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MODULATED FIBRING AND THE COLLAPSING PROBLEM

CRISTINA SERNADAS, JOÃO RASGA, AND WALTER A. CARNIELLI

Abstract. Fibring is recognized as one of the main mechanisms in combining logics, with great significance in the theory and applications of mathematical logic. However, an open challenge to fibring is posed by the *collapsing problem*: even when no symbols are shared, certain combinations of logics simply collapse to one of them, indicating that fibring imposes unwanted interconnections between the given logics. *Modulated fibring* allows a finer control of the combination, solving the collapsing problem both at the semantic and deductive levels. Main properties like soundness and completeness are shown to be preserved, comparison with fibring is discussed, and some important classes of examples are analyzed with respect to the collapsing problem.

§1. Introduction. Among the contemporary research on theory and application of logic, the topic of the combination of logics is one of the most interesting. Logicians, philosophers and computer scientists are finally emerging from the complexity and the perplexity of isolated logical systems, learning how to capitalize on the intricate characteristics of particular logic systems towards a general manner of investigating the way logics can be combined, the way such combinations can be applied and understanding the general properties (c.f. [1]). Among the several approaches for spelling out such combinations, the techniques of *fibring* (c.f. [9, 10, 11, 16, 18]) have been the most auspicious: the fibring of logics leads to a new logic where not only constructors are mixed, but proof methods are combined.

Although the fibring techniques can be defined in the context of quantificational logic (c.f. [17]), even if restricted to the propositional level (avoiding variables, terms, binding operators such as quantifiers, and the subtleties therein), fibring propositional based logics such as modal, intuitionistic and many-valued logics produces huge amounts of possibilities connected to real applications (in engineering and artificial intelligence) and is open to interesting philosophical interpretations. Furthermore, although most of the work on fibring has been restricted to logics endowed with truth-functional semantics, some steps have been taken towards encompassing logics (like paraconsistent logics [5]) with non truth-functional semantics [3].

Fibring can be presented from a proof-theoretical or from a model-theoretical perspective. From the proof-theoretical point of view, fibring is treated in a natural

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way over logic systems with a Hilbert-like (axiomatic) deductive style presentation, but this kind of deduction seems more appropriate for meta-mathematical investigation than for real applications, a practice sometimes requiring Gentzen (sequent) or tableau style presentation. An appropriate framework for fibring natural deduction systems by means of labeled (or annotated) deduction systems is given in [14].

Usually, soundness is preserved under the process of fibring in the sense that the fibring of a family of logics is sound, provided that the components are sound. However, a more difficult problem is to show that completeness is also preserved. This question was solved in [18] for a wide class of (truth-functional) propositional based logics, where it was shown that under certain reasonable requirements (to wit, that the component logics are complete under general frame semantics and endowed with congruence relations) a kind of transfer of completeness can be obtained, guaranteeing that the result of the fibring is complete.

The use of categorial language is very appropriate for defining fibring, because fibring appears as a universal construction in the appropriate category of logic systems [16], emphasizing the canonical nature of fibring. Furthermore, it is often useful to show that certain collections of objects together with certain appropriate transformations make up a category, such as the category of signatures of logics together with arity preserving maps. In this way, the underlying theory is held maximally uniform and general.

However, general as it is, the original notion of fibring is not yet broad enough to accommodate more subtle aspects of combinations of logics, for example to avoid the *collapsing problem*, which consists of the unexpected collapse of two logics when combined even by unconstrained fibring (no symbols are shared). A simple yet paradigmatic example was provided in [8, 10], where it is shown that, in our terms, the unconstrained fibring (sharing nothing!) of classical and intuitionistic logic collapses into classical logic. The result is that the original notion of fibring is not appropriate for controlling this kind of phenomenon, and in the present paper we extend the notion of fibring to a much more powerful notion of *modulated fibring*.

The main idea behind modulated fibring at the semantic level is as follows. In the original fibring (as clearly shown in [18]), even when no symbols are shared, an interconnection is imposed upon the two given logics (to wit, only pairs of models sharing the same algebra of truth values contribute to the resulting logic). In the novel notion of modulated fibring this imposition is relaxed by giving as input to the fibring a translation between the truth value algebras of the two given logics. This translation modulates the result. An appropriate choice of the translation recovers the original notion of fibring, but other translations are possible and in the example of [8] a manner of avoiding the collapsing is shown. The modulated fibring is also introduced at the deductive system level leading to some provisos when applying the inference rules.

Therefore, the main goal of this paper is to achieve a mechanism for combining logics both at the semantic and the deductive levels but avoiding when desired the collapsing phenomenon. Preservation of soundness and completeness are also investigated.

Besides the pioneering example of [8], other cases of collapsing are described and for each of them we show how modulated fibring avoids the collapsing under a specific choice of the truth values translation.

The remainder of the paper is structured as follows. Section 2 is dedicated to modulated fibring at the semantic level including the notions of interpretation system and morphism. In this section some interesting examples selected from intuitionistic and many-valued logics are presented. Moreover, it shows how to extract from an interpretation system the (local and global) notions of entailment and establishes some basic results. Finally, introduces the notion of modulated fibring as a pushout in the category of interpretation systems, indicates how to recover the original notion of fibring as a special case, and shows how to set up the the base diagram of the pushout from the intended translation between the truth value algebras. Section 3 concentrates on deductive aspects of modulated fibring starting with Hilbert systems and their morphisms. The modulated fibring at the deductive level appears as a pushout in the category of Hilbert systems. We conclude the section with examples and a comparison with fibring. Section 4 is dedicated to logic systems putting together interpretation systems on one hand and Hilbert systems on the other hand, fibring of logic systems and preservation of soundness. Section 5 concentrates on preservation of completeness namely establishing sufficient conditions. We conclude in Section 6 with some remarks and open problems.

§2. Interpretation systems.

In this section, we investigate modulated fibring from a semantic point of view. We start with signatures and proceed in Subsection 2.2 with the notions of interpretation system and morphism between interpretation systems. We conclude the section with several examples that will be used later on. In Subsection 2.3 we introduce (global and local) semantic entailments. Finally, in Subsection 2.4 we define modulated fibring as a pushout in the category of interpretation systems and give several examples showing that it is possible to choose bridges that lead to non-collapsing situations.

2.1. Signatures.

We introduce the basic symbols that we need in each signature. We start by identifying the notion of pre-signature.

DEFINITION 2.1. *A pre-signature is a triple $\Sigma = \langle C, \&, \Xi \rangle$ where C is an indexed family of sets over the natural numbers, & is a symbol and Ξ is a set.*

Elements of C_k are *constructors* of arity k, and elements of Ξ are *meta-variables*. The role of the symbol & will become clear when giving the semantics. Moreover this symbol is also essential for technical reasons in Section 5.

DEFINITION 2.2. *A pre-signature morphism $h : \langle C, \&, \Xi \rangle \to \langle C', \&', \Xi' \rangle$ is a pair $\langle h_1, h_2 \rangle$ such that $h_1 = \{h_{1_k}\}_{k \in \mathbb{N}}$ is a family of maps from C_k to C'_k for every $k \in \mathbb{N}$ and $h_2 : \Xi \to \Xi'$ is a map.*

Pre-signatures and their morphisms constitute the category *pSig*. This category has finite colimits and in particular pushouts.

DEFINITION 2.3. *A signature is a co-cone in pSig, that is $\Sigma = \langle C, \&, \Xi, S \rangle$.*

The set S contains the "safe-relevant" morphisms whose destination is $\langle C, \&, \Xi \rangle$. Safety will play an important role in the definition of the entailments by constraining the admissible assignments to meta-variables in the range of safe-relevant morphisms. This is also the reason why the meta-variables are local to signatures which was not the case of fibring in [18].

DEFINITION 2.4. *A signature morphism $h : \Sigma \to \Sigma'$ is a co-cone morphism, that is, h is a pre-signature morphism such that $h \circ f \in S'$ whenever $f \in S$.*

Signatures and their morphisms constitute the category *Sig*. Again this category has finite colimits, in particular pushouts.

2.2. Basic notions. The basic semantic unit is the structure for a signature. Typically in an algebraic setting, a structure is an algebra.

DEFINITION 2.5. *A Σ-structure $\mathscr{B} = \langle B, \leq, v\rangle$ is a pre-ordered algebra over C and & with finite meets*[1] *such that*

1. $v_2(\&)(b_1, b_2) = b_1 \sqcap b_2$;
2. $v_k(c)(b_1, \dots, b_k) \cong v_k(c)(d_1, \dots, d_k)$ *whenever* $b_i \cong d_i$ *for* $i = 1, \dots, k$.[2]

The elements in B are the *truth values* (or degrees) and $v_k(c)$ is the *denotation* of constructor c of arity k which is an operation in the algebra. The symbol & is the syntactical counterpart of 2-ary meets. Constraint 1. indicates that & behaves like a conjunction (whether or not such symbol is a constructor in the signature). Constraint 2. is congruence requirement: denotations of a constructor on "equivalent" truth values should be "equivalent".

In the fibring as presented in [18], structures were power set algebras based on sets of points (worlds). The more general setting of considering an algebra (not necessarily a power set algebra) also includes logics (like multi-valued logics) whose semantics is not provided in terms of points.

In the sequel we omit the reference to the arity of the constructors and the subscripts in signature morphisms in order to make the notation lighter. Sometimes we also use $\vec{b}$ as a short hand for $b_1, \dots, b_k$.

DEFINITION 2.6. *An interpretation system is a tuple $\mathscr{I} = \langle \Sigma, M, A\rangle$ where Σ is a signature, M is a class (of models), A is a map associating to each $m \in M$ a Σ-structure $\mathscr{B}_m$.*

The interpretation system could be a pair $\langle \Sigma, \mathscr{B}\rangle$. We include M because one can take the models of the logic at hand and use A to extract the underlying algebras and M also simplifies the notion of interpretation system morphism.

DEFINITION 2.7. *An interpretation system morphism $h : \mathscr{I} \to \mathscr{I}'$ is a tuple $\langle \hat{h}, \underline{h}, \dot{h}, \ddot{h}\rangle$ where:*

- $\hat{h} : \Sigma \to \Sigma'$ *is a morphism in Sig*;
- $\underline{h} : M' \to M$ *is a map*;
- $\dot{h} = \{\dot{h}_{m'}\}_{m' \in M'}$ *where* $\dot{h}_{m'} : \langle B_{\underline{h}(m')}, \leq_{\underline{h}(m')}\rangle \to \langle B'_{m'}, \leq'_{m'}\rangle$ *is a monotonic map*;
- $\ddot{h} = \{\ddot{h}_{m'}\}_{m' \in M'}$ *where* $\ddot{h}_{m'} : \langle B'_{m'}, \leq'_{m'}\rangle \to \langle B_{\underline{h}(m')}, \leq_{\underline{h}(m')}\rangle$ *is a monotonic map preserving finite meets*;[3]

[1] In a pre-order, meets are unique up to isomorphism. We use the notation $\sqcap\{b_1, \dots, b_k\}$ or even $b_1 \sqcap \dots \sqcap b_k$ for a choice of one of the meets for $\{b_1, \dots, b_k\}$ and $\top$ for $\sqcap\, \emptyset$. Observe that finite meets exist iff 0-ary and 2-ary meets exist.

[2] By $b_1 \cong b_2$ it is meant $b_1 \leq b_2$ and $b_2 \leq b_1$.

[3] Observe that a map preserves finite meets iff preserves 0-ary and 2-ary meets.

such that for every $m' \in M'$, $\vec{b} \in B^k_{\underline{h}(m')}$ *and* $\vec{b}' \in B^k_{m'}$:

1. $\ddot{h}_{m'}$ *is left adjoint of* $\dot{h}_{m'}$;
2. $v'_{m'}(\hat{h}(c))(\vec{b}') \cong_{m'} \dot{h}_{m'}(v_{\underline{h}(m')}(c)(\ddot{h}_{m'}(\vec{b}')))$ *for every* $c \in C_k$. △

Recall that $\ddot{h}_{m'}$ is left adjoint of $\dot{h}_{m'}$ iff for every $b' \in B_{m'}$ and $b \in B_{\underline{h}(m')}$:

$$b' \leq_{m'} \dot{h}_{m'}(\ddot{h}_{m'}(b')) \text{ and } \ddot{h}_{m'}(\dot{h}_{m'}(b)) \leq_{\underline{h}(m')} b.$$

As a consequence, $\dot{h}_{m'}$ also preserves meets for every $m' \in M'$. Observe that $\ddot{h}_{m'}(\dot{h}_{m'}(b)) \cong_{\underline{h}(m')} b$ whenever $\ddot{h}_{m'}$ is surjective. Moreover,

$$v'_{m'}(\&')(\dot{h}_{m'}(b_1), \dot{h}_{m'}(b_2)) \cong'_{m'} \dot{h}_{m'}(v_{\underline{h}(m')}(\&)(b_1, b_2)).$$

The map $\underline{h}$ is expected to be contravariant. The family of maps $\dot{h}_{m'}$ and $\ddot{h}_{m'}$ indicate that we need to represent the truth values of $B_{\underline{h}(m')}$ in the truth values of $B'_{m'}$ and vice versa. Clause 1. states constraints that the maps should fulfill. Clause 2. indicates that denotations of constructors from C in a model m' can be given for any truth values in $B'_{m'}$ by using the two maps.

The morphism between interpretation systems presented in [18] is a particular case of the one in Definition 2.7 with $\dot{h}_{m'} = \mathrm{id}_{B'_{m'}}$, $\ddot{h}_{m'} = \mathrm{id}_{B_{\underline{h}(m')}}$ and hence, $B_{\underline{h}(m')} = B'_{m'}$, etc.

PROP/DEFINITION 2.8. *Interpretation systems and their morphisms constitute the category Int.*

Some of the examples we consider are many-valued logics. For more details about these logics see [2, 13]. In all examples the signature is as follows: $\Sigma = \langle C, \&, \Xi, S\rangle$ where $\boldsymbol{t} \in C_0$ (in general in C_0 we also have propositional symbols), $C_1 = \{\neg\}$, $C_2 = \{\wedge, \vee, \Rightarrow\}$, $C_k = \emptyset$ for all $k \geq 3$, & is $\wedge$, $\Xi = \{\xi_i : i \in \mathbb{N}\}$ and $S = \emptyset$. Thus the interpretation systems in the examples only differ in the semantic part, that is in M and A.

EXAMPLE 2.9. Propositional interpretation system.

- *M is the class of all pairs $m = \langle \mathbb{B}, V\rangle$ where $\mathbb{B} = \langle B, \sqcap, \sqcup, \ominus, \top, \bot\rangle$ is a Boolean algebra and $V : C_0 \to B$ is a map such that $V(\boldsymbol{t}) = \top$;*
- *$A(m) = \langle B, \leq, v\rangle$ where*
  - $b_1 \leq b_2$ *iff* $b_1 \sqcap b_2 = b_1$;
  - $v_0(c) = V(c)$, $v_1(\neg) = \ominus$, $v_2(\wedge) = \sqcap$, *and* $v_2(\vee) = \sqcup$;
  - $v_2(\Rightarrow) = \lambda b_1 b_2.(\ominus b_1) \sqcup b_2$. △

EXAMPLE 2.10. Intuitionistic interpretation system.

- *M is the class of all pairs $m = \langle \mathbb{B}, V\rangle$ where $\mathbb{B} = \langle B, \sqcap, \sqcup, \sqsupset, \bot, \top\rangle$ is a Heyting algebra and $V : C_0 \to B$ such that $V(\boldsymbol{t}) = \top$;*
- *$A(m) = \langle B, \leq, v\rangle$ where*
  - $b_1 \leq b_2$ *iff* $b_1 \sqcap b_2 = b_1$;
  - $v_0(c) = V(c)$, $v_2(\wedge) = \sqcap$ *and* $v_2(\vee) = \sqcup$;
  - $v_2(\Rightarrow) = \sqsupset$;
  - $v_1(\neg) = \lambda b.b \sqsupset \bot$. △

EXAMPLE 2.11. (3-valued) Gödel interpretation system. *Gödel logics were introduced as approximations to intuitionistic logic, and extended the propositional intuitionistic calculus.*

- *M is the class of all pairs $m = \langle \mathbb{B}, V\rangle$ where $\mathbb{B} = \langle B, \sqcap, \sqcup, \sqsupset, \ominus, \bot, \top\rangle$ is a 3-valued Gödel algebra*[4] *and $V : C_0 \to B$ such that $V(t) = \top$;*
- *$A(m) = \langle B, \leq, v\rangle$ where*
  - *$b_1 \leq b_2$ iff $b_1 \sqcap b_2 = b_1$;*
  - *$v_0(c) = V(c)$, $v_2(\wedge) = \sqcap$, $v_2(\vee) = \sqcup$ and $v_1(\neg) = \ominus$;*
  - *$v(\Rightarrow) = \sqsupset$.* △

EXAMPLE 2.12. (3-valued) Łukasiewicz interpretation system. *Łukasiewicz logics, introduced in the twenties, were the first logics introducing a third truth value, designed to express linguistic modalities outside the scope of classical logic, like the possible (contingent) future.*

- *M is the class of all pairs $m = \langle \mathbb{B}, V\rangle$ where $\mathbb{B} = \langle B, \oplus, \ominus, \bot\rangle$ is a 3-valued multi-valued algebra*[5] *and $V : C_0 \to B$ is a map;*
- *$A(m) = \langle B, \leq, v\rangle$ where*
  - *$b_1 \leq b_2$ iff $b_1 \sqcap b_2 = b_1$;*
  - *$v_0(c) = V(c)$, $v_1(\neg) = \ominus$, $v_2(\wedge) = \sqcap$ and $v_2(\vee) = \sqcup$;*
  - *$v_2(\Rightarrow) = \sqsupset$.* △

2.3. Satisfaction and entailment. The objective of this section is to introduce the notion of entailment. As in other papers on fibring (e.g., [16]) we have two entailments: global entailment corresponding to proof and local entailment corresponding to derivation. We start by defining the languages over a given signature.

DEFINITION 2.13. *The set $L(\Sigma)$ of Σ-formulae is the free algebra over C, &, Ξ taking the elements of C_k as k-ary operations, & as a 2-ary operation and the elements of Ξ as 0-ary operations. We denote by $L(C, \&)$ the subset of $L(\Sigma)$ composed by ground formulae, that is formulae without meta-variables.*

We need the notion of assignment for defining the denotation of formulae and entailments. Assignments that give special values to schema variables that come from safe-relevant morphisms are referred to as safe.

Let $s : \check{\Sigma} \to \Sigma$ be a signature morphism and $\mathscr{B}$ a Σ-structure. Then, $\mathscr{B}(s)$ is the smallest subalgebra of $\mathscr{B}$ for signature $s(\check{\Sigma})$. Observe that $B'_{m'}(\hat{h}) \subseteq \hat{h}_{m'}(B_{\underline{h}(m')})$ whenever $\hat{h} \in S'$ and h is an interpretation system morphism.

DEFINITION 2.14. *An assignment over a Σ-structure $\mathscr{B}$ is a map $\alpha : \Xi \to B$. The assignment α is said to be safe for a set of formulae $\Gamma \subseteq L(\Sigma)$ iff $\alpha(s(\check{\xi})) \in B(s)$ for every $s : \check{\Sigma} \to \Sigma$ in S and $s(\check{\xi}) \in \Gamma$.*

[4]Recall that the typical 3-valued Gödel algebra has $B = \{\bot, 1/2, \top\}$ and operations $\ominus$ and $\sqsupset$ are defined as follows: $\ominus b = 1$ whenever $b = 0$ and 0 otherwise, and $b_1 \sqsupset b_2$ is $\top$ if $b_1 \leq b_2$ and b_2 otherwise.

[5]Recall that the operations $\otimes$, $\sqcap$, $\sqcup$ and $\sqsupset$ are defined as abbreviations: $b_1 \otimes b_2 = \ominus(\ominus b_1 \oplus \ominus b_2)$, $b_1 \sqsupset b_2 = (\ominus b_1) \oplus b_2$, $b_1 \sqcup b_2 = (b_1 \otimes (\ominus b_2)) \oplus b_2$, $b_1 \sqcap b_2 = (b_1 \oplus (\ominus b_2)) \otimes b_2$ and $\top = \ominus\bot$ The typical multi-valued algebra with three elements is $B = \{\bot, 1/2, \top\}$ and operations $\ominus$ and $\sqsupset$ are defined as: $\ominus b = 1 - b$ and $b_1 \sqsupset b_2$ is $1/2$ for the pairs $b_1 = 1/2$ and $b_2 = \bot$, $b_1 = \top$ and $b_2 = 1/2$, is $\bot$ for the pair $b_1 = \top$, $b_2 = \bot$ and is $\top$ otherwise.

Safe assignments show the relevance of having the component S in signatures and will be relevant when defining the entailment.

DEFINITION 2.15. *The interpretation of formulae over a Σ-structure $\mathscr{B}$ and an assignment α is a map $[\![.]\!]^{\mathscr{B}}_{\alpha} : L(\Sigma) \to B$ inductively defined as follows:*

- $[\![c]\!]^{\mathscr{B}}_{\alpha} = \nu(c)$, *whenever* $c \in C_0$;
- $[\![\xi]\!]^{\mathscr{B}}_{\alpha} = \alpha(\xi)$, *whenever* $\xi \in \Xi$;
- $[\![c(\gamma_1, \dots, \gamma_k)]\!]^{\mathscr{B}}_{\alpha} = \nu(c)([\![\gamma_1]\!]^{\mathscr{B}}_{\alpha}, \dots, [\![\gamma_k]\!]^{\mathscr{B}}_{\alpha})$, *whenever* $k \in \mathbb{N}$, $c \in C_k$ *and* $\gamma_1, \dots, \gamma_k \in L(\Sigma)$.

A formula γ is globally satisfied by $\mathscr{B}$ and a safe assignment α for γ, written $\mathscr{B}\alpha \Vdash \gamma$, iff $[\![\gamma]\!]^{\mathscr{B}}_{\alpha} \cong \top$. A formula γ is locally satisfied by $\mathscr{B}$, a safe assignment α for γ and $b \in B$, written $\mathscr{B}\alpha b \Vdash \gamma$, iff $b \leq [\![\gamma]\!]^{\mathscr{B}}_{\alpha}$.

In the context of an interpretation system, we can use $[\![\gamma]\!]^{m}_{\alpha}$ instead of $[\![\gamma]\!]^{A(m)}_{\alpha}$. Moreover, we write $m\alpha \Vdash \gamma$ and $m\alpha b \Vdash \gamma$ whenever $\mathscr{B}_m\alpha \Vdash \gamma$ and $\mathscr{B}_m\alpha b \Vdash \gamma$, respectively. Observe that local satisfaction of a formula at a truth value b indicates that a formula is at least as true as b. And we say that an assignment is over a model m iff $\alpha : \Xi \to B_m$.

DEFINITION 2.16. *A formula δ is a p-semantic consequence of a finite set of formulae Φ, written $\Phi \vDash_p \delta$, iff, for every model m and safe assignment α for $\Phi \cup \{\delta\}$, $m\alpha \Vdash \delta$ whenever $m\alpha \Vdash \varphi$ for every $\varphi \in \Phi$. A formula δ is a p-semantic consequence of a set of formulae Γ, written $\Gamma \vDash_p \delta$, iff there is a finite set Φ contained in Γ such that $\Phi \vDash_p \delta$.*

DEFINITION 2.17. *A formula δ is a d-semantic consequence of a finite set of formulae Φ, written $\Phi \vDash_d \delta$, iff $m\alpha b \Vdash \delta$ whenever $m\alpha b \Vdash \varphi$ for every $\varphi \in \Phi$, $m \in M$, safe assignment α over m for $\Phi \cup \{\delta\}$ and $b \in B_m$. A formula δ is a d-semantic consequence of a set of formulae Γ, written $\Gamma \vDash_d \delta$, iff there is a finite set Φ contained in Γ such that $\Phi \vDash_d \delta$.*

PROPOSITION 2.18. *Let Φ be a finite set of formulae and δ a formula. Then $\Phi \vDash_d \delta$ iff $\sqcap_{\varphi\in\Phi}[\![\varphi]\!]^{m}_{\alpha} \leq [\![\delta]\!]^{m}_{\alpha}$ for every model $m \in M$ and safe assignment α over m for $\Phi \cup \{\delta\}$.*

A signature morphism $\hat{h}$ can be extended to a map $\hat{h}^*$ between formulae: $\hat{h}^*(c) = \hat{h}(c)$ for $c \in C_0$, $\hat{h}^*(\xi) = \hat{h}(\xi)$, $\hat{h}^*(\varphi_1 \& \varphi_2)) = \hat{h}^*(\varphi_1)\&'\hat{h}^*(\varphi_2)$ and $\hat{h}^*(c(\varphi_1, \dots, \varphi_k)) = \hat{h}(c)(\hat{h}^*(\varphi_1), \dots, \hat{h}^*(\varphi_k))$. Below, $\hat{h}$ is used for the map $\hat{h}^*$.

We show below that p and d semantic entailments are preserved by some kind of morphisms. Before giving the result we need a lemma relating denotations of formulae in one signature with their counterparts in another signature.

LEMMA 2.19. *Let $h : \mathscr{I} \to \mathscr{I}'$ be an interpretation system morphism such that $\ddot{h}_{m'}$ is surjective for every $m' \in M'$ and α' is an assignment over m'. Then:*

- $[\![\hat{h}(\xi)]\!]^{m'}_{\alpha'} \cong_{m'} \dot{h}_{m'}([\![\xi]\!]^{h(m')}_{\underline{h}(\alpha')})$ *whenever α' is safe for $\hat{h}(\xi)$ and $\hat{h} \in S'$;*
- $[\![\hat{h}(\gamma)]\!]^{m'}_{\alpha'} \cong_{m'} \dot{h}_{m'}([\![\gamma]\!]^{h(m')}_{\underline{h}(\alpha')})$, *for every γ including at least a constructor from C where $\underline{h}(\alpha')(\xi) = \ddot{h}_{m'}(\alpha'(\hat{h}(\xi)))$.*

PROPOSITION 2.20. *Let $h : \mathcal{I} \to \mathcal{I}'$ be an interpretation system morphism such that $\ddot{h}_{m'}$ is surjective for every m' in M' and $\hat{h} \in S'$ whenever $\Gamma \cup \{\delta\}$ has meta-variables. Then* (1) $\hat{h}(\Gamma) \vDash'_p \hat{h}(\delta)$ *whenever* $\Gamma \vDash_p \delta$ *and* (2) $\hat{h}(\Gamma) \vDash'_d \hat{h}(\delta)$ *whenever* $\Gamma \vDash_d \delta$.

PROOF. Observe that if α' is a safe assignment over m' for $\hat{h}(\varphi)$ then the assignment $\underline{h}(\alpha')$ over $\underline{h}(m')$ as defined in Lemma 2.19 is safe for φ.
We only show claim (2). Let m' be in M' and α' be an assignment over m' safe for $\hat{h}(\Gamma \cup \{\delta\})$. Assume $\Gamma \vDash_d \delta$. Then there exists a finite set Φ of Γ with $\sqcap_{\varphi\in\Phi}[\![\varphi]\!]^m_\alpha \leq [\![\delta]\!]^m_\alpha$ for every model m in M and assignment α over m safe for $\Phi\cup\{\delta\}$. So $\sqcap_{\varphi\in\Phi}[\![\hat{h}(\varphi)]\!]^{m'}_{\alpha'} \cong \sqcap_{\varphi\in\Phi}\dot{h}_{m'}([\![\varphi]\!]^{\underline{h}(m')}_{\underline{h}(\alpha')}) = \dot{h}_{m'}(\sqcap_{\varphi\in\Phi}[\![\varphi]\!]^{\underline{h}(m')}_{\underline{h}(\alpha')}) \leq \dot{h}_{m'}([\![\delta]\!]^{\underline{h}(m')}_{\underline{h}(\alpha')}) \cong [\![\hat{h}(\delta)]\!]^{m'}_{\alpha'}$. Therefore $\hat{h}(\Gamma) \vDash_d \hat{h}(\delta)$. ⊣

As we shall see in Section 3, in the modulated fibring the morphisms that relate interpretation systems do have the required properties.

2.4. Modulated fibring of interpretation systems. The idea is that each model in the modulated fibring of $\mathcal{I}'$ and $\mathcal{I}''$ will be a pair $\langle m', m''\rangle$ where m' is a model of $\mathcal{I}'$ and m'' is a model of $\mathcal{I}''$. Moreover the truth values in the algebra of $\langle m', m''\rangle$ should be the union of the truth values in the algebras of m' and m''. However, for denotations of formulae we need some relationship between the truth values of m' and m'' for every m' and m''. Such a relationship is established by a bridge.

DEFINITION 2.21. *A bridge between interpretation systems $\mathcal{I}'$ and $\mathcal{I}''$ is a diagram $\beta = \langle f' : \check{\mathcal{I}} \to \mathcal{I}', f'' : \check{\mathcal{I}} \to \mathcal{I}''\rangle$ in Int such that $\hat{f}'$, $\hat{f}''$, $\dot{f}'_{m'}$ and $\dot{f}''_{m''}$ are injective maps and $\ddot{f}'_{m'}$ and $\ddot{f}''_{m''}$ are surjective maps for every $m' \in M'$ and $m'' \in M''$, respectively.*

Before defining modulated fibring, we introduce an auxiliary category and two functors.

PROP/DEFINITION 2.22. *The category poFam has pushouts. The objects are families of pre-orders with finite meets of the form $P = \{\langle P_i, \leq_i\rangle\}_{i\in I}$ and the morphisms $h : \{\langle P_i, \leq_i\rangle\}_{i\in I} \to \{\langle P'_{i'}, \leq'_{i'}\rangle\}_{i'\in I'}$ are pairs $\langle \underline{h}, \dot{h}\rangle$ such that $\underline{h} : I' \to I$ is a map and $\dot{h} = \{\dot{h}_{i'} : P_{\underline{h}(i')} \to P'_{i'}\}_{i'\in I'}$ is a family of monotonic maps.*

PROOF. Let $\beta = \langle\langle \underline{f}', \dot{f}'\rangle : \check{P} \to P', \langle \underline{f}'', \dot{f}''\rangle : \check{P} \to P''\rangle$. Then the pair $\langle\langle \underline{g}', \dot{g}'\rangle : P' \to P, \langle \underline{g}'', \dot{g}''\rangle : P'' \to P\rangle$, where $P = \{\langle P_i, \leq_i\rangle\}_{i\in I}$, $I = \{\langle i', i''\rangle : \underline{f}'(i') = \underline{f}''(i''), i' \in I', i'' \in I''\}$, $\underline{g}'(\langle i', i''\rangle) = i'$, $\underline{g}''(\langle i', i''\rangle) = i''$, and $\langle\langle P_i, \leq_i\rangle, \dot{g}'_i, \dot{g}''_i\rangle$ is a pushout of $\dot{f}'_{\underline{g}'(i)}$ and $\dot{f}''_{\underline{g}''(i)}$ in the category of pre-orders with finite meets for each $i \in I$, is a pushout of β in *poFam*. ⊣

Let $Sg : Int \to Sig$ be the functor such that $Sg(\mathcal{I}) = \Sigma$ and $Sg(h) = \hat{h}$ and $poF : Int \to poFam$ be the functor such that $poF(\mathcal{I}) = \{\langle B_m, \leq_m\rangle\}_{m\in M}$ and $poF(h) = \langle \underline{h}, \dot{h}\rangle$. We are now ready to show that the category *Int* has pushouts.

PROP/DEFINITION 2.23. *The modulated fibring of interpretation systems $\mathcal{I}'$ and $\mathcal{I}''$ by a bridge β is a pushout of β in Int.*

PROOF. Let $\beta = \langle f' : \check{\mathcal{I}} \to \mathcal{I}', f'' : \check{\mathcal{I}} \to \mathcal{I}''\rangle$. Consider $\langle g' : \mathcal{I}' \to \mathcal{I}, g'' : \mathcal{I}'' \to \mathcal{I}\rangle$ defined as follows:

- $\langle \hat{g}' : \Sigma' \to \Sigma, \hat{g}'' : \Sigma'' \to \Sigma\rangle$ is a pushout in *Sig* of $Sg(\beta)$;

- $\langle\langle \underline{g}', \dot{g}'\rangle : poF(\mathcal{J}') \to poF(\mathcal{J}), \langle \underline{g}'', \dot{g}''\rangle : poF(\mathcal{J}'') \to poF(\mathcal{J})\rangle$ is a pushout in *poFam* of $poF(\beta)$;
- $\langle B_{\langle m',m''\rangle}, \leq_{\langle m',m''\rangle}\rangle = (poF(\mathcal{J}))_{\langle m',m''\rangle}$;
- $A(\langle m', m''\rangle) = \langle B_{\langle m',m''\rangle}, \leq_{\langle m',m''\rangle}, \nu_{\langle m',m''\rangle}\rangle$;
- $\ddot{g}'_{\langle m',m''\rangle}(\dot{g}'_{\langle m',m''\rangle}(b')) = b'$;
- $\ddot{g}'_{\langle m',m''\rangle}(\dot{g}''_{\langle m',m''\rangle}(b'')) = \dot{f}'_{m'}(\ddot{f}''_{m''}(b''))$;
- $\ddot{g}'_{\langle m',m''\rangle}(\dot{g}'_{\langle m',m''\rangle}(b') \sqcap_{\langle m',m''\rangle} \dot{g}''_{\langle m',m''\rangle}(b'')) =$
$\ddot{g}'_{\langle m',m''\rangle}(\dot{g}'_{\langle m',m''\rangle}(b')) \sqcap'_{m'} \ddot{g}'_{\langle m',m''\rangle}(\dot{g}''_{\langle m',m''\rangle}(b''))$;
- $\nu_{\langle m',m''\rangle}(\hat{g}'(c'))(\vec{b}) = \dot{g}'_{\langle m',m''\rangle}(\nu'_{m'}(c')(\ddot{g}'_{\langle m',m''\rangle}(\vec{b})))$;
- $\ddot{g}''_{\langle m',m''\rangle}$ and $\nu_{\langle m',m''\rangle}(\hat{g}''(c''))$ defined in a similar way.

We have to check that $\langle \mathcal{J}, g', g''\rangle$ is a pushout in *Int* of f' and f''. For this purpose we consider $m' \in M'$ and $m'' \in M''$ and for the sake of simplification will omit the subscripts involving both m' and m''. Moreover we will consider that $\underline{f'}(m') = \underline{f''}(m'') = \check{m}$.

1. $\ddot{g}'$ (also $\ddot{g}''$) is well defined. On one hand, $\ddot{g}'(\dot{g}'(\dot{f}'(\check{b}))) \cong \dot{f}'(\check{b})$ using the definition of g' and on the other hand, $\ddot{g}'(\dot{g}''(\dot{f}''(\check{b}))) \cong \dot{f}'(\ddot{f}''(\dot{f}''(\check{b}))) \cong \dot{f}'(\check{b})$ using the same definition and surjectivity of $\ddot{f}''$.

2. $\ddot{g}'$ (also $\ddot{g}''$) is a monotonic map. Observe that $\leq$ is $lfp(\Delta, D_0)$ where D_0 includes:

- $\dot{g}'(\leq')$ and $\dot{g}''(\leq'')$;
- the pairs $\dot{g}'(b') \sqcap \dot{g}''(b'') \leq \dot{g}'(b')$ for every b' and b'';
- the pairs $\dot{g}'(b') \sqcap \dot{g}''(b'') \leq \dot{g}''(b'')$ for every b' and b'';
- $b \leq \dot{g}'(b') \sqcap \dot{g}''(b'')$ whenever $b \leq \dot{g}'(b')$, $b \leq \dot{g}''(b'')$ and b is $\dot{g}'(\dot{f}'(\check{b}))$;
- $\dot{g}'(b'_1) \sqcap \dot{g}''(b''_1) \leq \dot{g}'(b'_2) \sqcap \dot{g}''(b''_2)$ whenever $\dot{g}'(b'_1) \leq \dot{g}'(b'_2)$ and $\dot{g}''(b''_1) \leq \dot{g}''(b''_2)$;

and $\Delta : \wp B^2 \to \wp B^2$ is such that $\Delta(D)$ is the one-step transitive closure. Therefore Δ is extensive and monotonic. We prove that $\ddot{g}'(b_1) \leq' \ddot{g}'(b_2)$ whenever $b_1 \leq b_2 \in \Delta^{\mu}(D_0)$ by induction.

Base: $\mu = 0$.

(i) Assume that b_1 and b_2 are either $\dot{g}'(b'_1)$ and $\dot{g}'(b'_2)$ for some $b'_1, b'_2 \in B'$ or $\dot{g}''(b''_1)$ and $\dot{g}''(b''_2)$ for some $b''_1, b''_2 \in B''$. Then $\ddot{g}'(b_1) \leq' \ddot{g}'(b_2)$ by definition of $\leq$ and using the fact that $\ddot{g}'$ and $\ddot{g}''$ are surjective.

(ii) b_1 is $\dot{g}'(b') \sqcap \dot{g}''(b'')$ and b_2 is $\dot{g}'(b')$. Then $\ddot{g}'(b_2) = b'$ and $\ddot{g}'(b_1)$ is $b' \sqcap' \dot{f}'(\ddot{f}''(b''))$ and so $b' \sqcap' \dot{f}'(\ddot{f}''(b'')) \leq' b'$.

(iii) b_1 is $\dot{g}'(\dot{f}'(\check{b})) = \dot{g}''(\dot{f}''(\check{b}))$ and b_2 is $\dot{g}'(b') \sqcap \dot{g}''(b'')$ with $\dot{f}'(\check{b}) \leq' b'$ and $\dot{f}''(\check{b}) \leq'' b''$ (therefore $\check{b} \check{\leq} \ddot{f}''(b'')$). Then $\ddot{g}'(\dot{g}'(\dot{f}'(\check{b}))) \cong' \dot{f}'(\check{b})$ and $\ddot{g}'(\dot{g}''(b'')) \cong' \dot{f}'(\ddot{f}''(b''))$. Hence $\dot{f}'(\check{b}) \leq' b'$ and $\dot{f}'(\check{b}) \leq' \dot{f}'(\ddot{f}''(b''))$.

(iv) b_1 is $\dot{g}'(b'_1) \sqcap \dot{g}''(b''_1)$ and b_2 is $\dot{g}'(b'_2) \sqcap \dot{g}''(b''_2)$ with $\dot{g}'(b'_1) \leq \dot{g}'(b'_2)$ and $\dot{g}''(b''_1) \leq \dot{g}''(b''_2)$. So $b'_1 \leq' b'_2$, $b''_1 \leq'' b''_2$ and $\dot{f}'(\ddot{f}''(b''_1)) \leq' \dot{f}'(\ddot{f}''(b''_2))$. Then $\ddot{g}'(\dot{g}'(b'_1)) \leq' \ddot{g}'(\dot{g}'(b'_2))$ and $\ddot{g}'(\dot{g}''(b''_1)) \leq' \ddot{g}'(\dot{g}''(b''_2))$. Therefore $\ddot{g}'(\dot{g}'(b'_1)) \sqcap' \ddot{g}'(\dot{g}''(b''_1)) \leq' \ddot{g}'(\dot{g}'(b'_2)) \sqcap' \ddot{g}'(\dot{g}''(b''_2))$.

Step: $\mu = \varepsilon + 1$.

Let b be such that $b_1 \leq b, b \leq b_2 \in D_\varepsilon$. By the induction hypothesis $\ddot{g}'(b_1) \leq' \ddot{g}'(b)$ and $\ddot{g}'(b) \leq' \ddot{g}'(b_2)$ and so by transitivity of $\leq'$ we have $\ddot{g}'(b_1) \leq' \ddot{g}'(b_2)$.

Step: μ is a limit ordinal. Straightforward.

3. The preservation of meets by $\ddot{g}'$ and $\ddot{g}''$ is again straightforward.

4. $\ddot{f}'(\ddot{g}'(b)) \cong \ddot{f}''(\ddot{g}''(b))$: Let b be $\dot{g}'(b')$. Then $\ddot{f}'(\ddot{g}'(\dot{g}'(b'))) \cong \ddot{f}'(b')$ and $\ddot{f}''(\ddot{g}''(\dot{g}'(b'))) \cong \ddot{f}''(\dot{f}''(\ddot{f}'(b')))$ and so $\ddot{f}''(\dot{f}''(\ddot{f}'(b'))) \cong \ddot{f}'(b')$ since $\ddot{f}''$ is surjective. The other cases follow straightforwardly.

5. $v(\hat{g}'(\hat{f}'(\check{c})))(\vec{b}) \cong \dot{g}'(v'(\hat{f}'(\check{c}))(\ddot{g}'(\vec{b}))) \cong \dot{g}'(\dot{f}'(\check{v}(\check{c})(\ddot{f}'(\ddot{g}'(\vec{b}))))) \cong$
$$\dot{g}''(\dot{f}''(\check{v}(\check{c})(\ddot{f}''(\ddot{g}''(\vec{b}))))) \cong \dot{g}''(v''(\hat{f}''(\check{c}))(\ddot{g}''(\vec{b}))) \cong v(\hat{g}''(\hat{f}''(\check{c})))(\vec{b}).$$

6. $\ddot{g}'$ is left adjoint of $\dot{g}'$ ($\ddot{g}''$ is left adjoint of $\dot{g}''$).

(i) $b \leq \dot{g}'(\ddot{g}'(b))$: consider the case of b being $\dot{g}''(b'')$: $b'' \leq'' \dot{f}''(\ddot{f}''(b''))$, then $\dot{g}''(b'') \leq \dot{g}''(\dot{f}''(\ddot{f}''(b'')))$, so $\dot{g}''(b'') \leq \dot{g}'(\dot{f}'(\ddot{f}''(b'')))$ and $\dot{g}''(b'') \leq \dot{g}'(\ddot{g}'(\dot{g}''(b'')))$. (ii) $\ddot{g}'(\dot{g}'(b')) \leq b'$: straightforward.

7. Universal property. Let $h' : \mathcal{I}' \to \mathcal{I}'''$ and $h'' : \mathcal{I}'' \to \mathcal{I}'''$ be interpretation system morphisms such that $h' \circ f' = h'' \circ f''$.

Existence. $\hat{h}$ is the unique morphism in *Sig* such that $\hat{h} \circ \hat{g}' = \hat{h}'$ and $\hat{h} \circ \hat{g}'' = \hat{h}''$; $\underline{h} = \langle \underline{h}', \underline{h}'' \rangle$; $\dot{h}$ is the unique morphism in *poFam* such that $\dot{h} \circ \dot{g}' = \dot{h}'$ and $\dot{h} \circ \dot{g}'' = \dot{h}''$; and $\ddot{h}_{m'''}(b''') =^{def} \dot{g}'_{\underline{h}(m''')}(\ddot{h}'_{m'''}(b''')) \sqcap \dot{g}''_{\underline{h}(m''')}(\ddot{h}''_{m'''}(b'''))$. So, $\ddot{g}'(\ddot{h}(b''')) \cong \ddot{g}'(\dot{g}'(\ddot{h}'(b'''))) \sqcap \ddot{g}'(\dot{g}''(\ddot{h}''(b'''))) \cong \ddot{h}'(b''') \sqcap \dot{f}'(\ddot{f}''(\ddot{h}''(b'''))) \cong \ddot{h}'(b''') \sqcap \dot{f}'(\ddot{f}'(\ddot{h}'(b'''))) \cong \ddot{h}'(b''')$. We can also conclude that $\ddot{h}$ is monotonic and preserves finite meets and that $\ddot{h}$ is left adjoint to $\dot{h}$.

$$\begin{aligned}
v'''(\hat{h}(\hat{g}'(c')))(\vec{b}''') &\cong v'''(\hat{h}'(c'))(\vec{b}''') \\
&\cong \dot{h}'(v'(c')(\ddot{h}'(\vec{b}'''))) \\
&\cong \dot{h}(\dot{g}'(v'(c')(\ddot{g}'(\ddot{h}(\vec{b}'''))))) \\
&\cong \dot{h}(v(\hat{g}'(c'))(\ddot{h}(\vec{b}'''))).
\end{aligned}$$

Uniqueness. Assume that $k : \mathcal{I} \to \mathcal{I}'''$ is a morphism such that $k \circ g' = h'$ and $k \circ g'' = h''$. We want to show that $k = h$ that is $\ddot{k} = \ddot{h}$. We start by showing that $\ddot{k}(b''') = \dot{g}'(\ddot{g}'(\ddot{k}(b'''))) \sqcap \dot{g}''(\ddot{g}''(\ddot{k}(b''')))$. Assume that $\ddot{k}(b''') = \dot{g}'(b')$. Note that $\dot{g}'(b') \leq \dot{g}'(\dot{f}'(\ddot{f}'(b'))) \cong \dot{g}''(\dot{f}''(\ddot{f}'(b'))) \cong \dot{g}''(\ddot{g}''(\dot{g}'(b')))$. Then $\ddot{k}(b''') = \dot{g}'(\ddot{g}'(\ddot{k}(b'''))) \sqcap \dot{g}''(\ddot{g}''(\ddot{k}(b''')))$. The other cases follow in a straightforward manner. Since $\ddot{g}'(\ddot{k}(b''')) = \ddot{h}(b''')$ and $\ddot{g}''(\ddot{k}(b''')) = \ddot{h}(b''')$ then $k = h$. $\dashv$

Examples and the collapsing problem. We give some examples of modulated fibring namely showing how the collapse can be avoided. We start by a description of the most common collapse and then give a result stating how the bridge can be chosen to avoid the collapse when no constructors are shared.

DEFINITION 2.24. *In the modulated fibring $\langle g' : \mathcal{I}' \to \mathcal{I}, g'' : \mathcal{I}'' \to \mathcal{I} \rangle$ of $\mathcal{I}'$ and $\mathcal{I}''$ by a bridge β, $\mathcal{I}''$ collapses to $\mathcal{I}'$ iff there is a bijection $j_k : C''_k \to C'_k$ for all $k \in \mathbb{N}$ such that*

- $\hat{g}'(\Gamma') \vDash_p \hat{g}'(\varphi')$ *iff* $\hat{g}''(j^{-1}(\Gamma')) \vDash_p \hat{g}''(j^{-1}(\varphi'))$ *iff* $\Gamma' \vDash'_p \varphi'$ *and* $\hat{g}'(\Gamma') \vDash_d \hat{g}'(\varphi')$ *iff* $\hat{g}''(j^{-1}(\Gamma')) \vDash_d \hat{g}''(j^{-1}(\varphi'))$ *iff* $\Gamma' \vDash'_d \varphi'$;
- *there is a set $\Gamma'' \subseteq L(\Sigma'')$ and a formula $\varphi'' \in L(\Sigma'')$ such that $\Gamma'' \nvDash''_p \varphi''$ and $\hat{g}''(\Gamma'') \vDash_p \hat{g}''(\varphi'')$ or there is a set $\Gamma'' \subseteq L(\Sigma'')$ and a formula $\varphi'' \in L(\Sigma'')$ such that $\Gamma'' \nvDash''_d \varphi''$ and $\hat{g}''(\Gamma'') \vDash_d \hat{g}''(\varphi'')$.*

We now define a specific bridge that leads to a non-collapsing situation whenever there is no sharing of constructors.

PROPOSITION 2.25. *Let $\mathcal{I}'$, $\mathcal{I}''$ be interpretation systems such that $\boldsymbol{t}' \in C_0'$, $\boldsymbol{t}'' \in C_0''$, C' and C'' are in one to one correspondence and β a bridge such that $\check{C}_0 = \{\check{\boldsymbol{t}}\}$, $\check{C}_k = \emptyset$ for all $k \neq 0$, $\check{\Xi} = \emptyset$, $\check{S} = \emptyset$ $\check{M} = \{\check{m}\}$, $B_{\check{m}} = \{\check{\top}\}$, $id_{\Sigma'} \in S'$, $id_{\Sigma''} \in S''$, $\underline{f}'(m') = \underline{f}''(m'') = \check{m}$ and $\ddot{f}'_{m'}(b') = \ddot{f}''_{m''}(b'') = \check{\top}$ for every $m' \in M'$, $m'' \in M''$, $b' \in B'_{m'}$ and $b'' \in B''_{m''}$. Then the modulated fibring $\langle g' : \mathcal{I}' \to \mathcal{I}, g'' : \mathcal{I}'' \to \mathcal{I}\rangle$ of $\mathcal{I}'$ and $\mathcal{I}''$ by β does not collapse.*

PROOF. For every model $m'' \in M''$ all the pairs $\langle m', m''\rangle$ with $m' \in M'$ are in the modulated fibring. Therefore if $\Gamma'' \not\vDash''_p \varphi''$ then $\hat{g}''(\Gamma'') \not\vDash_p \hat{g}''(\varphi'')$ for every Γ'' and φ'' and if $\Gamma'' \not\vDash''_d \varphi''$ then $\hat{g}''(\Gamma'') \not\vDash_d \hat{g}''(\varphi'')$ for every Γ'' and φ''. ⊣

We say in this case that the interpretation system obtained is the *unconstrained modulated fibring* of $\mathcal{I}'$ and $\mathcal{I}''$. Thus, we can use this "universal" bridge for defining the modulated fibring whenever we do not want any symbols shared which is the case in most situations. Observe that in C_0' and C_0'' we can have propositional symbols.

Proposition 2.25 shows that for all cases of unconstrained modulated fibring (that is, only the verum is shared) it is possible to avoid the collapsing problem. Since $id_{\Sigma'} \in S'$, $id_{\Sigma''} \in S''$ using Proposition 2.20 we guarantee that the entailments of the component logics will be entailments in the modulated fibring. Observe also that the requirement $id_{\Sigma'} \in S'$, $id_{\Sigma''} \in S''$ does not change the entailments of $\mathcal{I}'$ and $\mathcal{I}''$. This requirement just prepares the interpretation systems for the combination. We can now instantiate Proposition 2.25 for several cases.

EXAMPLE 2.26. Modulated fibring of propositional and intuitionistic logics. *By choosing an adequate bridge as the one in Proposition 2.25 we can avoid the collapsing between propositional logic $\mathcal{I}'$ and intuitionistic logic $\mathcal{I}''$. Intuitionistic logic collapses into propositional logic when the formula $((\neg(\neg\varphi)) \Leftrightarrow \varphi)$ becomes valid which is not the case. Observe that in the modulated fibring, $\dot{g}'(B'_{m'})$ is a Boolean algebra "equivalent" to $B'_{m'}$ and $\dot{g}''(B''_{m''})$ is a Heyting algebra "equivalent" to $B''_{m''}$.* △

Similarly to Fariñas del Cerro and Herzig's C+J logic as presented in [8], in the modulated fibring of propositional logic $\mathcal{I}'$ and intuitionistic $\mathcal{I}''$ logic considered above, we have also no problems with the validity of the formula $\hat{g}'(\varphi' \Rightarrow' (\psi' \Rightarrow' \varphi'))$ since, according to our semantics, the formula is only valid for "intuitionistic values". Propositional values are converted to intuitionistic value "$\boldsymbol{t}$".

The following example is also an application of Proposition 2.25. Moreover it is also very interesting in showing the need for safe assignments.

EXAMPLE 2.27. Modulated fibring of propositional and Łukasiewicz logics. *Let $\mathcal{I}'$ and $\mathcal{I}''$ be the interpretation systems for propositional logic and 3-valued Łukasiewicz logic (see Examples 2.9 and 2.12, respectively). As a corollary of Proposition 2.25, the modulated fibring with no sharing does not collapse.*

In order to understand safe assignments consider the following case. We have $\{\xi_1', (\xi_1' \Rightarrow' \xi_2')\} \vDash'_d \xi_2'$ for propositional logic. In the unconstrained modulated fibring, we do not have $\{\hat{g}'(\xi_1'), (\hat{g}'(\xi_1')\hat{g}'(\Rightarrow')\hat{g}'(\xi_2'))\} \vDash_d \hat{g}'(\xi_2')$ if all assignments are possible. Let m' and m'' be such that $B'_{m'} = \{0', 1'\}$ and $B''_{m''} = \{0'', 1/2'', 1''\}$. Then $B_{\langle m', m''\rangle} = \{0', 0'', 1/2'', 1\}$. Consider an assignment α over $\langle m', m''\rangle$ such that $\alpha(\hat{g}'(\xi_1')) = 1$ and $\alpha(\hat{g}'(\xi_2')) = 1/2''$. Then

- $1 \leq [\![\hat{g}'(\xi_1')]\!]_\alpha^{\langle m',m''\rangle}$ *since* $[\![\hat{g}'(\xi_1')]\!]_\alpha^{\langle m',m''\rangle} = 1$;
- $1 \leq [\![(\hat{g}'(\xi_1')\hat{g}'(\Rightarrow')\hat{g}'(\xi_2'))]\!]_\alpha^{\langle m',m''\rangle}$ *since* $[\![(\hat{g}'(\xi_1')\hat{g}'(\Rightarrow')\hat{g}'(\xi_2'))]\!]_\alpha^{\langle m',m''\rangle} = 1$;
- *but not* $1 \leq [\![\hat{g}'(\xi_2')]\!]_\alpha^{\langle m',m''\rangle}$ *since* $[\![\hat{g}'(\xi_2')]\!]_\alpha^{\langle m',m''\rangle} = 1/2''$. △

The following example illustrates several possible combinations of propositional logic and Gödel logic through different bridges. In particular we introduce a specific bridge for sharing negation. The motivation for the sharing comes from the fact that the values of $(\neg\varphi)$ in 3-valued Gödel logic is always either $\perp$ or $\top$. That is, $1/2$ behaves as $\top$, and so negation has a classical flavor.

EXAMPLE 2.28. Modulated fibring of propositional and Gödel logics. *Let $\mathcal{I}'$ and $\mathcal{I}''$ be the interpretation systems for* 3*-valued Gödel logic and propositional logic (see Examples* 2.11 *and* 2.9*, respectively). For propositional logic only* 2*-valued algebras are included. Consider the fibring of propositional and Gödel logics modulated by three different bridges $\beta = \langle f' : \check{\mathcal{I}} \to \mathcal{I}', f'' : \check{\mathcal{I}} \to \mathcal{I}''\rangle$ as follows*:

Bridge 1:

- $\check{\mathcal{I}}$ *is such that*
  - $\check{M} = \{\check{m}\}$;
  - $\check{A}(\check{m}) = \langle\{\check{\top}\}, \{\langle\check{\top}, \check{\top}\rangle\}, \check{\nu}\rangle$;
- f' *and* f'' *are such that*
  - $\underline{f}'(m') = \underline{f}''(m'') = \check{m}$;
  - $\dot{f}'_{m'}(\check{\top}) = \top'_{m'}$ *and* $\dot{f}''_{m''}(\check{\top}) = \top''_{m''}$;
  - $\ddot{f}'_{m'}(b') = \check{\top}$ *and* $\ddot{f}''_{m''}(b'') = \check{\top}$ *for every* $b' \in B'_{m'}$, $b'' \in B''_{m''}$;

Bridge 2:

- $\check{\mathcal{I}}$ *is such that*
  - $\check{M} = \{\check{m}\}$;
  - $\check{A}(\check{m}) = \langle\{\check{\perp}, \check{\top}\}, \{\langle\check{\perp}, \check{\perp}\rangle, \langle\check{\perp}, \check{\top}\rangle, \langle\check{\top}, \check{\top}\rangle\}, \check{\nu}\rangle$;
- f' *and* f'' *are such that*
  - $\underline{f}'(m') = \underline{f}''(m'') = \check{m}$;
  - $\dot{f}'_{m'}(\check{\perp}_{\check{m}}) = \perp'_{m'}$, $\dot{f}'_{m'}(\check{\top}_{\check{m}}) = \top'_{m'}$;
  - $\dot{f}''_{m''}(\check{\perp}_{\check{m}}) = \perp''_{m''}$ *and* $\dot{f}''_{m''}(\check{\top}_{\check{m}}) = \top''_{m''}$;
  - $\ddot{f}'_{m'}(\perp'_{m'}) = \check{\perp}_{\check{m}}$ *and* $\ddot{f}'_{m'}(b') = \check{\top}_{\check{m}}$ *for every* $b' \neq \perp'_{m'}$;
  - $\ddot{f}''_{m''}(\perp''_{m''}) = \check{\perp}_{\check{m}}$ *and* $\ddot{f}''_{m''}(b'') = \check{\top}_{\check{m}}$ *for every* $b'' \neq \perp''_{m''}$;

Bridge 3:

- $\check{\mathcal{I}}$ *is such that*
  - $\check{M} = A'(M')|_{\check{C}} \cup A''(M'')|_{\check{C}}$;
  - $\check{A}$ *is the identity map*;
- f' *and* f'' *are such that*
  - $\underline{f}'(m') = A'(m')|_{\check{C}}$ *and* $\underline{f}''(m'') = A''(m'')|_{\check{C}}$;
  - $\dot{f}'_{m'} = id_{B'_{m'}}$ *and* $\dot{f}''_{m''} = id_{B''_{m''}}$;
  - $\ddot{f}'_{m'} = id_{\check{B}_{\underline{f}'(m')}}$ *and* $\ddot{f}''_{m''} = id_{\check{B}_{\underline{f}''(m'')}}$.

Bridges 1, 2 *and* 3 *can be used to modulate the fibring when* $\check{C}_0 = \{\check{t}\}$ *and* $\check{C}_k = \emptyset$, $\check{\Xi} = \emptyset$ *and* $\check{S} = \emptyset$. *Then* $\check{\nu}$ *is a family of empty maps except for* $\check{\nu}_0$ *and* $\hat{f}'$ *and* $\hat{f}''$ *are* *also empty maps except for* $k = 0$. *Bridges* 2 *and* 3 *can be used to modulate the fibring when* $\breve{C}_0 = \{\breve{\boldsymbol{f}}, \breve{\boldsymbol{t}}\}$, $\breve{C}_1 = \{\breve{\neg}\}$, $\breve{C}_k = \emptyset$ *for every* $k \geq 2$, $\breve{\Xi} = \emptyset$, $\breve{S} = \emptyset$, $\breve{v}(\breve{\neg})(\breve{\bot}) = \breve{\top}$, $\breve{v}(\breve{\neg})(\breve{\top}) = \breve{\bot}$ *and* $\hat{f}'$ *and* $\hat{f}''$ *are such that* $\hat{f}'(\breve{\neg}) = \neg'$ *and* $\hat{f}''(\breve{\neg}) = \neg''$. *Bridge* 3 *can be used to modulate the fibring when* $\breve{C} = C' = C''$, $\breve{\Xi} = \emptyset$, $\breve{S} = \emptyset$ *and* $\hat{f}'$ *and* $\hat{f}''$ *are such that* $\hat{f}'(\breve{\neg}) = \neg'$, $\hat{f}'(\breve{\wedge}) = \wedge'$, $\hat{f}''(\breve{\neg}) = \neg''$ *and* $\hat{f}''(\breve{\wedge}) = \wedge''$ (*corresponding to the collapse of Gödel logics into propositional logics since in the fibring we will only have Boolean algebras*). $\triangle$

We now turn our attention to the comparison at the semantic level between modulated fibring and the fibring as presented in [18] showing that the latter is a particular case of the former.

REMARK 2.29. Fibring. *Consider the subcategory fInt of Int whose objects are tuples* $\langle \Sigma, M, A \rangle$ *such that* $S = \emptyset$ *and the morphisms* $h : \langle \Sigma, M, A \rangle \to \langle \Sigma', M', A' \rangle$ *are such that* $\Xi' = \Xi$, $\dot{h}_{m'} = id_{m'}$ *and* $\ddot{h}_{m'} = id_{\underline{h}(m')}$ *for every* $m' \in M'$. *The objects and the morphisms of the subcategory fInt are the interpretation systems and the morphisms in the fibring as presented in* [18]. *The category fInt has pushouts that correspond to* (*unconstrained and constrained*) *fibring as presented in* [18] *by choosing the following bridge*:

- $\breve{C}$ *with the shared constructors if any*;
- $\breve{M} = A'(M')|_{\breve{C}} \cup A''(M'')|_{\breve{C}}$;
- $\breve{A}$ *is the identity map*;
- $\underline{f}'(m') = A'(m')|_{\breve{C}}$ *and* $\underline{f}''(m'') = A''(m'')|_{\breve{C}}$;
- $\dot{f}'_{m'} = id_{B'_{m'}}$, $\dot{f}''_{m''} = id_{B''_{m''}}$, $\ddot{f}'_{m'} = id_{\breve{B}_{\underline{f}'(m')}}$ *and* $\ddot{f}''_{m''} = id_{\breve{B}_{\underline{f}''(m'')}}$.

Thus, the class of models M *is composed by the pairs* $\langle m', m'' \rangle$ *that have the same underlying algebra. For instance when considering the fibring of propositional and intuitionistic logics the models to be considered in the fibring are those whose underlying algebra is Boolean. Therefore intuitionistic logic collapses into propositional logic even if no constructors are shared.* $\triangle$

§3. Hilbert systems.

§3. Hilbert systems. In this section we analyze the deductive component of modulated fibring. The basic deductive notion is Hilbert system. Hilbert systems are special pre-Hilbert systems. As in previous papers on fibring, we distinguish between proof and derivation rules. We go on giving the notion of morphism between Hilbert systems. Then, again modulated fibring appears as a pushout in the category of Hilbert systems.

3.1. Basic notions. We define the notion of inference rule in general. Proof rules and derivation rules are inference rules. The notion of substitution is a delicate one since we will work often with safe substitutions. Safe substitutions are the deductive counterpart of safe assignments. This means that instantiation of inference rules is sometimes restricted.

DEFINITION 3.1. *A* Σ*-inference rule is a pair* $\langle \Gamma, \varphi \rangle$ *where* $\Gamma \in \wp_{fin} L(\Sigma)$ *and* $\varphi \in L(\Sigma)$.

Given an inference rule $r = \langle \Gamma, \varphi \rangle$, the elements of Γ are the *premises* (*Prem*(r)) of r, and φ is the *conclusion* (*Conc*(r)) of r.

DEFINITION 3.2. *A pre-Hilbert system is a tuple* $\langle\Sigma, R_p, R_d\rangle$ *where* Σ *is a signature,* R_p *(proof rules) and* R_d *(derivation rules) are sets of* Σ*-inference rules such that* $R_d \subseteq R_p$ *and* R_d *does not include rules with no premises.*

We use the following notation: given $s : \check{\Sigma} \to \Sigma$, $L(\Sigma, s)$ is the set of formulae in $L(\Sigma)$ whose main constructor is from $s(\check{C})$ and $L(C, \&, s)$ is the subset of $L(\Sigma, s)$ composed by ground formulae whose main constructor is from $s(\check{C})$.

DEFINITION 3.3. *A* Σ*-substitution is a map* $\sigma : \Xi \to L(\Sigma)$. *A substitution* σ *is safe for a set of formulae* $\Gamma \subseteq L(\Sigma)$ *iff* $\sigma(s(\check{\xi})) \in L(\Sigma, s)$ *for every* $s : \check{\Sigma} \to \Sigma$ *in* S *and* $s(\check{\xi}) \in \Gamma$.

Therefore we should be careful whenever we have in a set of formulae images by safe-relevant signature morphisms of meta-variables that come from another signature. They have to be substituted by formulae whose main constructor belongs to that signature. Now we turn our attention to deductions. Since we distinguish between proof and derivation rules we have as deductions both proofs and derivations.

DEFINITION 3.4. *A formula* φ *is provable from a set of formulae* Γ *in a pre-Hilbert system, indicated by* $\Gamma \vdash_p \varphi$, *iff there is a sequence* $\varphi_1, \dots, \varphi_n$ *of formulae such that (i)* $\varphi_n = \varphi$ *and (ii) for each* $i = 1, \dots, n$ *either* $\varphi_i \in \Gamma$, *or there exist a rule* r *of* R_p *and a safe substitution* σ *for* $Prem(r) \cup \{Conc(r)\}$ *such that* $Conc(r)\sigma = \varphi_i$ *and* $Prem(r)\sigma \subseteq \{\varphi_1, \dots, \varphi_{i-1}\}$.

DEFINITION 3.5. *A formula* φ *is derivable from a set* Γ *of formulae in a pre-Hilbert system, in symbols* $\Gamma \vdash_d \varphi$, *iff there is a sequence* $\varphi_1, \dots, \varphi_m$ *of formulae such that (i)* $\varphi_m = \varphi$ *and (ii) for each* $i = 1, \dots, m$ *either* $\varphi_i \in \Gamma$, *or* φ_i *is provable from the empty set, or there exist a rule* r *of* R_d *and a safe substitution* σ *for* $Prem(r) \cup \{Conc(r)\}$ *such that* $Conc(r)\sigma = \varphi_i$ *and* $Prem(r)\sigma \subseteq \{\varphi_1, \dots, \varphi_{i-1}\}$.

A Hilbert system is a pre-Hilbert system with a conjunction like operator in what concerns deduction. This operator has a technical role in Section 5.

DEFINITION 3.6. *A Hilbert system is a pre-Hilbert system where* 1. *for* $i = 1, 2$, $\{(\varphi_1 \& \varphi_2)\} \vdash_d \varphi_i$ *(& elimination);* 2. $\{\varphi_1, \varphi_2\} \vdash_d (\varphi_1 \& \varphi_2)$ *(& introduction) for every formulae* φ_1 *and* φ_2.

We denote by $\gamma_1 \cong_\Gamma \gamma_2$ the fact that $\Gamma, \gamma_1 \vdash_d \gamma_2$ and $\Gamma, \gamma_2 \vdash_d \gamma_1$. When $\Gamma = \emptyset$ then we will omit the reference to the set. In the following examples the signature is as follows: $\Sigma = \langle C, \&, \Xi, S\rangle$ where $\boldsymbol{t}, \boldsymbol{f} \in C_0$, $C_1 = \{\neg\}$, $C_2 = \{\wedge, \vee, \Rightarrow\}$, $C_k = \emptyset$ for all $k \geq 3$ and & is $\wedge$; $\Xi = \{\xi_i : i \in \mathbb{N}\}$. Thus the Hilbert systems only differ in the inference rules.

EXAMPLE 3.7. (3-valued) Łukasiewicz Hilbert system. *We adapt from the axiomatic system in* [13].

- $R_d = \{\langle\{\xi_1, (\xi_1 \Rightarrow \xi_2)\}, \xi_2\rangle\}$;
- R_p *includes* R_d *plus:*
  - $\langle\emptyset, (\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1))\rangle$;
  - $\langle\emptyset, ((\xi_1 \Rightarrow \xi_2) \Rightarrow ((\xi_2 \Rightarrow \xi_3) \Rightarrow (\xi_1 \Rightarrow \xi_3)))\rangle$;
  - $\langle\emptyset, ((\neg\,\xi_1) \Rightarrow (\neg\,\xi_2)) \Rightarrow (\xi_2 \Rightarrow \xi_1))\rangle$;
  - $\langle\emptyset, (((\xi_1 \Rightarrow (\neg\,\xi_1)) \Rightarrow \xi_1) \Rightarrow \xi_1)\rangle$. △

EXAMPLE 3.8. (3-valued) Gödel Hilbert system. *We adapt from the axiomatic system in* [2].

- $R_d = \{\langle\{\xi_1, (\xi_1 \Rightarrow \xi_2)\}, \xi_2\rangle\}$;
- R_p *includes* R_d *plus*:
  - *the axiom schemata of propositional intuitionistic logic*;
  - *the axiom schema* $(((\neg\,\xi_1) \Rightarrow \xi_2) \Rightarrow (((\xi_2 \Rightarrow \xi_1) \Rightarrow \xi_2) \Rightarrow \xi_2))$. △

We now introduce the notion of morphism as a pair. The first component of the pair is a signature morphism. Of course we expect such a component to preserve inferences. The second component is specific and is basically needed in order to make things easier in Section 5.

DEFINITION 3.9. *A Hilbert system morphism from* $\mathscr{H}$ *to* $\mathscr{H}'$ *is a pair* $\langle\hat{h}, \check{h}\rangle$ *such that* $\hat{h} : \Sigma \to \Sigma'$ *is a signature morphism and* $\check{h} : L(C', \&') \to L(C, \&)$ *is a monotonic map*[6] *such that*:

1. $\hat{h}(r) \in R'_p$ *for every* $r \in R_p$;
2. $\hat{h}(r) \in R'_d$ *for every* $r \in R_d$;
3. $\check{h}$ *is left adjoint of* $\hat{h}$;[7]
4. $\hat{h}(c(\check{h}(\vec{\varphi}'))) \vdash'_d \hat{h}(c)(\vec{\varphi}')$.

A signature morphism $\hat{h} : \Sigma \to \Sigma'$ satisfying Clauses 1. and 2. is called a pre-Hilbert system morphism. In the fibring as presented in [18], the Hilbert system morphism is a pre-Hilbert system morphism in the present context where $\Xi = \Xi'$ and there is no need for the operator &. Moreover, in [18], there were no restrictions on substitutions either in proofs or derivations.

The more complex notion of Hilbert system morphism is the adequate one for fulfilling the requirements that are necessary for preserving congruence by fibring in Section 5. The contravariant map $\check{h}$ can be seen as a map relating truth values (formulae) in the Lindendaum-Tarski algebras that will be discussed in Section 5. Observe that in [18], preservation of congruence was obtained by sharing implication and equivalence. This cannot be the solution because sharing of implication and equivalence leads in most cases to collapse.

Observe that $\check{h}(\varphi'_1 \&' \varphi'_2) \cong \check{h}(\varphi'_1) \& \check{h}(\varphi'_2)$, for every morphism h. Observe also that $\check{h}(\Gamma') \vdash_p \check{h}(\delta')$ whenever $\Gamma' \vdash'_p \delta'$ and $\check{h}(\Gamma') \vdash_d \check{h}(\delta')$ whenever $\Gamma' \vdash'_d \delta'$ for every Γ' and δ' in $L(C', \&')$.

PROP/DEFINITION 3.10. *Hilbert systems and their morphisms constitute the category Hil.*

We now show that Hilbert system morphisms do preserve proofs and derivations.

PROPOSITION 3.11. *Let* $h : \mathscr{H} \to \mathscr{H}'$ *be a Hilbert system morphism such that* $\hat{h}$ *is injective for* Ξ *and* $\hat{h}(C) \subseteq \hat{s}'(C_{\hat{s}'})$ *whenever* $\hat{h}(\Xi) \cap \hat{s}'(\Xi_{\hat{s}'}) \neq \emptyset$ *for every* $\hat{s}' \in S'$. *Thus* $\hat{h}(\Gamma) \vdash'_p \hat{h}(\varphi)$ *whenever* $\Gamma \vdash_p \varphi$. *Similarly for derivations.*

PROOF. Observe that: (i) if σ is a safe substitution for Γ then the substitution $\hat{h}(\sigma)$, defined as $\hat{h}(\sigma)(\hat{h}(\xi)) = \hat{h}(\sigma(\xi))$ and $\hat{h}(\sigma)(\xi') = \xi'$ whenever $\xi' \in \Xi' \setminus \hat{h}(\Xi)$,

[6] That is, $\check{h}(\varphi') \vdash_p \check{h}(\psi')$ whenever $\varphi' \vdash'_p \psi'$ and $\check{h}(\varphi') \vdash_d \check{h}(\psi')$ whenever $\varphi' \vdash'_d \psi'$.

[7] Therefore $\varphi' \vdash'_d \hat{h}(\check{h}(\varphi'))$ and $\check{h}(\hat{h}(\varphi)) \vdash_d \varphi$.

is safe for $\hat{h}(\Gamma)$, and (ii) $\hat{h}(\phi\sigma) = \hat{h}(\phi)\hat{h}(\sigma)$ for any formula ϕ and substitution σ. Then the proof follows by induction on the length of a proof of φ from Γ. Base: if φ is in Γ then $\hat{h}(\varphi)$ is in $\hat{h}(\Gamma)$. Step: there is a rule $r = \langle\{\phi_1, \dots, \phi_k\}, \phi\rangle$ in R_p and substitution σ safe for $\{\phi_1, \dots, \phi_k, \phi\}$ with $\varphi_n = \varphi = \phi\sigma$ and $\{\phi_1, \dots, \phi_k\}\sigma \subseteq \{\varphi_i \mid i < n\}$. Then, by the induction hypothesis $\hat{h}(\Gamma) \vdash_p \hat{h}(\phi_i\sigma)$ for $i = 1, \dots, k$ and so $\hat{h}(\Gamma) \vdash_p \hat{h}(\phi_i)\hat{h}(\sigma)$ for $i = 1, \dots, k$. Since $\hat{h}(r) \in R'_p$ and $\hat{h}(\sigma)$ is safe for $\hat{h}(\{\phi_1, \dots, \phi_k, \phi\})$ then $\hat{h}(\phi_1)\hat{h}(\sigma), \dots, \hat{h}(\phi_k)\hat{h}(\sigma) \vdash_p \hat{h}(\phi)\hat{h}(\sigma)$. Then by transitivity we get $\hat{h}(\Gamma) \vdash'_p \hat{h}(\varphi)$. ⊣

3.2. Modulated fibring of Hilbert systems. As previously done for interpretation systems, we must start by defining a bridge for Hilbert systems. The bridge allows a mild relationship between the formulae in the Hilbert systems that we want to combine as well as between their consequence relations. Again modulated fibring appears as a pushout in the category of Hilbert systems.

DEFINITION 3.12. *A bridge between Hilbert systems $\mathscr{H}'$ and $\mathscr{H}''$ is a diagram $\beta = \langle f' : \check{\mathscr{H}} \to \mathscr{H}', f'' : \check{\mathscr{H}} \to \mathscr{H}''\rangle$ in Hil such that $\hat{f}'$, $\hat{f}''$ are injective and $\check{f}'$ and $\check{f}''$ are surjective.*

PROP/DEFINITION 3.13. *The modulated fibring of Hilbert systems $\mathscr{H}'$ and $\mathscr{H}''$ by a bridge β is a pushout of β in Hil.*

PROOF. Consider $\langle g' : \mathscr{H}' \to \mathscr{H}, g'' : \mathscr{H}'' \to \mathscr{H}\rangle$ defined as follows:

- $\langle \hat{g}' : \Sigma' \to \Sigma, \hat{g}'' : \Sigma'' \to \Sigma\rangle$ is a pushout in *Sig* of $Sg(\beta)$;
- define $\check{g}'$ and $\check{g}''$ inductively as follows:
  - $\check{g}'(\hat{g}'(c')) = c'$ and $\check{g}'(\hat{g}''(c'')) = \hat{f}'(\check{f}''(c''))$;
  - $\check{g}'(\hat{g}'(c')(\vec{\varphi})) = c'(\check{g}'(\vec{\varphi}))$ and $\check{g}'(\hat{g}''(c'')(\vec{\varphi})) = \hat{f}'(\check{f}''(c''(\check{g}''(\vec{\varphi}))))$;[8]
  - $\check{g}'(\varphi_1 \& \varphi_2) = \check{g}'(\varphi_1) \&' \check{g}'(\varphi_2)$;
- R_d includes $\hat{g}'(R'_d) \cup \hat{g}''(R''_d)$, & elimination and introduction plus the following rules, for any $\gamma \in L(C, \&)$ and $\vec{\gamma}$ a sequence over $L(C, \&)$:
  - $\langle\{\gamma\}, \hat{g}'(\check{g}'(\gamma))\rangle$;
  - $\langle\{\hat{g}'(c')(\hat{g}'(\check{g}'(\vec{\gamma})))\}, \hat{g}'(c')(\vec{\gamma})\rangle$;
  - similar rules for $\hat{g}''$ and $\check{g}''$;
- $R_p = \hat{g}'(R'_p) \cup \hat{g}''(R''_p) \cup R_d$.

It is straightforward to show that g' and g'' are indeed Hilbert system morphisms. Now we check that $\langle \mathscr{H}, g', g''\rangle$ is a pushout in *Hil* of f' and f''.

1. $\check{f}''(\check{g}''(\gamma)) \cong_d \check{f}'(\check{g}'(\gamma))$: by induction on the structure of γ.

Base: if γ is $\hat{g}'(p')$ then $\check{f}''(\check{g}''(\gamma)) = \check{f}''(\hat{f}''(\check{f}'(p'))) \cong_d \check{f}'(p') = \check{f}'(\check{g}'(\gamma))$.

Step: we only consider the case where γ is $\hat{g}'_k(c)(\gamma_1, \dots, \gamma_k)$ then $\check{f}''(\check{g}''(\gamma)) = \check{f}''(\hat{f}''(\check{f}'(\check{g}'(\gamma)))) \cong_d \check{f}'(\check{g}'(\gamma))$.

2. Universal property. Let $h' : \mathscr{H}' \to \mathscr{H}'''$ and $h'' : \mathscr{H}'' \to \mathscr{H}'''$ be Hilbert system morphisms such that $h' \circ f' = h'' \circ f''$.

Existence. Let $h : \mathscr{H} \to \mathscr{H}'''$ be as follows: $\hat{h}$ is the unique morphism in *Sig* such that $\hat{h} \circ \hat{g}' = \hat{h}'$ and $\hat{h} \circ \hat{g}'' = \hat{h}''$ and $\check{h}$ is such that $\check{h}(\varphi''') = \hat{g}'(\check{h}'(\varphi'''))\&\hat{g}''(\check{h}''(\varphi'''))$. It is straightforward to show that h is a Hilbert system morphism. We show that $\check{g}'(\check{h}(\varphi''')) \cong_d \check{h}'(\varphi''')$: $\check{g}'(\check{h}(\varphi''')) \vdash'_d \check{h}'(\varphi''')$ using &'

[8]Observe that $\check{g}'(\hat{g}'(\varphi'))$ is φ'.

elimination; $\check{h}'(\varphi''') \vdash_d' \hat{f}'(\check{f}'(\check{h}'(\varphi''')))$, thus $\hat{f}'(\check{f}'(\check{h}'(\varphi'''))) \vdash_d' \hat{f}'(\check{f}''(\check{h}''(\varphi''')))$ and so we can conclude $\check{h}'(\varphi''') \vdash_d' \check{g}'(\check{h}(\varphi'''))$.

Uniqueness. Let $k : \mathscr{H} \to \mathscr{H}'''$ be such that $k \circ g' = h'$ and $k \circ g'' = h''$. Then $k = h$ and in particular $\check{k}(\varphi''') \cong_d \check{h}(\varphi''')$. We start by proving that $\check{k}(\varphi''') = \hat{g}'(\check{g}'(\check{k}(\varphi''')))\&\hat{g}''(\check{g}''(\check{k}(\varphi''')))$. Assume that $\check{k}(\varphi''') = \hat{g}'(\varphi')$. Note that $\hat{g}'(\varphi') \vdash_d \hat{g}'(\hat{f}'(\check{f}'(\varphi')))$ and $\hat{g}'(\hat{f}'(\check{f}'(\varphi'))) \cong_d \hat{g}''(\hat{f}''(\check{f}'(\varphi'))) \cong_d \hat{g}''(\check{g}''(\hat{g}'(\varphi')))$. So, $\check{k}(\varphi''') \cong_d \hat{g}'(\check{g}'(\check{k}(\varphi''')))\&\hat{g}''(\check{g}''(\check{k}(\varphi''')))$. The other cases follow in a straightforward manner. ⊣

Examples and the collapsing problem. We now give some examples of modulated fibring of many-valued logics illustrating non-collapsing situations. We start by a general result which states how to choose a bridge without collapsing when there is no sharing of constructors. As we said in Section 2 this is the case in most situations because otherwise collapsing is inevitable.

PROPOSITION 3.14. *Let $\mathscr{H}'$, $\mathscr{H}''$ be Hilbert systems such that $\boldsymbol{t}' \in C_0'$, $\boldsymbol{t}'' \in C_0''$ and C' and C'' are in bijection and β a bridge such that $\check{C}_0 = \{\check{\boldsymbol{t}}\}$, $\check{C}_k = \emptyset$ for all $k \neq 0$, $\check{\Xi} = \emptyset$, $\check{S} = \emptyset$, $\check{R}_p = \check{R}_d$ include $\{\langle\emptyset, \check{\boldsymbol{t}}\rangle\}$ and the rules for $\check{\&}$ elimination and introduction, $id_{\Sigma'} \in S'$, $id_{\Sigma''} \in S''$, $\hat{f}'(\check{\boldsymbol{t}}) = \boldsymbol{t}'$, $\hat{f}''(\check{\boldsymbol{t}}) = \boldsymbol{t}''$, $\check{f}'(\varphi') = \check{\boldsymbol{t}}$ for every φ' and $\check{f}''(\varphi'') = \check{\boldsymbol{t}}$ for every φ''. Then the modulated fibring of $\mathscr{H}'$ and $\mathscr{H}''$ by β does not collapse.*

Proposition 3.14 shows that for all cases of unconstrained modulated fibring (that is only the verum is shared) it is possible to avoid the collapsing problem. Observe that in C_0' and C_0'' we can have propositional symbols. Since $id_{\Sigma'} \in S'$, $id_{\Sigma''} \in S''$ we guarantee that all proofs and derivations of the component logics will be proofs and derivations in the modulated fibring. Observe also that the requirement $id_{\Sigma'} \in S'$, $id_{\Sigma''} \in S''$ does not change the consequence relations of $\mathscr{H}'$ and $\mathscr{H}''$. This requirement only prepares the Hilbert systems for the combination. We can now instantiate Proposition 3.14 for several cases.

EXAMPLE 3.15. Modulated fibring of propositional and Łukasiewicz logics. *By choosing the bridge as in Proposition* 3.14 *we do not get the collapse between propositional and* 3*-valued Łukasiewicz logics.* △

EXAMPLE 3.16. Modulated fibring of propositional and Gödel logics. *By choosing the bridge as in Proposition* 3.14 *we do not get the collapse between propositional and* 3*-valued Gödel logics.* △

EXAMPLE 3.17. Modulated fibring of propositional and intuitionistic logics. *By choosing the bridge as in Proposition* 3.14 *we avoid the collapsing between propositional and intuitionistic logics.* △

We now show that the example of collapse of propositional and intuitionistic logics given by Gabbay in [10] can be avoided in the present context. The example also allows a better understanding of safe substitutions.

EXAMPLE 3.18. *Gabbay's example of collapse. Consider the modulated fibring of the Hilbert systems $\mathscr{H}'$ and $\mathscr{H}''$ for intuitionistic logic and propositional logic, respectively, with the bridge as in Proposition* 3.14. *Then* $(\hat{g}'(\xi_1')\hat{g}'(\Rightarrow')\hat{g}'(\xi_2')) \vdash_p$

$(\hat{g}'(\xi_1')\hat{g}''(\Rightarrow'')\hat{g}'(\xi_2'))$ *does not hold. In particular, the step*

$$((\hat{g}'(\xi_1')\hat{g}'(\Rightarrow')\hat{g}'(\xi_2'))\hat{g}''(\wedge'')\hat{g}'(\xi_1')) \vdash_p \hat{g}'(\xi_2')$$

is not possible because the underlying substitution for conjunction $\hat{g}''(\wedge'')$ *elimination is not safe since* $\hat{g}'(\xi_2')$ *does not start with a constructor from* $\mathscr{H}''$. △

We now analyze an example of modulated fibring of Hilbert systems sharing the negation constructor.

Example 3.19. Modulated fibring of propositional and Gödel logics sharing negation. *Let* $\mathscr{H}'$ *be the Hilbert system for* 3*-valued Gödel logic* (*see Example* 3.8) *and* $\mathscr{H}''$ *be the Hilbert system for propositional logic. Consider a bridge* β *such that*:

- $\check{C}_0 = \{\check{\boldsymbol{f}}, \check{\boldsymbol{t}}\}$, $\check{C}_1 = \{\check{\neg}\}$, $\check{C}_k = \emptyset$ *for* $k \geq 2$, $\check{\Xi} = \emptyset$, $\check{S} = \emptyset$;
- $S' = \{id_{\Sigma'}\}$ *and* $S'' = \{id_{\Sigma''}\}$;
- $\hat{f}'(\check{\boldsymbol{f}}) = \boldsymbol{f}'$, $\hat{f}'(\check{\boldsymbol{t}}) = \boldsymbol{t}'$ *and* $\hat{f}'(\check{\neg}) = \neg'$;
- $\check{f}'(\varphi') = \begin{cases} \check{\varphi} & \text{p.t. } \varphi' \text{ is } \hat{f}'(\check{\varphi}) \\ \check{\boldsymbol{f}} & \text{p.t. } \varphi' \vdash_d \boldsymbol{f}' \\ \check{\boldsymbol{t}} & \text{otherwise} \end{cases}$;
- $\hat{f}''$ *and* $\check{f}''$ *defined in a similar way*;
- $\check{R}_p$ *and* $\check{R}_d$ *are the translations of the ground instances of* R_p', R_p'', R_d', R_d'' *by* $\check{f}'$ *and* $\check{f}''$ *plus the rules* $\check{\&}$ *elimination and introduction.*

The pair $\langle \hat{f}', \check{f}' \rangle$ *is a morphism*: 1. $\check{f}'$ *is monotonic. Let* $\varphi' \vdash_d' \psi'$. *It can be proved that there are derivations for* $\varphi' \vdash_d' \psi'$ *involving only ground formulas. The proof follows by induction on the length of a ground derivation* $\varphi_1', \dots, \varphi_n'$ *for* $\varphi' \vdash_d' \psi'$. *Base. Straightforward. Step. There is a rule* $\langle \{\phi_1', \dots, \phi_k'\}, \phi' \rangle$ *and a ground substitution safe for* $\{\phi_1', \dots, \phi_k', \phi'\}$ *such that* $\{\phi_1', \dots, \phi_k'\}\sigma \subseteq \{\varphi_1', \dots, \varphi_{n-1}'\}$ *and* $\phi'\sigma = \varphi_n' = \psi'$. *So by induction hypothesis* $\check{f}'(\varphi')\check{\vdash}_d \check{f}'(\phi_i'\sigma)$ *for* $i = 1, \dots, k$. *Since* $\langle \{\check{f}'(\phi_1'\sigma), \dots, \check{f}'(\phi_k'\sigma)\}, \check{f}'(\phi'\sigma) \rangle \in \check{R}_d$ *then* $\check{f}'(\varphi')\check{\vdash}_d \check{f}'(\psi)$ *as desired.* 2. $\varphi' \vdash_d' \hat{f}'(\check{f}'(\varphi'))$. (*i*) φ' *is* $\hat{f}'(\check{\varphi})$. *Then* $\varphi' \vdash_d' \hat{f}'(\check{f}'(\varphi')) = \varphi'$. (*ii*) $\varphi' \vdash' \boldsymbol{f}'$. *Then* $\varphi' \vdash_d' \hat{f}'(\check{f}'(\varphi')) = \boldsymbol{f}'$. (*ii*) *Otherwise* $\varphi' \vdash' \hat{f}'(\check{f}'(\varphi')) = \boldsymbol{t}'$. 3. $\check{f}'(\hat{f}'(\check{\varphi}))\check{\vdash}_d \check{\varphi}$ *since* $\check{f}'(\hat{f}'(\check{\varphi})) = \check{\varphi}$. 4. $\hat{f}'(\check{\neg}(\check{f}'(\varphi'))) \vdash_d' (\neg' \varphi')$. (*i*) φ' *is* $\hat{f}'(\check{\varphi})$. *Therefore* $\hat{f}'(\check{\neg}(\check{f}'(\varphi'))) = (\neg' \varphi') \vdash_d' (\neg' \varphi')$. (*ii*) $\varphi' \vdash' \boldsymbol{f}'$. *Then* $\hat{f}'(\check{\neg}(\check{f}'(\varphi'))) \vdash_d' (\neg' \varphi') \cong_d' \boldsymbol{t}'$. (*iii*) *Otherwise* $\hat{f}'(\check{\neg}(\check{f}'(\varphi'))) \cong_d' \boldsymbol{f}'$ *and so* $\boldsymbol{f}' \vdash_d' (\neg' \varphi')$.

In the modulated fibring $C_k = \hat{g}'(C_k') \cup \hat{g}''(C_k'')$ *and* $\Xi = \hat{g}'(\Xi') \cup \hat{g}''(\Xi'')$, $R_p = \hat{g}'(R_p') \cup \hat{g}''(R_p'') \cup R_d$ *and* R_d *includes* $\hat{g}'(R_d') \cup \hat{g}''(R_d'')$, *the rules for* & *elimination and introduction and the rules for the modulated fibring. Note that we have two forms of detachment*: $\langle \{\hat{g}'(\xi_1'), (\hat{g}'(\xi_1')\hat{g}'(\Rightarrow')\hat{g}'(\xi_2'))\}, \hat{g}'(\xi_2') \rangle$, $\langle \{\hat{g}''(\xi_1''), (\hat{g}''(\xi_1'')\hat{g}''(\Rightarrow'')\hat{g}''(\xi_2''))\}, \hat{g}''(\xi_2'') \rangle$ *and that we do not have the inference* $\{\hat{g}'(\varphi'), (\hat{g}'(\varphi')\hat{g}''(\Rightarrow'')\hat{g}''(\varphi''))\} \vdash_p \hat{g}''(\varphi'')$ *for instance, because the substitution of* $\hat{g}''(\xi_1'')$ *by* $\hat{g}'(\varphi')$ *is not safe.* △

We proceed discussing how to recover unconstrained fibring as presented in [18] from modulated fibring with unconstrained fibring.

Remark 3.20. Unconstrained fibring. *Let* $\mathscr{H}'$ *and* $\mathscr{H}''$ *be Hilbert systems with* $S' = S'' = \emptyset$. *Consider the following bridge* β:

- $\check{C}_0 = \{\check{t}\}$, $\check{C}_k = \emptyset$ *for all* $k \neq 0$, $\check{\Xi} = \emptyset$, $\check{S} = \emptyset$;
- $\check{R}_p = \check{R}_d$ *include* $\{\langle \emptyset, \check{t} \rangle\}$ *plus* & *elimination and introduction*;
- $\hat{f}'(\check{t}) = t'$, $\hat{f}''(\check{t}) = t''$, $\check{f}'(\varphi') = \check{f}''(\varphi'') = \check{t}$ *for every* φ', φ'', *respectively.*

Then the modulated fibring of $\mathscr{H}'$ *and* $\mathscr{H}''$ *by a bridge* β *is a conservative extension of the unconstrained fibring (with no sharing of constructors except verum) as presented in* [18]. △

§4. Logic systems.

In this section we put together the semantic and the deductive components obtaining logic systems and morphisms between logic systems. Modulated fibring of logic systems is a pushout in the category of logic systems. We give new examples of modulated fibring and investigate preservation of soundness.

4.1. Basic notions.

DEFINITION 4.1. *A logic system is a tuple* $\langle \Sigma, M, A, R_p, R_d \rangle$ *such that* $\langle \Sigma, M, A \rangle$ *is an interpretation system and* $\langle \Sigma, R_p, R_d \rangle$ *is a Hilbert system.*

Among the properties of logic systems, we are interested in soundness and completeness. Since completeness is more complex we deal with it in Section 5. As expected we have the notions of soundness for proof and derivation.

DEFINITION 4.2. *A logic system is sound (with respect to proof and derivation) iff* $\Gamma \vDash_p \varphi$ *and* $\Gamma \vDash_d \varphi$ *whenever* $\Gamma \vdash_p \varphi$ *and* $\Gamma \vdash_d \varphi$, *respectively, for every* Γ *and* φ *in* $L(C, \&)$. *A logic system is complete with respect to proof and derivation iff* $\Gamma \vdash_p \varphi$ *and* $\Gamma \vdash_d \varphi$ *whenever* $\Gamma \vDash_p \varphi$ *and* $\Gamma \vDash_d \varphi$, *respectively, for every* Γ *and* φ *in* $L(C, \&)$.

DEFINITION 4.3. *A logic system morphism* $h : \mathscr{L} \to \mathscr{L}'$ *is a tuple* $\langle \hat{h}, \underline{h}, \dot{h}, \ddot{h}, \check{h} \rangle$ *such that* $\langle \hat{h}, \underline{h}, \dot{h}, \ddot{h} \rangle$ *is an interpretation system morphism from* $\mathscr{I}$ *to* $\mathscr{I}'$ *and* $\langle \hat{h}, \check{h} \rangle$ *is a Hilbert system morphism from* $\mathscr{H}$ *to* $\mathscr{H}'$ *such that* $\ddot{h}_{m'}([\![\gamma']\!]^{m'}) \cong_{\underline{h}(m')} [\![\check{h}(\gamma')]\!]^{\underline{h}(m')}$ *for every* $\gamma' \in L(C', \&')$ *and* $m' \in M'$.

A logic system morphism is an interpretation system morphism and a Hilbert system morphism plus a condition relating both. This additional requirement will be referred to as *soundness condition*.

PROP/DEFINITION 4.4. *Logic systems and their morphisms constitute the category Log.*

Let $I : Log \to Int$ and $H : Log \to Hil$ be the functors that associate to each logic system the underlying interpretation system and Hilbert system, respectively.

DEFINITION 4.5. *A bridge between logic systems* $\mathscr{L}'$ *and* $\mathscr{L}''$ *is a diagram* $\beta = \langle f' : \check{\mathscr{L}} \to \mathscr{L}', f'' : \check{\mathscr{L}} \to \mathscr{L}'' \rangle$ *in Log such that* $I(\beta) = \langle I(f') : I(\check{\mathscr{L}}) \to I(\mathscr{L}'), I(f'') : I(\check{\mathscr{L}}) \to I(\mathscr{L}'') \rangle$ *is a bridge in Int and* $H(\beta) = \langle H(f') : H(\check{\mathscr{L}}) \to H(\mathscr{L}'), H(f'') : H(\check{\mathscr{L}}) \to H(\mathscr{L}'') \rangle$ *is a bridge in Hil.*

As expected a bridge between logic systems constitutes a bridge between the underlying interpretation systems and a bridge between the underlying Hilbert systems. Again modulated fibring between logic systems appears as a pushout in the category of logic systems.

PROP/DEFINITION 4.6. *The modulated fibring of logic systems* $\mathscr{L}'$ *and* $\mathscr{L}''$ *by a bridge* β *is a pushout of* β *in Log.*

PROOF. The pushout $\langle g' : \mathscr{L}' \to \mathscr{L}, g'' : \mathscr{L}'' \to \mathscr{L}\rangle$ is such that:

- $\langle I(g') : I(\mathscr{L}') \to I(\mathscr{L}), I(g'') : I(\mathscr{L}'') \to I(\mathscr{L})\rangle$ is a modulated fibring in *Int* of $I(\beta)$;
- $\langle H(g') : H(\mathscr{L}') \to H(\mathscr{L}), H(g'') : H(\mathscr{L}'') \to H(\mathscr{L})\rangle$ is a modulated fibring in *Hil* of $H(\beta)$.

We show $\ddot{g}''_m(\llbracket \gamma \rrbracket^m) \cong_{\underline{g}''(m)} \llbracket \check{g}''(\gamma)\rrbracket^{\underline{g}'(m)}$ by induction on the structure of γ. We prove the base: (a) γ is $\hat{g}'(c')$ with $c' \in C'_0$:

$\ddot{g}''_m(\llbracket \hat{g}'(c')\rrbracket^m) \cong \ddot{g}''_m(\dot{g}'_m(\llbracket c'\rrbracket^{\underline{g}'(m)})) \cong \dot{f}''_{\underline{f}'(\underline{g}'(m))}(\ddot{f}'_{\underline{g}'(m)}(\llbracket c'\rrbracket^{\underline{g}'(m)})) \cong$
$\dot{f}''_{\underline{f}'(\underline{g}'(m))}(\llbracket \check{f}'(c')\rrbracket^{\underline{f}'(\underline{g}'(m))}) \cong \dot{f}''_{\underline{f}''(\underline{g}''(m))}(\llbracket \check{f}'(c')\rrbracket^{\underline{f}''(\underline{g}''(m))}) \cong \llbracket \hat{f}''(\check{f}'(c'))\rrbracket^{\underline{g}''(m)}$
$\cong \llbracket \check{g}''(\gamma)\rrbracket^{\underline{g}''(m)}$. ⊣

EXAMPLE 4.7. *Unconstrained modulated fibring. The diagram* $\langle f' : \check{\mathscr{L}} \to \mathscr{L}', f'' : \check{\mathscr{L}} \to \mathscr{L}''\rangle$ *such that* $\langle\langle \hat{f}', \underline{f}', \dot{f}', \ddot{f}'\rangle : \check{\mathscr{I}} \to \mathscr{I}', \langle \hat{f}'', \underline{f}'', \dot{f}'', \ddot{f}''\rangle : \check{\mathscr{I}} \to \mathscr{I}''\rangle$ *is the bridge in Proposition* 2.25 *and* $\langle\langle \hat{f}', \check{f}'\rangle : \check{\mathscr{H}} \to \mathscr{H}', \langle \hat{f}'', \check{f}''\rangle : \check{\mathscr{H}} \to \mathscr{H}''\rangle$ *is the bridge in Proposition* 3.14, *constitutes a bridge that defines the unconstrained modulated fibring of* $\mathscr{L}'$ *and* $\mathscr{L}''$. *This happens because the soundness condition is verified: let* m' *be in* M' *and* $\varphi' \in L(C', \&')$, *then* $\ddot{f}'_{m'}(\llbracket\varphi'\rrbracket^{m'}) = \check{\top} = \llbracket \check{f}'(\varphi')\rrbracket^{\underline{f}'(m')}$. *Similarly for* f''. △

In the Example 4.7 we proved that the soundness condition holds when considering the general bridge that can be used to avoid the collapsing when no sharing of symbols is wanted. We give below another example showing that the soundness condition holds.

EXAMPLE 4.8. *Modulated fibring of propositional and 3-valued Gödel logics sharing negation. Consider bridge* 2 *presented in Example* 2.28 *and the bridge in Example* 3.19 *assuming that* $C'_0 = \{\boldsymbol{f}', \boldsymbol{t}'\}$ *and* $C''_0 = \{\boldsymbol{f}'', \boldsymbol{t}''\}$. *We verify that* f' *(and similarly for* f''*) is a logic system morphism. Let* m' *be in* M' *and* $\varphi' \in L(C', \&')$, *then,*

- *if* $\varphi' = \hat{f}'(\check{\varphi})$ *then* $\ddot{f}'_{m'}(\llbracket\varphi'\rrbracket^{m'}) = \ddot{f}'_{m'}(\llbracket \hat{f}'(\check{\varphi})\rrbracket^{m'}) \cong \ddot{f}'_{m'}(\dot{f}'_{m'}(\llbracket\check{\varphi}\rrbracket^{\underline{f}'(m')})) \cong \llbracket\check{\varphi}\rrbracket^{\underline{f}'(m')} = \llbracket \check{f}'(\hat{f}'(\check{\varphi}))\rrbracket^{\underline{f}'(m')} = \llbracket \check{f}'(\varphi')\rrbracket^{\underline{f}'(m')}$;
- *if* $\varphi' \vdash'_d \boldsymbol{f}'$ *then* $\ddot{f}'_{m'}(\llbracket\varphi'\rrbracket^{m'}) = \ddot{f}'_{m'}(\bot') = \check{\bot} = \llbracket\check{\boldsymbol{f}}\rrbracket^{\underline{f}'(m')} = \llbracket \check{f}'(\varphi')\rrbracket^{\underline{f}'(m')}$ *using the fact that* $\mathscr{L}'$ *is sound;*
- *otherwise* $\varphi' \nvdash'_d \boldsymbol{f}'$ *and so* $\varphi' \nvDash'_d \boldsymbol{f}'$ *since* $\mathscr{L}'$ *is complete. So there exists* m' *such that* $\llbracket\varphi'\rrbracket^{m'} \neq \bot'$. *Since all constructors in* C' *have the same denotation in all models then* $\llbracket\varphi'\rrbracket^{m'} \neq \bot'$ *for all models* m'. *Therefore* $\ddot{f}'(\llbracket\varphi'\rrbracket^{m'}) = \check{\top}$ *for every* m'. *So* $\ddot{f}'_{m'}(\llbracket\varphi'\rrbracket^{m'}) = \check{\top} = \llbracket\check{\boldsymbol{t}}\rrbracket^{\underline{f}'(m')} = \llbracket \check{f}'(\varphi')\rrbracket^{\underline{f}'(m')}$. △

Now we establish a new way of considering modulated fibring of logic systems that satisfy certain requirements. Later we apply the general result to the modulated fibring of 3-valued Gödel and Łukasiewicz logics. In following proposition we consider a logic system with equivalence with the usual meaning.

PROP/DEFINITION 4.9. *Let* $\mathscr{L} = \langle \Sigma, M, A, R_d, R_p\rangle$ *be a sound and complete logic system with equivalence such that* M *is countable. Then, the logic system* $\tilde{L} = \langle \tilde{\Sigma}, M, \tilde{A}, \tilde{R}_d, \tilde{R}_p\rangle$ *defined as follows*

- *$\tilde{\Sigma}$ is equal to Σ except $\tilde{C}_0 = C_0 \cup G$ where G is composed by 0-ary constructors $c_{\vec{b}}$ for all possible $\vec{b}$ where $\vec{b}$ is a sequence $\langle b_1, b_2, \dots \rangle$ such that $b_i \in B_{m_i}$, assuming that $M = \{m_1, m_2, \dots\}$;*
- *$\tilde{A}(m_i)$ is equal to $A(m_i)$ except that $v_{m_i}(c_{\vec{b}}) = b_i$;*
- *$\tilde{R}_d$ includes R_d and*
  - $\{\langle\{c_{\vec{b}}\}, \varphi\rangle, \langle\{\varphi\}, c_{\vec{b}}\rangle \mid$ *for all* $\varphi, c_{\vec{b}}$ *with* $c_{\vec{b}} \tilde{\vDash}_d \varphi, \varphi \tilde{\vDash}_d c_{\vec{b}}\}$
  - $\{\langle\{c_{1\vec{b}}\}, c_{2\vec{b}}\rangle \mid$ *for all* $c_{1\vec{b}}, c_{2\vec{b}}$ *such that* $c_{1\vec{b}} \tilde{\vDash}_d c_{2\vec{b}}\}$
  - $\{\langle\{\vec{\varphi} \Leftrightarrow \vec{\delta}\}, c(\vec{\varphi}) \Leftrightarrow c(\vec{\delta})\rangle \mid$ *for all sequences of formulae* $\vec{\varphi}$ *and* $\vec{\delta}\}$;
- $\tilde{R}_p = \tilde{R}_d \cup R_p \cup \{\langle\{c_{1\vec{b}}\}, c_{2\vec{b}}\rangle \mid$ *for all* $c_{1\vec{b}}, c_{2\vec{b}}$ *such that* $c_{1\vec{b}} \tilde{\vDash}_p c_{2\vec{b}}\}$

is sound, complete, with congruence and is a conservative extension of $\mathscr{L}$.

The result in 4.9 can be applied to define the modulated fibring of an extension of Gödel logic and Łukasiewicz logic sharing conjunction and disjunction.

EXAMPLE 4.10. *Modulated fibring of Gödel and Łukasiewicz 3-valued logics. Both Gödel ($\mathscr{L}'$) and Łukasiewicz ($\mathscr{L}''$) 3-valued logics (see Examples 2.11, 3.8, 2.12, 3.7) are sound, complete, with equivalence and with a finite set of models (with the same truth values). We also assume that they have the same set of 0-ary constructors besides **t** and **f**. Let G be defined as in Prop/Definition 4.9. Consider $\tilde{\mathscr{L}}'$ and $\tilde{\mathscr{L}}''$, the extensions of $\mathscr{L}'$ and $\mathscr{L}''$, respectively. Consider the following bridge:*

- $\check{C}_0 = G \cup \{\boldsymbol{t}, \boldsymbol{f}\}$, $\check{C}_2 = \{\check{\wedge}, \check{\vee}\}$, $\check{C}_k = \emptyset$ *when* $k \geq 3$ *and* $k = 1$, $\check{\&}$ *is* $\check{\wedge}$, $\check{\Xi} = \emptyset$ *and* $\check{S} = \emptyset$;
- $\check{M} = A'(M')|_{\check{C}} \cup A''(M'')|_{\check{C}}$;
- $\check{R}_d$ *and* $\check{R}_p$ *are translations of all ground instances of* $\tilde{R}'_p$, $\tilde{R}'_d$, $\tilde{R}''_p$, $\tilde{R}''_d$ *by* $\check{f}'$ *and* $\check{f}''$;
- $\hat{f}'$ *and* $\hat{f}''$ *are injections*;
- $\check{f}'(\varphi') = \begin{cases} \check{\varphi} & \text{p.t. } \varphi' \text{ is } \hat{f}'(\check{\varphi}) \\ c_{\varphi'} & \text{otherwise} \end{cases}$, *similarly for* $\check{f}''$;
- $\underline{f}'$, $\dot{f}'_{m'}$ *and* $\ddot{f}'_{m'}$ *are identities, similarly for* $\underline{f}''$, $\dot{f}''_{m''}$ *and* $\ddot{f}''_{m''}$.

We show that f' is a morphism: 1. $\varphi' \cong'_d \hat{f}'(\check{f}'(\varphi'))$. *If* $\varphi' = \hat{f}'(\check{\varphi})$ *then* $\hat{f}'(\check{f}'(\varphi'))$ $= \varphi'$. *Otherwise* $\hat{f}'(\check{f}'(\varphi')) = c_{\varphi'}$ *and so* $\varphi' \cong'_d c_{\varphi'}$. 2. $\check{f}'(\hat{f}'(\check{\varphi})) \check{\vdash}_d \check{\varphi}$. *Straightforward.* 3. $\hat{f}'(\check{c})(\hat{f}'(\check{f}'(\varphi'_1)), \hat{f}'(\check{f}'(\varphi'_2))) \vdash'_d \hat{f}'(\check{c})(\varphi'_1, \varphi'_2)$ *because* $\varphi' \cong'_d \hat{f}'(\check{f}'(\varphi'))$ *and* $\mathscr{L}'$ *has congruence.* △

4.2. Preservation of soundness. We now concentrate our attention on soundness. The main objective is to obtain a result stating that if we start with sound logic systems then the logic system obtained by modulated fibring is again sound. After giving an auxiliary result about closure for safe substitutions of the entailment we provide a sufficient condition for a logic system to be sound.

LEMMA 4.11. *Let $\mathscr{I}$ be an interpretation system, δ a formula, Γ a set of formulae and σ a safe substitution for $\Gamma \cup \{\delta\}$. Then, $\Gamma\sigma \vDash_d \delta\sigma$ and $\Gamma\sigma \vDash_p \delta\sigma$ whenever $\Gamma \vDash_d \delta$ and $\Gamma \vDash_p \delta$, respectively.*

PROOF. Observe that $[\![\delta]\!]^m_{\alpha_\sigma} = [\![\delta\sigma]\!]^m_\alpha$, where α_σ is an assignment such that $\alpha_\sigma(\xi) = [\![\sigma(\xi)]\!]^m_\alpha$ which can be proved by a straightforward induction. Note also that α_σ is safe for $\Gamma \cup \{\delta\}$ whenever α is safe for $(\Gamma \cup \{\delta\})\sigma$.

(i) Assume that $\Gamma \vDash_d \delta$. Then, there is a finite set $\Phi \subseteq \Gamma$ such that $\sqcap_{\varphi\in\Phi}[\![\varphi]\!]^m_\alpha \leq [\![\delta]\!]^m_\alpha$ for every model m in M and assignment α safe for $\Phi \cup \{\delta\}$. Let m be a model in M and α an assignment over m safe for $(\Phi \cup \{\delta\})\sigma$. Hence $\sqcap_{\varphi\in\Phi}[\![\varphi]\!]^m_{\alpha_\sigma} \leq [\![\delta]\!]^m_{\alpha_\sigma}$. Then $\sqcap_{\varphi\in\Phi}[\![\varphi\sigma]\!]^m_\alpha \leq [\![\delta\sigma]\!]^m_\alpha$. Therefore $\Phi\sigma \vDash_d \delta\sigma$ and so $\Gamma\sigma \vDash_d \delta\sigma$. (ii) Assume that $\Gamma \vDash_p \delta$. It is easy to prove that $\Gamma\sigma \vDash_p \delta\sigma$. ⊣

The next result is easily shown by induction using Lemma 4.11. It states that in order to show that a logic system is sound it is enough to show that the inference rules are sound. We say that a model m is a *model for Hilbert system* $\mathscr{H}$ iff for every rule $\langle\Gamma,\delta\rangle \in R_p$, $m\alpha \Vdash_p \delta$ whenever $m\alpha \Vdash_p \gamma$ for every $\gamma \in \Gamma$ and safe assignment α for $\Gamma \cup \{\delta\}$ and for every rule $\langle\Gamma,\delta\rangle \in R_d$, $m\alpha b \Vdash_d \delta$ whenever $m\alpha b \Vdash_d \gamma$ for every $\gamma \in \Gamma$, safe assignment α for $\Gamma \cup \{\delta\}$ and $b \in B_m$.

PROPOSITION 4.12. *Let $\mathscr{L}$ be a logic system such that each $m \in M$ is a model for $H(\mathscr{L})$. Then $\mathscr{L}$ is sound with respect to proof and derivation.*

Note that we have to show that every model in the modulated fibring is a model for the additional rules that are not inherited from the component logic systems.

LEMMA 4.13. *Let $h : \mathscr{L} \to \mathscr{L}'$ be a logic system morphism such that $id_\Sigma \in S$. Then, for any $m' \in M'$,*

1. $[\![\gamma']\!]^{m'} \cong [\![\hat{h}(\check{h}(\gamma'))]\!]^{m'}$, *whenever $\gamma' \in L(C', \&', \hat{h})$ and $\ddot{h}_{m'}$ is surjective.*
2. $[\![\gamma']\!]^{m'} \leq' [\![\hat{h}(\check{h}(\gamma'))]\!]^{m'}$, *whenever γ' is a formula in $L(C', \&')$.*

PROOF.

1. $$\begin{aligned}&[\![\hat{h}(c)(\gamma'_1,\dots,\gamma'_k)]\!]^{m'} \cong\\ &v_{m'}(\hat{h}(c))([\![\gamma'_1]\!]^{m'},\dots,[\![\gamma'_k]\!]^{m'}) \cong\\ &\dot{h}_{m'}(v_{\underline{h}(m')}(c)(\ddot{h}_{m'}([\![\gamma'_1]\!]^{m'}),\dots,\ddot{h}_{m'}([\![\gamma'_k]\!]^{m'}))) \cong\\ &\dot{h}_{m'}(v_{\underline{h}(m')}(c)([\![\check{h}(\gamma'_1)]\!]^{\underline{h}(m')},\dots,[\![\check{h}(\gamma'_k)]\!]^{\underline{h}(m')})) \cong \text{since } \ddot{h}_{m'} \text{ is surjective}\\ &\dot{h}_{m'}(v_{\underline{h}(m')}(c)(\ddot{h}_{m'}(\dot{h}_{m'}([\![\check{h}(\gamma'_1)]\!]^{\underline{h}(m')})),\dots,\ddot{h}_{m'}(\dot{h}_{m'}([\![\check{h}(\gamma'_k)]\!]^{\underline{h}(m')})))) \cong\\ &\dot{h}_{m'}(v_{\underline{h}(m')}(c)(\ddot{h}_{m'}([\![\hat{h}(\check{h}(\gamma'_1))]\!]^{m'}),\dots,\ddot{h}_{m'}([\![\hat{h}(\check{h}(\gamma'_k))]\!]^{m'}))) \cong\\ &v_{m'}(\hat{h}(c))([\![\hat{h}(\check{h}(\gamma'_1))]\!]^{m'},\dots,[\![\hat{h}(\check{h}(\gamma'_k))]\!]^{m'}) \cong\\ &[\![\hat{h}(c)(\hat{h}(\check{h}(\gamma'_1)),\dots,\hat{h}(\check{h}(\gamma'_k)))]\!]^{m'}.\end{aligned}$$
2. $[\![\gamma']\!]^{m'} \leq' \dot{h}_{m'}(\ddot{h}_{m'}([\![\gamma']\!]^{m'})) \cong \dot{h}_{m'}([\![\check{h}(\gamma')]\!]^{\underline{h}(m')}) \cong [\![\hat{h}(\check{h}(\gamma'))]\!]^{m'}$. ⊣

We conclude the section with the main result on preservation of soundness.

THEOREM 4.14. *The modulated fibring $\langle g' : \mathscr{L}' \to \mathscr{L}, g'' : \mathscr{L}'' \to \mathscr{L}\rangle$ of $\mathscr{L}'$ and $\mathscr{L}''$ by a bridge β is sound, provided that $\mathscr{L}'$ and $\mathscr{L}''$ are sound, $id_{\Sigma'} \in S'$ and $id_{\Sigma''} \in S''$.*

PROOF. Taking into account Proposition 4.12 we just need to check that each model in M is a model for $\mathscr{H}$. Let $r \in R_d$. (i) r is $\hat{g}'(r')$ with r' in R'_d. Then $Prem(r) \vDash_d Conc(r)$ by Proposition 2.20 since $\ddot{h}_m$ is surjective for each $m \in M$, $Prem(r') \vDash'_d Conc(r')$ and $id_{\Sigma'} \in S'$. (ii) r is $\langle\{\gamma\}, \hat{g}'(\check{g}'(\gamma))\rangle$ with γ in $L(C,\&)$ then, using Lemma 4.13, we have $[\![\gamma]\!]^m \leq [\![\hat{g}'(\check{g}'(\gamma))]\!]^m$ for $m \in M$, so $\gamma \vDash_d \hat{g}'(\check{g}'(\gamma))$. (iii) r is $\langle\{\hat{g}'(c')(\hat{g}'(\check{g}'(\vec{\gamma})))\}, \hat{g}'(c')(\vec{\gamma})\rangle$ with $\vec{\gamma}$ a sequence over $L(C,\&)$ then, by Lemma 4.13, $[\![\hat{g}'(c')(\hat{g}'(\check{g}'(\vec{\gamma})))]\!]^m \cong [\![\hat{g}'(c')(\vec{\gamma})]\!]^m$ for each $m \in M$, so

$\hat{g}'(c')(\hat{g}'(\check{g}'(\vec{\gamma}))) \vDash_d \hat{g}'(c')(\vec{\gamma})$. For r in R_p we can conclude that $Prem(r) \vDash_p Conc(r)$ in a similar way. ⊣

§5. **Completeness.** Herein we study completeness with the objective of obtaining preservation results. The first main result is Theorem 5.6 giving a sufficient condition for completeness of a logic system. The second main result is Theorem 5.12 that provides sufficient conditions for preservation of completeness.

5.1. Sufficient conditions for completeness. For completeness purposes we will adopt the Lindenbaum-Tarski approach. Therefore we have to guarantee that the Hilbert systems we work with are with congruence.

DEFINITION 5.1. *A Hilbert system $\mathcal{H}$ is said to be with congruence iff for every p deductively closed set Γ, $c(\vec{\varphi}) \cong_\Gamma c(\vec{\delta})$ whenever $\vec{\varphi} \cong_\Gamma \vec{\delta}$ for every constructor c.*[9] *A logic system $\mathcal{L}$ is a logic system with congruence iff $H(\mathcal{L})$ is a Hilbert system with congruence.*

Observe that & is also congruent: assume that $\Gamma, \varphi_i \vdash_d \delta_i$, $i = 1, 2$: but $\Gamma, (\varphi_1 \& \varphi_2) \vdash_d \varphi_i$ with $i = 1, 2$, hence $\Gamma, (\varphi_1 \& \varphi_2) \vdash_d \delta_i$ with $i = 1, 2$ and so $\Gamma, (\varphi_1 \& \varphi_2) \vdash_d (\delta_1 \& \delta_2)$.

Another restriction is to be assumed: we will work with logic systems that have a special constructor $\boldsymbol{t}$ of 0-arity.

DEFINITION 5.2. *A logic system $\mathcal{L}$ is said to be with true iff $\boldsymbol{t} \in C_0$, $v_m(\boldsymbol{t}) = \top_m$ for every m in M and $\vdash_d \boldsymbol{t}$. A Hilbert system is with true iff $\boldsymbol{t} \in C_0$ and $\vdash_d \boldsymbol{t}$.*

We are now ready to introduce the Lindenbaum-Tarski algebra for each set of formulae closed under proof.

PROP/DEFINITION 5.3. *A Hilbert calculus $\mathcal{H}$ with congruence and true induces, for every p-deductively closed subset Γ of $L(C, \&)$, a Σ-structure $\lambda\tau_\Gamma$, called the Lindenbaum-Tarski algebra*[10] *for Γ, defined as follows:*

- $B_{\lambda\tau_\Gamma} = L(C, \&)$;
- $\varphi \leq_\Gamma \delta$ *iff* $\Gamma, \varphi \vdash_d \delta$;
- $\varphi \sqcap_\Gamma \delta = \varphi \& \delta$ *and* $\sqcap_\Gamma \emptyset = \boldsymbol{t}$;
- $v_{\lambda\tau_\Gamma}(c)(\varphi_1, \ldots, \varphi_k) = c(\varphi_1, \ldots, \varphi_k)$.

It is straightforward to check that the Lindenbaum-Tarski algebra satisfies the conditions in the definition of a Σ-structure. When there is no ambiguity with respect to the set of formulae, we can refer to a Lindenbaum-Tarski algebra for a set Γ as $\lambda\tau$.

LEMMA 5.4. *Let $\mathcal{L}$ be a logic system with congruence and true and Γ a p-deductively closed subset of $L(C, \&)$. Then*

1. $\varphi \cong_\Gamma \boldsymbol{t}$ *iff* φ *is in* Γ.
2. $[\![\varphi]\!]_\alpha^{\lambda\tau} = \varphi\sigma$ *where σ is such that $\sigma(\xi) = \alpha(\xi)$.*

Observe that, given a logic system $\mathcal{L}$ with congruence and true, a p-deductively closed set Γ contained in $L(C, \&)$ and a signature morphism $s : \check{\Sigma} \to \Sigma$ in S then

[9] Recall that $\varphi \cong_\Gamma \psi$ iff $\Gamma, \varphi \vdash_d \psi$ and $\Gamma, \psi \vdash_d \varphi$.

[10] Usually, the Lindenbaum-Tarski algebra is presented using equivalent classes of formulae because the underlying interpretation structures are partial orders.

$B_{\lambda\tau_\Gamma}(s)$ is the set $L(C, \&, s)$. Observe also that the Lindenbaum-Tarski algebra validates the rules in the Hilbert system at hand.

Now we have to guarantee that in a logic system, for each p-deductively closed set of formulae Γ, we have a model whose underlying structure is the Lindenbaum-Tarski algebra for Γ.

DEFINITION 5.5. *A logic system $\mathscr{L}$ with congruence and true is full iff, for every set of formulae Γ p-deductively closed, there is a model m_Γ such that $A(m_\Gamma)$ is isomorphic to the Lindenbaum-Tarski algebra for Γ.*

Observe that we can enrich the class of models of an interpretation system with one extra model for each p-deductively closed set Γ corresponding to the Lindenbaum-Tarski algebra for Γ. This was not done at that time because Hilbert systems were not yet defined. Now we can state the main result of this section.

THEOREM 5.6. *Every full logic system $\mathscr{L}$ with congruence and true is complete.*

PROOF. Let Γ_0 and δ be in $L(C, \&)$.
(i) Assume that $\Gamma_0 \nvdash_p \delta$. Then $\delta \notin \Gamma$ where Γ is the set $\Gamma_0^{\vdash_p}$. So $[\![\delta]\!]^{\lambda\tau_\Gamma} = \delta \ncong \boldsymbol{t}$ using Lemma 5.4. On the other hand $[\![\gamma]\!]^{\lambda\tau_\Gamma} = \gamma \cong \boldsymbol{t}$ for every γ in Γ. Therefore $\Gamma \nvDash_p^{\lambda\tau_\Gamma} \delta$. Let m_Γ be the model in M such that $A(m_\Gamma)$ is isomorphic to $\lambda\tau_\Gamma$. Then $\Gamma \nvDash_p^{A(m_\Gamma)} \delta$ and so $\Gamma \nvDash_p \delta$.
(ii) Assume $\Gamma_0 \vDash_d \delta$ and let m_Γ be the model in M whose structure is isomorphic to $\lambda\tau_\Gamma$ where Γ is the set $\emptyset^{\vdash_p}$. Then there is a finite set $\{\gamma_1, \dots, \gamma_k\} \subseteq \Gamma_0$ such that $\{\gamma_1, \dots, \gamma_k\} \vDash_d^{A(m_\Gamma)} \delta$. Hence $\{\gamma_1, \dots, \gamma_k\} \vDash_d^{\lambda\tau_\Gamma} \delta$ and $\sqcap_{i=1,\dots,k}[\![\gamma_i]\!]^{\lambda\tau_\Gamma} \leq_\Gamma [\![\delta]\!]^{\lambda\tau_\Gamma}$. So, using Lemma 5.4, $\sqcap_{i=1,\dots,k}\gamma_i \leq_\Gamma \delta$. Therefore $\{\gamma_1, \dots, \gamma_k\} \vdash_d \delta$ and so $\Gamma_0 \vdash_d \delta$. ⊣

5.2. Preservation of completeness. The main goal is to establish preservation of completeness by modulated fibring under reasonable conditions. According to Theorem 5.6 we can conclude that a logic system is complete provided that it is full and with congruence and true. Therefore we prove that congruence and true are preserved by modulated fibring. Moreover we also prove that fullness is preserved by modulated fibring provided that the bridge has additional properties.

LEMMA 5.7. *Let $h : \mathscr{H} \to \mathscr{H}'$ be a Hilbert system morphism such that $\hat{h}$ is injective and $\check{h}$ is surjective. Then, $\check{h}(\Gamma')$ is a p-deductively closed set of formulae whenever Γ' is a p-deductively closed set of ground formulae.*

PROOF. Let φ in $L(C, \&)$ be such that $\check{h}(\Gamma') \vdash_p \varphi$. Then $\hat{h}(\check{h}(\Gamma')) \vdash'_p \hat{h}(\varphi)$, so $\Gamma' \vdash'_p \hat{h}(\varphi)$, hence $\hat{h}(\varphi) \in \Gamma'$ and therefore $\varphi \in \check{h}(\Gamma')$ since $\varphi \cong \check{h}(\hat{h}(\varphi))$. ⊣

In the sequel, we need to work with the category of structures as well as the category of structures over the same signature.

PROP/DEFINITION 5.8. *The objects in St are tuples $\langle \Sigma, B, \leq, v\rangle$ where Σ is a signature and $\langle B, \leq, v\rangle$ is a Σ-structure. The morphisms in St are triples $\langle \hat{h}, \dot{h}, \ddot{h}\rangle$ such that $\hat{h}$ is a signature morphism, $\dot{h} : \langle B, \leq\rangle \to \langle B', \leq'\rangle$ is a monotonic map, $\ddot{h} : \langle B', \leq'\rangle \to \langle B, \leq\rangle$ is a monotonic map preserving finite meets, $\ddot{h}$ is left adjoint to $\dot{h}$ and $v'(\hat{h}(c))(\vec{b'}) = \dot{h}(v(c)(\ddot{h}(\vec{b'})))$.[11] The category $St(\Sigma)$ is the fiber of St over Σ.[12]*

[11] Compare with Definition 2.7.

[12] The objects are Σ-structures and the morphisms are pairs $\langle \dot{h}, \ddot{h}\rangle$.

We show that each Hilbert system morphism h induces a morphism between the Lindenbaum-Tarski algebra for each p-deductively closed set Γ and the Lindenbaum-Tarski algebra for $\check{h}(\Gamma)$. Observe that the conditions in the definition of the Hilbert system morphism were introduced with this purpose in mind.

PROPOSITION 5.9. *Let $\mathscr{H}$ and $\mathscr{H}'$ be Hilbert systems with congruence and true and $h : \mathscr{H} \to \mathscr{H}'$ a morphism such that $\hat{h}$ is injective and $\check{h}$ is surjective. Then, $\langle \hat{h}, \dot{h}_{\Gamma'}, \ddot{h}_{\Gamma'} \rangle : \langle \Sigma, \lambda\tau_{\check{h}(\Gamma')} \rangle \to \langle \Sigma', \lambda\tau_{\Gamma'} \rangle$ is a morphism in St where $\dot{h}_{\Gamma'}(\varphi) = \hat{h}(\varphi)$ and $\ddot{h}_{\Gamma'}(\varphi') = \check{h}(\varphi')$, for every p-deductively closed set Γ' over $L(C', \&')$.*

THEOREM 5.10. *The modulated fibring $\langle g' : \mathscr{L}' \to \mathscr{L}, g'' : \mathscr{L}'' \to \mathscr{L} \rangle$ of logic systems $\mathscr{L}'$ and $\mathscr{L}''$ with congruence and true by a bridge β is with congruence and true.*

PROOF. Let c be a constructor in C_k, $\Gamma, \vec{\delta} \vdash_d \vec{\varphi}$ and $\Gamma, \vec{\varphi} \vdash_d \vec{\delta}$. Then c is in $\hat{g}'(C_k')$ or in $\hat{g}''(C_k'')$. Suppose that there exists c' in C_k' with $c = \hat{g}_k'(c')$. Then $\check{g}'(\Gamma), \check{g}'(\vec{\delta}) \vdash_d' \check{g}'(\vec{\varphi})$ and $\check{g}'(\Gamma), \check{g}'(\vec{\varphi}) \vdash_d' \check{g}'(\vec{\delta})$. Since $\mathscr{L}'$ has congruence then $\check{g}'(\Gamma), c'(\check{g}'(\vec{\delta})) \vdash_d' c'(\check{g}'(\vec{\varphi}))$. Thus $\hat{g}'(\check{g}'(\Gamma)), \hat{g}'(c'(\check{g}'(\vec{\delta}))) \vdash_d \hat{g}'(c'(\check{g}'(\vec{\varphi})))$, $\Gamma, \hat{g}_k'(c')(\hat{g}'(\check{g}'(\vec{\delta}))) \vdash_d \hat{g}_k'(c')(\hat{g}'(\check{g}'(\vec{\varphi})))$ and finally $\Gamma, \hat{g}_k'(c')(\vec{\delta}) \vdash_d \hat{g}_k'(c')(\vec{\varphi})$. The proof of preservation of true is straightforward. ⊣

Observe that the more complex notion of Hilbert system morphism was essential for the preservation of congruence without the requirement of sharing implication and equivalence (as in [18] leading to the unwanted collapse). For the preservation of fullness by modulated fibring we need further constraints on the bridge.

DEFINITION 5.11. *A bridge $\langle f' : \check{\mathscr{L}} \to \mathscr{L}', f'' : \check{\mathscr{L}} \to \mathscr{L}'' \rangle$ is adequate iff $\mathscr{L}'$, $\mathscr{L}''$, $\check{\mathscr{L}}$ are full, with congruence and true and $\underline{f}'(m_{\Gamma'}') = m_{\check{f}(\Gamma')}$ and $\underline{f}''(m_{\Gamma''}'') = m_{\check{f}(\Gamma'')}$ for every p-deductively closed sets of ground formulae Γ' and Γ''.*

We would like to use Theorem 5.6 to conclude that the modulated fibring of full logic systems with congruence and true by an adequate bridge is complete. For this purpose we have to show that the modulated fibring is full. If Γ is a p-deductively closed set, then there is a model $\langle \check{g}'(\Gamma), \check{g}''(\Gamma) \rangle \in M$ such that $A'(\check{g}'(\Gamma))$ is isomorphic to $\lambda\tau_{\check{g}'(\Gamma)}$ and $A''(\check{g}''(\Gamma))$ is isomorphic to $\lambda\tau_{\check{g}''(\Gamma)}$. We show in Proposition 5.15 that $A(\langle \check{g}'(\Gamma), \check{g}''(\Gamma) \rangle)$ is isomorphic to $\lambda\tau_\Gamma$.

THEOREM 5.12. *The modulated fibring $\langle g' : \mathscr{L}' \to \mathscr{L}, g'' : \mathscr{L}'' \to \mathscr{L} \rangle$ of logic systems $\mathscr{L}'$ and $\mathscr{L}''$ by an adequate bridge β is complete.*

PROOF. We know, using Theorem 5.10, that $\mathscr{L}$ is with congruence and true. We also know that, for each set of formulae Γ closed for proof, the model $\langle \check{g}'(\Gamma), \check{g}''(\Gamma) \rangle$ is in M. Using Proposition 5.15 we can also conclude that the structure $A(\langle \check{g}'(\Gamma), \check{g}''(\Gamma) \rangle)$ is isomorphic to $\lambda\tau_\Gamma$. Therefore, $\mathscr{L}$ is full and using Theorem 5.6, $\mathscr{L}$ is complete. ⊣

EXAMPLE 5.13. *The following modulated fibrings are complete:*

- *Unconstrained modulated fibring of full logic systems with congruence and true by an adequate bridge. In particular, the unconstrained modulated fibring of full propositional and intuitionistic logics is complete. The same holds for the unconstrained modulated fibring of full propositional and Łukasiewicz logics.*

- *The modulated fibring of full propositional logic and Gödel logic sharing negation is complete.*
- *The modulated fibring of full Gödel logic and Łukasiewicz logic sharing conjunction and disjunction is complete.* $\triangle$

An "algebraic" version of the completeness result and the preservation of completeness as in [18] can be also be obtained. Of course in this case congruence is not always preserved by modulated fibring. As proved there, when the logics have implication and equivalence congruence is preserved.

5.3. Algebras $A(\langle \check{g}'(\Gamma), \check{g}''(\Gamma)\rangle)$ and $\lambda\tau_\Gamma$ are isomorphic. To conclude the section it remains to prove that the algebra obtained by a pushout of the Lindenbaum-Tarski algebras is isomorphic to the Lindenbaum-Tarski algebra for a p-deductively closed set of formulae in the pushout of the signatures.

Before we need a technical lemma. In order to make the notation lighter, we omit the subscripts of $\dot{g}'_{\langle \check{g}'(\Gamma),\check{g}''(\Gamma)\rangle}$, $\dot{g}''_{\langle \check{g}'(\Gamma),\check{g}''(\Gamma)\rangle}$, $\dot{f}'_{\check{g}'(\Gamma)}$ and $\dot{f}''_{\check{g}''(\Gamma)}$.

LEMMA 5.14. *Let $\langle g' : \mathscr{L}' \to \mathscr{L}, g'' : \mathscr{L}'' \to \mathscr{L}\rangle$ be the modulated fibring of logic systems $\mathscr{L}'$ and $\mathscr{L}''$ by an adequate bridge β. Then,*

1. $\ddot{g}'(\dot{g}''(\varphi'')) = \check{g}'(\hat{g}''(\varphi''))$.
2. $[\![\varphi]\!]_\alpha^{A(\langle \check{g}'(\Gamma),\check{g}''(\Gamma)\rangle)} = \dot{g}'(\check{g}'(\varphi\sigma_\alpha))$ *where $\varphi \in L(\Sigma, \hat{g}')$, α is safe for φ and σ_α is such that $\sigma_\alpha(\xi) = \hat{g}'(\varphi')$ if $\alpha(\xi) = \dot{g}'(\varphi')$; analogously if $\alpha(\xi) = \dot{g}''(\varphi'')$; and $\sigma_\alpha(\xi) = \hat{g}'(\varphi')\&\hat{g}''(\varphi'')$ whenever $\alpha(\xi)$ is not in the co-domain of either $\dot{g}'$ or $\dot{g}''$ and is equal to $\dot{g}'(\varphi') \sqcap \dot{g}''(\varphi'')$.*
3. $\dot{g}'(\check{g}'(\varphi)) \leq \dot{g}''(\check{g}''(\varphi))$ *for any $\varphi \in L(C, \&, \hat{g}')$.*

PROOF. 1. $\ddot{g}'(\dot{g}''(\varphi'')) = \dot{f}'(\ddot{f}''(\varphi'')) = \dot{f}'(\check{f}''(\varphi'')) = \hat{f}'(\check{f}''(\varphi'')) = \check{g}'(\hat{g}''(\varphi''))$.
2. We consider two cases:

- $[\![\xi]\!]_\alpha^{A(\langle \check{g}'(\Gamma),\check{g}''(\Gamma)\rangle)} = \alpha(\xi) = \dot{g}'(\varphi') = \dot{g}'(\check{g}'(\hat{g}'(\varphi'))) = \dot{g}'(\check{g}'(\xi\sigma_\alpha))$;
- $[\![\varphi]\!]_\alpha^{A(\langle \check{g}'(\Gamma),\check{g}''(\Gamma)\rangle)} = \dot{g}'(\check{g}'(\varphi\sigma_\alpha))$ ($\varphi \notin \Xi$): by induction. Base: if φ is $\hat{g}'(c')$ then $[\![\varphi]\!]_\alpha^{A(\langle \check{g}'(\Gamma),\check{g}''(\Gamma)\rangle)} = \dot{g}'(\nu_{\check{g}'(\Gamma)}(c')) = \dot{g}'(c') = \dot{g}'(\check{g}'(\hat{g}'(c'))) = \dot{g}'(\check{g}'(\varphi\sigma_\alpha))$. The rest of the proof follows straightforwardly.

3. Observe that $\dot{g}'(\check{g}'(\varphi)) = [\![\varphi]\!]_\alpha^{A(\langle \check{g}'(\Gamma),\check{g}''(\Gamma)\rangle)} \leq [\![\hat{g}''(\check{g}''(\varphi))]\!]_\alpha^{A(\langle \check{g}'(\Gamma),\check{g}''(\Gamma)\rangle)} = \dot{g}''(\check{g}''(\hat{g}''(\check{g}''(\varphi)))) = \dot{g}''(\check{g}''(\varphi))$. $\dashv$

PROPOSITION 5.15. *Let $\langle g' : \mathscr{L}' \to \mathscr{L}, g'' : \mathscr{L}'' \to \mathscr{L}\rangle$ be the modulated fibring of logic systems $\mathscr{L}'$ and $\mathscr{L}''$ by an adequate bridge β. Then $\lambda\tau$ is isomorphic to $A(\langle \check{g}'(\Gamma), \check{g}''(\Gamma)\rangle)$.*

PROOF. Consider the maps

- $\dot{k} : B_{\lambda\tau} \to B_{A(\langle \check{g}'(\Gamma),\check{g}''(\Gamma)\rangle)}$ such that
  - $\dot{k}(\varphi) = \dot{g}'(\check{g}'(\varphi))$ whenever φ is in $L(C, \hat{g}')$ (similarly for φ in $L(C, \hat{g}'')$);
  - $\dot{k}(\varphi_1\&\varphi_2) = \dot{k}(\varphi_1) \sqcap \dot{k}(\varphi_2)$.
- $\ddot{k} : B_{A(\langle \check{g}'(\Gamma),\check{g}''(\Gamma)\rangle)} \to B_{\lambda\tau}$ such that
  - $\ddot{k}(\dot{g}'(\varphi')) = \hat{g}'(\varphi')$ (similarly for $\ddot{k}(\dot{g}''(\varphi''))$);
  - $\ddot{k}(b_1 \sqcap b_2) = \ddot{k}(b_1)\&\ddot{k}(b_2)$ whenever $b_1 \sqcap b_2$ is not in the co-domain of either $\dot{g}'$ or $\dot{g}''$.

1. $\dot{k}$ is monotonic: $\dot{k}(\varphi_1) \leq \dot{k}(\varphi_2)$ whenever $\varphi_1 \leq \varphi_2$.

- $\varphi_1 \in L(C, \hat{g}')$ and $\varphi_2 \in L(C, \hat{g}')$. Straightforward.
- $\varphi_1 \in L(C, \hat{g}')$ and $\varphi_2 \in L(C, \hat{g}'')$. Then $\dot{k}(\varphi_1) = \dot{g}'(\breve{g}'(\varphi_1)) \leq \dot{g}''(\breve{g}''(\varphi_1)) \leq \dot{g}''(\breve{g}''(\varphi_2)) = \dot{k}(\varphi_2)$ by Lemma 5.14.
- φ_1 is in $L(C, \hat{g}')$ and φ_2 is $\varphi_{21} \& \varphi_{22}$, with φ_{2i} in $L(C, \hat{g}^{s_i})$ for $i = 1, 2$ and $s_i \in \{', ''\}$. Then $\varphi_1 \leq \varphi_{21}$ and $\varphi_1 \leq \varphi_{22}$. From the previous cases $\dot{k}(\varphi_1) \leq \dot{k}(\varphi_{21})$, $\dot{k}(\varphi_1) \leq \dot{k}(\varphi_{22})$ and so $\dot{k}(\varphi_1) \leq \dot{k}(\varphi_{21}) \sqcap \dot{k}(\varphi_{22}) = \dot{k}(\varphi_{21} \& \varphi_{22})$.

2. $\ddot{k}$ is monotonic: $\ddot{k}(b_1) \leq \ddot{k}(b_2)$ whenever $b_1 \leq b_2$.

- $b_1 = \dot{g}'(\varphi_1)$ and $b_2 = \dot{g}'(\varphi_2)$. Then $\varphi_1 \leq' \varphi_2$. So $\hat{g}'(\varphi_1) \leq \hat{g}'(\varphi_2)$. Therefore $\ddot{k}(b_1) \leq \ddot{k}(b_2)$.
- $b_1 = \dot{g}'(\varphi_1)$ and $b_2 = \dot{g}''(\varphi_2)$. Then there exists a $\breve{\varphi}$ in $L(\breve{C}, \breve{\&})$ with $\dot{g}'(\varphi_1) \leq \dot{g}'(\dot{f}'(\breve{\varphi})) = \dot{g}''(\dot{f}''(\breve{\varphi})) \leq \dot{g}''(\varphi_2)$. So using the previous case we have $\ddot{k}(b_1) \leq \ddot{k}(\dot{g}'(\dot{f}'(\breve{\varphi}))) = \ddot{k}(\dot{g}''(\dot{f}''(\breve{\varphi}))) \leq \ddot{k}(b_2)$.
- b_1 and b_2 are not in the co-domain of $\dot{g}'$ and $\dot{g}''$. Easy from cases 1. and 2.

3. $\ddot{k}$ preserves meets. Let $b_1, b_2 \in B_{A(\langle \breve{g}'(\Gamma), \breve{g}''(\Gamma)\rangle)}$ where $b_1 \sqcap b_2$ is $\dot{g}'(\varphi)$.

- b_1 is $\dot{g}'(\varphi_1)$ and b_2 is $\dot{g}''(\varphi_2)$. Note that $\ddot{g}'(\dot{g}'(\varphi)) = \ddot{g}'(\dot{g}'(\varphi_1)) \sqcap \ddot{g}'(\dot{g}''(\varphi_2))$. So $\varphi \cong \varphi_1 \&' \breve{g}'(\hat{g}''(\varphi_2))$ and $\breve{g}''(\hat{g}'(\varphi)) \cong \breve{g}''(\hat{g}'(\varphi_1)) \&'' \varphi_2$. Then $\hat{g}'(\varphi) \cong_\Gamma \hat{g}'(\varphi) \& \hat{g}''(\breve{g}''(\hat{g}'(\varphi))) \cong_\Gamma \hat{g}'(\varphi_1) \& \hat{g}''(\varphi_2)$ and so $\ddot{k}(b_1 \sqcap b_2) = \ddot{k}(b_1) \& \ddot{k}(b_2)$.
- b_1 is in the co-domain of $\dot{g}'$ (or $\dot{g}''$) and b_2 is not in the co-domain of $\dot{g}'$ or $\dot{g}''$. Then, $\ddot{k}(b_1 \sqcap b_2) = \ddot{k}(b_1 \sqcap b_2' \sqcap b_2'') = \ddot{k}(b_1 \sqcap b_2') \& \ddot{k}(b_2'') = \ddot{k}(b_1) \& \ddot{k}(b_2') \& \ddot{k}(b_2'') = \ddot{k}(b_1) \& \ddot{k}(b_2' \sqcap b_2'') = \ddot{k}(b_1) \& \ddot{k}(b_2)$.
- b_1 and b_2 are not in the co-domain of $\dot{g}'$ or in the co-domain of $\dot{g}''$. Straightforward, using the previous cases.

4. $\ddot{k}$ is a bijection with inverse $\dot{k}$.

- $\ddot{k} \circ \dot{k} \cong \mathrm{id}_{B_{\lambda\tau}}$. Let φ be in $B_{\lambda\tau}$.
  - φ is in $L(C, \hat{g}')$. Then $\ddot{k}(\dot{k}(\varphi)) = \ddot{k}(\dot{g}'(\breve{g}'(\varphi))) = \hat{g}'(\breve{g}'(\varphi)) \cong \varphi$.
  - φ is $\varphi_1 \& \varphi_2$ with φ_i in $L(C, \hat{g}^{j_i})$ for $i = 1, 2$ and $j_i \in \{', ''\}$. Then $\ddot{k}(\dot{k}(\varphi_1 \& \varphi_2)) = \ddot{k}(\dot{k}(\varphi_1) \sqcap \dot{k}(\varphi_2)) = \ddot{k}(\dot{k}(\varphi_1)) \sqcap \ddot{k}(\dot{k}(\varphi_2)) \cong \varphi_1 \sqcap \varphi_2 = \varphi_1 \& \varphi_2$.
- $\dot{k} \circ \ddot{k} \cong \mathrm{id}_{B_{A(\langle \breve{g}'(\Gamma), \breve{g}''(\Gamma)\rangle)}}$. Let b be in $B_{A(\langle \breve{g}'(\Gamma), \breve{g}''(\Gamma)\rangle)}$.
  - b is $\dot{g}'(\varphi')$. Therefore $\dot{k}(\ddot{k}(b)) = \dot{k}(\hat{g}'(\varphi')) = \dot{g}'(\breve{g}'(\hat{g}'(\varphi'))) \cong \dot{g}'(\varphi') = b$.
  - b is not in the co-domain of $\dot{g}'$ or $\dot{g}''$. Then b is $b_1 \sqcap b_2$. So, using the previous cases, $\dot{k}(\ddot{k}(b)) = \dot{k}(\ddot{k}(b_1 \sqcap b_2)) = \dot{k}(\ddot{k}(b_1) \sqcap \ddot{k}(b_2)) = \dot{k}(\ddot{k}(b_1)) \sqcap \dot{k}(\ddot{k}(b_2)) = b_1 \sqcap b_2 = b$.

5. $\dot{k}(\nu_{\lambda\tau}(c)(\ddot{k}(b_1), \dots, \ddot{k}(b_k))) = \nu_{A(\langle \breve{g}'(\Gamma), \breve{g}''(\Gamma)\rangle)}(c)(b_1, \dots, b_k)$. Straightforward.

So $\lambda\tau$ and $A(\langle \breve{g}'(\Gamma), \breve{g}''(\Gamma)\rangle)$ are isomorphic since $k \circ h \cong \mathrm{id}_{A(\langle \breve{g}'(\Gamma), \breve{g}''(\Gamma)\rangle)}$ and $h \circ k \cong \mathrm{id}_{\lambda\tau}$ where $h : A(\langle \breve{g}'(\Gamma), \breve{g}''(\Gamma)\rangle) \to \lambda\tau$ is a morphism in $St(\Sigma)$ such that $\dot{h} = \ddot{k}$ and $\ddot{h} = \dot{k}$. ⊣

§6. Final remarks. A general, universal theory for combinations of logics does not yet exist, and the conceptual machinery of fibring seems to be one of the most apt at our disposal. However, logics combine in very intricate ways, and pure fibring, general as it may be, is still rudimentary in expressing all subtleties, as evidenced by the collapsing problem: it may occur, especially when one of them extends the others, that the combinations of the logics involved just restores the differences between them, and the fibring product collapses.

The novel concept of modulated fibring, introduced in this paper, refines the fibring techniques permitting to gain a closer control over the combinations, while obtaining (pure) fibring as a particular case.

We have shown that several cases of (constrained and unconstrained) modulated fibring avoid the collapsing by means of appropriate bridges, and investigated important properties like preservation of soundness and completeness.

Some of the basic technical problems that were tackled were solved as follows. At the signature level we work with cones so that we could keep track of safe-relevant morphisms in order to restrict assignments and substitutions. At the semantic level we put together models with different algebras of truth values (by an adjunction between the pre-ordered algebras of truth values) and set up the the base diagram of the modulated fibring from the intended translation between the truth value algebras. At the deductive level we had to enrich the expected notion of morphism taking into account the envisaged results on the preservation of completeness. In what concerns preservation results, we provided sufficient conditions in order to obtain preservation of soundness and completeness, in the later case using a Lindenbaum-Tarski approach.

We close by suggesting some questions which deserve further study. The first problem is to investigate the modulated fibring of first-order based logics, that is logics with variables, terms and quantifiers, thus extending the work in [17]. Another issue is to define modulated fibring for other kinds of deductive systems such as labeled deduction systems extending the work in [4] and [14]. We would also like to investigate the case where we have several designated truth-values instead of just one: we would thus be able to deal, for instance, with the logic LFI_1 considered in [6]. Of course also related with this work is the extension to logics of contradiction [12] and to paraconsistent logics [5] and logics of formal inconsistency [7] in general. Finally, we also intend to investigate bi-Heyting algebras [15] in the context of modulated fibring.

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CENTER FOR LOGIC AND COMPUTATION - CLC
DEPARTMENT OF MATHEMATICS IST UTL
AV. ROVISCO PAIS, 1049-001 LISBOA PORTUGAL
E-mail: css@math.ist.utl.pt
URL: http://www.cs.math.ist.utl.pt/s84.www/cs/css.html

CENTER FOR LOGIC AND COMPUTATION - CLC
DEPARTMENT OF MATHEMATICS IST UTL
AV. ROVISCO PAIS, 1049-001 LISBOA PORTUGAL
E-mail: jfr@math.ist.utl.pt
URL: http://www.cs.math.ist.utl.pt/s84.www/cs/jfr.html

CENTER FOR LOGIC, EPISTEMOLOGY AND THE HISTORY OF SCIENCE - CLE
DEPARTMENT OF PHILOSOPHY IFCH UNICAMP
P.O. BOX 6133, 13083-970 CAMPINAS - SP - BRAZIL
E-mail: carniell@cle.unicamp.br
URL: http://www.cle.unicamp.br/prof/carnielli/