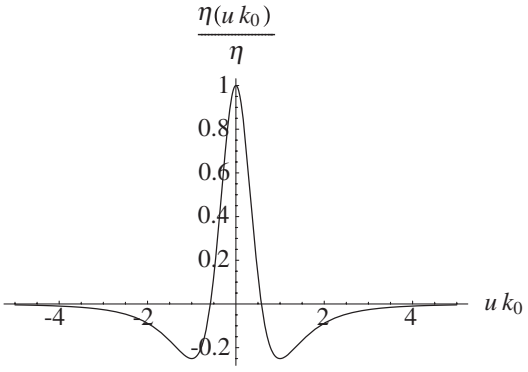

FIG. 1. Spatial correlation of F_1 and F_2.

ces an anomalous diffusive process in the system which will ultimately tend to normal diffusion once the particles are infinitely far apart.

In terms of the relative and center-of-mass coordinates, the equations of motion read

$$M\ddot{u}(t) + \eta\dot{u}(t) - \eta(u(t))\dot{u} + V'(u(t)) = F_u(t) \quad (23)$$

and

$$M\ddot{q}(t) + \eta\dot{q}(t) + \eta(u(t))\dot{q} = F_q(t), \quad (24)$$

where $F_u(t) = F_1(t) - F_2(t)$ and $F_q(t) = (F_1(t) + F_2(t))/2$. From the form of $V(u(t))$ and $\eta(u(t))$ and the statistical properties of the fluctuating forces $\langle F_u(t)\rangle = \langle F_q(t)\rangle = 0$, $\langle F_u(t)F_u(t')\rangle = 4kT(\eta - \eta(u))\delta(t - t')$, and $\langle F_q(t)F_q(t')\rangle = kT(\eta + \eta(u))\delta(t - t')$, it is evident that at large distances the equations of motion for the relative and center-of-mass coordinates represent the motion of Brownian particles with reduced mass $M/2$ and total mass $2M$, respectively. In general, the dissipation depends on the relative distance between the particles, and, for distances such that $uk_0 \ll 1$, we have up to first order in $u(t)$, $V'(u) \propto -u(t)$. In this approximation, both dissipation and fluctuations are negligible, and then we have an undamped oscillatory motion for $u(t)$.

In conclusion, we presented here a system-plus-reservoir model with a nonlinear coupling in the system coordinates which, in the adequate limit, allows us to reproduce the phenomenological results known for the dynamics of a Brownian particle. As the model is manifestly translation invariant, it turns out to become the natural candidate to describe the dissipative motion of a particle immersed in a homogeneous material medium. Actually, it can be shown that the exponential form employed here is the most general form of the interaction of a moving particle with the harmonic modes of the environment (see, for example, [8]). Moreover, in this generalization, the dissipation coefficient is still directly expressible in terms of a few parameters of the thermal bath that can be measured experimentally.

It should also be emphasized that the traditional bilinear model is the linearized version of the one we have just presented here, and, therefore, it is more appropriate to describe only local properties of the motion of a single particle in the environment.

Besides improving and generalizing a microscopic description of the Brownian motion, our model is still capable of inducing an effective coupling between two Brownian particles which arises from the nonlocal effects generated by the bath and depends on the choice of its dynamical susceptibility. Although the coupling between particles mediated by the medium where they are placed is quite common in many condensed matter systems (formation of Cooper pairs and bipolarons or RKKY interaction, just to name a few cases), there is no reason one should expect it to take place between two Brownian particles. So we have shown that the choice made for the behavior of the bath response function in order to reproduce the Brownian motion naturally generates this effective coupling. Actually, it is the separable k-dependent part of the susceptibility that gives the specific form of the static potential we have obtained here. Retarded effects will show up as one can no longer separate the k and ω dependence in that function. These nonlocal effects are also relevant for the single particle case but only in a non-Markovian approximation.

We should finally stress that we have reported only our findings for the classical or high temperature regime of the model. The quantum mechanical analysis of this system requires the evaluation of the reduced density operator for one or two particles of interest down to very low temperatures and will be presented elsewhere [11]. The extension of this study to deal with many Brownian particles might also shed some light on the behavior of complex many particle systems.

We kindly acknowledge Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and the Millenium Institute for Quantum Information for the financial support.

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