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Strong solution of the stochastic Burgers equation

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This work introduces a pathwise notion of solution for the stochastic Burgers equation, in particular, our approach encompasses the Cole–Hopf solution. The developments are based on regularization arguments from the theory of distributions.

Keywords: stochastic Burgers equation; generalized functions; generalized stochastic processes; Colombeau algebras; stochastic partial differential equations

AMS Subject Classifications: 60H15; 46F99

1. Introduction

The aim of this paper is to study the existence of solution to the stochastic Burgers equation of the following form:

$$\begin{cases} \partial_t U(t,x) = \triangle U(t,x) + \partial_x U^2(t,x) + \partial_x W(t,x), \\ U(0,x) = \partial_x f(x). \end{cases} \tag{1}$$

where $W(t,x)$ is a space-time white noise.

There are many motivations for studying the stochastic Burgers equation. Several authors have indeed suggested to use the stochastic Burgers equation as a simple model for turbulence, has also been proposed to study the dynamics of interfaces and numerous applications were found in astrophysics and statistical physics, see for example [1,2]. We refer to [3,4] for a more detailed historical account of the stochastic Burgers equation from a purely mathematical point of view.

The main difficulty with the stochastic Burgers Equation (1) is that the solutions do not take values in a function space but in a generalized function space. Thus, it is necessary to give meaning to the non-linear term $\partial_x U^2$, because the usual product makes no sense for arbitrary distributions. We recall that is not possible to define a product for arbitrary distributions with good properties, see [5,6].

In this article, we deal with product of distributions via regularizations. This is, we approximate the distributions to be multiplied by smooth functions, multiply the approximations and pass to the limit (see for instance [6–8]).

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Bertini and Giacomin in [9], proposed that $U(t, x) = \partial_x \ln Z(t, x)$ is the meaningful solution to the stochastic Burgers equation. Here $Z(t, x)$ denotes the solution of the stochastic heat equation with multiplicative noise. This is known as the Cole–Hopf solution for the stochastic Burgers equation. We recall that the stochastic heat equation with multiplicative noise is the Itô equation,

$$\begin{cases} dZ = \triangle Z\, dt + Z dW \\ Z(0, x) = e^{f(x)}. \end{cases} \tag{2}$$

The Hopf–Cole solution is believed to be the correct physical solution for (1). However up to recently a rigorous notion of solution to the stochastic Burgers equation was lacking. In [10], Assing introduces a weak solution in a probabilistic sense for the Equation (1). The idea is to approximate the Cole–Hopf solution by the density fluctuations in weakly asymmetric exclusion. In [11], Gonçalves and Jara considered a similar type of solution.

In this article we introduce a nice space of generalized stochastic processes in order to prove that the Cole–Hopf solution is a strong solution to the Equation (1). We also show that the Cole–Hopf solution satisfies a certain type of stability.

Our space of generalized stochastic processes looks like a generalized stochastic process space in the sense of Itô–Gelfand–Vilenkin, see [12,13]. A main ingredient in our approach is the use of regularization techniques to define non-linear operations in the theory of distributions, we refer the reader to [6,7] for the background material.

Finally, we mention that the Cole–Hopf solution does not make sense in the multidimensional case since the solution of the stochastic heat equation is not a standard stochastic process. It is realized as a generalized stochastic process in the space of stochastic Hida distribution, see for example [14]. Thus, we expect solutions of Stochastic Burgers equation only in the sense of Colombeau's generalized functions. We refer the reader to [15–19] for applications of this theory in stochastic analysis.

The article is organized as follows: Section 2 reviews some basic facts on the standard cylindrical Wiener process. Section 3, we give a notion of random generalized functions, we introduce a new concept of solution for the stochastic Burgers equation and we prove that the Cole–Hopf solution solves (1). Also we prove that its has certain property of stability.

2. Cylindrical Wiener process

Let $\{W(t, \cdot) : t \in [0, T]\}$ be a standard cylindrical Wiener process in $L^2(\mathbb{R})$; it is canonically realized as a family of continuous processes satisfying:

(1) For any $\varphi \in L^2(\mathbb{R})$, $\{W_t(\varphi), t \in [0, T]\}$ is a Brownian motion with variance $t \int \varphi^2(x)\, dx$,
(2) For any $\varphi_1, \varphi_2 \in L^2(\mathbb{R})$ and $s, t \in [0, T]$,

$$\mathbb{E}(W_s(\varphi_1)(W_t(\varphi_2)) = (s \wedge t) \int \varphi_1(x)\varphi_2(x)\, dx.$$

Let $\{\mathcal{F}_t : t \in [0, T]\}$ be the σ-field generated by the P-null sets and the random variables $W_s(\varphi)$, where $\varphi \in \mathcal{D}(\mathbb{R})$ and $s \in [0, t]$. The predictable σ-field is the σ-field in $[0, T] \times \Omega$ generated by the sets $(s, t] \times A$ where $A \in \mathcal{F}_s$ and $0 \le s < t \le T$.

Let $\{v_j : j \in \mathbb{N}\}$ be a complete orthonormal basis of $L^2(\mathbb{R})$. For any predictable process $g \in L^2(\Omega \times [0, T], L^2(\mathbb{R}))$ it turns out that the following series is convergent in

$L^2(\Omega, \mathcal{F}, P)$ and the sum does not depend on the chosen orthonormal system:

$$\int_0^T g_t W_t := \sum_{j=1}^{\infty} \int_0^T (g_t, v_j) dW_t(v_j). \tag{3}$$

We notice that each summand in the above series is a classical Itô integral with respect to a standard Brownian motion, and the resulting stochastic integral is a real-valued random variable. The independence of the terms in the series (3) leads to the isometry property

$$\mathbb{E}\left(\left|\int_0^T g_s \, dW_s\right|^2\right) = \mathbb{E}\left(\int_0^T \int |g_s(x)|^2 \, dx \, ds\right).$$

See [20] for properties of the cylindrical Wiener process and stochastic integration. The mollifier cylindrical Wiener process is defined by:

$$W_t^n(x) := W_t(n\rho(n(x - \cdot))) \tag{4}$$

where $\rho : \mathbb{R} \to \mathbb{R}$ is an infinitely differentiable function with compact support such that $\int \rho(x) \, dx = 1$.

The mollifier Wiener process satisfies:

(1) The covariance of the mollifier Wiener process is given by

$$\mathbb{E}[W_t^n(x) W_s^n(y)] = s \wedge t \int C_n(x - y) \tag{5}$$

where $C_n(z) = \int \delta_n(z - u)\delta_n(-u)du$ and $\delta_n(z) = n\rho(nz)$.

(2) The quadratic variation of $W_t^n(x)$ is given by

$$\langle W^n(x) \rangle_t = Cnt \tag{6}$$

where $C = \int \rho^2(-x)dx$.

(3) The mollifier Wiener process is an approximation to the cylindrical Wiener process. For all $\varphi \in L^2(\mathbb{R})$,

$$\lim_{n\to\infty} \int W_t^n(x)\varphi(x)dx = W_t(\varphi). \tag{7}$$

In the case that φ has compact support the above convergence is a.e..

3. Solving the stochastic Burgers equation

We denote by $\mathcal{D}((0, T)\times\mathbb{R})$ the space of the infinitely differentiable functions with compact support in $(0, T) \times \mathbb{R}$ and $\mathcal{D}'((0, T) \times \mathbb{R})$ its dual.

Definition 3.1 Let **D** be the space of functions $T : \Omega \to \mathcal{D}'((0, T) \times \mathbb{R})$ such that $\langle T, \varphi \rangle$ is a random variable for all $\varphi \in \mathcal{D}((0, T) \times \mathbb{R})$. The elements of **D** are called random generalized functions.

The initial condition of the stochastic Burgers equation requires the usage of the notion of section of a distribution in the sense of Lojasiewicz, see [6,21]. For convenience of the reader, we present the relevant definitions.

Definition 3.2 A strict delta net is a net $\{\rho_\varepsilon : \varepsilon > 0\}$ of $\mathcal{D}((0, T))$ such that it satisfies:

(1) $\lim_{\varepsilon\to 0} supp(\rho_\epsilon) = \{0\}$.
(2) For all $\varepsilon > 0$, $\int \rho_\varepsilon(t)\, dt = 1$.
(3) $\sup_{\varepsilon>0} \int |\rho^\varepsilon(t)|\, dt < \infty$.

Definition 3.3 A distribution $H \in \mathcal{D}'((0,T)\times\mathbb{R})$ has a section $U \in \mathcal{D}'(\mathbb{R})$ at $t = 0$ if for all $\varphi \in \mathcal{D}(\mathbb{R})$ and all strict delta net $\{\rho_\varepsilon : \varepsilon > 0\}$,

$$\lim_{\varepsilon\to 0} \langle H, \rho_\epsilon \varphi\rangle = \langle U, \varphi\rangle.$$

3.1. *Existence of generalized solution*

We say that a random field $\{S(t,x) : t \in [0,T],\ x \in \mathbb{R}\}$ is a spatially dependent semimartingale if for each $x \in \mathbb{R}$, $\{S(t,x) : t \in [0,T]\}$ is a semimartingale in relation to the same filtration $\{\mathcal{F}_t : t \in [0,T]\}$. If $S(t,x)$ is a C^∞-function of x and continuous in t almost everywhere, it is called a C^∞-semimartingale. See [22] for a rigorous study of spatially depend semimartingales and applications to stochastic differential equations. Now, following ideas of regularization and passage to the limit, we introduce a new concept of solution for the stochastic Burgers equation.

Definition 3.4 We say that $U \in \mathbf{D}$ is a *generalized solution* of the Equation (1) if

(1) There exists a sequence of C^∞-semimartingales $\{U_n : n \in \mathbb{N}\}$ such that $U = \lim_{n\to\infty} U_n$ and there exists $\lim_{n\to\infty} \partial_x U_n^2$ in $\mathcal{D}'((0,T)\times\mathbb{R})$ almost surely for $\omega \in \Omega$.
(2) For all $\varphi \in \mathcal{D}((0,T)\times\mathbb{R})$,

$$\langle U, \partial_t\varphi\rangle + \langle \triangle U, \varphi\rangle + \langle \partial_x U^2, \varphi\rangle + \int_0^T \partial_x\varphi(t,\cdot)dW_t = 0$$

where $\partial_x U^2 := \lim_{n\to\infty} \partial_x U_n^2$.
(3) There exists a section of U at $t = 0$ and is equal to $\partial_x f$.

THEOREM 3.1 *Let $f \in C_b^\infty(\mathbb{R})$ and Z be a solution of the stochastic heat Equation* (2)*. Then $U = \partial_x \ln Z$ is a generalized solution of the stochastic Burgers Equation* (1).

Proof Let us denote by $H_n(t,x)$ the process $\ln Z_n(t,x)$, where Z_n is the solution of the regularized stochastic heat equation in the Itô sense

$$\begin{cases} dZ_n &= \triangle Z_n\, dt + Z_n\, dW^n, \\ Z_n(0,x) &= e^{f(x)}. \end{cases} \tag{8}$$

We observe that the solution of the Equation (8) is understood in a mild sense, this is, Z_n satisfies the equation

$$Z_n = G_t * f + \int_0^t (G_{t-s} * Z_n)dW_t^n$$

where $G_t = G(t,.)$ is the fundamental solution the heat equation. This construction is due to Bertini and Giacomin [23] p. 1884. The solution $Z(t,x)$ of the stochastic heat Equation (2) is too understood in a mild sense, see Definition 2.1 and Theorem 2.2 of [23].

By Itô formula and (6), we have

$$H_n = f + \int_0^t \triangle H_n \, ds + \int_0^t (\partial_x H_n)^2 \, ds + W_t^n - \frac{C}{2} nt. \tag{9}$$

Let $U_n(t,x) = \partial_x \ln Z_n(t,x)$. Multiplying (9) by $\partial_x \partial_t \varphi(t,x)$, where $\varphi \in \mathcal{D}((0,T) \times \mathbb{R})$, and integrating in $(0,T) \times \mathbb{R}$ we obtain that

$$\langle U_n, \partial_t \varphi \rangle + \langle \triangle U_n, \varphi \rangle + \langle \partial_x U_n^2, \varphi \rangle + \int_0^T (\partial_x \varphi(t,\cdot) * \delta_n) \, dW_t = 0. \tag{10}$$

We observe that Z_n converge to Z uniformly on compacts of $(0,T) \times \mathbb{R}$, see Bertini and Cancrini [23], Theorem 2.2. Thus,

$$\lim_{n \to \infty} \langle U_n, \partial_t \varphi \rangle = \langle U, \partial_t \varphi \rangle \tag{11}$$

and

$$\lim_{n \to \infty} \langle \triangle U_n, \varphi \rangle = \langle \triangle U, \varphi \rangle. \tag{12}$$

We recall that $\int_0^T \varphi(t,\cdot) \, dW_t$ defines a continuous linear functional from $\mathcal{D}((0,T) \times \mathbb{R})$ to $\mathbb{R}$, see Schaumloffel [24]. Then

$$\lim_{n \to \infty} \int_0^T (\partial_x \varphi(t,\cdot) * \delta_n) \, dW_t = \int_0^T \partial_x \varphi \, dW_t. \tag{13}$$

From the Equation (10) and the convergences (11), (12) and (13) we have that for all $\varphi \in \mathcal{D}[0,T) \times \mathbb{R})$,

$$\int_0^T \int_{\mathbb{R}} \partial_x U_n^2 \, \varphi(t,x) \, dt dx$$

converges and defines a linear functional in $\mathcal{D}'((0,T) \times \mathbb{R})$. Thus, the nonlinearity

$$\langle \partial_x (U)^2, \varphi \rangle := \lim_{n \to \infty} \int_0^T \int_{\mathbb{R}} \partial_x (U_n)^2 \, \varphi(t,x) \, dt dx$$

is well defined. From the continuity of $Z(t,x)$, see for instance [23], we observe that

$$\begin{aligned} \lim_{\varepsilon \to 0} \int \int_0^T \partial_x \ln \; Z(t,x) \, \rho_\varepsilon(t) \, dt \, \varphi(x) \, dx &= - \int f(x) \, \partial_x \varphi(x) \, dx \\ &= \int \partial_x f(x) \, \varphi(x) \, dx \end{aligned}$$

for all $\varphi \in \mathcal{D}(\mathbb{R})$ and for all strict delta net $\{\rho_\varepsilon : \varepsilon > 0\}$. Thus, we conclude that U is a generalized solution for the problem (1). □

Remark 3.1 In our formulation of the Equation (1), the initial condition is $U|_{\{t=0\}} = \partial_x f(x)$. Thus, if f and g are functions such that $\partial_x f(x) = \partial_x g(x)$ the solution of Equation (1) are the same. This is, we are identifying inicial conditions that have the same derivative.

Remark 3.2 The uniqueness problem is interesting and probably impossible to solve in this setting. The problem is that different approximations of the stochatic heat equations could have different limits, we refer to the reader to Section 2.4 of [25].

3.2. *Stability of the Cole–Hopf solution*

A well-known statement is that if a PDE is well posed, then the solutions to every reasonable sequence of approximating PDEs should converge to it. However, in SPDEs this statement is ambiguous since stochastic integrals are too irregular to be defined pathwise in an unambiguous way. See [26] for interesting comments in relation to this issue in SPDEs. The Cole–Hopf solution of the stochastic Burgers equation has this property of stability for approximations obtained as solutions of the stochastic Burgers equation driven by a mollifier Wiener process.

THEOREM 3.2 *Let $f \in C_b^\infty(\mathbb{R})$. Then the Cole–Hopf solution $U = \partial_x \ln\ Z$ of the stochastic Burgers equation with initial condition $\partial_x f$ satisfies the following stability property: If V_n are semimartingales such that*

$$V_n = \partial_x f + \int_0^t \triangle V_n \, ds \int_0^t \partial_x V_n^2 \, ds + \partial_x W_t^n$$

then $\lim_{n\to\infty} V_n = U$ *in* $\mathcal{D}'((0,T)\times\mathbb{R})$.

Proof We observe that $V_n = \partial_x H_n$ where H_n satisfies

$$H_n = f + \int_0^t \triangle H_n \, ds \int_0^t (\partial_x H_n)^2 \, ds + W_t^n.$$

We have that $G_n = e^{H_n}$ verifies the following Stratonovich equation

$$\begin{cases} dG_n &= \triangle G_n \, dt + G_n \circ dW^n \\ G_n(0,x) &= e^{f(x)}, \end{cases} \tag{14}$$

or equivalently the Itô equation

$$\begin{cases} dG_n &= \triangle G_n \, dt + G_n dW^n + n\, C\, G_n dt \\ G_n(0,x) &= e^{f(x)}. \end{cases} \tag{15}$$

A trivial calculation shows that $G_n = Z_n e^{Cnt}$ where Z_n is the solution of the Equation (8). Then

$$H_n = \ln\left(Z_n e^{C\,nt}\right).$$

Thus,

$$V_n = \partial_x H_n = \partial_x \ln\ Z_n.$$

By continuity, we have that V_n converges to $U = \partial_x \ln\ Z$ in $\mathcal{D}'((0,T)\times\mathbb{R})$. □

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