Spinning strings and cosmic dislocations

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It is shown that the 1+2 gravity spinning particle metric, when lifted to 1+3 dimensions in a boost-covariant way, gives rise to a chiral conical space-time which includes as particular cases the space-time of a spinning string and two space-times that can be associated with the chiral string with a lightlike phase and the twisted string recently discovered by Bekenstein. Some gravitational effects are briefly discussed and a possibility for a new type of anyon is mentioned.

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The spinning point particle solution of 1+2 gravity [1, 2] received much attention recently in connection with anyons [3–6] and the violation of causality (see, e.g., [7–10] and references therein). In the framework of 1+2 gravity this solution is well motivated, being suggested by the Chern-Simons (CS) Poincaré gauge theory of gravity [11–13], where spin arises as one of the charges of the gauge group. Analogous smooth solutions are likely to be predicted in the Abelian Higgs model with the CS term [14] as well as in the framework of topologically massive gravity [15–17].

It is believed that the four-dimensional counterpart of this solution represents a spinning (rotating) cosmic string [18, 10]. To support the conjecture of a spinning cosmic string, various mechanisms of inducing an angular momentum on strings were discussed [19, 9]. Smooth rotating models were also considered both in the context of Einstein gravity [8] and Riemann-Cartan theory [20].

Recently a new class of *chiral strings*, with the phase of the complex scalar field possessing a helical structure both in space and time, were discussed by Bekenstein [21] in the framework of a global U(1) model in Minkowski space. They correspond to the extrema of the energy for fixed angular and linear momenta. In addition to the solutions with a rotating phase of the scalar field, the new class includes the configurations with the twisted surface of the constant phase of a scalar field, as well as the solutions for which this surface propagates along the string with the velocity of light. Although a global model does not provide a good setting for the coupling to gravity, the structure of the energy-momentum tensor of a timelike string is similar to that of the spinning cosmic string, giving new support to the rotating cosmic string conjecture. The gravitational counterparts of the *twisted* string and the string with *a lightlike* phase apparently were not discussed so far. Bekenstein actually found that lightlike strings, as well as the rotating ones, are stable against small perturbations, while twisted strings probably are not. However, they can be stabilized when forming loops.

In this paper we show that the relevant space-time structure, *chiral conical space-time*, arises naturally when starting from the same spinning particle solution of 1+2 gravity. It seems likely that chiral conical space-time provides the gravitational counterpart for the infinitely thin straight chiral strings in all the three cases mentioned above, in the same way that an ordinary conical space-time is associated with usual cosmic strings.

Recall that in the Einstein formulation the (1+2)-dimensional metric of a point particle endowed with a mass μ and a spin J^0 reads [1]

$$ds^2 = (dt + 4GJ^0\,d\varphi)^2 - r^{-8G\mu}(dr^2 + r^2 d\varphi^2) \quad . \tag{1}$$

Geometrically, (1) can be obtained by cutting a wedge of a three-dimensional Minkowski space-time and then time translating one of the two faces of the wedge before identifying them. Through the Einstein equations this metric produces the energy-momentum tensor

$$\begin{aligned} \sqrt{-g}\,T^{00} &= \mu\,\delta(x)\,\delta(y)\,, \\ \sqrt{-g}\,T^{0a} &= J^0\,\epsilon^{ab}\,\partial_b[\delta(x)\delta(y)], \end{aligned} \tag{2}$$

where $\epsilon^{12} = -\epsilon^{21} = 1$, and $a = 1, 2$. The space-time (1) can also be obtained as an asymptotic solution in the framework of some field-theoretical models, e.g., in the 1+2 Abelian Higgs model with the CS term coupled to Einstein gravity [14]. Alternatively, spinning particle solutions are likely to exist in the parity-violating topologically massive 1+2 gravity (with no matter), as was shown both in the linearized theory [3] and using the full nonlinear treatment [15].

The physical significance of the spinning 1+2 particle solution mostly comes from the hypothesis that these solutions have (cosmic) string four-dimensional counterparts. A lift from 1+2 to 1+3 dimensions usually is supposed to be performed simply by adding dz^2 to (1) [18, 10], that gives

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$$ds^2 = (dt + 4GJ_0\, d\varphi)^2 - r^{-8G\mu}(dr^2 + r^2 d\varphi^2) - dz^2. \tag{3}$$

In the spinless case an important feature of the metric of an infinitely thin straight cosmic string is the *boost invariance* along the symmetry axis. However, for $J_0 \neq 0$, this property is lost. Indeed, under the Lorentz transformation in the z-t plane the interval (3) becomes

$$\begin{aligned} ds^2 = {} & (dt' + 4GJ'^0\, d\varphi)^2 - r^{-8G\mu}(dr^2 + r^2 d\varphi^2) \\ & - (dz' + 4GJ'^z\, d\varphi)^2, \end{aligned} \tag{4}$$

where

$$J'^0 = \gamma J^0, \quad J'^z = \gamma v J^0, \quad \gamma = 1/\sqrt{1-v^2}. \tag{5}$$

Hence, an observer moving along the spinning string will see a twisted metric with the *spacelike* helical structure in the z direction. Also, geometrically, (4) can be obtained by cutting a wedge of a four-dimensional Minkowski space-time and then making a boost in one of the faces before regluing them.

To describe this situation in a more symmetric way we pass to Cartesian coordinates $x = r\cos\varphi$, $y = r\sin\varphi$ and introduce a new notation $(x^A, x^a) = (t, z, x, y)$, $A = 0, 3$, $a = 1, 2$. Then Eq. (4) will read

$$ds^2 = \eta_{AB}\, \omega^A \omega^B - \delta_{ab}\, \omega^a \omega^b\,, \tag{6}$$

where $\eta_{AB} = \mathrm{diag}(1, -1)$ and

$$\omega^{(A)} = dx^A + 4GJ^A(W_1 dy - W_2 dx), \tag{7a}$$

$$W_1 = \frac{x}{r^2}, \quad W_2 = \frac{y}{r^2}, \tag{7b}$$

$$\omega^{(a)} = r^{-4G\mu}\, dx^a. \tag{7c}$$

The raising (lowering) of the tetrad indices (A) and (a) is performed with the Minkowski metric $\mathrm{diag}(1, -1, -1, -1)$.

Now, to make the boost-invariance manifest, one merely has to admit that the parameters $J^A = (J^0, J^z)$ form a *two-dimensional vector* under the 1+1 Lorentz group acting on the t-z plane:

$$J^0 = \gamma(J'^0 - vJ'^z), \quad J^z = \gamma(J'^z - vJ'^0). \tag{8}$$

This relation can be considered a generalization of Eq. (5) for the case $J^z \neq 0$. In accordance with this assumption, the raising and lowering of the index A, from here on, will be performed with the metric η_{AB}.

Alternatively, one can think of the vector J^A as a world-sheet quantity which transforms as a two-dimensional vector under reparametrization. With a slight abuse of notation we can write the string world sheet as $x^\mu = x^\mu(\xi^A)$ with $\xi^A = (\tau, \sigma)$. Now, under the reparametrization $\xi^A \rightarrow \xi^{\overline{A}}$ the quantity J^A is demanded to transform as

$$J^A \rightarrow J^{\overline{A}} = J^B \frac{\partial \xi^{\overline{A}}}{\partial \xi^B}\,. \tag{9}$$

Then Eq. (8) can be seen as representing a particular case of the subgroup SO(1,1) of the reparametrization group. The generalization (9) opens a way to develop a theory of infinitely thin chiral cosmic strings of an arbitrary configuration (this work is in preparation).

The vector J^A in the initial formulation was supposed to be timelike $J^2 = J^A J^B \eta_{AB} > 0$, so there exists a Lorentz frame in which $J^z = 0$. Now, two more options arise: $J^2 < 0$ and $J^2 = 0$. In the first case, a Lorentz frame exists in which $J^0 = 0$ and the corresponding metric is static $ds^2 = dt^2 - dl^2$ with the three-dimensional element

$$dl^2 = (dz + 4GJ^z\, d\varphi)^2 + r^{-8G\mu}(dr^2 + r^2 d\varphi^2). \tag{10}$$

These three-dimensional spaces are studied in the context of the geometric theory of continuum media. They represent screw dislocations, or more precisely (for nonzero μ), the superposition of an aligned screw dislocation and a disclination [22, 23]. This is the reason why we call the corresponding space-time as generated by a "cosmic dislocation." The quantity $2GJ^z/\pi$ is analogous to Burgers vector of a dislocation.

The last option is an isotropic J^A. In this case in any Lorentz frame $|J^0| = |J^z| = J$, and the metric of the space-time is better represented by

$$ds^2 = du\, dv + 4GJ du\, d\varphi - r^{-8G\mu}(dr^2 + r^2 d\varphi^2), \tag{11}$$

where $u = t - z$, $v = t + z$. This lightlike string metric has helical structure both in space and time in equal amounts. It can also be considered as a limiting case of the space-time that represents a usual cosmic string interacting with a circular polarized plane-fronted gravitational wave [24].

Therefore, a Lorentz invariance in the t-z plane formally predicts a wider class of space-time topological defects that contains the spinning cosmic string as a particular case. The generalized metric (7) is locally flat elsewhere except for the symmetry axis. To clarify the nature of the singularity, by using Cartan formalism, we compute the connection one-forms $d\omega^{(\mu)} = -\omega^{(\mu)}{}_{(\nu)} \wedge \omega^{(\nu)}$ associated with (7):

$$\omega^{(x)}{}_{(y)} = 2Gr^{4G\mu}(r^{4G\mu}\, w\, J_A\, \omega^{(A)} + 2\mu\epsilon^{ab}\omega_{(a)}\partial_b \ln r), \tag{12a}$$

$$\omega^{(A)}{}_{(a)} = -2Gr^{8G\mu}\, J^A\, w\, \epsilon_{ab}\, \omega^{(b)}\,, \tag{12b}$$

where

$$w = \partial_x W_1 + \partial_y W_2. \tag{13}$$

Substituting here Eq. (7b) we get

$$w = (\partial_x^2 + \partial_y^2)\ln r = 2\pi\delta^2(\mathbf{r}), \tag{14}$$

and hence the first term in (12a) is the product of $r^{4G\mu}$ with the two-dimensional δ function. For $\mu > 0$ such a quantity is zero in the sense of the distributions. However, one has to keep terms of the same structure in (12b) since the corresponding factor in front of the δ function will be compensated when Einstein tensor is computed.

This treatment of singular structures associated with the metric (6) and (7) exactly corresponds to the Hamiltonian treatment in [1]. We note that an additional analysis is desirable in order to achieve more rigorous description of the singularity involved. In fact, for $\mu < 0$ the above argument fails.

The connection one-forms produce the curvature two-forms

$$R^{(x)}{}_{(y)} = 2G[4\pi\mu\delta^2(\mathbf{r})\, dx \wedge dy - r^{8G\mu} J^A \partial_a w\, \omega^{(A)} \wedge dx^a]\,, \tag{15a}$$

$$R^{(A)}{}_{(a)} = 2G r^{4G\mu} J^A \partial_a\, w dx \wedge dy. \tag{15b}$$

We have omitted ambiguous squares of δ functions multiplied by some regular function vanishing on their support. Finally, taking into account (13), we obtain the nonzero contravariant components of the Einstein tensor:

$$\begin{aligned} \sqrt{-g}\; G^{AB} &= 8\pi G\mu\eta^{AB}\delta^2(\mathbf{r}), \\ \sqrt{-g}\; G^{Aa} &= 8\pi G J^A \epsilon^{ab}\partial_b\delta^2(\mathbf{r}). \end{aligned} \tag{16}$$

The corresponding energy-momentum tensor is

$$\begin{aligned} \sqrt{-g}\, T^{AB} &= \eta^{AB}\mu\delta(x)\delta(y), \\ \sqrt{-g}\, T^{Aa} &= J^A\, \epsilon^{ab}\, \partial_b[\delta(x)\,\delta(y)]. \end{aligned} \tag{17}$$

One can easily generalize the above to the case of multiple parallel chiral strings with the parameters μ_i, J_i^A, $i = 1, ..., N$ located at the points $\mathbf{r}_i$ (for pure spinning strings see [2, 16]). The metric has the form (6) with

$$\omega^{(A)} = dx^A + 4G\sum_{i=1}^{N} J_i^A\, \frac{(x - x_i)dy - (y - y_i)dx}{|\mathbf{r} - \mathbf{r}_i|^2}, \tag{18a}$$

$$\omega^{(a)} = \prod_{i=1}^{N} |\mathbf{r} - \mathbf{r}_i|^{-4G\mu_i} dx^a. \tag{18b}$$

Let us now discuss the geodesic structure of the chiral conical space-time. The Hamilton-Jacoby equation for the metric (6) and (7) reads

$$\eta^{AB}\, \frac{\partial S}{\partial x^A}\frac{\partial S}{\partial x^B} \;-\; r^{8G\mu}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi} - 4GJ^A\frac{\partial S}{\partial x^A}\right)^2\right] \;=\; M^2\,. \tag{19}$$

The solution has the form

$$S \;=\; -\, P_A x^A \;+\; L\varphi \pm \int (p_\perp^2 - L_{\text{eff}}^2/R^2)^{\frac{1}{2}} dR \quad, \tag{20}$$

where $P_A = (P_0, P_z)$ and L are arbitrary constants, $R = r^{(1-4G\mu)}$ and

$$p_\perp^2 = (P_A P^A - M^2)/(1 - 4G\mu)^2\,, \tag{21a}$$

$$L_{\text{eff}} = (L + 4GP_A J^A)/(1 - 4G\mu)\,. \tag{21b}$$

The corresponding solution of the equations of motion is obtained by a differentiation with respect to P_A and L:

$$(1 - 4G\mu)(\varphi - \varphi_0) = \mp \int \frac{L_{\text{eff}}/R^2}{(p_\perp^2 - L_{\text{eff}}^2/R^2)^{\frac{1}{2}}} dR\,, \tag{22a}$$

$$(1 - 4G\mu)(x^A - x_0^A) = \pm \int \frac{P^A - 4GJ^A\, L_{\text{eff}}/R^2}{(p_\perp^2 - L_{\text{eff}}^2/R^2)^{\frac{1}{2}}} dR\,. \tag{22b}$$

Equation (22a) describes the deflection of the geodesic in the chiral conical space-time. Note that the canonical angular momentum L is shifted by the amount $4GP_AJ^A$ and enlarged by a conical factor. Equation (22b) gives a time shift and also a z shift. The first is due to the momentum P_A (the first term in the integrand) and the second to the helical structure of the metric (the second term in the integrand). When $P^A = 0$ we have $\Delta x^A = 4GJ^A\Delta\varphi$. The motion is restricted to the plane $z =$const only for the pure radial case $L_{\text{eff}} = 0$. For the timelike J^A (spinning string), the second term in the integrand in (22b) produces a time delay associated with two images of a radiating object split by the string. In the case of cosmic dislocation, it gives the z splitting of two images, $\Delta z = 8\pi GJ^z$. Hence, a cosmic dislocation produces not only a transversal, but also a longitudinal shift of the images. If the direction of the string is unknown, two objects behind the string allow us to distinguish between the usual cosmic string and the cosmic dislocation, in the latter case, the typical picture of the images being a parallelogram instead of a rectangle.

Let us discuss briefly the effect of quantization. The Klein-Gordon equation in the metric (6) and (7) is equivalent to

$$(\eta^{AB} - 16G^2R^{-2}J^AJ^B)\partial_A\partial_B\Phi + 8GR^{-2}J^A\partial_A\partial_\varphi\Phi - R^{-1}(1-4G\mu)^2\partial_R(R\partial_R\Phi)(1-4G\mu) - R^{-2}(\partial_\varphi)^2\Phi + M^2\Phi = 0, \tag{23}$$

and has the regular solution at the origin

$$\Phi = \sum_{m=-\infty}^{\infty} \int K(P_t, P_z, m) J_{m_{\text{eff}}}(kR) \times e^{-i(P_A x^A + m\varphi)} dP_t dP_z, \tag{24}$$

where

$$\begin{aligned} k &= p_\perp/(1 - 4G\mu), \\ m_{\text{eff}} &= (m + 4GP_AJ^A)/(1 - 4G\mu), \end{aligned} \tag{25}$$

and $m = 0, \pm 1, \pm 2, \ldots$ (the φ variable takes values between 0 and 2π); J is the Bessel function and K an arbitrary function in the indicated arguments. Thus the shift of the angular momentum due to Burgers vectors J^A translates into the shift of the magnetic quantum number m after a quantization ($\hbar = 1$),

$$m \to (m + 4GP_A J^A)/(1 - 4G\mu), \tag{26}$$

enlarged by the factor $(1 - 4G\mu)^{-1}$. This shift is analogous to the shift $m \to m - \frac{e\Phi}{2\pi}$ for the charge in the Aharonov-Bohm effect, where Φ is the magnetic flux. In fact, as was argued in [5], in the case of a spinning string this shift is responsible for producing gravitational anyons [3, 4, 6]. The role of the charge is played by the energy constant P_0, while the magnetic flux corresponds to the rotation parameter $8\pi G J^0$. Here we see that a similar phenomenon exists in the (essentially four-dimensional) case of a cosmic dislocation, with the role of a charge being played by the longitudinal component of the linear momentum P_z. Another interesting case is that of the lightlike string, this time the effective charge being $P_0 - P_z$. For the massless particle moving along the string the angular momentum shift vanishes.

Higher spin fields can be treated along the lines of [25]. A new feature is the nontrivial self-adjoint extension of the operators involved, similar to the case of a spinning string [26].

To summarize, we have found that a boost-invariance can be preserved for the spinning string by introducing a two-dimensional Burgers parameter on the world sheet. This suggests in a natural way a spacelike and a lightlike helical structure for the strings in addition to the usual timelike one. The corresponding metric is likely to be interpreted as describing the gravitational field of infinitely thin chiral strings. The spacelike helical structure suggests the possibility of anyonic string-particle composites with the longitudinal linear momentum acting as a charge.

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