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Superluminal localized solutions to Maxwell equations propagating along a waveguide: The finite-energy case

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In a previous paper we have shown localized (nonevanescent) solutions to Maxwell equations to exist, which propagate without distortion with superluminal speed along normal-sized waveguides, and consist in trains of "X-shaped" beams. Those solutions possessed infinite energy. In this paper we show how to obtain, by contrast, *finite*-energy solutions, with the same localization and superluminality properties.

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I. INTRODUCTION: LOCALIZED SOLUTIONS TO THE WAVE EQUATION

As early as 1915 Bateman [1] showed that the Maxwell equations admit (besides the ordinary solutions endowed in vacuum with speed c) wavelet-type solutions, endowed in vacuum with group-velocities $0 \leqslant v \leqslant c$. But Bateman's work went practically unnoticed. Only a few authors, such as Barut *et al.* [2], followed such a research line; incidentally, Barut *et al.* constructed even a wavelet-type solution traveling with superluminal group velocity [3] $v > c$.

In more recent times, however, many authors discussed the fact that all (homogeneous) wave equations admit solutions with $0 < v < \infty$: see, e.g., Ref. [4]. Most of those authors confined themselves to investigating (subluminal or superluminal) *localized* nondispersive solutions in vacuum; namely, those solutions that were called "undistorted progressive waves" by Courant and Hilbert. Among localized solutions, the most interesting appeared to be the so-called "X-shaped" waves, which—predicted to exist even by special relativity in its extended version [5]—had been mathematically constructed by Lu and Greenleaf [6] for acoustic waves, and by Ziolkowski *et al.* [6], and later Recami [6], for electromagnetism. Let us recall that such X-shaped localized solutions are superluminal (i.e., travel with a group-velocity $v > c$ in the vacuum) in the electromagnetic case; and are "supersonic" (i.e., travel with a speed larger than the sound speed in the medium) in the acoustic case. The first authors to produce X-shaped waves *experimentally* were Lu and Greenleaf [7] for acoustics, Saari *et al.* [7] for optics, and Mugnai *et al.* for microwaves [7].

In a recent paper of ours, which appeared in this journal [8], we showed that solutions to the Maxwell equations exist that displace themselves with superluminal group velocity even along a normal waveguide, where one ordinarily expects to meet propagating, subluminal modes only. Actually, a segment of "undersized" waveguide constitutes an evanescence region [9], and evanescent waves are known to travel superluminally [5,9–11]; however, it was rather unexpected that (localized) waves could propagate superluminally down a normal-sized waveguide. In fact, the dispersion relation in undersized guides is $\omega^2/c^2 - \beta^2 = -K^2$, so that the standard formula $v \simeq d\omega/d\beta$ yields a $v > c$ group velocity [12]; by contrast, in normal guides the dispersion relation becomes $\omega^2/c^2 - \beta^2 = +K^2$, so that it seems to yield values $v < c$ only. Instead, in our paper [8] we have shown that localized solutions to the Maxwell equations do exist, propagating with $v > c$ even in normal waveguides; but their group-velocity v cannot be given by the approximate formula $v \simeq d\omega/d\beta$. (Let us recall that the group velocity is well defined only when the pulse has a clear bump in space; but it can be calculated by the approximate, elementary relation $v \simeq d\omega/d\beta$ only when ω as a function of β is also clearly bumped.)

II. THE INFINITE-ENERGY SOLUTIONS

In Ref. [8] we construed localized solutions to the Maxwell equations (which propagate undistorted, with superluminal group velocity down a *cylindrical* waveguide located along the z direction) for the TM (transverse magnetic) case and for a dispersion-free medium. The case with dispersion has been treated elsewhere [13], as well as the case of a coaxial cable [14]. Here, let us call attention to two points, which received just a mention in Ref. [8], with regard to Eq. (9) and Fig. 2 therein: (i) those solutions consist of *trains* of pulses (similar to the one depicted in Fig. 2 of Ref. [1]); (ii) each of such pulses is X-*shaped*, see our Fig. 1 below. Let us note, incidentally, that we are referring to the electromagnetic case, but the same would hold for all situations in which a fundamental role is played by the wave equation (as

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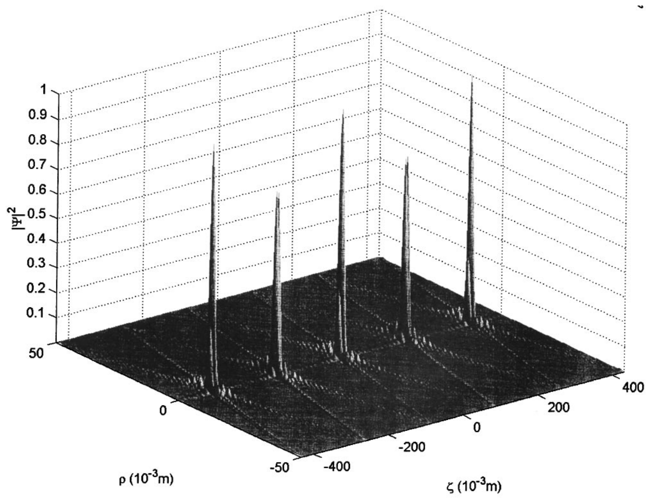

FIG. 1. This figure depicts one of our (infinite total-energy) localized solutions, given in Eq. (1). It consists, as expected, in a *train* of X-shaped waves; and propagates rigidly along the considered circular, normal-sized waveguide, with radius $r=5$ cm, with the superluminal speed $V=c/\cos\theta$. [Let us recall that any solution that depends on z (and t) only through the variable $\zeta\equiv z-Vt$ is rigidly moving with speed V.] In this figure $\theta=\pi/3$ has been chosen, while $\zeta\equiv z-Vt$. At last, we have adopted $N=22$, that is, the same value of N used by us in Ref. [8] (cf., for instance, Fig. 3 therein).

in acoustics, geophysics, gravitational waves, elementary particle physics, etc.).

In the case of cylindrical symmetry, let us consider a metallic waveguide [8] with radius r, and use the notations $\boldsymbol{\rho}\equiv(x,y)$, and $\rho=|\boldsymbol{\rho}|$, as well as the boundary condition $\Psi(\rho=r,z;t)=0$. In the previous paper [8] we constructed the axisymmetric solution

$$\Psi(\rho,z;t)=\sum_{n=1}^{N}\left(\frac{2}{r^2\sin^2\theta J_1^2(\lambda_n)}\right)J_0(K_n\rho)\cos\left[\frac{\omega_n}{V}(z-Vt)\right], \tag{1}$$

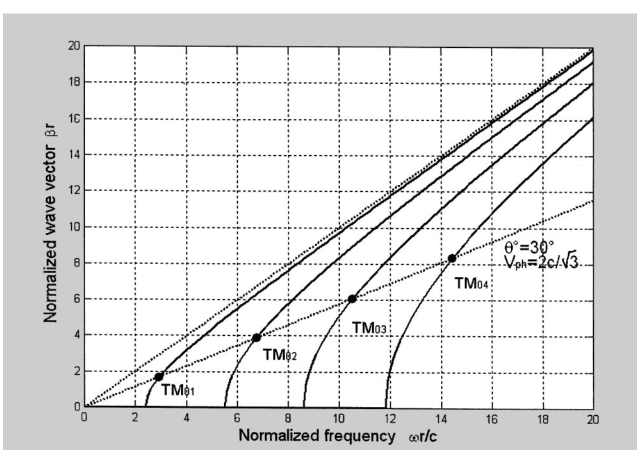

FIG. 2. Dispersion curves for the symmetrical TM_{0l} modes in a perfect cyclindrical waveguide, and location of the frequencies whose corresponding modes have equal phase velocity. See the text.

where Ψ represents the longitudinal component of the electric field E_z, while N is an integer, the quantities λ_n are the roots of the Bessel function, $K_n=\lambda_n/r$ and $\omega_n=K_n c/\sin\theta$, while $V=c/\cos\theta$. These solutions are therefore Fourier-Bessel–type sums over different propagating modes with angular frequencies ω_n.

Let us recall the obvious circumstance that any solution that depends on z (and t) only through the variable $\zeta\equiv z-Vt$ will be "rigidly" moving with speed V.

One can moreover notice that, in Eq. (1), the quantity θ is an arbitrary angle: by varying it, one obtains different train speeds and a different distance between the pulses. Actually, our solutions propagate rigidly along the guide with (superluminal) group velocity $V=c/\cos\theta$. In Fig. 1, we depict one of the trains of X-shaped waves, obtained by numerical evaluation of Eq. (1) for a waveguide radius $r=5$ cm, with $\theta=\pi/3$ (and, therefore, group-velocity $V=2c$). It is interesting to mention also that the integer N determines the space-time width of the pulses: the higher the N is, the smaller the pulse "spatiotemporal" width will be. Figure 1 has been obtained by choosing $N=22$, that is, by using the same value of N adopted by us in Ref. [8] (cf., for instance, Fig. 3 therein).

Let us emphasize that Eq. (1) represents a *multimodal* (but *localized*) propagation, as if the geometric dispersion compensated for the multimodal dispersion.

We mentioned that Ψ represents the electric field component E_z. Let us add that, by following the procedure adopted by us in Ref. [8], the other eletromagnetic field components in the considered TM case can be written as

$$\boldsymbol{E}_\perp=i\frac{cV}{V^2-c^2}\sum_{n=1}^{\infty}\frac{c}{\omega_n}\boldsymbol{\nabla}_\perp\Psi, \tag{2}$$

where

$$\frac{cV}{V^2-c^2}\equiv\frac{\cos\theta}{\sin^2\theta},$$

and

$$\boldsymbol{H}_\perp=\varepsilon_0 V\,\hat{\boldsymbol{z}}\times\boldsymbol{E}_\perp\,. \tag{3}$$

Equation (1) allows for a physical interpretation, which suggests a very simple way to get it. Each pulse train is a sum of the first N modes of our expansion (and for each N we get a different train, at our choice), whose frequencies have been suitably chosen as corresponding to the *intersections* of the modal curves (i.e., the various branches of the dispersion-relation) with the single *straight* line $\omega=V\beta$ whose slope depends on θ only; see Fig. 2. In such a case, all the modes correspond to the same (superluminal) *phase-velocity* $V_{\rm ph}$, it being independent of the mode index n; but, when the phase velocity is independent of the frequency, it becomes the *group velocity*, which is the velocity *tout court* of the considered pulse. Let us repeat once more that we thus got (nonevanescent) solutions to the Maxwell equations, which are waves propagating undistorted along normal waveguides with superluminal group velocity, *even if* in

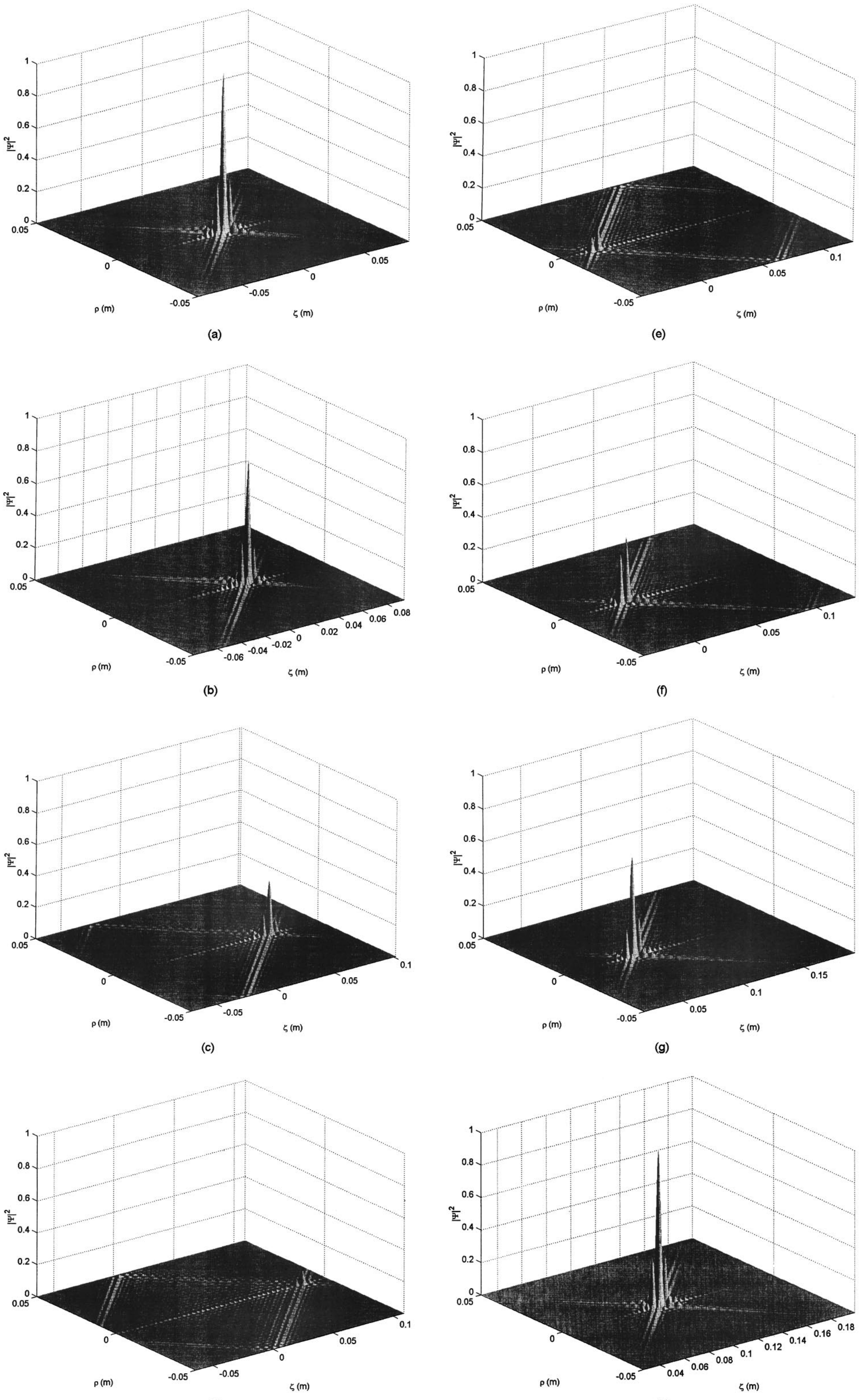

FIG. 3. This set of eight figures depicts one of our *finite* total-energy localized solutions, given by Eqs. (5) and (8). Indeed, they show the time evolution of a finite total-energy solution: Choosing $q=2.041\times10^{-10}$ and $\theta=\pi/3$ (and normalized units), there is only *one* X-shaped pulse inside the subluminal envelope: see the text. The pulse and envelope; velocities are given by $V=c/\cos\theta$ and $v_g=c^2/V$, respectively. The (global) speed v of the envelope is therefore subluminal in the finite-energy case, while superluminal speeds V are met only *locally* (internally). Nevertheless, in the present case the superluminal speed $V=c^2/v_{\rm g}$ of such a "single" pulse might be regarded as the actual velocity of the wave. (a)–(h) show a complete cycle of the pulse; the first set of four figures corresponds to the time instants $t=0$, $t=5$, $t=10$, $t=15\times10^{-11}$ s, respectively, while the (symmetrical) second set of four figures corresponds to the time instants $t=25$, $t=30$, $t=35$, $t=40\times10^{-11}$ s, respectively. Quantity ρ is the radial coordinate, while $\zeta\equiv z-Vt$. We have used, once more, the value $N=22$.

normal-sized waveguides the dispersion relation for each mode, i.e., for each term of our Fourier-Bessel expansion, is the ordinary "subluminal" one, $\omega^2/c^2-\beta^2=+K^2$. Let us repeat that the *global* velocity v (or group-velocity $v_{\rm g}\equiv v$) of the pulses corresponding to Eq. (1) is not to be evaluated by the ordinary formula $v_{\rm g}\simeq d\omega/d\beta$, valid for quasimonochromatic signals. This is at variance with the common situation in optical and microwave communications, when the signal is usually superimposed to a carrier wave whose frequency is generally much higher than the signal bandwidth. In *that* case the standard formula for $v_{\rm g}$ yields the correct velocity to deal with (e.g., when propagation delays are studied). Our case, on the contrary, is much more reminiscent of a baseband modulated signal, as those studied in ultrasonics: the very concept of a carrier becomes meaningless here, as the elementary "harmonic" components have widely different frequencies.

The fact that our superluminal trains travel rigidly, down the waveguide (i.e., with a spatial limitation only), is at variance with what happens to other (superluminal) solutions [15,7], which are truncated in time: the latter travel almost rigidly only along their finite "field depth," and then abruptly decay.

Finally, it may be underscored that the coefficients in Eq. (1) can be varied so as to keep the pulse spectrum inside the desired frequency range. This point will be discussed again soon.

III. THE FINITE-ENERGY SOLUTIONS

In this paper, we have called attention to the fact that solutions (1) are infinite trains of pulses, with *infinity* energy. This is not a real problem (plane waves too have infinite energy), provided that we are able to truncate them in space and time withount destroying their good properties. We shall go on following the previous assumptions: what we are going to do holds, however, for both the TM and the TE case. Let us *anticipate* that, in order to get finite total-energy solutions (FTESs), we shall have to replace each characteristic frequency ω_n [cf. Eq. (1), or Fig. 2] by a *small* frequency band $\Delta\omega$ centered at ω_n, always choosing the same $\Delta\omega$ independently of n. In fact, since all the modes entering the Fourier-type expansion (1) possess the same phase velocity $V_{\rm ph}\equiv V=c/\cos\theta$, each small bandwidth packet associated with ω_n will possess *the same group velocity* $v_{\rm g}=c^2/V_{\rm ph}$, so that we shall have as a result a wave whose *envelope* travels with the *subluminal* group velocity $v_{\rm g}$. *However*, inside that subluminal envelope, one or more *pulses* will be traveling with the dual (*superluminal*) speed $V=c^2/v_{\rm g}$. Such well-localized peaks will have nothing to do with the ordinary (sinusoidal) carrier wave, and will be regarded as constituting the relevant wave. When integrating each term of expansion (1) over its corresponding frequency band, one may choose, e.g., Gaussiam spectra.

Before going on, let us mention that previous work related to FTESs can be found—as far as we know—only in Refs. [16,14,17].

More formally, let us consider our ordinary solutions for a metallic waveguide, written in the form

$$\psi_n(\rho,z;t)=A_nR_n(\rho)\cos[\beta(\omega)z-\omega t],$$

where coefficients A_n and functions R_n are given by the coefficients and the (transverse) functions entering Eq. (1); namely,

$$A_n=\frac{2}{r^2\sin^2\theta J_1^2(\lambda_n)},\quad R_n(\rho)=J_0(K_n\rho),\quad K_n=\frac{\lambda_n}{r}.$$

Then, let us adopt the spectral functions

$$W_n\equiv\exp[-q^2(\omega-\omega_n)^2], \tag{4}$$

where the weight parameter q is always the same, so that $\Delta\omega$ too is independent of n [in fact, it is $\Delta\omega=2/q$], and where

$$\omega_n\equiv\frac{K_nc}{\sin\theta},$$

the quantity $\sin\theta$ having a fixed but otherwise arbitrary value. Notice that the last relation implies the wave numbers β_n of the longitudinal waves to be given, in terms of the corresponding ω_n, by $\beta_n=\omega_n\cos\theta/c$. We can construct FTESs, $\mathcal{F}(\rho,z;t)$, of the type[1]

$$\mathcal{F}(\rho,z;t)=\sum_{n=1}^{N}\int_{-\infty}^{\infty}d\omega\,\psi_nW_n \tag{5}$$

with arbitrary N. Notice that we are not using a single Gaussian weight, but a different Gaussian function W_n for each ω_n value. Such weights W_n are well localized around the corresponding ω_n, so that one can expand (for each value of n, in the above sum) the function $\beta(\omega)$ in the neighborhood of the corresponding ω_n value as follows:

$$\beta(\omega)\simeq\beta(\omega_n)+\frac{\partial\beta}{\partial\omega}\Big|_{\omega_n}(\omega-\omega_n)+\cdots, \tag{6}$$

where $\beta(\omega_n)=\omega_n\cos\theta/c$, and the further terms are neglected [since, let us repeat, $\Delta\omega/\omega$ has been assumed to be small]. Therefore, we are now facing no longer a set of phase velocities, but the set of group velocities

$$\frac{1}{v_{{\rm g}n}}=\frac{\partial\beta}{\partial\omega}\bigg|_{\omega_n},$$

which result to be *independent* of n, all of them possessing therefore the same value

$$v_{{\rm g}n}\equiv v_{\rm g}=c\cos\theta. \tag{7}$$

By performing the integration in Eq. (5), we eventually obtain

$$\mathcal{F}(\rho,z;t)=\frac{\sqrt{\pi}}{q}\exp\left[-\frac{(z-v_{\rm g}t)^2}{4q^2v_{\rm g}^2}\right]\Psi(\rho,z-Vt), \tag{8}$$

where $\Psi(\rho,z-Vt)$ is any pulse train given by Eq. (1); and we had recourse to the identity

[1]When integrating over ω from $-\infty$ to $+\infty$, also the nonphysical (noncausal) components that travel backwards in space could contribute [17,14]. But their actual contribution is totally negligible, since the weight functions W_n are strongly localized in the vicinity of the ω_n values (which are all positive; see, e.g., Fig. 2). In any case, one could integrate from 0 to ∞ at the price of increasing a little the mathematical complexity: we are preferring the present formalism for simplicity's sake.

$$\int_{-\infty}^{\infty} df \exp[-q^2 f^2] \cos[f(z - v_g t)/v_g]$$

$$= \frac{\sqrt{\pi}}{q} \exp\left[-\frac{(z - v_g t)^2}{4 q^2 v_g^2} \right].$$

It is rather interesting that our FTESs are related to the X-shaped waves, since Eq. (5) has been written in the form (8), where the function $\Psi(\rho, z - Vt)$ is any one of our previous solutions in Eq. (1) above, at our choice.

Let us notice that our finite-energy solutions, Eqs. (5) and (8), have got a finite depth of field, Z. To evaluate Z, one has to consider also the second-order term in the Taylor expansion, Eq. (6), of quantity $\beta(\omega)$, and then estimate the distance after which the solutions start deforming. The relevant calculations appear in the Appendix. One finds that

$$Z \approx \frac{2\sqrt{3}\, q^2}{\beta_{21}}, \tag{9}$$

where

$$\beta_{21} \equiv \left. \frac{\partial^2 \beta}{\partial \omega^2} \right|_{\omega_1} = \frac{\tan^2 \theta}{\omega_1 v_g}.$$

IV. CONCLUSIONS

In conclusion, looking for *finite* total-energy solutions, we have found a Gaussian envelope that travels with a *subluminal* velocity $v = c \cos \theta$. However, inside it, we have got a train of pulses traveling superluminally (with $V = c^2/v = c/\cos \theta$). And we can control the number of pulses inside the envelope just by varying the value of q.

We have actually shown that, if we choose the ω_n values as in Fig. 2, all the small-bandwidth packets centered at the ω_n's will have the same phase velocity $V > c$, and therefore the same group velocity $v_g < c$ (since for metallic waveguides the quantities $K_n^2 = \omega_n^2/c^2 - \beta^2$ are constant for each mode, and $v_g \equiv \partial\omega/\partial\beta$, so that it is $V v_g = c^2$). This means that the envelope of solution (5)–(8) moves with slower-than-light speed, the envelope length[2] $\Delta\ell$ depending on the chosen $\Delta\omega$, and being therefore proportional to $q v_g$. However, inside such an envelope, one has a train of (X-shaped) pulses—having nothing to do with the ordinary carrier wave,[3]—traveling with the superluminal speed V.

According to the standard theory of waveguides [18], the energy and signal propagation velocities coincide with the group velocity (i.e., with the envelope speed), which has been seen to be subluminal. From this point of view—even if a lively debate is still in progress [19], in connection with the rather delicate and general questions just mentioned—no particular problem is met, therefore, with our finite-energy solutions. An interesting point is that we can choose, however, the envelope length so that it contains *only one* (X-shaped wave) peak. Even if the (global) speed of the envelope is subluminal in the finite-energy case, while superluminal speeds are met only "internally," nevertheless in the present case the superluminal speed $V = c^2/v_g$ of such a "single" pulse might be regarded as the actual *local* velocity of the wave. In order to have just one peak inside the envelope, the envelope length is to be chosen smaller than the distance between two successive peaks of the (infinite total-energy) train (1).

The amplitude of such a single X-shaped pulse (which remains confined inside the envelope boundary) first increases, and afterwards decreases, while traveling, till when it practically disappears. As soon as the considered pulse vanishes on the right (i.e., under the right tail of the envelope), a second pulse starts to be created on the left, and so on [from Eq. (8) it is clear, in fact, that our finite-energy solutions are nothing but an (infinite-energy) solution of the type in Eq. (1), multiplied by a Gaussian function].

We illustrate such behavior (again with $N = 22$) in Fig. 3, namely, in the set of eight figures from Fig. 3(a) to Fig. 3(h). We have found a similar behavior in Ref. [13], and depicted it in the last set of figures of that reference, when studying the case of a coaxial guide.

Let us moreover remark that similar considerations could be extended to all the situations where a waveguide supports several modes. Tests at microwave frequencies, for instance, should be rather easy to perform; by contrast, experiments in the optical domain are made difficult, at present, by the limited extension of the spectral windows corresponding to not too large attenuations. We shall discuss this point elsewhere.

It is rather interesting that our FTESs are related to the X-shaped waves, since in Eq. (8) the function $\Psi(\rho, z - Vt)$ can be any one of our previous solutions in Eq. (1) above, at our choice.

Let us finally recall that such superluminal beams—even in the case of (infinite-energy) trains of undistorted pulses—do *not* imply causality problems, both in principle [5,12], and in consideration of the important fact that all Bessel beams, and superpositions of them, are generated by interference among ordinary (c-speed) waves, so as to be apparently uncapable of carrying superluminal information [15]—even if some debate is still in progress on this point too. Even more, the finite-energy solutions are obtained, generally speaking, by truncating in time the mentioned localized waves; the finite-energy solutions, therefore, keep localized only along a finite depth of field: namely, as long as they are fed by the incoming, interfering (ordinary) waves. It is not without meaning, actually, that all the solutions gotten in this paper, as well as in previous papers, have been obtained on the basis of the Maxwell equations (or of the wave equation) *only*, and can do nothing but comply with the fundamental postulates of special relativity [5,12].

[2]One may call "envelope length" the distance between the two points at which the envelope height is, for instance, 10% of its maximum height.

[3]Actually, they can be regarded as a sum of carrier waves.

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APPENDIX

Estimating the field depth of the finite-energy solutions

As announced at the end of Sec. III, we show in this appendix how to evaluate the field depth of our finite-energy solutions. Let us add the second-order term in Eq. (6), which is the Taylor expansion (for each value of n) of function $\beta(\omega)$ in the neighborhood of the corresponding ω_n,

$$\beta(\omega)\simeq\beta(\omega_n)+\frac{\partial\beta}{\partial\omega}\bigg|_{\omega_n}(\omega-\omega_n)+\frac{1}{2}\frac{\partial^2\beta}{\partial\omega^2}\bigg|_{\omega_n}(\omega-\omega_n)^2+\cdots, \tag{A1}$$

where it is easy to show that

$$\frac{\partial^2\beta}{\partial\omega^2}\Big|_{\omega_n}=\frac{\tan^2\theta}{\omega_n v_g}. \tag{A2}$$

On defining

$$\frac{\partial^2\beta}{\partial\omega^2}\Big|_{\omega_n}\equiv\beta_{2n}, \tag{A3}$$

our solution, Eq. (5), after some manipulations can be written as

$$\mathcal{F}(\rho,z;t)=\mathrm{Re}\Bigg\{\sum_{n=1}^{N}\sqrt{\frac{2\pi}{2q^2-i\beta_{2n}z}}A_nR_n(\rho)\times\exp\left[\frac{-q^2(z-v_gt)^2}{v_g^2(4q^4+\beta_{2n}^2z^2)}\right]\times\exp\left[-\frac{i\beta_{2n}z(z-v_gt)^2}{2v_g^2(4q^4+\beta_{2n}^2z^2)}\right]\times\exp\left[i\frac{\omega_n}{V}(z-Vt)\right]\Bigg\}. \tag{A4}$$

Equation (A4) shows that the factors responsible for the field depth being finite are

$$\mathcal{E}_n\equiv\exp\left[\frac{-q^2(z-v_gt)^2}{v_g^2(4q^4+\beta_{2n}^2z^2)}\right]. \tag{A5}$$

Such factors, Eq. (A5), imply a different alteration for each ω_n, that is, for each term of the series (A4). This fact makes it somewhat difficult to evaluate the depth of field. We can calculate the distance after which the first distortions start appearing; let us calculate, namely, the field depth of the first term in the series (A4), which is the term that suffers more distortion [cf. Eq. (A2)]. By considering the ''envelope'' Eq. (A4) with $n=1$, we may choose the distance at which its width doubles as an acceptable value for the field depth Z. One straightforwardly obtains

$$Z=\frac{2\sqrt{3}\,q^2}{\beta_{21}}. \tag{A6}$$

Therefore, Eq. (A6) of this appendix, and Eq. (9) of the text, allow evaluation of the distance after which the shape of the finite-energy solutions start deforming.

Let us moreover observe that, while the mentioned (mathematical) envelope suffers a spreading, the pulse (or pulses) traveling inside it do not suffer any temporal spread. As they propagate, they meet the following changes: (i) their amplitude decreases (together with the envelope amplitude); and (ii) more superluminal pulses are born, which propagate inside the envelope, since the width of the latter increases as time elapses (and more pulses find room inside it).

[1] H. Bateman, *Electrical and Optical Wave Motion* (Cambridge University Press, Cambridge, 1915).
[2] A.O. Barut, and A. Grant, Found. Phys. Lett. **3**, 303 (1990); A.O. Barut, and A.J. Bracken, Found. Phys. **22**, 1267 (1992); see also Refs. [14,19] below.
[3] A.O. Barut and H.C. Chandola, Phys. Lett. A **180**, 5 (1993); see also A.O. Barut, *ibid.* **189**, 277 (1994).
[4] R. Donnelly and R.W. Ziolkowski, Proc. R. Soc. London, Ser. A **440**, 541 (1993); I.M. Besieris, A.M. Shaarawi, and R.W. Ziolkowski, J. Math. Phys. **30**, 1254 (1989); S. Esposito, Phys. Lett. A **225**, 203 (1997); W.A. Rodrigues, Jr. and J. Vaz, Jr., Adv. Appl. Cliff. Alg. **S-7**, 457 (1997).
[5] See, e.g., E. Recami, Riv. Nuovo Cimento **9**(16), 1 (1986), and references therein.
[6] Jian-yu Lu and J.F. Greenleaf, IEEE Trans. Ultrason. Ferroelectr. Freq. Control **39**, 19 (1992); R.W. Ziolkowski, I.M. Besieris, and A.M. Shaarawi, J. Opt. Soc. Am. A **10**, 75 (1993); E. Recami, Physica A **252**, 586 (1998).
[7] Jian-yu Lu and J.F. Greenleaf, IEEE Trans. Ultrason. Ferroelectr. Freq. Control **39**, 441 (1992); P. Saari and K. Reivelt, Phys. Rev. Lett. **79**, 4135 (1997); D. Mugnai, A. Ranfagni, and R. Ruggeri, *ibid.* **84**, 4830 (2000).
[8] M. Zamboni Rached, E. Recami, and F. Fontana, Phys. Rev. E **64**, 066603 (2001); e-print physics/0001039; cf. also Jian-yu Lu (unpublished).

[9] Cf., e.g., A.M. Steinberg, P.G. Kwiat, and R.Y. Chiao, Sci. Am. **269**, 38 (1993); A. Enders and G. Nimtz, J. Phys. (France) I **3**, 1089 (1993); Phys. Rev. E **48**, 632 (1993).

[10] Cf., e.g., A.P.L. Barbero, H.E. Hernández F., and E. Recami, Phys. Rev. E **62**, 8628 (2000), and referemces therein; H.M. Brodowsky, W. Heitmann, and G. Nimtz, Phys. Lett. A **222**, 125 (1996).

[11] V.S. Olkhovsky and E. Recami, Phys. Rep. **214**, 339 (1992), and references therein; V.S. Olkhovsky, E. Recami, F. Raciti, and A.K. Zaichenko, J. Phys. (France) I **5**, 1351 (1995); T.E. Hartman, J. Appl. Phys. **33**, 3427 (1962).

[12] E. Recami, F. Fontana, and G. Garavaglia, Int. J. Mod. Phys. A **15**, 2793 (2000), and references therein; cf. also R. Garavaglia, thesis work, Dip. Sc. Informazione, Università Statale di Milano, Milan, 1998 (unpublished).

[13] M. Zamboni Rached, K.Z. Nóbrega, E. Recami, and H.E. Hernández F., e-print physics/0209101.

[14] M. Zamboni Rached, K.Z. Nóbrega, E. Recami, and H.E. Hernández F., Phys. Rev. E **66**, 046617 (2002); e-print physics/0209104.

[15] See, e.g., R.W. Ziolkowski, I.M. Besieris, and A.M. Shaarawi, J. Opt. Soc. Am. A **10**, 75 (1993); cf. also E. Recami, Ref. [6].

[16] I.M. Besieris, M. Abdel-Rahman, A. Shaarawi, and A. Chatzipetros, Prog. Electromagn. Res. **19**, 1 (1998).

[17] M. Zamboni Rached, E. Recami, and H.E. Hernández F., Eur. Phys. J. D **21**, 217 (2002); e-print physics/0109062.

[18] See, e.g., R. Collins, *Field Theory of Guided Waves* (IEEE Press, New York, 1991), Chap. 3.

[19] Besides the classical works by A. Sommerfeld, Z. Physik **8**, 841 (1907), and by L. Brillouin, Ann. Phys. (Paris) **44**, 203 (1914); A. Sommerfeld, *Wave Propagation and Group Velocity* (Academic Press, New York, 1969), see, e.g., P.W. Milonni, J. Phys. B **35**, R31 (2002); R.W. Ziolkowski, Phys. Rev. E **63**, 046604 (2001); A. M. Shaarawi and I.H. Besieris, J. Phys. A **33**, 7227 (2000); **33**, 7255 (2002); G. Nimtz, Eur. Phys. J. B **7**, 523 (1999); E. Recami *et al.*, Ref. [12]; A. P. L. Barbero *et al.*, Ref. [10].