PHYSICAL REVIEW E **84**, 066210 (2011)

Lyapunov statistics and mixing rates for intermittent systems

Carlos J. A. Pires,[1,*] Alberto Saa,[2,†] and Roberto Venegeroles[1,‡]
[1] *Centro de Matemática, Computação e Cognição, UFABC, 09210-170 Santo André, São Paulo, Brazil*
[2] *Departamento de Matemática Aplicada, UNICAMP, 13083-859 Campinas, São Paulo, Brazil*
(Received 11 July 2011; published 27 December 2011)

We consider here a recent conjecture stating that correlation functions and tail probabilities of finite time Lyapunov exponents would have the same power law decay in weakly chaotic systems. We demonstrate that this conjecture fails for a generic class of maps of the Pomeau-Manneville type. We show further that, typically, the decay properties of such tail probabilities do not provide significant information on key aspects of weakly chaotic dynamics such as ergodicity and instability regimes. Our approaches are firmly based on rigorous results, particularly the Aaronson-Darling-Kac theorem, and are also confirmed by exhaustive numerical simulations.

DOI: 10.1103/PhysRevE.84.066210 PACS number(s): 05.45.Ac, 02.50.Ga, 74.40.De

I. INTRODUCTION

It has been well known since the seminal works of Sinai, Ruelle, and Bowen (SRB) [1–3] that the strongest chaotic systems (Smale's axiom A and Anosov systems) have SRB measures with exponentially decaying mixing rates (see also Ref. [4]). For these systems, the difference between temporal and spatial averages is statistically described by a Gaussian distribution (the central limit theorem) and the convergence of both averages toward a unique value is assured. On the other hand, there is a wide range of systems where mixing rates and other related correlation functions decay as power laws. Such a class of dynamical systems, dubbed weakly chaotic in the physics literature, typically exhibits weak statistical properties when compared with the chaotic ones. Examples of weakly chaotic systems include maps with indifferent fixed points [5–8], billiards [9], and Hamiltonian systems with sticky islands in phase space [10], among others. These systems have in common an intermittent dynamical behavior, exhibiting a transition from regular to chaotic regimes which has attracted the attention of physicists and mathematicians in the last 20 years. We recall that the so-called mixing rate of a pair of phase space observable functions ϕ and ψ for maps of the type

$$x_{t+1} = f(x_t) \tag{1}$$

is defined as being the correlation function

$$C_t(\phi,\psi) = \left| \int \phi(x)\psi[f^t(x)]\,d\mu(x) - \int \phi(x)\,d\mu(x) \int \psi(x)\,d\mu(x) \right|, \tag{2}$$

where $\mu(x)$ is the invariant measure under the map f. A map is said to be mixing if $C_t \to 0$ as $t \to \infty$ for any pair of phase space smooth observables (ϕ,ψ).

Since correlation functions such as the mixing rate (2) might characterize the transition from strong to weak chaos, the attempt to relate them to more fundamental dynamical quantities is a goal that looks, at first glance, really promising. In Ref. [11], for instance, the finite time Lyapunov exponents

$$\Lambda_t(x) = \frac{1}{t}\sum_{k=0}^{t-1} \ln|f'[f^k(x)]|, \tag{3}$$

were considered for one-dimensional maps like Eq. (1) for which $\Lambda_\infty > 0$, i.e., for strongly chaotic cases. Essentially, they show that if there exist two positive constants Λ_0 and $\gamma > 1$ such that

$$M_t(\Lambda_0) = \int_{\Lambda_0}^{\infty} \eta(\Lambda_t)d\Lambda_t \leqslant a_1 t^{-\gamma}, \tag{4}$$

where $\eta(\Lambda_t)$ stands for the distribution of finite time Lyapunov exponents for the system in question, then we also have the following upper bound,

$$C_t(\phi,\psi) \leqslant a_2 t^{-(\gamma-1)}, \tag{5}$$

for the mixing rates of any pair of Hölder continuous observables (ϕ,ψ). In other words, they have proved rigorously that if the tail probabilities of the finite time Lyapunov exponents are bounded by $t^{-\gamma}$ for large t, then correlations will be also bounded asymptotically by $t^{-(\gamma-1)}$. These important results have inspired a recent work [12] in which a related conjecture is made for weakly chaotic systems, i.e., irrespective of having $\Lambda_\infty > 0$. The main results of Ref. [12] can be summarized as follows.

(1) They argue that scrutinizing the way in which $1 - M_t$ decays for large t provides an "extremely efficient way" of studying quantitatively the decay of correlations.

(2) They conjecture, in view of the recent results of Ref. [11], that the estimates of Eqs. (4) and (5) are not optimal and that the decay properties of $1 - M_t$ and of $C_t - \int \phi d\mu \int \psi d\mu$ should be, in fact, both polynomial with the same exponent.

(3) They check numerically such a conjecture for a one-dimensional intermittent map with two indifferent fixed points of the Pomeau-Manneville type [13], for which the polynomial decay rates of correlations are known exactly. Two other types of two-dimensional intermittent maps are also numerically considered to support the conjecture.

Here, we show that this conjecture is false by presenting an explicit class of counterexamples. We consider a general

*carlosamado@ig.com.br
†asaa@ime.unicamp.br
‡roberto.venegeroles@ufabc.edu.br

class of Pomeau-Manneville maps [13] and show that their Lyapunov exponents' tail probabilities (4) decay faster than any power law, whereas their correlations (2) do exhibit a power law decay. Our approaches are firmly based on rigorous results, particularly on the Aaronson-Darling-Kac theorem [14], and are also confirmed by exhaustive numerical simulations. For all maps in this class, correlation functions decay more slowly than the tail probabilities of the Lyapunov exponents, suggesting that bounds of the types (4) and (5) can be physically relevant also for weakly chaotic systems.

II. LYAPUNOV STATISTICS

Our counterexamples consist of a general class of Pomeau-Manneville (PM) intermittent dynamical systems of type (1) with $f:[0,1]\to[0,1]$, where

$$f(x)\sim x(1+ax^{z-1}) \tag{6}$$

for $x\to 0$, with $a>0$ and $z>1$. The global form of f is irrelevant, provided it respects Adler's, finite image, and nonuniformly expansion (AFN) axioms [15]. For maps of type (6), $x=0$ is an indifferent (neutral) fixed point; i.e., $f(0)=0$ and $f'(0)=1$. Such systems are known to have power law invariant measures near their indifferent fixed points. More specifically, we have $d\mu(x)=\omega(x)\,dx$, where $\omega(x)\sim bx^{-1/\alpha}$ near the fixed point $x=0$, with $\alpha=(z-1)^{-1}$ [16]. As a consequence, such systems have a diverging invariant measure near this point for $z>2$. Moreover, a finite invariant measure ($1<z<2$) implies ergodicity and the usual Lyapunov exponential instability, whereas the diverging case ($z>2$) implies nonergodicity and subexponential instability. We consider each of these cases separately and show that the conjecture proposed in Ref. [12] fails for both.

A. Exponential instability

Let us first consider the statistics of finite time Lyapunov exponents (3) for randomly distributed initial conditions $x\in[0,1]$, in the case of finite invariant measure cases ($1<z<2$). It is well known that ergodicity properties can determine completely such statistics. For instance, Birkhoff theorem [17] states that, in an ergodic regime, the time average of an arbitrary observable function ϑ, $t^{-1}\sum_{k=0}^{t-1}\vartheta[f^k(x)]$, converges uniformly to the spatial average $\int\vartheta\,d\mu$. Then, for almost all initial conditions $x\in[0,1]$, the local expansion rate $\Lambda_t(x)$ in Eq. (3) converges to the unique positive Lyapunov exponent Λ_∞ as $t\to\infty$. On the other hand, if t is finite, $\Lambda_t(x)$ assumes different values depending on the initial condition x. The corresponding probability density function $\eta(\Lambda_t)=\eta(\Lambda,t)$ is given by

$$\eta(\Lambda,t)=\int\delta[\Lambda_t(x)-\Lambda]d\mu(x). \tag{7}$$

For large t, $\eta(\Lambda,t)$ takes the scaling form [18]

$$\eta(\Lambda,t)\sim\eta(\Lambda_\infty,t)\exp[-t\Omega(\Lambda)], \tag{8}$$

where $\Omega(\Lambda)\geqslant 0$ is a concave function with minimum at $\Omega(\Lambda_\infty)=0$. Then we have $\Omega(\Lambda)\sim c_1(\Lambda-\Lambda_\infty)^2$ and $\eta(\Lambda_\infty,t)\sim(c_1t/\pi)^{1/2}$, with $c_1>0$. Now, a simple calculation by using Laplace's method leads to

$$M_t\sim\tfrac{1}{2}\mathrm{erfc}(\sqrt{\Omega_0 t}), \tag{9}$$

where $\Omega_0=\Omega(\Lambda_0)$. The decaying properties of Eq. (9) are definitively different than those ones predict by Ref. [12]. In fact, one has

$$M_t\sim\frac{1}{2\sqrt{\pi}}\frac{\exp(-\Omega_0 t)}{\sqrt{\Omega_0 t}}, \tag{10}$$

for $\Lambda_0\neq\Lambda_\infty$ and $M_t(\Lambda_\infty)\sim 1/2$. It is important to stress that there are many rigorous results in the literature establishing polynomial bounds for the decay of correlations of PM maps in the regime $1<z<2$ (see Ref. [5] and references therein). Most notably, Sarig [6] and Gouëzel [7] have achieved optimal polynomial bounds for such correlations in this regime. Therefore, contrary to the conjecture proposed in Ref. [12], polynomial decay of correlations can occur simultaneously with exponential decay of Lyapunov tail probability distributions. In fact, any conjecture stating that M_t should decay as a power law is generically violated for ergodic regimes. For the PM map with $1<z<2$, this is indeed predicted by Theorem 2 (exponential level I result) of Ref. [8].

B. Subexponential instability

Let us consider now the cases for which the invariant measure diverges locally at the indifferent fixed point $x=0$, i.e., $z>2$. For such cases, the system typically exhibits a nonergodic behavior and, hence, time averages do not converge to a unique constant value. Nevertheless, the Aaronson-Darling-Kac (ADK) theorem [14,19,20] ensures that a suitable time-weighted average does converge uniformly in distribution terms toward a Mittag-Leffler distribution of unit first moment. More specifically, for a positive function ϑ and a random variable x with an absolutely continuous measure with respect to the Lebesgue measure on the interval $[0,1]$, there is a (return) sequence $\{a_t\}$ for which

$$\frac{1}{a_t}\sum_{k=0}^{t-1}\vartheta[f^k(x)]\xrightarrow{d}\xi_\alpha\int\vartheta\,d\mu, \tag{11}$$

for $t\to\infty$, where ξ_α is a non-negative Mittag-Leffler random variable of index $\alpha\in(0,1)$ and with unit expected value. The return sequence $\{a_t\}$ for PM systems like Eq. (6) and $0<\alpha<1$ is given by [15,21]

$$a_t\sim\frac{1}{b}\frac{1}{a}\left(\frac{a}{\alpha}\right)^\alpha\frac{\sin(\pi\alpha)}{\pi\alpha}t^\alpha, \tag{12}$$

for $t\to\infty$. What the ADK theorem is really pointing out here is the explicit necessity of dealing with finite time subexpoential Lyapunov exponents,

$$\lambda_t^{(\alpha)}(x)=\frac{1}{t^\alpha}\sum_{k=0}^{t-1}\ln|f'[f^k(x)]|, \tag{13}$$

instead of the usual exponents (3) for PM systems of the AFN type (see also Ref. [22]). From Eq. (11), we have

$$\frac{\lambda_t^{(\alpha)}}{\langle\lambda\rangle} \xrightarrow{d} \xi_\alpha, \tag{14}$$

for $t \to \infty$, where the ADK average value $\langle\lambda\rangle$ is given by

$$\langle\lambda\rangle = \frac{1}{ba}\left(\frac{a}{\alpha}\right)^\alpha \frac{\sin(\pi\alpha)}{\pi\alpha}\int_0^1 \ln|f'(x)|\omega(x)\,dx. \tag{15}$$

The ADK theorem completely determines the correlations and the tail probability of Lyapunov exponents for the maps of type (6), as one can see by considering a randomly distributed initial condition $x \in [0,1]$ with probability density $h(x) > 0$ in Eq. (11), leading to

$$\frac{1}{a_t}\sum_{k=0}^{t-1}\int \vartheta[f^k(x)]h(x)\,dx = \int \vartheta\, d\mu, \tag{16}$$

for $t \to \infty$. We can rearrange this expression and write

$$C_t(\phi,\vartheta) - \int \phi\, d\mu \int \vartheta\, d\mu \sim \alpha\langle\vartheta\rangle t^{\alpha-1} \tag{17}$$

for $t \to \infty$, where $\phi(x) = h(x)/\omega(x)$ and the ADK average $\langle\vartheta\rangle$ is given by an expression analogous to Eq. (15). It remains now to show that the tail probability of Lyapunov exponents for systems of type (6) does not decay as predicted by Eq. (17). From Eq. (14), one can obtain M_t for the map (6) by recalling that $\lambda_t^{(\alpha)} = t^{1-\alpha}\Lambda_t$, implying the following distribution of finite time Lyapunov exponents for systems of type (6),

$$\eta(\Lambda_t) = \frac{t^{1-\alpha}}{\langle\lambda\rangle}\rho_\alpha^{(r)}\left(\frac{t^{1-\alpha}\Lambda_t}{\langle\lambda\rangle}\right), \tag{18}$$

where $\rho_\alpha^{(r)}$ is a Mittag-Leffler probability density function with unit first moment, which corresponds to choice $r^\alpha = \alpha\Gamma(\alpha)$, according to the definitions of Ref. [23]. Then, we have finally from Eqs. (4) and (18)

$$M_t = \int_{u(t)}^{\infty} \rho_\alpha^{(r)}(s)ds, \tag{19}$$

for $t \to \infty$, where $u(t) = t^{1-\alpha}\Lambda_0/\langle\lambda\rangle$. The behavior of $\rho_\alpha^{(r)}(x)$ for large x was recently discussed in Ref. [23], based on the known relation between Mittag-Leffler and one-sided Lévy distributions [24] and the Mikusiński's asymptotic analysis [25] of the latter. In particular, one has

$$\rho_\alpha^{(r)}(x) \sim \sqrt{\frac{A}{2\pi\alpha}\frac{x^{(2\alpha-1)/(2-2\alpha)}}{1-\alpha}}\exp\left(-Ax^{1/(1-\alpha)}\right), \tag{20}$$

for $r^\alpha = \alpha\Gamma(\alpha)$, valid for $x \to \infty$, where

$$A = \frac{1-\alpha}{\alpha}\Gamma(\alpha)^{1/(\alpha-1)}. \tag{21}$$

The integral of Eq. (20) can be written in terms of the complementary error function, leading simply to

$$M_t \sim \frac{1}{\sqrt{2\alpha}}\operatorname{erfc}(\sqrt{Bt}), \tag{22}$$

for large t, where

$$B = \frac{1-\alpha}{\alpha}\left(\frac{\Lambda_0\Gamma(\alpha)}{\langle\lambda\rangle}\right)^{1/(\alpha-1)}. \tag{23}$$

The decaying properties of Eq. (22) are also definitively different than those predicted by Eq. (17), in the context of conjecture proposed in Ref. [12]. Once more we have

$$M_t \sim \frac{1}{\sqrt{2\pi\alpha}}\frac{\exp(-Bt)}{\sqrt{Bt}}, \tag{24}$$

for large t, demonstrating finally that the conjecture presented in Ref. [12] is false. It is noteworthy that the tail probabilities of Lyapunov exponents given by Eq. (24) are essentially the same as those we would expect from finite measure cases, i.e., Eq. (10). This shows that the way in which M_t decays does not provide significant information on key aspects of weakly chaotic dynamics.

III. NUMERICAL SIMULATIONS

In order to test and illustrate the conclusions of the last section, we perform an exhaustive numerical analysis of two particular AFN maps of type (6), namely, the Thaler map [16], defined for $z > 2$ as

$$f(x) = x\left[1 + \left(\frac{x}{1+x}\right)^{z-2} - x^{z-2}\right]^{-1/(z-2)}, \tag{25}$$

mod 1, and the modified Bernoulli map [22], defined also for $z > 2$ as

$$f(x) = \begin{cases} x + 2^{z-1}x^z, & 0 \leqslant x \leqslant \frac{1}{2}, \\ x - 2^{z-1}(1-x)^z, & \frac{1}{2} < x \leqslant 1. \end{cases} \tag{26}$$

The Thaler map (25) is very convenient here because its invariant measure density is explicitly known, namely [16],

$$\omega(x) = x^{-1/\alpha} + (1+x)^{-1/\alpha}, \tag{27}$$

where $\alpha = (z-1)^{-1}$, allowing in this way the explicit evaluation of the ADK averages like Eq. (15). In contrast with the Thaler map, there is no explicit expression for the invariant measure of the modified Bernoulli map, but it is known to have the form $\omega(x) \sim b_k|x - x_k|^{-1/\alpha}$, also with $\alpha = (z-1)^{-1}$, in the neighborhood of each of the two indifferent fixed points $x_0 = 0$ and $x_1 = 1$ [26]. Note that the ADK theorem is also valid for systems with more than one indifferent fixed point [15]. For the Bernoulli map (26), we also have return rates in the form $a_t \sim t^\alpha$ for $0 < \alpha < 1$ [15]. However, since its corresponding explicit expression for the invariant measure is lacking, we cannot evaluate the ADK averages for the modified Bernoulli map. We show that this problem can be circumvented by exploiting the numerical data.

A. Tail probabilities of Lyapunov exponents

Our first task here is to determine if the tail probabilities of finite time Lyapunov exponents (4) for the maps (25) and (26) do indeed decay as predicted by Eq. (22). From Sec. II, we known that such decaying behavior is assured if the distribution of finite time Lyapunov exponents is effectively given by Eq. (18). We compute numerically the distribution of finite time Lyapunov exponents Λ_t for the maps (25) and (26) for random initial condition and large t and confront the obtained numerical data with the theoretically predicted distribution (18). The algorithm for the numerical computation of Mittag-Leffler distributions with arbitrary index α introduced in Ref. [23] is instrumental to performing such a task. The key point of our analysis is to check if a given distribution is well described by a generic Mittag-Leffler probability density function. We recall that a Mittag-Leffler probability density $\rho_\alpha^{(r)}(x)$ is defined from its Laplace transform as

$$\int_0^\infty e^{-sx}\rho_\alpha^{(r)}(x)\,dx = \sum_{n=0}^{\infty}\frac{(-sr^\alpha)^n}{\Gamma(1+n\alpha)}, \tag{28}$$

for $s \geqslant 0$, with $0 < \alpha < 1$. The choice $r^\alpha = \alpha\Gamma(\alpha)$ assures that $\langle x\rangle = 1$, where the average here is evaluated with respect to $\rho_\alpha^{(r)}(x)$. From Eqs. (18) and (28), we have the following constraints on the high-order moments

$$\frac{\langle\Lambda^n\rangle}{\langle\Lambda\rangle^n} = \frac{n!\alpha^{n-1}\Gamma(\alpha)^n}{n\Gamma(n\alpha)} \tag{29}$$

of the probability density (18). One can evaluate $\langle\Lambda^n\rangle$ easily from the numerical data and the constraints (29) can be objectively used to decide if a given distribution is well described by a Mittag-Leffler probability density. In particular, notice that one can determine the two free parameters of the distribution (18), $\langle\lambda\rangle$ and α, by considering, for instance, $\langle\Lambda\rangle = t^{\alpha-1}\langle\lambda\rangle$ and

$$\frac{\langle\Lambda^2\rangle}{\langle\Lambda\rangle^2} = \frac{\alpha\Gamma(\alpha)^2}{\Gamma(2\alpha)}. \tag{30}$$

It is very instructive to inspect the graphics of $\langle\Lambda^2\rangle/\langle\Lambda\rangle^2$ as a function of α (see Fig. 1). For Mittag-Leffler distributions with unit first moment, one has necessarily $1 \leqslant \langle\Lambda^2\rangle/\langle\Lambda\rangle^2 \leqslant 2$, with the boundaries corresponding, respectively, to $\alpha = 1$ and $\alpha = 0$. For such values of α, the Mittag-Leffler probability density function approaches, respectively, a δ function centered in $x = 1$ and a simple exponential e^{-x} (see Ref. [23]). The violation of such boundaries would point out unequivocally that one is not leading with Mittag-Leffler distributions with first unit moment. Analogous bounds hold also for higher-order moments (29), $1 \leqslant \langle\Lambda^n\rangle/\langle\Lambda\rangle^n \leqslant n!$.

For the case of the Thaler map, both parameters $\langle\lambda\rangle$ and α in the distribution (18) are predicted theoretically by the ADK theorem, allowing the inspection of the convergence rate of Eq. (14) with respect to t and to the number of initial conditions used to evaluate the Lyapunov exponents. On the other hand, for the modified Bernoulli map one cannot determine exactly the average $\langle\lambda\rangle$, but it is possible to infer its value by computing $\langle\Lambda\rangle$ from the numerical data and then using $\langle\lambda\rangle = t^{1-\alpha}\langle\Lambda\rangle$. Figure 2 depicts the distribution of finite time Lyapunov exponents for the Thaler map (25). The plots show clearly that the distribution of Lyapunov exponents becomes peaked around the origin for $t\to\infty$, leading to $\langle\Lambda^n\rangle \to 0$ for large t. In particular, one has $\Lambda_\infty = 0$, in perfect agreement with the predicted distribution (18) and the fact that the Thaler map is known to be weakly chaotic. Figure 3 illustrates the case of the modified Bernoulli map (26). For all cases, we see, graphically and according to the higher-order moments constraints (29), that the distribution of finite time Lyapunov exponents is very well described by a Mittag-Leffler probability density according to the prediction of Eq. (18). The tail probability (22) is then guaranteed for these maps.

Our numerical examples are, in fact, illustrating the convergence of Eq. (14), which is a consequence of ADK theorem. As expected, for large values of t and for large numbers of initial conditions, the histograms of both Figs. 2 and 3 approach the Mittag-Leffler probability density with the theoretical predicted values of α and $\langle\lambda\rangle$. We could, however, detect another very interesting property. For a given value of t and a given number of initial conditions, the corresponding histograms are already very well described by a Mittag-Leffler probability density! With the increasing of t and the number of initial conditions, such "instantaneous" Mittag-Leffler probability densities approach the ADK ones,

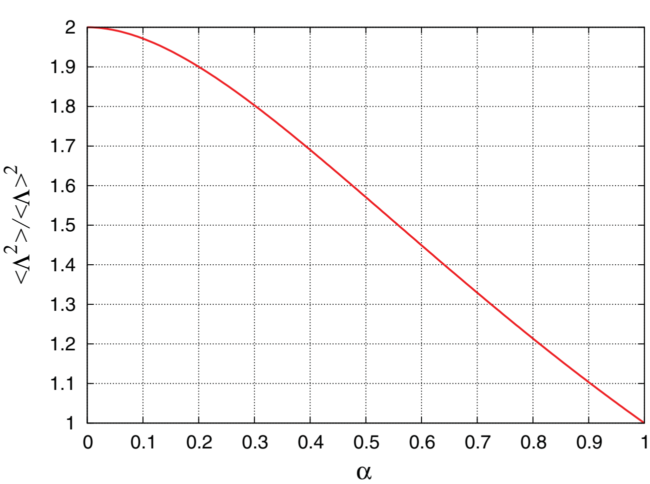

FIG. 1. (Color online) Graphics of $\langle\Lambda^2\rangle/\langle\Lambda\rangle^2$ as a function of α for Mittag-Leffler distributions with unit first moment [see Eq. (30)]. The Mittag-Leffler index α can be determined from the value of $\langle\Lambda^2\rangle/\langle\Lambda\rangle^2 \in [1,2]$.

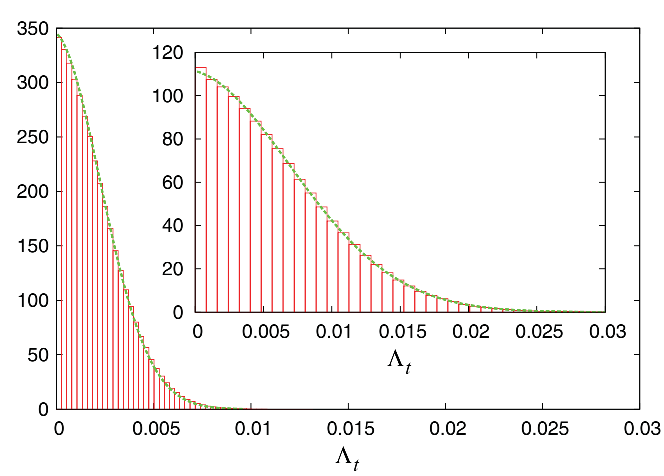

FIG. 2. (Color online) Distribution of finite time Lyapunov exponents (3) for the Thaler map, determined from the iteration of Eq. (25), with $z = 22/7$ ($\alpha = 7/15$), for 2.5×10^5 initial conditions uniformly distributed on the interval [0,1]. The histograms are built directly from the numerical data, while the solid lines are the corresponding Mittag-Leffler probability density (18), computed by means of the algorithm of Ref. [23]. The inset and the background plots correspond, respectively, to $t = 6\times10^4$ and $t = 5\times10^5$.

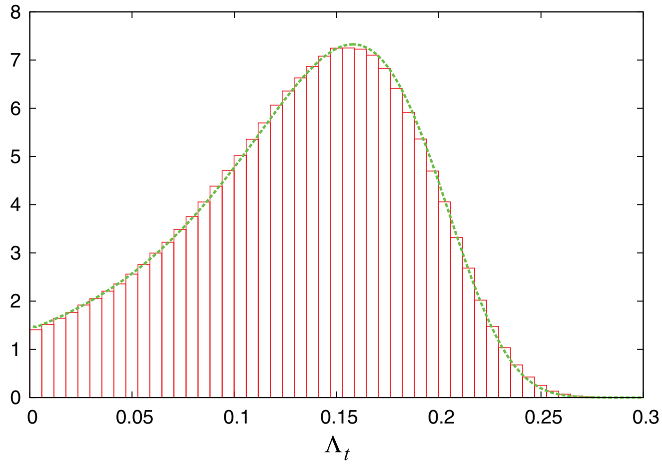

FIG. 3. (Color online) Distribution of finite time Lyapunov exponents (3) for the modified Bernoulli map, determined from the iteration of Eq. (26), with $z = 28/13$ ($\alpha = 13/15$) and $t = 6 \times 10^4$, for 2.5×10^5 initial conditions uniformly distributed on the interval $[0,1]$. The histogram was built directly from the numerical data; the solid line is the Mittag-Leffler probability density computed with the algorithm of Ref. [23]. The behavior of the distribution for large t is identical to that of the Thaler map case depicted in Fig. 3. In particular, one also has $\langle \Lambda^n \rangle \to 0$ for $t \to \infty$ and $\Lambda_\infty = 0$, in agreement with the predictions of Eq. (18).

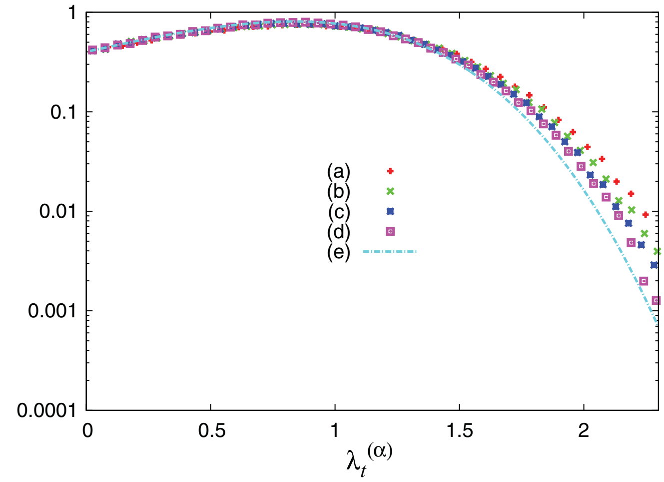

FIG. 4. (Color online) Log plot of the distribution of finite time subexponential Lyapunov exponents (13) for the Thaler map, determined from the iteration of Eq. (25), with $z = 26/11$ ($\alpha = 11/15$), for 2.5×10^5 initial conditions uniformly distributed on the interval $[0,1]$. The set of points (a), (b), (c), and (d) correspond, respectively, to the histograms built from the numerical data obtained for $t = 10^4$, $t = 5 \times 10^4$, $t = 25 \times 10^4$, and $t = 10^6$. Each one of these data sets is very well described by a Mittag-Leffler probability density (see Table I). The line (e) corresponds to the Mittag-Leffler probability density with the ADK values for α and $\langle \lambda \rangle$. As one can see, the numerically obtained distributions converge toward the prediction of the ADK theorem with the increasing of t. The increasing of the number of initial conditions does not alter considerably such a convergence, but a better description of the density tail requires a larger number of initial conditions, as expected.

as is illustrated, for the Thaler map, in Fig. 4 and in Table I.

The solid lines in both Figs. 2 and 3, for instance, are the instantaneous Mittag-Leffler probability densities, i.e., their parameters α and $\langle \lambda \rangle$, although close to the theoretically predicted values, were calculated from the numerical data by using $\langle \lambda \rangle = t^{1-\alpha} \langle \Lambda \rangle$ and Eq. (30). Table I shows the values of the higher-order moment constraints (29) for the data sets presented in Fig. 4.

B. Correlation functions

The numerical computation of the correlation functions (2) is rather tricky for the maps in question due to the highly discontinuous nature of the iterated maps $f^t(x)$ for large t. For both cases (25) and (26), for instance, the iterated map $f^t(x)$ has $2^t - 1$ discontinuous points. An accurate numerical computation for large t of a correlation function such as C_t would require an extremely fine subdivision of the interval $[0,1]$, rendering the task practically and computationally unviable. Nevertheless, in order to establish the correlation function decaying (17), it is enough to assure that, for some value of $0 < \alpha < 1$, the quantity

$$\theta_t^{(\alpha)} = \frac{1}{t^\alpha} \sum_{k=0}^{t-1} \vartheta[f^k(x)], \tag{31}$$

where $\vartheta(x)$ is an integrable function, converges uniformly in distribution terms toward a random variable for large t. The ADK theorem does assure such a convergence with $\alpha = (z-1)^{-1}$ for the maps in question, and, as demonstrated in Sec. II, the correlation decaying (17) is also firmly based on the ADK theorem. As an example, let us consider the observable function $\vartheta(x) = \sin^z \pi x$ for the case of the modified Bernoulli map. According to the discussion of Sec. II, the correlation $C_t(h,\vartheta)$ for any smooth observable function $h(x) > 0$ will exhibit a power lay decay as predicted by Eq. (17) provided $\theta_t^{(\alpha)}$ does converge in distribution terms to a Mittag-Leffler random variable. Figure 5 depicts such a distribution and one can confirm the very good agreement with the predictions of the ADK theorem, assuring the validity of the correlation

TABLE I. Statistical data for the graphics in Fig. 4. For each data set, the higher-order moment constraints (29) are respected, showing that each "instantaneous" histogram of Fig. 4 is indeed well described by a Mittag-Leffler probability density. The values of the Mittag-Leffler index α for each one of the data sets (last row) were calculated from $\langle \lambda^2 \rangle / \langle \lambda \rangle^2$ [see Eq. (30) and also Fig. 1]. The last column corresponds to the probability density predicted from the ADK theorem, namely, the curve (e) in Fig. 4.

	(a)	(b)	(c)	(d)	ADK
$\langle \lambda \rangle$	0.853	0.840	0.829	0.822	0.807
$\langle \lambda^2 \rangle / \langle \lambda \rangle^2$	1.305	1.303	1.300	1.296	1.290
$\langle \lambda^3 \rangle / \langle \lambda \rangle^3$	1.956	1.947	1.935	1.921	1.899
$\langle \lambda^4 \rangle / \langle \lambda \rangle^4$	3.216	3.189	3.155	3.117	3.051
$\langle \lambda^5 \rangle / \langle \lambda \rangle^5$	5.674	5.598	5.510	5.414	5.242
$\langle \lambda^6 \rangle / \langle \lambda \rangle^6$	10.589	10.383	10.168	9.927	9.497
15α	10.81	10.84	10.88	10.92	11

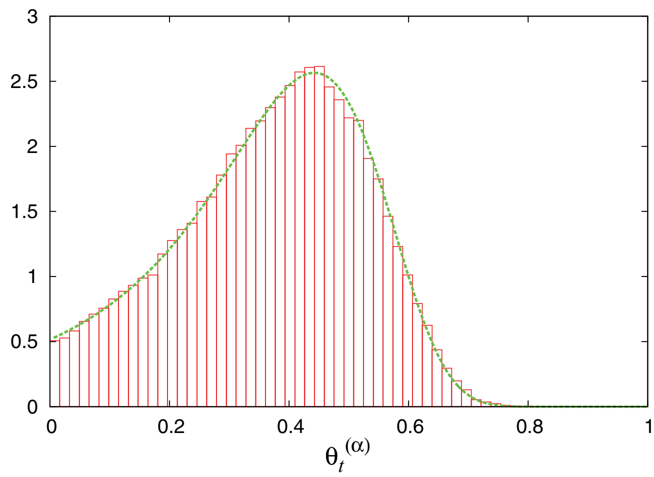

FIG. 5. (Color online) Distribution of the quantity $\theta_t^{(\alpha)}$, given by Eq. (31), for $\vartheta(x) = \sin^z \pi x$, determined from the iteration of the modified Bernoulli map (26), with $z = 28/13$ ($\alpha = 13/15$) and $t = 6 \times 10^4$, for 2.5×10^5 initial conditions uniformly distributed on the interval $[0,1]$. The histogram is built directly from the numerical data and the solid line is the corresponding Mittag-Leffler probability density, computed by means of the numerical algorithm of Ref. [23].

decaying (17). Similar results hold also for the Thaler map (25) and for other observables.

We notice that the validity of Eq. (17) is stronger than the ADK theorem, in the sense that the convergence to a Mittag-Leffler distribution given by Eq. (11) is not a necessary condition to establish Eq. (16). In fact, the existence of a sequence $a_t \sim t^\alpha$ such that $a_t^{-1} \sum_{k=0}^{t-1} \vartheta[f^k(x)]$ does converge in distribution terms toward a random variable, not necessarily of the Mittag-Leffler type, is enough to assure the decaying (17).

IV. FINAL REMARKS

We close by noticing that the first map presented in Ref. [12] to support the conjecture we have just proved to be false is also a map with indifferent fixed points, namely, the so-called Pikovsky map, which is defined implicitly by [27]

$$x = \begin{cases} \frac{1}{2z}[1 + f(x)]^z, & 0 < x < \frac{1}{2z}, \\ f(x) + \frac{1}{2z}[1 - f(x)]^z, & \frac{1}{2z} < x < 1. \end{cases} \tag{32}$$

The Pikovsky map is defined on the interval $[-1,1]$. For negative x, one has simply $f(x) = -f(-x)$. This map has two indifferent fixed points located at $x = \pm 1$ for $z > 1$. The correlation functions for the Pikovsky map are known to decay as a power law [27,28]. The authors of Ref. [12] present some numerical evidence suggesting that the tail probability M_t for the map (32) would also decay with the same power law. This fact seems to contradict our results of Sec. II. However, a closer inspection of Eq. (32) reveals that the Pikovsky map is not an AFN map [15] and, hence, the ADK theorem cannot be invoked here to determine the distribution of finite time Lyapunov exponents. From the first equation of map (32), we have

$$\frac{f''}{(f')^2} = (1 - z)(2zx)^{-1/z}, \tag{33}$$

showing that the axiom A (Adler's condition) [15] is not satisfied for $x = 0$ and positive z. The violation of Adler's condition here is related to the infinity slope of the map at the origin, and this is known to be capable of inducing some new dynamical properties such as, for instance, the existence of a regular invariant measure despite the indifferent fixed points; see Example 1 of Ref. [29], for instance. The failure of Adler's condition might explain why the authors of Ref. [12] have arrived at the conclusion that M_t decays as a power law for the map (32), but certainly a deeper investigation of the Pikvosky map would be interesting and revealing.

ACKNOWLEDGMENTS

This work was partially supported by CNPq (AS and RV), FAPESP (AS), and UFABC (CJAP). RV thanks V. Pinheiro for enlightening discussions.

[1] Ya. Sinai, Russ. Math. Surv. **27**, 21 (1972).
[2] R. Bowen, *Lecture Notes in Mathematics*, Vol. 470 (Springer-Verlag, New York, 1975).
[3] D. Ruelle, Am. J. Math. **98**, 619 (1976).
[4] A. Bunimovich and Y. Sinai, Commun. Math. Phys. **78**, 247 (1981); **78**, 479 (1981).
[5] L.-S. Young, Isr. J. Math. **110**, 153 (1999); I. Melbourne, Proc. Am. Math. Soc. **137**, 1735 (2009).
[6] O. M. Sarig, Invent. Math. **150**, 629 (2002).
[7] S. Gouëzel, Isr. J. Math. **139**, 29 (2004).
[8] M. Pollicott and R. Sharp, Nonlinearity **22**, 2079 (2009).
[9] N. Chernov and H.-K. Zhang, Nonlinearity **18**, 1527 (2005); J. Machta, J. Stat. Phys. **32**, 555 (1983); J. Machta and B. Reinhold, *ibid.* **42**, 949 (1986).
[10] C. F. F. Karney, Phys. D **8**, 360 (1983); R. Venegeroles, Phys. Rev. Lett. **102**, 064101 (2009); Phys. Rev. E **77**, 027201 (2008).
[11] J. F. Alves, S. Luzzatto, and V. Pinheiro, Ergod. Th. Dynam. Syst. **24**, 637 (2004).
[12] R. Artuso and C. Manchein, Phys. Rev. E **80**, 036210 (2009).
[13] Y. Pomeau and P. Manneville, Commun. Math. Phys. **74**, 189 (1980); P. Manneville, J. Phys. (Paris) **41**, 1235 (1980).
[14] J. Aaronson, *An Introduction to Infinite Ergodic Theory* (American Mathematical Society, Providence, 1997).
[15] R. Zweimüller, Ergod. Th. Dynam. Syst. **20**, 1519 (2000).
[16] M. Thaler, Studia Math. **143**, 103 (2000).
[17] G. D. Birkhoff, Proc. Natl. Acad. Sci. USA **17**, 656 (1931).
[18] R. S. Ellis, *Entropy, Large Deviations, and Statistical Mechanics* (Springer-Verlag, New York, 1985).
[19] J. Aaronson, J. Anal. Math. **39**, 203 (1981).
[20] D. A. Darling and M. Kac, Trans. Am. Math. Soc. **84**, 444 (1957).
[21] M. Thaler and R. Zweimüller, Probab. Theory Relat. Fields **135**, 15 (2006).

[22] P. Gaspard and X.-J. Wang, Proc. Natl. Acad. Sci. USA **85**, 4591 (1988); T. Akimoto and Y. Aizawa, J. Korean Phys. Soc. **50**, 254 (2007); Chaos **20**, 033110 (2010); N. Korabel and E. Barkai, Phys. Rev. Lett. **102**, 050601 (2009); Phys. Rev. E **82**, 016209 (2010).
[23] A. Saa and R. Venegeroles, Phys. Rev. E **84**, 026702 (2011).
[24] W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. II (Wiley, New York, 1971).
[25] J. Mikusiński, Studia Math. **18**, 191 (1959).
[26] M. Thaler, Isr. J. Math. **37**, 303 (1980).
[27] A. S. Pikovsky, Phys. Rev. A **43**, 3146 (1991).
[28] S. Grossmann and H. Horner, Z. Phys. B **60**, 79 (1985).
[29] R. Zweimüller, Nonlinearity **11**, 1263 (1998).