SIAM J. DISCRETE MATH.
Vol. 19, No. 4, pp. 848–880

CHAIN DECOMPOSITIONS OF 4-CONNECTED GRAPHS*

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Abstract. In this paper we give a decomposition of a 4-connected graph G into nonseparating chains, which is similar to an ear decomposition of a 2-connected graph. We also give an $O(|V(G)|^2|E(G)|)$ algorithm that constructs such a decomposition. In applications, the asymptotic performance can often be improved to $O(|V(G)|^3)$. This decomposition will be used to find four independent spanning trees in a 4-connected graph.

Key words. connectivity, nonseparating, good chain, chain decomposition, algorithm

AMS subject classifications. 05C40, 05C85, 05C38, 05C75

DOI. 10.1137/S0895480103434592

1. Introduction. In [1], Cheriyan and Maheshwari gave an $O(|V(G)|^2)$ algorithm for finding a "nonseparating ear decomposition" of a 3-connected graph G, and they used this decomposition to construct three independent spanning trees in a 3-connected graph.

In this paper we give a 4-connected version of the nonseparating ear decomposition of Cheriyan and Maheshwari and an $O(|V(G)|^2|E(G)|)$ algorithm for finding such a decomposition. This will be used in a forthcoming paper to find four independent spanning trees in an arbitrary 4-connected graph G, where the asymptotic performance can be improved to $O(|V(G)|^3)$.

We use the definitions and notation given in [2]. Some of those definitions are quite long, so we simply refer the readers to [2]. In particular, see [2] for definitions of *chain* (Definition 1.3 of [2]), *planar chain* (Definition 1.4 of [2]), *cyclic chain* (Definition 4.2 of [2]), and *planar cyclic chain* (Definition 4.3 of [2]). Intuitively, the roles of planar chains and planar cyclic chains in our decompsitions of 4-connected graphs are similar to those of paths and cycles in ear decompositions of 2-connected graphs.

In [2], we showed how to find the first planar chain in our decomposition of 4-connected graphs. The other chains in our decomposition can be classified into four types, as described below. The first three types are planar chains as defined in Definition 1.1. The fourth type is not a planar chain (but almost planar as we will see), and it is defined in Definition 1.2. See Figure 1 for illustrations of Definitions 1.1 and 1.2.

DEFINITION 1.1. *Let G be a graph, let F be a subgraph of G, and let $r \in V(F)$. Let H be a planar x-y chain in G such that $V(H) - \{x,y\} \subseteq V(G) - V(F)$.*

*Received by the editors September 17, 2003; accepted for publication (in revised form) June 3, 2005; published electronically December 30, 2005.
http://www.siam.org/journals/sidma/19-4/43459.html

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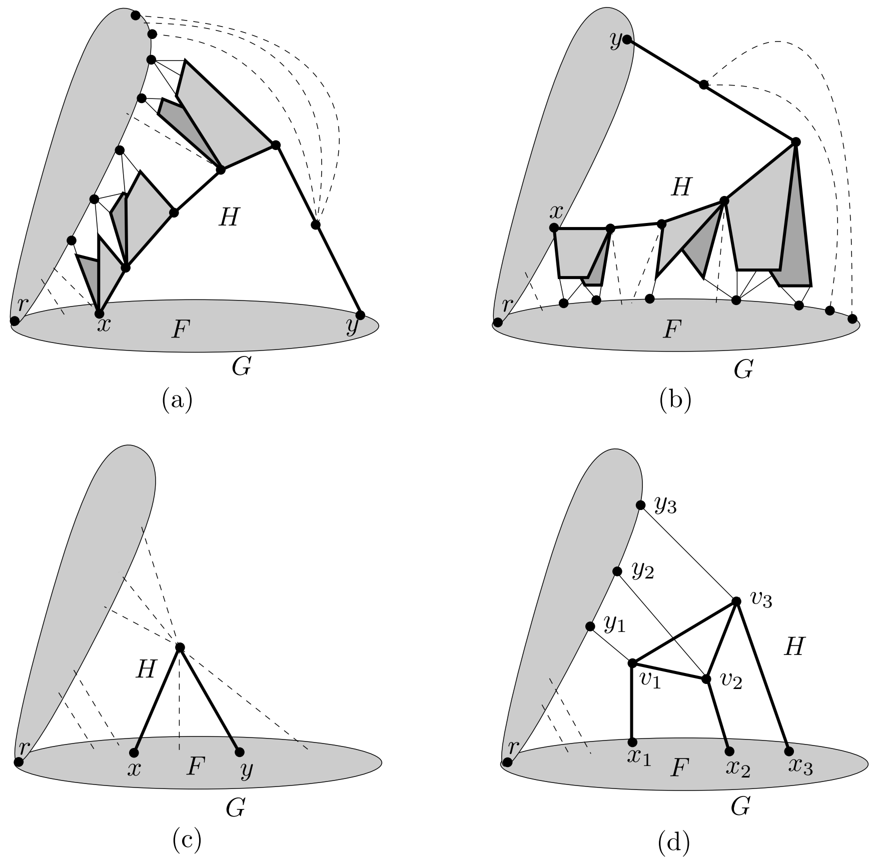

FIG. 1. (a) *An up F-chain,* (b) *a down F-chain,* (c) *an elementary F-chain, and* (d) *a triangle F-chain. The dashed edges need not exist.*

We say that

(a) *H is an up F-chain if $\{x, y\} \subseteq V(F)$ and $N_G(H - \{x, y\}) \subseteq (V(G) - V(F - r)) \cup \{x, y\}$;*

(b) *H is a down F-chain if $\{x, y\} \subseteq V(G) - V(F - r)$ and $N_G(H - \{x, y\}) \subseteq V(F - r) \cup \{x, y\}$; and*

(c) *H is an elementary F-chain if $\{x, y\} \subseteq V(F)$ and H is an x-y path of length two.*

In any of the three cases above we say that H is a planar *x-y F-chain in G (or simply, a* planar F-chain*). For an x-y chain H we let $I(H) := V(H) - \{x, y\}$, and for a cyclic chain H we let $I(H) := V(H)$.*

For a graph G, a subgraph H of G, and $S \subseteq V(G) \cup E(G)$, we let $H + S$ denote the graph with vertex set $V(H) \cup (S \cap V(G))$ and edge set $E(H) \cup (S \cap E(G))$.

DEFINITION 1.2. *Let G be a graph, let F be a subgraph of G, and let $r \in V(F)$. Suppose that $\{v_1, v_2, v_3\} \subseteq V(G) - V(F)$ induces a triangle T in G, and for each $1 \le i \le 3$, v_i has exactly one neighbor x_i in $V(F - r)$ and exactly one neighbor y_i in $V(G) - (V(F) \cup V(T))$ (thus, each v_i has degree four in G). Moreover, assume that x_1, x_2, x_3 are distinct and y_1, y_2, y_3 are distinct. Then we say that $H := T + \{x_1, x_2, x_3, v_1x_1, v_2x_2, v_3x_3\}$ is a* triangle F-chain *in G. We let $I(H) := \{v_1, v_2, v_3\}$.*

The definitions above depend on the choice of r and F, but in spite of this, whenever we use these concepts in this paper, it should be clear which pair r, F we refer to.

DEFINITION 1.3. *Let G be a graph, let F be a subgraph of G, and let $r \in V(F)$. By a* good F-chain in G, *we mean an up F-chain, a down F-chain, an elementary F-chain, or a triangle F-chain.*

We are now ready to describe a chain decomposition, which is similar to an ear decomposition.

DEFINITION 1.4. *Let G be a graph, let $r \in V(G)$, and let $H_1, \ldots, H_t$ be chains in G, where $t \geq 2$. We say that $(H_1, \ldots, H_t)$ is a* nonseparating chain decomposition of G rooted at r *if the following conditions hold:*

(i) *H_1 is a planar cyclic chain in G rooted at r;*

(ii) *for each $i = 2, \ldots, t-1$, H_i is a good $G[\bigcup_{j=1}^{i-1} I(H_j)]$-chain in G;*

(iii) *$H_t = G - (\bigcup_{j=1}^{t-1} I(H_j) - \{r\})$ is a planar cyclic chain in G rooted at r; and*

(iv) *for each $i = 1, \ldots, t-1$, both $G[\bigcup_{j=1}^{i} I(H_j)]$ and $G - (\bigcup_{j=1}^{i} I(H_j) - \{r\})$ are 2-connected.*

The chains $H_2, \ldots, H_{t-1}$ are called internal chains *of the nonseparating chain decomposition. If ra is a piece of H_1, then we say that $H_1, \ldots, H_t$ is a nonseparating chain decomposition of G* starting at ra.

The main result of this paper is the following.

THEOREM 1.5. *Let G be a 4-connected graph, let $r \in V(G)$, and let $ra \in E(G)$. Then G has a nonseparating chain decomposition rooted at r starting at ra, and such a decomposition can be found in $O(|V(G)|^2|E(G)|)$ time.*

The existence of the first chain H_1 of the chain decomposition is guaranteed by the next result which corresponds to Theorem 4.4 of [2].

THEOREM 1.6. *Let G be a 4-connected graph, and let $ra \in E(G)$. Then there exists a planar cyclic chain H in G rooted at r such that ra is a piece of H and $G - (V(H) - \{r\})$ is 2-connected. Moreover, such a chain can be found in $O(|V(G)||E(G)|)$ time.*

In order to construct the internal chains of the chain decomposition in Theorem 1.5, we need the following result which is Theorem 1.6 of [2].

THEOREM 1.7. *Let G be a graph, let $\{a, b\} \subseteq V(G)$, and let P be a nonseparating induced a-b path in G. Let B_P be a nontrivial block of $G - V(P)$, and let $X_P := N_G(G - V(B_P))$. Suppose $G - (V(B_P) - X_P)$ is $(4, X_P \cup \{a, b\})$-connected. Then there exists a planar a-b chain H in G such that $G - V(H)$ is 2-connected and $B_P \subseteq G - V(H)$. Moreover, such a chain can be found in $O(|V(G)||E(G)|)$ time.*

The rest of this paper is organized as follows. In section 2 we recall some lemmas proved in [2] and provide some new auxiliary lemmas concerning nonseparating induced paths. In section 3 we prove a technical result, which will be used to find the internal chains of a nonseparating chain decomposition. Finally, in section 4 we complete the proof of Theorem 1.5.

2. Nonseparating paths.

In this section we state and prove some results concerning nonseparating induced paths which will be used later. First, we state two lemmas without proof, which are Lemmas 2.3 and 2.4 of [2], respectively.

LEMMA 2.1. *Let G be a connected graph, $S \subseteq V(G)$, $\{a, a'\} \subseteq S$, and let P be an a-a' path in G. Suppose*

(i) *G is $(3, S)$-connected, and*

(ii) *$S - \{a, a'\}$ is contained in a component U of $G - V(P)$.*

Then there exists a nonseparating induced a-a' path P' in G such that $V(P') \cap V(U) = \emptyset$. Moreover, such a path can be found in $O(|V(G)| + |E(G)|)$ time.

LEMMA 2.2. *Let G be a graph and $S := \{a, a', b, b'\} \subseteq V(G)$. Suppose that G is $(4, S)$-connected. Then exactly one of the following holds:*

(1) *there exists a nonseparating induced a-a' path P' in G such that $V(P') \cap \{b, b'\} = \emptyset$;*

(2) *(G, a, b, a', b') is planar.*

Moreover, one can in $O(|V(G)| + |E(G)|)$ time find a path as in (i) *or certify that* (ii) *holds.*

Note our use of "prime" notation in the statements of the lemmas. The reader should not infer that the paths labeled P' are derived from an assumed path P. We reserve P to denote a particular path specified in section 3, and we therefore label paths P' in the statements of our lemmas. We hope this will sidestep any source of confusion when these lemmas are applied.

The next lemma is a variation of Lemma 2.1 (and Lemma 2.2 as well) in which we prove the existence of a specific nonseparating induced path. However, here it is not possible to specify the ends of the desired path. Moreover, in the hypotheses of Lemma 2.3 there are some technical conditions which arise when we try to produce an internal chain. Note that conditions (iii), (iv), and (v) of Lemma 2.3 are automatically satisfied if G is $(4, S \cup \{b, b'\})$-connected. Actually, this is the case in all applications of Lemma 2.3 with the exception of the proof of Lemma 3.15, where the more complicated conditions are required.

LEMMA 2.3. *Let G be a graph, let $S \subset V(G)$, and let $\{b, b'\} \subseteq V(G) - S$. Suppose*

(i) *$G - S$ is 2-connected,*

(ii) *every element of S has a neighbor in $V(G) - (S \cup \{b, b'\})$,*

(iii) *G is $(3, S \cup \{b, b'\})$-connected,*

(iv) *if $|S| = 2$, then G is $(4, S \cup \{b, b'\})$-connected, and*

(v) *if $|S| \geq 3$, then there exists some component of $G - (S \cup \{b, b'\})$ which has at least two neighbors in S.*

Then exactly one of the following holds:

(1) *there exist $a, a' \in S$ and an induced a-a' path P' in G such that $V(P') \cap \{b, b'\} = \emptyset$, $V(P') \cap S = \{a, a'\}$, and $G - (V(P') \cup S)$ is connected;*

(2) *$|S| = 2$, and the elements of S can be labeled as a, a' such that (G, a, b, a', b') is planar.*

Moreover, one can in $O(|V(G)| + |E(G)|)$ time find a path as in (1) *or certify that* (2) *holds.*

Proof. First, suppose that $|S| = 2$. Let a, a' denote the vertices in S. By (iv) G is $(4, \{a, a', b, b'\})$-connected. Thus, by Lemma 2.2 exactly one of the following holds:

(a) there exists a nonseparating induced a-a' path P' such that $V(P') \cap \{b, b'\} = \emptyset$; or

(b) (G, a, b, a', b') is planar.

Moreover, one can in $O(|V(G)| + |E(G)|)$ time find a path as in (a) or certify that (b) holds. If (a) holds, then P' is the required path in (1). If (b) holds, then (2) holds.

Thus, we may assume that $|S| \geq 3$. First, we prove the following.

Claim. There exist $a, a^* \in S$ and an a-a^* path Q in $G - (S - \{a, a^*\})$ such that b and b' are contained in a component of $G - V(Q)$. Moreover, such a path can be found in $O(|V(G)| + |E(G)|)$ time.

Proof of Claim. We consider two cases. See Figure 2 for an illustration of the outcomes of Lemma 2.3.

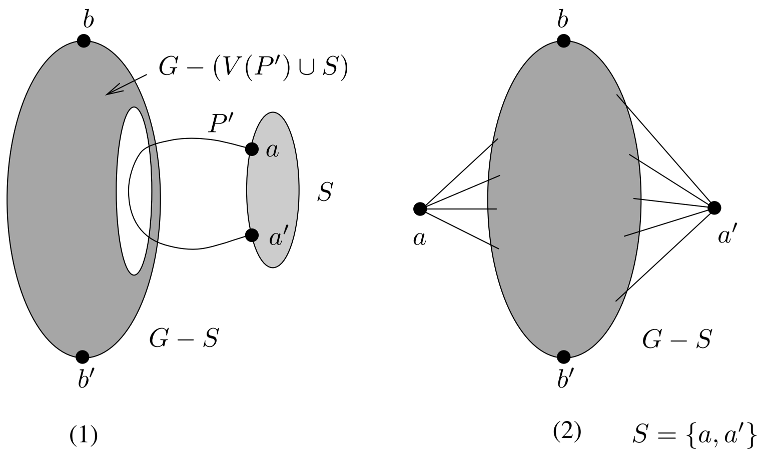

FIG. 2. *Outcomes in Lemma* 2.3.

Case 1. $G-(S\cup\{b,b'\})$ is not connected.

In this case, there exist edge-disjoint subgraphs G_1, G_2 of $G-S$ such that $G_1 \cup G_2 = G-S$, $V(G_1)\cap V(G_2)=\{b,b'\}$, and $|V(G_1)|\geq 3\leq |V(G_2)|$. Note that such a partition can be found in $O(|V(G)|+|E(G)|)$ time. Since $G-S$ is 2-connected (by (i)), both G_1 and G_2 are connected.

By (v) there exists some component K of $G-(S\cup\{b,b'\})$ which has at least two neighbors in S. Note that such a component can be found in $O(|V(G)|+|E(G)|)$ time. We may assume that $V(K)\subseteq V(G_1)$. Let $a,a^*\in N_G(K)\cap S$, and let Q be an a-a^* path in $G[V(K)\cup\{a,a^*\}]$. Since G_2 is connected, b,b' are contained in a component of $G-V(Q)$. Moreover, such a path can be found in $O(|V(G)|+|E(G)|)$ time.

Case 2. $G-(S\cup\{b,b'\})$ is connected.

Since $G-S$ is 2-connected by (i), one can find in $O(|V(G)|+|E(G)|)$ time two internally disjoint b-b' paths P_1,P_2 in $G-S$. Let a_1,a_2,a_3 be distinct vertices in S. For $i=1,2,3$, let $v_i\in N_G(a_i)\subseteq V(G)-(S\cup\{b,b'\})$ (they exist by (ii)). Since $G-(S\cup\{b,b'\})$ is connected by assumption, for each $i=1,2,3$, there exists a path Q_i from v_i to some vertex u_i in $(V(P_1)\cup V(P_2))-\{b,b'\}$ internally disjoint from $V(P_1)\cup V(P_2)$. Moreover, such paths can be found in $O(|V(G)|+|E(G)|)$ time. Note that at least two (not necessarily distinct) vertices in u_1,u_2,u_3 lie on the same path $P_1-\{b,b'\}$ or $P_2-\{b,b'\}$. By symmetry, we may assume that $u_1,u_2\in V(P_1)-\{b,b'\}$. Then there exist disjoint paths in $G-(S-\{a_1,a_2\})$ from a_1 to a_2 (the path contained in $Q_1\cup Q_2\cup(P_1-\{b,b'\})$) and from b to b' (the path P_2), respectively. Thus, the result follows by taking $a=a_1$ and $a^*=a_2$. Moreover, it is not hard to see that such paths can be found in $O(|V(G)|+|E(G)|)$ time. ∎

Now given a,a^* and Q, we will describe how to find $a'\in S$ and an induced a-a' path P' such that $V(P')\cap\{b,b'\}=\emptyset$, $V(P')\cap S=\{a,a'\}$, and $G-(V(P')\cup S)$ is connected. Let G' be the graph obtained from G by identifying the vertices in $S-\{a\}$ to a single vertex a'' and removing the resulting multiple edges. Let $S':=\{a,a'',b,b'\}$.

We claim that G' is $(3,S')$-connected. Suppose for a contradiction that there exists $T\subseteq V(G')$ such that $|T|\leq 2$ and $G'-T$ has a component K with $V(K)\cap S'=\emptyset$. Clearly $a''\in T$ because G is $(3,S\cup\{b,b'\})$-connected (by (iii)); then either $a\in T$ or

$T - \{a''\}$ is a vertex cut of $G - S$, which is a contradiction since $G - S$ is 2-connected (by (i)). Thus, G' is $(3, S')$-connected.

Note that the a-a^* path Q in G corresponds to an a-a'' path P in G', and $S' - \{a, a''\} = \{b, b'\}$ is contained in a component U of $G' - V(P)$. Thus, the hypotheses of Lemma 2.1 are satisfied with G', S', P, a, a'', U as G, S, P, a, a', U, respectively. Hence, there exists a nonseparating induced a-a'' path P'' in G' such that $V(P'') \cap V(U) = \emptyset$. Moreover, such a path P'' can be found in $O(|V(G')| + |E(G')|)$ time (hence, in $O(|V(G)| + |E(G)|)$ time). The path P'' corresponds to an induced a-a' path P' in G for some $a' \in S - \{a\}$ such that $V(P') \cap \{b, b'\} = \emptyset$ and $V(P') \cap S = \{a, a'\}$. Since P'' is nonseparating in G', $G - (V(P') \cup S)$ is connected. Therefore, a, a' and P' satisfy (1), and they can be found in $O(|V(G)| + |E(G)|)$ time. ◻

The following lemma is a variation of Lemma 2.3 (by letting $b = b'$), and its proof is essentially the same. For the sake of completeness, we include it here.

LEMMA 2.4. *Let G be a graph, let $S \subseteq V(G)$, and let $b \in V(G) - S$. Suppose*

(i) *$G - S$ is 2-connected,*

(ii) *every element of S has a neighbor in $V(G) - (S \cup \{b\})$, and*

(iii) *G is $(3, S \cup \{b\})$-connected.*

Then there exist $a, a' \in S$ and an induced a-a' path P' in G such that $V(P') \cap \{b\} = \emptyset$, $V(P') \cap S = \{a, a'\}$, and $G - (V(P') \cup S)$ is connected. Moreover, such a path can be found in $O(|V(G)| + |E(G)|)$ time.

Proof. Since G is $(3, S \cup \{b\})$-connected (by (iii)), $|S| \geq 2$, so let $a, a^* \in S$. Since $G - S$ is 2-connected (by (i)), $G - (S \cup \{b\})$ is connected. Since a and a^* have a neighbor in $V(G) - (S \cup \{b\})$ (by (ii)), there exists an a-a^* path Q in $G - ((S - \{a, a^*\}) \cup \{b\})$. Clearly, such a path can be found in $O(|V(G)| + |E(G)|)$ time.

Let G' be the graph obtained from G by identifying the vertices in $S - \{a\}$ to a single vertex a'' and removing the resulting multiple edges. Let $S' := \{a, a'', b\}$.

We claim that G' is $(3, S')$-connected. Suppose for a contradiction that there exists $T \subseteq V(G')$ such that $|T| \leq 2$ and $G' - T$ has a component K with $V(K) \cap S' = \emptyset$. Clearly, $a'' \in T$ because G is $(3, S \cup \{b\})$-connected (by (iii)). But then either $a \in T$ or $T - \{a''\}$ is a vertex cut of $G - S$, which is a contradiction since $G - S$ is 2-connected (by (i)). Thus, G' is $(3, S')$-connected.

Note that the a-a^* path Q in G corresponds to an a-a'' path P in G', and $S' - \{a, a''\} = \{b\}$ is contained in a component U of $G' - V(P)$. Thus, by Lemma 2.1 (with G', S', P, a, a'', U as G, S, P, a, a', U, respectively), there exists a nonseparating induced a-a'' path P'' in G' such that $V(P'') \cap V(U) = \emptyset$. Moreover, such a path P'' can be found in $O(|V(G')| + |E(G')|)$ time (and hence, in $O(|V(G)| + |E(G)|)$ time). The path P'' corresponds to an induced a-a' path P' in G for some $a' \in S - \{a\}$ such that $V(P') \cap \{b\} = \emptyset$ and $V(P') \cap S = \{a, a'\}$. Since P'' is nonseparating in G', $G - (V(P') \cup S)$ is connected. So a, a' and P' are as required, and they can be found in $O(|V(G)| + |E(G)|)$ time. ◻

Some results and algorithms which we use here require that we find an embedding of a planar graph (G, a, b, c, d) in a closed disk such that a, b, c, d occur on the boundary of the disk in that cyclic order. This can be done in linear time using an algorithm of Hopcroft and Tarjan [4] (or a more recent algorithm by Hsu and Shih [5]). For convenience, we state this result as our next lemma.

LEMMA 2.5. *Let (G, a, b, c, d) be a planar graph. Then one can find in $O(|V(G)| + |E(G)|)$ time an embedding of G in a closed disk such that a, b, c, d occur on the boundary of the disk in that cyclic order.*

Let (G, a, b, a', b') be a planar graph. Then any a-a' path in $G - \{b, b'\}$ separates b from b'. The next lemma shows that one can find efficiently an a-a' path P' in $G - \{b, b'\}$ such that $G - V(P')$ has exactly two components. This will be used in section 3.

LEMMA 2.6. *Let (G, a, b, a', b') be a planar graph with $|V(G)| \geq 5$ and suppose G is $(4, \{a, a', b, b'\})$-connected. Then there exists an induced a-a' path P' in G such that $G - V(P')$ has exactly two components K and K' with $b \in V(K)$ and $b' \in V(K')$. Moreover, such a path can be found in $O(|V(G)| + |E(G)|)$ time.*

Proof. Take an embedding of G in a closed disk such that a, b, a', b' occur on the boundary of the disk in the cyclic order listed. By Lemma 2.5, this can be done in $O(|V(G)| + |E(G)|)$ time. Let $G' := (G - b') + \{ab, a'b\}$.

We claim that G' is 2-connected. Suppose for a contradiction that G' is not 2-connected. Let x be a cut vertex of G'. Since $|V(G)| \geq 5$ and G is $(4, \{a, a', b, b'\})$-connected, $G - \{b, b'\}$ contains an a-a' path, and hence, $\{a, a', b\}$ is contained in a cycle in G'. Therefore, $\{a, a', b\}$ is contained in an x-bridge of G', and G' has another x-bridge B such that $(V(B) - \{x\}) \cap \{a, a', b\} = \emptyset$. Hence, $B - x$ is a component of $G - T$, where $T := \{x, b'\}$ and $(V(B) - \{x\}) \cap \{a, a', b, b'\} = \emptyset$, which contradicts the assumption that G is $(4, \{a, a', b, b'\})$-connected.

Thus, we can assume that $ab, a'b$ are in the cycle bounding the infinite face of G'. Let P' be the a-a' subpath of this cycle which avoids b. Note that $N_G(b') \subseteq V(P')$ and P' can be computed in $O(|V(G)| + |E(G)|)$ time.

We claim that $G' - V(P')$ is connected. Suppose for a contradiction that $G' - V(P')$ is not connected. Let $\mathcal{K}$ be the set of components of $G' - V(P')$ which do not contain b. For any $K \in \mathcal{K}$, let $u_K, u'_K \in V(P')$ such that $N_{G'}(K) \cap V(P') \subseteq V(P'[u_K, u'_K])$ and $P'[u_K, u'_K]$ is minimal with respect to this property. If $|\mathcal{K}| \geq 2$, choose $K \in \mathcal{K}$ such that for any $K' \neq K$, if $E(P[u_K, u'_K]) \cap E(P[u_{K'}, u'_{K'}]) \neq \emptyset$, then $P[u_K, u'_K] \subseteq P[u_{K'}, u'_{K'}]$; such a component must exist because of planarity. If $|\mathcal{K}| = 1$, let $\mathcal{K} = \{K\}$. In either case, $N_G(P'(u_K, u'_K)) \subseteq V(K) \cup \{u_K, u'_K, b'\}$. Thus, $K \cup P'(u_K, u'_K)$ is contained in a component of $G - \{u_K, u'_K, b'\}$ that does not contain any vertex in $\{a, a', b, b'\}$, which contradicts the assumption that G is $(4, \{a, a', b, b'\})$-connected.

So $G' - V(P') = G - (V(P') \cup \{b'\})$ is connected. Hence, $G - V(P')$ has exactly two components K and K' with $b \in V(K)$ and $b' \in V(K')$.

We now show that P' is an induced path in G. Suppose on the contrary that P' is not induced. Let $e = xy \in E(G) - E(P')$ with $x, y \in V(P')$. Then $V(P'(x, y)) \neq \emptyset$. Moreover, by planarity $N_G(P'(x, y)) \subseteq \{x, y, b'\}$. Then $P'(x, y)$ is contained in a component of $G - \{x, y, b'\}$ that does not contain any vertex in $\{a, a', b, b'\}$, which contradicts again the assumption that G is $(4, \{a, a', b, b'\})$-connected.

Thus, P' is a path as required. Moreover, it is easy to see that such a path can be found in $O(|V(G)| + |E(G)|)$ time. ∎

We conclude this section with another lemma which concerns nonseparating induced paths in planar graphs.

LEMMA 2.7. *Let (G, a, a', b, b') be a planar graph with $|V(G)| \geq 5$ and suppose G is $(4, \{a, a', b, b'\})$-connected and $G \not\cong K_{1,4}$. Then there exists a nonseparating induced a-a' path P' in G such that $V(P') \cap \{b, b'\} = \emptyset$. Moreover, such a path can be found in $O(|V(G)| + |E(G)|)$ time.*

Proof. For convenience, let $S := \{a, a', b, b'\}$. Take an embedding of G in a closed disk such that a, a', b, b' occur on the boundary of the disk in the cyclic order listed. By Lemma 2.5, this can be done in $O(|V(G)| + |E(G)|)$ time. Let $G' := G + \{ab', a'b\}$.

We claim that G' is 2-connected. Suppose for a contradiction that G' is not 2-connected. Let x be a cut vertex of G'. Since $|V(G)| \geq 5$ and G (and hence, G') is $(4, S)$-connected, it follows that any component of $G' - x$ either contains vertices only in S or contains at least one vertex in $V(G) - S$ and at least three vertices in S. Since $a'b, ab' \in E(G')$, $G' - x$ cannot have both kinds of components. Therefore, every component of $G' - x$ contains vertices only in S. Moreover, since $|V(G)| \geq 5$, $x \notin S$. But then, it is easy to see that (G, a, a', b, b') must be isomorphic to $K_{1,4}$ with x as the vertex of degree four, which contradicts the hypothesis. Hence, G' is 2-connected.

Thus, we can assume that $ab', a'b$ are in the cycle bounding the infinite face of G'. Let P' be the a-a' subpath of this cycle which avoids b and b'. Note that P' is an a-a' path in G and such a path can be found in $O(|V(G)| + |E(G)|)$ time.

We claim that P' is nonseparating in G. Suppose for a contradiction that $G' - V(P')$ is not connected. Note that b and b' are contained in a component of $G - V(P')$. Let $\mathcal{K}$ be the set of components of $G' - V(P')$ which contain neither b nor b'. For any $K \in \mathcal{K}$, let $u_K, u'_K \in V(P')$ such that $N_{G'}(K) \cap V(P') \subseteq V(P'[u_K, u'_K])$ and $P'[u_K, u'_K]$ is minimal with respect to this property. If $|\mathcal{K}| \geq 2$, choose $K \in \mathcal{K}$ such that for any $K' \neq K$, if $E(P[u_K, u'_K]) \cap E(P[u_{K'}, u'_{K'}]) \neq \emptyset$, then $P[u_K, u'_K] \subseteq P[u_{K'}, u'_{K'}]$; such a component must exist because of planarity. If $|\mathcal{K}| = 1$, let $\mathcal{K} = \{K\}$. In either case, $N_G(P'(u_K, u'_K)) \subseteq V(K) \cup \{u_K, u'_K\}$. Thus, $K \cup P'(u_K, u'_K)$ is contained in a component of $G - \{u_K, u'_K\}$ that does not contain any vertex in S, which contradicts the assumption that G is $(4, S)$-connected. Thus, $G - V(P')$ is connected.

Next we show that P' is an induced path in G. Suppose by contradiction that P' is not induced. Let $e = xy \in E(G) - E(P')$ such that $x, y \in V(P')$. Then $V(P'(x, y)) \neq \emptyset$. Moreover, by planarity $N_G(P'(x, y)) \subseteq \{x, y\}$. Then $P'(x, y)$ is contained in a component of $G - \{x, y\}$ that does not contain any vertex in S, which again contradicts the assumption that G is $(4, S)$-connected.

Thus, P' is a nonseparating induced a-a' path in G such that $V(P') \cap \{b, b'\} = \emptyset$ as required. ∎

3. Internal chains. In this section, we prove the following theorem, which will be used to construct internal chains in a nonseparating chain decomposition. See Figure 3 for an illustration of the statement of the result. Recall that, for a graph K and $u, v \in V(K)$, $K - uv$ denotes the graph with vertex set $V(K)$ and edge set $E(K) - \{uv\}$ (note that uv need not be an edge of K).

DEFINITION 3.1. *Let G be a 4-connected graph, let F be a subgraph of G, and let $r \in V(F)$ such that $G_F := G - (V(F) - \{r\})$ is 2-connected. For any distinct $a, a' \in V(F)$, an a-a' path in $G - aa'$ is said to be a* feasible *F-path if the following hold:*

(i) *$V(P) \cap V(F) = \{a, a'\}$ and P is an induced path in $G - aa'$;*
(ii) *$P(a, a')$ is a non-separating path in G_F;*
(iii) *r is contained in a nontrivial block B_P of $G_F - V(P(a, a'))$; and*
(iv) *if $r \in \{a, a'\}$, then r is not a cut vertex of $G_F - V(P(a, a'))$.*

Remark 1. Condition (iv) in Definition 3.1 is necessary for a technical reason, and the reader may want to assume in a first reading that $r \notin \{a, a'\}$ to become familiar with the proof of the next result.

THEOREM 3.2. *Let G be a 4-connected graph, let F be a subgraph of G, and let $r \in V(F)$ such that $G_F := G - (V(F) - \{r\})$ is 2-connected. Suppose that G has a* feasible *a-a'* F-path *P for some $a, a' \in V(F)$. Then there exists a good F-chain*

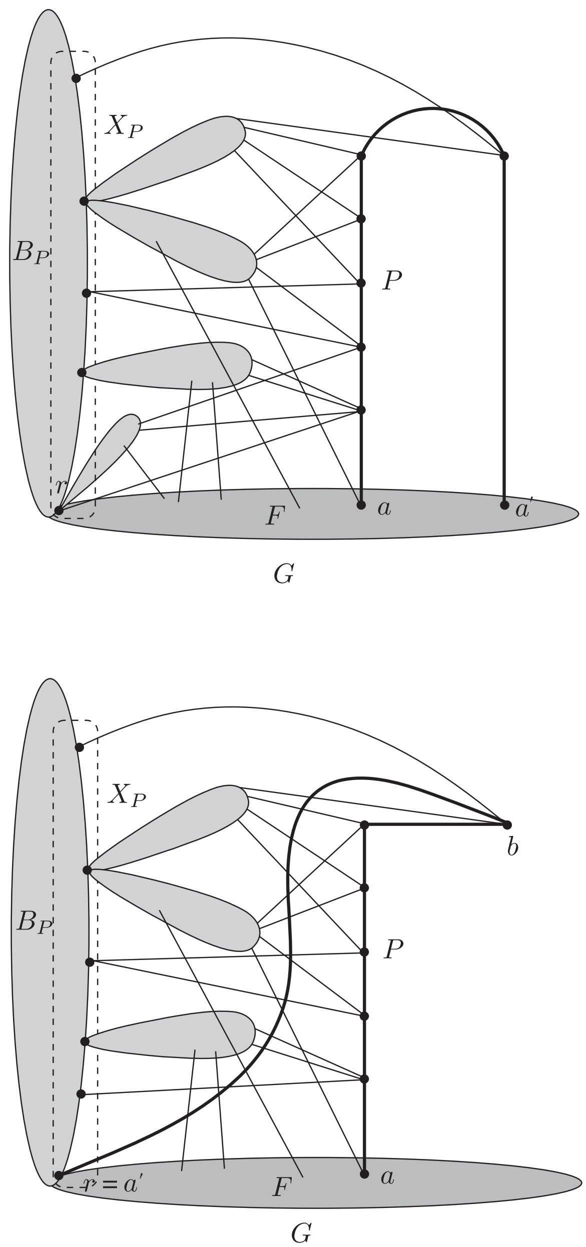

FIG. 3. *Illustration for Theorem* 3.2 *and Notation and Definition* 3.3*: one with* $r \notin \{a, a'\}$ *and the other with* $r \in \{a, a'\}$.

H in G such that $G_F - I(H)$ is 2*-connected, $G[V(F) \cup I(H)]$ is* 2*-connected, and $B_P \subseteq G_F - I(H)$. Moreover, such a chain can be found in $O(|V(G)||E(G)|)$ time.*

Throughout the rest of this section, we fix the following notation.

NOTATION AND DEFINITION 3.3. *Let G be a* 4*-connected graph, let F be a subgraph of G, and let $r \in V(F)$ such that $G_F := G - (V(F) - \{r\})$ is* 2*-connected.*

Suppose G has a feasible a-a' F-path P and r is contained in a nontrivial block B_P of $G_F - V(P(a, a'))$.

*Let $\mathcal{P}_P$ be the set of feasible F-paths P' (with ends, say u, u') in G such that $B_P \subseteq G_F - V(P'(u, u'))$. For each $P' \in \mathcal{P}_P$ with ends, say u, u', let $B_{P'}$ denote the block of $G_F - V(P'(u, u'))$ which contains B_P. We say that $P' \in \mathcal{P}_P$ is a B_P-*augmenting path *if* $|V(B_P)| < |V(B_{P'})|$.

We will describe an algorithm for finding a good F-chain as required in Theorem 3.2. The idea of the algorithm is roughly the following. At the beginning of each iteration we have vertices $a, a' \in V(F)$ and a feasible a-a' F-path P in G. The algorithm iteratively tries to find a B_P-augmenting path P' with ends u, u', and start a new iteration with u, u', P' as a, a', P, respectively. Note that r, u, u', P', F and G (as r, a, a', P, F and G, respectively) satisfy the hypotheses of Theorem 3.2 with B_P enlarged to $B_{P'}$. When the algorithm does not find such a path, it finds a good F-chain as required in Theorem 3.2.

The next lemma says that (assuming G has a feasible a-a' F-path P) one can find in $O(|V(G)| + |E(G)|)$ time a feasible u-u' F-path P' such that $|V(P')| = 3$ or $N_G(P'(u, u')) \cap V(F) \subseteq \{u, u'\} \cup \{r\}$. The latter condition is equivalent to requiring that $N_G(P'(u, u')) \cap V(F) = \{u, u'\}$ when $r \in \{u, u'\}$ or that $N_G(P'(u, u')) \cap V(F) \subseteq \{u, u', r\}$ when $r \notin \{u, u'\}$ (see Figure 3).

LEMMA 3.4. *There exist $u, u' \in V(F)$ and a feasible u-u' F-path P' such that*

(1) $|V(P')| = 3$ *or* $N_G(P(u, u')) \cap V(F) \subseteq \{u, u'\} \cup \{r\}$*, and*

(2) $B_P \subseteq B_{P'}$.

Moreover, such a path can be found in $O(|V(G)| + |E(G)|)$ time.

Proof. If either $|V(P)| = 3$ or $N_G(P(a, a')) \cap V(F) \subseteq \{a, a'\} \cup \{r\}$, then the result follows with $P' := P$.

Thus, assume that $|V(P)| \geq 4$ and $(N_G(P(a, a')) \cap V(F)) - (\{a, a'\} \cup \{r\}) \neq \emptyset$. By symmetry, we may assume that $a \neq r$. Let $v \in V(P(a, a'))$ such that v has a neighbor in $V(F) - (\{a, a'\} \cup \{r\})$, and subject to this, $P[a, v]$ is minimal. If v has two neighbors in $V(F) - \{r, a\}$, say u and u', let $P' := (u, v, u')$. In this case, (1) holds with $|V(P')| = 3$. If v has exactly one neighbor in $V(F) - \{r, a\}$, say u, then let $P' := P[a, v] + \{u, vu\}$ and $u' := a$. Note that in both cases $r \notin \{u, u'\}$. By the choice of v, $N_G(P(u, u')) \cap V(F) \subseteq \{u, u'\} \cup \{r\}$, and hence, (1) holds. Moreover, since $G_F - V(P(a, a')) \subseteq G_F - V(P'(u, u'))$, we have $B_P \subseteq G_F - V(P'(u, u'))$, and hence, (2) holds.

Finally, we show that P' is a feasible u-u' F-path. Since P is induced in $G - aa'$, P' is induced in $G - uu'$. Clearly $V(P') \cap V(F) = \{u, u'\}$, so (i) of Definition 3.1 holds. Since G_F is 2-connected and $P(a, a')$ is an induced path in $G_F - aa'$, if $V(P(v, a')) \neq \emptyset$, then $N_{G_F}(P(v, a')) \cap (V(G_F) - V(P(a, a'))) \neq \emptyset$. Thus, since $G_F - V(P(a, a'))$ is connected, $P'(u, u')$ is nonseparating in G_F, so (ii) of Definition 3.1 holds. Also, r is contained in a nontrivial block of $G_F - V(P'(u, u'))$ because $r \in B_P \subseteq G_F - V(P'(u, u'))$, so (iii) of Definition 3.1 holds. Since $r \notin \{u, u'\}$, we do not need to verify (iv) of Definition 3.1.

Therefore, P' is a feasible F-path as required, and it is not hard to see that such a path P' can be found in $O(|V(G)| + |E(G)|)$ time. ☐

Assumption 1. Using Lemma 3.4, we can preprocess a feasible F-path at the beginning of each iteration (in $O(|V(G)| + |E(G)|)$ time). Henceforth, we may assume that for the (current) feasible F-path P, $|V(P)| = 3$ or $N_G(P(a, a')) \cap V(F) \subseteq \{a, a'\} \cup \{r\}$. We may also assume that $G_F - V(P(a, a'))$ is not 2-connected; otherwise, $H := P$ gives an F-chain as required in Theorem 3.2: H is an up F-chain (where each of its blocks is trivial), or H is an elementary F-chain. Moreover, $G_F - I(H) = G_F - V(P(a, a'))$ is 2-connected.

NOTATION 3.5. *Let $X_P := N_{G_F}(G_F - V(B_P))$. For each B_P-bridge B of $G_F - V(P(a, a'))$, let r_B denote the unique vertex in $V(B) \cap V(B_P)$. Note that $r_B \in X_P$. Also, if $r \in \{a, a'\}$, then $r \in X_P$.*

Remark 2. Note that since G_F is 2-connected, we have $|X_P| \geq 2$. Moreover, if B is a B_P-bridge of $G_F - V(P(a, a'))$, then $V(B) - \{r_B\}$ has a neighbor in $V(P(a, a'))$.

The next lemma shows that if, for every B_P-bridge B of $G_F - V(P(a, a'))$, $N_G(B - r_B) \subseteq V(P)$, then one can find efficiently a good F-chain (in fact, an up F-chain) H as required in Theorem 3.2 by invoking Theorem 1.7.

LEMMA 3.6. *Suppose that for every B_P-bridge B of $G_F - V(P(a, a'))$, $N_G(B - r_B) \subseteq V(P)$. Then there exists an a-a' up F-chain H in G such that $G_F - I(H)$ is 2-connected, $G[V(F) \cup I(H)]$ is 2-connected, and $B_P \subseteq G_F - I(H)$. Moreover, such a chain can be found in $O(|V(G)||E(G)|)$ time.*

Proof. Suppose first that $r \notin \{a, a'\}$ (see Figure 3). Let G' be the graph obtained from G_F by adding $\{a, a'\}$ and the edges of G from $\{a, a'\}$ to $V(G_F) - \{r\}$. Note that P is a nonseparating induced a-a' path in G'. Note also that B_P is a nontrivial block of $G' - V(P)$. Let $X'_P = N_{G'}(G' - V(B_P))$.

We claim that $G' - (V(B_P) - X'_P)$ is $(4, X'_P \cup \{a, a'\})$-connected. For convenience, let $K := G' - (V(B_P) - X'_P)$. Since, for any B_P-bridge B of $G' - V(P) = G_F - V(P(a, a'))$, $V(B) - \{r_B\}$ has a neighbor in $V(P(a, a'))$, it follows that K is connected and $K - (X'_P \cup \{a, a'\})$ is a component of $G - (X'_P \cup \{a, a'\})$. Hence, because G is 4-connected, K is $(4, X'_P \cup \{a, a'\})$-connected.

Thus, the hypotheses of Theorem 1.7 are satisfied with G', a, a', P, B_P, X'_P as G, a, b, P, B_P, X_P, respectively. Hence, there exists a planar a-a' chain H in G' such that $G' - V(H) = G_F - I(H)$ is 2-connected and $B_P \subseteq G' - V(H) = G_F - I(H)$. Moreover, such a chain can be found in $O(|V(G')||E(G')|)$ time (and hence, in $O(|V(G)||E(G)|)$ time). Note also that H is an up F-chain in G. Hence, $G[V(F) \cup I(H)]$ is 2-connected, so the result follows.

Now suppose that $r \in \{a, a'\}$, and without loss of generality, let $r = a'$ (see Figure 3). Let b be the neighbor of r in P. Let G' be the graph obtained from G_F by adding a and the edges of G from a to $V(G_F) - \{r\}$. Note that $b \in V(G')$ and $P[a, b]$ is a nonseparating induced path in G'. Note also that B_P is a nontrivial block of $G' - V(P[a, b]) = G_F - V(P(a, r))$. Let $X'_P = N_{G'}(G' - V(B_P))$. Since P is a feasible a-r F-path, r is not a cut vertex of $G' - V(P[a, b]) = G_F - V(P(a, r)$ (in particular, there is no B_P-bridge in $G' - V(P[a, b])$ containing r).

We claim that $G' - (V(B_P) - X'_P)$ is $(4, X'_P \cup \{a, b\})$-connected. For convenience, let $K := G' - (V(B_P) - X'_P)$. Since, for any B_P-bridge B of $G_F - V(P(a, r))$, $V(B) - \{r_B\}$ has at least two neighbors in $V(P(a, r))$ (because G is 4-connected), it follows that $V(B) - \{r_B\}$ has at least one neighbor in $V(P(a, b))$. Hence, K is connected and $K - (X'_P \cup \{a, b\})$ is a component of $G - (X'_P \cup \{a, b\})$. Since G is 4-connected, K is $(4, X'_P \cup \{a, b\})$-connected.

Thus, the hypotheses of Theorem 1.7 are satisfied with $G', a, b, P[a, b], B_P, X'_P$ as G, a, b, P, B_P, X_P, respectively. Hence, there exists a planar a-b chain H' in G' such that $G' - V(H')$ is 2-connected and $B_P \subseteq G' - V(H')$. Moreover, such a chain can be found in $O(|V(G')||E(G')|)$ time (and hence, $O(|V(G)||E(G)|)$ time). Since b is the only neighbor of r in $V(P) - \{a, r\}$ and no B_P-bridge in $G' - V(P[a, b])$ contains r, $r \notin N_G(V(H') - \{a, b\})$. Thus, $H := H' + rb$ is an up a-r F-chain in G (recall

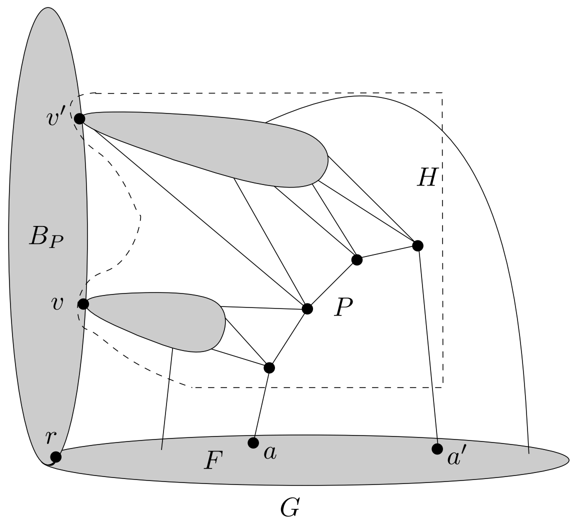

FIG. 4. *Graph H in the proof of Lemma* 3.7.

$a' = r$), so $G[V(F) \cup I(H)]$ is 2-connected. Note also that $G_F - I(H) = G' - V(H')$ is 2-connected, and hence, the result follows. $\square$

Next, we show that if $|X_P| = 2$, then one can find efficiently either a B_P-augmenting path or a good F-chain as required in Theorem 3.2.

LEMMA 3.7. *Suppose that $|X_P| = 2$, and let v, v' be the vertices in X_P. Then exactly one of the following holds:*

(1) *there exists a B_P-augmenting path; or*

(2) *$H := (G_F - (V(B_P) - X_P)) - vv'$ is a down v-v' F-chain in G such that $G_F - I(H)$ is 2-connected and $G[V(F) \cup I(H)]$ is 2-connected.*

Moreover, one can in $O(|V(G)| + |E(G)|)$ time either find a path as in (1) *or certify that* (2) *holds.*

Proof. Let $H := (G_F - (V(B_P) - X_P)) - vv'$. Since G_F is 2-connected and $X_P = \{v, v'\}$, H is a v-v' chain in G and $N_G(H - \{v, v'\}) \subseteq V(F - r) \cup \{v, v'\}$. See Figure 4 for an example. Let $H := v_0B_1v_1 \ldots v_{k-1}B_kv_k$, where $v_0 = v$ and $v_k = v'$. This decomposition of H into blocks can be computed in $O(|V(G)| + |E(G)|)$ time. If every block of H is trivial, then H is a down F-chain, $G_F - I(H) = B_P$ is 2-connected, and $G[V(F) \cup I(H)]$ is 2-connected, so (2) holds.

Thus, we may assume that H contains a nontrivial block. For each nontrivial block B_i, let $S_i := V(F - r) \cap N_G(B_i - \{v_{i-1}, v_i\})$, and let G_i be the graph obtained from B_i by adding S_i and the edges of G from S_i to $V(B_i) - \{v_{i-1}, v_i\}$. Note that $G_i - S_i = B_i$ is 2-connected and $B_i - \{v_{i-1}, v_i\}$ is a union of components of $G - (S_i \cup \{v_{i-1}, v_i\})$. Because G is 4-connected, G_i is $(4, S_i \cup \{v_{i-1}, v_i\})$-connected, and every component of $B_i - \{v_{i-1}, v_i\}$ has at least two neighbors in S_i. Thus, the hypotheses of Lemma 2.3 are satisfied with G_i, S_i, v_{i-1}, v_i as G, S, b, b', respectively.

Hence, either (a) there exist $u_i, u_i' \in S_i$ and an induced u_i-u_i' path P_i' in G_i such that $V(P_i) \cap \{v_{i-1}, v_i\} = \emptyset$, $V(P_i) \cap S_i = \{u_i, u_i'\}$, and $G_i - (V(P_i) \cup S_i)$ is connected, or (b) $|S_i| = 2$ and the elements of S_i can be labeled as u_i, u_i' such that $(G_i, v_{i-1}, u_i, v_i, u_i')$ is planar. Moreover, one can in $O(|V(G_i)| + |E(G_i)|)$ time find a path as in (a) or certify that (b) holds. If (a) holds for some nontrivial block B_i, then P_i' is a B_P-augmenting path for the following reasons: (i)–(iii) of Definition 3.1 hold,

$r \notin \{u, u'\}$ (so (iv) of Definition 3.1 holds), and there exists a v-v' path contained in $H - V(P'_i(u_i, u'_i))$ (so B_P is properly contained in $B_{P'_i}$). If (b) holds for every nontrivial block B_i, then H is clearly a down F-chain, $G[V(F) \cup I(H)]$ is 2-connected (because G is 4-connected, and so $G_i - \{v_{i-1}, v_i\}$ is a u_i-u'_i chain), and $G_F - I(H)$ is 2-connected.

One can verify that either (1) or (2) holds in $O(|V(G)| + |E(G)|)$ time because if (b) holds for a nontrivial block B_i, then $|V(G_i)| + |E(G_i)| = O(|V(B_i)| + |E(B_i)|)$, and if (a) holds for some G_i, then $|V(G_i)| + |E(G_i)| = O(|V(G)| + |E(G)|)$. In the latter case, we find a B_P-augmenting path and we stop. Thus, this verification can be carried out in $O(|V(G)| + |E(G)|)$ time. ∎

The following lemma shows that if $|X_P| \geq 3$ and $|V(P)| = 3$, then one can find efficiently a B_P-augmenting path.

LEMMA 3.8. *Suppose that* $|X_P| \geq 3$ *and* $|V(P)| = 3$*. Then exactly one of the following holds:*

(1) *there exists a* B_P*-augmenting path; or*

(2) P *is an elementary* F*-chain in* G *such that* $G_F - I(P)$ *is* 2*-connected and* $G[V(F) \cup I(P)]$ *is* 2*-connected.*

Moreover, one can in $O(|V(G)| + |E(G)|)$ *time either find a path as in* (1) *or certify that* (2) *holds.*

Proof. If $G_F - V(P(a, a'))$ is 2-connected, then P is an elementary F-chain in G, $G_F - I(P)$ is 2-connected, and $G[V(F) \cup I(P)]$ is 2-connected, so (2) holds. Note, this can be checked in $O(|V(G)| + |E(G)|)$ time.

So we may assume that $G_F - V(P(a, a'))$ is not 2-connected. Let K be a B_P-bridge of $G_F - V(P(a, a'))$, and let v denote the unique vertex in $V(P(a, a'))$. If K is 2-connected, then let $B := K$ and $b := r_K$. Otherwise let B be an endblock of K not containing r_K, and let b denote the cut vertex of K contained in $V(B)$. Since G_F is 2-connected, $v \in N_G(B - b)$. Note that B can be computed in $O(|V(G)| + |E(G)|)$ time.

First, suppose that B is trivial, and let w be the unique vertex in $V(B - b)$. Since G is 4-connected, w has at least three neighbors in $V(F - r) \cup \{v\}$, and hence, it has two neighbors u, u' in $V(F - r)$. Let $P' := (u, w, u')$. We claim that P' is a feasible F-path. Clearly, P' is an induced path in $G - uu'$ and $V(P') \cap V(F) = \{u, u'\}$. Since G_F is 2-connected, $G_F - V(P'(u, u')) = G_F - w$ is connected. Thus, $P'(u, u')$ is nonseparating in G_F. Also $r \in V(B_P)$ and $B_P \subseteq G_F - V(P'(u, u'))$. Therefore, since $r \notin \{u, u'\}$, P' is a feasible F-path. Since $|X_P| \geq 3$, there exists a path (containing v) with ends in $X_P - \{r_B\}$ which is internally disjoint from $V(B_P) \cup V(B)$. Therefore, B_P is properly contained in $B_{P'}$, and hence, P' is a B_P-augmenting path.

Thus, we may assume that B is nontrivial, so B is 2-connected. Let $S := N_G(B - b) - \{b, v\}$, and let G' be obtained from B by adding S and the edges of G from S to $V(B) - \{b\}$. Note that $S \subseteq V(F - r)$ and $G' - S = B$ is 2-connected. Since G is 4-connected, $G[V(G') \cup \{v\}]$ is $(4, S \cup \{b, v\})$-connected, and hence, G' is $(3, S \cup \{b\})$-connected. By Lemma 2.4 (with G', b, S as G, b, S, respectively) there exist $u, u' \in S$ and an induced u-u' path P' in G' such that $V(P') \cap \{b\} = \emptyset$, $V(P') \cap S = \{u, u'\}$, and $G' - (V(P') \cup S)$ is connected. Moreover, such a path can be found in $O(|V(G')| + |E(G')|)$ time (and hence, in $O(|V(G)| + |E(G)|)$ time).

We claim that P' is a feasible F-path. Clearly, P' is an induced path in $G - uu'$ and $V(P') \cap V(F) = \{u, u'\}$. Since $G' - (V(P') \cup S) = B - V(P'(u, u'))$ is connected and $b \notin V(P')$, we have that $G_F - V(P'(u, u'))$ is connected. Thus, $P'(u, u')$ is nonseparating in G_F. Also $r \in V(B_P)$, and $B_P \subseteq G_F - V(P'(u, u'))$. Since $r \notin S$,

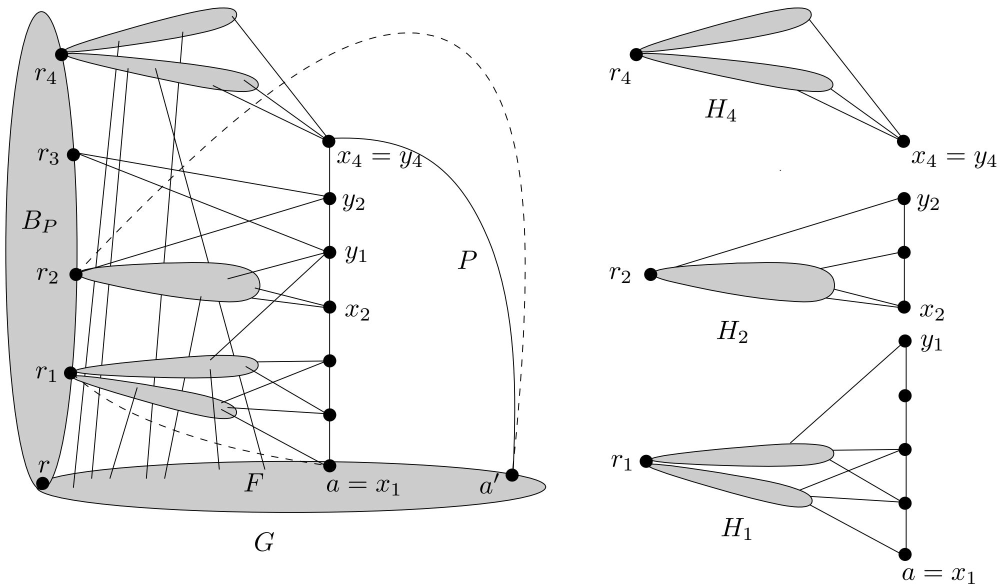

FIG. 5. *Example for Notation* 3.9 *with* $X_P = \{r_1, r_2, r_3, r_4\}$. *Note that the edges* r_1a, r_2a' *are not contained in any* H_i.

$r \notin \{u, u'\}$, so P' is a feasible F-path. Furthermore, since $|X_P| \geq 3$, there exists a path (containing v) with ends in $X_P - \{r_B\}$ which is internally disjoint from $V(B_P) \cup V(B)$. Therefore, B_P is properly contained in $B_{P'}$, and hence, P' is a B_P-augmenting path. □

By Lemmas 3.6, 3.7, and 3.8, we need to deal only with the case where $|X_P| \geq 3$, $|V(P)| \geq 4$, and for some B_P-bridge B of $G_F - V(P(a, a'))$, $B - r_B$ has a neighbor in $V(F - r) - \{a, a'\}$. Our aim is to prove that we can find either a B_P-augmenting path or a triangle F-chain H such that $G_F - I(H)$ is 2-connected. In order to do this, we need to introduce some notation and prove auxiliary results.

NOTATION 3.9. *For any* $x, y \in V(P)$, *we denote* $x \leq y$ *if* $x \in V(P[a, y])$. *If* $x \leq y$ *and* $x \neq y$, *then we write* $x < y$. *In this case, we say that* x *is* lower *than* y, *or* y *is* higher *than* x.

Let $X_P := \{r_1, \ldots, r_p\}$. *For each* i, $1 \leq i \leq p$, *if* r_i *is a cut vertex of* $G_F - V(P(a, a'))$, *then let* $V_i := \bigcup V(B)$, *where the union is taken over all the* B_P*-bridges* B *of* $G_F - V(P(a, a'))$ *with* $r_B = r_i$; *if* r_i *is not a cut vertex of* $G_F - V(P(a, a'))$, *then let* $V_i := \{r_i\}$.

For each i *such that* $V_i \neq \{r_i\}$, *let* $x_i, y_i \in V(P)$ *with* $x_i \leq y_i$ *such that* G *has an edge from* x_i *(*y_i*, respectively) to* V_i *which is not an edge from* $\{a, a'\}$ *to* r_i, *and subject to this,* $P[x_i, y_i]$ *is maximal. Note that we may have* $x_i = a$ *or* $y_i = a'$, *but* $r \notin \{x_i, y_i\}$ *because* B_P *is a block of* $G_F - V(P(a, a'))$.

Let $P_i := P[x_i, y_i]$, *and let* H_i *be the graph obtained from* $G[V_i \cup V(P_i)]$ *by removing all edges from* $\{a, a'\}$ *to* r_i. *Let* $\mathcal{H} := \{H_i : 1 \leq i \leq p, V_i \neq \{r_i\}\}$. *We say that* $H_i \in \mathcal{H}$ *is* adjacent *to* F *if* $N_G(V_i - \{r_i\}) \cap (V(F - r) - \{a, a'\}) \neq \emptyset$. *See Figure* 5 *for an example.*

LEMMA 3.10. *Every* $H_i \in \mathcal{H}$ *is an* r_i*-*x_i *(and also an* r_i*-*y_i*) chain. Moreover, no vertex of* P_i *is a cut vertex of* H_i, *and* P_i *is contained in an endblock of* H_i.

Proof. Since $G[V_i] = H_i - V(P_i)$ is connected and because H_i has edges from both x_i and y_i to V_i, no vertex of P_i is a cut vertex of H_i, and hence, P_i is contained in a block of H_i. We claim that if B is an endblock of H_i, then $r_i \in V(B)$ or $V(P_i) \subseteq V(B)$ (and hence, we have Lemma 3.10). Suppose for a contradiction that B is an endblock of H_i and B contains neither r_i nor any vertex in $V(P_i)$. Let v be the cut vertex of H_i contained in $V(B)$. Then $B - v$ is a component of $G_F - v$, which is a contradiction, since G_F is 2-connected. Similarly, we can show that H_i is an r_i-y_i chain. ∎

NOTATION 3.11. *For each $H_i \in \mathcal{H}$ with $x_i \neq y_i$, let A_i denote the block of H_i containing P_i. If $A_i \neq H_i$, then let b_i denote the cut vertex of H_i contained in A_i. If $A_i = H_i$, then let $b_i := r_i$.*

The next lemma illustrates two situations when we can find a B_P-augmenting path.

LEMMA 3.12. *Assume that $|X_P| \geq 3$, and let $H_i \in \mathcal{H}$. Suppose that one of the following holds:*

(i) *$x_i = y_i$; or*

(ii) *$x_i \neq y_i$, and H_i contains at least three blocks or H_i contains a nontrivial block other than A_i.*

Then one can find a B_P-augmenting path in $O(|V(G)| + |E(G)|)$ time.

Proof. If $x_i = y_i$, then let $H := H_i$. If $x_i \neq y_i$, then let $H := H_i - (V(A_i) - \{b_i\})$. Note that H is an r_i-x_i chain if $x_i = y_i$, and H is an r_i-b_i chain if $x_i \neq y_i$. Moreover, since (i) or (ii) holds, H is not induced by an edge.

Let $H := v_0B_1v_1 \dots v_{k-1}B_kv_k$ with $v_0 = r_i$, $v_k = x_i$ if $x_i = y_i$, and $v_k = b_i$ if $x_i \neq y_i$. This decomposition of H_i into blocks can be computed in $O(|V(G)|+|E(G)|)$ time.

Case 1. There exists $j \in \{1, \dots, k\}$ such that B_j is nontrivial.

Let $S := N_G(B_j - \{v_{j-1}, v_j\}) - \{v_{j-1}, v_j\}$. Note that $S \subseteq V(F - r) - \{a, a'\}$ because B_P is a block of $G_F - V(P(a, a'))$. Let G' be the graph obtained from B_j by adding S and the edges of G from S to $V(B_j) - \{v_{j-1}, v_j\}$. Note that $G' - S = B_j$ is 2-connected and G' is $(4, S \cup \{v_{j-1}, v_j\})$-connected (because G is 4-connected). Therefore, the hypotheses of Lemma 2.3 are satisfied with G', S, v_{j-1}, v_j as G, S, b, b', respectively. Then by Lemma 2.3 exactly one of the following occurs:

(1) there exist $u, u' \in S$ and an induced u-u' path P' in G' such that $V(P') \cap \{v_{j-1}, v_j\} = \emptyset$, $V(P') \cap S = \{u, u'\}$, and $G' - (V(P') \cup S)$ is connected; or

(2) $|S| = 2$, and the elements of S can be labeled as u, u' such that $(G', v_{j-1}, u, v_j, u')$ is planar.

Moreover, one can in $O(V(G')| + |E(G')|)$ time (and hence, in $O(|V(G)| + |E(G)|)$ time) find a path as in (1) or certify that (2) holds.

Note that since $|X_P| \geq 3$, there exists a path W with ends in $X_P - \{r_i\}$ which is internally disjoint from $V(B_P) \cup V_i$.

Suppose (1) holds. We claim that P' is a feasible F-path. Clearly, $V(P') \cap V(F) = \{u, u'\}$, and P' is an induced path in $G - uu'$. Since $B_j - V(P'(u, u')) = G' - (V(P') \cup S)$ is connected and $v_{j-1}, v_j \notin V(P')$, we have that $G_F - V(P'(u, u'))$ is connected. Thus, $P'(u, u')$ is nonseparating in G_F. Also $r \in V(B_P)$, and $B_P \subseteq G_F - V(P'(u, u'))$. Therefore, since $r \notin \{u, u'\}$, P' is a feasible F-path. Moreover, since W is also a path in $G_F - V(P'(u, u'))$, $B_P \cup W \subseteq B_{P'}$. Therefore, P' is a B_P-augmenting path.

Now assume (2) holds. By Lemma 2.6 one can find in $O(|V(G')| + |E(G')|)$ time (and hence, in $O(|V(G)| + |E(G)|)$ time) an induced u-u' path Q in G' such that $G' - V(Q)$ has exactly two components K, K' with $v_{j-1} \in V(K)$ and $v_j \in V(K')$. We claim that Q is a feasible F-path. Clearly, $V(Q) \cap V(F) = \{u, u'\}$, and Q is an induced path in $G - uu'$. Note that $B - Q(u, u') = G' - V(Q)$ has exactly two components (namely K and K'), there exists a path in H_i from $v_{j-1} \in V(K)$ to $r_i \in X_P$ disjoint from Q, and there exists a path from $v_j \in V(K')$ to X_P in $G_F - V(Q(u,u'))$ (because $|X_P| \geq 2$). It follows that $G_F - V(Q(u,u'))$ is connected. Also $r \in V(B_P)$, and $B_P \subseteq G_F - V(Q(u,u'))$. Since $r \notin \{u,u'\}$, Q is a feasible F-path. Moreover, W is a path in $G_F - V(Q(u,u'))$, and hence, $B_P \cup W \subseteq B_Q$. Therefore, Q is a B_P-augmenting path.

Case 2. All blocks of H are trivial.

By (ii), H_i contains at least two blocks other than A_i, and hence, $k \geq 3$. So B_1 and B_2 are trivial blocks of H. Since G is 4-connected, v_1 has at least two neighbors in $V(F-r)$, say u, u'. Let $P' := (u, v_1, u')$. We claim that P' is a feasible F-path. Clearly, $V(P') \cap V(F) = \{u,u'\}$, and P' is an induced path in $G - uu'$. Since G_F is 2-connected, $G_F - V(P'(u,u')) = G_F - v_1$ is connected. Also since $B_P \subseteq G_F - V(P'(u,u'))$ and $r \notin \{u,u'\}$, it follows that P' is a feasible F-path. Moreover, one can see that $B_P \cup W \subseteq B_{P'}$. Therefore, P' is a B_P-augmenting path. ∎

Now we study the case where, for every $H_i \in \mathcal{H}$, $x_i \neq y_i$, H_i has at most two blocks, and if H_i has exactly two blocks, then A_i is the only nontrivial block of H_i. We give three lemmas which deal with this case. The arguments used for many cases in the proofs are similar, but unfortunately it seems necessary to cover all of those cases. We frequently produce a B_P-augmenting path P' in the following way. We first exhibit a nontrivial path W in G_F with ends in B_P such that W is internally disjoint from B_P. We then produce a feasible F-path P' disjoint from W such that $V(B_P) \cup V(W) \subseteq V(B_{P'})$, so P' is B_P-augmenting. For the sake of brevity, when we state a result occurs "because of the path W," we are implicitly using this technique.

Recall that by Assumption 1 we may assume that if $|V(P)| \geq 4$, then $N_G(P(a,a')) \cap V(F) \subseteq \{a,a'\} \cup \{r\}$.

LEMMA 3.13. *Assume that $|X_P| \geq 3$, $|V(P)| \geq 4$, and, for every $H_j \in \mathcal{H}$, $x_j \neq y_j$. Suppose that, for every $H_j \in \mathcal{H}$, $V(A_j) - \{b_j, x_j, y_j\}$ has no neighbor in $V(F-r) - \{a,a'\}$. Assume that for some $H_i \in \mathcal{H}$, H_i is adjacent to F. Then exactly one of the following holds:*

(1) *there exists a B_P-augmenting path; or*

(2) *there exists a triangle F-chain H in G such that $I(H) = V(G_F) - V(B_P)$, $G_F - I(H)$ is 2-connected, and $G[V(F) \cup I(H)]$ is 2-connected.*

Moreover, one can in $O(|V(G)|+|E(G)|)$ time find either a path as in (1) *or a triangle F-chain as in* (2).

Proof. Let us first show that (1) and (2) are mutually exclusive. Suppose that (2) holds. It is not hard to see that there exists no B_P-augmenting path because every feasible F-path must use exactly two vertices of $V(G_F) - V(B_P)$. Thus, it remains to show that either (1) or (2) holds and that one can determine in $O(|V(G)| + |E(G)|)$ time which of them occurs.

We consider two cases.

Case 1. There exist distinct $m, n \in \{1, \ldots, p\} - \{i\}$ such that both V_m and V_n have a neighbor in $V(P(x_i, a'))$ or both V_m and V_n have a neighbor in $V(P(a, y_i))$.

Without loss of generality, assume that both V_m and V_n have a neighbor in $V(P(x_i, a'))$.

We claim that A_i contains a nonseparating induced b_i-x_i path Q such that $V(Q) \cap (V(P_i) - \{x_i\}) = \emptyset$. This is obvious if $V(A_i) - V(P_i) = \{b_i\}$ because then b_i must be adjacent to x_i, and the result follows by taking Q as the path induced by the edge $b_i x_i$. Thus, we may assume that $V(A_i) - V(P_i) \neq \{b_i\}$. Let S_i denote the set of vertices in $V(P(x_i, y_i))$ which have a neighbor in $(\bigcup_{j=1}^{p} V_j) - V_i$. Since G is 4-connected, A_i

is $(4, S_i \cup \{b_i, x_i, y_i\})$-connected. Moreover, $A_i - (V(P_i) - \{x_i\})$ is connected and $S_i \cup \{y_i\} \subseteq V(P_i) - \{x_i\}$, so there exists a b_i-x_i path Q' in A_i such that $V(P_i) - \{x_i\}$ (and hence, $S_i \cup \{y_i\}$) is contained in a component U of $A_i - V(Q')$. Therefore, the hypotheses of Lemma 2.1 are satisfied with $A_i, S_i \cup \{b_i, x_i, y_i\}, b_i, x_i, Q', U$ as G, S, a, a', P, U, respectively. By Lemma 2.1 one can find in $O(|V(G)| + |E(G)|)$ time a nonseparating induced b_i-x_i path Q in A_i such that $V(Q) \cap V(U) = \emptyset$. Since $V(P_i) - \{x_i\} \subseteq V(U)$, we have $V(Q) \cap (V(P_i) - \{x_i\}) = \emptyset$, and thus, Q is a path as required.

By hypothesis, $x_i \neq y_i$, so by Lemma 3.12, we can in $O(|V(G)| + |E(G)|)$ time either find a B_P-augmenting path or certify that H_i has at most two blocks. Hence, we may assume that H_i has at most two blocks. Since H_i is adjacent to F and $V(A_i) - \{b_i, x_i, y_i\}$ has no neighbors in $V(F - r) - \{a, a'\}$, it follows that H_i has exactly two blocks, and b_i is adjacent to some vertex $u \in V(F - r) - \{a, a'\}$. Let $P' := (Q \cup P[a, x_i]) + \{u, b_i u\}$. By assumption, both V_m and V_n have a neighbor on $P(x_i, a')$. Since P' is disjoint from $V(P_i) - \{x_i\}$, there exists an r_m-r_n path W in $G_F - V(P'(a, u))$ which is internally disjoint from $V(B_P) \cup V_i \cup \{x_i\}$.

Next we show that P' is a B_P-augmenting path. Since $N_G(P(a, a')) \cap V(F) \subseteq \{a, a'\} \cup \{r\}$ (by Assumption 1) and P is induced in $G - aa'$, we have that P' is an induced u-a path in $G - au$. Also, since $A_i - V(Q)$ is connected, $P'(a, u)$ is nonseparating in G_F. Note also that if r is an end of P', then $a = r$, and r is not a cut vertex of $G_F - V(P(a, a'))$. Then, because of the path W, r is not a cut vertex of $G_F - V(P'(a, u))$. Thus, P' is a feasible F-path. Since $B_P \cup W \subseteq B_{P'}$, P' is a B_P-augmenting path and (1) holds.

Case 2. For any distinct $m, n \in \{1, \dots, p\} - \{i\}$, V_m and V_n do not both have a neighbor in $V(P(x_i, a'))$, nor do both V_m and V_n have a neighbor in $V(P(a, y_i))$.

By hypothesis, $x_i \neq y_i$, so by Lemma 3.12, we can in $O(|V(G)| + |E(G)|)$ time either find a B_P-augmenting path or certify that H_i has at most two blocks. Hence, we may assume that H_i has at most two blocks. Since H_i is adjacent to F and $A_i - \{b_i, x_i, y_i\}$ has no neighbor in $V(F - r) - \{a, a'\}$, it follows that H_i has exactly two blocks, and b_i has at least one neighbor in $V(F - r) - \{a, a'\}$. Moreover, since we are in Case 2, we must have $|X_P| = 3$. Without loss of generality, we may assume that $i = 3$, V_1 has a neighbor in $V(P(a, x_3])$, and V_2 has a neighbor in $V(P[y_3, a'))$. Moreover, V_1 has no neighbor in $V(P(x_3, a'))$, and V_2 has no neighbor in $V(P(a, y_3))$.

Suppose b_3 has two neighbors in $V(F - r) - \{a, a'\}$, say u, u'. Let $P' := (u, b_3, u')$. Clearly, $G_F - V(P'(u, u')) = G_F - b_3$ is connected. Since $r \notin \{u, u'\}$, it is not hard to see that P' is a feasible F-path. Moreover, there exists an r_1-r_2 path which is internally disjoint from $V(B_P) \cup V_i$. Hence, P' is a B_P-augmenting path, and (1) holds. Clearly, P' can be found in $O(|V(G)| + |E(G)|)$ time.

Thus, we may assume that b_3 has exactly one neighbor in $V(F - r) - \{a, a'\}$. We consider two subcases.

Subcase 2.1. For some $j \in \{1, 2\}$, say $j = 1$, $V_1 \neq \{r_1\}$.

Let $H_1 := w_0 B'_1 w_1 \dots w_{s-1} B'_s w_s$ where $w_0 = r_1$, and $B'_s = A_1$. Since $x_1 \neq y_1$ (by assumption), then from Lemma 3.12 either $s = 1$ or $s = 2$ and B'_1 is trivial.

We claim that $V(A_1) = \{b_1, x_1, y_1\}$. Suppose for a contradiction that $V(A_1) - \{b_1, x_1, y_1\} \neq \emptyset$. Then $A_1 - \{b_1, x_1, y_1\}$ is a component of $G - \{b_1, x_1, y_1\}$ for the following reasons: $V(A_1) - \{b_1, x_1, y_1\}$ has no neighbor in $V(F - r) - \{a, a'\}$ (by hypothesis), $V(P(x_1, y_1))$ has no neighbor in $V_3 \cup V_2$ (by assumption in Case 2), and

P is an induced path in $G-aa'$. But then $\{b_1, x_1, y_1\}$ is a 3-cut in G which contradicts the assumption that G is 4-connected. Thus, $V(A_1) = \{b_1, x_1, y_1\}$.

Therefore, $\{b_1, x_1, y_1\}$ induces a triangle in G. Since $H_1 \in \mathcal{H}$, $V_1 \neq \{r_1\}$. This implies that $s = 2$ and B_1' is a trivial block of H_1 (and hence, r_1 is adjacent to b_1). Since b_1 has degree at least four in G, b_1 must have some neighbor in $V(F-r)$. Hence, H_1 is adjacent to F, and V_2 and V_3 have neighbors in $V(P(x_1, a'))$, so we can proceed as in Case 1 and find a B_P-augmenting path in $O(|V(G)| + |E(G)|)$ time.

Subcase 2.2. For every $j \in \{1, 2\}$, $V_j = \{r_j\}$.

Thus, r_1 has a neighbor in $V(P(a, x_3])$, and hence, $x_3 \neq a$. Similarly, $y_3 \neq a'$.

We claim that $V(A_3) = \{b_3, x_3, y_3\}$. Suppose for a contradiction that $V(A_3) - \{b_3, x_3, y_3\} \neq \emptyset$. Then $A_3 - \{b_3, x_3, y_3\}$ is a component of $G - \{b_3, x_3, y_3\}$ for the following reasons: $V(A_3) - \{b_3, x_3, y_3\}$ has no neighbor in $V(F - r) - \{a, a'\}$ (by hypothesis), $V(P(x_3, y_3))$ has no neighbor in $V_1 \cup V_2$ (by assumption in Case 2), and P is an induced path in $G - aa'$. But then $\{b_3, x_3, y_3\}$ is a 3-cut in G, which contradicts the assumption that G is 4-connected. Thus, $V(A_3) = \{b_3, x_3, y_3\}$, and A_3 is a triangle.

Since G_F is 2-connected and P is an induced path in $G - aa'$, and because $N_G(P(a, a')) \cap V(F) \subseteq \{a, a'\} \cup \{r\}$, it follows that $V(P) = V(P_3) \cup \{a, a'\}$, r_1 is adjacent to x_3, and r_2 is adjacent to y_3. Let u denote the only neighbor of b_3 in $V(F-r)-\{a, a'\}$. Note that $a \neq r$; otherwise $r_1 = r$ (because $|X_P| = 3$), and x_3 would have degree three in G which is a contradiction because G is 4-connected. Similarly, $a' \neq r$. If $r = r_1$, then (r_1, x_3, a) is a B_P-augmenting path. If $r = r_2$, then (r_2, y_3, a') is a B_P-augmenting path. If $r = r_3$, then (r_3, b_3, u) is a B_P-augmenting path. Thus, we may assume that $r \notin \{r_1, r_2, r_3\}$. Therefore, $H := A_i + \{u, a, a', b_iu, x_ia, y_ia'\}$ is a triangle F-chain in G with $b_3, x_3, y_3, u, a, a', r_3, r_1, r_2$ as $v_1, v_2, v_3, u_1, u_2, u_3, w_1, w_2, w_3$, respectively, in Definition 1.2. It is easy to see that $G_F - I(H) = B_P$ is 2-connected and $G[V(F) \cup I(H)]$ is 2-connected. So (2) holds. ∎

LEMMA 3.14. *Assume that* $|X_P| \geq 3$, $|V(P)| \geq 4$, *and for every* $H_j \in \mathcal{H}$, $x_j \neq y_j$. *Suppose that* $H_i \in \mathcal{H}$ *and* $V(A_i) - \{b_i, x_i, y_i\}$ *has a neighbor in* $V(F - r) - \{a, a'\}$. *Assume that* $V(P(x_i, y_i))$ *has no neighbor in* $(\bigcup_{j=1}^{p} V_j) - V_i$. *Then a* B_P*-augmenting path can be found in* $O(|V(G)| + |E(G)|)$ *time.*

Proof. Since G_F is 2-connected and $V(P(x_i, y_i))$ has no neighbor in $(\bigcup_{j=1}^{p} V_j) - V_i$, there exists $m \in \{1, \dots, p\} - \{i\}$ such that V_m has a neighbor in $V(P(a, x_i])$ or in $V(P[y_i, a'))$.

By symmetry we may assume that V_m has a neighbor in $V(P[y_i, a'))$. Then $y_i \neq a'$.

First, we find an endblock of $A_i - \{x_i, y_i\}$ in $O(|V(G)| + |E(G)|)$ time as follows. If $A_i - \{x_i, y_i\}$ is 2-connected, then let $B := A_i - \{x_i, y_i\}$, and let $b := b_i$. Otherwise, let B be an endblock of $A_i - \{x_i, y_i\}$, and let b denote the cut vertex of $A_i - \{x_i, y_i\}$ contained in B so that $b_i \notin V(B - b)$. Note that $V(P(x_i, y_i))$ has no neighbors in $(\bigcup_{j=1}^{p} V_j) - V_i$, and $N_G(B - b) \subseteq V(F - r) \cup \{x_i, y_i, b\}$. Since $r \notin \{x_i, y_i\}$ (by the definition of x_i, y_i in Notation 3.9), $r \notin N_G(B - b) - \{b\}$. Moreover, since G is 4-connected, $|N_G(B - b)| \geq 4$. Note that such an endblock B can be found in $O(|V(G)| + |E(G)|)$ time.

Next, we consider two cases.

Case 1. y_i has a neighbor in $V(A_i) - (\{x_i, y_i\} \cup V(B - b))$.

Then, since V_m has a neighbor in $V(P[y_i, a'))$, there exists an r_i-r_m path W in $G_F - V(P(a, x_i])$ which is internally disjoint from $V(B_P)$ and intersects $P[y_i, a')$.

Subcase 1.1. B is trivial.

Let v denote the unique vertex in $V(B)-\{b\}$. Then $N_G(v) \subseteq V(F-r)\cup\{x_i, y_i, b\}$. Since G is 4-connected, v has at least three neighbors in $V(F-r)\cup\{x_i, y_i\}$, and hence, it has two neighbors in $V(F-r)\cup\{x_i\}$. Let u, u' be distinct neighbors of v in $V(F-r)\cup\{x_i\}$, and assume that $u \neq x_i$. By the definition of x_i, y_i in Notation 3.9, one can see that $\{u, u'\}\cap\{a'\} = \emptyset$ and $u \neq a$ (because $y_i \neq a'$ and $x_i \neq u$). If $u' \neq x_i$, then let $P' := (u, v, u')$; otherwise, let $P' := P[a, x_i] + \{u, v, uv, vx_i\}$. Clearly, P' is a path with ends in $V(F)$ which is internally disjoint from $V(B_P)\cup V(F)$.

Next we show that P' is a B_P-augmenting path. Let u, u'' denote the ends of P'. By assumption, $N_G(P(a, a'))\cap V(F) \subseteq \{a, a'\}\cup\{r\}$ (Assumption 1), and P is induced in $G - aa'$. Then since $N_G(v) \subseteq V(F-r)\cup\{x_i, y_i, b\}$ and $V(P[a, x_i))$ has no neighbor in $V(B)$ (by the definition of x_i in Notation 3.9), it follows that P' is induced in $G-uu''$. Because of the path W, and since $P(a, a')$ is nonseparating in G_F, $G_F - V(P'(u, u''))$ is connected. So $P'(u, u'')$ is nonseparating in G_F. If $r \in \{u, u''\}$, then since $r \notin \{u, u'\}$, $r = u'' = a$ and r is not a cut vertex of $G_F - V(P(a, a'))$. Then, because of the path W, r is not a cut vertex of $G_F - V(P'(u, u''))$. Thus, P' is a feasible F-path. Since $B_P \cup W \subseteq B_{P'}$, P' is a B_P-augmenting path. Clearly, P' can be found in $O(|V(G)| + |E(G)|)$ time.

Subcase 1.2. B is nontrivial.

Let $S := N_G(B-b) - \{b, y_i\}$, and let G' be obtained from B by adding S and the edges of G from S to $V(B)-\{b\}$. Since $r \notin N_G(B-b)-\{b\}$, $r \notin S$. Since G is 4-connected, $|S| \geq 2$ and G' is $(3, S\cup\{b\})$-connected (if $y_i \notin N_G(B-b)$, then actually $|S| \geq 3$ and G' is $(4, S\cup\{b\})$-connected). Moreover, $G'-S = B$ is 2-connected. Thus, the hypotheses of Lemma 2.4 are satisfied with G', S, b as G, S, b, respectively. Then there exist $u, u' \in S$ and an induced u-u' path Q in G' such that $V(Q)\cap\{b\} = \emptyset$, $V(Q)\cap S = \{u, u'\}$, and $G' - (V(Q)\cup S)$ is connected. Moreover, such a path Q can be found in $O(|V(G')| + |E(G')|)$ time (and hence, in $O(|V(G)| + |E(G)|)$ time).

By the definition of x_i, y_i in Notation 3.9 and because $y_i \neq a'$, $\{u, u'\}\cap\{a'\} = \emptyset$. By symmetry we may assume that $u \neq x_i$. If $u' \neq x_i$, then let $P' := Q$; otherwise, let $P' := P[a, x_i]\cup Q$. Clearly, P' is a path with ends in $V(F)$ which is internally disjoint from $V(B_P)\cup V(F)$.

Next we show that P' is a B_P-augmenting path. Let u, u'' denote the ends of P'. By assumption, $N_G(P(a, a'))\cap V(F) \subseteq \{a, a'\}\cup\{r\}$ (Assumption 1), and P is induced in $G - aa'$. Then since Q is induced in G' and $P[a, x_i)$ has no neighbor in $V(B)$ (by the definition of x_i in Notation 3.9), it follows that P' is induced in $G-uu''$. Since $B - V(Q(u, u')) = G' - (V(Q)\cup S)$ is connected and because of the path W, $P'(u, u'')$ is nonseparating in G_F. If $r \in \{u, u''\}$, then since $r \notin S$, $r = u'' = a$, and r is not a cut vertex of $G_F - V(P(a, a'))$. Then, because of the path W, r is not a cut vertex of $G_F - V(P'(u, u''))$. Thus, P' is a feasible F-path. Since $B_P\cup W \subseteq B_{P'}$, P' is a B_P-augmenting path. Clearly, P' can be found in $O(|V(G)| + |E(G)|)$ time.

Case 2. y_i has no neighbor in $V(A_i) - (\{x_i, y_i\}\cup V(B-b))$ (and hence, $y_i \in N_G(B-b)$).

Subcase 2.1. B is trivial.

Let v denote the unique vertex in $V(B)-\{b\}$. Then $N_G(v) \subseteq V(F-r)\cup\{x_i, y_i, b\}$, and y_i is adjacent to v. Since G is 4-connected, v has at least four neighbors in $V(F-r)\cup\{x_i, y_i, b\}$, and hence, it has at least two neighbors in $V(F-r)\cup\{x_i\}$. Let $u, u' \in N_G(v) - \{b, y_i\}$, and assume that $u \neq x_i$. By the definition of x_i, y_i in Notation 3.9, one can see that $\{u, u'\}\cap\{a'\} = \emptyset$ (because $y_i \neq a'$) and $u \neq a$ (because $u \neq x_i$).

Suppose that there exists $n \in \{1, \dots, p\} - \{i, m\}$ such that V_n has a neighbor in $V(P[y_i, a'))$. Then there exists an r_m-r_n path W in $G_F - V(P(a, y_i))$ which is internally disjoint from $V(B_P)$ and intersects $P[y_i, a')$. If $u' \neq x_i$, then let $P' := (u, v, u')$; otherwise, let $P' := P[a, x_i] + \{u, v, uv, vx_i\}$. Then P' is a path with ends in $V(F)$ which is internally disjoint from $V(B_P) \cup V(F)$. Next we show that P' is a B_P-augmenting path. Let u, u'' denote the ends of P'. By assumption, $N_G(P(a, a')) \cap V(F) \subseteq \{a, a'\} \cup \{r\}$ (Assumption 1) and P is induced in $G - aa'$. Then, since $N_G(v) \subseteq V(F-r) \cup \{x_i, y_i, b\}$ and $V(P[a, x_i))$ has no neighbor in $V(B)$ (by the definition of x_i in Notation 3.9), it follows that P' is an induced path in $G - uu''$. Because of the path W and since $P(a, a')$ is nonseparating in G_F, $G_F - V(P'(u, u''))$ is connected, and so $P'(u, u'')$ is nonseparating in G_F. If $r \in \{u, u''\}$, then since $r \notin \{u, u'\}$, $r = u'' = a$, and r is not a cut vertex of $G_F - V(P(a, a'))$. Then, because of the path W, r is not a cut vertex of $G_F - V(P'(u, u''))$. Thus, P' is a feasible F-path. Since $B_P \cup W \subseteq B_{P'}$, P' is a B_P-augmenting path.

Thus, we may assume that there exists no $n \in \{1, \dots, p\} - \{i, m\}$ such that V_n has a neighbor in $V(P[y_i, a'))$. Since $|X_P| \geq 3$ and $V(P(x_i, y_i))$ has no neighbor in $(\bigcup_{j=1}^{p} V_j) - V_i$, we have that $x_i \neq a$, and there exists $n \in \{1, \dots, p\} - \{i, m\}$ such that V_n has a neighbor in $V(P(a, x_i])$. Furthermore, x_i has a neighbor in $V(A_i) - \{x_i, y_i, v\}$; otherwise, since y_i has no neighbor in $V(A_i) - (\{x_i, y_i\} \cup V(B - b))$, v would be a cut vertex of A_i. Therefore, there exists an r_i-r_n path W in $G_F - V(P'[y_i, a'))$ which is internally disjoint from $V(B_P)$ and intersects $P(a, x_i]$.

Let $P' := P[y_i, a'] + \{u, v, uv, vy_i\}$. Then P' is a path with ends in $V(F)$ which is internally disjoint from $V(B_P) \cup V(F)$. One can show that P' is an induced path in $G - ua'$, and because of the path W, $P'(u, a')$ is nonseparating in G_F. Since $u \neq x_i \neq r$, r is an end of P' only if $a' = r$. In this case, r is not a cut vertex of $G_F - V(P(a, a'))$, and because of the path W, r is not a cut vertex of $G_F - V(P'(u, a'))$. Thus, P' is a feasible F-path. Since $B_P \cup W \subseteq B_{P'}$, P' is a B_P-augmenting path. Note that in all above cases, such a path P' can be found in $O(|V(G)| + |E(G)|)$ time.

Subcase 2.2. B is nontrivial.

First, we define a graph G' from B. If y_i has at least two neighbors in $V(B)$, then let $S := N_G(B - b) - \{b, y_i\}$, let G' be obtained from B by adding $S \cup \{y_i\}$ and the edges of G from $S \cup \{y_i\}$ to $V(B) - \{b\}$, and let $y^* := y_i$. If y_i has exactly one neighbor in $V(B)$, then let y^* denote this vertex (note that $y^* \neq b$ because $y_i \in N_G(B - b)$ by assumption), let $S := N_G(B - \{b, y^*\}) - \{b, y^*\}$, and let G' be obtained from B by adding S and the edges of G from S to $V(B) - \{b, y^*\}$. Note that in either case $S \subseteq V(F - r) \cup \{x_i\}$. Moreover, $G' - S = B$ is 2-connected, and G' is $(4, S \cup \{b, y^*\})$-connected (because G is 4-connected). Thus, the hypotheses in Lemma 2.3 are satisfied with G', S, b, y^* as G, S, b, b', respectively. By Lemma 2.3 exactly one of the following holds:

(1) there exist $u, u' \in S$ and an induced u-u' path Q in G' such that $V(Q) \cap \{b, y^*\} = \emptyset$, $V(Q) \cap S = \{u, u'\}$, and $G' - (V(Q) \cup S)$ is connected; or

(2) $|S| = 2$, and the elements of S can be labeled as u, u' such that (G', u, b, u', y^*) is planar.

Moreover, one can in $O(|V(G')| + |E(G')|)$ time (and hence, in $O(|V(G)| + |E(G)|)$ time) find a path as in (1) or certify that (2) holds. Without loss of generality, we may assume that $u \neq x_i$.

Suppose (1) occurs. If $u' \neq x_i$, then let $P' := Q$; otherwise let $P' := P[a, x_i] \cup Q$. Then P' is a path with ends in $V(F)$ which is internally disjoint from $V(B_P) \cup V(F)$. Since y^* and b are in $G' - (V(Q) \cup S)$ which is connected, and because V_m has a neighbor in $V(P[y_i, a'))$, there exists an r_i-r_m path W in $G_F - V(P(a, y_i))$ which is internally disjoint from $V(B_P) \cup V(F)$ and intersects $P[y_i, a')$.

Next we show that P' is a B_P-augmenting path. Let u, u'' denote the ends of P'. Since Q is induced in G' and $N_G(P(a, a')) \cap V(F) \subseteq \{a, a'\} \cup \{r\}$, and because P is induced in $G - aa'$ and $P[a, x_i)$ has no neighbor in $V(B)$ (by the definition of x_i in Notation 3.9), one can see that P' is an induced path in $G - uu''$. Because of the path W, and since $P(a, a')$ is nonseparating in G_F, $P'(u, u'')$ is nonseparating in G_F. Since $r \notin S$, if $r \in \{u, u''\}$, then $r = u'' = a$, and r is not a cut vertex of $G_F - V(P(a, a'))$. Then, because of the path W, r is not a cut vertex of $G_F - V(P'(u, u''))$. Thus, P' is a feasible F-path. Since $B_P \cup W \subseteq B_{P'}$, P' is a B_P-augmenting path.

So we may assume (2) occurs. We consider two cases.

First, assume there exists $n \in \{1, \dots, p\} - \{i, m\}$ such that V_n has a neighbor in $V(P[y_i, a'))$. Then there exists an r_m-r_n path W in $G_F - V(P(a, y_i))$ which is internally disjoint from $V(B_P) \cup V(F)$ and intersects $P[y_i, a')$. By Lemma 2.6 (with G', u, u', b, y^* as G, a, a', b, b', respectively), there exists an induced u-u' path Q in G' such that $G' - V(Q)$ has exactly two components K and K' with $b \in V(K)$ and $y^* \in V(K')$. Moreover, such a path can be found in $O(|V(G')| + |E(G')|)$ time (and hence, in $O(|V(G)| + E(G)|)$ time). If $u' \neq x_i$, then let $P' := Q$; otherwise let $P' := P[a, x_i] \cup Q$. So P' is a path with ends in $V(F)$ which is internally disjoint from $V(B_P) \cup V(F)$.

Next we show that P' is a B_P-augmenting path. Let u, u'' denote the ends of P'. Since Q is induced in G' and $N_G(P(a, a')) \cap V(F) \subseteq \{a, a'\} \cup \{r\}$ (by Assumption 1), and because P is induced in $G - aa'$ and $P[a, x_i)$ has no neighbor in $V(B)$ (by the definition of x_i in Notation 3.9), one can see that P' is an induced path in $G - uu''$. Since $G' - V(Q)$ has exactly two components, one containing b and the other containing y^*, and because of the path W, it follows that $P'(u, u'')$ is nonseparating in G_F. If $r \in \{u, u''\}$, then $r = u'' = a$, and r is not a cut vertex of $G_F - V(P(a, a'))$. Then, because of the path W, r is not a cut vertex of $G - V(P'(u, u''))$. Thus, P' is a feasible F-path. Since $B_P \cup W \subseteq B_{P'}$, P' is a B_P-augmenting path.

Now assume that there exists no $n \in \{1, \dots, p\} - \{i, m\}$ such that V_n has a neighbor in $V(P[y_i, a'))$. Since $|X_P| \geq 3$ and $V(P(x_i, y_i))$ has no neighbor in $(\bigcup_{j=1}^{p} V_j) - V_i$, there exists $n \in \{1, \dots, p\} - \{i, m\}$ such that V_n has a neighbor in $V(P(a, x_i])$, and hence, $x_i \neq a$. Note that $G' \ncong K_{1,4}$ because B is nontrivial. By Lemma 2.7 (with G', u, y^*, u', b as G, a, a', b, b', respectively), there exists a nonseparating induced u-y^* path Q in G' such that $V(Q) \cap \{u', b\} = \emptyset$. Moreover, such a path can be found in $O(|V(G')| + |E(G')|)$ time (and hence, in $O(|V(G)| + |E(G)|)$ time). Note that either x_i has a neighbor in $V(A_i) - V(B - b)$ or x_i is in $G' - V(Q)$. Since V_n has a neighbor in $V(P(a, x_i])$, there exists an r_i-r_n path W in $G_F - V(P(x_i, a'))$ which is internally disjoint from $V(B_P) \cup V(F)$ and intersects $P(a, x_i]$. If $y^* = y_i$, then let $P' := Q \cup P[y_i, a']$; otherwise, let $P' := (Q \cup P[y_i, a']) + \{y_i, y_i y^*\}$. One can show that P' is an induced path in $G - ua'$, and because of the path W, $P'(u, a')$ is nonseparating in G_F. If $r \in \{u, a'\}$, then $a' = r$, r is not a cut vertex of $G_F - V(P'(u, a'))$ because of the path W, and because r is not a cut vertex of $G_F - V(P(a, a'))$. Thus, P' is a feasible F-path. Since $B_P \cup W \subseteq B_{P'}$, P' is a B_P-augmenting path. ∎

LEMMA 3.15. *Assume that $|X_P| \geq 3$, $|V(P)| \geq 4$, and for every $H_j \in \mathcal{H}$, $x_j \neq y_j$. Suppose that $H_i \in \mathcal{H}$ and $V(A_i) - \{b_i, x_i, y_i\}$ has a neighbor in $V(F - r) - \{a, a'\}$. Assume that $V(P(x_i, y_i))$ has a neighbor in $(\bigcup_{j=1}^{p} V_j) - V_i$. Then a B_P-augmenting path can be found in $O(|V(G)| + |E(G)|)$ time.*

Proof. Since $|X_P| \geq 3$ and $V(P(x_i, y_i))$ has a neighbor in $(\bigcup_{j=1}^{p} V_j) - V_i$, there exist $m, n \in \{1, \ldots, p\} - \{i\}$ such that both V_m and V_n have a neighbor in $V(P(a, y_i))$, or both V_m and V_n have a neighbor in $V(P(x_i, a'))$.

By symmetry we may assume that both V_m and V_n have a neighbor in $V(P(x_i, a'))$. Therefore, there exists an r_m-r_n path W in $G_F - V(P(a, x_i])$ which is internally disjoint from $V(B_P) \cup V(F)$ and intersects $P(x_i, a')$.

Let D be the graph obtained from $A_i - \{x_i, y_i\}$ by adding a new vertex b' and new edges from b' to each $v \in V(P(x_i, y_i))$ such that v has a neighbor in some V_j, $j \in \{1, \ldots, p\} - \{i\}$. Since $P(x_i, y_i) \subseteq A_i - \{x_i, y_i\}$, $N_D(b') \cup \{b'\}$ is contained in a block of D, and b' is not a cut vertex of D. Note also that if D is not connected, then D has exactly two components, one containing b_i and the other induced by $V(P(x_i, y_i)) \cup \{b'\}$, and the component containing b' is a block of D since every vertex in $V(P(x_i, y_i))$ has a neighbor in some V_j, $j \neq i$ (because $N_G(P(a, a')) \cap V(F) \subseteq \{a, a'\} \cup \{r\}$ by Assumption 1). We consider two cases.

Case 1. D is not a b_i-b' chain.

Then there exists an endblock B of D such that one of the following holds: (1) $b' \notin V(B)$, and if $b_i \in V(B)$, then b_i is a cut vertex of D, or (2) D has exactly two components and B is the component of D containing b_i (and hence, $V(B) \cap (V(P(x_i, y_i)) \cup \{b'\}) = \emptyset$ by the argument in the last paragraph). Note that such an endblock can be found in $O(|V(G)| + |E(G)|)$ time. If (1) holds, then let b denote the cut vertex of D contained in B. If (2) holds, then let $b := b_i$. Since $|X_P| \geq 3$ and B_P is a block of $G_F - V(P(a, a'))$, it follows from the definition of x_i, y_i in Notation 3.9 that $r \notin \{x_i, y_i\}$. Note that $N_D(b') \cap V(B - b) = \emptyset$ and $r \notin N_G(B - b)$.

Subcase 1.1. B is trivial.

Let v denote the only vertex in $V(B) - \{b\}$. Note that $N_G(v) \subseteq V(F - r) \cup \{x_i, y_i, b\}$. Since G is 4-connected and $N_D(b') \cap V(B - b) = \emptyset$, v has at least three neighbors in $V(F - r) \cup \{x_i, y_i\}$. Let u, u' be two distinct neighbors of v in $V(F - r) \cup \{x_i\}$. By symmetry, we may assume that $u \neq x_i$. If $u' \neq x_i$, then let $P' := (u, v, u')$. If $u' = x_i$, then let $P' := P[a, x_i] + \{u, v, uv, vx_i\}$. Then P' is a path with ends in $V(F)$ which is internally disjoint from $V(B_P) \cup V(F)$.

Next we show that P' is a B_P-augmenting path. Let u, u'' denote the ends of P'. By assumption, $N_G(P(a, a')) \cap V(F) \subseteq \{a, a'\} \cup \{r\}$, and P is induced in $G - aa'$. Since $N_G(v) \subseteq V(F - r) \cup \{x_i, y_i, b\}$ and $V(P[a, x_i))$ has no neighbor in $V(B)$ (by the definition of x_i in Notation 3.9), it follows that P' is induced in $G - uu''$. Because of the path W and since $P(a, a')$ is nonseparating in G_F, $P'(u, u'')$ is nonseparating in G_F. Moreover, if $r \in \{u, u''\}$, then $r = u'' = a$, and r is not a cut vertex of $G_F - V(P(a, a'))$. Then, because of the path W, r is not a cut vertex of $G_F - V(P'(u, a))$. Thus, P' is a feasible F-path. Since $B_P \cup W \subseteq B_{P'}$, P' is a B_P-augmenting path.

Subcase 1.2. B is nontrivial.

Let $S := N_G(B - b) - \{y_i, b\}$, and let G' be obtained from B by adding S and the edges of G from S to $V(B) - \{b\}$. Note that since $r \notin \{x_i, y_i\}$ and $r \notin N_G(B - b)$, $r \notin S$. Since G is 4-connected and y_i is the only possible neighbor of $V(B - b)$ not in $S \cup \{b\}$, G' is $(3, S \cup \{b\})$-connected. By Lemma 2.4 (with G', S, b as G, S, b, respectively) there exist $u, u' \in S$ and an induced u-u' path Q in G' such that $V(Q) \cap \{b\} = \emptyset$, $V(Q) \cap S = \{u, u'\}$, and $G' - (V(Q) \cup S)$ is connected. Moreover, such a path can be found in $O(|V(G')| + |E(G')|)$ time (and hence, in $O(|V(G)| + |E(G)|)$ time). Without loss of generality, we may assume that $u \neq x_i$. If $u' \neq x_i$, then let $P' := Q$; otherwise let $P' := P[a, x_i] \cup Q$. Then P' is a path with ends in $V(F)$ which is internally disjoint from $V(B_P) \cup V(F)$.

Next we prove that P' is a B_P-augmenting path. Let u, u'' denote the ends of P'. Note that Q is induced in $G - uu''$, $N_G(P(a,a')) \cap V(F) \subseteq \{a,a'\} \cup \{r\}$ (by Assumption 1), and P is induced in $G - aa'$. Then since $N_G(v) \subseteq V(F-r) \cup \{x_i, y_i, b\}$ and $V(P[a,x_i))$ has no neighbor in $V(B)$ (by the definition of x_i in Notation 3.9), it follows that P' is induced in $G - uu''$. Because of the path W and since $G' - (V(Q) \cup S)$ is connected, $P'(u,u'')$ is nonseparating in G_F. If $r \in \{u, u''\}$, then $r = u'' = a$, and r is not a cut vertex of $G_F - V(P(a,a'))$. Then, because of the path W, r is not a cut vertex of $G_F - V(P'(u,u''))$. Thus, P' is a feasible F-path. Since $B_P \cup W \subseteq B_{P'}$, P' is a B_P-augmenting path.

Case 2. D is a b_i-b' chain.

Let $D := w_0 B_1 w_1 \dots w_{l-1} B_l w_l$ where $w_0 := b_i$ and $w_l = b'$. Note that this block decomposition can be found in $O(|V(G)| + |E(G)|)$ time.

For each nontrivial block B_j with $1 \leq j \leq l-1$, let $S_j := N_G(B_j - \{w_{j-1}, w_j\})$. If B_l is nontrivial, then let $S_l := N_G(B_l - \{w_{l-1}, w_l\}) - X_P$, namely, S_l contains all neighbors of $V(B_l - \{w_{l-1}, w_l\})$ except the neighbors of $N_D(b')$ contained in X_P. For each nontrivial block B_j with $1 \leq j \leq l$, let G_j be obtained from B_j by adding S_j and the edges of G from S_j to $V(B_j)$. Note that $N_D(b') \cup \{b'\} \subseteq B_l$, and for $1 \leq j \leq l-1$, $V(B_j) \cap (N_D(b') \cup \{b'\}) \subseteq \{w_{l-1}\}$. Hence, for $1 \leq j \leq l-1$, $S_j \subseteq V(F-r) \cup \{x_i, y_i\}$. Moreover, $r \notin \{x_i, y_i\}$ by Notation 3.9. Thus, $r \notin S_j$ for $1 \leq j \leq l-1$. Also if B_l is nontrivial, then no vertex in $V(B_l - N_D(b'))$ is adjacent to r, and by the definition of S_l, $r \notin S_l$. First, we prove the following.

Claim. One can in $O(|V(G)| + |E(G)|)$ time either find a B_P-augmenting path or certify that the following statements hold:

(I) for each nontrivial block B_j with $1 \leq j \leq l-1$, $|S_j| = 2$, $y_i \in S_j$, and if u denotes the vertex in $S_j - \{y_i\}$, then $(G_j, y_i, w_{j-1}, u, w_j)$ is planar;

(II) for each $1 \leq j \leq l-2$ for which both B_j, B_{j+1} are trivial, $|N_G(w_j) - \{w_{j-1}, w_{j+1}\}| = 2$, and $y_i \in N_G(w_j)$; and

(III) if B_l is nontrivial, then $S_l \cap (V(F-r) - \{a,a'\}) = \emptyset$.

Proof of Claim. We will show that if one of (I)–(III) does not hold, then one can find in $O(|V(G)| + |E(G)|)$ time a B_P-augmenting path.

Proof of (I). Suppose that $j \in \{1, \dots, l-1\}$ and B_j is nontrivial. Note that $G_j - S_j = B_j$ is 2-connected. Moreover, since G is 4-connected, G_j is $(4, S_j \cup \{w_{j-1}, w_j\})$-connected. Thus, the hypotheses of Lemma 2.3 are satisfied with G_j, S_j, w_{j-1}, w_j as G, S, b, b', respectively. By Lemma 2.3 exactly one of the following holds:

(1) there exist $u, u' \in S_j$ and an induced u-u' path Q such that $V(Q) \cap \{w_{j-1}, w_j\} = \emptyset$, $V(Q) \cap S_j = \{u, u'\}$, and $G_j - (V(Q) \cup S_j)$ is connected; or

(2) $|S_j| = 2$, and the elements of S_j can be labeled as u, u' such that $(G_j, u, w_{j-1}, u', w_j)$ is planar.

Moreover, one can in $O(|V(G_j)| + |E(G_j)|)$ time (and hence, in $O(|V(G)| + |E(G)|)$ time) find a path as in (1) or certify that (2) holds.

Suppose that (1) holds. Define P' as follows.

(a) if $\{u, u'\} \cap \{x_i, y_i\} = \emptyset$, then let $P' := Q$;

(b) if $\{u, u'\} = \{x_i, y_i\}$, then let $P' := (P - V(P(x_i, y_i))) \cup Q$;

(c) if $\{u, u'\} \cap \{x_i, y_i\} = \{x_i\}$, then let $P' := P[a, x_i] \cup Q$; and

(d) if $\{u, u'\} \cap \{x_i, y_i\} = \{y_i\}$, then let $P' := P[y_i, a'] \cup Q$.

We claim that P' is a path with ends in $V(F)$ which is internally disjoint from $V(B_P) \cup V(F)$. If (a) or (b) occurs, then clearly P' is a path as claimed. Suppose (c) occurs, that is, $\{u, u'\} \cap \{x_i, y_i\} = \{x_i\}$. If $a \notin \{u, u'\}$, then clearly P' is a path as claimed;

if $a \in \{u, u'\}$, then by the definition of x_i in Notation 3.9, $x_i = a$, and hence, P' is a path as claimed. Similarly, if (d) occurs, then P' is a path as claimed.

Next we show that P' is a B_P-augmenting path. Let u_1, u_2 denote the ends of P'. Since Q is induced in G_j and $N_G(P(a, a')) \cap V(F) \subseteq \{a, a\} \cup \{r\}$ (by Assumption 1), and because P is induced in $G - aa'$ and $P[a, x_i) \cup P(y_i, a']$ has no neighbor in $V(B_j)$ (by the definition of x_i and y_i in Notation 3.9), one can show that P' is an induced path in $G - u_1u_2$. Since $G_j - (V(Q) \cup S_j)$ is connected, it is easy to see that $P'(u_1, u_2)$ is nonseparating in G_F. If $r \in \{u_1, u_2\}$, then since $r \notin \{u, u'\} \subseteq S_j$, (b), (c), or (d) occurs and either $r = a$ or $r = a'$. In this case, r is not a cut vertex of $G_F - V(P(a, a'))$, and since $|X_P| \geq 3$, r is not a cut vertex of $G_F - V(P'(u_1, u_2))$. Thus, P' is a feasible F-path. Since there exists a w_{j-1}-w_j path in $G_j - (V(Q) \cup S_j)$, there exists an r_i-b' path in $D - V(P'(u_1, u_2))$. By the definition of b', the vertex adjacent to b' in this path has a neighbor in V_t for some $t \in \{1, \dots, p\} - \{i\}$. Hence, B_P is properly contained in a block of $G_F - V(P'(u_1, u_2))$, and therefore, P' is a B_P-augmenting path.

So assume that (2) holds. If $y_i \in \{u, u'\}$, then (I) holds, so we may assume that $y_i \notin \{u, u'\}$. By Lemma 2.6 with G_j, u, u', w_{j-1}, w_j as G, a, a', b, b', respectively, one can find in $O(|V(G_j)| + |E(G_j)|)$ time an induced u-u' path Q such that $G_j - V(Q)$ has exactly two components K and K' with $w_{j-1} \in V(K)$ and $w_j \in V(K)$. Without loss of generality, we may assume that $u \neq x_i$. If $u' \neq x_i$, then let $P' := Q$; otherwise, let $P' := P[a, x_i] \cup Q$. Clearly, P' is a path with ends in $V(F)$ which is internally disjoint from $V(B_P) \cup V(F)$.

Next we show that P' is a B_P-augmenting path. Let u, u'' denote the ends of P'. Since Q is induced in G_j and $N_G(P(a, a')) \cap V(F) \subseteq \{a, a'\} \cup \{r\}$, and because P is an induced path in $G - aa'$ and $P[a, x_i)$ has no neighbor in $V(B_j) - \{w_{j-1}, w_j\}$ (by the definition of x_i in Notation 3.9), one can show that P' is an induced path in $G - uu''$. Since $G_j - V(Q)$ has exactly two components, one containing w_{j-1} and the other containing w_j, and because of the path W, it follows that $G - V(P'(u, u''))$ is connected, so P' is nonseparating in G_F. If $r \in \{u, u''\}$, then $r = u'' = a$, and r is not a cut vertex of $G_F - V(P(a, a'))$. Then, because of the path W, r is not a cut vertex of $G_F - V(P'(u, u''))$. Thus, P' is a feasible F-path. Since $B_P \cup W \subseteq B_{P'}$, P' is a B_P-augmenting path.

Proof of (II). Suppose that for some $j \in \{1, \dots, l-1\}$ both B_j and B_{j+1} are trivial. If $y_i \in N_G(w_j)$ and $|N_G(w_j) - \{w_{j-1}, w_j\}| = 2$, then (II) holds, so we may assume that $y_i \notin N_G(w_j)$ or $|N_G(w_j) - \{w_{j-1}, w_j\}| \neq 2$. Therefore, $|N_G(w_j) - \{w_{j-1}, w_j, y_i\}| \geq 2$. Let u, u' be distinct vertices in $N_G(w_j) - \{w_{j-1}, w_j, y_i\}$. Note that $r \notin \{u, u'\}$ because B_P is a block of $G_F - V(P(a, a'))$. Without loss of generality we may assume that $u \neq x_i$. If $u' \neq x_i$, then let $P' := (u, w_j, u')$. If $u' = x_i$, then let $P' := P[a, x_i] + \{u, w_j, w_jx_i, uw_j\}$. By the definition of x_i, y_i in Notation 3.9, $u \neq a$ when $u' = x_i$. So P' is a path with ends in $V(F)$ which is internally disjoint from $V(B_P) \cup V(F)$. Note that $V(P') \cap V(P(x_i, a')) = \emptyset$.

Next we show that P' is a B_P-augmenting path. Let u, u'' denote the ends of P. Since P is an induced path in $G - aa'$, and because w_j has no neighbor in $P[a, x_i)$ (by the definition of x_i in Notation 3.9), one can see that P' is induced in $G - uu''$. Because of the path W and since $P(a, a')$ is nonseparating in G_F, P' is nonseparating in G_F. If $r \in \{u, u''\}$, then $r = u'' = a$, and r is not a cut vertex of $G_F - V(P(a, a'))$. Then, because of the path W, r is not a cut vertex of $G_F - V(P'(u, u''))$. Thus, P' is a feasible F-path. Since $B_P \cup W \subseteq B_{P'}$, P' is a B_P-augmenting path.

Proof of (III). Suppose B_l is nontrivial. If $S_l \cap (V(F-r)-\{a,a'\}) = \emptyset$, then (III) holds, so we may assume that $S_l \cap (V(F-r)-\{a,a'\}) \neq \emptyset$. We want to apply Lemma 2.3 to find a B_P-augmenting path, so we need to show that $G_l, S_l, w_{l-1}, w_l = b'$ (as G, S, b, b', respectively) satisfy the hypotheses in the statement of Lemma 2.3. Clearly, $G_l - S_l = B_l$ is 2-connected and by definition, every vertex in S_l has a neighbor in $V(B_l) - \{w_{l-1}, w_l\}$. Since P is an induced path in $G - aa'$ and G is 4-connected, $G_l - b' \subseteq G$ is $(4, S_l \cup \{w_{l-1}\} \cup N_D(b'))$-connected. Hence, G_l is $(3, S_l \cup \{w_{l-1}, b'\})$-connected. Recall that $r \notin S_l$ (see the definition of S_l), $V(P(x_i,y_i))$ has no neighbor in $V(F-r)-\{a,a'\}$ unless $x_i = a$ or $y_i = a'$ (by Assumption 1), and $A_i - V(P_i)$ is connected. Thus, since $S_l \cap (V(F-r)-\{a,a'\}) \neq \emptyset$, $V(B_l) - (\{w_{l-1}, w_l\} \cup V(P(x_i,y_i))) \neq \emptyset$. This implies that $V(P(x_i,y_i)) \subseteq V(B_l) - \{w_{l-1}, w_l\}$ (and hence, $x_i, y_i \in S_l$); otherwise, $w_{l-1} \in V(P(x_i,y_i))$, contradicting the fact that $A_i - V(P_i)$ is connected. Thus, $|S_l| \geq 3$, and there exists a component K of $G_l - (S_l \cup \{w_{l-1}, w_l\}) = B_l - \{w_{l-1}, w_l\}$ which contains $V(P(x_i,y_i))$. Note that K has at least two neighbors in S_l, namely, x_i, y_i. Thus, the hypotheses of Lemma 2.3 are satisfied with G_l, S_l, w_{l-1}, w_l as G, S, b, b', respectively.

Therefore, by Lemma 2.3 there exist $u, u' \in S_l$ and an induced path Q in G_l such that $V(Q) \cap \{w_{l-1}, w_l\} = \emptyset$, $V(Q) \cap S_l = \{u, u'\}$, and $G_l - (V(Q) \cup S_l)$ is connected. Define P' as follows:

(a) if $\{u,u'\} \cap \{x_i,y_i\} = \emptyset$, then let $P' := Q$;
(b) if $\{u,u'\} = \{x_i,y_i\}$, then let $P' := (P - V(P(x_i,y_i))) \cup Q$;
(c) if $\{u,u'\} \cap \{x_i,y_i\} = \{x_i\}$, then let $P' := P[a,x_i] \cup Q$; and
(d) if $\{u,u'\} \cap \{x_i,y_i\} = \{y_i\}$, then let $P' := P[y_i,a'] \cup Q$.

We claim that P' is a path with ends in $V(F)$ which is internally disjoint from $V(B_P) \cup V(F)$. Clearly, this is true if (a) or (b) occurs. Suppose (c) occurs, that is, $\{u,u'\} \cap \{x_i,y_i\} = \{x_i\}$. If $a \notin \{u,u'\}$, then P' is a path as claimed. If $a \in \{u,u'\}$, then by the definition of x_i in Notation 3.9, $a = x_i$. Again, P' is a path as claimed. Similarly, if (d) occurs, then P' is a path as claimed.

Next we show that P' is a B_P-augmenting path. Let u_1, u_2 denote the ends of P'. Since Q is induced in G_l and $N_G(P(a,a')) \cap V(F) \subseteq \{a,a\} \cup \{r\}$, and because P is induced in $G - aa'$ and $P[a,x_i) \cup P(y_i,a']$ has no neighbor in B_l (by the definition of x_i and y_i in Notation 3.9), one can see that P' is an induced path in $G - u_1u_2$. Since $G_l - (V(Q) \cup S_l)$ is connected and $V(P(x_i,y_i))$ has a neighbor in $(\bigcup_{j=1}^p V_j) - V_i$, it is easy to see that $P'(u_1,u_2)$ is nonseparating in G_F. If $r \in \{u_1,u_2\}$, then since $r \notin S_l$, (b), (c), or (d) occurs, and either $r = a$ or $r = a'$. In this case, r is not a cut vertex of $G_F - V(P(a,a'))$, and since $|X_P| \geq 3$, r is not a cut vertex of $G_F - V(P'(u_1,u_2))$. Thus, P' is a feasible F-path. Moreover, since there exists a w_{l-1}-w_l path W' in $G_l - (V(Q) \cup S_l)$, there exists an r_i-b' path W'' in $D - V(P'(u_1,u_2))$. By the definition of b', the vertex adjacent to b' in W'' has a neighbor in V_t for some $t \in \{1,\ldots,p\} - \{i\}$. Hence, B_P is properly contained in a block of $G_F - V(P'(u_1,u_2))$, and therefore, P' is a B_P-augmenting path.

This concludes the proof of the claim. □

By the above claim, we may assume that (I), (II), and (III) hold. Therefore, by (III) and since $V(A_i) - \{b_i, x_i, y_i\}$ has a neighbor in $V(F-r) - \{a,a'\}$, we have $l \geq 2$. We consider three subcases.

Subcase 2.1. x_i has at least two neighbors in $V(B_l)$.

Thus, B_l is nontrivial (because x_i is not adjacent to b' in D). We claim that $P(x_i,y_i) \subseteq B_l - w_{l-1}$. Suppose for a contradiction that $P(x_i,y_i) \not\subseteq B_l - w_{l-1}$. Then $w_{l-1} \in V(P(x_i,y_i))$. Since $G_F - V(P(a,a'))$ is connected, $B_l - b' \subseteq P(x_i,y_i)$. But then x_i has at most one neighbor in $V(B_l)$ because P is an induced path in $G_F - aa'$, a contradiction. Therefore, $P(x_i, y_i) \subseteq B_l - w_{l-1}$.

Since $V(A_i) - \{b_i, x_i, y_i\}$ has a neighbor in $V(F - r) - \{a, a'\}$ and $S_l \cap (V(F - r) - \{a, a'\}) = \emptyset$ by (III), there exists $q \in \{1, \dots, l-1\}$ such that $V(B_q - w_{q-1})$ has a neighbor in $V(F - r) - \{a, a'\}$. Choose q to be maximum with this property, and let u be a neighbor of $V(B_q - w_{q-1})$ in $V(F - r) - \{a, a'\}$.

Next we define a u-w_q path Q_q in G_q. If B_q is trivial or u is adjacent to w_q, then let Q_q be the path induced by the edge uw_q. Otherwise, B_q is nontrivial, $S_q = \{u, y_i\}$, and $(G_q, y_i, w_{q-1}, u, w_q)$ is planar (by (I)). By Lemma 2.7 (with $G_q, u, w_q, y_i, w_{q-1}$ as G, a, a', b, b', respectively), there exists a nonseparating induced u-w_q path Q_q in G_q such that $V(Q_q) \cap \{y_i, w_{q-1}\} = \emptyset$. Moreover, such a path can be found in $O(|V(G_q)| + |E(G_q)|)$ time.

By the maximality of q, for $q + 1 \leq j \leq l - 1$, the following holds: If B_j is nontrivial, then $S_j = \{x_i, y_i\}$ and $(G_j, y_i, w_{j-1}, x_i, w_j)$ is planar (by (I)), and if B_j and B_{j+1} are trivial, then $N_G(w_j) - \{w_{j-1}, w_{j+1}\} = \{x_i, y_i\}$ (by (II)). Note also that $x_i \in S_l$ because $P(x_i, y_i) \subseteq B_l - w_{l-1}$.

Choose the minimum $t \in \{q + 1, \dots, l\}$ such that $x_i \in N_G(B_t - w_t)$. Thus, by the choice of q and t, B_j is trivial for every $j \in \{q + 1, \dots, t - 1\}$. For each $j \in \{q + 1, \dots, t - 1\}$, let Z_j denote the path induced by the edge $w_{j-1}w_j$.

If B_t is trivial, then let Q_t denote the path induced by the edge $w_{t-1}x_i$. If B_t is nontrivial, then we define a path Q_t according to the following two cases.

- $t < l$. Then $S_t = \{x_i, y_i\}$, and $(G_t, w_{t-1}, x_i, w_t, y_i)$ is planar. By Lemma 2.7 with $G_t, w_{t-1}, x_i, w_t, y_i$ as G, a, a', b, b', respectively, there exists a nonseparating induced w_{t-1}-x_i path Q_t in G_t such that $V(Q_t) \cap \{w_t, y_i\} = \emptyset$. Moreover, such a path can be found in $O(|V(G_t)| + |E(G_t)|)$ time.
- $t = l$. Since P is induced in $G - aa'$ and x_i has at least two neighbors in $V(B_l)$, x_i has a neighbor in $V(B_l) - V(P(x_i, y_i))$. Moreover, $B_l - V(P(x_i, y_i))$ is connected because $A_i - V(P_i)$ is connected, and hence, there exists a w_{l-1}-x_i path Q' in $B_l - V(P(x_i, y_i))$. Let $G' := G_l - b'$, and let $S' := N_D(b') \cup S_l \cup \{w_{l-1}\}$. Then G' is $(4, S')$-connected, and $S' - \{w_{l-1}, x_i\}$ is contained in a component U of $G' - V(Q')$. By Lemma 2.1 (with G', S', w_{l-1}, x_i, U as G, S, a, a', U, respectively) there exists a nonseparating induced w_{l-1}-x_i path Q_l in G' such that $V(Q_l) \cap V(U) = \emptyset$ (and hence, $V(Q_l) \cap V(P(x_i, y_i)) = \emptyset$). Moreover, such a path can be found in $O(|V(G')| + |E(G')|)$ time (and hence, in $O(|V(G)| + |E(G)|)$ time).

Let $P' := Q_q \cup Z_{q+1} \cup \cdots \cup Z_{t-1} \cup Q_t \cup P[a, x_i]$. Then P' is a u-a path in G such that $V(P') \cap V(F) = \{u, a\}$. Moreover, it is not hard to see that such a path can be found in $O(|V(G)| + |E(G)|)$ time.

Next we show that P' is a B_P-augmenting path. It is not hard to see that P' is an induced path in $G - ua$. Because of the path W and since $P(a, a')$ is nonseparating in G_F, $P'(u, a)$ is nonseparating in G_F. If $a = r$, then r is not a cut vertex of $G_F - V(P(a, a'))$, and because of the path W, r is not a cut vertex of $G_F - V(P'(u, a))$. Thus, P' is a feasible F-path. Moreover, since $V(P') \cap V(P(x_i, a']) = \emptyset$, $B_P \cup W \subseteq B_{P'}$. Therefore, P' is a B_P-augmenting path.

Subcase 2.2. x_i has at most one neighbor in $V(B_l)$, and x_i has a neighbor in $V(A_i) - (V(P(x_i, y_i)) \cup \{b_i\})$.

Then since A_i is 2-connected, x_i has a neighbor in $V(D) - (V(B_l) \cup \{b_i\})$. Therefore, since $V(A_i) - \{b_i, x_i, y_i\}$ has a neighbor in $V(F - r) - \{a, a'\}$ and by (I), (II), and (III), there exist $u \in V(F-r) - \{a, a'\}$ and $q, t \in \{1, \dots, l-1\}$ with $q \le t$ such that one of the following holds:

(a) $u \in N_G(B_q - w_{q-1})$, and $x_i \in N_G(B_t - w_t)$; or

(b) $x_i \in N_G(B_q - w_{q-1})$, and $u \in N_G(B_t - w_t)$.

Choose q, t so that $t-q$ is minimum and (a) or (b) holds. Note that $q < t$ because in (I) we must have $y_j \in S_j$ and in (II) we must have $y_j \in N_G(w_j)$.

We may assume that (a) holds because the other case is symmetric.

By the minimality of $t-q$ and by (I), B_j is trivial for every $j \in \{q+1, \dots, t-1\}$. Using (II), one can also show that $t-q \le 2$. For $q+1 \le j \le t-1$, let Z_j denote the path induced by the edge $w_{j-1}w_j$.

If B_q is trivial, then let Q_q be the path induced by the edge uw_q. Otherwise (by (I)) B_q is nontrivial, $S_q = \{u, y_i\}$, and $(G_q, y_i, w_{q-1}, u, w_q)$ is planar . By Lemma 2.7 (with $G_q, u, w_q, y_i, w_{q-1}$ as G, a, a', b, b', respectively), there exists a nonseparating induced u-w_q path Q_q in G_q such that $V(Q_q) \cap \{y_i, w_{q-1}\} = \emptyset$. Moreover, such a path can be found in $O(|V(G_q)| + |E(G_q)|)$ time.

Similarly, if B_t is trivial, then let Q_t be the path induced by the edge x_iw_{t-1}. Otherwise (by (I)) B_t is nontrivial, $S_t = \{x_i, y_i\}$, and $(G_t, y_i, w_{t-1}, x_i, w_t)$ is planar. By Lemma 2.7 (with $G_t, x_i, w_{t-1}, y_i, w_t$ as G, a, a', b, b', respectively) there exists a nonseparating induced x_i-w_{t-1} path Q_t in G_t such that $V(Q_t) \cap \{y_i, w_t\} = \emptyset$. Moreover, such a path can be found in $O(|V(G_t)| + |E(G_t)|)$ time.

Let $P' := Q_q \cup Z_{q+1} \cup \cdots \cup Z_{t-1} \cup Q_t \cup P[a, x_i]$. Then P' is a u-a path which is internally disjoint from $V(B_P) \cup V(F)$. Moreover, it is not hard to see that such a path can be found in $O(|V(G)| + |E(G)|)$ time.

Next we show that P' is a B_P-augmenting path. Since Q_q, Q_t are nonseparating and induced in G_q, G_t, respectively, it is not hard to see that P' is an induced path in $G - ua$. Because of the path W and since $P(a, a')$ is nonseparating in G_F, $P'(u, a)$ is non-separating in G_F. If $a = r$, then r is not a cut vertex of $G_F - V(P(a, a'))$, and because of the path W, r is not a cut vertex of $G_F - V(P'(u, a))$. Thus, P' is a feasible F-path. Since $V(P') \cap V(P_i - x_i) = \emptyset$, $B_P \cup W \subseteq B_{P'}$. Therefore, P' is a B_P-augmenting path.

Subcase 2.3. x_i has at most one neighbor in $V(B_l)$, and x_i has no neighbor in $V(A_i) - (V(P(x_i, y_i)) \cup \{b_i\})$.

In this case, since A_i is 2-connected, b_i is the only neighbor of x_i in A_i not contained in $V(P(x_i, y_i))$. We consider two cases according to whether $x_i = a$ or $x_i \ne a$.

(A) $x_i = a$.

Then by the definition of x_i in Notation 3.9, $b_i \ne r_i$. Since $V(A_i) - \{b_i, x_i, y_i\}$ has a neighbor in $V(F-r) - \{a, a'\}$ and (III) holds, there exists some $q \in \{1, \dots, l-1\}$ such that $V(B_q - w_{q-1})$ has a neighbor in $V(F-r) - \{a, a'\}$. Choose q to be minimum with this property.

Therefore, since b_i is the only neighbor of x_i in A_i not contained in $V(P_i)$ and (I) holds, B_j is trivial for every $j \in \{1, \dots, q-1\}$. Using (II), one can show that either $q = 1$ or $q = 2$. For each $j \in \{1, \dots, q-1\}$ let Z_j be the path induced by the edge $w_{j-1}w_j$.

If B_q is trivial (in this case $q = 1$), then, by the choice of q, w_q has a neighbor u in $V(F-r)$, and let $Q_q := (w_{q-1}, w_q, u)$. If B_q is nontrivial, then by (I) $S_q = \{u, y_i\}$ for some $u \in V(F-r) - \{a, a'\}$, and $(G_q, y_i, w_{q-1}, u, w_q)$ is planar. Note that $u \ne a$ because x_i has no neighbor in $V(A_i) - (V(P_i) \cup \{b_i\})$. By Lemma 2.7 (with $G_q, u, w_{q-1}, y_i, w_q$ as G, a, a', b, b', respectively), there exists a nonseparating induced

u-w_{q-1} path Q_q in G_q such that $V(Q_q) \cap \{y_i, w_q\} = \emptyset$. Moreover, such a path can be found in $O(|V(G_q)| + |E(G_q)|)$ time.

Let $P' := (Z_1 \cup \cdots \cup Z_{q-1} \cup Q_q) + \{x_i, x_i b_i\}$. Then P' is a u-a path which is internally disjoint from $V(B_P) \cup V(F)$. Moreover, it is not hard to see that such a path can be found in $O(|V(G)| + |E(G)|)$ time.

Next we show that P' is a B_P-augmenting path. It is not hard to see that P' is an induced path in $G - ua$. Because of the path W and since $P(a, a')$ is nonseparating in G_F and Q_q is nonseparating in G_q, $P'(u, a)$ is nonseparating in G_F. If $a = r$, then r is not a cut vertex of $G_F - V(P(a, a'))$, and because of the path W, r is not a cut vertex of $G_F - V(P'(u, a))$. Thus, P' is a feasible F-path. Since $B_P \cup W \subseteq B_{P'}$, P' is a B_P-augmenting path.

(B) $x_i \neq a$.

In this case, it is possible that $b_i = r_i$. Note that x_i has degree at least four in G (because G is 4-connected), P is induced in $G - aa'$, and x_i has no neighbor in $V(A_i) - (V(P_i) \cup \{b_i\})$ (by assumption in this subcase). So x_i has a neighbor in $(\bigcup_{j=1}^{p} V_j) - V_i$. Let $t \in \{1, \ldots, p\} - \{i\}$ such that x_i has a neighbor in V_t.

Suppose that for some $j \in \{1, \ldots, l-1\}$, B_j is nontrivial. Then by (I) and by our assumption that x_i has no neighbor in $V(A_i) - (V(P(x_i, y_i)) \cup \{b_i\})$, $S_j = \{u, y_i\}$ for some $u \in V(F - r)$, and $(G_j, y_i, w_{j-1}, u, w_j)$ is planar. Note that $u \neq a'$ by the definition of y_i in Notation 3.9 and because $u \neq y_i$. Also $u \neq a$ because $x_i \neq a$. By Lemma 2.6 (with $G_j, y_i, u, w_{j-1}, w_j$ as G, a, a', b, b', respectively), there exists a nonseparating induced u-y_i path Q in G_j such that $G_j - V(Q)$ has exactly two components K and K' with $w_{j-1} \in V(K)$ and $w_j \in V(K')$. Moreover, such a path can be found in $O(|V(G_j)| + |E(G_j)|)$ time (and hence, in $O(|V(G)| + |E(G)|)$ time). Let $P' := Q \cup P[y_i, a']$. Then P' is a u-a' path in G such that $V(P') \cap V(F) = \{u, a'\}$. Moreover, it is not hard to see that such a path can be found in $O(|V(G)| + |E(G)|)$ time.

Next we show that P' is a B_P-augmenting path. Since Q is induced in G_j and $N_G(P(a, a')) \cap V(F) \subseteq \{a, a'\} \cup \{r\}$ (by Assumption 1), and because P is an induced path in $G - aa'$ and $P((y_i, a'])$ has no neighbor in $V(B_j)$ (by the definition of y_i in Notation 3.9), one can see that P' is an induced path in $G - ua'$. Since $G_j - V(Q)$ has exactly two components, one containing w_{j-1} and the other containing w_j, and because x_i has a neighbor in V_t, it is not hard to show that P' is nonseparating in G_F. If $r \in \{u, a'\}$, then $r = a'$ and r is not a cut vertex of $G_F - V(P(a, a'))$. In this case, because x_i has a neighbor in V_t, r is not a cut vertex of $G_F - V(P'(u, a'))$. Thus, P' is a feasible F-path. Moreover, since b_i is adjacent to x_i and x_i has a neighbor in V_t, it follows that P' is a B_P-augmenting path.

Thus, we may assume that B_j is trivial for every $j \in \{1, \ldots, l-1\}$. If $l \geq 3$, then B_1 and B_2 are trivial, and by (II), $N_G(w_1) - \{w_0, w_2\} = \{u, y_i\}$ for some $u \in V(F - r)$. Note that $u \notin \{a, a'\}$ because $x_i \neq a$ and $y_i \neq u$. By an argument similar to the above paragraph, one can show that $P' := (u, w_1, y_i) \cup P[y_i, a']$ is a B_P-augmenting path.

So we may assume that $l = 2$ and B_1 is trivial. This implies that $V(P(x_i, y_i)) \subseteq V(B_2)$. Hence, B_2 is nontrivial, so $S_2 = \{x_i, y_i\}$ (by (III)). Since $V(A_i) - \{b_i, x_i, y_i\}$ has a neighbor in $V(F - r) - \{a, a'\}$ (by assumption in this lemma) and (III) holds, w_1 is adjacent to some $u \in V(F - r) - \{a, a'\}$. Let x', y' denote the vertices in $N_D(b')$ (see Notation 3.9) which are the lowest and the highest in P, respectively. Since B_2 is 2-connected, $V(B_2) - (V(P(x_i, y_i)) \cup \{b'\})$ has a neighbor in $V(P(x', y_i))$. Since $B_2 - (V(P(x_i, y_i)) \cup \{b'\})$ is connected (because $A_i - V(P_i)$ is connected), there exists a w_1-y_i path Q' in G_2 such that x_i and b' are contained in a component U of $G_2 - V(Q')$.

Moreover, recall that G_2 is $(3, \{w_1, x_i, y_i, b'\})$-connected. Thus, the hypotheses of Lemma 2.1 are satisfied with $G_2, \{w_1, x_i, y_i, b'\}, w_1, y_i, U$ as G, S, a, a', U, respectively. By Lemma 2.1 there exists a nonseparating induced w_1-y_i path Q in G_2 such that $V(Q) \cap V(U) = \emptyset$ (and hence, $V(Q) \cap \{x_i, b'\} = \emptyset$). Moreover, such a path can be found in $O(|V(G_2)| + |E(G_2)|)$ time (and hence, in $O(|V(G)| + |E(G)|)$ time). Let $P' := (P[y_i, a'] \cup Q) + \{u, uw_1\}$. Then P' is a u-a' path in G such that $V(P') \cap V(F) = \{u, a'\}$. Moreover, it is not hard to see that such a path can be found in $O(|V(G)| + |E(G)|)$ time.

We conclude the proof by showing that P' is a B_P-augmenting path. Since Q is induced in G_2 and $N_G(P(a, a')) \cap V(F) \subseteq \{a, a'\} \cup \{r\}$ (by Assumption 1), and because P is an induced path in $G - aa'$ and $P((y_i, a'])$ has no neighbor in $V(B_2)$ (by the definition of y_i in Notation 3.9), one can see that P' is an induced path in $G - ua'$. Since $G_2 - V(Q)$ is connected, and because x_i has a neighbor in V_t, it is not hard to see that P' is nonseparating in G_F. If $r \in \{u, a'\}$, then $r = a'$, and r is not a cut vertex of $G_F - V(P(a, a'))$. In this case, because x_i has a neighbor in V_t, r is not a cut vertex of $G_F - V(P'(u, a'))$. Thus, P' is a feasible F-path. Moreover, since b_i is adjacent to x_i and x_i has a neighbor in V_t, it follows that P' is a B_P-augmenting path. ∎

We are now ready to prove the main result of this section, which implies Theorem 3.2. Consider Algorithm 1.

THEOREM 3.16. *Algorithm* 1 *is correct and runs in* $O(|V(G)||E(G)|)$ *time.*

Proof. First, we will prove the correctness of the algorithm.

At the start of each iteration of the main loop, P is a feasible a-a' F-path, and B_P is a nontrivial block of $G_F := G - V(F - r)$ containing r. As the algorithm progresses, $|V(B_P)|$ increases.

If $G_F - V(P(a, a'))$ is 2-connected, then the algorithm stops at line 5. Since P is an induced path in $G_F - aa'$, $H := P$ is either an elementary F-chain or an up a-a' F-chain whose blocks are all trivial. Moreover, $G_F - I(H) = G_F - V(P(a, a'))$ and $G[V(F) \cup I(H)] = F \cup P$ are 2-connected.

If for every B_P-bridge B of $G_F - V(P(a, a'))$, $N_G(B - r_B) \subseteq V(P)$, then by Lemma 3.6 the a-a' F-chain H in line 8 exists, and $G_F - I(H)$ and $G[V(F) \cup I(H)]$ are 2-connected. Thus, if the algorithm stops at line 9, it returns a correct answer.

If $|X_P| = 2$, then by Lemma 3.7 either the subgraph H defined in line 12 is a down F-chain or there exists a B_P-augmenting path. Thus, if the algorithm stops at line 14, then H is a down F-chain and $G_F - I(H) = B_P$ and $G[V(F) \cup I(H)]$ are 2-connected. Otherwise, the algorithm increases B_P by executing lines 16 and 17.

In line 19, if $|V(P)| = 3$ (and hence, $|X_P| \geq 3$), then $G_F - V(P(a, a'))$ is not 2-connected; for otherwise, Algorithm 1 would have stopped at line 5. By Lemma 3.8 a B_P-augmenting path exists, and the algorithm increases B_P.

Suppose then that $|X_P| \geq 3$ and $|V(P)| \geq 4$. Let $H_i \in \mathcal{H}$ be adjacent to F (see Notation 3.9). If $x_i = y_i$, then by Lemma 3.12 the B_P-augmenting path in line 24 exists, and the algorithm increases B_P. If $x_i \neq y_i$, then by Lemmas 3.12, 3.13, 3.14, and 3.15 either the subgraph H defined in line 26 is a triangle chain, or there exists a B_P-augmenting path. Thus, if the algorithm stops at line 28, then H is a triangle F-chain such that $G_F - I(H) = B_P$ and $G[V(F) \cup I(H)]$ are 2-connected. Otherwise, the algorithm increases B_P by executing lines 30 and 31.

Since $|V(B_P)|$ increases at each iteration, the main loop at line 1 eventually stops and a good F-chain in G is returned. Hence, Algorithm 1 is correct.

ALGORITHM 1. INTERNAL CHAIN.

Require: G, r, F, a, a', P, B_P satisfying the hypotheses of Theorem 3.2.

Return: A good F-chain H in G such that $G_F - I(H)$ and $G[V(F) \cup I(H)]$ are 2-connected.

1: **loop**
2: Apply Lemma 3.4 to P, and let P denote the resulting path;
3: Let a, a' denote the ends of P;
4: **if** $G_F - V(P(a, a'))$ is 2-connected **then**
5: Return $H := P$ and stop;
6: Compute X_P (as defined in Notation 3.5);
7: **if** for every B_P-bridge B of $G_F - V(P(a, a'))$, $N_G(B - r_B) \subseteq V(P)$ **then**
8: Find an up a-a' F-chain H by applying Lemma 3.6;
9: Return H and stop;
10: **if** $|X_P| = 2$ **then**
11: Let v, v' be the vertices in X_P;
12: $H \leftarrow (G_F - (V(B_P) - X_P)) - vv'$;
13: **if** H is a down F-chain in G **then**
14: Return H and stop;
15: **else**
16: Find a B_P-augmenting path P' as in Lemma 3.7;
17: Set $P \leftarrow P'$ and start a new iteration;
18: **if** $|V(P)| = 3$ **then**
19: Find a B_P-augmenting path P' as in Lemma 3.8;
20: Set $P \leftarrow P'$ and start a new iteration;
21: Compute $\mathcal{H}$;
22: Let $H_i \in \mathcal{H}$ be adjacent to F;
23: **if** $x_i = y_i$ **then**
24: Find a B_P-augmenting path P' as in Lemma 3.12
25: $P \leftarrow P'$ and start a new iteration;
26: Let H be obtained from A_i by adding $N_G(A_i) \cap V(F)$ and all the edges of G from $V(A_i)$ to $V(F)$;
27: **if** $G_F - V(B_P) = A_i$ and H is a triangle chain of F **then**
28: Return H and stop;
29: **else**
30: Find a B_P-augmenting path P' as in Lemmas 3.12, 3.13, 3.14, and 3.15;
31: Set $P \leftarrow P'$ and start a new iteration;

Now we discuss the running time of the algorithm.

The loop in line 1 is executed at most $|V(G)|$ times since $|V(B_P)|$ increases at each iteration.

By Lemma 3.4, the step in line 2 can be performed in $O(|V(G)| + |E(G)|)$ time.

The test in line 4 and the steps in line 6 can be executed in $O(|V(G)| + |E(G)|)$ time by standard graph search techniques [6].

The steps in lines 7–9 can be executed in $O(|V(G)|+|E(G)|)$ time by Lemma 3.6.

The steps in lines 10–17 can be executed in $O(|V(G)|+|E(G)|)$ time by Lemma 3.7.

The steps in lines 18–20 can be executed in $O(|V(G)|+|E(G)|)$ time by Lemma 3.8.

The steps in lines 21–22 can be executed in $O(|V(G)|+|E(G)|)$ time by standard graph search techniques [6].

The steps in lines 23–25 can be executed in $O(|V(G)|+|E(G)|)$ time by Lemma 3.12.

Finally, the steps in lines 26–31 can be executed in $O(|V(G)| + |E(G)|)$ time by Lemmas 3.12, 3.13, 3.14, and 3.15.

Therefore, the running time of the Algorithm 1 is $O(|V(G)||E(G)|)$. ∎

4. Chain decomposition. In this section, we describe how to construct a nonseparating chain decomposition of a 4-connected graph G.

The idea is the following. Suppose we have found a partial chain decomposition $H_1, H_2, \ldots, H_{i-1}$ of G and we want to find the next chain H_i. Let $F := G[\bigcup_{j=1}^{i-1} I(H_j)]$, and assume that $G_F := G - (V(F) - \{r\})$ is 2-connected. If G_F is a planar cyclic chain rooted at r, then we obtain our desired decomposition by taking $H_i := G_F$ and $t := i$. If G_F is not a planar cyclic chain, then we want to use Theorem 3.2. In order to apply it, we need to efficiently find vertices $a, a' \in V(F)$ and a feasible a-a' F-path P. This will follow from Lemma 4.2 below.

We need the following result, proved in [7] and [1], which was used in [2].

THEOREM 4.1. *Let G be a 3-connected graph, let $e \in E(G)$, and let $u \in V(G)$ be nonincident to e. Then there exists a nonseparating induced cycle in G through e and avoiding u. Moreover, such a cycle can be found in $O(|V(G)| + |E(G)|)$ time.*

LEMMA 4.2. *Let G be a 4-connected graph, let $r \in V(G)$, and let F be a connected subgraph of G such that $r \in V(F)$, $|V(F)| \geq 2$, and $G_F := G - (V(F) - \{r\})$ is 2-connected. Then one of the following holds:*

(1) *G_F is a planar cyclic chain in G rooted at r; or*

(2) *there exists a feasible a-a' F-path P in G, that is,*

(i) *$V(P) \cap V(F) = \{a, a'\}$ and P is an induced path in $G - aa'$;*

(ii) *$P(a, a')$ is nonseparating in G_F;*

(iii) *r is contained in a nontrivial block of $G_F - V(P(a, a'))$; and*

(iv) *if $r \in \{a, a'\}$, then r is not a cut vertex of $G_F - V(P(a, a'))$.*

Moreover, one can in $O(|V(G)| + |E(G)|)$ time certify that (1) holds or find a path as in (2).

Proof. First, suppose that G_F is 3-connected. Let G' be obtained from G by contracting $F - r$ to a single vertex, say v'. Then G' is 4-connected; otherwise, there exists a 3-cut T in G'. Since G is 4-connected, $v' \in T$. But then $T - \{v'\}$ is a 2-cut in G_F, which is a contradiction. By Theorem 4.1, we can find a nonseparating induced cycle C in G' through rv' in $O(|V(G)| + |E(G)|)$ time. The path $C - rv'$ in G' corresponds to an induced path P in G from r to some vertex $a' \in V(F - r)$. Since G is 4-connected, r has at least two neighbors in $G_F - V(P(r, a'))$. Moreover, since C is nonseparating in G', r is not a cut vertex of $G_F - V(P(r, a'))$, and r is contained in a nontrivial block of $G_F - V(P(r, a'))$. Thus, P, $a := r$, and a' satisfy (2).

So we may assume that G_F is 2-connected but not 3-connected. Let $\{b, b'\}$ be a 2-cut of G_F. Let H_1, H_2 be edge-disjoint subgraphs of G_F such that $r \in V(H_1)$, $V(H_1) \cap V(H_2) = \{b, b'\}$, $H_1 \cup H_2 = G_F$, $|V(H_1)| \geq 3$, and $|V(H_2)| \geq 3$. Choose H_1, H_2 such that H_2 is minimal. Note that b, b', H_1, H_2 can be found in $O(|V(G)| + |E(G)|)$ time using the algorithm in [3] for finding the 3-connected components of G_F. Let $S := N_G(H_2 - \{b, b'\}) - \{b, b'\}$, and let G' be obtained from H_2 by adding S and the edges of G from S to $V(H_2) - \{b, b'\}$. Note that $S \subseteq V(F)$, $|S| \geq 2$, and $r \notin S$ because $\{b, b'\}$ is a 2-cut of G_F and $r \notin V(H_2) - \{b, b'\}$. Moreover, G' is $(4, S \cup \{b, b'\})$-connected.

Suppose that $|V(H_2)| \geq 4$. Then by minimality of H_2, H_2 is 2-connected and G', b, b', S satisfy (i)–(v) of Lemma 2.3 (with G' as G). Therefore, we can in $O(|V(G')| + |E(G')|)$ time either

(I) find $a, a' \in S$ and an induced a-a' path P' in G' such that $V(P') \cap \{b, b'\} = \emptyset$, $V(P') \cap S = \{a, a'\}$, and $G - (V(P') \cup S)$ is connected, or

(II) certify that $|S| = 2$, and the vertices in S can be labeled as a, a' such that (G', a, b, a', b') is planar.

If (I) occurs, then r is contained in a nontrivial block of $G - V(P')$ since there exists a b-b' path in $H_2 - V(P(a, a'))$. Since $r \notin S$, we have $r \notin \{a, a'\}$. Hence, $P := P'$ is a path that satisfies (2).

So we may assume that one of the following holds: $|V(H_2)| \geq 4$ and (II) occurs, or $|V(H_2)| = 3$.

We claim that one can find in $O(|V(G')|+|E(G')|)$ time a path P in G' with ends a, a' in S such that $G' - (V(P) \cup S)$ has exactly two components K, K' with $b \in V(K)$ and $b' \in V(K')$. If $|V(H_2)| \geq 4$ and (II) occurs, then this follows from Lemma 2.6. If $|V(H_2)| = 3$, then let v be the only vertex in $V(H_2) - V(H_1)$. Then v has degree two in G_F, and since G is 4-connected, v has at least two neighbors in $V(F)$, say a, a'. Then $P := (a, v, a')$ is the required path.

Therefore, $G_F - V(P(a, a'))$ is connected. If r is contained in a nontrivial block of H_1, then r is contained in a nontrivial block of $G_F - V(P(a, a'))$, and since $r \notin S$, $r \notin \{a, a'\}$. In this case, P satisfies (2).

So assume that r is contained only in trivial blocks of H_1.

Since G_F is 2-connected, H_1 is a b-b' chain. Moreover, either r is a cut vertex of H_1, or $r \in \{b, b'\}$. In both cases, G_F is a cyclic chain rooted at r. Let $G_F := v_0 B_1 v_1 \cdots v_{k-1} B_k v_k$ for some integer $k \geq 2$ (where $v_0 = v_k = r$). Note that either $H_2 = B_j$ for some $1 \leq j \leq k$ (when $|V(H_2)| \geq 4$), or $H_2 = B_j \cup B_{j+1}$ for some $1 \leq j \leq k-1$ where B_j, B_{j+1} are trivial (when $|V(H_2)| = 3$).

If all the B_i's are trivial, then G_F is a planar cyclic chain and (2) holds. So assume that not all B_i's are trivial. For each 2-connected B_i, let $S_i := N_G(B_i - \{v_{i-1}, v_i\}) - \{v_{i-1}, v_i\}$, and let G_i be obtained from B_i by adding S_i and the edges of G from S_i to $V(B_i)$. Then $S_i \subseteq V(F - r)$, because $\{v_{i-1}, v_i\}$ is a 2-cut of G_F, and $r \notin V(B_i) - \{v_{i-1}, v_i\}$. Note that G_i, S_i, v_{i-1}, v_i (as G, S, b, b', respectively) satisfy (i)–(v) of Lemma 2.3 because $G_i - S_i = B_i$ is 2-connected and G_i is $(4, S_i \cup \{v_{i-1}, v_i\})$-connected. Thus, one can in $O(|V(G_i)| + |E(G_i)|)$ time either (a) find $a_i, a'_i \in S_i$ and an induced a_i-a'_i path P_i in G such that $V(P_i) \cap \{v_{i-1}, v_i\} = \emptyset$, $V(P_i) \cap S_i = \{a_i, a'_i\}$, and $G_i - (V(P_i) \cup S_i) = B_i - V(P_i(a_i, a'_i))$ is connected, or (b) certify that $|S_i| = 2$, and the vertices in S_i can be labeled as a_i, a'_i such that $(G_i, v_{i-1}, a_i, v_i, a'_i)$ is planar. Since G is 4-connected, if (b) occurs, then $B_i - \{v_{i-1}, v_i\} = G_i - (S_i \cup \{v_{i-1}, v_i\})$ is connected.

If G_F is not a planar cyclic chain rooted at r, then (a) must hold for some 2-connected B_i, and hence, $P := P_i$ satisfies (2) (because $r \notin S_i$). Otherwise, (1) holds.

It is not hard to see that all the steps described above can be executed in $O(|V(G)| + |E(G)|)$ time. □

Thus, combining Lemma 4.2 and Theorem 3.2 we obtain the following.

THEOREM 4.3. *Let G be a 4-connected graph, let F be a subgraph of G, and let $r \in V(F)$ such that $G_F := G - (V(F) - \{r\})$ is 2-connected. Then one of the following holds:*

(1) *there exists a good F-chain H in G such that $G_F - I(H)$ and $G[V(F) \cup I(H)]$ are 2-connected; or*

(2) *G_F is a planar cyclic chain rooted at r.*

Moreover, one can in $O(|V(G)|+|E(G)|)$ *time find a good* F*-chain as in* (1) *or certify that* (2) *holds.*

We are now ready to prove the main result in this paper.

Proof of Theorem 1.5. A nonseparating chain decomposition of G starting at ra can be found as follows. The first chain H_1 can be found in $O(|V(G)||E(G)|)$ time by Theorem 1.6. The internal chains can be found iteratively as follows. Suppose we have found a partial chain decomposition $H_1, \ldots, H_{i-1}$ $(i \geq 2)$ of G and we want to find H_i. Let $F := G[\bigcup_{j=1}^{i-1} I(H_j)]$. Apply Theorem 4.3 to G, F, and r. Then one of the following holds:

(1) there exists a good F-chain H in G such that $G_F - I(H)$ and $G[V(F) \cup I(H)]$ are 2-connected; or

(2) G_F is a planar cyclic chain rooted at r.

Moreover, one can in $O(|V(G)| + |E(G)|)$ time find a planar chain as in (1) or certify that (2) holds. If (1) holds, then let $H_i := H$ and set $i \leftarrow i + 1$. If (2) holds, then $H_1, \ldots, H_i := G_F$ is the desired chain decomposition.

Since the number of chains is at most $|V(G)|$, the above algorithm has time complexity $O(|V(G)|^2|E(G)|)$. ∎

Acknowledgment. The authors thank the referees for carefully reading the manuscript and for their helpful suggestions.

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