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Accretion of nonminimally coupled scalar fields into black holes

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By using a quasistationary approach, we consider the mass evolution of Schwarzschild black holes in the presence of a nonminimally coupled cosmological scalar field. The mass evolution equation is analytically solved for generic coupling, revealing a qualitatively distinct behavior from the minimal coupling case. In particular, for black hole masses smaller than a certain critical value, the accretion of the scalar field can lead to mass decreasing even if no phantom energy is involved. The physical validity of the adopted quasistationary approach and some implications of our result for the evolution of primordial and astrophysical black holes are discussed. More precisely, we argue that black hole observational data could be used to place constraints on the nonminimally coupled energy content of the Universe.

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I. INTRODUCTION

The accretion of matter is one of the most studied physical processes involving black holes. Assuming the validity of certain energy conditions for the accreting matter, the black hole mass will never decrease. In fact, if the null energy condition holds, no classical process can lead to mass decreasing for black holes [1]. The situation changes completely if quantum processes are allowed: a black hole can, in fact, shrink due to the emission of Hawking radiation [2]. Such processes are particularly relevant, for instance, to primordial black holes (PBHs) [3]. One of the most striking features of PBHs is that they could indeed evaporate completely due to the emission of Hawking radiation. It is known, in particular, that a PBH with mass smaller than the so called Hawking mass $M_{\rm H} = 10^{15}$ g should have already evaporated by now. PBHs with masses close to that limit are specially relevant because their emitted Hawking radiation might, in principle, produce observable effects in the present-day Universe [4].

The interest in these problems has increased considerably in the last years due to the many dark energy phenomenological models that have been proposed to describe the recent accelerated expansion of the Universe [5]. Such models [6] typically involve a scalar field pervading all the Universe that could, in principle, be absorbed by any black hole, implying consequently new channels for black hole mass accretion [7]. It is interesting to notice that the study of black holes growth in the presence of scalar fields was initiated before [8] the discovery of the recent acceleration of the Universe and, thus, before the proposal of any dark energy model.

The mass evolution of any black hole is governed by two competing processes. The first one is Hawking radiation, which decreases the black hole mass due to the emission of a thermal radiation. The other one, which tends to increase the black hole mass, is the accretion of the surrounding available matter and energy. The survival or not of a PBH, for instance, was believed to depend on the detailed balance of these processes. The unexpected possibility that black hole masses could effectively decrease due to the accretion of exotic (phantom) dark energy [9] was received with great interest because, mainly, it could alter qualitatively the evolution of any black hole, implying, occasionally, observational consequences for both astrophysical and primordial black holes. Since phantom dark energy violates the usual energy conditions, there is no contradiction between these results and the classical theory of black holes. Nevertheless, one should keep in mind that the physical viability of models involving phantom energy has been constantly challenged by their severe inherent classical and quantum instabilities [10].

In this paper, we study the mass evolution of Schwarzschild black holes in the presence of a nonminimally coupled scalar field. A quasistationary approach is introduced and the mass evolution equation is analytically solved for generic coupling. Our main conclusion is that, for black hole initial masses smaller than a certain critical value, one could indeed have mass decreasing even in the absence of the Hawking evaporation mechanism and without any component of phantom energy in the model. This is a more robust scenario for mass decreasing of black holes due to the accretion of exotic matter since it is not plagued by the phantom energy instabilities. Moreover, one could have, in principle, mass decreasing for considerably larger black holes than the minimally coupled case, with possible implications for primordial and astrophysical black holes, which could be explored in order to place observation constraints on the nonminimally coupled energy content of the Universe.

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II. NONMINIMALLY COUPLED SCALAR FIELDS AROUND BLACK HOLES

We are concerned here with a scalar field ϕ governed by the action

$$S = \frac{1}{2}\int d^4x\sqrt{-g}[F(\phi)R - \partial_a\phi\partial^a\phi - 2V(\phi)], \quad (1)$$

surrounding a Schwarzschild black hole. Nonminimally coupled cosmological models of the type (1) have been intensively used in modern cosmology [11]. Models for which it is indeed possible to reach $F(\phi) = 0$ are known to be plagued with singularities [12]. The hypersurface $F(\phi) = 0$ marks, in a sense, the boundary between standard $[F(\phi) > 0]$ and phantomlike $[F(\phi) < 0]$ behavior for the scalar field ϕ [13]. We are mainly interested here in models such that $F(\phi) > 0$ everywhere since, in such cases, phantomlike behavior is excluded by construction.

Since Schwarzschild spacetime is Ricci-flat, the equation of motion for ϕ obtained from (1) reads simply

$$\Box\phi = V'(\phi), \quad (2)$$

and the associated energy momentum tensor is given by

$$T_{ab} = \partial_a\phi\partial_b\phi - \frac{g_{ab}}{2}(\partial_c\phi\partial^c\phi + 2V) + \nabla_a\nabla_b F - g_{ab}\Box F. \quad (3)$$

Note that, due to the Ricc flatness of Schwarzschild spacetime, we have $\nabla_b T_a{}^b = 0$. By adopting the usual Schwarzschild coordinates, the spherically symmetrical version of Eq. (2) will be given by

$$-\frac{\partial^2\phi}{\partial t^2} + \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\frac{\partial}{\partial r}\left[r^2\left(1 - \frac{2M}{r}\right)\frac{\partial\phi}{\partial r}\right] = \left(1 - \frac{2M}{r}\right)V'(\phi). \quad (4)$$

The standard formulation of the stationary Bondi accretion process [14] for this problem consists in considering solutions of (4) with the following boundary condition:

$$\lim_{r\to\infty}\phi(t, r) = \phi_c(t), \quad (5)$$

where $\phi_c(t)$ corresponds to the cosmological homogeneous and isotropic solution of the model (1), with cosmological and Schwarzschild time coordinates identified. Since no back reaction of the scalar field is taken into account, our approach requires that the energy content of the scalar field must remain bounded and small around the black hole. Once we have a solution $\phi(t, r)$ of (4) with bounded energy and obeying the boundary condition (5), we assume that its energy flux on the black hole horizon is completely absorbed by the black hole, implying that

$$\frac{dM}{dt} = \oint_{r=2M} r^2 T_t{}^r d\Omega. \quad (6)$$

This problem was solved, for $F(\phi) = 1$ and $V(\phi) = 0$, in [15]. In the Eddington-Finkelstein coordinates (v, r), with $v = t + r + 2M\log(r/2M - 1)$ corresponding to incoming light geodesics, the pertinent solution corresponds to the stationary configuration

$$\phi(v, r) = \beta + \gamma\left(v - r + 2M\log\frac{2M}{r}\right), \quad (7)$$

with β and γ constant. We do not expect to have stationary solutions like this for the generic model (1). In fact, stationary solutions are possible only for actions that are invariant under shifts $\phi \to \phi + \lambda$ (see [16]). We can, however, adopt a quasistationary approach based on the observation [17] that, for slowly varying cosmological solutions $\phi_c(t)$, the "delayed" field configuration given by

$$\phi(v, r) = \phi_c\left(v - r + 2M\log\frac{2M}{r}\right) \quad (8)$$

is an approximated solution of (4) for certain potentials $V(\phi)$. The validity of this approximation will ensure, of course, the validity of our quasistationary approach. By substituting (8) in (4) one gets

$$\left(1 + \frac{2M}{r} + \left(\frac{2M}{r}\right)^2 + \left(\frac{2M}{r}\right)^3\right)\ddot{\phi}_c + V'(\phi_c) = 0, \quad (9)$$

with the dots standing for the derivative with respect to t. Hence, our approximation is valid if $\ddot{\phi}_c \approx 0$ and $V'(\phi_c) \approx 0$. Because of the typical cosmological time scales, the assumption of a quasistationary ($\ddot{\phi}_c \approx 0$) evolution around the black hole is not, in fact, too restrictive. The same is true for the assumption $V'(\phi_c) \approx 0$, but the argument is more involved. Assuming a small variation of ϕ_c, the potential can be linearized as $V(\phi_c) = \mu\phi_c$, since the constant factor is irrelevant here. In this case, Eq. (4) will be a linear equation, and it is possible to find a stationary solution obeying the Bondi boundary condition (5). The approximation will be valid provided ϕ_c is small and r is kept smaller than the cosmological horizon scale (see [7] for the details). It is interesting to notice that the explicit examples of failure of the approximation (8) presented in [17] correspond clearly to situations where one cannot ensure $\ddot{\phi}_c \approx 0$ or $V'(\phi_c) \approx 0$.

For the solution (8), one has

$$T_t^r = \left(\frac{2M}{r}\right)^2\left((1 + F'')\dot{\phi}_c^2 + F'\ddot{\phi}_c - \frac{F'}{4M}\dot{\phi}_c\right). \quad (10)$$

Also from (8), we see that, on the black hole horizon, the field ϕ assumes the value of ϕ_c, propagated along an incoming light geodesic, but arriving with a certain "delay"[17]. Our quasistationary analysis neglects also such delay and, hence, in the quasistationary approximation

$$\phi_c(t) \approx \phi_\infty + \dot{\phi}_\infty(t - t_0), \quad (11)$$

with ϕ_∞ and $\dot{\phi}_\infty$ constants, we have

$$\dot{M} = 16\pi M^2(1 + F'')\dot{\phi}_\infty^2 - 4\pi MF'\dot{\phi}_\infty. \quad (12)$$

For the minimal coupling case, $F(\phi) = 1$ and (12) reduce to the usual scalar field accretion rate [15]. It is clear, however, that for the nonminimally coupled case one could have, in principle, $\dot{M} < 0$ even in the absence of phantom modes. The rate (12) corresponds only to the accretion of the scalar field. The complete mass evolution equation is obtained by adding to the right-hand side a term $\propto M^{-2}$ corresponding to the Hawking radiation. As we will see in the next section, the fact that the two accretion terms in (12) have different signs and different powers of M implies the existence of a critical mass $M_{\rm cr}$ delimiting the mass increasing and decreasing accretion regimes.

We finish this section by noticing that the possibility of negative energy fluxes for nonminimally coupled scalar fields and their implications for a mass decreasing process involving black holes has been already considered previously in another context, namely, in the investigation of the generalized second law of thermodynamics [18].

III. MASS EVOLUTION

For a generic coupling function $F(\phi)$, the complete mass evolution equation has the general form

$$\dot{M} = f(t)M^2 - g(t)M - \frac{\alpha}{M^2}, \tag{13}$$

where $f(t)$ and $g(t)$ are smooth functions and α is a characteristic constant for Hawking radiation. Let us consider, initially, only the accretion process ($\alpha = 0$). By introducing $M(t) = G(t)P(t)$, with

$$G(t) = e^{-\int_{t_0}^{t} g(s)ds}, \tag{14}$$

we obtain a separable equation for $P(t)$, which can be easily solved, leading to the following solution for (13) with $\alpha = 0$:

$$M(t) = \frac{M_0 G(t)}{1 - M_0 H(t)}, \tag{15}$$

where $M(t_0) = M_0$ and

$$H(t) = \int_{t_0}^{t} f(s)G(s)ds. \tag{16}$$

Typically, if the denominator of (15) does not vanish, the mass $M(t)$ decreases according to (14) for positive $g(t)$. Mass increasing solutions appear when the denominator vanishes. For positive and well behaved $f(t)$ and $g(t)$, the function $H(t)$ will be monotonically increasing and bounded by $H_\infty = \lim_{t\to\infty} H(t)$, leading to a critical mass $M_{\rm cr} = H_\infty^{-1}$. Any black hole with initial mass M_0 such that $0 < M_0 < M_{\rm cr}$, even in the absence of Hawking radiation, will disappear due to the accretion of the scalar field, but such a process typically will take an infinite amount of time. On the other hand, those black holes with initial masses $M_0 > M_{\rm cr}$ will grow by accreting the scalar field. In fact, in this case, the denominator of (15) vanishes for $t = t_{\rm cr}$, with $H(t_{\rm cr}) = M_0^{-1}$, implying that the black hole grows up to infinite mass in a finite time. The larger the black hole initial mass M_0, the shorter $t_{\rm cr}$ is. In contrast to the $0 < M_0 < M_{\rm cr}$ case, such behavior for $M_0 > M_{\rm cr}$ is similar to that observed for the minimally coupled case $F = 1$. The qualitative evolution for the case $M_0 = M_{\rm cr}$ will depend on the details of the functions $f(t)$ and $g(t)$.

For situations with large $M_{\rm cr}$, the inclusion of Hawking radiation will alter qualitatively only the final instants of the mass decreasing process. In such a case, for $M_0 < M_{\rm cr}$, the black hole also disappears, but now in a finite time, since Hawking radiation dominates the process for $M(t) \ll 1$. In fact, for $M > M_{\rm cr}$, the Hawking radiation term can be neglected and the dynamics are essentially that described by (15). Let us now consider some explicit examples of the coupling function $F(\phi)$ in order to elucidate these points.

A. $F(\phi) = 1 + \xi\phi$

In this linear coupling case, Eq. (13) is autonomous, with $f(t) = 16\pi\dot{\phi}_\infty^2$ and $g(t) = 4\pi\xi\dot{\phi}_\infty$, and can be integrated by quadrature for any value of α. We do not need, however, the exact solution here. We assume ξ and ϕ to both be positive in order to avoid possible singularities [12] and, without loss of generality, $t_0 = 0$. The functions $G(t)$ and $H(t)$ are in this case

$$G(t) = e^{-4\pi\xi\dot{\phi}_\infty t} \tag{17}$$

and

$$H(t) = \frac{4\dot{\phi}_\infty}{\xi}(1 - G(t)). \tag{18}$$

For positive $\dot{\phi}_\infty$, we have

$$M_{\rm cr} = \frac{\xi}{4}\dot{\phi}_\infty^{-1}, \tag{19}$$

and

$$t_{\rm cr} = \frac{1}{4\pi\xi\dot{\phi}_\infty}\log\frac{M_0}{M_0 - M_{\rm cr}}. \tag{20}$$

Notice that, for typical cosmological situations, $\dot{\phi}_\infty$ is small, implying large values of $M_{\rm cr}$ for ξ of the order of unity (in Planck units). In these cases, the Hawking radiation is important only in the final instants of the mass decreasing phase.

B. $F(\phi) = 1 + \xi\phi^2$

We assume $\xi > 0$. We have $f(t) = 16\pi(1 + 2\xi)\dot{\phi}_\infty^2$ and $g(t) = 8\pi\xi(\phi_\infty\dot{\phi}_\infty + \dot{\phi}_\infty^2 t)$ in this case. The pertinent functions are, for $t_0 = 0$,

$$G(t) = e^{-4\pi\xi(2\phi_\infty\dot{\phi}_\infty t + \dot{\phi}_\infty^2 t^2)} \tag{21}$$

and

$$H(t) = 16\pi(1+2\xi)\dot{\phi}_\infty^2 \int_0^t e^{-4\pi\xi(2\phi_\infty\dot{\phi}_\infty s + \dot{\phi}_\infty^2 s^2)} ds. \quad (22)$$

The critical mass is given by $M_{\rm cr} = H_\infty^{-1}$, with

$$H_\infty = 4\pi \frac{1+2\xi}{\sqrt{\xi}} |\dot{\phi}_\infty| e^{4\pi\xi\phi_\infty^2}[1 - \sigma {\rm erf}(2\sqrt{\pi\xi}\phi_\infty)], \quad (23)$$

where $\sigma = {\rm sgn}\dot{\phi}_\infty$ and ${\rm erf}(x)$ is the error function [19]. For the typical cosmological situations we have that ϕ_∞ is very small, leading to

$$M_{\rm cr} \approx \frac{\sqrt{\xi}}{4\pi(1+2\xi)} |\dot{\phi}_\infty|^{-1}. \quad (24)$$

Notice that, as in the previous case, $M_{\rm cr} \propto \dot{\phi}_\infty^{-1}$.

C. $F(\phi) = e^{\xi\phi}$

In this case, we have $f(t) = 16\pi(1 + \xi^2 e^{\xi(\phi_\infty + \dot{\phi}_\infty t)})\dot{\phi}_\infty^2$ and $g(t) = 4\pi\xi\dot{\phi}_\infty e^{\xi(\phi_\infty+\dot{\phi}_\infty t)}$, leading, for $t_0 = 0$, to

$$G(t) = \exp(-4\pi e^{\xi\phi_\infty}(e^{\xi\dot{\phi}_\infty t} - 1)) \quad (25)$$

and

$$H(t) = 16\pi\dot{\phi}_\infty^2 \int_0^t (1 + \xi^2 e^{\xi(\phi_\infty + \dot{\phi}_\infty s)}) G(s) ds. \quad (26)$$

The critical mass is given by

$$M_{\rm cr}^{-1} = \frac{16\pi\dot{\phi}_\infty}{\xi}\left[\frac{\xi^2}{4\pi} + \exp(4\pi e^{\xi\phi_\infty})\Gamma(0, 4\pi e^{\xi\phi_\infty})\right], \quad (27)$$

where $\Gamma(z, x)$ is the incomplete gamma function [19]. For small ϕ_∞, we have

$$M_{\rm cr} \approx \frac{\xi}{a + 4\xi^2}\dot{\phi}_\infty^{-1}, \quad (28)$$

where a is a numerical constant of the order of unity, namely, $a = 16\pi e^{4\pi}\Gamma(0, 4\pi) \approx 3.72$. Again, we observe the same behavior $M_{\rm cr} \propto \dot{\phi}_\infty^{-1}$.

D. Radiation era with $F(\phi) = 1 + \xi\phi$

The previous examples involve only the nonminimally scalar field in the quasistationary approximation. This is not enough, for instance, to describe PBHs, since they were created in the primordial Universe and have existed for eras where dark energy was not the gravitationally dominant content of the Universe. In the radiation dominated era, in particular, the Universe was filled and dominated by ultra-relativistic matter whose energy density is described in Planck units by

$$\varepsilon_\gamma = \frac{3}{32\pi t^2}. \quad (29)$$

Such an energy density has also been available to be accreted by the black hole and should be incorporated in our analysis. The case of linear coupling $F(\phi) = 1 + \xi\phi$ in the presence of radiation with energy density (29) corresponds to the choices $f(t) = 16\pi\dot{\phi}_\infty^2 + (3/2)t^{-2}$ and $g(t) = 4\pi\xi\dot{\phi}_\infty$. The $G(t)$ and $H(t)$ functions in this case are

$$G(t) = e^{-4\pi\xi\dot{\phi}_\infty(t-t_0)} \quad (30)$$

and

$$H(t) = \frac{4\dot{\phi}_\infty}{\xi}(1 - G(t)) + \frac{3}{2}\int_{t_0}^t s^{-2} e^{-4\pi\xi\dot{\phi}_\infty(s-t_0)} ds, \quad (31)$$

leading to

$$H_\infty = \frac{4\dot{\phi}_\infty}{\xi}\left(1 + \frac{3\xi e^\beta}{8\dot{\phi}_\infty t_0}\beta\Gamma(-1,\beta)\right), \quad (32)$$

with $\beta = 4\pi\xi\dot{\phi}_\infty t_0$. Since

$$\lim_{x\to 0} x\Gamma(-1, x) = 1, \quad (33)$$

we have in the present case

$$M_{\rm cr} = H_\infty^{-1} \approx \frac{\xi}{4\dot{\phi}_\infty}\left(1 + \frac{3\xi}{8\dot{\phi}_\infty t_0}\right)^{-1}, \quad (34)$$

if β is small.

IV. DISCUSSION

If we assume that $\dot{\phi}_\infty^2$ is of the same order as the critical density of the Universe today ($\rho_0 \approx 10^{-29}$ g/cm^3), we have $M_{\rm cr} \approx 10^{56}$ g for coupling constants ξ of the order of unity (in Planck units) in the first three cases considered in the last section, allowing all the black holes in the Universe to be in the shrinking phase today. In fact, even the galactic supermassive black holes with $M \approx 10^6 M_\odot \approx 10^{39}$ g are far below such a limit. These black holes would be shrinking today according to (15). The exact characteristic decaying time will depend on the particular coupling function. For the case of the linear coupling, the characteristic time is, according to (17), 10^{17} s, similar to the Universe's age. Notice that all the other coupling functions considered in the last section lead, typically, to faster decreasing mass regimes.

The fact that there are likely many black holes around us might be used to constrain the nonminimally coupled energy content of the Universe during the cosmological history. Let us consider, for simplicity, the last example of the previous section: the linear coupling case during the radiation dominated era. Suppose that the dark energy content of the Universe has changed slightly after, say, $t_0 = 1$ s. In this case, $\dot{\phi}_\infty t_0 \approx 10^{-18}$ in Planck units, justifying taking $\beta \approx 0$ in (32) and leading to $M_{\rm cr} \approx 10^{38}$ g for a coupling constant ξ of the order of unity. Thus, only PBHs with mass greater than 10^{38} g would escape from the shrink phase. Notice that this mass is extremely large if compared with the usual Hawking mass $M_{\rm H} = 10^{15}$ g. Observational constraints on the PBH mass cutoff [4] could be used, in principle, to establish constraints on the nonminimal coupling parameter ξ, although the details depend on the coupling function $F(\phi)$. If we take $t_0 = 10^{11}$ s, corresponding to the radiation-matter equality era, we will have $\dot{\phi}_\infty t_0 \approx 10^{-7}$, leading to $M_{\rm cr} \approx 10^{49}$ g. This is, again, a huge mass and implies that virtually all black holes present at the end of the radiation era have existed during all the matter dominated era in a shrinking regime. They should have lost two-thirds of their mass by now, suggesting that observational data about supermassive black holes could also be used to constrain the nonminimally coupled energy content of the Universe.

We finish by noticing two points. First, one knows that it is not expected, in general, to have constant values for ϕ_∞ and $\dot{\phi}_\infty$ along the cosmological history. Equation (13) accommodates also situations where ϕ_∞ and $\dot{\phi}_\infty$ are functions of t. However, we should keep in mind that our formalism is based on the assumption of a quasistationary evolution, requiring $\ddot{\phi}_c(t) \approx 0$ in order to work properly. One needs to take backreaction into account in order to treat nonstationary situations (see, for instance, [20] for a recent discussion).

The second point is related to the hypothesis that ϕ is a field test around a Schwarzschild black hole. This is a good approximation provided that the energy content of the scalar field (dark energy) is negligible when compared with the black hole physics scale. For the much larger cosmological scale, on the other hand, the scalar field is indeed the dominant energy content, being solely responsible for the accelerated expansion of the Universe, usually described by a quasi–de Sitter solution. In our Universe, these two scales are very different. Since the dark energy content is so small, in order to probe the quasi–de Sitter properties of the spacetime one needs to consider length scales of the same order as the Hubble radius. It is perfectly possible, in particular, to apply condition (9) in a region far from the black hole (large r), but still far from the cosmological horizon. Furthermore, provided that the effective cosmological constant of the accelerated expansion is small, the dynamics near the black hole horizon are essentially the same as the Schwarzschild case, implying that (12) is still valid. From a theoretical point of view, however, it is certainly interesting to consider the problem of accretion onto Schwarzschild–de Sitter black holes as is done, for instance, in [21] for the case of perfect fluids and minimally coupled fields. We already know, however, that our present analysis should arise naturally in the limit of small Λ. These points are now under investigation.

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