OPTIMAL CONTROL APPLICATIONS AND METHODS
Optim. Control Appl. Meth. 2011; **32**:1–13
Published online 9 November 2009 in Wiley Online Library (wileyonlinelibrary.com). DOI: 10.1002/oca.916

Robust $\mathscr{H}_2$ static output feedback design starting from a parameter-dependent state feedback controller for time-invariant discrete-time polytopic systems

Heber R. Moreira, Ricardo C. L. F. Oliveira and Pedro L. D. Peres*, †

School of Electrical and Computer Engineering, University of Campinas – UNICAMP, 13081-970 Campinas, São Paulo, Brazil

SUMMARY

This paper investigates the problem of computing robust $\mathscr{H}_2$ static output feedback controllers for discrete-time uncertain linear systems with time-invariant parameters lying in polytopic domains. A two stages design procedure based on linear matrix inequalities is proposed as the main contribution. First, a *parameter-dependent* state feedback controller is synthesized and the resulting gains are used as an input condition for the second stage, which designs the desired *robust* static output feedback controller with an $\mathscr{H}_2$ guaranteed cost. The conditions are based on parameter-dependent Lyapunov functions and, differently from most of existing approaches, can also cope with uncertainties in the output control matrix. Numerical examples, including a mass–spring system, illustrate the advantages of the proposed procedure when compared with other methods available in the literature. Copyright © 2009 John Wiley & Sons, Ltd.

Received 18 May 2009; Revised 7 July 2009; Accepted 14 September 2009

KEY WORDS: static output feedback; uncertain discrete-time systems; linear matrix inequalities; $\mathscr{H}_2$ norm

1. INTRODUCTION

Static output feedback design is a classic problem in control theory [1]. The implementation, in practice, is simpler than in the state feedback case, but the computation of the output gain is a non-convex problem. Several optimization techniques have been proposed in the last decades to solve the problem, mainly based on Lyapunov functions whose existence provides the output feedback gain [2–9].

In general, these methods are the extensions of state feedback control strategies that have a convex parameterization for the controller [10]. These parameterizations, obtained by means of change of variables, provide only sufficient conditions in the output feedback case. Approaches that neither use change of variables nor are based on Lyapunov functions can be found in [11, 12].

The situation is more involved in the case of linear systems with uncertain parameters. Moreover, the use of quadratic stability (i.e. the same Lyapunov function guaranteeing stability) is already a source of conservativeness [7, 8]. Within the class of methods that use parameter-dependent

*Correspondence to: Pedro L. D. Peres, School of Electrical and Computer Engineering, University of Campinas – UNICAMP, 13081-970 Campinas, São Paulo, Brazil.
†E-mail: peres@dt.fee.unicamp.br

Contract/grant sponsor: CAPES
Contract/grant sponsor: FAPESP
Contract/grant sponsor: CNPq

Lyapunov functions to reduce the conservatism is worth mentioning [13–16]. Moreover, differently from most of the methods, the approaches of [13, 15, 16] can also cope with uncertainties affecting the output matrix.

The problem to be dealt with in this paper is the design of $\mathscr{H}_2$ robust static output feedback for discrete-time linear systems affected by time-invariant parameters belonging to a polytope. Similar to the approaches in [8, 13, 15], a two stages linear matrix inequalities (LMIs) based procedure is proposed. The main novelty with respect to the previous works is that in the first stage, instead of a robust gain, a *parameter-dependent* state feedback controller is designed. The resulting gains that compose the state feedback controller are then used as the input matrices for the second stage, which synthesizes the desired $\mathscr{H}_2$ *robust* output feedback gain. A feasible solution in the second stage provides an affine parameter-dependent Lyapunov function that certifies the closed-loop stability for both the static output feedback and the parameter-dependent state feedback control laws. As shown by means of numerical examples, this class of functions can provide robust static output feedback stabilizing controllers when other methods available in the literature fail. Some strategies to improve the associated $\mathscr{H}_2$ guaranteed costs are also discussed.

2. PRELIMINARIES

Consider the discrete-time time-invariant linear system described by

$$\begin{aligned} x(k+1) &= A(\alpha)x(k)+B_1(\alpha)w(k)+B_2(\alpha)u(k),\\ z(k) &= C_1(\alpha)x(k)+D_2(\alpha)u(k),\\ y(k) &= C_2(\alpha)x(k), \end{aligned} \tag{1}$$

where $x(k)\in\mathbb{R}^n$ is the state vector, $w(k)\in\mathbb{R}^r$ the exogenous input, $u(k)\in\mathbb{R}^m$ the control input, $z(k)\in\mathbb{R}^p$ the controlled output and $y(k)\in\mathbb{R}^q$ is the measured output. The system matrices $A(\alpha)\in\mathbb{R}^{n\times n}$, $B_1(\alpha)\in\mathbb{R}^{n\times r}$, $B_2(\alpha)\in\mathbb{R}^{n\times m}$, $C_1(\alpha)\in\mathbb{R}^{p\times n}$, $C_2(\alpha)\in\mathbb{R}^{q\times n}$ and $D_2(\alpha)\in\mathbb{R}^{p\times m}$ are not precisely known and belong to the polytope

$$\begin{aligned} \mathscr{D} = \{&(A, B_1, B_2, C_1, C_2, D_2)(\alpha) : (A, B_1, B_2, C_1, C_2, D_2)(\alpha)\\ &= \sum_{i=1}^{N} \alpha_i (A, B_1, B_2, C_1, C_2, D_2)_i, \alpha\in\Lambda_N\}, \end{aligned}$$

where $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_N)'$ is the vector of time-invariant uncertain parameters lying in the unit simplex Λ_N given by

$$\Lambda_N = \left\{ \delta\in\mathbb{R}^N : \sum_{i=1}^{N} \delta_i = 1, \delta_i \geqslant 0, i=1,\ldots,N \right\}. \tag{2}$$

This uncertainty model is largely used in the literature and it is also known as *polytopic model*. Matrices A_i, B_{1i}, B_{2i}, C_{1i}, C_{2i} and D_{2i} are known as the vertices of the system and are given *a priori*.

The problem investigated in this paper is to determine a static gain K associated with the output feedback control law $u=Ky$ such that the closed-loop system is robustly stable for all $\alpha\in\Lambda_N$ with a prescribed $\mathscr{H}_2$ guaranteed cost.

For the open-loop system, the transfer function from the input vector w to the controlled output vector z is denoted by

$$H_{wz}(\xi)=C_1(\alpha)(\xi\mathbf{I}-A(\alpha))^{-1}B_1(\alpha),$$

where ξ is the time-shift operator. In a stable single-input single-output system, the $\mathscr{H}_2$ norm can be interpreted as the root mean-square response of the system when the input is a white noise or as the energy of the impulse response of the system [17]. Using observability and controllability Gramians, the $\mathscr{H}_2$ norm of system (1) can be characterized in the state space representation through robust LMIs, as shown in the following lemma [18].

DOI: 10.1002/oca

Lemma 1
Assume that the matrix $A(\alpha)$ is Schur stable. The inequality $\|H_{wz}(\xi)\|_2^2 < \mu^2$ holds for all $\alpha \in \Lambda_N$ if and only if there exists a symmetric positive-definite parameter-dependent matrix $P(\alpha) \in \mathbb{R}^{n \times n}$ such that

$$\text{trace}(W(\alpha)) < \mu^2, \tag{3a}$$

$$B_1(\alpha)' P(\alpha) B_1(\alpha) - W(\alpha) < \mathbf{0}, \tag{3b}$$

$$A(\alpha)' P(\alpha) A(\alpha) - P(\alpha) + C_1(\alpha)' C_1(\alpha) < \mathbf{0}, \tag{3c}$$

hold for all $\alpha \in \Lambda_N$ with $\mathbf{0} \leqslant W(\alpha) = W(\alpha)' \in \mathbb{R}^{r \times r}$. Or, equivalently, by duality,

$$\text{trace}(W(\alpha)) < \mu^2, \tag{4a}$$

$$C_1(\alpha) P(\alpha) C_1(\alpha)' - W(\alpha) < \mathbf{0}, \tag{4b}$$

$$A(\alpha) P(\alpha) A(\alpha)' - P(\alpha) + B_1(\alpha) B_1(\alpha)' < \mathbf{0}, \tag{4c}$$

hold for all $\alpha \in \Lambda_N$ with $\mathbf{0} \leqslant W(\alpha) = W(\alpha)' \in \mathbb{R}^{p \times p}$.

The minimum value of μ under the conditions of Lemma 1 is the worst-case $\mathscr{H}_2$ norm (or the optimal guaranteed cost) of system (1). The robust LMIs of Lemma 1 are explored in the sequel for state and output feedback design by considering particular structures for the matrices $P(\alpha)$ and $W(\alpha)$, yielding finite-dimensional LMI relaxations.

3. MAIN RESULTS

First, an LMI condition that provides a state feedback parameter-dependent gain is presented.

Theorem 1
If there exist symmetric matrices $P_i \in \mathbb{R}^{n \times n}$, $W_i \in \mathbb{R}^{p \times p}$, matrices $G_i \in \mathbb{R}^{n \times n}$, $Z_i \in \mathbb{R}^{m \times n}$, $i = 1, \ldots, N$, such that the following LMIs are satisfied:[‡]

$$\text{trace}(W_i) < \mu^2, \quad i = 1, \ldots, N, \tag{5}$$

$$\begin{bmatrix} W_i & C_{1i} G_i + D_{2i} Z_i \\ \star & G_i + G_i' - P_i \end{bmatrix} > \mathbf{0}, \quad i = 1, \ldots, N, \tag{6}$$

$$\begin{bmatrix} W_i + W_j & C_{1i} G_j + C_{1j} G_i + D_{2i} Z_j + D_{2j} Z_i \\ \star & G_i + G_j + G_i + G_j' - (P_i + P_j) \end{bmatrix} > \mathbf{0}, \quad i = 1, \ldots, N-1, \quad j = i+1, \ldots, N, \tag{7}$$

$$\begin{bmatrix} P_i & A_i G_i + B_{2i} Z_i & B_{1i} \\ \star & G_i + G_i' - P_i & \mathbf{0} \\ \star & \star & \mathbf{I} \end{bmatrix} > \mathbf{0}, \quad i = 1, \ldots, N, \tag{8}$$

$$\begin{bmatrix} P_i + P_j & A_i G_j + A_j G_i + B_{2i} Z_j + B_{2j} Z_i & B_{1i} + B_{1j} \\ \star & G_i + G_j + G_i' + G_j' - (P_i + P_j) & \mathbf{0} \\ \star & \star & 2\mathbf{I} \end{bmatrix} > \mathbf{0},$$

$$i = 1, \ldots, N-1, \quad j = i+1, \ldots, N, \tag{9}$$

[‡]The symbol $\star$ means symmetric blocks in the LMIs.

DOI: 10.1002/oca

then the parameter-dependent state feedback control law $u = Z(\alpha)G(\alpha)^{-1}x$, with

$$Z(\alpha) = \sum_{i=1}^{N} \alpha_i Z_i, \quad G(\alpha) = \sum_{i=1}^{N} \alpha_i G_i, \quad \alpha \in \Lambda,$$

stabilizes system (1) with a guaranteed $\mathscr{H}_2$ performance bounded by μ for all $\alpha \in \Lambda$.

Proof
Using the technique of de Oliveira *et al.* [19] to deal with products of parameter-dependent matrices, multiply (5) by α_i, (6) and (8) by α_i^2 and sum for $i = 1, \ldots, N$. Multiply (7) and (9) by $\alpha_i \alpha_j$, and sum for $i = 1, \ldots, N-1$, $j = i+1, \ldots, N$. Summing the results one has

$$\text{trace}(W(\alpha)) < \mu^2, \tag{10}$$

$$\begin{bmatrix} W(\alpha) & C_{\text{cl}}(\alpha)G(\alpha) \\ \star & G(\alpha) + G(\alpha)' - P(\alpha) \end{bmatrix} > \mathbf{0}, \tag{11}$$

$$\begin{bmatrix} P(\alpha) & A_{\text{cl}}(\alpha)G(\alpha) & B_1(\alpha) \\ \star & G(\alpha) + G(\alpha)' - P(\alpha) & \mathbf{0} \\ \star & \star & \mathbf{I} \end{bmatrix} > \mathbf{0}, \tag{12}$$

with $A_{\text{cl}}(\alpha) = A(\alpha) + B_2(\alpha)Z(\alpha)G(\alpha)^{-1}$ and $C_{\text{cl}}(\alpha) = C_1(\alpha) + D_2(\alpha)Z(\alpha)G(\alpha)^{-1}$. Multiply (11) on the left by $[\mathbf{I} \ \ -C_{\text{cl}}(\alpha)]$ and on the right by the transpose, to obtain

$$C_{\text{cl}}(\alpha)P(\alpha)C_{\text{cl}}(\alpha)' - W(\alpha) < \mathbf{0}. \tag{13}$$

Now apply the Schur complement in (12) to obtain

$$\begin{bmatrix} P(\alpha) - B_1(\alpha)B_1(\alpha)' & A_{\text{cl}}(\alpha)G(\alpha) \\ \star & G(\alpha) + G(\alpha)' - P(\alpha) \end{bmatrix} > \mathbf{0}. \tag{14}$$

Multiply (14) on the left by $[\mathbf{I} \ \ -A_{\text{cl}}(\alpha)]$ and on the right by the transpose to obtain

$$A_{\text{cl}}(\alpha)P(\alpha)A_{\text{cl}}(\alpha)' - P(\alpha) + B_1(\alpha)B_1(\alpha)' < \mathbf{0}. \tag{15}$$

Inequalities (10), (13) and (15) guarantee (4) and, consequently, the parameter-dependent control law $u = Z(\alpha)G(\alpha)^{-1}x$ stabilizes system (1) with an $\mathscr{H}_2$ guaranteed cost bounded by μ for all $\alpha \in \Lambda_N$. □

The optimal value of μ such that the conditions in Theorem 1 hold can be obtained through the following convex optimization problem:

$$\mu^\star = \min \mu \quad \text{s.t. (5)–(9) hold.} \tag{16}$$

Note that $\mu^\star$ in (16) is suboptimal with respect to the global optimum of Lemma 1 due to the particular structures imposed to $P(\alpha)$ and $W(\alpha)$ and to the sufficient polynomial positivity test proposed. As discussed in [19], the relaxation of Theorem 1 has shown to be efficient and has a polynomial increase in the complexity with respect to the dimensions of the system. Observe also that the gain $Z(\alpha)G(\alpha)^{-1}$ (i.e. matrices Z_i and G_i, $i = 1, \ldots, N$) was obtained using a convex state feedback parametrization for (4a). Different gains, probably yielding distinct upper bounds μ, could be obtained from convex conditions based on (3) as well.

If a robust gain (parameter-independent) is desired, the following corollary can be used.

Corollary 1
If there exist symmetric matrices $P_i \in \mathbb{R}^{n \times n}$, $W_i \in \mathbb{R}^{p \times p}$, $i = 1, \ldots, N$ and matrices $G \in \mathbb{R}^{n \times n}$, $Z \in \mathbb{R}^{m \times n}$, such that the LMIs (5), (6) and (8) are feasible with $G_i = G$, $Z_i = Z$, $i = 1, \ldots, N$, then the

DOI: 10.1002/oca

robust control law $u = ZG^{-1}x$ stabilizes system (1) with an $\mathscr{H}_2$ guaranteed performance bounded by μ.

Proof
Similar to the proof of Theorem 1. □

The condition of Corollary 1, originally published in [20, Theorem 9], is used in the numerical experiments section for comparison. In what follows, the main contribution of the paper is presented, that is, LMI relaxations for the design of a static output feedback gain, using the state feedback gain matrices Z_i and G_i, $i = 1, \ldots, N$ synthesized through the conditions of Theorem 1 as a starting point.

Theorem 2
Let Z_i and G_i, $i = 1, \ldots, N$, be the solution matrices of Theorem 1. If there exist symmetric matrices $P_i \in \mathbb{R}^{n \times n}$, matrices $F_i \in \mathbb{R}^{n \times n}$, $H_i \in \mathbb{R}^{p \times p}$, $i = 1, \ldots, N$, and matrices $R \in \mathbb{R}^{m \times m}$, $L \in \mathbb{R}^{m \times q}$ such that the following LMIs are verified:

$$\operatorname{trace}(W_i) < \mu^2, \quad i = 1, \ldots, N, \tag{17}$$

$$B_{1i}' P_i B_{1i} - W_i < \mathbf{0}, \quad i = 1, \ldots, N, \tag{18}$$

$$B_{1i}' P_j B_{1i} + B_{1i}' P_i B_{1j} + B_{1j}' P_i B_{1i} - 2W_i - W_j < \mathbf{0}, \quad i = 1, \ldots, N, j \neq i, \quad j = 1, \ldots, N, \tag{19}$$

$$\begin{gathered} B_{1i}' P_j B_{1\ell} + B_{1\ell}' P_j B_{1i} + B_{1j}' P_i B_{1\ell} + B_{1\ell}' P_i B_{1j} + B_{1i}' P_\ell B_{1j} + B_{1j}' P_\ell B_{1i} - 2(W_i + W_j + W_\ell) < \mathbf{0}, \\ i = 1, \ldots, N-2, \quad j = i+1, \ldots, N-1, \quad \ell = j+1, \ldots, N, \end{gathered} \tag{20}$$

$$\begin{bmatrix} -G_i' P_i G_i & G_i' A_i' F_i + Z_i' B_{2i}' F_i & G_i' C_{1i}' H_i + Z_i' D_{2i}' H_i & G_i' C_{2i}' L' - Z_i' R' \\ \star & P_i - F_i - F_i' & \mathbf{0} & F_i' B_{2i} \\ \star & \star & \mathbf{I} - H_i - H_i' & H_i' D_{2i} \\ \star & \star & \star & -R - R' \end{bmatrix} < \mathbf{0}, \quad i = 1, \ldots, N, \tag{21}$$

$$\begin{gathered} \begin{bmatrix} \begin{array}{c} -G_i' P_i G_j \\ -G_j' P_i G_i - G_i' P_j G_i \end{array} & \begin{array}{c} G_i' A_i' F_j + G_j' A_i' F_i + G_i' A_j' F_i \\ + Z_i' B_i' F_j + Z_j' B_i' F_i + Z_i' B_j' F_i \end{array} & \begin{array}{c} G_i' C_{1i}' H_j + G_j' C_{1i}' H_i \\ + G_i' C_{1j}' H_i + Z_i' D_{2i}' H_j \\ + Z_j' D_{2i}' H_i + Z_i' D_{2j}' H_i \end{array} & \begin{array}{c} (G_i' C_{2i}' + G_i' C_{2j}' \\ + G_j' C_{2i}') L' \\ - (2Z_i' + Z_j') R' \end{array} \\ \star & \begin{array}{c} 2P_i + P_j - 2F_i - F_j \\ -2F_i' - F_j' \end{array} & \mathbf{0} & F_i' B_i + F_i' B_j + F_j' B_i \\ \star & \star & 3\mathbf{I} - 2H_i - H_j - 2H_i' - H_j' & \begin{array}{c} H_i' D_{2i} + H_i' D_{2j} \\ + H_j' D_{2i} \end{array} \\ \star & \star & \star & -3R - 3R' \end{bmatrix} < \mathbf{0}, \\ i = 1, \ldots, N, \quad j \neq i, \quad j = 1, \ldots, N, \end{gathered} \tag{22}$$

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} \\ \star & \Theta_{22} & \mathbf{0} & \Theta_{24} \\ \star & \star & \Theta_{33} & \Theta_{34} \\ \star & \star & \star & -6R - 6R' \end{bmatrix} < \mathbf{0}, \quad \begin{array}{c} i = 1, \ldots, N-2, \\ j = i+1, \ldots, N-1, \\ \ell = j+1, \ldots, N, \end{array} \tag{23}$$

with

$$\Theta_{11} = -(G_j' P_i G_\ell + G_\ell' P_i G_j + G_i' P_j G_\ell + G_\ell' P_j G_i + G_i' P_\ell G_j + G_j' P_\ell G_i),$$

$$\begin{aligned} \Theta_{12} = {} & G_j' A_i' F_\ell + G_\ell' A_i' F_j + G_i' A_j' F_\ell + G_\ell' A_j' F_i \\ & + G_i' A_\ell' F_j + G_j' A_\ell' F_i + Z_j' B_{2i}' F_\ell + Z_\ell' B_{2i}' F_j + Z_i' B_{2j}' F_\ell + Z_\ell' B_{2j}' F_i + Z_i' B_{2\ell}' F_j + Z_j' B_{2\ell}' F_i, \end{aligned}$$

DOI: 10.1002/oca

$$\Theta_{13} = G_j' C_{1i}' H_\ell + G_\ell' C_{1i}' H_j + G_i' C_{1j}' H_\ell + G_\ell' C_{1j}' H_i + G_i' C_{1\ell}' H_j + G_j' C_{1\ell}' H_i + Z_j' D_{2i}' H_\ell + Z_\ell' D_{2i}' H_j + Z_i' D_{2j}' H_\ell + Z_\ell' D_{2j}' H_i + Z_i' D_{2\ell}' H_j + Z_j' D_{2\ell}' H_i,$$

$$\Theta_{14} = (G_i' C_{2j}' + G_j' C_{2i}' + G_i' C_{2\ell}' + G_\ell' C_{2i}' + G_j' C_{2\ell}' + G_\ell' C_{2j}') L' - 2(Z_i' + Z_j' + Z_\ell') R',$$

$$\Theta_{22} = 2(P_i + P_j + P_\ell - F_i - F_i' - F_j - F_j' - F_\ell - F_\ell'),$$

$$\Theta_{24} = F_i' B_{2j} + F_j' B_{2i} + F_i' B_{2\ell} + F_\ell' B_{2i} + F_j' B_{2\ell} + F_\ell' B_{2j},$$

$$\Theta_{33} = 6\mathbf{I} - 2(H_i + H_j + H_\ell + H_i' + H_j' + H_\ell'),$$

$$\Theta_{34} = H_i' D_{2j} + H_j' D_{2i} + H_i' D_{2\ell} + H_\ell' D_{2i} + H_j' D_{2\ell} + H_\ell' D_{2j},$$

then the robust static output feedback control gain $K = R^{-1} L$ stabilizes system (1) with an $\mathscr{H}_2$ guaranteed performance bounded by μ.

Proof
Using the technique of de Oliveira *et al.* [19] to deal with triple products of parameter-dependent matrices, multiply (17) by α_i, (18) and (21) by α_i^3 and sum for $i=1,\dots,N$. Multiply (19) and (22) by $\alpha_i^2 \alpha_j$ and sum for $i=1,\dots,N$, $j \neq i$, $j=1,\dots,N$. Multiply (20) and (23) by $\alpha_i \alpha_j \alpha_k$ and sum for $i=1,\dots,N-2$, $j=i+1,\dots,N-1$, $k=j+1,\dots,N$. Adding the results yields

$$\text{trace}(W(\alpha)) < \mu^2, \tag{24}$$

$$B_1(\alpha)' P(\alpha) B_1(\alpha) - W(\alpha) < \mathbf{0}, \tag{25}$$

$$\begin{bmatrix} -G(\alpha)' P(\alpha) G(\alpha) & \begin{matrix} G(\alpha)' A(\alpha)' F(\alpha) \\ +Z(\alpha)' B_2(\alpha)' F(\alpha) \end{matrix} & \begin{matrix} G(\alpha)' C_1(\alpha)' H(\alpha) \\ +Z(\alpha)' D_2(\alpha)' H(\alpha) \end{matrix} & G(\alpha)' C_2(\alpha)' L' - Z(\alpha)' R' \\ \star & P(\alpha) - F(\alpha) - F(\alpha)' & \mathbf{0} & F(\alpha)' B_2(\alpha) \\ \star & \star & \mathbf{I} - H(\alpha) - H(\alpha)' & H(\alpha)' D_2(\alpha) \\ \star & \star & \star & -R - R' \end{bmatrix} < \mathbf{0}. \tag{26}$$

Multiply (26) on the right by $R_1(\alpha)$ and on the left by $R_1(\alpha)'$, with

$$R_1(\alpha) = \begin{bmatrix} G(\alpha)^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \star & \star & \mathbf{I} & \mathbf{0} \\ \star & \star & \star & \mathbf{I} \end{bmatrix},$$

to obtain

$$\begin{bmatrix} -P(\alpha) & \begin{matrix} A(\alpha)' F(\alpha) \\ +G(\alpha)^{-1'} Z(\alpha)' B_2(\alpha)' F(\alpha) \end{matrix} & \begin{matrix} C_1(\alpha)' H(\alpha) \\ +G(\alpha)^{-1'} Z(\alpha)' D_2(\alpha)' H(\alpha) \end{matrix} & \begin{matrix} C_2(\alpha)' L' \\ -G(\alpha)^{-1'} Z(\alpha)' R' \end{matrix} \\ \star & P(\alpha) - F(\alpha) - F(\alpha)' & \mathbf{0} & F(\alpha)' B_2(\alpha) \\ \star & \star & \mathbf{I} - H(\alpha) - H(\alpha)' & H(\alpha)' D_2(\alpha) \\ \star & \star & \star & -R - R' \end{bmatrix} < \mathbf{0}. \tag{27}$$

DOI: 10.1002/oca

Multiply (27) on the left by $R_2(\alpha)$ and on the right by $R_2(\alpha)'$, with

$$R_2(\alpha)=\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & S(\alpha)' \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix},$$

and $S(\alpha)=R^{-1}LC(\alpha)-Z(\alpha)G(\alpha)^{-1}$, to obtain

$$\begin{bmatrix} -P(\alpha) & A_{\text{cl}}(\alpha)'F(\alpha) & C_{\text{cl}}(\alpha)'H(\alpha) \\ \star & P(\alpha)-F(\alpha)-F(\alpha)' & \mathbf{0} \\ \star & \star & \mathbf{I}-H(\alpha)-H(\alpha)' \end{bmatrix}<\mathbf{0}, \tag{28}$$

with

$$A_{\text{cl}}(\alpha)=A(\alpha)+B_2(\alpha)R^{-1}LC_2(\alpha),$$

$$C_{\text{cl}}(\alpha)=C_1(\alpha)+D_2(\alpha)R^{-1}LC_2(\alpha).$$

Multiply (28) on the left by $[\mathbf{I}\ \ A_{\text{cl}}(\alpha)'\ \ C_{\text{cl}}(\alpha)']$ and on the right by the transpose, to obtain

$$A_{\text{cl}}(\alpha)'(\alpha)P(\alpha)A_{\text{cl}}(\alpha)-P(\alpha)+C_{\text{cl}}(\alpha)'C_{\text{cl}}(\alpha)<\mathbf{0}, \tag{29}$$

which, together with (24) and (25), guarantee that (3a) holds for all $\alpha\in\Lambda_N$ and, consequently, the closed-loop system $A_{\text{cl}}(\alpha)$ is robustly stable with an $\mathscr{H}_2$ guaranteed cost bounded by μ. □

The novelty of Theorem 2 is the possibility of synthesizing a robust static output feedback gain starting from a *parameter-dependent* state feedback gain. A feasible solution yields the desired robust gain and a parameter-dependent Lyapunov function that certifies simultaneously the closed-loop matrix stabilized with a parameter-dependent state feedback gain and with a robust static output feedback gain. The ability to search for such a Lyapunov function is, to the best of the authors' knowledge, being explored by the first time. Note that similar methods presented in [8, 13, 15] require a *robust* state feedback gain as the starting point.

An immediate byproduct of Theorem 2 is also the possibility of designing robust state feedback gains, as presented by the next corollary.

Corollary 2

Let Z_i and G_i, $i=1,\dots,N$, be the solution matrices of Theorem 1. If there exist symmetric matrices $P_i\in\mathbb{R}^{n\times n}$, matrices $F_i\in\mathbb{R}^{n\times n}$, $H_i\in\mathbb{R}^{p\times p}$, $i=1,\dots,N$, and matrices $R\in\mathbb{R}^{m\times m}$, $L\in\mathbb{R}^{m\times n}$ such that the LMIs (17)–(23) are verified with $C_i=I_n$, $i=1,\dots,N$, then the robust state feedback control gain $K=R^{-1}L$ stabilizes system (1) with an $\mathscr{H}_2$ guaranteed performance bounded by μ.

Proof

Follows the same steps of the proof of Theorem 2. □

If the objective is to design a robust state feedback gain, the conditions of Corollary 2 can be used as an alternative when the conditions from [20] fail.

The justification for this fact is that both approaches are only sufficient, presenting different levels of conservativeness. The particular characteristic of proposed conditions is that if no feasible solution is found in the second stage, the method can be performed again by choosing different stabilizing parameter-dependent state feedback gains. Note that both Theorem 1 and Corollary 2 only require that the matrices Z_i and G_i, $i=1,\dots,N$, must be associated with a *stabilizing* parameter-dependent state feedback gain and not necessarily with a prescribed $\mathscr{H}_2$ guaranteed cost. Other performance requirements associated with the state feedback design as the $\mathscr{H}_\infty$ norm, pole location, real positivity, larger bounds for the prescribed $\mathscr{H}_2$ guaranteed cost, or dual conditions based on (4a), could be used to generate the gains, but nothing can be said with respect to the best possible output feedback $\mathscr{H}_2$ guaranteed cost obtained by Theorem 2 or Corollary 2 in the second

DOI: 10.1002/oca

stage. For instance, one could use the conditions of Theorem 1 only considering the stabilization problem, selecting the first two blocks of LMIs (8) and (9).

4. NUMERICAL EXPERIMENTS

The numerical complexity associated with an optimization problem based on LMIs can be estimated from the number V of scalar variables and the number L of LMI rows. All the experiments have been performed in an Athlon 64 X2 6000+ (3.0 GHz), 2 GB RAM (800 MHz), Linux (Ubuntu 9.04), using SeDuMi [21] and Yalmip [22] under Matlab 7.0.1.

Example I

Consider the system (1) with $n=2$, $N=2$ and the following matrices:

$$[A_1 \quad A_2] = \begin{bmatrix} 0.4 & 0.7 & 0.9 & 0.6 \\ 0.7 & 0.4 & -0.7 & -1.3 \end{bmatrix}, \quad B_{11}=B_{12}=\begin{bmatrix} 0.7 \\ 0.6 \end{bmatrix}, \quad [B_{21} \quad B_{22}] = \begin{bmatrix} 0.5 & 0.4 \\ 2.1 & 0.2 \end{bmatrix},$$

$$C_{11}=C_{12}=[1.3 \quad 0], \quad C_{21}=C_{22}=[1 \quad 0], \quad D_{21}=0.8, \quad D_{22}=-0.9.$$

The aim is to design a robust stabilizing *state* feedback gain. Using the conditions of Corollary 1 (similar to [20, Theorem 9]) it is not possible to find a feasible solution. However, using the conditions of Theorem 1, a parameter-dependent state feedback gain $K(\alpha)=Z(\alpha)G(\alpha)^{-1}$ that stabilizes the system ($\mu=2.6594, V=21, L=26, \text{Time}=0.06\,\text{s}$) can be computed. The matrices that compose the gain are

$$[Z_1 \quad Z_2] = [-0.9948 \quad -1.4455 \quad -0.3619 \quad 1.8595],$$

$$[G_1 \quad G_2] = \begin{bmatrix} 2.8616 & -0.9005 & 3.0302 & -1.7254 \\ 0.1574 & 4.6547 & -1.7254 & 3.0528 \end{bmatrix}.$$

Using these matrices as input parameters for the conditions of Corollary 2, the following robust state feedback gain was obtained ($\mu=3.7117, V=20, L=28, \text{Time}=0.09\,\text{s}$):

$$K=[-0.4387 \quad 0.2801].$$

Figure 1 shows the eigenvalues of the close-loop system $A_{\text{cl}}(\alpha)=A(\alpha)+B_2(\alpha)K$ computed through an exhaustive grid on the space of the parameters. The maximum eigenvalue is given by $|\lambda_{\max}|=0.8984$, which proves that the synthesized robust state feedback gain does stabilize the system. This example shows that there may exist uncertain linear systems where the LMI conditions from [20] fail but the proposed approach can find a feasible solution for the design of a robust state feedback gain.

Example II

This example considers system (1) with $n=3$, $N=2$ and the following matrices:

$$[A_1 \quad A_2] = \begin{bmatrix} -0.2 & 0.3 & 0.1 & 1.2 & 0.2 & -0.9 \\ 0.3 & 0.5 & -0.4 & -0.1 & -0.7 & 1.1 \\ 0.4 & -0.5 & 0.7 & 1.2 & 0.4 & 0.1 \end{bmatrix}, \quad B_{1i}=\begin{bmatrix} 0.6 \\ 0.9 \\ 0.4 \end{bmatrix}, \quad B_{2i}=\begin{bmatrix} 0.9 \\ -1.2 \\ 0.4 \end{bmatrix},$$

$$D_{2i}=-1.3, \; C_{1i}=\begin{bmatrix} -0.6 \\ -1.0 \\ 0.1 \end{bmatrix}', \quad i=1,2, \quad C_{21}=\begin{bmatrix} 0.1 \\ -0.5 \\ 0.5 \end{bmatrix}', \quad C_{22}=\begin{bmatrix} 0.7 \\ 0.5 \\ -0.5 \end{bmatrix}'.$$

DOI: 10.1002/oca

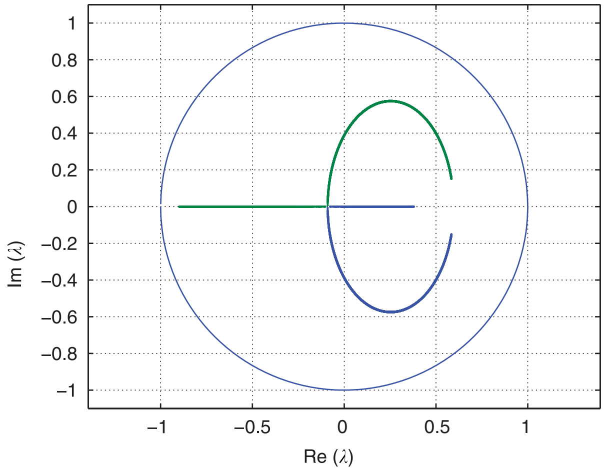

Figure 1. Eigenvalues for the closed-loop system in Example I.

The aim is to control this system by means of a robust *output* feedback gain. As the output control matrix $C_2(\alpha)$ is affected by uncertainties, the approach of [20, Section 4.2] cannot be used. Moreover, the conditions of Corollary 1 do not find a feasible solution. As an immediate consequence, the method of Mehdi *et al*. [15], which needs an initial robust state feedback gain, cannot be applied. On the other hand, Theorem 1 provides a parameter-dependent state-feedback gain, making possible the use of Theorem 2. For comparison purposes, the LMI conditions from [16, Theorem 4] were also implemented. The results are shown in Table I, with the associated numerical complexities.

As can be seen, the conditions of Theorem 2 stabilized the system with the robust output feedback gain $K=-1.0231$ and presented an $\mathscr{H}_2$ guaranteed cost less conservative than the one from [16]. The computational time of the proposed approach is slightly greater since the elapsed times of both Theorem 1 and Theorem 2 must be considered, yielding 0.26 s.

Example III

An application with practical appealing is investigated in this example. The system represents a mechanical system (borrowed from [23]) with two-mass–spring and whose graphical illustration is depicted in Figure 2. The transfer function to be considered is from the input force d applied to mass m_1 to the error signal $e=x_2$ (position of mass m_2). The following discrete-time equation is obtained using Euler's first-order approximation for the derivative and a sampling time of 0.1 s:

$$x(k+1)=Ax(k)+Bu(k),$$

$$A=\begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ \dfrac{-0.1(k_1+k_2)}{m_1} & \dfrac{0.1k_2}{m_1} & 1-\dfrac{0.1c_0}{m_1} & 0 \\ \dfrac{0.1k_2}{m_2} & \dfrac{-0.1k_2}{m_2} & 0 & 1-\dfrac{0.1c_0}{m_2} \end{bmatrix}, \quad B=\begin{bmatrix} 0 \\ 0 \\ \dfrac{0.1}{m_1} \\ 0 \end{bmatrix}.$$

The masses and the stiffness of the second spring are assumed to be constant as $m_1=2$, $m_2=1$, $k_2=0.5$. The friction forces f_1 and f_2 are associated with the viscous friction coefficient c_0. The stiffness of the first spring and the viscous friction coefficient are assumed to be uncertain and belonging to the ranges

$$1\leqslant k_1\leqslant 4, \quad 1\leqslant c_0\leqslant 4.$$

DOI: 10.1002/oca

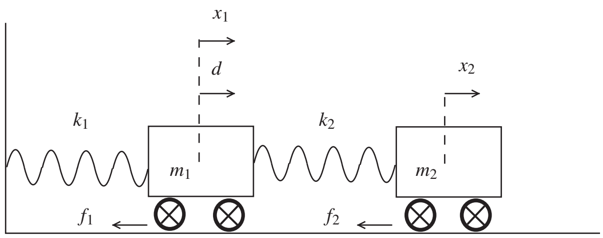

Figure 2. Mass–spring system.

Table I. Comparison in the output feedback design of Example II.

Method	μ	K	V	L	Time (s)
[16, Theorem 4]	3.0702	−1.1348	141	64	0.25
Theorem 1	1.4548	—	39	35	0.19
Theorem 2	1.5713	−1.0231	35	36	0.07

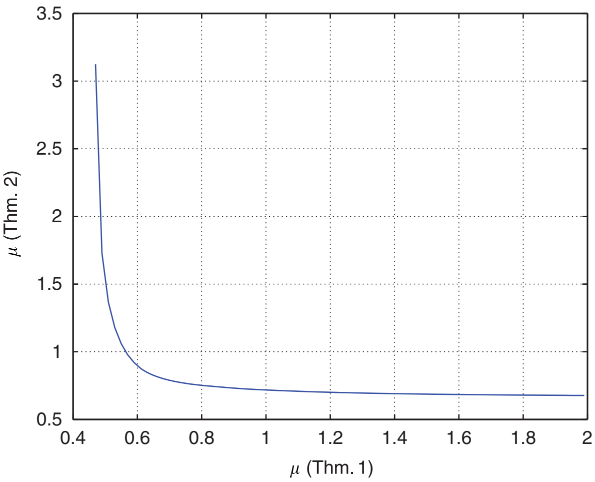

Figure 3. $\mathscr{H}_2$ guaranteed costs provided by Theorem 2 using state feedback controllers designed by Theorem 1 with $\mathscr{H}_2$ prescribed bounds given in the range $\mu \in [0.47 \;\; 2.0]$.

Evaluating the dynamic matrix at the extreme values of the parameters, one obtain a polytope of $N=4$ vertices. The other system matrices are given by:

$$C_{1i}=[0\;\;1\;\;0\;\;0],\quad B_{1i}=[0\;\;0.1\;\;0.1\;\;0]',\quad C_{2i}=\begin{bmatrix}0&0&1&0\\0&0&0&1\end{bmatrix},\quad D_{2i}=0,\quad i=1,\ldots,N.$$

The third and fourth states, which are the velocities of masses m_1 and m_2, respectively, are the only states available for feedback and the aim is to compute a static output feedback gain minimizing the $\mathscr{H}_2$ norm of the closed-loop system. As the output control matrix $C_2(\alpha)$ is not affected by uncertainties, the method from [20, Theorem 9] can be adapted to cope with the output feedback design and it is used in the numerical comparisons. With respect to the proposed conditions, the conditions of Theorem 2 are tested using the relaxations of Theorem 1 with $\mathscr{H}_2$ prescribed bounds given in the range $\mu \in [0.47 \;\; 2]$. The results, depicted in Figure 3, show that the more relaxed the $\mathscr{H}_2$ upper bound used in Theorem 1, the less conservative the $\mathscr{H}_2$ guaranteed cost obtained by Theorem 2.

DOI: 10.1002/oca

Table II. Comparison in the design of Example III.

Method	μ	V	L	Time (s)
[16, Theorem 4]	—	803	268	6.59
[20, Theorem 9]	—	59	60	0.12
Theorem 1	0.47	124	144	0.09
Theorem 2	3.1251	112	220	0.57
Theorem 1	2.00	124	144	0.11
Theorem 2	0.6770	112	220	0.34

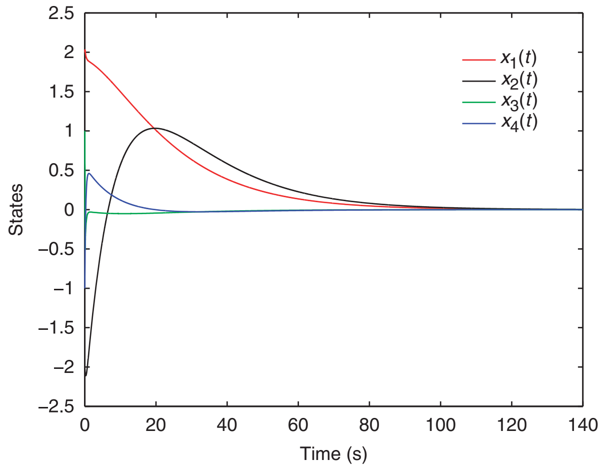

Figure 4. Trajectories of states for original continuous-time system investigated in Example III using the gain (30).

For numerical complexity comparison purposes, the results obtained by the methods [16, Theorem 4], [20, Theorem 9] and Theorem 2 using two initial state feedback controllers (for $\mu=0.47$ and $\mu=2.00$ in Theorem 1) are shown in Table II.

As can be seen, the other methods available in the literature are not capable to provide a stabilizing controller. On the other hand, the proposed approach stabilizes the system and presents flexibility in the search for less conservative $\mathscr{H}_2$ guaranteed costs. For illustration, the designed output feedback gain that yields an upper bound for the $\mathscr{H}_2$ normal equal to $\mu=0.6770$ is given by

$$K=[-18.5369 \;\; 5.3400]. \tag{30}$$

Finally, the original continuous-time system was simulated with the gain (30) and initial condition $x(0)=[2 \;\; -2 \;\; 1 \;\; -1]'$. The results are shown in Figure 4. The trajectories of the states converging to zero also guarantee that the Euler discretization (sampling time $T_s=0.1\,\text{s}$) employed in this example was reliable.

Example IV

The aim of this example is to evaluate the level of conservativeness of the proposed approach when compared with the methods [15, 16] by means of a statistical analysis for the robust output feedback stabilization problem. The comparison is similar to the one presented in [15], where uncertain linear systems of different dimensions were considered.

A database of 100 polytopes[§] $(A(\alpha), B_2(\alpha), C_2(\alpha))$ that can be stabilized by static output feedback for the dimensions

$$n=\{3,6\}, \quad m=\{1,2,3\}, \quad p=\{1,2,3\}, \quad N=\{3,4\}, \tag{31}$$

[§]Available for download in `www.dt.fee.unicamp.br/~ricfow/robust.htm`.

DOI: 10.1002/oca

Table III. Positive evaluations and computational times (in seconds) presented by the methods [15, 16] and Theorem 2 in the statistical analysis of Example IV for uncertain systems with the dimensions given in (31).

			$N=3$						$N=4$					
n	m	p	Dong and Yang [16]	Time	Mehdi *et al.* [15]	Time	T2	Time	Dong and Yang [16]	Time	Mehdi *et al.* [15]	Time	T2	Time
		1	4	12.4	75	5.3	92	7.4	0	35.0	69	5.7	96	10.3
	1	2	0	10.9	61	5.0	90	7.5	0	31.3	58	5.3	92	10.6
		3	0	10.7	64	4.9	89	7.4	0	28.3	53	4.9	92	10.4
		1	4	12.0	62	6.6	77	8.4	0	35.3	56	6.6	74	12.4
3	2	2	0	10.8	52	6.3	67	8.4	0	30.7	47	5.7	75	12.3
		3	0	11.0	52	5.8	74	8.4	0	29.0	36	5.3	64	12.6
		1	2	12.3	48	7.1	44	9.4	1	35.5	51	7.9	53	14.8
	3	2	0	11.1	43	7.0	54	9.4	0	30.4	39	7.1	52	14.7
		3	0	10.7	36	6.7	49	9.5	0	27.3	26	6.2	39	14.4
		1	1	275.4	62	11.2	90	24.8	0	1587.0	53	12.8	87	49.7
	1	2	0	246.6	49	10.3	79	25.4	0	1353.8	36	10.3	79	50.9
		3	0	234.4	44	9.4	76	25.0	0	1231.9	37	10.0	76	47.8
		1	2	279.1	47	13.4	67	29.3	0	1575.9	34	14.2	62	60.6
6	2	2	0	251.1	30	12.2	59	29.0	0	1349.9	26	11.6	49	7.4
		3	0	238.8	11	9.0	46	30.2	0	1225.7	6	8.1	33	51.3
		1	1	277.8	22	17.7	47	33.3	0	1570.6	24	19.1	52	69.4
	3	2	0	254.0	15	13.9	39	33.7	0	1351.4	10	11.0	25	63.0
		3	0	235.5	10	10.7	31	32.4	0	1213.3	4	7.2	9	46.6

is generated. Similar to [15], the system matrices are randomly generated such that $A(\alpha)$ is unstable and there exists a gain K such that $A(\alpha)+B_2(\alpha)KC_2(\alpha)$ is stable, using an exhaustive grid procedure. The methods are applied and the number of positive evaluations and the computational times are stored. The details of how the methods are applied are briefly summarized in the sequel:

- *Proposed*: First, Theorem 1 is applied considering only the first two blocks of LMIs (8) and (9). Then the resulting gains are used as input conditions of Theorem 2 considering the LMIs (21)–(23) removing the third columns and the third rows. The computational time is the total time to solve the LMIs in both stages.
- *Mehdi et al.* [15]: First, an initial stabilizing robust state feedback controller is obtained using [24, Theorem 3]. Then, the stabilizing gain is used as initial condition in [15, Theorem 4.1]. The computational time is the total time to solve the LMIs in both stages.
- *Dong and Yang* [16]: The similarity transformations required by Dong and Yang [16, Theorem 1] are obtained following the suggestion given in [16, Equation (4)], i.e. $T_i = [C'_{2i}(C_{2i}C'_{2i})^{-1} \;\; C_{2i}^{\perp}]$.

The results are shown in Table III. The approach of Dong and Yang [16] showed to be the most conservative and more expensive from a computational point of view. With the increase in the dimensions, the proposed approach found approximately the double of feasible solutions when compared with [15]. With respect to the computational burden, however, the proposed approach is always costlier than [15].

5. CONCLUSION

A new strategy to design $\mathscr{H}_2$ robust static output feedback controllers for discrete-time linear systems with time-invariant parameters was proposed. The novelty relies on the fact that first, a

DOI: 10.1002/oca

parameter-dependent state feedback controller is designed and then, the resulting gains are used as an input condition for the design of the robust output feedback gain. Numerical examples, including a physical system shows different situations where the proposed approach outperforms the other methods available in the literature. A statistical analysis also indicates that, in general, the proposed approach is less conservative than the other methods.

ACKNOWLEDGEMENTS

The authors wish to thank Olivier Bachelier, from LAII-ESIP, Université de Poitiers, for the helpful discussions on the subject of this paper.

REFERENCES

1. Syrmos VL, Abdallah CT, Dorato P, Grigoriadis K. Static output feedback—a survey. *Automatica* 1997; **33**(2): 125–137.
2. Anderson BDO, Bose NK, Jury EI. Output feedback stabilization and related problems—solution via decision methods. *IEEE Transactions on Automatic Control* 1975; **20**(1):53–66.
3. Geromel JC, Peres PLD, Souza SR. Convex analysis of output feedback control problems: robust stability and performance. *IEEE Transactions on Automatic Control* 1996; **41**(7):997–1003.
4. El Ghaoui L, Oustry F, Aitrami M. A cone complementarity linearization algorithm for static output feedback and related problems. *IEEE Transactions on Automatic Control* 1997; **42**(8):1171–1176.
5. Cao YY, Lam J, Sun YX. Static output feedback stabilization: an ILMI approach. *Automatica* 1998; **34**(12): 1641–1645.
6. Geromel JC, de Souza CC, Skelton RE. Static output feedback controllers: stability and convexity. *IEEE Transactions on Automatic Control* 1998; **43**(1):120–125.
7. Crusius CAR, Trofino A. Sufficient LMI conditions for output feedback control problems. *IEEE Transactions on Automatic Control* 1999; **44**(5):1053–1057.
8. Peaucelle D, Arzelier D. An efficient numerical solution for $\mathscr{H}_2$ static output feedback synthesis. *Proceedings of the 2001 European Control Conference*, Porto, Portugal, September 2001.
9. Garcia G, Pradin B, Zeng F. Stabilization of discrete time linear systems by static output feedback. *IEEE Transactions on Automatic Control* 2001; **46**(12):1954–1958.
10. Bernussou J, Peres PLD, Geromel JC. A linear programming oriented procedure for quadratic stabilization of uncertain systems. *Systems and Control Letters* 1989; **13**(1):65–72.
11. Camino JF, de Oliveira MC, Skelton RE. Convexifying linear matrix inequality methods for integrating structure and control design. *Journal of Structural Engineering* (ASCE) 2003; **129**(7):978–988.
12. Henrion D, Lasserre JB. Convergent relaxations of polynomial matrix inequalities and static output feedback. *IEEE Transactions on Automatic Control* 2006; **51**(2):192–202.
13. Arzelier D, Peaucelle D, Salhi S. Robust static output feedback stabilization for polytopic uncertain systems: improving the guaranteed performance bound. *Proceedings of the 4th IFAC Symposium on Robust Control Design* (*ROCOND 2003*), Milan, Italy, June 2003.
14. Shaked U. An LPD approach to robust $\mathscr{H}_2$ and $\mathscr{H}_\infty$ static output-feedback design. *IEEE Transactions on Automatic Control* 2003; **48**(5):866–872.
15. Mehdi D, Boukas EK, Bachelier O. Static output feedback design for uncertain linear discrete time systems. *IMA Journal of Mathematical Control and Information* 2004; **21**(1):1–13.
16. Dong J, Yang G-H. Static output feedback control synthesis for linear systems with time-invariant parametric uncertainties. *IEEE Transactions on Automatic Control* 2007; **52**(10):1930–1936.
17. Boyd SP, Barratt CH. *Linear Control Design*: *Limits of Performance*. Prentice-Hall: Englewood Cliffs, NJ, U.S.A., 1991.
18. Zhou K, Doyle JC, Glover K. *Robust and Optimal Control*. Prentice-Hall: Upper Saddle River, NJ, U.S.A., 1996.
19. de Oliveira PJ, Oliveira RCLF, Leite VJS, Montagner VF, Peres PLD. $\mathscr{H}_2$ guaranteed cost computation by means of parameter dependent Lyapunov functions. *International Journal of Systems Science* 2004; **35**(5):305–315.
20. de Oliveira MC, Geromel JC, Bernussou J. Extended $\mathscr{H}_2$ and $\mathscr{H}_\infty$ characterization and controller parametrizations for discrete-time systems. *International Journal of Control* 2002; **75**(9):666–679.
21. Sturm JF. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software* 1999; **11–12**:625–653. Available from: http://sedumi.mcmaster.ca/.
22. Löfberg J. YALMIP: A toolbox for modeling and optimization in MATLAB. *Proceedings of the 2004 IEEE International Symposium on Computer Aided Control Systems Design*, Taipei, Taiwan, September 2004; 284–289. Available from: http://control.ee.ethz.ch/∼joloef/yalmip.php.
23. Iwasaki T. Robust performance analysis for systems with structured uncertainty. *International Journal of Robust and Nonlinear Control* 1996; **6**:85–99.
24. de Oliveira MC, Bernussou J, Geromel JC. A new discrete-time robust stability condition. *Systems and Control Letters* 1999; **37**(4):261–265.

DOI: 10.1002/oca